Absolute Anabelian Cuspidalizations of Proper Hyperbolic Curves

By

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Abstract

In this paper, we develop the theory of *"cuspidalizations"* of the étale fundamental group of a proper hyperbolic curve over a finite or nonarchimedean mixed-characteristic local field. The ultimate goal of this theory is the *group-theoretic reconstruction* of the étale fundamental group of an arbitrary open subscheme of the curve from the étale fundamental group of the full proper curve. We then apply this theory to show that a certain *absolute p-adic version of the Grothendieck Conjecture* holds for *hyperbolic curves "of Belyi type"*. This includes, in particular, affine hyperbolic curves over a nonarchimedean mixed-characteristic local field which are defined over a number field and isogenous to a hyperbolic curve of genus zero. Also, we apply this theory to prove the *analogue for proper hyperbolic curves over finite fields* of the version of the Grothendieck Conjecture that was shown in [Tama].

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Introduction

Let X be a *proper hyperbolic curve* over a field k which is either *finite* or *nonarchimedean local of mixed characteristic*; let $U \subseteq X$ be an *open subscheme* of X. Write Π_X for the *étale fundamental group* of X. In this paper, we study the extent to which the étale fundamental group of U may be *group-theoretically reconstructed* from Π_X.

In §1, we show that the *abelian portion* of the extension of Π_X determined by the étale fundamental group of U may be *group-theoretically reconstructed* from Π_X [cf. Theorem 1.1, (iii)], and, moreover, that this construction has certain *remarkable rigidity* properties [cf. Propositions 1.10, (i); 2.3, (i)].

In §2, we show that this abelian portion of the extension is sufficient to reconstruct [in essence] the multiplicative group of the *function field* of X [cf. Theorem 2.1, (ii)]. In the case of *nonarchimedean [mixed-characteristic] local fields*, this already implies various interesting consequences in the context of the *absolute anabelian geometry* studied in [Mzk5], [Mzk6], [Mzk8]. In particular, it implies that the *absolute p-adic version of the Grothendieck Conjecture* [i.e., an absolute version of [a certain portion of] the *relative* result that appears as the main result of [Mzk4]] holds for *hyperbolic curves "of Belyi type"* [cf. Definition 2.3; Corollary 2.3]. This includes, in particular, hyperbolic curves "of strictly Belyi type", i.e., affine hyperbolic curves over a nonarchimedean [mixed-characteristic] local field which are defined over a number field and isogenous to a hyperbolic curve of genus zero. In particular, we obtain a new countable class of *"absolute curves"* [in the terminology of [Mzk6]], whose absoluteness is, in certain respects, reminiscent of the absoluteness of the *canonical curves of p-adic Teichmüller theory* discussed in [Mzk6] [cf. Remark 30], but [in contrast to the class of canonical curves] appears [at least from the point of view of certain circumstantial evidence] unlikely to be Zariski dense in most moduli spaces [cf. Remark 31].

Finally, in §3, we apply the theory of the *weight filtration* [cf., e.g., [Kane], [Mtm]], together with various generalities concerning *free Lie algebras* [cf. the Appendix], to develop various *"higher order generalizations"* of the theory of §1, 2. In particular, we obtain various "higher order generalizations" of the *"remarkable rigidity"* referred to above [cf. Propositions 3.4, 3.6, especially Proposition 3.6, (iii)], which we apply to show that, relative to the notation introduced above, the *geometrically pro-l portion* [where l is a prime number invertible in k] of the étale fundamental group of U may be *recovered from* Π_X, at least when U is obtained from X by removing a *single k-rational point* [cf. Theorem 3.1]. This, along with the theory of §2, allows one to verify the *analogue for proper hyperbolic curves over finite fields* of the version of the Grothendieck Conjecture that was shown in [Tama] [cf. Theorem 3.2].

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0. Notations and Conventions

Numbers:

We shall denote by $\widehat{\mathbb{Z}}$ the *profinite completion* of the additive group of rational integers $\mathbb{Z}$. If p is a prime number, then $\mathbb{Z}_p$ denotes the ring of *p-adic integers*; $\mathbb{Q}_p$ denotes its quotient field. We shall refer to as a *p-adic local field* (respectively, *nonarchimedean local field*) any finite field extension of $\mathbb{Q}_p$ (respectively, a p-adic local field, for some p). A *number field* is defined to be a finite extension of the field of rational numbers. If Σ is a *set of prime numbers*, then we shall refer to a positive integer each of whose prime factors belongs to Σ as a *Σ-integer*. We shall refer to a *finite étale covering* of schemes whose degree is a Σ-integer as a *Σ-covering*. Also, we shall write $\mathfrak{Primes}$ for the *set of all prime numbers*.

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$G^{\mathrm{ab}}$$

for the *abelianization* of G [i.e., the quotient of G by the closed subgroup of G topologically generated by the commutators of G]. Let us write

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall\, h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H,\ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H,\ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$. If $H = N_G(H)$ (respectively, $H = C_G(H)$), then we shall say that H is *normally terminal* (respectively, *commensurably terminal*) in G. Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G.

If G_1, G_2 are *Hausdorff topological groups*, then an *outer homomorphism* $G_1 \to G_2$ is defined to be an equivalence class of continuous homomorphisms $G_1 \to G_2$, where two such homomorphisms are considered equivalent if they differ by composition with an inner automorphism of G_2. The group of *outer* *automorphisms* of G [i.e., bijective bicontinuous outer homomorphisms $G \to G$] will be denoted $\mathrm{Out}(G)$. If G is *center-free*, then there is a *natural exact sequence*:

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

[where the homomorphism $G \to \mathrm{Aut}(G)$ is defined by letting G act on G by *conjugation*].

If G is a *profinite group* such that, for every open subgroup $H \subseteq G$, we have $Z_G(H) = \{1\}$, then we shall say that G is *slim*. One verifies immediately that G is slim if and only if every open subgroup of G is center-free [cf. [Mzk5], Remark 0.1.3].

If G is a *profinite group* and Σ is *set of prime numbers*, then we shall say that G is a *pro-*Σ *group* if the order of every finite quotient group of G is a Σ*-integer*. If $\Sigma = \{l\}$ is of *cardinality one*, then we shall refer to a pro-Σ group as a *pro-*l *group*.

Curves:

Suppose that $g \geq 0$ is an *integer*. Then if S is a scheme, a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of* r*-pointed stable curves of genus* g (where we assume the points to be *unordered*) by $\overline{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\overline{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points]. The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth* r*-pointed stable curves of genus* g or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r).

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism.* (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the* l*-torsion points of the Jacobian of* Y. Note that

$S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D) since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbb{Z}[\frac{1}{l}]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D) as the *compactification* (respectively, *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above. A family of hyperbolic curves $X \to S$ of type $(0, 3)$ will be referred to as a *tripod*.

If X is a *hyperbolic curve* over a field K with *compactification* $X \subseteq \overline{X}$, then we shall write

$$X^{\text{cl}}; \quad X^{\text{cl}+}$$

for the *sets of closed points* of X and $\overline{X}$, respectively.

If X_K (respectively, Y_L) is a *hyperbolic curve over a field* K (respectively, L), then we shall say that X_K is *isogenous* to Y_L if there exists a hyperbolic curve Z_M over a field M together with *finite étale morphisms* $Z_M \to X_K$, $Z_M \to Y_L$. Note that in this situation, the morphisms $Z_M \to X_K$, $Z_M \to Y_L$ induce *finite separable inclusions of fields* $K \hookrightarrow M$, $L \hookrightarrow M$. [Indeed, this follows immediately from the easily verified fact that every subgroup $G \subseteq \Gamma(Z, \mathcal{O}_Z^\times)$ such that $G \bigcup \{0\}$ determines a *field* is necessarily contained in $M^\times$.]

If X is a *generically scheme-like algebraic stack* [i.e., an algebraic stack which admits a "scheme-theoretically" dense open that is isomorphic to a scheme] over a field K of *characteristic zero* that admits a [surjective] *finite étale* [or, equivalently, *finite étale Galois*] *covering* $Y \to X$, where Y is a hyperbolic curve over a finite extension of K, then we shall refer to X as a *hyperbolic orbicurve* over K. [Although this definition differs from the definition of a "hyperbolic orbicurve" given in [Mzk6], Definition 2.2, (ii), it follows immediately from a theorem of Bundgaard-Nielsen-Fox [cf., e.g., [Namba], Theorem 1.2.15, p. 29] that these two definitions are equivalent.] If $X \to Y$ is a dominant morphism of hyperbolic orbicurves, then we shall refer to $X \to Y$ as a *partial coarsification morphism* if the morphism induced by $X \to Y$ on *associated coarse spaces* [cf., e.g., [FC], Chapter I, §4.10] is an *isomorphism*.

Let X be a *hyperbolic orbicurve* over an algebraically closed field of characteristic zero; denote its *étale fundamental group* by Δ_X. We shall refer to the order of the [manifestly finite!] decomposition group of a closed point x of X as the *order of* x. We shall refer to the [manifestly finite!] *least common multiple* of the orders of the closed points of X as the *order of* X. Thus, it follows immediately from the definitions that X is a *hyperbolic curve* if and only if the order of X is equal to 1.

1. Maximal Abelian Cuspidalizations

Let X be a *proper hyperbolic curve* over a *field* k which is either *finite* or *nonarchimedean local*. Write

$$\underline{d}_k$$

for the *cohomological dimension* of k. Thus, if k is finite (respectively, nonarchimedean local), then $\underline{d}_k = 1$ (respectively, $\underline{d}_k = 2$ [cf., e.g., [NSW], Chapter 7, Theorem 7.1.8, (i)]). If k is finite (respectively, nonarchimedean local), we shall denote the characteristic of k (respectively, of the residue field of k) by p and the number p (respectively, 1) by $p^\dagger$. Also, we shall write

$$\mathfrak{Primes}^\dagger \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus (\mathfrak{Primes} \bigcap \{p^\dagger\})$$

[where $\mathfrak{Primes}$ is the set of all prime numbers [cf. §0]; the intersection is taken in the "ambient set" $\mathbb{Z}$].

Let Σ be a *set of prime numbers* that contains at least one prime number that is *invertible* in k. Write

$$\Sigma' \stackrel{\mathrm{def}}{=} \Sigma \setminus (\Sigma \bigcap \{p\}); \quad \Sigma^\dagger \stackrel{\mathrm{def}}{=} \Sigma \setminus (\Sigma \bigcap \{p^\dagger\})$$

[where the intersections are taken in the "ambient set" $\mathbb{Z}$]. Denote by $\widehat{\mathbb{Z}}'$ the *maximal pro-*Σ' *quotient* of $\widehat{\mathbb{Z}}$ and by $\widehat{\mathbb{Z}}^\dagger$ the *maximal pro-*$\Sigma^\dagger$ *quotient* of $\widehat{\mathbb{Z}}$.

If $\overline{k}$ is an *algebraic closure* of k, then we shall denote the result of base-changing objects over k to $\overline{k}$ by means of a subscript "$\overline{k}$". Any choice of a basepoint of X determines an algebraic closure $\overline{k}$ of k, and hence an *exact sequence*

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to G_k \to 1$$

where $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $\pi_1(X)$, $\pi_1(X_{\overline{k}})$ are the *étale fundamental groups* of X, $X_{\overline{k}}$, respectively. Write Δ_X for the *maximal pro-*Σ *quotient* of $\pi_1(X_{\overline{k}})$ and $\Pi_X \stackrel{\mathrm{def}}{=} \pi_1(X)/\mathrm{Ker}(\pi_1(X_{\overline{k}}) \twoheadrightarrow \Delta_X)$. Thus, we have an *exact sequence*:

$$1 \to \Delta_X \to \Pi_X \to G_k \to 1$$

Similarly, if we write $X \times X \stackrel{\mathrm{def}}{=} X \times_k X$, then we obtain [by considering the *maximal pro-*Σ *quotient* of $\pi_1((X \times X)_{\overline{k}})$] an *exact sequence*

$$1 \to \Delta_{X \times X} \to \Pi_{X \times X} \to G_k \to 1$$

where $\Pi_{X \times X}$ (respectively, $\Delta_{X \times X}$) may be identified with $\Pi_X \times_{G_k} \Pi_X$ (respectively, $\Delta_X \times \Delta_X$). Let $\Pi_Z \subseteq \Pi_{X \times X}$ be an open subgroup that surjects onto G_k. Write $Z \to X \times X$ for the corresponding covering; $\Delta_Z \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_Z \twoheadrightarrow G_k)$.

Proposition 1.1. **(Group-theoreticity of Étale Cohomology)** *Let* $\widehat{\mathbb{Z}}^\dagger \twoheadrightarrow A$ *be a* **finite quotient**, *and* N *a finite* A*-module equipped with a continuous* Δ_X*- (respectively,* Π_X*-;* Δ_Z*-;* Π_Z*-) action. Then for* $i \in \mathbb{Z}$*, the natural homomorphism*

$$H^i(\Delta_X, N) \to H^i_{\text{ét}}(X_{\overline{k}}, N) \quad \textit{(respectively, } H^i(\Pi_X, N) \to H^i_{\text{ét}}(X, N);$$

$$H^i(\Delta_Z, N) \to H^i_{\text{ét}}(Z_{\overline{k}}, N); \quad H^i(\Pi_Z, N) \to H^i_{\text{ét}}(Z, N))$$

is an **isomorphism**.

Proof. In light of the Leray spectral sequence for the surjections $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \text{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "Im(−)" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$], it suffices to verify the asserted isomorphism in the case of Δ_X. If $Y \to X_{\overline{k}}$ is a *connected finite étale Galois* Σ*-covering*, then the associated Leray spectral sequence has "E_2-term" given by the cohomology of $\text{Gal}(Y/X)$ with coefficients in the étale cohomology of Y and abuts to the étale cohomology of $X_{\overline{k}}$. By allowing Y to vary, one then verifies immediately that it suffices to verify that every étale cohomology class of Y [with coefficients in N] vanishes upon pull-back to some [connected] finite étale Σ-covering $Y' \to Y$. Moreover, by passing to an appropriate Y, one reduces immediately to the case where $N = A$, equipped with the trivial Π_X-action. Then the vanishing assertion in question is a tautology for "H^1"; for "H^2", it suffices to take $Y' \to Y$ so that the degree of $Y' \to Y$ *annihilates* A [cf., e.g., the discussion at the bottom of [FK], p. 136]. ☐

Set:

$$M_X \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathbb{Z}}^\dagger}(H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger), \widehat{\mathbb{Z}}^\dagger); \quad M_k \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathbb{Z}}^\dagger}(H^{\underline{d}_k}(G_k, M_X^{\otimes \underline{d}_k - 1}), M_X^{\otimes \underline{d}_k - 1})$$

Thus, M_k, M_X are *free* $\widehat{\mathbb{Z}}^\dagger$*-modules of rank one*; M_X is isomorphic as a G_k-module to $\widehat{\mathbb{Z}}^\dagger(1)$ [where the "(1)" denotes a "Tate twist" — i.e., G_k acts on $\widehat{\mathbb{Z}}^\dagger(1)$ via the cyclotomic character]; M_k is isomorphic as a G_k-module to $\widehat{\mathbb{Z}}^\dagger(\underline{d}_k - 1)$. [Indeed, this follows from Proposition 1.1; *Poincaré duality* [cf., e.g., [FK], Chapter II, Theorem 1.13]; the fact, in the finite field case, that $G_k \cong \widehat{\mathbb{Z}}$ [together with an easy computation of the group cohomology of $\widehat{\mathbb{Z}}$]; the well-known theory of the cohomology of nonarchimedean local fields [cf., e.g., [NSW], Chapter 7, Theorem 7.2.6].]

Remark 1. Note that for any open subgroup $\Pi_{X'} \subseteq \Pi_X$ [which we think of as corresponding to a finite étale covering $X' \to X$], we obtain a *natural isomorphism*

$$M_X \xrightarrow{\sim} M_{X'}$$

by applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}^\dagger}(-, \widehat{\mathbb{Z}}^\dagger)$ to the induced morphism on group cohomology $H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger) \to H^2(\Delta_{X'}, \widehat{\mathbb{Z}}^\dagger)$ [where $\Delta_{X'} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{X'} \to G_k)$] and *dividing* by $[\Delta_X : \Delta_{X'}]$. [One verifies easily that this does indeed yield an isomorphism — cf., e.g., the discussion at the bottom of [FK], p. 136.] Moreover, relative to the tautological isomorphisms $H^2(\Delta_X, M_X) \cong \widehat{\mathbb{Z}}^\dagger$, $H^2(\Delta_{X'}, M_{X'}) \cong \widehat{\mathbb{Z}}^\dagger$, the isomorphism $M_X \xrightarrow{\sim} M_{X'}$ just constructed induces [via the restriction morphism on group cohomology] the morphism $\widehat{\mathbb{Z}}^\dagger \to \widehat{\mathbb{Z}}^\dagger$ given by multiplication by $[\Delta_X : \Delta_{X'}]$. Similarly, if k' is the base field of X', then we obtain a *natural isomorphism*

$$M_k \xrightarrow{\sim} M_{k'}$$

by applying the natural isomorphism $M_X \xrightarrow{\sim} M_{X'}$ just constructed and the dual of the natural pull-back morphism on group cohomology and then *dividing* by $[k' : k]$ [cf., e.g., [NSW], Chapter 7, Corollary 7.1.4].

Proposition 1.2. **(Top Cohomology Modules)**

(i) There are natural isomorphisms:

$$H^{\underline{d}_k}(G_k, M_k) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^2(\Delta_X, M_X) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^{\underline{d}_k+2}(\Pi_X, M_X \otimes M_k) \cong \widehat{\mathbb{Z}}^\dagger$$

$$H^4(\Delta_Z, M_X^{\otimes 2}) \cong \widehat{\mathbb{Z}}^\dagger; \quad H^{\underline{d}_k+4}(\Pi_Z, M_X^{\otimes 2} \otimes M_k) \cong \widehat{\mathbb{Z}}^\dagger$$

(ii) There is a **unique** *isomorphism $M_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger(1)$ such that the image of $1 \in \widehat{\mathbb{Z}}^\dagger$ maps via the composite of the isomorphism $\widehat{\mathbb{Z}}^\dagger \cong H^2(\Delta_X, M_X)$ of (i) with the isomorphism $H^2(\Delta_X, M_X) \xrightarrow{\sim} H^2(\Delta_X, \widehat{\mathbb{Z}}^\dagger(1))$ induced by the isomorphism $M_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger(1)$ in question to the [first] Chern class of a line bundle of degree 1 on $X_{\overline{k}}$.*

Proof. Assertion (i) follows from the definitions; the Leray spectral sequence for the surjections $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \mathrm{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "$\mathrm{Im}(-)$" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$]. Assertion (ii) is immediate from the definitions. $\square$

Proposition 1.3. **(Duality)** *For $i \in \mathbb{Z}$, let $\widehat{\mathbb{Z}}^\dagger \twoheadrightarrow A$ be a* **finite quotient***, and N a finite A-module.*

(i) Suppose that N is equipped with a continuous G_k-action. Then the pairing

$$H^i(G_k, N) \times H^{\underline{d}_k - i}(G_k, \mathrm{Hom}_A(N, M_k \otimes A)) \to A$$

determined by the cup product in group cohomology and the natural isomorphisms of Proposition 1.2*, (i), determines an isomorphism as follows:*

$$H^i(G_k, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k - i}(G_k, \mathrm{Hom}_A(N, M_k \otimes A)), A)$$

(ii) Suppose that N *is equipped with a continuous* Π_X*- (respectively,* Δ_X*-;* Π_Z*-;* Δ_Z*-) action. Then the pairing*

$$H^i(\Pi_X, N) \times H^{\underline{d}_k + 2 - i}(\Pi_X, \mathrm{Hom}_A(N, M_X \otimes M_k \otimes A)) \to A$$

$$\text{(respectively, } H^i(\Delta_X, N) \times H^{2-i}(\Delta_X, \mathrm{Hom}_A(N, M_X \otimes A)) \to A;$$

$$H^i(\Pi_Z, N) \times H^{\underline{d}_k + 4 - i}(\Pi_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes M_k \otimes A)) \to A;$$

$$H^i(\Delta_Z, N) \times H^{4-i}(\Delta_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes A)) \to A\text{)}$$

determined by the cup product in group cohomology and the natural isomorphisms of Proposition 1.2*, (i), determines an isomorphism as follows:*

$$H^i(\Pi_X, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k + 2 - i}(\Pi_X, \mathrm{Hom}_A(N, M_X \otimes M_k \otimes A)), A)$$

$$\text{(respectively, } H^i(\Delta_X, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{2-i}(\Delta_X, \mathrm{Hom}_A(N, M_X \otimes A)), A);$$

$$H^i(\Pi_Z, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{\underline{d}_k + 4 - i}(\Pi_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes M_k \otimes A)), A);$$

$$H^i(\Delta_Z, N) \xrightarrow{\sim} \mathrm{Hom}_A(H^{4-i}(\Delta_Z, \mathrm{Hom}_A(N, M_X^{\otimes 2} \otimes A)), A)\text{)}$$

Proof. Assertion (i) follows immediately from the fact that $G_k \cong \widehat{\mathbb{Z}}$ [together with an easy computation of the group cohomology of $\widehat{\mathbb{Z}}$] in the finite field case; [NSW], Chapter 7, Theorem 7.2.6, in the nonarchimedean local field case. Assertion (ii) then follows from assertion (i); the Leray spectral sequences associated to $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow \mathrm{Im}(\Pi_Z) \subseteq \Pi_X$ [i.e., where "Im(−)" denotes the image via the natural homomorphism associated to one of the projections $Z \to X \times X \to X$]; Proposition 1.1; *Poincaré duality* [cf., e.g., [FK], Chapter II, Theorem 1.13]. ☐

Proposition 1.4. (Automorphisms of Cyclotomic Extensions)

(i) We have: $H^0(G_k, H^1(\Delta_X, M_X)) = 0$.

(ii) There are natural isomorphisms

$$H^1(\Pi_X, M_X) \xrightarrow{\sim} H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$$

$$H^1(\Pi_Z, M_X) \xrightarrow{\sim} H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$$

— where the first isomorphisms in each line are induced by the surjections $\Pi_X \twoheadrightarrow G_k$*,* $\Pi_Z \twoheadrightarrow G_k$*; the second isomorphisms in each line are induced by the isomorphism of Proposition* 1.2*, (ii), and the Kummer exact sequence;* $(k^\times)^\wedge$ *is the* **maximal pro-$\Sigma^\dagger$-quotient** *of* $k^\times$*.*

Proof. Assertion (i) follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" [cf., e.g., [Mumf], p. 206] in the finite field case; [Mzk8], Lemma 4.6, in the nonarchimedean local field case. The first isomorphisms of assertion (ii) follow immediately from assertion (i) and the Leray spectral sequences associated to $\Pi_X \twoheadrightarrow G_k$, $\Pi_Z \twoheadrightarrow G_k$; the second isomorphisms follow immediately from consideration of the Kummer exact sequence for Spec(k). ☐

Definition 1.1.

(i) Let H be a profinite group equipped with a homomorphism $H \to \Pi_X$. Then we shall refer to the kernel I_H of $H \to \Pi_X$ as the *cuspidal subgroup* of H [relative to $H \to \Pi_X$]. We shall say that H is *cuspidally abelian* (respectively, *cuspidally pro-*Σ^* [where Σ^* is a set of prime numbers]) [relative to $H \to \Pi_X$] if I_H is abelian (respectively, a pro-Σ^* group). If H is cuspidally abelian, then observe that H/I_H acts naturally [by conjugation] on I_H; we shall say that H is *cuspidally central* [relative to $H \to \Pi_X$] if this action of H/I_H on I_H is trivial. Also, we shall use similar terminology to the terminology just introduced for $H \to \Pi_X$ when Π_X is replaced by Δ_X, $\Pi_{X\times X}$, $\Delta_{X\times X}$.

(ii) Let H be a profinite group; $H_1 \subseteq H$ a closed subgroup. Then we shall refer to as an H_1*-inner automorphism* of H an inner automorphism induced by conjugation by an element of H_1. If H' is also a profinite group, then we shall refer to as an H_1*-outer homomorphism* $H' \to H$ an equivalence class of homomorphisms $H' \to H$, where two such homomorphisms are considered equivalent if they differ by composition by an H_1-inner automorphism. If H is equipped with a homomorphism $H \to G_k$ [cf., e.g., the various groups introduced above], and $H_1 \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H \to G_k)$, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *geometrically inner automorphism* (respectively, *geometrically outer homomorphism*). If H is equipped with a structure of extension of some other profinite group H_0 by a finite product H_1 of copies of M_X, or, more generally, a projective limit H_1 of such finite products, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *cyclotomically inner automorphism* (respectively, *cyclotomically outer homomorphism*). If H is equipped with a homomorphism to Π_X, Δ_X, $\Pi_{X\times X}$, or $\Delta_{X\times X}$ [cf. the situation of (i)], and H_1 is the kernel of this homomorphism, then we shall refer to an H_1-inner automorphism (respectively, H_1-outer homomorphism) as a *cuspidally inner automorphism* (respectively, *cuspidally outer homomorphism*).

Next, let

$$\Pi_{X'} \subseteq \Pi_X$$

be an *open normal subgroup*, corresponding to a finite étale Galois covering $X' \to X$. Set

$$\Pi_{Z'} \stackrel{\text{def}}{=} \Pi_{X'\times X'} \cdot \Pi_X \subseteq \Pi_{X\times X}$$

[where we regard Π_X as a subgroup of $\Pi_{X\times X}$ via the diagonal map]; write $Z' \to X \times X$ for the covering determined by $\Pi_{Z'}$. Thus, it is a tautology that the diagonal morphism $\iota : X \hookrightarrow X \times X$ lifts to a morphism

$$\iota' : X \hookrightarrow Z'$$

which induces the inclusion $\Pi_X \hookrightarrow \Pi_{Z'}$ on fundamental groups. If $Z \to X \times X$ is a connected finite étale covering arising from an open subgroup of $\Pi_{X\times X}$, write:

$$U_{X\times X} \stackrel{\text{def}}{=} (X \times X)\backslash \iota(X); \quad U_Z \stackrel{\text{def}}{=} (U_{X\times X}) \times_{(X\times X)} Z$$

Denote by $\Delta_{U_{X\times X}}$ the *maximal cuspidally* [i.e., relative to the natural map to $\pi_1((X\times X)_{\overline{k}})$] *pro-*$\Sigma^\dagger$ *quotient* of the maximal pro-Σ quotient of the tame fundamental group of $(U_{X\times X})_{\overline{k}}$ [where "tame" is with respect to the divisor $\iota(X) \subseteq X \times X$] and by $\Pi_{U_{X\times X}}$ the quotient $\pi_1(U_{X\times X})/\mathrm{Ker}(\pi_1((U_{X\times X})_{\overline{k}}) \twoheadrightarrow \Delta_{U_{X\times X}})$; write $\Pi_{U_Z} \subseteq \Pi_{U_{X\times X}}$ for the open subgroup corresponding to the finite étale covering $U_Z \to U_{X\times X}$.

Proposition 1.5. (Characteristic Class of the Diagonal)

(i) The pull-back morphism arising from the natural inclusion

$$\Pi_X \hookrightarrow \Pi_{Z'} \ (\subseteq \Pi_{X\times X} = \Pi_X \times_{G_k} \Pi_X)$$

composed with the natural isomorphism of Proposition 1.2*, (i), determines a homomorphism*

$$H^{\underline{d}_k+2}(\Pi_{Z'}, M_X \otimes M_k) \to H^{\underline{d}_k+2}(\Pi_X, M_X \otimes M_k) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\dagger$$

hence [by Proposition 1.3*, (ii)] a class*

$$\eta^{\text{diag}}_{Z'} \in H^2(\Pi_{Z'}, M_X)$$

which is equal to the étale cohomology class associated to $\iota'(X) \subseteq Z'$*, or, alternatively, the [first] Chern class of the line bundle* $\mathcal{O}_{Z'}(\iota'(X))$*.*

(ii) Denote by

$$\mathbb{L}^{\times}_{\text{diag}}[Z'] \to Z'$$

the complement of the zero section in the geometric line bundle [i.e., $\mathbb{G}_m$*-torsor] determined by* $\mathcal{O}_{Z'}(\iota'(X))$*, by* $\Delta_{\mathbb{L}^{\times}_{\text{diag}}[Z']}$ *the maximal cuspidally pro-*$\Sigma^{\dagger}$ *quotient of the maximal pro-*Σ *quotient of the tame fundamental group of* $(\mathbb{L}^{\times}_{\text{diag}}[Z'])_{\overline{k}}$ *[where "tame" is with respect to the divisor determined by the complement of the* $\mathbb{G}_m$*-torsor* $\mathbb{L}^{\times}_{\text{diag}}[Z']$ *in the naturally associated* $\mathbb{P}^1$*-bundle], and by* $\Pi_{\mathbb{L}^{\times}_{\text{diag}}[Z']}$ *the quotient* $\pi_1(\mathbb{L}^{\times}_{\text{diag}}[Z'])/\text{Ker}(\pi_1((\mathbb{L}^{\times}_{\text{diag}}[Z'])_{\overline{k}}) \twoheadrightarrow \Delta_{\mathbb{L}^{\times}_{\text{diag}}[Z']})$*. Then [in light of the isomorphism of Proposition* 1.2*, (ii)] we have a natural exact sequence*

$$1 \to M_X \to \Pi_{\mathbb{L}^{\times}_{\text{diag}}[Z']} \to \Pi_{Z'} \to 1$$

whose associated extension class is equal to the class $\eta^{\text{diag}}_{Z'}$*.*

(iii) The global section of $\mathcal{O}_{Z'}(\iota'(X))$ *over* Z' *determined by the natural inclusion* $\mathcal{O}_{Z'} \hookrightarrow \mathcal{O}_{Z'}(\iota'(X))$ *defines a morphism*

$$U_{Z'} \to \mathbb{L}^{\times}_{\text{diag}}[Z']$$

over Z' *which induces a* **surjective** *homomorphism of groups over* $\Pi_{Z'}$*:*

$$\Pi_{U_{Z'}} \twoheadrightarrow \Pi_{\mathbb{L}^{\times}_{\text{diag}}[Z']}$$

Proof. Assertion (i) follows immediately from Propositions 1.1, 1.2, 1.3, together with well-known facts concerning Chern classes and associated cycles in étale cohomology [cf., e.g., [FK], Chapter II, Definition 1.2, Proposition 2.2]. Assertion (ii) follows from Proposition 1.1; [Mzk7], Definition 4.2, Lemmas 4.4, 4.5. Assertion (iii) follows from [Mzk8], Lemma 4.2, by considering fibers over one of the two natural projections $\Pi_{Z'} \to \Pi_{X\times X} \twoheadrightarrow \Pi_X$. [Here, we note that although in [Mzk7], §4; [Mzk8], the base field is assumed to be of characteristic zero, one verifies immediately that the same arguments as those applied in *loc. cit.* yield the corresponding results in the finite field case — so long as we restrict the coefficients of the cohomology modules in question to modules over $\widehat{\mathbb{Z}}^{\dagger}$.] $\square$

Definition 1.2.

(i) We shall refer to a covering $Z' \to X \times X$ as in the above discussion as the *diagonal covering associated to the covering* $X' \to X$. We shall refer to an extension of profinite groups

$$1 \to M_X \to \mathcal{D}' \to \Pi_{Z'} \to 1$$

whose associated extension class is the class $\eta^{\mathrm{diag}}_{Z'}$ of Proposition 1.5, (i), as a *fundamental extension* [of $\Pi_{Z'}$]. In the following (ii) — (iv), we shall assume that $1 \to M_X \to \mathcal{D} \to \Pi_{X\times X} \to 1$ is a fundamental extension.

(ii) Let $x, y \in X(k)$; write $D_x, D_y \subseteq \Pi_X$ for the associated decomposition groups [which are well-defined up to conjugation by an element of Δ_X — cf. Remark 2 below]. Now set:

$$\mathcal{D}_x \stackrel{\mathrm{def}}{=} \mathcal{D}|_{D_x \times_{G_k} \Pi_X}; \quad \mathcal{D}_{x,y} \stackrel{\mathrm{def}}{=} \mathcal{D}|_{D_x \times_{G_k} D_y}$$

Thus, $\mathcal{D}_x$ (respectively, $\mathcal{D}_{x,y}$) is an extension of Π_X (respectively, G_k) by M_X. Similarly, if $D = \sum_i m_i \cdot x_i, E = \sum_j n_j \cdot y_j$ are divisors on X supported on points that are rational over k, then set:

$$\mathcal{D}_D \stackrel{\mathrm{def}}{=} \sum_i m_i \cdot \mathcal{D}_{x_i}; \quad \mathcal{D}_{D,E} \stackrel{\mathrm{def}}{=} \sum_{i,j} m_i \cdot n_j \cdot \mathcal{D}_{x_i,y_j}$$

[where the sums are to be understood as sums of extensions of Π_X or G_k by M_X — i.e., the sums are induced by the additive structure of M_X]. Also, we shall write $\mathcal{C} \stackrel{\mathrm{def}}{=} -\mathcal{D}|_{\Pi_X}$ [where we regard Π_X as a subgroup of $\Pi_{X\times X}$ via the diagonal map]. [Thus, $\mathcal{C}$ is an extension of Π_X by M_X whose extension class is the Chern class of the *canonical bundle* of X.]

(iii) Let $S \subseteq X(k)$ be a finite subset. Then we shall write

$$\mathcal{D}_S \stackrel{\mathrm{def}}{=} \prod_{x\in S} \mathcal{D}_x$$

[where the product is to be understood as the fiber product over Π_X]. Thus, $\mathcal{D}_S$ is an extension of Π_X by a product of copies of M_X indexed by elements of S. We shall refer to $\mathcal{D}_S$ as a *maximal abelian* S*-cuspidalization [of* Π_X *at* S*]*. Observe that if $T \subseteq X(k)$ is a finite subset such that $S \subseteq T$, then we obtain a natural *projection morphism* $\mathcal{D}_T \to \mathcal{D}_S$.

(iv) We shall refer to a homomorphism

$$\Pi_{U_{X\times X}} \to \mathcal{D}$$

over $\Pi_{X\times X}$ as a *fundamental section* if, for some isomorphism of $\mathcal{D}$ with $\Pi_{\mathbb{L}^\times_{\mathrm{diag}}}$ that induces the identity on $\Pi_{X\times X}$, M_X, the resulting composite homomorphism $\Pi_{U_{X\times X}} \to \Pi_{\mathbb{L}^\times_{\mathrm{diag}}}$ is the homomorphism of Proposition 1.5, (iii).

Remark 2. Relative to the situation in Definition 1.2, (ii), conjugation by elements $\delta \in \Delta_X$ induces isomorphisms between the different possible choices of "D_x", all of which lie over the isomorphism between any of these choices and G_k induced by the projection $\Pi_X \twoheadrightarrow G_k$. Moreover, by lifting $(\delta, 1) \in \Delta_{X\times X} \subseteq \Pi_{X\times X}$ to an element $\delta_{\mathcal{D}} \in \mathcal{D}$, and conjugating by $\delta_{\mathcal{D}}$, we obtain natural isomorphisms between the various resulting "$\mathcal{D}_x$'s" which induce the *identity* on the quotient group $\mathcal{D}_x \twoheadrightarrow \Pi_X$, as well as on the subgroup $M_X \subseteq \mathcal{D}_x$. Note that this last property [i.e., of inducing the identity on Π_X, M_X] holds *precisely* because we are working with $\delta \in \Delta_X \subseteq \Pi_X$, as opposed to an arbitrary "$\delta \in \Pi_X$".

Remark 3. By Proposition 1.4, (ii), if $\mathcal{E}$ is any profinite group extension of Π_X (respectively, G_k; an open subgroup $\Pi_Z \subseteq \Pi_{X\times X}$ that surjects onto G_k) by M_X, then the *group of cyclotomically outer automorphisms of the extension* $\mathcal{E}$ [i.e., that induce the identity on Π_X (respectively, G_k; Π_Z) and M_X] may be naturally identified with $(k^\times)^\wedge$. In particular, in the context of Definition 1.2, (iv), any two fundamental sections of $\mathcal{D}$ differ, up to composition with a cyclotomically inner automorphism of $\mathcal{D}$, by a "$(k^\times)^\wedge$-multiple".

Next, if k is *nonarchimedean local*, then set $G_k^\dagger \stackrel{\mathrm{def}}{=} G_k$; if k is *finite*, then write $G_k^\dagger \subseteq G_k$ for the *maximal pro-*$\Sigma^\dagger$ *subgroup of* G_k [so $G_k^\dagger \cong \widehat{\mathbb{Z}}^\dagger$]. Also, we shall use the notation

$$\Pi^\dagger_{(-)} \stackrel{\mathrm{def}}{=} \Pi_{(-)} \times_{G_k} G_k^\dagger \subseteq \Pi_{(-)}$$

[where "$(-)$" is any *smooth, geometrically connected scheme* over k, with arithmetic fundamental group $\Pi_{(-)} \twoheadrightarrow G_k$].

Proposition 1.6. (Basic Properties of Maximal Abelian Cuspidalizations) *Let*

$$1 \to M_X \to \mathcal{D} \to \Pi_{X\times X} \to 1$$

be a **fundamental extension**; $\phi : \Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{D}$ *a* **fundamental section**; $S \subseteq X(k)$ *a finite subset. Then:*

(i) The profinite groups $\Delta_{X\times X}$, Δ_X, as well as any profinite group extension of $\Pi^\dagger_{X\times X}$ or $\Pi^\dagger_X$ by a [possibly empty] finite product of copies of M_X is **slim** *[cf. §0]. In particular, the profinite group $\mathcal{D}_S^\dagger \stackrel{\mathrm{def}}{=} \mathcal{D}_S \times_{G_k} G_k^\dagger$ is* **slim**.

(ii) For $x \in X(k)$, write $U_x \stackrel{\mathrm{def}}{=} X \backslash \{x\}$. Denote by Δ_{U_x} the maximal cuspidally [i.e., relative to the natural map to $\pi_1((U_x)_{\overline{k}})$] pro-$\Sigma^\dagger$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(U_x)_{\overline{k}}$ [where "tame" is

with respect to the complement of U_x in X] and by Π_{U_x} the quotient given by $\pi_1(U_x)/\mathrm{Ker}(\pi_1((U_x)_{\overline{k}}) \twoheadrightarrow \Delta_{U_x})$. Then the inverse image via either of the natural projections $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_X$ of the decomposition group $D_x \subseteq \Pi_X$ is naturally isomorphic to Π_{U_x}. In particular, $\Delta_{U_{X\times X}}$, $\Pi^{\dagger}_{U_{X\times X}}$ are **slim**.

(iii) For $S \subseteq X(k)$ a finite subset, write:

$$U_S \stackrel{\mathrm{def}}{=} \prod_{x\in S} U_x$$

[where the product is to be understood as the fiber product over X]. Denote by Δ_{U_S} the maximal cuspidally [i.e., relative to the natural map to $\pi_1((U_S)_{\overline{k}})$] pro-$\Sigma^{\dagger}$ quotient of the maximal pro-Σ quotient of the tame fundamental group of $(U_S)_{\overline{k}}$ [where "tame" is with respect to the complement of U_S in X], and by Π_{U_S} the quotient $\pi_1(U_S)/\mathrm{Ker}(\pi_1((U_S)_{\overline{k}}) \twoheadrightarrow \Delta_{U_S})$. Then Δ_{U_S}, $\Pi^{\dagger}_{U_S}$ are **slim**. *Forming the product of the specializations of ϕ to the various $D_x \times_{G_k} \Pi_X \subseteq \Pi_{X\times X}$ yields homomorphisms*

$$\Pi_{U_S} \to \prod_{x\in S} \Pi_{U_x} \to \mathcal{D}_S$$

[where the product is to be understood as the fiber product over Π_X]. Moreover, the composite morphism $\Pi_{U_S} \to \mathcal{D}_S$ is **surjective**; *the resulting quotient of $\Delta_{U_S} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S} \twoheadrightarrow G_k)$ is the* **maximal cuspidally central quotient** *of Δ_{U_S}, relative to the surjection $\Delta_{U_S} \twoheadrightarrow \Delta_X$.*

(iv) The quotient of $\Delta_{U_{X\times X}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_{X\times X}} \twoheadrightarrow G_k)$ determined by $\phi : \Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{D}$ is the **maximal cuspidally central quotient** *of $\Delta_{U_{X\times X}}$, relative to the surjection $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta_{X\times X}$.*

Proof. Assertion (i) follows immediately from the *slimness* of $\Pi^{\dagger}_X$, Δ_X [cf., e.g., [Mzk5], Theorem 1.1.1, (ii); the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10], together with the [easily verified] fact that $G^{\dagger}_k$ acts *faithfully* on M_X via the *cyclotomic character*. Next, we consider assertion (ii). The portion of assertion (ii) concerning Π_{U_x} follows immediately from the existence of the "homotopy exact sequence associated to a family of curves" [cf., e.g., [Stix], Proposition 2.3]. The *slimness* assertion then follows from assertion (i) [applied to $\Pi^{\dagger}_X$] and the slimness of Δ_{U_x} [cf. the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10]. As for assertion (iii), the *slimness* of Δ_{U_S}, $\Pi^{\dagger}_{U_S}$ follows via the arguments given in the proofs of [Mzk5], Lemmas 1.3.1, 1.3.10. The existence of homomorphisms $\Pi_{U_S} \to \prod_{x\in S} \Pi_{U_x} \to \mathcal{D}_S$ as asserted is immediate from the definitions, assertion (ii). For $x \in S$, write

$$D_x[U_S] \subseteq \Pi_{U_S}$$

for the *decomposition group* of x; $I_x[U_S] \subseteq D_x[U_S]$ for the *inertia subgroup*. Now it is immediate from the definitions that $I_x[U_S]$ maps isomorphically onto the copy M_X in $\mathcal{D}_S$ corresponding to the point x. This implies the desired *surjectivity*. Since, moreover, it is immediate from the definitions that the cuspidal subgroup of any cuspidally central quotient of Δ_{U_S} is generated by the image of the $I_x[U_S]$, as x ranges over the elements of S, the final assertion concerning the *maximal cuspidally central quotient* of Δ_{U_S} follows immediately. Assertion (iv) follows by a similar argument to the argument applied to the final portion of assertion (iii). $\square$

Next, let $Z' \to X \times X$ (respectively, $Z'' \to X \times X$; $Z^* \to X \times X$) be the *diagonal covering* associated to a covering $X' \to X$ (respectively, $X'' \to X$; $X^* \to X$) arising from an open subgroup of Π_X; denote by $\iota' : X \hookrightarrow Z'$ (respectively, $\iota'' : X \hookrightarrow Z''$; $\iota^* : X \hookrightarrow Z^*$) the tautological lifting of the diagonal embedding $\iota : X \hookrightarrow X \times X$ and by k' (respectively, k''; k^*) the extension of k determined by X' (respectively, X''; X^*). Assume, moreover, that the covering $X'' \to X$ factors as follows:

$$X'' \to X' \to X^* \to X$$

Thus, we obtain a factorization $Z'' \to Z' \to Z^* \to X \times X$. Let

$$1 \to M_X \to \mathcal{D}'' \to \Pi_{Z''} \to 1$$

be a *fundamental extension* of $\Pi_{Z''}$.

Write

$$1 \to M_X \to \mathcal{D}''_{X'' \times X''} \to \Pi_{X'' \times X''} \to 1$$

for the *pull-back* of the extension $\mathcal{D}''$ via the inclusion $\Pi_{X'' \times X''} \subseteq \Pi_{Z''}$. Now if we think of $\Pi_{X \times X}$ or $\Pi_{X'' \times X''}$ as only being defined *up to* $\Delta_{X''} \times \{1\}$*-inner automorphisms*, then it makes sense, for $\delta \in \Delta_X/\Delta_{X''}$ to speak of the *pull-back of the extension* $\mathcal{D}''_{X'' \times X''}$ *via* $\delta \times 1$:

$$1 \to M_X \to (\delta \times 1)^* \mathcal{D}''_{X'' \times X''} \to \Pi_{X'' \times X''} \to 1$$

In particular, we may form the *fiber product* over $\Pi_{X'' \times X''}$:

$$\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''} \stackrel{\mathrm{def}}{=} \prod_{\delta \in \Delta_{X^*}/\Delta_{X''}} (\delta \times 1)^* \mathcal{D}''_{X'' \times X''}$$

Thus, $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''}$ is an extension of $\Pi_{X'' \times X''}$ by a product of copies of M_X indexed by $\Delta_{X^*}/\Delta_{X''}$; $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''}$ admits a *tautological* $\Delta_{X''} \times \{1\}$*-outer* [more precisely: a $(\Delta_{X''} \times \{1\}) \times_{\Pi_{X'' \times X''}} \mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''}$-outer] *action* by the finite group $\Delta_{X^*}/\Delta_{X''} \cong (\Delta_{X^*}/\Delta_{X''}) \times \{1\}$. Moreover, the natural outer action of $\mathrm{Gal}(X''/X) \cong \mathrm{Gal}((X'' \times X'')/Z'') \cong \Pi_X/\Pi_{X''}$ on $\Pi_{X'' \times X''}$ [arising from the diagonal embedding $\Pi_X \hookrightarrow \Pi_{Z''}$] clearly lifts to an outer action of $\mathrm{Gal}(X''/X)$ on $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X'' \times X''}$, which is *compatible*, relative to the natural action of $\mathrm{Gal}(X''/X)$ on $\Delta_{X^*}/\Delta_{X''}$ by conjugation, with the $\Delta_{X''} \times \{1\}$-outer action of $\Delta_{X^*}/\Delta_{X''}$ on $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$. Thus, in summary, the natural isomorphism

$$\Big\{(\Delta_{X^*}/\Delta_{X''}) \times \{1\}\Big\} \rtimes \mathrm{Gal}(X''/X) \cong \mathrm{Gal}((X'' \times X'')/Z^*)$$

determines a homomorphism $\mathrm{Gal}((X'' \times X'')/Z^*) \to \mathrm{Out}(\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''})$ via which we may pull-back the extension "$1 \to (-) \to \mathrm{Aut}(-) \to \mathrm{Out}(-) \to 1$" [cf. §0; Proposition 1.6, (i)] for $\mathbb{S}_{X''/X^*}(\mathcal{D}'')_{X''\times X''}$ to obtain an extension

$$1 \to \prod_{\Delta_{X^*}/\Delta_{X''}} M_X \to \mathbb{S}_{X''/X^*}(\mathcal{D}'') \to \Pi_{Z^*} \to 1$$

in which Π_{Z^*} is only determined up to $\Delta_{X''} \times \{1\}$-inner automorphisms. Note, moreover, that every cyclotomically outer automorphism of the extension $\mathcal{D}''$ — i.e., an element of $(k^\times)^\wedge$ [cf. Remark 3] — induces a *cyclotomically outer automorphism* of $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$. In particular, we have a *natural cyclotomically outer action of* $(k^\times)^\wedge$ *on* $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$.

Next, let us *push-forward* the extension $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$ just constructed via the *natural surjection*

$$\prod_{\Delta_{X^*}/\Delta_{X''}} M_X \twoheadrightarrow \prod_{\Delta_{X^*}/\Delta_{X'}} M_X$$

[which induces the identity morphism $M_X \to M_X$ between the various factors of the domain and codomain], so as to obtain an extension $\mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'')$ as follows:

$$1 \to \prod_{\Delta_{X^*}/\Delta_{X'}} M_X \to \mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'') \to \Pi_{Z^*} \to 1$$

[in which Π_{Z^*} is only determined up to $\Delta_{X''} \times \{1\}$-inner automorphisms].

Proposition 1.7. **(Symmetrizations and Traces)** *In the notation of the discussion above:*

(i) The extension $\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')$ *of* $\Pi_{Z'}$ *by* M_X *is a* **fundamental extension** *of* $\Pi_{Z'}$.

(ii) There is a **natural commutative diagram***:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \prod_{\Delta_X/\Delta_{X''}} M_X & \longrightarrow & \mathbb{S}_{X''/X}(\mathcal{D}'') & \longrightarrow & \Pi_{X\times X} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
1 & \longrightarrow & \prod_{\Delta_X/\Delta_{X'}} M_X & \longrightarrow & \mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')) & \longrightarrow & \Pi_{X\times X} & \longrightarrow & 1
\end{array}$$

[which is well-defined up to $\Delta_{X'} \times \{1\}$-inner automorphisms — cf. Remark 4 below].

(iii) Relative to the commutative diagram of (ii), the natural cyclotomically outer action of $(k^\times)^\wedge$ on $\mathbb{S}_{X''/X}(\mathcal{D}'')$ lies over the composite of the map $(k^\times)^\wedge \to (k^\times)^\wedge$ given by **raising to the** $[\Delta_{X'} : \Delta_{X''}]$**-power** *with the natural cyclotomically outer action of $(k^\times)^\wedge$ on $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$. In particular, if N is a positive integer that divides $[\Delta_{X'} : \Delta_{X''}]$, then the natural cyclotomically outer action of an element of $(k^\times)^\wedge$ on $\mathbb{S}_{X''/X}(\mathcal{D}'')$ lies over the cyclotomically outer action of an element of $\{(k^\times)^\wedge\}^N$ on $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$.*

Proof. To verify assertion (i), *observe* that it is immediate from the definitions that

$$\iota'(X) \times_{Z'} (X'' \times X'') \subseteq X'' \times X''$$

is equal to the $\Delta_{X'}/\Delta_{X''}$-*orbit* of $\iota''(X) \times_{Z''} (X'' \times X'') \subseteq X'' \times X''$. Now assertion (i) follows by translating this observation into the language of étale cohomology classes associated to subvarieties; assertions (ii), (iii) follow formally from assertion (i) and the definitions of the various objects involved. ☐

Remark 4. Relative to the commutative diagram of Proposition 1.7, (ii), note that, although $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$ is, by definition, only well-defined *up to* $\Delta_{X'} \times \{1\}$-*inner automorphisms*, the *push-forward* of $\mathbb{S}_{X''/X}(\mathcal{D}'')$ by

$$\prod_{\Delta_X/\Delta_{X''}} M_X \to \prod_{\Delta_X/\Delta_{X'}} M_X$$

is well-defined *up to* $\Delta_{X''} \times \{1\}$-*inner automorphisms*. That is to say, the push-forward extension implicit in this commutative diagram furnishes a *canonically more rigid* version of the extension $\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$.

Definition 1.3.

(i) We shall refer to the extension $\mathbb{S}_{X''/X^*}(\mathcal{D}'')$ [of Π_{Z^*}] constructed from the fundamental extension $\mathcal{D}''$ as the *[X''/X^*-]symmetrization* of $\mathcal{D}''$, or, alternatively, as a *symmetrized fundamental extension*. We shall refer to the extension $\mathrm{Tr}_{X''/X':X^*}(\mathcal{D}'')$ [of Π_{Z^*}] constructed from the fundamental extension $\mathcal{D}''$ as the *[$X''/X' : X^*$-]trace* of $\mathcal{D}''$, or, alternatively, as a *trace-symmetrized fundamental extension*.

(ii) If $\mathcal{D}'$ is a fundamental extension of $\Pi_{Z'}$, then we shall refer to as a *morphism of trace type* any morphism

$$\mathbb{S}_{X''/X}(\mathcal{D}'') \to \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by composing the morphism

$$\mathbb{S}_{X''/X}(\mathcal{D}'') \to \mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}''))$$

of Proposition 1.7, (ii), with a morphism

$$\mathbb{S}_{X'/X}(\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'')) \to \mathbb{S}_{X'/X}(\mathcal{D}')$$

arising [by the *functoriality* of the construction of "$\mathbb{S}_{X'/X}(-)$"] from an isomorphism of [fundamental] extensions $\mathrm{Tr}_{X''/X':X'}(\mathcal{D}'') \xrightarrow{\sim} \mathcal{D}'$ of $\Pi_{Z'}$ by M_X [which induces the identity on $\Pi_{Z'}$, M_X].

(iii) We shall refer to as a *pro-symmetrized fundamental extension* any compatible system [indexed by the natural numbers]

$$\ldots \twoheadrightarrow \mathcal{S}_i \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{S}_j \twoheadrightarrow \ldots \twoheadrightarrow \Pi_{X\times X}$$

of morphisms of trace type [up to inner automorphisms of the appropriate type] between symmetrized fundamental extensions, where $\mathcal{S}_i$ is the X_i/X-symmetrization of a fundamental extension of Π_{Z_i}; Z_i is the diagonal covering associated to an open normal subgroup $\Pi_{X_i} \subseteq \Pi_X$; the intersection of the Π_{X_i} is trivial. In this situation, we shall refer to the inverse limit profinite group

$$\mathcal{S}_\infty \stackrel{\mathrm{def}}{=} \varprojlim_i \ \mathcal{S}_i$$

as the *limit of the pro-symmetrized fundamental extension* $\{\mathcal{S}_i\}$; any profinite group $\mathcal{S}_\infty$ arising in this fashion will be referred to as a *pro-fundamental extension [of* $\Pi_{X\times X}$*]*.

(iv) Let $S \subseteq X(k)$ be a finite subset; $\mathcal{S}'$ an X'/X-symmetrization of a fundamental extension $\mathcal{D}'$ of $\Pi_{Z'}$. Then we shall write

$$\mathcal{S}'_S \stackrel{\mathrm{def}}{=} \prod_{x\in S} \mathcal{S}'_{D_x \times_{G_k} \Pi_X}$$

[where the product is to be understood as the fiber product over Π_X]. Thus, $\mathcal{S}'_S$ is an extension of Π_X by a product of copies of M_X. Similarly, given a projective system $\{\mathcal{S}_i\}$ as in (iii), we obtain a projective system $\{(\mathcal{S}_i)_S\}$, with inverse limit:

$$(\mathcal{S}_\infty)_S$$

We shall refer to $(\mathcal{S}_\infty)_S$ as a *maximal abelian* S*-pro-cuspidalization [of* Π_X *at* S*]*. Observe that if $T \subseteq X(k)$ is a finite subset such that $S \subseteq T$, then we obtain a natural *projection morphism* $(\mathcal{S}_\infty)_T \to (\mathcal{S}_\infty)_S$.

Remark 5. Let $\mathcal{D}$ be as in Definition 1.2, (iii); $\mathcal{S}'$, $\{\mathcal{S}_i\}$, $\mathcal{S}_\infty$ as in Definition 1.3, (iii), (iv). Then observe that it follows from Proposition 1.6, (i), that the *"daggered versions"* $\mathcal{D}^\dagger$, $(\mathcal{S}')^\dagger$, $\mathcal{S}_i^\dagger$, and $\mathcal{S}_\infty^\dagger$ [i.e., the result of applying "$\times_{G_k} G_k^\dagger$" to $\mathcal{D}$, $\mathcal{S}'$, $\mathcal{S}_i$, and $\mathcal{S}_\infty$] are *slim*. In particular, if $S \subseteq X^{\text{cl}}$ is *any finite set of closed points* of X, then we may form the objects

$$\mathcal{D}_S^\dagger; \quad (\mathcal{S}')_S^\dagger; \quad (\mathcal{S}_i)_S^\dagger; \quad (\mathcal{S}_\infty)_S^\dagger$$

by passing to a Galois covering $X_{k_S} \to X$ [i.e., the result of base-changing X to some finite Galois extension k_S of k] such that the closed points of X_{k_S} that lie over points of S are *rational* over k_S; forming the various objects in question over X_{k_S} [cf. Definition 1.2, (iii); Definition 1.3, (iv)]; and, finally, *"descending to X"* via the *natural outer action* of $G_k/G_{k_S}^\dagger$ on the various objects in question [cf. the exact sequence "$1 \to (-) \to \text{Aut}(-) \to \text{Out}(-) \to 1$" of §0; the *slimness* mentioned above]. Thus, in the remainder of this paper, we shall often speak of the various objects defined in Definition 1.2, (iii); Definition 1.3, (iv), *even when the points of the finite set S are not necessarily rational over k.*

Before proceeding, we note the following:

Lemma 1.1. **(Conjugacy Estimate)** *Let $H \subseteq \Delta_X$ be a normal open subgroup; $a \in \Delta_X/H$ an element not equal to the identity; N a $\Sigma^\dagger$-integer [cf. §0]. Then there exists a normal open subgroup $H' \subseteq \Delta_X$ contained in H such that for any normal open subgroup $H'' \subseteq \Delta_X$ contained in H' and any $a'' \in \Delta_X/H''$ that lifts a, the* **cardinality of the H-conjugacy class** $\text{Conj}(a'', H'') \subseteq \Delta_X/H''$ *of a'' in Δ_X/H'' is* **divisible** *by N.*

Proof. In the notation of the statement of Lemma 1.1, let us denote by $Z(a'', H'') \subseteq H$ the subgroup of elements $\delta \in H$ such that $\delta \cdot a'' \cdot \delta^{-1} = a''$ in Δ_X/H''. Then it is immediate that if a' is the image of a'' in Δ_X/H', then $Z(a'', H'') \subseteq Z(a', H')$, so the cardinality of $\text{Conj}(a'', H'') \cong H/Z(a'', H'')$ is *divisible* by the cardinality of $\text{Conj}(a', H') \cong H/Z(a', H')$. Thus, it suffices to find a normal open subgroup $H' \subseteq H$ such that for any $a' \in \Delta_X/H'$ that lifts a, the cardinality of $\text{Conj}(a', H')$ is divisible by N.

To this end, let us consider, for some prime number $l \in \Sigma^\dagger$, the *maximal pro-l quotient* $H[l]$ of the *abelianization* H^{ab} of H. Note that Δ_X/H acts by conjugation on H^{ab}, $H[l]$. Now I *claim* that there exists a [nonzero] $h_l \in H[l]$ such that $a(h_l) \neq h_l$. Indeed, if this claim were false, then it would follow that a acts trivially on $H[l]$. But since a induces a nontrivial automorphism of the covering of $X_{\overline{k}}$ determined by H, it follows that a induces a nontrivial automorphism of the l-power torsion points of the Jacobian of $X_{\overline{k}}$ [since these points are Zariski dense in this Jacobian] — a contradiction. This completes the proof of the *claim*.

Now let $j \in H$ be an element that lifts the various h_l obtained above for the [finite collection of] primes l that divide N; let $a_X \in \Delta_X$ be an element that lifts a. Then observe that for some integer power M of N that is *independent* of the choice of a_X, the image of $j^n \cdot a_X \cdot j^{-n} \cdot a_X^{-1}$ in $H^{\mathrm{ab}} \otimes (\mathbb{Z}/M\mathbb{Z})$ is *nonzero*, for $n \in \widehat{\mathbb{Z}}$ with nonzero image in $\widehat{\mathbb{Z}}/N \cdot \widehat{\mathbb{Z}}$. Thus, if we take H' equal to the inverse image of $M \cdot H^{\mathrm{ab}}$ in $H(\subseteq \Delta_X)$, we obtain that the intersection of the subgroup $j^{\widehat{\mathbb{Z}}} \subseteq H$ with $Z(a', H')$ [where $a' \in \Delta_X/H'$ lifts a] *does not contain* j^n, for $n \in \widehat{\mathbb{Z}}$ with nonzero image in $\widehat{\mathbb{Z}}/N \cdot \widehat{\mathbb{Z}}$. But this implies that the intersection $(j^{\widehat{\mathbb{Z}}}) \bigcap Z(a', H') \subseteq j^{N \cdot \widehat{\mathbb{Z}}}$, hence that $[H : Z(a', H')]$ is divisible by N, as desired. $\square$

Next, we consider the following *fundamental extensions* of $\Pi_{Z''}$, $\Pi_{Z'}$:

$$\underline{\mathcal{D}}'' \stackrel{\mathrm{def}}{=} \Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z'']}; \quad \underline{\mathcal{D}}' \stackrel{\mathrm{def}}{=} \mathrm{Tr}_{X''/X':X'}(\underline{\mathcal{D}}'')$$

[cf. Proposition 1.5, (ii)]. Note that in this situation, it follows immediately from the definitions that we obtain a natural isomorphism $\underline{\mathcal{D}}' \xrightarrow{\sim} \Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z']}$, which we shall use in the following discussion to *identify* $\underline{\mathcal{D}}'$, $\Pi_{\mathbb{L}^{\times}_{\mathrm{diag}}[Z']}$. Thus, we have *fundamental sections*:

$$\Pi_{U_{Z''}} \twoheadrightarrow \underline{\mathcal{D}}''; \quad \Pi_{U_{Z'}} \twoheadrightarrow \underline{\mathcal{D}}'$$

[cf. Proposition 1.5, (iii)]. In particular, by pulling back from Z'' to $X'' \times X''$, we obtain a surjection:

$$\Pi_{U_{X'' \times X''}} \twoheadrightarrow \underline{\mathcal{D}}''_{X'' \times X''}$$

Now if we apply the natural *outer* $(\Delta_X/\Delta_{X''}) \times \{1\}$*-action* on $\Pi_{U_{X'' \times X''}}$ to this surjection, it follows from the definition of "$\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$" that we obtain a natural homomorphism

$$\Pi_{U_{X'' \times X''}} \twoheadrightarrow \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X'' \times X''}$$

which is easily verified [cf. Proposition 1.6, (ii), (iii)] to be *surjective*. Since, moreover, the construction of this surjective homomorphism is manifestly compatible with the outer actions of $\mathrm{Gal}(X''/X)$ on both sides, we thus obtain a *natural surjection*:

$$\Pi_{U_{X \times X}} \twoheadrightarrow \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$$

Now let us denote by

$$D_X \subseteq \Pi_{U_{X \times X}}$$

the *decomposition group* of the subvariety $\iota(X) \subseteq X \times X$. [Thus, D_X is well-defined up to conjugation; here, we assume that we have chosen a conjugate that maps to the image of the diagonal embedding $\Pi_X \hookrightarrow \Pi_{X \times X}$ via the natural surjection $\Pi_{U_{X \times X}} \twoheadrightarrow \Pi_{X \times X}$.] Observe that we have a *natural exact sequence*

$$1 \to I_X \to D_X \to \Pi_X \to 1$$

[where I_X — i.e., the *inertia subgroup* of D_X — is defined so as to make the sequence exact], together with a natural isomorphism $I_X \cong M_X$. Also, we shall write $D_{X'} \stackrel{\text{def}}{=} D_X \bigcap \Pi_{U_{X' \times X'}}$; $D_{X''} \stackrel{\text{def}}{=} D_X \bigcap \Pi_{U_{X'' \times X''}}$. Since the construction just carried out for *double primed* objects may also be carried out for *single primed* objects, we thus obtain the following:

Proposition 1.8. **(Symmetrized Fundamental Sections)** *In the notation of the discussion above:*

(i) There is a natural commutative diagram:

$$\begin{array}{ccccc} D_X & \subseteq & \Pi_{U_{X \times X}} & \twoheadrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'') \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \\ D_X & \subseteq & \Pi_{U_{X \times X}} & \twoheadrightarrow & \mathbb{S}_{X'/X}(\underline{\mathcal{D}}') \end{array}$$

[where the vertical arrow on the right is the morphism in the diagram of Proposition 1.7*, (ii)].*

(ii) Denote by means of a subscript X'' the result of pulling back extensions of $\Pi_{X \times X}$, $\Pi_{Z''}$, $\Pi_{X'' \times X''}$ to $\Pi_{X''}$ [via the diagonal inclusion]. Then the projection [cf. the fiber product defining $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$] to the factor labeled "$\Delta_{X''}/\Delta_{X''}$" detemines a **natural surjection**

$$\zeta'' : \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X''} \twoheadrightarrow \underline{\mathcal{D}}''_{X''}$$

whose **restriction** *to $D_{X''}$ [i.e., relative to the arrows in the first line of the commutative diagram of (i)] defines an* **isomorphism** *$D_{X''} \xrightarrow{\sim} \underline{\mathcal{D}}''_{X''}$. Moreover, the cuspidal subgroup of $D_{X''}$ maps isomorphically onto the factor of M_X in $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$ labeled "$\Delta_{X''}/\Delta_{X''}$". In particular, if we denote by*

$$\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}$$

the quotient of $\mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')$ by this factor of M_X, then ζ'' determines a surjection

$$\zeta''_{\neq} : \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}_{X''} \twoheadrightarrow \Pi_{X''}$$

whose **restriction** *to the quotient $D_{X''} \twoheadrightarrow \Pi_{X''}$ is equal to the identity $\Pi_{X''} \xrightarrow{\sim} \Pi_{X''}$ [up to geometric inner automorphisms]. Thus, we have a* **natural commutative diagram** *[well-defined up to geometric inner automorphisms]*

$$\begin{array}{ccccc} D_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')_{X''} & \stackrel{\zeta''}{\longrightarrow} & \underline{\mathcal{D}}''_{X''} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}')^{\neq}_{X''} & \stackrel{\zeta''_{\neq}}{\longrightarrow} & \Pi_{X''} \end{array}$$

in which the two horizontal composites are isomorphisms; the vertical arrows are surjections; both squares are **cartesian**.

(iii) If we carry out the construction of (ii) for the single primed objects, then the commutative diagram of (i) induces a **natural commutative diagram** *[well-defined up to geometric inner automorphisms]:*

$$\begin{array}{ccccc} \Pi_{X''} & \hookrightarrow & \mathbb{S}_{X''/X}(\underline{\mathcal{D}}'')^{\neq}_{X''} & \stackrel{\zeta''_{\neq}}{\longrightarrow} & \Pi_{X''} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{X'} & \hookrightarrow & \mathbb{S}_{X'/X}(\underline{\mathcal{D}}')^{\neq}_{X'} & \stackrel{\zeta'_{\neq}}{\longrightarrow} & \Pi_{X'} \end{array}$$

Moreover, there is a natural outer action of $\mathrm{Gal}(X''/X)$ *(respectively,* $\mathrm{Gal}(X'/X)$*) on the first (respectively, second) line of this diagram; these outer actions are compatible with one another.*

(iv) When considered up to cyclotomically inner automorphisms, the sections of $\zeta''_{\neq}$ *form a* **torsor** *over the group*

$$\prod_{(\Delta_X/\Delta_{X''})\backslash(\Delta_{X''}/\Delta_{X''})} ((k'')^{\times})^{\wedge}$$

[where the "$\backslash$" denotes the set-theoretic complement]. The $\mathrm{Gal}(X''/X)$**-equivariant sections** *of* $\zeta''_{\neq}$ *form a torsor over the* $\mathrm{Gal}(X''/X)$*-invariant subgroup of this group. Similar statements hold for the single primed objects.*

(v) The double and single primed torsors of equivariant sections of (iv) are related, via the right-hand square of the diagram of (iii), by a homomorphism

$$\left\{ \prod_{(\Delta_X/\Delta_{X''})\backslash(\Delta_{X''}/\Delta_{X''})} ((k'')^{\times})^{\wedge} \right\}^{\mathrm{Gal}(X''/X)} \to \left\{ \prod_{(\Delta_X/\Delta_{X'})\backslash(\Delta_{X'}/\Delta_{X'})} ((k')^{\times})^{\wedge} \right\}^{\mathrm{Gal}(X'/X)}$$

[where the superscripts denote the result of taking invariants with respect to the action of the superscripted group] that satisfies the following property:

> *An element* ξ'' *of the domain maps to an element of the codomain whose component in the factor labeled* $a' \in \Delta_X/\Delta_{X'}$ *is a product of elements of* $((k')^{\times})^{\wedge}$ *of the form* $\mathcal{N}_{k'_{a''}/k'}(\lambda'')^{n''}$.

Here, $a'' \in (\Delta_X/\Delta_{X''})\backslash(\Delta_{X'}/\Delta_{X''})$ *maps to* a' *in* $\Delta_X/\Delta_{X'}$*;* $\lambda'' \in ((k'')^{\times})^{\wedge}$ *is the component of* ξ'' *in the factor labeled* a''*;* $k'_{a''}$ *is an intermediate field* *extension between k' and k'' such that $\lambda'' \in ((k'_{a''})^\times)^\wedge$; $\mathcal{N}_{k'_{a''}/k'} : ((k'_{a''})^\times)^\wedge \to ((k')^\times)^\wedge$ is the norm map; n'' is the* **cardinality of the $\Delta_{X'}$-conjugacy class** *of a'' in $(\Delta_X/\Delta_{X''})$. In particular, by Lemma 1.1 [where we take "H" to be $\Delta_{X'}$, "H''" to be $\Delta_{X''}$], for a given $\Delta_{X'}$, if, for a given positive integer N, $\Delta_{X''}$ is "sufficiently small", then an* **arbitrary** $\mathrm{Gal}(X''/X)$**-equivariant section** *of $\zeta''_{\neq}$ lies over the* **canonical section** *of $\zeta'_{\neq}$ given in (iii), up to the cyclotomically outer action of some N-th power of an element of the single primed version of the group exhibited in the display of (iv).*

Proof. All of these assertions follow immediately from the definitions [and, in the case of assertion (iv), Proposition 1.4, (ii)]. □

Definition 1.4. Let $\mathcal{D}'$ be a *fundamental extension* of $\Pi_{Z'}$; $\{\mathcal{S}_i\}$ a *pro-symmetrized fundamental extension*, with limit $\mathcal{S}_\infty$ [cf. Definition 1.3, (iii)].

(i) We shall refer to as a *symmetrized fundamental section* a homomorphism

$$\Pi_{U_{X\times X}} \twoheadrightarrow \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by composing the surjection $\Pi_{U_{X\times X}} \twoheadrightarrow \mathbb{S}_{X'/X}(\underline{\mathcal{D}}')$ of Proposition 1.8, (i), with the isomorphism $\mathbb{S}_{X'/X}(\underline{\mathcal{D}}') \xrightarrow{\sim} \mathbb{S}_{X'/X}(\mathcal{D}')$ induced by an isomorphism $\underline{\mathcal{D}}' \xrightarrow{\sim} \mathcal{D}'$ of fundamental extensions of $\Pi_{Z'}$ by M_X [which induces the identity on $\Pi_{Z'}$, M_X]. We shall refer to an inclusion

$$D_X \hookrightarrow \mathbb{S}_{X'/X}(\mathcal{D}')$$

obtained by restricting a symmetrized fundamental section to $D_X \subseteq \Pi_{U_{X\times X}}$ [cf. Proposition 1.8, (i)] as a *fundamental inclusion*.

(ii) We shall refer to a compatible system of symmetrized fundamental sections $\Pi_{U_{X\times X}} \to \mathcal{S}_i$ as a *pro-symmetrized fundamental section* and to the resulting limit homomorphism $\Pi_{U_{X\times X}} \to \mathcal{S}_\infty$ as a *pro-fundamental section*. Similarly, we have a notion of *"pro-fundamental inclusions"*.

Remark 6. Thus, by the above discussion, if we take the "$\mathcal{S}_i$" to be the symmetrizations of the $\Pi_{\mathbb{L}^\times_{\mathrm{diag}}[Z']}$ as in Proposition 1.5, (ii), then we obtain natural *pro-fundamental sections* and *pro-fundamental inclusions* [cf. Proposition 1.8, (i), (ii), (iii)].

Proposition 1.9. **(Maximal Cuspidally Abelian Quotients)** *Let* $\{\mathcal{S}_i\}$ *be a* **pro-symmetrized fundamental extension***, with* **limit** $\mathcal{S}_\infty$ *[cf. Definition* 1.3*, (iii)] and* **pro-fundamental section** $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ *[cf. Definition* 1.4*, (ii)];* $S \subseteq X^{\mathrm{cl}}$ *a finite set of closed points [cf. Remark* 5*]. Then:*

(i) The pro-fundamental section $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ *determines a* **surjection**

$$\Pi_{U_S} \twoheadrightarrow (\mathcal{S}_\infty)_S$$

[cf. Proposition 1.6*, (iii)]. The resulting quotient of* Δ_{U_S} *(respectively,* Π_{U_S}*) is the* **maximal cuspidally abelian quotient** $\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-ab}}$ *(respectively,* $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab}}$*) of* Δ_{U_S} *(respectively,* Π_{U_S}*).*

(ii) The quotient of $\Delta_{U_{X\times X}}$ *(respectively,* $\Pi_{U_{X\times X}}$*) induced by the pro-fundamental section* $\Pi_{U_{X\times X}} \twoheadrightarrow \mathcal{S}_\infty$ *is the* **maximal cuspidally abelian quotient** *[which we shall denote by]* $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta_{U_{X\times X}}^{\text{c-ab}}$ *(respectively,* $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{U_{X\times X}}^{\text{c-ab}}$*) of* $\Delta_{U_{X\times X}}$ *(respectively,* $\Pi_{U_{X\times X}}$*).*

Proof. Indeed, this follows as in the proof of Proposition 1.6, (iii), (iv), by observing that the cuspidal subgroup of the maximal cuspidally abelian quotient of Δ_{U_S} (respectively, $\Delta_{U_{X\times X}}$) is naturally isomorphic to the inverse limit of the cuspidal subgroups of the maximal cuspidally *central* quotients of the $\Delta_{U_S} \times_{\Delta_X} \Delta_{X'}$ ($\subseteq \Delta_{U_S}$) (respectively, $\Delta_{U_{X'\times X'}}$) [as $\Delta_{X'} \subseteq \Delta_X$ ranges over the open normal subgroups of Δ_X]. ☐

Proposition 1.10. **(Automorphisms and Commensurators)** *Let* $\{\mathcal{S}_i\}$ *be a* **pro-symmetrized fundamental extension***, with* **limit** $\mathcal{S}_\infty$ *[cf. Definition* 1.3*, (iii)] and* **pro-fundamental inclusion** $D_X \hookrightarrow \mathcal{S}_\infty$ *[cf. Definition* 1.4*, (ii)]. Then:*

(i) Any automorphism α *of the profinite group* $\Pi_{U_{X\times X}}^{\text{c-ab}}$ *which*

(a) is compatible with the natural surjection $\Pi_{U_{X\times X}}^{\text{c-ab}} \twoheadrightarrow \Pi_{X\times X}$ *and induces the identity on* $\Pi_{X\times X}$*;*

(b) preserves the image of $M_X \cong I_X \subseteq D_X$ *via the natural inclusion* $D_X \hookrightarrow \Pi_{U_{X\times X}}^{\text{c-ab}}$

is **cuspidally inner***.*

(ii) Π_X *(respectively,* Δ_X*) is* **commensurably terminal** *[cf. §0] in* $\Pi_{X\times X}$

(respectively, $\Delta_{X\times X}$).

(iii) D_X *is* **commensurably terminal** *in* $\mathcal{S}_i$*,* $\mathcal{S}_\infty \cong \Pi^{\text{c-ab}}_{U_{X\times X}}$.

Proof. First, we verify assertion (i). By Proposition 1.9, (ii), we have a natural isomorphism $\Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \mathcal{S}_\infty$, so we may think of α as an automorphism of $\mathcal{S}_\infty$. In light of (a); Proposition 1.6, (iii), it follows that α is compatible with the natural surjections $\mathcal{S}_\infty \twoheadrightarrow \mathcal{S}_i$. Write α_i for the automorphism of $\mathcal{S}_i$ induced by α. By (a), (b), it follows that α_i is an automorphism of the *extension* $\mathcal{S}_i$ of $\Pi_{X\times X}$ by a product of copies of M_X which induces the *identity* on both $\Pi_{X\times X}$ and the product of copies of M_X [cf. the definition by a certain fiber product of the symmetrized fundamental extension $\mathcal{S}_i$]. [Here, we note that the fact that α_i induces the identity on each copy of M_X follows by considering the *non-torsion* [cf. Propositions 1.2, (ii); 1.5, (i), (ii)] *extension class* determined by that copy of M_X [which is preserved by α_i!], together with the fact that α_i induces the identity on the second cohomology groups of open subgroups of $\Delta_{X\times X}$ with coefficients in M_X.] Thus, up to cyclotomically inner automorphisms, α_i arises from a collection of elements of $(k_i^\times)^\wedge$, where k_i is some finite Galois extension of k [cf. Proposition 1.4, (ii)], one corresponding to each copy of M_X. Moreover, since these copies of M_X are *permuted* by the action of $\Pi_{X\times X}$ by conjugation, it follows that [up to cyclotomically inner automorphisms] α_i arises from a *single element* of $(k_i^\times)^\wedge$, which in fact belongs to $(k^\times)^\wedge$ ($\subseteq (k_i^\times)^\wedge$) [as one sees by considering the conjugation action via the "G_k portion" of $\Pi_{X\times X}$]. On the other hand, since the α_i form a *compatible system* of automorphisms of the $\mathcal{S}_i$, it follows from Proposition 1.7, (iii), that this element of $(k^\times)^\wedge$ must be equal to 1, as desired.

Next, to verify assertion (ii), let us observe that it suffices to show that Δ_X is commensurably terminal in $\Delta_{X\times X}$. But this follows immediately from the fact that Δ_X is *slim* [cf. Proposition 1.6, (i)]. Finally, we consider assertion (iii). Clearly, it suffices to show that D_X is commensurably terminal in $\mathcal{S}_i$. By assertion (ii), to verify this commensurable terminality, it suffices to show that the [manifestly abelian] cuspidal subgroup $H_i \subseteq \mathcal{S}_i$ [i.e., relative to the natural surjection $\mathcal{S}_i \twoheadrightarrow \Pi_{X\times X}$] satisfies the following property: Every $h \in H_i$ such that $h^\delta - h \in D_X$, for all δ in some open subgroup J of D_X, satisfies $h \in D_X$. But this property follows immediately [cf. the definition by a certain fiber product of the symmetrized fundamental extension $\mathcal{S}_i$] from the fact that, for J sufficiently small, the J-module $H_i/(D_X \bigcap H_i)$ is isomorphic to a direct product of a finite number of copies of M_X. $\square$

The following result is the *main result* of the present §1:

Theorem 1.1. **(Reconstruction of Maximal Cuspidally Abelian Quotients)** *Let X, Y be* **hyperbolic curves** *over a* **finite** *or* **nonarchimedean local field***; denote the base fields of X, Y by k_X, k_Y, respectively. Let Σ_X (respectively, Σ_Y) be a* **set of prime numbers** *that contains at least one prime number that is invertible in k_X (respectively, k_Y); write Δ_X (respectively, Δ_Y) for the* **maximal cuspidally pro-$\Sigma_X^\dagger$** *(respectively,* **pro-$\Sigma_Y^\dagger$***)* **quotient of the maximal pro-Σ_X** *(respectively,* **pro-Σ_Y***)* **quotient** *of the* **tame fundamental group** *of $X_{\overline{k}_X}$ (respectively, $Y_{\overline{k}_Y}$) [where "tame" is with respect to the complement of $X_{\overline{k}_X}$ (respectively, $Y_{\overline{k}_Y}$) in its canonical compactification], and Π_X (respectively, Π_Y) for the corresponding quotient of the étale fundamental group of X (respectively, Y). Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an **isomorphism** *of profinite groups. Then:*

(i) We have $\Sigma_X^\dagger = \Sigma_Y^\dagger$; write $\Sigma^\dagger \stackrel{\text{def}}{=} \Sigma_X^\dagger = \Sigma_Y^\dagger$. Moreover, k_X is a **finite field** *if and only if k_Y is; α preserves the* **decomposition groups of cusps***; X is of* **type** *(g, r) [where $g, r \geq 0$ are integers such that $2g-2+r > 0$] if and only if Y is of type (g, r). Finally, if k_X, k_Y are* **nonarchimedean local***, then their* **residue characteristics coincide***.*

(ii) α is **compatible** *with the natural* **quotients** *$\Pi_X \twoheadrightarrow G_{k_X}$, $\Pi_Y \twoheadrightarrow G_{k_Y}$.*

(iii) Assume that X, Y are **proper***. Denote by $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ the* **maximal cuspidally** *[i.e., relative to the natural surjections $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{X\times X}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi_{Y\times Y}$]* **abelian quotients** *[cf. Proposition 1.9]. Then there is a commutative diagram [well-defined up to* **cuspidally inner automorphisms***]*

$$\begin{array}{ccc}
\Pi^{\text{c-ab}}_{U_{X\times X}} & \xrightarrow{\alpha^{\text{c-ab}}} & \Pi^{\text{c-ab}}_{U_{Y\times Y}} \\
\downarrow & & \downarrow \\
\Pi_{X\times X} & \xrightarrow{\alpha\times\alpha} & \Pi_{Y\times Y}
\end{array}$$

— where, the horizontal arrows are **isomorphisms** *which are* **compatible** *with the* **natural inclusions** *$D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ [cf. Proposition 1.8, (i)]; the vertical arrows are the natural surjections. Finally, the correspondence*

$$\alpha \mapsto \alpha^{\text{c-ab}}$$

is **functorial** *[up to cuspidally inner automorphisms] with respect to α.*

Proof. First, we consider assertions (i), (ii). Note that k_X is *finite* if and only if, for every open subgroup $H \subseteq \Pi_X$, the quotient of the abelianization

H^{ab} by the closure of the torsion subgroup of H^{ab} is *topologically cyclic* [cf. [Tama], Proposition 3.3, (ii)]; a similar statement holds for k_Y, Π_Y. Thus, k_X is *finite* if and only if k_Y is. Now suppose that k_X, k_Y are *finite*. Then assertion (ii) also follows from [Tama], Proposition 3.3, (ii). The fact that $\Sigma^\dagger_X = \Sigma^\dagger_Y$ then follows from the following observation: The subset $\Sigma^\dagger_X \subseteq \mathfrak{Primes}$ is the subset on which the function

$$\mathfrak{Primes} \ni l \mapsto \dim_{\mathbb{Q}_l}((\Delta_X)^{\mathrm{ab}} \otimes \mathbb{Q}_l)$$

attains its *maximum* value [cf. [Tama], Proposition 3.1]; a similar statement holds for Y. Now by considering the respective outer actions of G_{k_X}, G_{k_Y} on the *maximal pro-l quotients* of Δ_X, Δ_Y, for some $l \in \Sigma^\dagger$, we obtain that α preserves the *decomposition groups of cusps* [hence that X is of type (g, r) if and only if Y is of type (g, r)], by [Mzk9], Corollary 2.7, (i). This completes the proof of assertions (i), (ii) in the *finite field case*.

Next, let us assume that k_X, k_Y are *nonarchimedean local*. Then the portion of assertion (i) concerning $\Sigma_X = \Sigma^\dagger_X$, $\Sigma_Y = \Sigma^\dagger_Y$ follows by considering the *cohomological dimension* of Π_X, Π_Y — cf., e.g., Proposition 1.3, (ii) [in the proper case]. As for assertion (ii), if the cardinality of $\Sigma \stackrel{\mathrm{def}}{=} \Sigma^\dagger$ is ≥ 2, then assertion (ii) follows from the evident pro-Σ analogue of [Mzk5], Lemma 1.3.8; if the cardinality of Σ is 1, then assertion (ii) follows from Lemma 1.2, (c), (d) below. Now the portion of assertion (i) concerning the residue characteristics of k_X, k_Y follows from assertion (ii) and [Mzk5], Proposition 1.2.1, (i); the fact that α preserves the *decomposition groups of cusps* [hence that X is of type (g, r) if and only if Y is of type (g, r)] follows from [Mzk9], Corollary 2.7, (i). This completes the proof of assertions (i), (ii) in the *nonarchimedean local field case*.

Finally, we consider assertion (iii). It follows from the definitions that α induces an isomorphism $M_X \xrightarrow{\sim} M_Y$. If, moreover, $Z'_X \to X$, $Z'_Y \to Y$ are *diagonal coverings* corresponding to [connected] finite étale Galois coverings $X' \to X$, $Y' \to Y$ that arise from open subgroups of Π_X, Π_Y that correspond via α, then α induces an isomorphism of group cohomology modules

$$H^2(\Pi_{Z'_X}, M_X) \xrightarrow{\sim} H^2(\Pi_{Z'_Y}, M_Y)$$

that preserves the extension classes associated to *fundamental extensions* of $\Pi_{Z'_X}$, $\Pi_{Z'_Y}$ [cf. Proposition 1.5, (i)]. In particular, if $\mathcal{D}'$ (respectively, $\mathcal{E}'$) is a fundamental extension of $\Pi_{Z'_X}$ (respectively, $\Pi_{Z'_Y}$), then α induces an isomorphism

$$\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$$

which is compatible with the morphisms $M_X \xrightarrow{\sim} M_Y$, $\Pi_{Z'_X} \xrightarrow{\sim} \Pi_{Z'_Y}$ already induced by α, and, moreover, *uniquely determined*, up to cyclotomically inner automorphisms, and the action of $(k_X^\times)^\wedge$ (respectively, $(k_Y^\times)^\wedge$) [cf. Proposition 1.4, (ii)]. On the other hand, by allowing X', Y' to *vary*, taking *symmetrizations* of the fundamental extensions involved [which may be constructed entirely *group-theoretically*!], and making use of the vertical morphism in the center of the diagram of Proposition 1.7, (ii) [again an object which may be constructed entirely *group-theoretically*!], it follows from Proposition 1.7, (iii), that the indeterminacy of the isomorphism $\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$ arising from the action of $(k_X^\times)^\wedge$, $(k_Y^\times)^\wedge$ *"converges to the identity indeterminacy"* [i.e., by taking $\mathcal{D}' \xrightarrow{\sim} \mathcal{E}'$ to arise as just described from an isomorphism of fundamental extensions $\mathcal{D}'' \xrightarrow{\sim} \mathcal{E}''$ associated to [connected] finite étale coverings $X'' \to X'$, $Y'' \to Y'$ [that arise from open subgroups of Π_X, Π_Y that correspond via α], where the open subgroups $\Pi_{X''} \subseteq \Pi_{X'}$, $\Pi_{Y''} \subseteq \Pi_{Y'}$ are *sufficiently small*]. Thus, in light of the manifest *functoriality* of the vertical morphism in the center of the diagram of Proposition 1.7, (ii) [the detailed explication of which, in terms of various commutative diagrams, is a routine task which we leave to the reader!], we obtain an isomorphism

$$\{\mathcal{S}_i\} \xrightarrow{\sim} \{\mathcal{T}_j\}$$

of *pro-symmetrized fundamental extensions* [cf. Definition 1.3, (iii)] of $\Pi_{X\times X}$, $\Pi_{Y\times Y}$, respectively, which arises from α and is *completely determined up to cyclotomically inner automorphisms*. Here, we pause to note that although in the construction of the symmetrization of a fundamental extension $\mathcal{D}'$ (respectively, $\mathcal{E}'$), one must, a priori, contend with a certain *indeterminacy* with respect to $\Delta_{X'} \times \{1\}$- (respectively, $\Delta_{Y'} \times \{1\}$-)inner automorphisms [cf., e.g., Proposition 1.7, (ii)], in fact, by allowing X', Y' to vary, this indeterminacy also *"converges to the identity indeterminacy"* [cf. Remark 4].

Thus, in summary, α induces an *isomorphism* [well-defined up to *cyclotomically* [or, alternatively, *cuspidally*] *inner automorphisms*]

$$\mathcal{S}_\infty \xrightarrow{\sim} \mathcal{T}_\infty$$

of *pro-fundamental extensions* of $\Pi_{X\times X}$, $\Pi_{Y\times Y}$, respectively. Moreover, by applying the fact that the left-hand square of the commutative diagram of Proposition 1.8, (ii), is *cartesian*, together with the fact that the *"canonical section"* of "$\zeta'_{\neq}$" that appears in Proposition 1.8, (iii), is *completely determined* [cf. Proposition 1.8, (v); Lemma 1.1] by the condition that it *lie under an arbitrary "equivariant section"* [cf. Proposition 1.8, (iv)] of the "$\zeta''_{\neq}$" associated to coverings "$X'' \to X'$" arising from arbitrarily small open subgroups $\Pi_{X''} \subseteq \Pi_X$, it follows that the isomorphism $\mathcal{S}_\infty \xrightarrow{\sim} \mathcal{T}_\infty$ just obtained is *compatible* with the *pro-fundamental inclusions* $D_X \hookrightarrow \mathcal{S}_\infty$, $D_Y \hookrightarrow \mathcal{T}_\infty$. In particular, by Proposition 1.9, (ii) [cf. also Proposition 1.8, (i)], we conclude that α induces an *isomorphism* [well-defined up to *cuspidally inner automorphisms*]

$$(\mathcal{S}_\infty \cong\ ) \quad \Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{U_{Y\times Y}} \quad (\ \cong \mathcal{T}_\infty)$$

which is compatible with *the natural inclusions* $D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$. Finally, the *functoriality* of this isomorphism follows from the naturality of its construction. $\square$

Remark 7. It follows immediately from the *naturality* of the constructions used in the proof of Theorem 1.1, (iii), that when "α" arises from an *isomorphism of schemes* $X \xrightarrow{\sim} Y$, the resulting $\alpha^{\text{c-ab}}$ of Theorem 1.1, (iii), coincides with the morphism induced on fundamental groups by the resulting isomorphism of schemes $U_{X\times X} \xrightarrow{\sim} U_{Y\times Y}$.

Lemma 1.2. **(Normal Subgroups of the Absolute Galois Group of a Nonarchimedean Local Field)** *Let k be a* **nonarchimedean local field** *of residue characteristic p; write G_k for the* **absolute Galois group** *of k. Also, let us write $I \subseteq G_k$ for the* **inertia subgroup** *of G_k and $W \subseteq I$ for the* **wild inertia subgroup***. [Here, we recall that W is the unique Sylow pro-p subgroup of I.] Let $H \subseteq G_k$ be a* **closed subgroup** *that satisfies [at least] one of the following four conditions:*

(a) H is a **finite group***.*

(b) H **commutes** *with W.*

(c) H is a **pro-prime-to-p group** *[i.e., the order of every finite quotient group of H is prime to p] that is* **normal** *in G_k.*

(d) H is a **topologically finitely generated pro-p group** *that is* **normal** *in G_k.*

Then $H = \{1\}$.

Proof. Indeed, suppose that H satisfies *condition (a)*. Then the fact that $H = \{1\}$ follows from [NSW], Corollary 12.1.3, Theorem 12.1.7. Now suppose that H satisfies *condition (b)*. Then by the well-known functorial isomorphism [arising from *local class field theory*] between the additive group underlying a finite field extension of k that corresponds to an open subgroup $J \subseteq G_k$ and the tensor product with $\mathbb{Q}_p$ of the image of $W \bigcap J$ in the abelianization J^{ab}, it follows immediately that the conjugation action of H on W is *nontrivial*, whenever H is *nontrivial*. Thus we conclude again that $H = \{1\}$. Next, suppose that H satisfies *condition (c)*. Then since H, W are both *normal* in G_k, it follows [by considering commutators of elements of H with elements of W] that arbitrary elements of H *commute* with arbitrary elements of W. In particular, H satisfies condition (b), so we conclude yet again that $H = \{1\}$.

Finally, we assume that H is *nontrivial* and satisfies *condition (d)*. Then I *claim* that H has *trivial image* $\text{Im}(H)$ in G_k/W. Indeed, since I/W, $\text{Im}(H)$ are *normal* in G_k/W, and, moreover, I/W is *pro-prime-to-p*, it follows that these two groups *commute*. On the other hand, since, as is well-known, G_k/I *acts faithfully* [by conjugation, via the cyclotomic character] on I/W, it thus follows that $\text{Im}(H)$ is *trivial*, as asserted. Thus, $H \subseteq W$. Since [as in well-known —

cf., e.g., the proof of [Mzk4], Lemma 15.6] W is a *free pro-p group of infinite rank*, we thus conclude that there exists an open subgroup $U \subseteq W$ [so U is also a *free pro-p group of infinite rank*] containing H such that the natural map

$$H^{\text{ab}} \otimes \mathbb{F}_p \to U^{\text{ab}} \otimes \mathbb{F}_p$$

is *injective*, but *not surjective*. Then it follows immediately from the well-known *theory of free pro-p groups* that there exists a set of *free topological generators* $\{\xi_i\}_{i \in I}$ [so the index set I is *infinite*] of U such that for some *finite* subset $J \subseteq I$, the elements $\{\xi_j\}_{j \in J}$ lie in and topologically generate H. On the other hand, since H is *normal* in U, it follows from the well-known structure of free pro-p groups that we obtain a contradiction. This completes the proof of Lemma 1.2. □

Remark 8. The author would like to thank *A. Tamagawa* for informing him of the content of Lemma 1.2.

Definition 1.5. In the situation of Theorem 1.1, (i), (ii), suppose further that $\Sigma_X = \Sigma_Y$; write $\Sigma \stackrel{\text{def}}{=} \Sigma_X = \Sigma_Y$.

(i) If, for every finite étale covering $X' \to X$ of X arising from an open subgroup $\Pi_{X'} \subseteq \Pi_X$, it holds that the map from $(X')^{\text{cl}+}$ [cf. §0] to conjugacy classes of closed subgroups of $\Pi_{X'}$ given by assigning to a closed point its associated decomposition group is *injective*, then we shall say that X is Σ-*separated*.

(ii) If the map induced by α on closed subgroups of Π_X, Π_Y induces a bijection between the decomposition groups of the points of $X^{\text{cl}+}$, $Y^{\text{cl}+}$, then we shall say that α is *quasi-point-theoretic*. If α is *quasi-point-theoretic*, and, moreover, X, Y are Σ-*separated* — in which case α induces *bijections*

$$X^{\text{cl}} \xrightarrow{\sim} Y^{\text{cl}}; \quad X^{\text{cl}+} \xrightarrow{\sim} Y^{\text{cl}+}$$

— then we shall say that α is *point-theoretic*.

(iii) Suppose further that we are in the *finite field case*. Then we shall say that α is *Frobenius-preserving* if the isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ induced by α [cf. Theorem 1.1, (ii)] maps the Frobenius element of G_{k_X} to the Frobenius element of G_{k_Y}.

Remark 9. In the *finite field case*, when $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, the Frobenius element of G_{k_X} may be characterized as in [Tama], Proposition 3.4, (i), (ii); a similar statement holds for the Frobenius element of G_{k_Y}. [Moreover, in the *proper* case, the Frobenius element of G_{k_X} may be characterized as the element of G_{k_X} that acts on M_X via multiplication by the cardinality of k_X, i.e., the cardinality of $H^1(G_{k_X}, M_X)$ plus 1.] Thus, when $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, any α as in Theorem 1.1, (i), (ii), is *automatically Frobenius-preserving*.

Remark 10. Let us suppose that we are in the situation of Definition 1.5, and that the base fields k_X, k_Y are *finite*. Let us refer to as a *quasi-section* [of $\Pi_X \twoheadrightarrow G_{k_X}$] any closed subgroup $D \subseteq \Pi_X$ [i.e., such as a *decomposition group* of a point $\in X^{\mathrm{cl}}$] that maps isomorphically onto an open subgroup of G_{k_X}. Let us refer to a quasi-section of $\Pi_X \twoheadrightarrow G_{k_X}$ as a *subdecomposition group* if it is contained in some decomposition group of a point $\in X^{\mathrm{cl}}$. Then:

(i) Since X is *not necessarily* Σ*-separated*, it is not necessarily the case that *decomposition groups* of points $\in X^{\mathrm{cl}}$ are *commensurably terminal* in Π_X [cf. Proposition 2.3, (ii), below]. On the other hand, if $D \subseteq \Pi_X$ is a *quasi-section*, and we write $E \stackrel{\mathrm{def}}{=} C_{\Pi_X}(D) \subseteq \Pi_X$ for the *commensurator* of D in Π_X [cf. §0], then one verifies immediately E is also a *quasi-section*. [Indeed, by considering the projection $\Pi_X \twoheadrightarrow G_{k_X}$, it follows immediately that every element of E *centralizes* some open subgroup $D' \subseteq D$; on the other hand, by considering the well-known properties of the action of open subgroups of G_k on abelianizations of open subgroups of Δ_X [i.e., more precisely, the "Riemann hypothesis for abelian varieties over finite fields" — cf., e.g., [Mumf], p. 206], it follows that every *centralizer* of D' in Δ_X is *trivial*, i.e., that $E \bigcap \Delta_X = \{1\}$.]

(ii) It is immediate that any *maximal subdecomposition group* of Π_X is, in fact, a *decomposition group* of some point $\in X^{\mathrm{cl}}$. On the other hand, since X is *not necessarily* Σ*-separated*, it is not clear whether or not every decomposition group of a point $\in X^{\mathrm{cl}}$ is necessarily a maximal subdecomposition group. If X, Y are Σ*-separated*, then the arguments of [Tama], Corollary 2.10, Proposition 3.8, yield a "group-theoretic" characterization of the *subdecomposition groups* [hence also of the maximal subdecomposition groups, i.e., the decomposition groups of points $\in X^{\mathrm{cl}}$] of Π_X, Π_Y in terms of the actions of the Frobenius elements. That is to say, if X, Y are Σ*-separated*, then any *Frobenius-preserving* isomorphism α is *[quasi-]point-theoretic*.

(iii) Nevertheless, as was pointed out to the author by *A. Tamagawa*, even if X, Y are *not necessarily* Σ*-separated*, it is still possible to conclude, essentially from the arguments of [Tama], Corollary 2.10, Proposition 3.8, that:

> Any *Frobenius-preserving* isomorphism α is *quasi-point-theoretic*.

Indeed, it suffices to give a *"group-theoretic" characterization* of the quasi-sections $D \subseteq \Pi_X$ which are decomposition groups of points $\in X^{\mathrm{cl}}$. We may assume [for simplicity] without loss of generality that X, Y are *proper*. Write $E \stackrel{\mathrm{def}}{=} C_{\Pi_X}(D)$; k_D, k_E for the finite extension fields of k_X determined by D, E. Let $H \subseteq \Delta_X$ be a characteristic open subgroup; denote by $Y \to X$ the covering determined by the open subgroup $E \cdot H \subseteq \Pi_X$. Then it follows immediately from the definition of a "decomposition group" that it suffices to give a "group-theoretic" criterion for the condition that $Y(k_D)$ contain a point whose *field of definition* [which is, a priori, some subextension in k_D of k_E] is equal to k_D. In [Tama], the *Lefschetz trace formula* is applied to compute the cardinality of $Y(k_D)$. On the other hand, if we use the superscript "fld-def" to denote the subset of points whose field of definition is equal to the field given in parentheses, and "$|-|$" to denote the cardinality of a finite set, then for any subextension $k' \subseteq k_D$ of k_E, we have

$$|Y(k')| = \sum_{k''} |Y(k'')^{\text{fld-def}}|$$

[where $k'' \subseteq k'$ ranges over the subextensions of k_E]. In particular, by applying *induction* on $[k' : k_E]$, one concludes immediately from the above formula that $|Y(k')^{\text{fld-def}}|$ may be computed from $|Y(k'')|$ for subextensions $k'' \subseteq k'$ of k_E [while $|Y(k'')|$ may be computed, as in [Tama], from the *Lefschetz trace formula*]. This yields the desired "group-theoretic" characterization of the decomposition groups of Π_X.

Remark 11. Note that in the *finite field case*, if α as in Theorem 1.1, (i), (ii), is *Frobenius-preserving*, then the *cardinalities*, hence also the *characteristics*, of k_X, k_Y *coincide*. Indeed, this follows immediately by reducing to the *proper* case via Theorem 1.1, (i), and considering the actions of G_{k_X}, G_{k_Y} [cf. Theorem 1.1, (ii)] on M_X, M_Y [which are compatible relative to the isomorphism $M_X \xrightarrow{\sim} M_Y$ induced by α].

Now we return to the notation of the *discussion preceding Theorem* 1.1. Observe that the automorphism

$$\tau : X \times X \to X \times X$$

given by *switching the two factors* induces an outer automorphism of $\Pi_{U_{X\times X}}$. Moreover, by choosing the basepoints used to form the various fundamental groups involved in an appropriate fashion, it follows that there exists an automorphism

$$\Pi_\tau : \Pi_{U_{X\times X}} \to \Pi_{U_{X\times X}}$$

among those automorphisms induced by τ [i.e., all of which are related to one another by composition with an inner automorphism] which induces the automorphism on $\Pi_{X\times X} = \Pi_X \times_{G_k} \Pi_X$ given by *switching the two factors*; *preserves* the subgroup $D_X \subseteq \Pi_{U_{X \times X}}$; and *preserves and induces the identity automorphism* on the subgroup $I_X \subseteq D_X$ $(\subseteq \Pi_{U_{X \times X}})$. Note that by the *slimness* of Proposition 1.6, (i), together with the well-known *commensurable terminality* of $D_X \subseteq \Pi_{U_{X \times X}}$ in $\Pi_{U_{X \times X}}$ [cf., e.g., [the proof of] [Mzk5], Lemma 1.3.12], it follows that, at least when $\Sigma = \mathfrak{Primes}$, these three conditions [are more than sufficient to] *uniquely determine* Π_τ, up to composition with an inner automorphism arising from I_X; one then obtains a natural Π_τ for *arbitrary* Σ [well-defined up to composition with an inner automorphism arising from I_X] by taking the automorphism *induced on the appropriate quotients* by "Π_τ in the case $\Sigma = \mathfrak{Primes}$".

Proposition 1.11. (Switching the Two Factors) *The automorphism*

$$\Pi_\tau^{\text{c-ab}} : \Pi_{U_{X \times X}}^{\text{c-ab}} \to \Pi_{U_{X \times X}}^{\text{c-ab}}$$

induced by Π_τ *is the* **unique** *automorphism of the profinite group* $\Pi_{U_{X \times X}}^{\text{c-ab}}$*, up to composition with a cuspidally inner automorphism, that satisfies the following two conditions: (a) it preserves the quotient* $\Pi_{U_{X \times X}}^{\text{c-ab}} \twoheadrightarrow \Pi_{X \times X}$ *and induces on this quotient the automorphism on* $\Pi_{X \times X} = \Pi_X \times_{G_k} \Pi_X$ *given by switching the two factors; (b) it preserves the image of* $I_X \subseteq D_X \hookrightarrow \Pi_{U_{X \times X}}^{\text{c-ab}}$.

Proof. This follows immediately from Proposition 1.10, (i). ☐

2. Points and Functions

We maintain the notation of §1 [i.e., the discussion preceding Theorem 1.1]. If $x \in X^{\text{cl}}$, then we shall denote by

$$D_x \subseteq \Pi_X$$

the *decomposition group* of x [well-defined up to conjugation in Π_X]. If $x \in X(k)$, then D_x determines a section $s_x : G_k \to \Pi_X$ [which is well-defined as a geometrically outer homomorphism].

Next, let $S \subseteq X^{\text{cl}}$ be a *finite set.* If n is a $\Sigma^\dagger$-integer [cf. §0], then the *Kummer exact sequence*

$$1 \to \boldsymbol{\mu}_n \to \mathbb{G}_{\text{m}} \to \mathbb{G}_{\text{m}} \to 1$$

[where $\mathbb{G}_{\text{m}} \to \mathbb{G}_{\text{m}}$ is the n-th power map; $\boldsymbol{\mu}_n$ is defined so as to make the sequence exact] on the étale site of X determines a homomorphism $\text{Pic}(X) \to H^2(\Delta_X, \boldsymbol{\mu}_n)$ [where $\text{Pic}(X)$ is the Picard group of X]. Now there is a *unique isomorphism*

$$\boldsymbol{\mu}_n \xrightarrow{\sim} M_X / n \cdot M_X$$

such that the homomorphism $\mathrm{Pic}(X) \to H^2(\Delta_X, \boldsymbol{\mu}_n)$ sends line bundles of degree 1 to the element determined by $1 \in \mathbb{Z}/n\mathbb{Z}$ via the composite of the induced isomorphism $H^2(\Delta_X, \boldsymbol{\mu}_n) \xrightarrow{\sim} H^2(\Delta_X, M_X/n \cdot M_X)$ with the *tautological isomorphism* $H^2(\Delta_X, M_X/n \cdot M_X) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ [cf. Proposition 1.2, (i)]. In the following discussion, we shall identify $\boldsymbol{\mu}_n$ with $M_X/n \cdot M_X$ via this isomorphism.

If we consider the Kummer exact sequence on the étale site of $U_S \subseteq X$ [and pass to the inverse limit with respect to n], then we obtain a *natural homomorphism*

$$\Gamma(U_S, \mathcal{O}_{U_S}^{\times}) \to H^1(\Pi_{U_S}, M_X)$$

[where we note that here, it suffices to consider the group cohomology of Π_{U_S} [i.e., as opposed to the étale cohomology of U_S], since the extraction of n-th roots of an element of $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$ yields finite étale coverings of U_S that correspond to open subgroups of Π_{U_S}] which is *injective* [since the abelian topological group $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$ is clearly topologically finitely generated and free of $p^{\dagger}$-torsion, hence *injects* into its prime-to-$p^{\dagger}$ completion] whenever $\Sigma^{\dagger} = \mathfrak{Primes}^{\dagger}$. In particular, by allowing S to *vary*, we obtain a *natural homomorphism*

$$K_X^{\times} \to \varinjlim_{S} H^1(\Pi_{U_S}, M_X)$$

[where K_X is the *function field* of X; the direct limit is over all finite subsets S of X^{cl}] which is *injective* whenever $\Sigma^{\dagger} = \mathfrak{Primes}^{\dagger}$.

Proposition 2.1. **(Kummer Classes of Functions)** *If $S \subseteq X^{\mathrm{cl}}$ is a finite subset, write*

$$\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{\text{c-ab}} \twoheadrightarrow \Delta_{U_S}^{\text{c-cn}}$$

for the **maximal cuspidally abelian** *and* **maximal cuspidally central** *quotients, respectively, and*

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\text{c-ab}} \twoheadrightarrow \Pi_{U_S}^{\text{c-cn}}$$

for the corresponding quotients of Π_{U_S}. If $x \in X^{\mathrm{cl}}$, then let us write

$$D_x[U_S] \subseteq \Pi_{U_S}$$

for the **decomposition group** *of x in Π_{U_S} [which is well-defined up to conjugation in Π_{U_S}] and $I_x[U_S] \subseteq D_x[U_S]$ for the inertia subgroup. [Thus, when $x \in S$, we obtain [cf. Proposition* 1.5, *(ii), (iii)] a natural isomorphism of M_X with $I_x[U_S] \stackrel{\mathrm{def}}{=} D_x[U_S] \bigcap \Delta_{U_S}$.]*

(i) The natural surjections induce **isomorphisms** *as follows:*

$$H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X)$$

In particular, we obtain **natural homomorphisms** *as follows:*

$$\Gamma(U_S, \mathcal{O}_{U_S}^{\times}) \to H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} H^1(\Pi_{U_S}, M_X)$$

$$K_X^{\times} \to \varinjlim_{S} H^1(\Pi_{U_S}^{\text{c-cn}}, M_X) \xrightarrow{\sim} \varinjlim_{S} H^1(\Pi_{U_S}^{\text{c-ab}}, M_X) \xrightarrow{\sim} \varinjlim_{S} H^1(\Pi_{U_S}, M_X)$$

These natural homomorphisms are **injective** *whenever* $\Sigma^{\dagger} = \mathfrak{Primes}^{\dagger}$.

(ii) Suppose that $S \subseteq X(k)$ *is a finite subset. Then restricting cohomology classes of* Π_{U_S} *to the various* $I_x[U_S]$, *for* $x \in S$, *yields a natural exact sequence*

$$1 \to (k^{\times})^{\wedge} \to H^1(\Pi_{U_S}, M_X) \to \Big(\bigoplus_{x \in S} \widehat{\mathbb{Z}}^{\dagger}\Big)$$

[where we identify $\text{Hom}_{\widehat{\mathbb{Z}}^{\dagger}}(I_x[U_S], M_X)$ *with* $\widehat{\mathbb{Z}}^{\dagger}$*]. Moreover, the image [via the natural homomorphism given in (i)] of* $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$ *in* $H^1(\Pi_{U_S}, M_X)/(k^{\times})^{\wedge}$ *is equal to the inverse image in* $H^1(\Pi_{U_S}, M_X)/(k^{\times})^{\wedge}$ *of the submodule of*

$$\Big(\bigoplus_{x \in S} \mathbb{Z}\Big) \subseteq \Big(\bigoplus_{x \in S} \widehat{\mathbb{Z}}^{\dagger}\Big)$$

determined by the **principal divisors** *[with support in S]. A similar statement holds when "*Π_{U_S}*" is replaced by "*$\Pi_{U_S}^{\text{c-ab}}$*" or "*$\Pi_{U_S}^{\text{c-cn}}$*".*

(iii) If $f \in \Gamma(U_S, \mathcal{O}_{U_S}^{\times})$, *write*

$$\kappa_f^{\text{c-cn}} \in H^1(\Pi_{U_S}^{\text{c-cn}}, M_X); \quad \kappa_f^{\text{c-ab}} \in H^1(\Pi_{U_S}^{\text{c-ab}}, M_X); \quad \kappa_f \in H^1(\Pi_{U_S}, M_X)$$

for the associated Kummer classes. If $x \in X^{\text{cl}} \backslash S$, *then* $D_x[U_S]$ *maps, via the natural surjection* $\Pi_{U_S} \twoheadrightarrow G_k$, *isomorphically onto the open subgroup* $G_{k(x)} \subseteq G_k$ *[where* $k(x)$ *is the residue field of* X *at* x*]. Moreover, the images of the pulled back classes*

$$\kappa_f^{\text{c-cn}}|_{D_x[U_S]} = \kappa_f^{\text{c-ab}}|_{D_x[U_S]} = \kappa_f|_{D_x[U_S]} \in H^1(D_x[U_S], M_X) \xrightarrow{\sim} H^1(G_{k(x)}, M_X) \xrightarrow{\sim} (k(x)^{\times})^{\wedge}$$

in $(k(x)^{\times})^{\wedge}$ *are equal to the image in* $(k(x)^{\times})^{\wedge}$ *of the* **value** *of* f *at* x.

Proof. Assertion (i) follows immediately from the definitions. The exact sequence of assertion (ii) follows immediately from Proposition 1.4, (ii). The characterization of the image of $\Gamma(U_S, \mathcal{O}_{U_S}^{\times})$ is immediate from the definitions and the exact sequence of assertion (ii). Assertion (iii) follows immediately from the definitions and the functoriality of the Kummer class. $\square$

Remark 12. If, in the situation of Proposition 2.1, (iii), we think of the extension of $\Pi^{\text{c-cn}}_{U_S}$ of Π_X as being given by the extension $\mathcal{D}_S$ [cf. Proposition 1.6, (iii)], where $\mathcal{D}$ is a *fundamental extension* of $\Pi_{X\times X}$ that appears as a *quotient* of $\Pi_{U_{X\times X}}$ [hence is *"rigid"* with respect to the action of $(k^\times)^\wedge$ — cf. Proposition 1.7, (iii); the proof of Theorem 1.1, (iii)], then it follows that the image of $D_x[U_S]$ in $\Pi^{\text{c-cn}}_{U_S}$ may be thought of as the image of $D_x[U_S]$ in $\mathcal{D}_S$. If, moreover, we assume, for simplicity, that $x \in X(k)$, $S \subseteq X(k)$, then this image of $D_x[U_S]$ in $\mathcal{D}_S$ amounts to a *section* of $\mathcal{D}_S \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$ lying over the section s_x of $\Pi_X \twoheadrightarrow G_k$. Since $\mathcal{D}_S$ is defined as a certain *fiber product*, this section is equivalent to a *collection of sections* [regarded as cyclotomically outer homomorphisms]

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$$

[where y ranges over the points of S]. [Here, we note that it is immediate from the definitions that, as the notation suggests, $\gamma_{y,x}$ depends only on x, y — i.e., that $\gamma_{y,x}$ is *independent* of the choice of S.] That is to say, from this point of view, Proposition 2.1, (iii), may be regarded as stating that:

> The image in $(k^\times)^\wedge = (k(x)^\times)^\wedge$ of the *value of a function* $\in \Gamma(U_S, \mathcal{O}^\times_{U_S})$ at $x \in X(k)$ may be computed from its *Kummer class*, as soon as one knows the *sections* $\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$, for $y \in S$.

Also, before proceeding, we note that an arbitrary section of $\mathcal{D}_{y,x} \twoheadrightarrow G_k$ differs [as a cyclotomically outer homomorphism] from $\gamma_{y,x}$ by the action of an element of $H^1(G_k, M_X) \xrightarrow{\sim} (k^\times)^\wedge$. Thus, the datum of "$\gamma_{y,x}$" may be regarded as a *trivialization of a certain* $(k^\times)^\wedge$*-torsor*.

Remark 13. The finite field portion of Proposition 2.1 may be regarded as the evident finite field analogue of [a certain portion of] the theory of [Mzk8], §4. Also, we observe that the approach of "reconstructing the function field of the curve via *Kummer theory*, as opposed to *class field theory* [as was done in [Tama], [Uchi]]" has the advantage of being applicable to *nonarchimedean local fields*, as well as to finite fields.

Definition 2.1. For $x, y \in X(k)$, we shall refer to the section [regarded as a cyclotomically outer homomorphism]

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$$

as the *Green's trivialization* of $\mathcal{D}$ at (y, x). If D is a divisor on X supported in the subset of k-rational points $X(k) \subseteq X^{\text{cl}}$, then multiplication of the various Green's trivializations for the points in the support of D determines a section [regarded as a cyclotomically outer homomorphism]

$$\gamma_{D,x} : G_k \to \mathcal{D}_{D,x}$$

which we shall refer to as the *Green's trivialization* of $\mathcal{D}$ at (D, x). [Note that the definition of $\gamma_{D,x}$ generalizes immediately to the case where the divisor D, but not necessarily the points in its support, is rational over k — cf. Remark 5.]

Remark 14. The terminology of Definition 2.1, is intended to suggest the similarity between the $\gamma_{y,x}$ of the present discussion and the *"Green's functions"* that occur in the theory of *bipermissible metrics* — cf., e.g., [MB], §4.11.4.

Remark 15. Note that the Green's trivializations are *symmetric* with respect to the involution of $\mathcal{D}$ induced by the automorphism $\Pi_\tau^{\text{c-ab}}$ of Proposition 1.11. Indeed, relative to the natural projections

$$\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_{U_{X\times X}}^{\text{c-ab}} \twoheadrightarrow \mathcal{D}$$

the Green's trivialization at (y, x) is simply the section of $\mathcal{D} \twoheadrightarrow G_k$ arising [by composition] from the section of $\Pi_{U_{X\times X}} \twoheadrightarrow G_k$ determined by the *decomposition group* of the point $(y, x) \in U_{X\times X}(k)$. Thus, the asserted symmetry of the Green's trivializations follows from the fact that $\Pi_\tau^{\text{c-ab}}$ is compatible with Π_τ, together with the evident fact that [by "transport of structure"] Π_τ maps the decomposition group of $(y, x) \in U_{X\times X}(k)$ isomorphically onto the decomposition group of $(x, y) \in U_{X\times X}(k)$.

If $d \in \mathbb{Z}$, denote by J^d the subscheme of the *Picard scheme* of X that parame-trizes line bundles of *degree* d; write $J \overset{\text{def}}{=} J^0$. Thus, J^d is a *torsor over* J. Note that there is a natural morphism $X \to J^1$ [given by assigning to a point of X the line bundle of degree 1 determined by the point]. Thus, the basepoint of X [already chosen in §1] determines a basepoint of J^1. At the level of *"geometrically pro-Σ" étale fundamental groups*, this morphism induces a surjective homomorphism

$$\Pi_X \twoheadrightarrow \Pi_{J^1}$$

whose kernel is the kernel of the maximal abelian quotient $\Delta_X \twoheadrightarrow \Delta_X^{\text{ab}}$. In particular, for $x \in X(k)$, the section s_x determines a section $t_x : G_k \to \Pi_{J^1}$. Note that applying the "change of structure group" given by the "multiplication by d map" on J to the J-torsor J^1 yields the J-torsor J^d. [Indeed, this follows by considering the group structure of the Picard scheme.] Thus, we obtain a morphism $J^1 \to J^d$ whose induced morphism on fundamental groups

$$\Pi_{J^1} \to \Pi_{J^d}$$

determines an *isomorphism* of Π_{J^d} with the *push-forward* of the extension Π_{J^1} [i.e., of G_k by $\Delta_{J^1} \cong \Delta_X^{\mathrm{ab}}$] via the homomorphism $\Delta_X^{\mathrm{ab}} \to \Delta_X^{\mathrm{ab}}$ given by *multiplication by* d. When $d \geq 1$, the group structure on the Picard scheme also determines a morphism

$$\prod \Pi_{J^1} \to \Pi_{J^d}$$

[where the product is a fiber product over G_k of d factors of Π_{J^1}] which determines an *isomorphism* of Π_{J^d} with the *push-forward* of the extension constituted by the fiber product via the homomorphism $\prod \Delta_X^{\mathrm{ab}} \to \Delta_X^{\mathrm{ab}}$ [i.e., from a product of d copies of Δ_X^{ab} to Δ_X^{ab} given by adding up the d components]. Moreover, one verifies immediately that when $d \geq 1$, these two constructions of "Π_{J^d}" from Π_{J^1} yield groups that are *naturally isomorphic*.

Thus, by applying the various homomorphisms induced on fundamental groups by the group structure of the Picard scheme, it follows that if D is any *divisor of degree* d on X whose support lies in the set of k-rational points $X(k) \subseteq X^{\mathrm{cl}}$, then D determines a section

$$t_D : G_k \to \Pi_{J^d}$$

which may be constructed *entirely group-theoretically* from the "t_x", where $x \in X(k)$ ranges over the points in the support of D. In particular, if D is of *degree* 0, then the section $t_D : G_k \to \Pi_J$ may be compared with the *identity section* of Π_J to obtain a cohomology class:

$$\eta_D \in H^1(G_k, \Delta_X^{\mathrm{ab}})$$

Now we have the following well-known result:

Proposition 2.2. **(Points and Galois Sections)** *Suppose that* $\Sigma = \mathfrak{Primes}$. *Then, in the notation of the above discussion:*

(i) The divisor D *is* **principal** *if and only if* $\eta_D = 0$.

(ii) The map $x \mapsto D_x$ *from* X^{cl} *to conjugacy classes of closed subgroups of* Π_X *is* **injective***, i.e.,* X *is* **$\mathfrak{Primes}$-separated**.

Proof. First, we consider assertion (i). By well-known general nonsense [cf., e.g., [Naka], Claim (2.2); [NTs], Lemma (4.14); [Mzk4], the Remark preceding Definition 6.2], there is a *natural isomorphism*

$$H^1(k, \Delta_X^{\mathrm{ab}}) \xrightarrow{\sim} J(k)^\wedge \ (\supseteq J(k))$$

[where the "$\wedge$" denotes the profinite completion] which maps η_D to the element of $J(k)$ determined by D. [Here, we recall that this natural isomorphism arises by considering the long exact sequence obtained by applying the functors $H^*(G_k, -)$ to the short exact sequence of G_k-modules

$$1 \to J(\overline{k})[n] \to J(\overline{k}) \to J(\overline{k}) \to 1$$

— where n is a positive integer; the morphism $J(\overline{k}) \to J(\overline{k})$ is the "multiplication by n map"; $J(\overline{k})[n]$ is defined so as to make the sequence exact.] Thus, assertion (i) follows immediately.

To prove assertion (ii), it suffices [by possibly base-changing to a finite extension of k] to verify that two points $x_1, x_2 \in X(k)$ that induce Δ_X-conjugate sections s_{x_1}, s_{x_2} are necessarily equal [cf. also [Tama], Corollary 2.10]. But this follows formally from assertion (i), by considering the divisor $x_1 - x_2$ [and the well-known fact that the natural morphism $X \to J^1$ considered above is an *embedding*]. $\square$

Remark 16. From the point of view of Definition 1.2, (ii), the reader may feel tempted to expect that [still under the assumption that $\Sigma = \mathfrak{Primes}$] D is principal if and only if the extension $\mathcal{D}_D$ of Π_X [by M_X] is *trivial* [i.e., determines the zero class in $H^2(\Pi_X, M_X)$]. When k is *nonarchimedean local*, it is not difficult to verify, using Proposition 2.2, (i), that this is indeed the case. On the other hand, when k is *finite*, although this condition for principality is easily verified to be *necessary*, it is *not*, however, *sufficient*, since it only involves the *"prime-to-$p^\dagger$ portion"* of the point of $J(k)$ determined by D.

Definition 2.2. In the situation of Theorem 1.1, (iii), suppose further that ($\Sigma \stackrel{\text{def}}{=}$) $\Sigma_X = \Sigma_Y$, and that α is *point-theoretic*. Let $S \subseteq X^{\text{cl}}$ be a [not necessarily finite] subset that corresponds via the bijection $X^{\text{cl}} \xrightarrow{\sim} Y^{\text{cl}}$ induced by [the point-theoreticity of] α to a subset $T \subseteq Y^{\text{cl}}$.

(i) Write $\mathcal{D}$ (respectively, $\mathcal{E}$) for the fundamental extension of $\Pi_{X\times X}$ (respectively, $\Pi_{Y\times Y}$) that arises as the quotient of $\Pi^{\text{c-ab}}_{U_{X\times X}}$ (respectively, $\Pi^{\text{c-ab}}_{U_{Y\times Y}}$) by the kernel of the *maximal cuspidally central quotient* $\Delta^{\text{c-ab}}_{U_{X\times X}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{X\times X}}$ (respectively, $\Delta^{\text{c-ab}}_{U_{Y\times Y}} \twoheadrightarrow \Delta^{\text{c-cn}}_{U_{Y\times Y}}$) [cf. Proposition 1.6, (iv)]. Thus, $\alpha^{\text{c-ab}}$ induces an isomorphism:

$$\alpha^{\text{c-cn}} : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$$

We shall say that α is *(S,T)-locally Green-compatible* if, for every pair of points $(x_1, x_2) \in X(k_X) \times X(k_X)$ corresponding via the bijection induced by α to a pair of points $(y_1, y_2) \in Y(k_Y) \times Y(k_Y)$, such that $x_2 \in S$, $y_2 \in T$, the isomorphism

$$\mathcal{D}_{x_1,x_2} \xrightarrow{\sim} \mathcal{E}_{y_1,y_2}$$

[obtained by restricting $\alpha^{\text{c-cn}}$] is compatible with the *Green's trivializations*. We shall say that α is *(S,T)-locally degree zero* (respectively, *(S,T)-locally* *principally) Green-compatible* if, for every $x \in X(k_X) \bigcap S$ and every divisor of degree zero (respectively, principal divisor) D supported in $X(k_X) \subseteq X^{\mathrm{cl}}$ corresponding via the bijection induced by α to a pair (y, E) of Y [so $y \in Y(k_Y) \bigcap T$], the isomorphism

$$\mathcal{D}_{D,x} \xrightarrow{\sim} \mathcal{E}_{E,y}$$

is compatible with the *Green's trivializations*.

(ii) We shall say that α is *totally (S,T)-locally Green-compatible* (respectively, *totally (S,T)-locally degree zero Green-compatible*; *totally (S,T)-locally principally Green-compatible*) if, for all pairs of connected finite étale coverings $X' \to X$, $Y' \to Y$ that arise from open subgroups of Π_X, Π_Y that correspond via α, the isomorphism

$$\Pi_{X'} \xrightarrow{\sim} \Pi_{Y'}$$

induced by α is (S',T')-locally Green-compatible (respectively, (S',T')-locally degree zero Green-compatible; (S',T')-locally principally Green-compatible), where $S' \subseteq (X')^{\mathrm{cl}}$, $T' \subseteq (Y')^{\mathrm{cl}}$ are the inverse images in X', Y' of S, T, respectively.

(iii) With respect to the terminology introduced in (i), (ii), when $S = X^{\mathrm{cl}}$, $T = Y^{\mathrm{cl}}$, then we shall replace the phrase "(S,T)-locally" by the phrase *"globally"*.

Remark 17. In the situation of Definition 2.2, if $X' \to X$, $Y' \to Y$ are connected finite étale coverings that arise from open subgroups of Π_X, Π_Y that correspond via α; $\mathcal{D} \xrightarrow{\sim} \mathcal{E}$ is the isomorphism of fundamental extensions of $\Pi_{X\times X}$, $\Pi_{Y\times Y}$ that arises from the isomorphism $\alpha^{\text{c-ab}}$ of Theorem 1.1, (iii); and the points x_1, x_2 (respectively, y_1, y_2) are Δ_X- (respectively, Δ_Y-) *conjugate*, then it follows immediately from the *compatibility* of $\alpha^{\text{c-ab}}$ with the natural inclusions $D_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$, $D_Y \hookrightarrow \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ [cf. Theorem 1.1, (iii)] that the isomorphism $\mathcal{D}_{x_1,x_2} \xrightarrow{\sim} \mathcal{E}_{y_1,y_2}$ is *automatically* compatible with the *Green's trivializations*. [Indeed, this follows from the easily verified fact that the Green's trivializations in this case are, in essence, specializations of conjugates of the *"canonical sections of $\zeta'_{\neq}$"* of Proposition 1.8.] Unfortunately, however, the author is unable, at the time of writing, to see how to generalize the argument applied in the proof of Theorem 1.1, (iii), involving Lemma 1.1; Proposition 1.8, (v), so as to cover the case where the points x_1, x_2 (respectively, y_1, y_2) *fail to be* Δ_X- (respectively, Δ_Y-) *conjugate*.

Remark 18. It is immediate that (S,T)-local Green-compatibility (respectively, (S,T)-local degree zero Green-compatibility) implies (S,T)-local degree zero Green-compatibility (respectively, (S,T)-local principal Green-compatibility), and that total (S,T)-local Green-compatibility (respectively, total

(S,T)-local degree zero Green-compatibility) implies total (S,T)-local degree zero Green-compatibility (respectively, total (S,T)-local principal Green-compatibility).

Theorem 2.1. **(Reconstruction of Functions)** *In the situation of Theorem* 1.1, *(iii), suppose further that* $(\Sigma \stackrel{\mathrm{def}}{=})\ \Sigma_X = \Sigma_Y$, *and that* α *is* **point-theoretic**. *Then:*

(i) Let $S \subseteq X^{\mathrm{cl}}$, $T \subseteq Y^{\mathrm{cl}}$ *be finite subsets that correspond via the bijection* $X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ *induced by* α. *Then* α, $\alpha^{\text{c-ab}}$ *induce* **isomorphisms** *[well-defined up to* **cuspidally inner automorphisms***]*

$$\Pi^{\text{c-ab}}_{U_S} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_T}$$

[where $V_T \stackrel{\mathrm{def}}{=} Y \backslash T$*] lying over* α, *which are* **functorial** *with respect to* α *and* S, T, *as well as with respect to passing to* **connected finite étale coverings of** X, Y *[that do not necesarily arise from open subgroups of* Π_X, Π_Y*!].*

(ii) Suppose that $\Sigma = \mathfrak{Primes}$. *Then the bijection* $X^{\mathrm{cl}} \xrightarrow{\sim} Y^{\mathrm{cl}}$ *induced by* α *induces a bijection between the groups of* **principal divisors** *on* X, Y. *This bijection, together with the isomorphisms of (i), induces a* **compatible isomorphism**

$$K_X^{\times} \cdot (k_X^{\times})^{\wedge} \xrightarrow{\sim} K_Y^{\times} \cdot (k_Y^{\times})^{\wedge}$$

between the push-forwards of the multiplicative groups associated to the **function fields** *of* X, Y, *relative to the homomorphisms* $k_X^{\times} \hookrightarrow (k_X^{\times})^{\wedge}$, $k_Y^{\times} \hookrightarrow (k_Y^{\times})^{\wedge}$.

Proof. Assertion (i) follows immediately by *"specializing to* S, T*"* the isomorphism of Theorem 1.1, (iii) [cf. also Proposition 1.9, (i), (ii); the definitions of the various objects involved]. [Here, we note that the *functoriality* asserted in assertion (i), which is somewhat *stronger* than the functoriality asserted in Theorem 1.1, (iii), follows from the definitions, together with the *naturality* of the constructions applied in the proof of Theorem 1.1, (iii) — cf., e.g., the diagram of Proposition 1.7, (ii).] Assertion (ii) follows immediately from assertion (i); Proposition 2.2, (i); Proposition 2.1, (i), (ii). ☐

Remark 19. In fact, the *crucial isomorphism* $\Pi^{\text{c-ab}}_{U_S} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_T}$ of Theorem 2.1, (i), may also be constructed, in the *finite field case*, via the techniques to be introduced in §3 [although we shall not discuss this approach in detail;

cf., however, the proof of Theorem 3.1]. On the other hand, observe that unlike the techniques of §3, the techniques of §1 [in particular, the proof of Theorem 1.1, (iii), via Propositions 1.7, 1.8] apply to situations [e.g., the case of *nonarchimedean local fields*!] where the *weight filtration* [cf. §3] *does not admit a Galois-invariant splitting*. Indeed, the techniques of §1, essentially only require that the Galois cohomology of the base field admit a natural *duality pairing*. Moreover, even in the finite field case, in light of the importance of this isomorphism $\Pi^{\text{c-ab}}_{U_S} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_T}$ in the theory of the present paper, it is of interest to see that this isomorphism may be constructed via *two fundamentally different approaches*. Finally, although the techniques of §3 are better suited to the reconstruction of the *Green's trivializations*, they have the drawback that they depend essentially on the *choice of a "basepoint"* $x_* \in X(k)$. Thus, it is of interest to know that this isomorphism may be constructed [i.e., via the techniques of §1] *"cohomologically"* [cf. Proposition 1.5, (i)] without making such a choice.

Remark 20. In the case of *nonarchimedean local fields*, it is natural to ask, in the style of [Mzk8], §4, whether or not various *"canonical integral structures"* on the extensions $\mathcal{D}_{x,y}$ [where $x, y \in X(k)$] of G_k by M_X are preserved by arbitrary isomorphisms of arithmetic fundamental groups. When $x \neq y$, such a canonical integral structure is determined by the *Green's trivialization*; when $x = y$, such a canonical integral structure is determined by the integral structure [in the usual sense of scheme theory] on the canonical sheaf of the stable model of the curve [when the curve has stable reduction] — cf. [Mzk8], §4.

Before proceeding, we note the following *"analogue for $\Pi^{\text{c-ab}}_{U_S}$"* of Proposition 1.10, (i):

Proposition 2.3. **(Automorphisms and Commensurators)** *Let $\Pi^{\text{c-ab}}_{U_S}$ be as in Proposition* 2.1. *For $x \in S$, write $D_x[U_S] \hookrightarrow \Pi^{\text{c-ab}}_{U_S}$ for the natural inclusion. Then:*

(i) Any automorphism α of the profinite group $\Pi^{\text{c-ab}}_{U_S}$ which

(a) is compatible with the natural surjection $\Pi^{\text{c-ab}}_{U_S} \twoheadrightarrow \Pi_X$ and induces the identity on Π_X;

(b) for each $x \in S$, preserves the image of $M_X \cong I_x[U_S] \subseteq D_x[U_S]$ via the natural inclusion $D_x[U_S] \hookrightarrow \Pi^{\text{c-ab}}_{U_S}$

is **cuspidally inner**.

(ii) Suppose that X *is* **Σ-separated**. *Then for* $x \in S$, D_x *is* **commensurably terminal** *in* Π_X.

(iii) Suppose that X *is* **Σ-separated**. *Then the image of* $D_x[U_S] \hookrightarrow \Pi_{U_S}^{\text{c-ab}}$ *is* **commensurably terminal** *in* $\Pi_{U_S}^{\text{c-ab}}$.

Proof. First, we observe that assertion (ii) follows formally from the definition of a "decomposition group" and "Σ-separated". Thus, assertion (i) (respectively, (iii)) follows by an argument which is entirely similar to the argument that was used to prove assertion (i) (respectively, (iii)) of Proposition 1.10. □

Remark 21. In the situation of Definition 2.2, suppose that S, T are *finite*, and that α arises from an *isomorphism*

$$\Pi_{U_S} \xrightarrow{\sim} \Pi_{V_T}$$

which is *point-theoretic* [or, equivalently, *quasi-point-theoretic*] — a condition that is automatically satisfied in the *finite field case* whenever α is *Frobenius-preserving* [cf. Remark 10]. Then observe that, [in light of our *point-theoreticity* assumption] it follows from Proposition 2.3, (i), that the resulting induced isomorphism

$$\Pi_{U_S}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab}}$$

coincides [up to cuspidally inner automorphisms] with the isomorphism of Theorem 2.1, (i). Thus, in light of Remark 15, it follows formally from the definitions that α is *totally* (S, T)*-locally Green-compatible.*

Corollary 2.1. (Point-theoretic Totally Locally Principally Green-compatible Isomorphisms) *In the situation of Theorem* 1.1, *(iii), assume further that* $(\Sigma \stackrel{\text{def}}{=})\ \Sigma_X = \Sigma_Y = \mathfrak{Primes}$, *and that* α *is* **point-theoretic** *and* **totally** (S, T)**-locally principally Green-compatible**, *for some* **nonempty** *subsets* $S \subseteq X^{\text{cl}}$, $T \subseteq Y^{\text{cl}}$ *which correspond via the bijection* $X^{\text{cl}} \xrightarrow{\sim} Y^{\text{cl}}$ *induced by* α. *Then* α *arises from a* **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\sim} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are **isomorphisms***; the vertical arrows are the pro-finite étale coverings determined by the profinite groups* Π_X*,* Π_Y*.*

Proof. Corollary 2.1 follows immediately — i.e., by *"specializing functions to points"* — from the definitions; Theorem 2.1, (ii); Proposition 2.1, (iii); Remark 12; and [Tama], Lemma 4.7. Here, we note that, in the present situation, the isomorphism

$$K_X^\times \cdot (k_X^\times)^\wedge \xrightarrow{\sim} K_Y^\times \cdot (k_Y^\times)^\wedge$$

of Theorem 2.1, (ii), necessarily induces an isomorphism $K_X^\times \xrightarrow{\sim} K_Y^\times$ [cf. the assumption that $\Sigma^\dagger = \mathfrak{Primes}^\dagger$]. Indeed, this is immediate in the *finite field case*. In the *nonarchimedean local field case*, it follows via the arguments applied in the proof of [Mzk8], Theorem 4.10: That is to say, we assume for simplicity that $S \subseteq X(k_X)$; then if $f \in K_X^\times$, and $x \in S$ is a point that does not lie in the divisor of zeroes and poles of f, then let us *observe* that the subset

$$f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$$

may be characterized as the subset of elements whose *values* [cf. Proposition 2.1, (iii)] at x lie in $k_X^\times \subseteq (k_X^\times)^\wedge$. Note that since, for a given $x_1 \in S$, there clearly exist $f \in K_X^\times$ [at least after possibly passing to an appropriate connected finite étale covering of X] that have a zero or pole at x_1 but not at some other $x \in S$, this *observation* allows us to recover the *canonical discrete structure* [cf. [Mzk8], Definition 4.1, (iii); the proof of [Mzk8], Theorem 4.10] on the decomposition groups in $\Pi_{U_{S_1}}^{\text{c-ab}}$ [where $S_1 \subseteq X^{\text{cl}}$ is an arbitrary finite subset containing S, which corresponds, say, to a subset $T_1 \subseteq Y^{\text{cl}}$ that contains T] at arbitrary points [i.e., arbitrary "x_1"] of S. Thus, by applying this canonical discrete structure [as in the proof of [Mzk8], Theorem 4.10], we may recover the subset

$$f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$$

for arbitrary $f \in K_X^\times$ [i.e., even f that have a zero or pole at every point of S] as the subset of elements for which the restriction to each point x of S *either* lies in $k_X^\times \subseteq (k_X^\times)^\wedge$ *or* [when the element in question has a zero or pole at x] is compatible with the canonical discrete structure at x. Since this characterization of the subset $f \cdot k_X^\times \subseteq f \cdot (k_X^\times)^\wedge$ is *manifestly compatible* [in light of the *Green-compatibility* assumption on α] with the isomorphisms $\Pi_{U_{S_1}}^{\text{c-ab}} \xrightarrow{\sim} \Pi_{V_{T_1}}^{\text{c-ab}}$ induced by α, we thus conclude that the isomorphism

$$K_X^\times \cdot (k_X^\times)^\wedge \xrightarrow{\sim} K_Y^\times \cdot (k_Y^\times)^\wedge$$

of Theorem 2.1, (ii), maps the subset $K_X^\times \subseteq K_X^\times \cdot (k_X^\times)^\wedge$ onto the subset $K_Y^\times \subseteq K_Y^\times \cdot (k_Y^\times)^\wedge$, as desired. $\square$

Remark 22. Suppose, in the situation of Corollary 2.1, that $S = X^{\mathrm{cl}}$, $T = Y^{\mathrm{cl}}$. Then unlike the situation discussed in [Tama], one has the freedom to evaluate functions at arbitrary points of the *entire sets* X^{cl}, Y^{cl}, as opposed to just certain restricted subsets $S \subseteq X^{\mathrm{cl}}$, $T \subseteq Y^{\mathrm{cl}}$. Thus, instead of applying [Tama], Lemma 4.7, one may instead apply the somewhat *easier* argument *implicit* in [Uchi], §3, Lemmas 8-11 [which is used to treat the function field case].

Thus, in light of Remark 21 [together with the portion of Theorem 1.1, (i), concerning the preservation of *decomposition groups of cusps*], Corollary 2.1 implies the following result, in the *affine* case:

Corollary 2.2. (Point-theoretic Isomorphisms in the Affine Case) *Let* U, V *be* **affine hyperbolic curves** *over a* **finite** *or* **nonarchimedean local field**. *Suppose that* $\Sigma = \mathfrak{Primes}$. *Write* Δ_U *(respectively,* Δ_V*) for the* **maximal cuspidally pro-**$\Sigma^\dagger$ **quotient of the maximal pro-**Σ **quotient** *of the tame geometric fundamental group of* U *(respectively,* V*) [where "tame" is with respect to the complement of* U *(respectively,* V*) in its canonical compactification], and* Π_U *(respectively,* Π_V*) for the corresponding quotient of the étale fundamental group of* U *(respectively,* V*). Then any* **point-theoretic isomorphism**

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

arises from a **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\sim} & \widetilde{V} \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are **isomorphisms***; the vertical arrows are the pro-finite étale coverings determined by the profinite groups* Π_U, Π_V.

Remark 23. In light of the results of [Tama] [cf. Remarks 9, 10], Corollary 2.2 is only truly of interest in the case of *nonarchimedean local fields*.

Definition 2.3. Suppose that k is a *nonarchimedean local field*.

(i) A [necessarily affine] hyperbolic curve U over k will be said to be of *strictly*

Belyi type if it is defined over a number field and isogenous [cf. §0] to a hyperbolic curve of genus zero.

(ii) A [necessarily affine] hyperbolic curve U over k will be said to be of *Belyi type* if it is defined over a number field, and, moreover, for some positive integer m, there exists a finite sequence

$$U = U_1 \rightsquigarrow U_2 \rightsquigarrow \ldots \rightsquigarrow U_{m-1} \rightsquigarrow U_m$$

of *hyperbolic orbicurves* [cf. §0] U_j such that U_m is a *tripod* [cf. §0], and, moreover, for each $j = 1, \ldots, m-1$, U_{j+1} is related to U_j in one of the following ways:

(a) there exists a *finite étale morphism* $U_{j+1} \to U_j$ [i.e., "U_{j+1} is a *finite étale covering* of U_j"];

(b) there exists a *finite étale morphism* $U_j \to U_{j+1}$ [i.e., "U_{j+1} is a *finite étale quotient* of U_j"];

(c) there exists an *open immersion* $U_j \hookrightarrow U_{j+1}$ [i.e., in the terminology of [Mzk8], "U_{j+1} is a *[hyperbolic] partial compactification* of U_j"];

(d) there exists a *partial coarsification morphism* [cf. §0] $U_j \to U_{j+1}$ [i.e., "U_{j+1} is a *partial coarsification* of U_j"].

(iii) A [necessarily affine] hyperbolic curve U over k will be said to be of *quasi-Belyi type* if it is defined over a number field and admits a connected finite étale covering $V \to U$ such that V admits a [not necessarily finite or étale!] dominant morphism $V \to W$ to a tripod W.

Remark 24. It is immediate that every hyperbolic curve of strictly Belyi type is also of Belyi type [as the terminology suggests]. Moreover, one verifies easily by "induction on m" [where "m" is as in Definition 2.3, (ii)] that every hyperbolic curve of Belyi type is also of quasi-Belyi type [as the terminology suggests]. It is not difficult to see that there exist [multiply] punctured elliptic curves that are of Belyi type, but not of strictly Belyi type [cf. Remark 31 below]. On the other hand, it is not clear to the author at the time of writing whether or not there exist hyperbolic curves of quasi-Belyi type that are not of Belyi type.

Remark 25. Hyperbolic curves of strictly Belyi type are precisely the sort of curves considered in [Mzk8], Corollaries 2.8, 3.2.

Remark 26. The author would like to thank *A. Tamagawa* for useful discussions concerning Definition 2.3, (ii), especially Definition 2.3, (ii), (d).

Proposition 2.4. **(Decomposition Groups of Curves of Quasi-Belyi Type)** *Let* U *(respectively,* V*) be a* **hyperbolic curve** *over a nonarchimedean local field. Denote the base field of* U *(respectively,* V*) by* k_U *(respectively,* k_V*), the étale fundamental group of* U *(respectively,* V*) by* Π_U *(respectively,* Π_V*) [i.e., "we take* $\Sigma = \mathfrak{Primes}$*"]. Let*

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Then:

(i) If U *is of* **quasi-Belyi type***, then the closed points of "*DLoc*-type" [in the sense of [Mzk8], Definition 2.4] are* $\boldsymbol{p_U}$**-adically dense** *[where* p_U *is the residue characteristic of* k_U*] in* $U(k_U)$*.*

(ii) If U *is of* **quasi-Belyi type***, then* β *maps every decomposition group of a closed point of* U *isomorphically onto a decomposition group of a closed point of* V*.*

(iii) If **both** U*,* V *are of* **quasi-Belyi type***, then* β *is* **point-theoretic***.*

(iv) If U *is of* **Belyi type***, then so is* V*.*

Proof. The proof of assertion (i) is similar to the proof of [Mzk8], Corollary 2.8: That is to say, in the terminology of *loc. cit.*, it follows formally from the fact that U is of *quasi-Belyi type* that the *"algebraic" closed points* [i.e., closed points defined over a number field, which are manifestly p_U*-adically dense* in $U(k_U)$] of U are of "DLoc-type" [cf. the proof of [Mzk8], Corollary 2.8]: Indeed, it suffices to consider the following commutative diagram of hyperbolic curves, whose existence follows from the assumption that U is of *quasi-Belyi type*:

$$\begin{array}{ccccccc} & & V' & \longrightarrow & W' & \hookrightarrow & U' & \longrightarrow & U \\ & & \downarrow & & \downarrow & & & & \\ U & \longleftarrow & V & \longrightarrow & W & & & & \end{array}$$

Here, the "hooked arrow $\hookrightarrow$" is an *open immersion*; all of the "non-hooked arrows" *except for* $V \to W$, $V' \to W'$ are *finite étale morphisms*; $V \to W$, $V' \to W'$ are *dominant*; the finite étale morphism $U' \to U$ is obtained by a *base-change* to a finite extension of the base field k_U; and W is a *tripod* [so $W' \to W$ is a *"Belyi map"*]. Note that the composite arrow $V' \to W' \hookrightarrow U' \to U$ may be thought of as an arrow in the category $\mathrm{DLoc}_{k_U}(U)$ of [Mzk8], §2. Observe, moreover, that the arrow $W' \hookrightarrow U'$ may be chosen to have *arbitrarily designated algebraic closed points* in the complement of its image. Thus, we conclude that this diagram exhibits the [arbitrarily designated] algebraic closed points in the complement of the image of $W' \hookrightarrow U' \to U$ as *points of* DLoc-*type*, as desired. This completes the proof of assertion (i).

In light of assertion (i) [applied to the various connected finite étale coverings of U], the proof of assertion (ii) is entirely similar to the proof of [Mzk8], Corollary 3.2: That is to say, by [Mzk8], Corollary 2.5, it follows that β *maps decomposition groups of* DLoc-*type of* U *to decomposition groups of* DLoc-*type of* V. Thus, assertion (ii) follows by applying [Mzk8], Lemma 3.1 [where the density statement of assertion (i) concerning points of DLoc-type allows one to replace the "algebraicity" condition of [Mzk8], Lemma 3.1, (iii), by the condition that the points in question be of DLoc-type]. Finally, assertion (iii) follows formally from assertion (ii) [and Proposition 2.2, (ii)].

Finally, we consider assertion (iv). First, I *claim* that by applying the isomorphism β [and thinking of hyperbolic orbicurves as being represented by their associated étale fundamental groups], one may *transform* the sequence

$$U = U_1 \rightsquigarrow U_2 \rightsquigarrow \ldots \rightsquigarrow U_{m-1} \rightsquigarrow U_m$$

of Definition 2.3, (ii), into a sequence

$$V = V_1 \rightsquigarrow V_2 \rightsquigarrow \ldots \rightsquigarrow V_{m-1} \rightsquigarrow V_m$$

that also satisfies the conditions of Definition 2.3, (ii), in such a way that we also obtain *compatible isomorphisms* $\beta_j : \Pi_{U_j} \xrightarrow{\sim} \Pi_{V_j}$ [where $j = 1, \ldots, m$; $\beta_1 = \beta$]. Indeed, we reason by induction on m. If [for $j = 1, \ldots, m-1$] U_{j+1} is related to U_j as in (a) [of Definition 2.3, (ii)], then it is immediate [by thinking in terms of open subgroups of Π_{U_j}, Π_{V_j}] that one may construct [from V_j] a V_{j+1} related to V_j as in (a). If U_{j+1} is related to U_j as in (b) (respectively, (c)), then it follows from [Mzk6], Theorem 2.4 (respectively, [Mzk8], Theorem 1.3, (iii) [cf. also [Mzk8], Theorem 2.3]), that one may construct [from V_j] a V_{j+1} related to V_j as in (b) (respectively, (c)). If U_{j+1} is related to U_j as in (d), then $\Pi_{U_{j+1}}$ is obtained from Π_{U_j} by forming the quotient of Π_{U_j} by the closed normal subgroup of Π_{U_j} generated by some finite collection of elements of Δ_{U_j} that belong to the *decomposition groups* of points of U_j in Δ_{U_j}. Thus, by Lemma 2.1, (v), below, we conclude that the quotient $\Pi_{U_j} \twoheadrightarrow \Pi_{U_{j+1}}$ determines a quotient $\Pi_{V_j} \twoheadrightarrow \Pi_{V_{j+1}}$ that corresponds to a *partial coarsification* $V_j \to V_{j+1}$, as desired. Finally, if U_m is a tripod, the existence of the isomorphism $\Pi_{U_m} \xrightarrow{\sim} \Pi_{V_m}$ implies that V_m is also a *tripod* [cf. [Mzk5], Lemma 1.3.9]. This completes the proof of the *claim*.

Thus, to complete the proof of assertion (iv), it suffices to verify that V is *defined over a number field*. But observe that since U is defined over a number field, there exists a *diagram of hyperbolic curves* [i.e., in essence, a "Belyi map"]

$$U_m \longleftarrow U'_m \hookrightarrow U' \longrightarrow U$$

where the "hooked arrow $\hookrightarrow$" is an *open immersion*; the "non-hooked arrows" are *finite étale morphisms*; and the finite étale morphism $U' \to U$ is obtained by a *base-change* to a finite extension of the base field k_U. Now the isomorphisms $\Pi_{U_m} \xrightarrow{\sim} \Pi_{V_m}$, $\Pi_U \xrightarrow{\sim} \Pi_V$ allow us to *transform* [cf. [Mzk8], Theorem 2.3 and its proof] this diagram into a similar diagram

$$V_m \longleftarrow V'_m \hookrightarrow V' \longrightarrow V$$

whose existence [since V_m is also a *tripod*!] shows that V is also *defined over a number field*, as desired. This completes the proof of assertion (iv). $\square$

Remark 27. Note that the essential reason that the author is unable to prove the stronger statement of Proposition 2.4, (iv), in the *quasi-Belyi* case is that, in the notation of the proof of Proposition 2.4, (i), it is *unclear how to construct* [at the level of arithmetic fundamental groups] *the dominant morphism* $V \to W$ *from* V. That is to say, unlike the situation involving the operations of Definition 2.3, (ii), (a), (b), (c), (d), it is by no means clear how to construct, via *purely group-theoretic operations*, the quotient of an arithmetic fundamental group arising from an arbitrary dominant morphism.

Lemma 2.1. (Finite Subgroups of Fundamental Groups of Hyperbolic Orbicurves) *Let* W *be a* **hyperbolic orbicurve** *over an algebraically closed field of characteristic zero;* Σ_W *a nonempty set of prime numbers. Denote the maximal pro-*Σ_W *quotient of the* **étale fundamental group** *of* W *by* Δ_W*; suppose that* W *admits a finite étale covering by a* **hyperbolic curve** *that arises from an open subgroup of* Δ_W*. Let* $A \subseteq \Delta_W$ *(respectively,* $B \subseteq \Delta_W$*) be the* **decomposition group** *[well-defined up to conjugation in* Δ_W*] of a closed point* w_A *(respectively,* w_B*) of* W*; suppose that* $w_A \neq w_B$*. Then:*

(i) A, B *are* **cyclic**.

(ii) $A \bigcap B = \{1\}$*. In particular, if* $A \neq \{1\}$*, then* A *is* **normally terminal** *in* Δ_W.

(iii) The order of **every finite cyclic closed subgroup** $C \subseteq \Delta_W$ *divides the* **order of** W *[cf. §0].*

(iv) Every **finite nontrivial closed subgroup** $C \subseteq \Delta_W$ *is contained in a* **decomposition group of a unique closed point** *of* W.

(v) The nontrivial decomposition groups of closed points of W *may be characterized as the* **maximal finite nontrivial closed subgroups** *of* Δ_W.

Proof. Assertion (i) follows immediately from the well-known [and easily verified] fact that the absolute Galois group of a complete discrete valuation field with algebraically closed residue field of characteristic zero is *cyclic*.

Next, we consider assertion (ii). Let $C \subseteq A \bigcap B$ be a subgroup of *prime order* $l \in \Sigma_W$. Now consider a normal open subgroup $H \subseteq \Delta_W$ such that the covering $W_H \to W$ determined by H is a *hyperbolic curve*. Note that this implies that $A \bigcap H = B \bigcap H = C \bigcap H = \{1\}$ [cf., e.g., assertion (iii), which will be proven below without applying the present assertion (ii)]. Write $W_H \to W_C \to W$ for the covering determined by the open subgroup $C \cdot H \subseteq \Delta_W$. Observe that there exist closed points w'_A, w'_B of W_C that lift w_A, w_B, respectively, and whose decomposition groups [well-defined up to conjugation in $C \cdot H$] are equal to C. Note that since W_H is a *hyperbolic curve*, and C is of prime order l, it follows that the order of every closed point of W_C is equal to either 1 or l. Now if W_C is *affine*, then let v be a *cusp* of W_C. If W_C is *proper* and admits ≥ 3 *points of order* l, then let v be a point of W_C of order l such that $v \neq w'_A, w'_B$. Note that if W_C is *proper* and admits ≤ 2 *points of order* l, then it follows from the *hyperbolicity* assumption that the *coarsification* of W_C is a proper smooth curve of genus ≥ 1; thus, by *replacing* H by an appropriate open subgroup of H, one verifies immediately that one may assume without loss of generality that *either* W_C is affine *or* W_C admits ≥ 3 points of order l. Now observe that W_C admits a finite étale cyclic covering $W'_C \to W_C$ of degree l which is *étale* over the compactification of the coarsification of W_C, *except* over the points in the compactification of the coarsification of W_C corresponding to v, w'_B, over which W'_C is *totally ramified*. In particular, it follows that any point of W'_C lying over w'_A (respectively, w'_B) is of order l (respectively, 1), thus contradicting the observation that the decomposition groups [well-defined up to conjugation in $C \cdot H$] of w'_A, w'_B are equal to C. This completes the proof that $A \bigcap B = \{1\}$. By applying this fact to arbitrary finite étale coverings of W, it follows formally [cf. Proposition 2.3, (ii)] that A is *normally terminal* in Δ_W, whenever $A \neq \{1\}$.

Next, we consider assertion (iii). Denote the *order of* W by n. Now if $C \subseteq \Delta_W$ is a *nontrivial finite cyclic closed subgroup*, then there exists a normal open subgroup $N \subseteq \Delta_W$ such that $C \bigcap N = \{1\}$. In particular, it follows that if we take $H \stackrel{\text{def}}{=} C \cdot N$ [so $H \subseteq \Delta_W$ is an open subgroup], then the natural map $C \to H^{\text{ab}}$ is *injective*. On the other hand, if we denote by $W_H \to W$ the covering determined by H, then it is clear that the order of W_H divides n, hence that H^{ab} is an extension of a *torsion-free* profinite abelian group by a *finite abelian group annihilated by* n. Thus, we conclude from the injection

$C \hookrightarrow H^{\mathrm{ab}}$ that the *order of* C *divides* n, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, let us observe that *uniqueness* follows formally from assertion (ii). Next, let us verify assertion (iv) under the further assumption that C is *solvable*. By induction on the order of C, we may assume that [at least] *one* of the following conditions is satisfied: (a) C is an extension of a group of prime order by a nontrivial subgroup $C_1 \subseteq C$ which is contained in the decomposition group A; (b) C is of prime order $l \in \Sigma_W$. If (a) is satisfied, then by replacing W by a finite étale covering of W determined by a suitable open subgroup containing C, we may assume that $(C_1 \subseteq)\ A \subseteq C$. Thus, if $A \neq C$, then $A = C_1$ is normal in C. But this implies, by the *normal terminality* portion of assertion (ii), that $A = C$, a contradiction. Thus, (a) implies that $C \subseteq A$. If (b) is satisfied, then we argue as follows: Observe that by assertion (iii), every open subgroup $H \subseteq \Delta_W$ that contains C determines a finite étale covering $W_H \to W$ such that the *order of* W_H *is divisible by* l. Write

$$\mathrm{Stack}_l(W_H)$$

for the set of closed points of W_H whose order is *divisible by* l. Now observe that since the order of W_H is divisible by the *prime number* l, it follows that $\mathrm{Stack}_l(W_H)$ is *nonempty*. Since the set $\mathrm{Stack}_l(W_H)$ is *finite* and *nonempty*, we thus conclude that, if we allow H to *vary* [among open subgroups $H \subseteq \Delta_W$ that contain C], then the inverse limit

$$\varprojlim_{H} \mathrm{Stack}_l(W_H)$$

is *nonempty*. But, unraveling the definitions, this means precisely that C contains the decomposition group D associated to some compatible system of points of the sets $\mathrm{Stack}_l(W_H)$. Since D is of order divisible by l, we thus conclude that $D = C$, as desired. This completes the proof of assertion (iv) for C *solvable*. On the other hand, a well-known theorem from the theory of finite groups asserts that a finite group in which every Sylow subgroup is cyclic is *solvable* [cf. [Scott], p. 356]. Thus, in light of assertion (i), we conclude that assertion (iv) for C solvable implies assertion (iv) for C *arbitrary*.

Finally, we observe that assertion (v) follows formally from assertions (ii), (iv). ☐

Remark 28. The author would like to thank *A. Tamagawa* for informing him of Lemma 2.1 and, in particular, of the theorem on finite groups that was applied in the proof of Lemma 2.1, (iv).

We are now ready to state the following *"absolute p-adic version of the Grothendieck Conjecture"* for hyperbolic curves of Belyi or quasi-Belyi type:

Corollary 2.3. **(Curves of Belyi or Quasi-Belyi Type)** *Let U (respectively, V) be a* **hyperbolic curve** *over a nonarchimedean local field. Denote the base field of U (respectively, V) by k_U (respectively, k_V), the étale fundamental group of U (respectively, V) by Π_U (respectively, Π_V) [i.e., "we take $\Sigma = \mathfrak{Primes}$"]. Suppose further that* **at least one** *of the following conditions holds:*

(a) **both** *U and V are of* **quasi-Belyi type***;*

(b) **either** *U* **or** *V [but not necessarily both!] is of* **Belyi type**.

Then any isomorphism of profinite groups

$$\beta : \Pi_U \xrightarrow{\sim} \Pi_V$$

arises from a **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\sim} & \widetilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are **isomorphisms***; the vertical arrows are the pro-finite étale coverings determined by the profinite groups Π_U, Π_V.*

Proof. In light of Proposition 2.4, (iii), (iv) [cf. also Remark 24], Corollary 2.3 follows formally from Corollary 2.2. □

Remark 29. Note that in the proof of Proposition 2.4, Corollary 2.3, it is necessary, in the *quasi-Belyi* case, to apply the full "Hom version" of [Mzk4], Theorem A. This differs from the situation of [Mzk8], Corollaries 2.8, 3.2 — i.e., where one only treats hyperbolic curves of *strictly* Belyi type — or, indeed, of the portion of Proposition 2.4, Corollary 2.3, that concerns curves of *Belyi* type, in which the *"isomorphism version"* of [Mzk4], Theorem A, suffices [cf. [Mzk8], Remark 2.8.1].

Thus, in the terminology of [Mzk6], Definition 3.7, the portion of Corollary 2.3 concerning hyperbolic curves of Belyi type admits the following formal consequence:

Corollary 2.4. **(Absoluteness of Curves of Belyi Type)** *Every hyperbolic curve of* **Belyi type** *over a nonarchimedean local field is* **absolute**.

Remark 30. It is interesting to note that the essential property that underlies the absoluteness of Corollary 2.4 is the existence of a *Belyi map* [since the curve is defined over a number field], which, in the context of the theory of [Mzk8], §2, may be regarded as a sort of *endomorphism* of the curve. From this point of view, Corollary 2.4 is *reminiscent* of [Mzk6], Corollary 3.8, which states that the *"canonical curves" of p-adic Teichmüller theory* are absolute. Indeed, from the point of view of the theory of [Mzk2], this canonicality may be regarded as the existence of a sort of *"Frobenius endomorphism"* of the curve. It is also interesting to note that both of these results assert that every member of some *countable* collection of nonarchimedean hyperbolic curves is absolute.

Remark 31. In the context of Remark 30, it is interesting to note that, unlike the canonical curves discussed in [Mzk6], §3, the set of points determined by the hyperbolic curves of *strictly* Belyi type *fails*, for all pairs (g, r) such that $2g-2+r \geq 3$, $g \geq 1$, to be *Zariski dense* in the moduli stack of hyperbolic curves of type (g, r). Indeed, this follows immediately from [Mzk1], Theorem B. On the other hand, it is not clear to the author at the time of writing whether or not the set of points determined by the hyperbolic curves of Belyi (respectively, quasi-Belyi) type is Zariski dense in the moduli stack of hyperbolic curves of type (g, r) [when, say, $2g - 2 + r \geq 3$, $g \geq 2$]. Note, however, that when $g = 0, 1$, [one verifies easily that] every hyperbolic curve of type (g, r) that is defined over a number field is *automatically of Belyi type*.

3. Maximal Pro-l Cuspidalizations

In this §, we apply the theory of the *weight filtration* [cf. [Kane], [Mtm]], together with various generalities concerning *free Lie algebras* [cf. the Appendix], to construct, in the *finite field* case, *"maximal cuspidally pro-l cuspidalizations"* [cf. Theorem 3.1], whose existence implies, under quite general conditions [cf. Corollary 3.1 below], that an isomorphism "α" as in Theorem 1.1, (iii), is always *totally globally Green-compatible*.

In the following discussion, we maintain the notation of §2, and assume further throughout the present §3 that we are in the *finite field* case.

Definition 3.1. Let l be a prime number; G, H, A *topologically finitely generated pro-l groups*; $\phi : H \to A$ a [continuous] homomorphism. Suppose further that A is *abelian*, and that G is an *l-adic Lie group* [cf., e.g., [Serre], Chapter V, §7, §9, for basic facts concerning l-adic Lie groups].

(i) We shall refer to as the *ϕ-central filtration* on H the filtration defined as follows:

$$H(1) \stackrel{\text{def}}{=} H$$
$$H(2) \stackrel{\text{def}}{=} \text{Ker}(\phi)$$
$$H(m) \stackrel{\text{def}}{=} \Big(\text{the subgroup topologically generated by the commutators } [H(a), H(b)], \text{ where } a + b = m, \ \forall\, m \geq 3\Big)$$

Thus, in words, this filtration on H is the *"fastest decreasing central filtration among those central filtrations whose top quotient factors through ϕ"*. We shall say that H is *ϕ-nilpotent* if $H(m) = \{1\}$ for sufficiently large m. If H is ϕ-nilpotent when ϕ is taken to be the natural surjection $H \twoheadrightarrow H^{\text{ab}}$ to its abelianization H^{ab}, then we shall say that H is *nilpotent*. In the following, for $a, b, n \in \mathbb{Z}$ such that $1 \leq a \leq b$, $n \geq 1$, we shall write

$$H(a/b) \stackrel{\text{def}}{=} H(a)/H(b)$$

and

$$\text{Gr}(H)(n) \stackrel{\text{def}}{=} \bigoplus_{m \geq n} H(m/m+1) \subseteq \text{Gr}(H) \stackrel{\text{def}}{=} \text{Gr}(H)(1)$$
$$\text{Gr}(H)(a/b) \stackrel{\text{def}}{=} \text{Gr}(H)(a)/\text{Gr}(H)(b)$$

and append a *subscript* $\mathbb{Q}_l$ (respectively, $\mathbb{F}_l$) to these objects to denote the result of tensoring over $\mathbb{Z}_l$ with $\mathbb{Q}_l$ (respectively, $\mathbb{F}_l$). Thus, $\text{Gr}(H)$, $\text{Gr}_{\mathbb{Q}_l}(H)$, $\text{Gr}_{\mathbb{F}_l}(H)$ are *graded Lie algebras* over $\mathbb{Z}_l$, $\mathbb{Q}_l$, $\mathbb{F}_l$, respectively; $\text{Gr}(H)(n) \subseteq \text{Gr}(H)$ is a [Lie algebra-theoretic] *ideal*. Also, if $\mathbb{Z} \ni a \geq 1$, then we shall write:

$$H(a/\infty) \stackrel{\text{def}}{=} \varprojlim_{b} H(a/b)$$

[where b ranges over the integers $\geq a + 1$].

(ii) We shall denote by $\text{Lie}(G)$ the *Lie algebra over* $\mathbb{Q}_l$ determined by G. If G is *nilpotent*, then $\text{Lie}(G)$ is a nilpotent Lie algebra over $\mathbb{Q}_l$, hence determines a connected, unipotent linear algebraic group $\text{Lin}(G)$, which we shall refer to as the *linear algebraic group associated to G*. In this situation, there exists [cf., e.g., Remark 33 below] a *natural [continuous] homomorphism* [with *open image*]

$$G \to \text{Lin}(G)(\mathbb{Q}_l)$$

[from G to the l-adic Lie group determined by the $\mathbb{Q}_l$-valued points of $\text{Lin}(G)$] which is *uniquely determined* [since $\text{Lin}(G)$ is connected and unipotent!] by the condition that it induce the identity morphism on the associated Lie algebras. In the situation of (i), if $\mathbb{Z} \ni a \geq 1$, then we shall write:

$$\mathrm{Lie}(H(a/\infty)) \stackrel{\mathrm{def}}{=} \varprojlim_{b} \mathrm{Lie}(H(a/b)); \quad \mathrm{Lin}(H(a/\infty)) \stackrel{\mathrm{def}}{=} \varprojlim_{b} \mathrm{Lin}(H(a/b))$$

[where b ranges over the integers $\geq a+1$; we recall that it is well-known [or easily verified] that each $H(a/b)$ is an l-adic Lie group].

Now let us *fix* a prime number $l \in \Sigma^\dagger$. For $S \subseteq X(k)$ a finite subset, let us denote by

$$\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{(l)}; \quad \Delta_X \twoheadrightarrow \Delta_X^{(l)}$$

the *maximal pro-l quotients* and by

$$\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{(l)}; \quad \Pi_X \twoheadrightarrow \Pi_X^{(l)}$$

the quotients of Π_{U_S}, Π_X by the kernels of $\Delta_{U_S} \twoheadrightarrow \Delta_{U_S}^{(l)}$, $\Delta_X \twoheadrightarrow \Delta_X^{(l)}$. [Here, we recall that Δ_{U_S}, Π_{U_S} are as defined in Proposition 1.6, (ii), (iii).] Also, for $x \in X^{\mathrm{cl}}$, let us write

$$D_x^{(l)}[U_S] \subseteq \Pi_{U_S}^{(l)}; \quad I_x^{(l)}[U_S] \subseteq \Delta_{U_S}^{(l)}$$

for the images of $D_x[U_S]$, $I_x[U_S]$ [notation as in Proposition 2.1], respectively, in $\Pi_{U_S}^{(l)}$.

Note that we have a natural surjection:

$$\Delta_{U_S}^{(l)} \twoheadrightarrow \Delta_X^{(l)} \twoheadrightarrow (\Delta_X^{(l)})^{\mathrm{ab}}$$

The cup product on the group cohomology of $\Delta_X^{(l)}$ determines an *isomorphism* [cf. Proposition 1.3, (ii)]

$$\mathrm{Hom}((\Delta_X^{(l)})^{\mathrm{ab}}, M_X^{(l)}) \xrightarrow{\sim} (\Delta_X^{(l)})^{\mathrm{ab}}$$

[where we write $M_X^{(l)} \stackrel{\mathrm{def}}{=} M_X \otimes \mathbb{Z}_l$], hence a *natural G_k-equivariant injection*

$$M_X^{(l)} \hookrightarrow \wedge^2 \, (\Delta_X^{(l)})^{\mathrm{ab}}$$

whose image we denote by $I_{\mathrm{cup}}^{(l)}$.

Definition 3.2. We shall refer to the central filtration

$$\{\Delta_{U_S}^{(l)}(m)\}$$

on $\Delta_{U_S}^{(l)}$ with respect to the natural surjection $\Delta_{U_S}^{(l)} \twoheadrightarrow (\Delta_X^{(l)})^{\mathrm{ab}}$ as the *weight filtration* on $\Delta_{U_S}^{(l)}$ [cf., e.g., [Mtm], §3, p. 200].

Proposition 3.1. **(Freeness and Centralizers)** *Let $x \in S$. Write $S_x \overset{\text{def}}{=} S \backslash \{x\}$; r for the cardinality of S, g for the genus of X. For $x' \in S$, let $\zeta_{x'}$ be a generator of $I_{x'}^{(l)}[U_S]$. By abuse of notation, we shall also denote by $\zeta_{x'}$ the image of $\zeta_{x'}$ in $\Delta_{U_S}^{(l)}(2/3)$. Then:*

(i) $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ *is a* **free Lie algebra** *over $\mathbb{Z}_l$ [hence, in particular, is torsion-free as a $\mathbb{Z}_l$-module] which is freely generated by $2g$ elements*

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in \Delta_{U_S}^{(l)}(1/2)$$

together with the $\zeta_{x'} \in \Delta_{U_S}^{(l)}(2/3)$, for $x' \in S_x$. Alternatively, for an appropriate choice of the elements $\zeta_{x'}$, $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ is the quotient of the free Lie algebra generated by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, together with the $\zeta_{x'} \in \Delta_{U_S}^{(l)}(2/3)$, for $x' \in S$, by the **single relation***:*

$$\sum_{x' \in S} \zeta_{x'} + \sum_{n=1}^{g} [\alpha_n, \beta_n] = 0$$

At a more intrinsic level, this relation is a generator of the image of the **natural G_k-equivariant morphism**

$$M_X^{(l)} \hookrightarrow \Big(\bigoplus_{x' \in S} I_{x'}^{(l)}[U_S] \Big) \oplus I_{\mathrm{cup}}^{(l)}$$

[determined by the various natural isomorphisms $M_X^{(l)} \overset{\sim}{\to} I_{x'}^{(l)}[U_S]$, $M_X^{(l)} \overset{\sim}{\to} I_{\mathrm{cup}}^{(l)}$]], whose codomain maps to $\mathrm{Gr}(\Delta_{U_S}^{(l)})$ via the **natural G_k-equivariant morphism**

$$\Big(\bigoplus_{x' \in S} I_{x'}^{(l)}[U_S] \Big) \oplus I_{\mathrm{cup}}^{(l)} \to \Delta_{U_S}^{(l)}(2/3)$$

[determined by the natural inclusions $I_{x'}^{(l)}[U_S] \hookrightarrow \Delta_{U_S}^{(l)}(2/3)$ and the bracket operation $\wedge^2 (\Delta_X^{(l)})^{\mathrm{ab}} \to \Delta_{U_S}^{(l)}(2/3)$].

(ii) Let ξ be any of the elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$; $\zeta_{x'}$, where $x' \in S_x$, of (i). Then the **centralizer** *in $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$ of [the image of] ξ [in $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$] is equal to $\mathbb{Q}_l \cdot \xi$. In particular, the Lie algebra $\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}^{(l)})$ is* **center-free***.*

(iii) Let ξ be as in (ii). Then for $m \geq 1$, the **centralizer** *in $\Delta_{U_S}^{(l)}(1/m+2)$ of [the image of] ξ [in $\Delta_{U_S}^{(l)}(1/m+2)$] is contained in the subgroup of $\Delta_{U_S}^{(l)}(1/m+2)$ generated by [the image of] ξ and $\Delta_{U_S}^{(l)}(m/m+2)$.*

(iv) Let $S_* \subseteq S$ be a subset of S. Write

$$\mathrm{New}^{(l)}_{S_*} \subseteq \mathrm{Gr}(\Delta^{(l)}_{U_S})$$

for the sub-Lie algebra over $\mathbb{Z}_l$ generated by the image of the restriction

$$\Big(\bigoplus_{x'\in S_*} I^{(l)}_{x'}[U_S]\Big) \subseteq \Big(\bigoplus_{x'\in S} I^{(l)}_{x'}[U_S]\Big) \to \Delta^{(l)}_{U_S}(2/3)$$

to the direct summands indexed by elements of S_* of the morphism of (i), and $\mathrm{New}^{(l)}_{S_*}(a) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\Delta^{(l)}_{U_S})(a) \bigcap \mathrm{New}^{(l)}_{S_*}$*;* $\mathrm{New}^{(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}^{(l)}_{S_*}(a)/\mathrm{New}^{(l)}_{S_*}(b)$ *for $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$. Then, in the notation of (i),* $\mathrm{New}^{(l)}_{S_*}$ *is a* **free Lie algebra** *over $\mathbb{Z}_l$ generated by the elements $\zeta_{x'}$, for $x' \in S_*$. Moreover, the [***"new"*** and ***"co-new"***] $\mathbb{Z}_l$-modules*

$$\mathrm{New}^{(l)}_{S_*}(a/b); \quad \mathrm{Cnw}^{(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\Delta^{(l)}_{U_S})(a/b)/\mathrm{New}^{(l)}_{S_*}(a/b)$$

are **free***. In the following discussion, we shall write* $\mathrm{New}^{\mathrm{tor},(l)}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}^{(l)}_{S_*}(a/b) \otimes \mathbb{Q}/\mathbb{Z}$.

Proof. Assertion (i) (respectively, (ii)) is, in essence, the content of [Kane], Proposition 1 (respectively, Proposition A.1, (ii), (iii)). Assertion (iii) follows formally from assertion (ii). Finally, we consider assertion (iv). By Proposition A.1, (iii), it follows that any free Lie algebra over $\mathbb{F}_l$ with ≥ 2 generators is *center-free*. Thus, let M be the module determined by any *faithful* representation [e.g., when the cardinality of S_* is ≥ 2, the *adjoint representation*] of the free Lie algebra $\mathcal{F}$ over $\mathbb{F}_l$ in the formal generators $\zeta_{x'}$, where $x' \in S_*$. Now observe that we obtain an action of $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ on $M' \stackrel{\mathrm{def}}{=} M \oplus M$ as follows: We let $\alpha_2, \ldots \alpha_g$; $\beta_2, \ldots \beta_g$; $\zeta_{x'}$, where $x' \in S_0 \stackrel{\mathrm{def}}{=} S \backslash S_*$, act by multiplication by 0 on M'. We let α_1, β_1 act on $M' = M \oplus M$ via the matrices

$$\begin{pmatrix} 0 & \sum_{x'\in S_*} \zeta_{x'} \\ 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

respectively. Finally, we let $\zeta_{x'}$, where $x' \in S_*$, act on M' via the following matrix:

$$\begin{pmatrix} \zeta_{x'} & 0 \\ 0 & -\zeta_{x'} \end{pmatrix}$$

Thus, [by assertion (i)] M' determines a representation of $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ whose restriction to the image of $\mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ in $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$ determines [via the natural surjection $\mathcal{F} \twoheadrightarrow \mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$] a *faithful* representation of $\mathcal{F}$. Thus, we conclude that the natural surjection $\mathcal{F} \twoheadrightarrow \mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ is an *isomorphism*, and that $\mathrm{New}^{(l)}_{S_*} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ *injects* into $\mathrm{Gr}_{\mathbb{F}_l}(\Delta^{(l)}_{U_S})$. Assertion (iv) now follows formally. □

Remark 32. The author wishes to thank A. Tamagawa for pointing out to him the content of Proposition 3.1, (i).

Remark 33. One way to verify the *existence* of the homomorphism "$G \to \mathrm{Lin}(G)(\mathbb{Q}_l)$" of Definition 3.1, (ii), is to think of G as a *quotient* of a free pro-l group of finite *even* rank F, whose associated "$\mathrm{Gr}_{\mathbb{Q}_l}(-)$" is a *center-free free Lie algebra* [cf. Proposition 3.1, (i), (ii), in the case of $r = 1$], hence determines an [infinite-dimensional, over $\mathbb{Q}_l$] *faithful* [cf. Proposition 3.1, (iii)] *unipotent representation* [i.e., the adjoint representation — cf. the proof of Proposition 3.1, (iv)] of F. More precisely, by Proposition 3.1, (iii), it follows that there exists a unipotent linear representation $\rho_F : F \to GL(V)$ on a finite-dimensional $\mathbb{Q}_l$-vector space V such that $\mathrm{Ker}(\rho_F) \subseteq \mathrm{Ker}(F \twoheadrightarrow G)$. But this implies that $F \twoheadrightarrow G$ factors through a quotient $F \twoheadrightarrow Q \twoheadrightarrow G$ such that Q is *nilpotent* and admits an injective homomorphism of topological groups $\rho_Q : Q \hookrightarrow Q_{\mathrm{alg}}(\mathbb{Q}_l)$ [induced by ρ_F], where Q_{alg} is a connected, unipotent algebraic group over $\mathbb{Q}_l$, such that ρ_Q is a *local isomorphism*, and $\mathrm{Ker}(\rho_Q) \subseteq \mathrm{Ker}(Q \twoheadrightarrow G)$. Thus, ρ_Q determines a structure of *l-adic Lie group* on Q such that the morphism $\mathrm{Lie}(\rho_Q)$ induced by ρ_Q on Lie algebras is an *isomorphism*. Moreover, the morphism induced by $Q \twoheadrightarrow G$ on Lie algebras *factors* through $\mathrm{Lie}(\rho_Q)$, thus determining a homomorphism of [connected, unipotent] algebraic groups $Q_{\mathrm{alg}} \to \mathrm{Lin}(G)$ such that the resulting composite homomorphism $Q \to Q_{\mathrm{alg}}(\mathbb{Q}_l) \to \mathrm{Lin}(G)(\mathbb{Q}_l)$ *factors* [cf. the induced morphisms on Lie algebras, together with the fact that $\mathrm{Lin}(G)(\mathbb{Q}_l)$ has no torsion!] though G, thus yielding a homomorphism $G \to \mathrm{Lin}(G)(\mathbb{Q}_l)$, as desired.

Next, let us *fix* an $x_* \in S$, as well as a *choice of decomposition group*

$$D_{x_*}[U_S] \subseteq \Pi_{U_S}$$

[i.e., among the various Π_{U_S}-conjugates of this subgroup] associated to x_*. [Thus, $D_{x_*}[U_S]$ determines a *specific* subgroup [i.e., not just a conjugacy class of subgroups] $D^{(l)}_{x_*}[U_S] \subseteq \Pi^{(l)}_{U_S}$.] Recall that the natural exact sequences

$$1 \to I_{x_*}[U_S] \to D_{x_*}[U_S] \to G_k \to 1; \quad 1 \to I^{(l)}_{x_*}[U_S] \to D^{(l)}_{x_*}[U_S] \to G_k \to 1$$

split. [Indeed, extracting roots of any local uniformizer of X at x_* determines such a splitting — cf., e.g., the discussion at the beginning of [Mzk8], §4.] In the following discussion, we shall *fix a splitting*

$$G_k \to D_{x_*}[U_S]$$

of this exact sequence. Thus, this splitting determines a natural action of G_k [by conjugation] on $\Delta^{(l)}_{U_S}$, hence also on

$$\mathrm{Lin}^{(l)}_{U_S}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lin}(\Delta^{(l)}_{U_S}(a/b))(\mathbb{Q}_l); \quad \mathrm{Lie}^{(l)}_{U_S}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lie}(\Delta^{(l)}_{U_S}(a/b))$$
$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(a/b)$$

[where $a, b \in \mathbb{Z}$; $1 \leq a \leq b$]. Write

$$F_k \in G_k$$

for the *Frobenius element* of G_k. In the following, we shall denote the *cardinality of* k by q_k.

Proposition 3.2. **(Galois Invariant Splitting)** *Let* $a, b \in \mathbb{Z}$, $1 \leq a \leq b$.

(i) The **eigenvalues** *of the action of* F_k *on* $\mathrm{Lie}^{(l)}_{U_S}(a/a{+}1)$ *are algebraic numbers all of whose complex absolute values are equal to* $q_k^{a/2}$ *[i.e., "of* **weight** a*"].*

(ii) There is a **unique** G_k**-equivariant isomorphism** *of Lie algebras*

$$\mathrm{Lie}^{(l)}_{U_S}(a/b) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(a/b)$$

which induces the identity isomorphism $\mathrm{Lie}^{(l)}_{U_S}(c/c+1) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta^{(l)}_{U_S})(c/c+1)$, *for all* $c \in \mathbb{Z}$ *such that* $a \leq c \leq b-1$.

(iii) The isomorphism of (ii) together with the natural inclusions $I^{(l)}_x[U_S] \hookrightarrow \Delta^{(l)}_{U_S}$ *for* $x \in S$ *[which are well-defined up to* $\Delta^{(l)}_{U_S}$*-conjugation] determine a* G_k**-equivariant morphism**

$$\Big(\bigoplus_{x \in S} I^{(l)}_x[U_S] \otimes \mathbb{Q}_l\Big) \oplus \mathrm{Lie}^{(l)}_{U_S}(1/2) \to \mathrm{Lie}^{(l)}_{U_S}(1/\infty)$$

which exhibits, in a G_k*-equivariant fashion,* $\mathrm{Lie}^{(l)}_{U_S}(1/\infty)$ *as the quotient of the completion [with respect to the filtration topology] of the* **free Lie algebra** *generated by the finite dimensional* $\mathbb{Q}_l$*-vector space*

$$\Big(\bigoplus_{x \in S} I^{(l)}_x[U_S] \otimes \mathbb{Q}_l\Big) \oplus \mathrm{Lie}^{(l)}_{U_S}(1/2)$$

[equipped with a **natural grading***, hence also a* **filtration***, by taking the* $I^{(l)}_x[U_S] \otimes \mathbb{Q}_l$ *to be of weight* 2, $\mathrm{Lie}^{(l)}_{U_S}(1/2)$ *to be of weight* 1*], by the* **single relation** *determined by the image of the morphism*

$$M^{(l)}_X \otimes \mathbb{Q}_l \hookrightarrow \Big(\bigoplus_{x \in S} I^{(l)}_x[U_S] \otimes \mathbb{Q}_l\Big) \oplus (I^{(l)}_{\mathrm{cup}} \otimes \mathbb{Q}_l)$$

of Proposition 3.1*, (i), tensored with* $\mathbb{Q}_l$.

(iv) For each $g \in \mathrm{Lin}^{(l)}_{U_S}(1/\infty)$*, there exists a* **unique** $h \in \mathrm{Lin}^{(l)}_{U_S}(1/\infty)$ *such*

that

$$F_k \circ \mathrm{Inn}_g = \mathrm{Inn}_h \circ F_k \circ \mathrm{Inn}_{h^{-1}}$$

*[where "*Inn*" denotes the inner automorphism of* $\mathrm{Lin}^{(l)}_{U_S}(1/\infty)$ *defined by conjugation by the subscripted element]. Moreover, when* g *lies in the image of* $I^{(l)}_{x_*} \otimes \mathbb{Q}_l$ *[which is stabilized by the action of* F_k*],* h *also lies in the image of* $I^{(l)}_{x_*} \otimes \mathbb{Q}_l$*.*

Proof. Assertion (i) follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" — cf., e.g., [Mumf], p. 206. Assertion (ii) (respectively, (iii); (iv)) follows formally from assertion (i) (respectively, and Proposition 3.1, (i); and successive approximation of h with respect to the natural filtration $\mathrm{Lin}^{(l)}_{U_S}(a/\infty) \subseteq \mathrm{Lin}^{(l)}_{U_S}(1/\infty)$). ☐

Next, let

$$S_* \subseteq S$$

be a subset such that $x_* \in S_*$; $S_0 \stackrel{\mathrm{def}}{=} S \backslash S_*$. In the following, we shall regard $\mathrm{Lin}^{(l)}_{U_S}(a/b)$ as being equipped with its natural *l-adic topology*. Thus, G_k acts continuously on $\mathrm{Lin}^{(l)}_{U_S}(a/b)$, $\mathrm{Lie}^{(l)}_{U_S}(a/b)$, and we have *natural G_k-equivariant surjections*:

$$\mathrm{Lin}^{(l)}_{U_S}(a/b) \twoheadrightarrow \mathrm{Lin}^{(l)}_{U_{S_0}}(a/b); \quad \mathrm{Lie}^{(l)}_{U_S}(a/b) \twoheadrightarrow \mathrm{Lie}^{(l)}_{U_{S_0}}(a/b)$$

Let us write

$$\mathrm{Lin}^{(l)}_{U_S/U_{S_0}}(a/b); \quad \mathrm{Lie}^{(l)}_{U_S/U_{S_0}}(a/b)$$

for the *kernels* of these surjections. In the following, to *simplify the notation*, we shall often *omit the superscript* (l) from the objects "$\mathrm{Lin}^{(l)}$", "$\mathrm{Lie}^{(l)}$", "$\mathrm{New}^{(l)}$", "$\mathrm{New}^{\mathrm{tor},(l)}$" introduced above and write:

$$\mathrm{Lin}_{U_S}(a/b); \quad \mathrm{Lie}_{U_S}(a/b); \quad \mathrm{Lin}_{U_{S_0}}(a/b); \quad \mathrm{Lie}_{U_{S_0}}(a/b)$$
$$\mathrm{Lin}_{U_S/U_{S_0}}(a/b); \quad \mathrm{Lie}_{U_S/U_{S_0}}(a/b); \quad \mathrm{New}_{S_*}(a/b); \quad \mathrm{New}^{\mathrm{tor}}_{S_*}(a/b)$$

Also, we shall write:

$$\mathrm{New}^{\mathbb{Q}}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{New}_{S_*}(a/b) \otimes \mathbb{Q}; \quad \Delta^{\mathrm{Lie}}_{U_S} \stackrel{\mathrm{def}}{=} \mathrm{Lin}_{U_S}(1/\infty) \times_{\mathrm{Lin}_{U_{S_0}}(1/\infty)} \Delta_{U_{S_0}}$$

Note that, for $\mathbb{Z} \ni b \geq 1$, we have a *natural G_k-equivariant inclusion*

$$\begin{aligned}\mathrm{Lin}_{U_S/U_{S_0}}(b+1/\infty) &\stackrel{\sim}{\to} \mathrm{Lin}_{U_S/U_{S_0}}(b+1/\infty) \times_{\{1\}} \{1\} \\ &\hookrightarrow \mathrm{Lin}_{U_S}(1/\infty) \times_{\mathrm{Lin}_{U_{S_0}}(1/\infty)} \Delta_{U_{S_0}} = \Delta^{\mathrm{Lie}}_{U_S}\end{aligned}$$

whose image forms a normal subgroup of $\Delta_{U_S}^{\mathrm{Lie}}$; write

$$\Delta_{U_S}^{\mathrm{Lie}} \twoheadrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b}$$

for the quotient of $\Delta_{U_S}^{\mathrm{Lie}}$ by this normal subgroup. Also, we have a *natural G_k-equivariant [composite] inclusion*

$$\mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2) \hookrightarrow \mathrm{Lie}_{U_S/U_{S_0}}(b+1/b+2) \xrightarrow{\sim} \mathrm{Lin}_{U_S/U_{S_0}}(b+1/b+2) \hookrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b+1}$$

whose image forms a normal subgroup of $\Delta_{U_S}^{\mathrm{Lie}\leq b+1}$; write

$$\Delta_{U_S}^{\mathrm{Lie}\leq b+1} \twoheadrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b+}$$

for the quotient of $\Delta_{U_S}^{\mathrm{Lie}\leq b+1}$ by this normal subgroup. Thus, we have *natural G_k-equivariant homomorphisms of topological groups*:

$$\Delta_{U_S} \to \Delta_{U_S}^{\mathrm{Lie}} \twoheadrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b+} \twoheadrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b} \twoheadrightarrow \Delta_{U_{S_0}}$$

[the last three of which are easily verified to be *surjective*]. Moreover, forming the *semi-direct product* with G_k [via the natural actions of G_k] yields topological groups and homomorphisms as follows:

$$\Pi_{U_S} \to \Pi_{U_S}^{\mathrm{Lie}} \twoheadrightarrow \Pi_{U_S}^{\mathrm{Lie}\leq b+} \twoheadrightarrow \Pi_{U_S}^{\mathrm{Lie}\leq b} \twoheadrightarrow \Pi_{U_{S_0}}$$

Also, we note that we have *natural exact sequences*:

$$1 \to \mathrm{Lin}_{U_S/U_{S_0}}(1/\infty) \to \Delta_{U_S}^{\mathrm{Lie}} \to \Delta_{U_{S_0}} \to 1$$

$$1 \to \mathrm{Lin}_{U_S/U_{S_0}}(1/\infty) \to \Pi_{U_S}^{\mathrm{Lie}} \to \Pi_{U_{S_0}} \to 1$$

Definition 3.3.

(i) We shall refer to $\Delta_{U_S}^{\mathrm{Lie}}$ (respectively, $\Pi_{U_S}^{\mathrm{Lie}}$; $\Delta_{U_S}^{\mathrm{Lie}\leq b}$; $\Pi_{U_S}^{\mathrm{Lie}\leq b}$; $\Delta_{U_S}^{\mathrm{Lie}\leq b+}$; $\Pi_{U_S}^{\mathrm{Lie}\leq b+}$) as the [$l$-adic] *Lie-ification* (respectively, *Lie-ification*; *Lie-ification, truncated to order* b; *Lie-ification, truncated to order* b; *Lie-ification, truncated to order* $b+$; *Lie-ification, truncated to order* $b+$) of Δ_{U_S} (respectively, Π_{U_S}; Δ_{U_S}; Π_{U_S}; Δ_{U_S}; Π_{U_S}) [over $\Delta_{U_{S_0}}$ (respectively, $\Pi_{U_{S_0}}$; $\Delta_{U_{S_0}}$; $\Pi_{U_{S_0}}$; $\Delta_{U_{S_0}}$; $\Pi_{U_{S_0}}$)].

(ii) Observe that it follows immediately from the definitions that, for $\mathbb{Z} \ni b \geq 1$, we have *natural exact sequences*

$$1 \to \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2) \to \Delta_{U_S}^{\mathrm{Lie}\leq b+1} \to \Delta_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

$$1 \to \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2) \to \Pi_{U_S}^{\mathrm{Lie}\leq b+1} \to \Pi_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

on which $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ acts naturally by conjugation. [Here, we note in passing that it is immediate from the definitions that the submodule

$$\mathrm{New}_{S_*}(b+1/b+2) \subseteq \mathrm{New}_{S_*}^{\mathbb{Q}}(b+1/b+2)$$

is contained in the image of Δ_{U_S}.] In particular, we obtain a *natural inclusion*:

$$\mathrm{New}_{S_*}(b+1/b+2) \hookrightarrow \Delta_{U_S}^{\mathrm{Lie}\leq b+1} \ (\subseteq \Pi_{U_S}^{\mathrm{Lie}\leq b+1})$$

We shall refer to the quotients of $\Delta_{U_S}^{\mathrm{Lie}\leq b+1}$, $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ by the image of this natural inclusion as the *toral Lie-ifications* $\Delta_{U_S}^{\mathrm{tor}\leq b+1}$, $\Pi_{U_S}^{\mathrm{tor}\leq b+1}$ of Δ_{U_S}, Π_{U_S} [over $\Delta_{U_{S_0}}$, $\Pi_{U_{S_0}}$]. Thus, we have *natural exact sequences*

$$1 \to \mathrm{New}_{S_*}^{\mathrm{tor}}(b+1/b+2) \to \Delta_{U_S}^{\mathrm{tor}\leq b+1} \to \Delta_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

$$1 \to \mathrm{New}_{S_*}^{\mathrm{tor}}(b+1/b+2) \to \Pi_{U_S}^{\mathrm{tor}\leq b+1} \to \Pi_{U_S}^{\mathrm{Lie}\leq b+} \to 1$$

on which $\Pi_{U_S}^{\mathrm{Lie}\leq b+1}$ acts naturally by conjugation.

(iii) Suppose that $U'_{S'_0} \to U_{S_0}$ is a *connected finite étale covering* that arises from an open subgroup $\Pi_{U'_{S'_0}} \subseteq \Pi_{U_{S_0}}$; write $X' \to X$ for the *normalization* of X in $U'_{S'_0}$. Then we shall say that the [ramified] covering $X' \to X$ is (S, S_0, Σ)-*admissible* if every closed point of X' that lies over a point of S is rational over the base field k' of X', and, moreover, $\Pi_{U'_{S'_0}}$ is a *characteristic* subgroup of $\Pi_{U_{S_0}}$.

Remark 34. Note that it follows immediately from the definition of $\Pi_{U_S}^{\mathrm{Lie}}$ [cf. also Proposition 3.2, (iii)] that we obtain a *natural subgroup*

$$D_{x_*}^{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} \left(I_{x_*}^{(l)}[U_S] \otimes \mathbb{Q}\right) \rtimes G_k \subseteq \Pi_{U_S}^{\mathrm{Lie}}$$

which contains the image of the decomposition group $D_{x_*}[U_S] \subseteq \Pi_{U_S}$ via the natural homomorphism $\Pi_{U_S} \to \Pi_{U_S}^{\mathrm{Lie}}$. Let us write, for $\mathbb{Z} \ni b \geq 1$, $D_{x_*}^{\mathrm{Lie}\leq b} \subseteq \Pi_{U_S}^{\mathrm{Lie}\leq b}$ for the image of $D_{x_*}^{\mathrm{Lie}}$ in $\Pi_{U_S}^{\mathrm{Lie}\leq b}$; $I_{x_*}^{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} D_{x_*}^{\mathrm{Lie}} \bigcap \Delta_{U_S}^{\mathrm{Lie}}$; $I_{x_*}^{\mathrm{Lie}\leq b} \stackrel{\mathrm{def}}{=} D_{x_*}^{\mathrm{Lie}\leq b} \bigcap \Delta_{U_S}^{\mathrm{Lie}\leq b}$. [Also, we shall use similar notation when "b" is replaced by "$b+$".]

Proposition 3.3. **(Center-freeness of Lie-ification)** $\Delta_{U_S}^{\mathrm{Lie}}$ *is center-free.*

Proof. Since $\Delta_{U_{S_0}}$ is *center-free* [cf. Proposition 1.6, (iii)], and the natural morphism $\Delta^{\mathrm{Lie}}_{U_S} \to \Delta_{U_{S_0}}$ is *surjective*, it suffices to verify that the centralizer in $\mathrm{Lin}_{U_S}(1/\infty)$ of the image of $\Delta^{\mathrm{Lie}}_{U_S}$ is trivial. But the image of $\Delta^{\mathrm{Lie}}_{U_S}$ in $\mathrm{Lin}_{U_S}(1/\infty)$ contains the image of Δ_{U_S} in $\mathrm{Lin}_{U_S}(1/\infty)$. In particular, it follows that the centralizer in question lies in the *center* of $\mathrm{Lin}_{U_S}(1/\infty)$. Thus, Proposition 3.3 follows from Proposition 3.1, (ii) [or, alternatively, (iii)]. ☐

Remark 35. Observe that changing the *choice of splitting*

$$G_k \to D_{x_*}[U_S]$$

affects the image of the element $F_k \in G_k$ via the composite of the inclusion $G_k \hookrightarrow \Pi_{U_S}$ with the morphisms

$$\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b+}_{U_S}$$

by *conjugation by an element* $h \in I^{\mathrm{Lie}}_{x_*}$, which, up to a denominator dividing $q_k - 1$, lies in the image of $I_{x_*}[U_S] \subseteq \Delta_{U_S}$ — cf. Proposition 3.2, (iv); Proposition 3.3. In particular, it follows that changing the choice of splittings $G_k \to D_{x_*}[U_S]$ affects the *Galois invariant splittings* of Proposition 3.2, (ii), by *conjugation by* h. Put another way, if we *identify* the "$\mathrm{Lin}_{U_S}(1/\infty)$", "$\mathrm{Lin}_{U_{S_0}}(1/\infty)$" portions of $\Delta^{\mathrm{Lie}}_{U_S}$ [cf. the definition of $\Delta^{\mathrm{Lie}}_{U_S}$] with the [$l$-adic points of the pro-unipotent algebraic groups determined by the] corresponding graded Lie objects "$\mathrm{Gr}_{\mathbb{Q}_l}(-)(1/\infty)$" via the Galois invariant splittings of Proposition 3.2, (ii), then it follows that: *Changing the choice of splitting* $G_k \to D_{x_*}[U_S]$ *affects the images of the morphisms*

$$\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b}_{U_S}; \quad \Pi_{U_S} \to \Pi^{\mathrm{Lie}\leq b+}_{U_S}$$

[where $\mathbb{Z} \ni b \geq 1$*] by conjugation by* h.

In light of Proposition 3.3, we may apply the exact sequence "$1 \to (-) \to \mathrm{Aut}(-) \to \mathrm{Out}(-) \to 1$" [cf. §0] to construct the following *topological group*:

$$\Delta^{\mathrm{LIE}}_{U_S} \stackrel{\mathrm{def}}{=} \varprojlim_{X'} \ \mathrm{Aut}(\Delta^{\mathrm{Lie}}_{U'_{S'}}) \times_{\mathrm{Out}(\Delta^{\mathrm{Lie}}_{U'_{S'}})} \mathrm{Gal}(X'_{\overline{k}}/X_{\overline{k}})$$

[where $X' \to X$ ranges over the (S, S_0, Σ)-*admissible coverings* of X; $U'_{S'} \subseteq X'$ is the open subscheme determined by the complement of the set S' of closed points of X' that lie over points of S]. Note that G_k *acts naturally* on $\Delta^{\mathrm{LIE}}_{U_S}$; thus, we may form the *semi-direct product* of $\Delta^{\mathrm{LIE}}_{U_S}$ with G_k to obtain a *topological group* $\Pi^{\mathrm{LIE}}_{U_S}$. Also, since the various $\Delta_{U'_{S'_0}}$ [where $U'_{S'_0} \subseteq X'$ is the open subscheme determined by the complement of the set S'_0 of closed points of X'

that lie over points of S_0] arising from the $X' \to X$ that appear in this inverse limit are *center-free* [cf. Proposition 1.6, (iii)], the natural isomorphism

$$\varprojlim_{X'} \mathrm{Aut}(\Delta_{U'_{S'_0}}) \times_{\mathrm{Out}(\Delta_{U'_{S'_0}})} \mathrm{Gal}(X'_{\overline{k}}/X_{\overline{k}}) \xrightarrow{\sim} \Delta_{U_{S_0}}$$

determines surjections $\Delta^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}$, $\Pi^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Pi_{U_{S_0}}$.

Next, let us observe that, for $\mathbb{Z} \ni b \geq 1$, the various quotients $\Delta^{\mathrm{Lie}}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{tor}\leq b+1}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b+}_{U'_{S'}} \twoheadrightarrow \Delta^{\mathrm{Lie}\leq b}_{U'_{S'}}$ determine *quotients of topological groups* $\Delta^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b}_{U_S}$, $\Pi^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$. Thus, we obtain *natural homomorphisms of topological groups*:

$$\Delta_{U_S} \to \Delta^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Delta^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq b}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}$$

$$\Pi_{U_S} \to \Pi^{\mathrm{LIE}}_{U_S} \twoheadrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S} \twoheadrightarrow \Pi_{U_{S_0}}$$

We shall denote by

$$\Delta^{\leq b+}_{U_S} \subseteq \Delta^{\mathrm{LIE}\leq b+}_{U_S}; \quad \Pi^{\leq b+}_{U_S} \subseteq \Pi^{\mathrm{LIE}\leq b+}_{U_S}; \quad \Delta^{\leq b}_{U_S} \subseteq \Delta^{\mathrm{LIE}\leq b}_{U_S}; \quad \Pi^{\leq b}_{U_S} \subseteq \Pi^{\mathrm{LIE}\leq b}_{U_S}$$

the respective images of Δ_{U_S}, Π_{U_S} via these natural homomorphisms. Thus, one may think of $\Delta^{\leq b}_{U_S}$, $\Pi^{\leq b}_{U_S}$ as being a sort of *"canonical integral structure"* on the "inverse limit truncated Lie-ifications" $\Delta^{\mathrm{LIE}\leq b}_{U_S}$, $\Pi^{\mathrm{LIE}\leq b}_{U_S}$.

Here, we note in passing, relative to the theory of §1, 2, that [it is immediate from the definitions that] when $S = S_*$ [so $U_{S_0} = X$], the quotient $\Pi_{U_S} \twoheadrightarrow \Pi^{\leq 2}_{U_S}$ is the *maximal cuspidally pro-l abelian quotient* of Π_{U_S} [cf. Proposition 1.9, (i)].

Next, let us observe that in the inverse limit used to define $\Delta^{\mathrm{LIE}}_{U_S}$, $\Pi^{\mathrm{LIE}}_{U_S}$, the various "$I^{\mathrm{Lie}}_{x_*}$", "$D^{\mathrm{Lie}}_{x_*}$" [cf. Remark 34] form a *compatible system*, hence give rise to subgroups

$$I^{\mathrm{LIE}}_{x_*} \subseteq D^{\mathrm{LIE}}_{x_*} \subseteq \Pi^{\mathrm{LIE}}_{U_S}; \quad I^{\mathrm{LIE}\leq b}_{x_*} \subseteq D^{\mathrm{LIE}\leq b}_{x_*} \subseteq \Pi^{\mathrm{LIE}\leq b}_{U_S}$$

together with natural exact sequences and isomorphisms [when $b \geq 2$]

$$1 \to I^{\mathrm{LIE}}_{x_*} \to D^{\mathrm{LIE}}_{x_*} \to G_k \to 1$$

$$1 \to I^{\mathrm{LIE}\leq b}_{x_*} \to D^{\mathrm{LIE}\leq b}_{x_*} \to G_k \to 1$$

$$I^{\mathrm{LIE}}_{x_*} \cong I^{\mathrm{LIE}\leq b}_{x_*} \cong I^{(l)}_{x_*}[U_S] \otimes \mathbb{Q}$$

[and similarly when "b" is replaced by "$b+$"]. Also, the images of the subgroups $I_{x_*}[U_S]$, $D_{x_*}[U_S]$ of Π_{U_S} determine subgroups

$$I^{\leq b}_{x_*} \subseteq D^{\leq b}_{x_*} \subseteq \Pi^{\leq b}_{U_S}$$

[and similarly when "b" is replaced by "$b+$"].

In the following, let us write [cf. Proposition 3.1, (iv)]

$$\mathrm{Cnw}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Cnw}^{(l)}_{S_*}(a/b); \quad \mathrm{Cnw}^{\mathbb{Q}}_{S_*}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Cnw}^{(l)}_{S_*}(a/b) \otimes \mathbb{Q}$$

[where $a, b \in \mathbb{Z}$, $1 \leq a \leq b$].

Before proceeding, let us observe that [it is immediate from the definitions that] the natural surjections

$$\Delta^{\mathrm{LIE}\leq 1+}_{U_S} \twoheadrightarrow \Delta^{\mathrm{LIE}\leq 1}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}; \quad \Pi^{\mathrm{LIE}\leq 1+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq 1}_{U_S} \twoheadrightarrow \Pi_{U_{S_0}}$$

are *isomorphisms*. On the other hand, for $b \geq 2$, we have the following result:

Proposition 3.4. **(Plus Liftings of Canonical Integral Structures)** *For $\mathbb{Z} \ni b \geq 2$:*

(i) The natural surjections $\Delta^{\leq b+}_{\overline{U}_S} \twoheadrightarrow \Delta^{\leq b}_{\overline{U}_S}$, $\Pi^{\leq b+}_{\overline{U}_S} \twoheadrightarrow \Pi^{\leq b}_{\overline{U}_S}$ are **isomorphisms**.

(ii) Any two liftings of the natural inclusion $\Pi^{\leq b}_{\overline{U}_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$ to inclusions $\Pi^{\leq b}_{\overline{U}_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ differ by **conjugation** *in $\Pi^{\mathrm{LIE}\leq b+}_{U_S}$ by a* **unique** *element of the kernel of $\Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$.*

(iii) Any two liftings of the natural inclusion $\Pi^{\leq b}_{\overline{U}_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$ to inclusions $\Pi^{\leq b}_{\overline{U}_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ whose images **contain** *$D^{\leq b+}_{x_*}$ in fact* **coincide**.

Proof. First, we consider assertion (i). It follows immediately from the definitions that the kernel in question

$$\mathrm{Ker}(\Delta^{\leq b+}_{\overline{U}_S} \twoheadrightarrow \Delta^{\leq b}_{\overline{U}_S}) = \mathrm{Ker}(\Pi^{\leq b+}_{\overline{U}_S} \twoheadrightarrow \Pi^{\leq b}_{\overline{U}_S})$$

is contained in [and, in fact, equal to] the inverse limit

$$\varprojlim_{X'} \mathrm{Cnw}_{S'_*}(b+1/b+2)$$

[where $X' \to X$ ranges over the (S, S_0, Σ)-*admissible coverings* of X; S'_* (respectively, S') is the set of closed points of X' that lie over points of S_* (respectively, S)]. On the other hand, it follows from the definition of "$\mathrm{Cnw}_{S'_*}(b+1/b+2)$" that $\mathrm{Cnw}_{S'_*}(b+1/b+2)$ is generated by certain successive brackets of the various generators of the Lie algebra $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ [cf. Proposition 3.1, (i)] with the property that at least one of the generators appearing in the successive bracket is [in the notation of Proposition 3.1, (i)] either one of the [analogue for X' of the] "$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$" or one of the "$\zeta_{x'}$", where $x' \in S'_0 \stackrel{\mathrm{def}}{=} S' \backslash S'_*$.

Moreover, since, by taking $\Pi_{U''_{S''_0}} \subseteq \Pi_{U'_{S'_0}}$ to be *sufficiently small*, one may arrange that the image of $\Delta^{(l)}_{U''_{S''_0}}(1/3)$ in $\Delta^{(l)}_{U'_{S'_0}}(1/3)$ be contained in an *arbitrarily small* open subgroup of $\Delta^{(l)}_{U'_{S'_0}}(1/3)$, it thus follows that the above inverse limit *vanishes*. This completes the proof of assertion (i).

Next, let us observe that to prove assertion (ii), it suffices — in light of the *natural isomorphism*

$$\mathrm{Ker}(\Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}) \xrightarrow{\sim} \varprojlim_{X'} \mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)$$

[where X', S'_* are as above] — to show that

$$H^i(\Pi^{\leq b}_{U_S}, \mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)) = 0$$

for $i = 0, 1$, each S'_* as above. Since the action of $\Delta^{\leq b}_{U_S}$ on $\mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)$ clearly factors through a *finite quotient* of $\Delta^{\leq b}_{U_S} \twoheadrightarrow \Delta_{U_{S_0}}$, it thus suffices to observe [by considering the Leray spectral sequence associated to the surjection $\Pi^{\leq b}_{U_S} \twoheadrightarrow G_k$] that the action of F_k on $\mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)$ is *"of weight* $b+1 \geq 3$*"*, while the action of F_k on $(\Delta^{(l)}_{U'_{S'}})^{\mathrm{ab}}$ is *"of weight* ≤ 2*"* [cf. Proposition 3.2, (i)]. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, let us observe that any two liftings of the natural inclusion $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}$ to inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ whose images contain $D^{\leq b+}_{x_*} \xrightarrow{\sim} D^{\leq b}_{x_*}$ [since $b \geq 2$] in fact *coincide* on $D^{\leq b}_{x_*} \subseteq \Pi^{\leq b}_{U_S}$. Thus, by assertion (ii), it suffices to verify that the submodule of F_k*-invariants* of

$$\mathrm{Ker}(\Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S})$$

is zero. But in light of the natural isomorphism

$$\mathrm{Ker}(\Pi^{\mathrm{LIE}\leq b+}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b}_{U_S}) \xrightarrow{\sim} \varprojlim_{X'} \mathrm{Cnw}^{\mathbb{Q}}_{S'_*}(b+1/b+2)$$

[where X', S'_* are as above], this follows from Proposition 3.2, (i). This completes the proof of assertion (iii). ☐

Next, for $\mathbb{Z} \ni b \geq 1$, let us denote by

$$\Delta^{\leq b++}_{U_S} \subseteq \Delta^{\mathrm{TOR}\leq b+1}_{U_S}; \quad \Pi^{\leq b++}_{U_S} \subseteq \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$$

the respective images of Δ_{U_S}, Π_{U_S} via the natural homomorphisms considered above and by

$$I^{\leq b++}_{x_*} \subseteq D^{\leq b++}_{x_*} \subseteq \Pi^{\leq b++}_{U_S}$$

the images of the subgroups $I_{x_*}[U_S]$, $D_{x_*}[U_S]$ of Π_{U_S}. Observe that it follows from the definition of $\Delta_{U_S}^{\mathrm{TOR}\leq b+1}$, $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ [cf. also Proposition 3.1, (iv)] that the natural surjections $\Delta_{U_S}^{\leq b++} \twoheadrightarrow \Delta_{U_S}^{\leq b+}$, $\Pi_{U_S}^{\leq b++} \twoheadrightarrow \Pi_{U_S}^{\leq b+}$ are, in fact, *isomorphisms*. Thus, by Proposition 3.4, (i), we obtain a *commutative diagram of natural homomorphisms*

$$\begin{array}{ccccccc} \Pi_{U_S}^{\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\leq b++} & \stackrel{\sim}{\to} & \Pi_{U_S}^{\leq b+} & \stackrel{\sim}{\to} & \Pi_{U_S}^{\leq b} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Pi_{U_S}^{\mathrm{LIE}\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{TOR}\leq b+1} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{LIE}\leq b+} & \twoheadrightarrow & \Pi_{U_S}^{\mathrm{LIE}\leq b} \end{array}$$

[where the vertical arrows are the *natural inclusions*; all of the horizontal arrows are surjections; the second two upper horizontal arrows are isomorphisms]. Moreover, it follows immediately from the definitions that the *first square* in this commutative diagram is *cartesian*. That is to say, the subgroup $\Pi_{U_S}^{\leq b+1} \subseteq \Pi_{U_S}^{\mathrm{LIE}\leq b+1}$ may be thought of as the *inverse image* via the natural surjection $\Pi_{U_S}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$ of the image of a certain *lifting* of the natural inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{LIE}\leq b+}$ [cf. Proposition 3.4, (i)] to an inclusion $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\mathrm{TOR}\leq b+1}$. Also, let us write:

$$\Pi_{U_S}^{\leq b}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_X)$$

$$\Pi_{U_S}^{\leq b++}[\mathrm{csp}] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{U_S}^{\leq b++} \twoheadrightarrow \Pi_X)$$

for the *cuspidal subgroups* of $\Pi_{U_S}^{\leq b}$, $\Pi_{U_S}^{\leq b++}$.

Next, following the pattern of §1, we relate the constructions made so far to the fundamental groups $\Delta_{U_{X\times X}}$, $\Pi_{U_{X\times X}}$ [cf. the discussion preceding Proposition 1.5]. For simplicity, we *assume* from now on that:

$$S = S_* = \{x_*\}$$

[so $S_0 = \emptyset$]. Write $D_{x_*}[X] \subseteq \Pi_X$ for the image of $D_{x_*}[U_S]$ via the natural surjection $\Pi_{U_S} \twoheadrightarrow \Pi_X$. Then the projection $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi_X$ to the *second factor* determines a *natural isomorphism*

$$\Pi_{U_S} \stackrel{\sim}{\to} \Pi_{U_{X\times X}} \times_{\Pi_X} D_{x_*}[X]$$

[cf. Proposition 1.6, (ii)]. Moreover, this isomorphism determines a *natural isomorphism*

$$(\Pi_{U_S} \supseteq) \quad D_{x_*}[U_S] \stackrel{\sim}{\to} D_X \times_{\Pi_X} D_{x_*}[X] \quad (\subseteq D_X \subseteq \Pi_{U_{X\times X}})$$

[where "D_X" is as in the discussion preceding Proposition 1.8] which is *compatible* with the natural inclusions $D_{x_*}[U_S] \hookrightarrow \Pi_{U_S}$, $D_X \hookrightarrow \Pi_{U_{X\times X}}$. Put another way, $D_{x_*}[U_S]$ [hence also $I_{x_*}[U_S], G_k \subseteq D_{x_*}[U_S]$] may be thought of as being "simultaneously" a *subgroup of both* Π_{U_S} *and* D_X. Thus, we obtain a *natural exact sequence*

$$1 \longrightarrow \Delta_{U_S} \longrightarrow \Pi_{U_{X\times X}} \longrightarrow \Pi_X \longrightarrow 1$$

together with *compatible inclusions*

$$\Delta_{U_S} \supseteq \Delta_{U'_{S'}} \supseteq I_{x_*}[U_S] \subseteq D_{x_*}[U_S] \subseteq D_X \subseteq \Pi_{U_{X\times X}}$$

[where $X' \to X$ is an $(S, \emptyset, \Sigma)$-*admissible covering* of X; $U'_{S'} \subseteq X'$ is the open subscheme determined by the complement of the set S' of closed points of X' that lie over x_*]. Also, we shall write:

$$D_X^{\Delta} \stackrel{\text{def}}{=} D_X \bigcap \Delta_{U_{X\times X}} \subseteq \Pi_{U_{X\times X}}$$

In particular, we obtain *natural actions* [by conjugation] of D_X on Δ_{U_S}, $\Delta_{U'_{S'}}$ [as well as on the various objects naturally constructed from Δ_{U_S}, $\Delta_{U'_{S'}}$ in the above discussion], which we shall refer to as *diagonal actions*.

Proposition 3.5. **(Characterization of the Diagonal Action)** *Suppose that* $S = S_* = \{x_*\}$. *Then in the notation and terminology of the above discussion, the* **diagonal action** *of* D_X *on* $\text{Lin}_{U'_{S'}}(1/\infty)$ *is* **completely determined** *[i.e., as a continuous action of the topological group* D_X *on the topological group* $\text{Lin}_{U'_{S'}}(1/\infty)$*] by the following conditions:*

(a) the action is **compatible** *with the* **natural action** *of* $D_{x_*}[U_S] \subseteq D_X$ *on* $\text{Lin}_{U'_{S'}}(1/\infty)$*;*

(b) the action is **compatible** *with the* **filtration** $\{\text{Lin}_{U'_{S'}}(a/\infty)\}$ *[where* $a \geq 1$ *is an integer] on* $\text{Lin}_{U'_{S'}}(1/\infty)$.

(c) the action **coincides** *with the* **diagonal action** *of* D_X *on the quotient* $\text{Lin}_{U'_{S'}}(1/4)$ *[cf. condition (b)] of* $\text{Lin}_{U'_{S'}}(1/\infty)$.

Proof. First, I *claim* that it suffices to show that these conditions determine the action of the subgroup $D^{\Delta}_{X'/X} \stackrel{\text{def}}{=} D^{\Delta}_X \times_{\Delta_X} \Delta_{X'} \subseteq D^{\Delta}_X \subseteq D_X$ on $\text{Lin}_{U'_{S'}}(1/\infty)$. Indeed, once the action of $D^{\Delta}_{X'/X}$ is determined, it follows that the action of

$$D_{X'/X} \stackrel{\text{def}}{=} D_X \times_{\Pi_X} \Pi_{X'} \subseteq D_{x_*}[U_S] \cdot D^{\Delta}_{X'/X} \subseteq D_X$$

is determined [cf. condition (a)]. On the other hand, since $\Pi_{X'}$ is an open normal subgroup of Π_X, it follows that $D_{X'/X}$ is an *open normal subgroup* of

D_X. Thus, by considering the *conjugation actions* of D_X on $D_{X'/X}$ and of $\mathrm{Im}(D_X) \subseteq \mathrm{Lin}_{U'_{S'}}(1/\infty)$ on $\mathrm{Im}(D_{X'/X}) \subseteq \mathrm{Lin}_{U'_{S'}}(1/\infty)$ [i.e., of the group of automorphisms of $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ induced by elements of D_X on the group of automorphisms of $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ induced by elements of $D_{X'/X}$], we conclude that the action of D_X on $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ is determined up to composition with automorphisms of $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ that *commute* with the action of $D_{X'/X}$ and [cf. condition (c)] induce the *identity* on the quotient $\mathrm{Lin}_{U'_{S'}}(1/4)$. Now let α be an automorphism of $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ that commutes with the action of $D_{X'/X}$ and induces the identity on the quotient $\mathrm{Lin}_{U'_{S'}}(1/4)$. Then α commutes with some open subgroup of $G_k \subseteq D_{x_*}[U_S] \subseteq D_X$, so α induces an automorphism of $\mathrm{Lie}_{U'_{S'}}(1/\infty)$ that is *compatible* with the splittings of Proposition 3.2, (ii). Since $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ is *generated* by its elements *"of weight ≤ 2"* [cf. Proposition 3.1, (i)], we thus conclude that α induces the identity automorphism of $\mathrm{Lie}_{U'_{S'}}(1/\infty)$, hence that α itself is the identity automorphism. This completes the proof of the *claim*.

Next, let us observe that by condition (c) [cf. also Proposition 3.1, (i)], the action of $D^{\Delta}_{X'/X}$ on $\mathrm{Lin}_{U'_{S'}}(1/\infty)$ is *unipotent*, relative to the filtration of condition (b). Thus, it follows [from the definition of "Lie(−)"] that the induced action of $D^{\Delta}_{X'/X}$ on $\mathrm{Lie}_{U'_{S'}}(1/\infty)$ determines an *action of the Lie algebra*

$$\mathrm{Lie}(D^{\Delta}_{X'/X}) \stackrel{\mathrm{def}}{=} \mathrm{Lie}((D^{\Delta}_{X'/X})^{(l)}(1/\infty))$$

[where we write $(D^{\Delta}_{X'/X})^{(l)}$ for the maximal pro-l quotient of $(D^{\Delta}_{X'/X})^{(l)}$] on the Lie algebra $\mathrm{Lie}_{U'_{S'}}(1/\infty)$. Moreover, to complete the proof of Proposition 3.5, it suffices to show that this *Lie algebra action* is the action arising from the *diagonal action*. In fact, since this Lie algebra action is *compatible* [cf. condition (a)] with the actions of G_k on $\mathrm{Lie}(D^{\Delta}_{X'/X})$, $\mathrm{Lie}_{U'_{S'}}(1/\infty)$, it follows, by considering the *induced eigenspace splittings* [cf. Proposition 3.2, (ii)], that [to complete the proof of Proposition 3.5] it suffices to show that the *Lie algebra action* of $\mathrm{Gr}(D^{\Delta}_{X'/X}) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\mathrm{Lie}(D^{\Delta}_{X'/X}))$ on $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ is the action arising from the *diagonal action*. On the other hand, since $\mathrm{Gr}(D^{\Delta}_{X'/X})$, $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ are generated by elements *"of weight ≤ 2"* [cf. Proposition 3.1, (i)], this follows by observing that the Lie algebra action of the unique generator of $\mathrm{Gr}(D^{\Delta}_{X'/X})$ "of weight 2" [which arises from $I_{x_*}[U_S] \subseteq D^{\Delta}_{X'/X}$] is determined by condition (a), while the Lie algebra action of the generators of $\mathrm{Gr}(D^{\Delta}_{X'/X})$ "of weight 1" [which send elements of $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ "of weight ≤ 2" to elements of $\mathrm{Gr}(\Delta^{(l)}_{U'_{S'}})$ "of weight ≤ 3"] is determined by condition (c). This completes the proof of Proposition 3.5. □

Remark 36. Note that the conditions of Proposition 3.5 allow one to characterize not only the *diagonal action* of D_X on $\mathrm{Lin}_{U'_{S'}}(1/\infty)$, but also on $\Delta^{\mathrm{Lie}}_{U'_{S'}}$, $\Pi^{\mathrm{Lie}}_{U'_{S'}}$, hence also on $\Delta^{\mathrm{LIE}}_{U_S}$, $\Pi^{\mathrm{LIE}}_{U_S}$ [where we note that the diagonal action of D_X on $\mathrm{Gal}(X'_{\overline{k}}/X_{\overline{k}})$ is simply the conjugation action arising from the quotients $D_X \twoheadrightarrow \Pi_X$, $\Delta_X \twoheadrightarrow \mathrm{Gal}(X'_{\overline{k}}/X_{\overline{k}})$].

Remark 37. Note that the groups $\mathrm{Lin}_{U'_{S'}}(1/4)$ of condition (c) of Proposition 3.5 are, as groups equipped with the surjection $\mathrm{Lin}_{U'_{S'}}(1/4) \twoheadrightarrow \mathrm{Lin}_{X'}(1/4)$, *cuspidally abelian* [i.e., the kernel of this surjection is *abelian*], hence may be constructed from the *maximal cuspidally abelian quotients* $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab}}_{U_{X\times X}}$ of Theorem 1.1.

Proposition 3.6. **(Extensions of Canonical Integral Structures)** *Suppose that $S = S_* = \{x_*\}$ [cf. Remark 39 below]. Let $b \geq 1$ be an integer. Then:*

(i) Suppose that $b = 1$. Then any two liftings of the natural inclusion $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ to inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ differ by **conjugation** *in $\Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ by an element of the kernel of $\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$.*

(ii) Suppose that $b \geq 2$. Then any two liftings of the natural inclusion $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ to inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ whose images **contain** *$I^{\leq b++}_{x_*}$ differ by* **conjugation** *in $\Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ by an element of the kernel of $\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$.*

(iii) Let β be an automorphism of the profinite group $\Pi^{\leq b+1}_{U_S}$ that satisfies the following two conditions: (a) β preserves and induces the identity on the quotient $\Pi^{\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\leq b}_{U_S}$; (b) β preserves the subgroup $I^{\leq b+1}_{x_*} \subseteq \Pi^{\leq b+1}_{U_S}$. Then β is a $\mathrm{Ker}(\Pi^{\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\leq b}_{U_S})$**-inner automorphism***.*

(iv) Let $\delta \in \mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S})$ be an element that is **invariant** *under the* **diagonal action** *of D_X. Then if $b = 1$, then δ lies in the image of $I_{x_*}[U_S] \otimes (\mathbb{Q}_l/\mathbb{Z}_l)$; if $b \geq 2$, then δ is the* **identity** *element.*

(v) Write

$$\Pi_{U_S} \twoheadrightarrow \Pi^{\leq \infty}_{U_S} \stackrel{\mathrm{def}}{=} \varprojlim_{b} \Pi^{\leq b}_{U_S}; \quad \Delta_{U_S} \twoheadrightarrow \Delta^{\leq \infty}_{U_S} \stackrel{\mathrm{def}}{=} \varprojlim_{b} \Delta^{\leq b}_{U_S}$$

for the the quotients of Π_{U_S}, Δ_{U_S} defined by the inverse limit of the $\Pi^{\leq b}_{U_S}$, $\Delta^{\leq b}_{U_S}$ and

$$\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\leq \infty}_{U_{X\times X}}; \quad \Delta_{U_{X\times X}} \twoheadrightarrow \Delta^{\leq \infty}_{U_{X\times X}}$$

for the quotients of $\Pi_{U_{X\times X}}$, $\Delta_{U_{X\times X}}$ *determined by the kernel in* $\Delta_{U_S} \subseteq \Delta_{U_{X\times X}} \subseteq \Pi_{U_{X\times X}}$ *[cf. the discussion preceding Proposition* 3.5*] of* $\mathrm{Ker}(\Delta_{U_S} \twoheadrightarrow \Delta^{\leq\infty}_{U_S})$. *Then* $\Pi_{U_S} \twoheadrightarrow \Pi^{\leq\infty}_{U_S}$ *(respectively,* $\Delta_{U_S} \twoheadrightarrow \Delta^{\leq\infty}_{U_S}$; $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\leq\infty}_{U_{X\times X}}$; $\Delta_{U_{X\times X}} \twoheadrightarrow \Delta^{\leq\infty}_{U_{X\times X}}$*) is the* **maximal cuspidally pro-*l* quotient** *of* Π_{U_S} *(respectively,* Δ_{U_S}; $\Pi_{U_{X\times X}}$; $\Delta_{U_{X\times X}}$*); moreover,* $(\Pi^{\leq\infty}_{U_S})^\dagger$, $\Delta^{\leq\infty}_{U_S}$, $(\Pi^{\leq\infty}_{U_{X\times X}})^\dagger$, $\Delta^{\leq\infty}_{U_{X\times X}}$ *[where the daggers denote the result of applying the operation "*$\times_{G_k} G_k^\dagger$*"] are* **slim**.

Proof. First, we consider assertions (i), (ii). Observe that, for $\mathbb{Z} \ni b \geq 1$, the *difference* of any two liftings of the natural inclusion $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}$ to inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ determines a *compatible collection of cohomology classes*

$$\eta_{S'} \in H^1(\Pi^{\leq b}_{U_S}, \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$$

[where $X' \to X$ ranges over the $(S, \emptyset, \Sigma)$-*admissible coverings* of X; S' is the set of closed points of X' that lie over x_*]. Since $\mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2) = 0$ whenever b is even, we may assume for the remainder of the proof of assertions (i), (ii) that b is *odd*.

Next, let us observe that by Proposition 3.2, (i), the zeroth cohomology module

$$H^0(\Pi^{\leq b}_{U_S}, \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$$

is *finite*. This finiteness implies that any [not necessarily compatible!] system of sections of a compatible system of torsors over $H^0(\Pi^{\leq b}_{U_S}, \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$ always admits a *compatible cofinal subsystem*. In light of the natural isomorphism

$$\mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S}) \xrightarrow{\sim} \varprojlim_{X'} \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$$

[where X', S' are as described above], we thus conclude that in order to show that the two inclusions $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ differ by *conjugation* by an element of $\mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S})$, it suffices to show that the $\eta_{S'} = 0$.

Note that $\Pi^{\leq b}_{U_S}[\mathrm{csp}]$ acts *trivially* on $\mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$. Now I *claim* that:

> *Each* $\eta_{S'}$ *arises from a unique class [which, by abuse of notation, we shall also denote by* $\eta_{S'}$*] in* $H^1(\Pi_X, \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$.

Indeed, if $b = 1$, this claim follows from the fact that $\Pi^{\leq b}_{U_S}[\mathrm{csp}] = \{1\}$ [cf. the discussion preceding Proposition 3.4], so assume that $b \geq 2$, and that we are in the situation of assertion (ii). Now observe that since $S = S_*$ is of *cardinality one*, it follows that $\Pi^{\leq b}_{U_S}[\mathrm{csp}]$ (respectively, $\Pi^{\leq b++}_{U_S}[\mathrm{csp}]$) is *topologically generated* by the $\Pi^{\leq b}_{U_S}$- (respectively, $\Pi^{\leq b++}_{U_S}$-) *conjugates* of $I^{\leq b}_{\bar{x}_*}$ (respectively, $I^{\leq b++}_{\bar{x}_*}$). Note, moreover, that it is immediate from the definitions that every element of $\mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S})$ *commutes* with $I^{\leq b++}_{\bar{x}_*}$. In particular, it follows that the images of $\Pi_{U_S}^{\leq b}[\text{csp}]$ via the two inclusions $\Pi_{U_S}^{\leq b} \hookrightarrow \Pi_{U_S}^{\text{TOR}\leq b+1}$ under consideration necessarily *coincide*. But this implies that each $\eta_{S'}$ arises from a unique class in $H^1(\Pi_X, \text{New}_{S'}^{\text{tor}}(b+1/b+2))$, thus completing the proof of the *claim*.

Next, [returning to the general situation involving *both* assertions (i) and (ii)] let

$$X'' \to X'$$

be a morphism of $(S, \emptyset, \Sigma)$-*admissible coverings* of X. Write $U''_{S''} \subseteq X''$ for the open subscheme determined by the complement of the set S'' of closed points of X'' that lie over points of S. Also, let us assume that the open subgroup $\Delta_{X''} \subseteq \Delta_{X'}$ *arises from some open subgroup* $H'' \subseteq \Delta_{X'}^{\text{ab}}$ that is *preserved by the action of* Π_X. Thus, it follows that the covering $X''_{\overline{k}} \to X'_{\overline{k}}$ is *abelian*; write $\underline{\text{Gal}}(X''/X') \stackrel{\text{def}}{=} \text{Gal}(X''_{\overline{k}}/X'_{\overline{k}})$. For c a positive integer, set:

$$R' \stackrel{\text{def}}{=} \mathbb{Z}_l; \quad R''_c \stackrel{\text{def}}{=} \mathbb{Z}_l[c \cdot \underline{\text{Gal}}(X''/X')] \subseteq R'' \stackrel{\text{def}}{=} \mathbb{Z}_l[\underline{\text{Gal}}(X''/X')]$$

[where we write $c \cdot \underline{\text{Gal}}(X''/X') \subseteq \underline{\text{Gal}}(X''/X')$ for the subgroup of the abelian group $\underline{\text{Gal}}(X''/X')$ that arises as the image of *multiplication by* c]. Thus, R'' (respectively, R''; R''_c) is a commutative ring with unity whose underlying R'- (respectively, R''_c-; R'-) module is *finite and free*; moreover, R'', R''_c admit a natural Π_X-action [induced by the conjugation action of Π_X on the subquotient $\underline{\text{Gal}}(X''/X')$ of Π_X]. Also, we shall denote by

$$\epsilon''_c : R''_c \twoheadrightarrow R'; \quad \epsilon'' : R'' \twoheadrightarrow R'$$

the *augmentations* obtained by mapping all of the elements of $\underline{\text{Gal}}(X''/X')$ to 1.

Next, let us observe that S', S'' admit *natural* Π_X*-actions* with respect to which we have *natural isomorphisms of* Π_X*-modules* [cf. Proposition 3.1, (i), (iv)]

$$\text{New}_{S'}(2/3) \xrightarrow{\sim} R'[S'] \otimes M_X^{(l)}; \quad \text{New}_{S''}(2/3) \xrightarrow{\sim} R'[S''] \otimes M_X^{(l)}$$

which determine *natural isomorphisms of* Π_X*-modules*

$$\text{New}_{S'}(2c/2c+1) \xrightarrow{\sim} \mathfrak{Lie}^c_{R'}(R'[S'] \otimes M_X^{(l)})$$

$$\text{New}_{S''}(2c/2c+1) \xrightarrow{\sim} \mathfrak{Lie}^c_{R'}(R'[S''] \otimes M_X^{(l)})$$

[cf. the notation of Proposition A.1] for integers $c \geq 1$. In the following, we shall *identify* the domains and codomains of these isomorphisms via these isomorphisms.

Next, let us observe that the R'-module $R'[S'']$ admits a natural R''*-module structure* that is compatible with the Π_X-action on R'', $R'[S'']$. Note, moreover, that $R'[S'']$ is a *free* R''*-module*, and that we have a natural isomorphism

$$R'[S''] \otimes_{R'', \epsilon''} R' \xrightarrow{\sim} R'[S']$$

induced by the augmentation $\epsilon'' : R'' \twoheadrightarrow R'$. Also, we observe that any *choice of representatives* in S'' of the $\Delta_{X'}/\Delta_{X''} = \underline{\mathrm{Gal}}(X''/X')$-orbits of S'' [where we note that the set of such orbits may be naturally identified with S'] determines an R''-*basis* of $R'[S'']$, hence [by considering *"Hall bases"* — cf., e.g., [Bour], Chapter II, §2.11] an R''-*basis* of $\mathfrak{Lie}^c_{R''}(R'[S''])$. Note that since the natural action of $\underline{\mathrm{Gal}}(X''/X')$ on $\mathfrak{Lie}^c_{R''}(R'[S''])$ is *compatible* with the Lie algebra structure, it follows that:

> This *natural action* of $\underline{\mathrm{Gal}}(X''/X')$ on $\mathfrak{Lie}^c_{R''}(R'[S''])$ is given by *composing* the R''-module structure action $\underline{\mathrm{Gal}}(X''/X') \hookrightarrow R''$ with the morphism $c\cdot : \underline{\mathrm{Gal}}(X''/X') \to \underline{\mathrm{Gal}}(X''/X')$ given by *multiplication by* c.

In particular, this natural action of $\underline{\mathrm{Gal}}(X''/X')$ on $\mathfrak{Lie}^c_{R''}(R'[S''])$ *factors* through the quotient $\underline{\mathrm{Gal}}(X''/X') \twoheadrightarrow c\cdot\underline{\mathrm{Gal}}(X''/X')$ and hence determines on $\mathfrak{Lie}^c_{R''}(R'[S''])$ a structure of *"induced"* $c \cdot \underline{\mathrm{Gal}}(X''/X')$-module [in the terminology of the cohomology theory of finite groups]. Thus, we obtain *natural,* Π_X*-equivariant isomorphisms*

$$R'[S'] \xrightarrow{\sim} R'[S'']^{\underline{\mathrm{Gal}}(X''/X')}$$

$$\mathfrak{Lie}^c_{R''}(R'[S'']) \otimes_{R''_c,\epsilon''_c} R' \xrightarrow{\sim} \mathfrak{Lie}^c_{R''}(R'[S''])^{\underline{\mathrm{Gal}}(X''/X')} = \mathfrak{Lie}^c_{R''}(R'[S''])^{c\cdot\underline{\mathrm{Gal}}(X''/X')}$$

[where we use superscripts to denote the submodules of invariants with respect to the action of the superscripted group]. Moreover, we observe that relative to these natural isomorphisms, the restriction of the natural surjection $\mathfrak{Lie}^c_{R''}(R'[S'']) \twoheadrightarrow \mathfrak{Lie}^c_{R''}(R'[S'']) \otimes_{R''_c,\epsilon''_c} R'$ to the submodule of $\underline{\mathrm{Gal}}(X''/X')$-invariants induces the endomorphism of the module $\mathfrak{Lie}^c_{R''}(R'[S'']) \otimes_{R''_c,\epsilon''_c} R'$ given by *multiplication by the order of* $c \cdot \underline{\mathrm{Gal}}(X''/X')$.

Now let us write:

$$\mathrm{New}^{\mathrm{tor}}_{S''/S'}(2c/2c+1) \stackrel{\mathrm{def}}{=} \mathfrak{Lie}^c_{R''}(R'[S''] \otimes M^{(l)}_X) \otimes (\mathbb{Q}_l/\mathbb{Z}_l)$$

$$\underline{\mathrm{New}}^{\mathrm{tor}}_{S''/S'}(2c/2c+1) \stackrel{\mathrm{def}}{=} \mathrm{New}^{\mathrm{tor}}_{S''/S'}(2c/2c+1) \otimes_{R''_c,\epsilon''_c} R'$$

[where $c \geq 1$ is an integer]. Then in light of the above observations [together with Propositions A.1, (iv); 3.1, (iv)], we conclude the following:

(A) The *natural surjection* of Π_X-modules

$$\mathrm{New}^{\mathrm{tor}}_{S''}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)$$

admits a *factorization*

$$\mathrm{New}^{\mathrm{tor}}_{S''}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2) \twoheadrightarrow \underline{\mathrm{New}}^{\mathrm{tor}}_{S''/S'}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)$$

[via morphisms of Π_X-modules]. Moreover, the natural action of $\Delta_{X'}$ on the module $\mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$ *factors* through the quotient $\Delta_{X'} \twoheadrightarrow \underline{\mathrm{Gal}}(X''/X') \twoheadrightarrow c \cdot \underline{\mathrm{Gal}}(X''/X')$ and determines on $\mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$ a structure of *induced* $c \cdot \underline{\mathrm{Gal}}(X''/X')$-module.

(B) The induced morphism on $\Delta_{X'}$*-invariants*

$$\mathrm{New}^{\mathrm{tor}}_{S''}(b+1/b+2)^{\Delta_{X'}} \to \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)^{\Delta_{X'}} = \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)$$

of the [first] natural surjection of (A) *factors*, in a Π_X*-equivariant* fashion, through the endomorphism

$$\underline{\mathrm{New}}^{\mathrm{tor}}_{S''/S'}(b+1/b+2) \to \underline{\mathrm{New}}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$$

[hence also through the endomorphism $\mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2) \to \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)$] given by *multiplication by the order of* $c \cdot \underline{\mathrm{Gal}}(X''/X')$.

Also, before proceeding, we make the following elementary observation concerning the group cohomology of induced modules:

(C) Suppose that $H'' = l^{n''} \cdot \Delta^{\mathrm{ab}}_{X'} \subseteq \Delta^{\mathrm{ab}}_{X'}$, where n'' is a positive integer. For M a finitely generated $\mathbb{Z}_l$-module [which we regard as equipped with the *trivial* $\Delta_{X'}$-action], write:

$$\mathcal{H}_{X'} \stackrel{\mathrm{def}}{=} H^1(\Delta_{X'}, M \otimes M^{(l)}_X)$$

$$\mathcal{H}_{X''} \stackrel{\mathrm{def}}{=} H^1(\Delta_{X''}, M \otimes M^{(l)}_X) \xrightarrow{\sim} H^1(\Delta_{X'}, M[\underline{\mathrm{Gal}}(X''/X')] \otimes M^{(l)}_X)$$

Then the *"trace map"*

$$\mathrm{Tr}_{\mathcal{H}} : \mathcal{H}_{X''} \to \mathcal{H}_{X'}$$

— i.e., the map induced by the morphism of coefficients $M[\underline{\mathrm{Gal}}(X''/X')] \twoheadrightarrow M$ that maps each element of $\underline{\mathrm{Gal}}(X''/X')$ to 1 — *factors* through the endomorphism of $\mathcal{H}_{X'}$ given by *multiplication by* $l^{n''}$ [cf. Remark 39 below].

[Indeed, to verify (C), we recall that this trace map $\mathrm{Tr}_{\mathcal{H}}$ is well-known to be *dual* [via Poincaré duality — cf., e.g., [FK], pp. 135-136] to the *pull-back morphism*; we thus conclude that, relative to the natural isomorphisms $\mathcal{H}_{X''} \xrightarrow{\sim} \Delta^{\mathrm{ab}}_{X''} \otimes M$, $\mathcal{H}_{X'} \xrightarrow{\sim} \Delta^{\mathrm{ab}}_{X'} \otimes M$ [arising from Poincaré duality — cf., e.g., Proposition 1.3, (ii)], the trace map corresponds to the natural morphism

$$\mathcal{H}_{X''} = \Delta^{\mathrm{ab}}_{X''} \otimes M \to \Delta^{\mathrm{ab}}_{X'} \otimes M = \mathcal{H}_{X'}$$

induced by the inclusion $\Delta_{X''} \subseteq \Delta_{X'}$ — hence, by the definition of H'', *factors* through the endomorphism of $\mathcal{H}_{X'}$ given by *multiplication by* $l^{n''}$. This completes the proof of (C).]

Next, let us suppose that we have been given morphisms of $(S, \emptyset, \Sigma)$-*admissible coverings* of X

$$X''' \to X^{'''*} \to X'' \to X^{''*} \to X'$$

and write $U'''_{S'''} \subseteq X'''$, $U^{'''*}_{S'''*} \subseteq X^{'''*}$, $U^{''*}_{S''*} \subseteq X^{''*}$ for the open subscheme determined, respectively, by the complements of the sets S''', $S^{'''*}$, $S^{''*}$ of closed points of X''', $X^{'''*}$, $X^{''*}$ that lie over points of S. Also, let us assume that the open subgroups $\Delta_{X'''} \subseteq \Delta_{X'}$, $\Delta_{X'''*} \subseteq \Delta_{X'}$, $\Delta_{X''} \subseteq \Delta_{X'}$, $\Delta_{X''*} \subseteq \Delta_{X'}$ *arise, respectively, from open subgroups*

$$H''' = l^{n'''} \cdot \Delta^{\mathrm{ab}}_{X''*} \ \subseteq \ H^{'''*} = l^{n^{'''*}} \cdot \Delta^{\mathrm{ab}}_{X''*} \ \subseteq \ \Delta^{\mathrm{ab}}_{X''*}$$

$$H'' = l^{n''} \cdot \Delta^{\mathrm{ab}}_{X'} \ \subseteq \ H^{''*} = l^{n^{''*}} \cdot \Delta^{\mathrm{ab}}_{X'} \ \subseteq \ \Delta^{\mathrm{ab}}_{X'}$$

— where $n^{'''*} \stackrel{\mathrm{def}}{=} n''' - c$, $n^{''*} \stackrel{\mathrm{def}}{=} n'' - c$; we suppose that $n''' > 2c$, $n'' > c$ are *"sufficiently large"* positive integers, to be chosen below. Then we wish to *apply the theory developed above* [in particular, the observations (A), (B), (C)] by taking "$X'' \to X'$" in this theory to be various subcoverings of $X''' \to X'$.

Now let us compute the cohomology of Π_X via the *Leray spectral sequence* associated to the surjection $\Pi_X \twoheadrightarrow \Pi_X/\Delta_{X''*}$. Suppose that c has been chosen so that $b + 1 = 2c$. Then by applying (A) to the covering "$X'' \to X^{''*}$" (respectively, "$X''' \to X^{''*}$"), we conclude that $\Delta_{X''*}$ (respectively, $\Delta_{X'''*}$) acts *trivially* on $\mathrm{New}^{\mathrm{tor}}_{S''/S''*}(b+1/b+2)$ (respectively, $\mathrm{New}^{\mathrm{tor}}_{S'''/S''*}(b+1/b+2)$). Also, it follows immediately from the definitions that we have a *natural* Π_X-*equivariant surjection* $\mathrm{New}^{\mathrm{tor}}_{S'''/S''*}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S''/S''*}(b+1/b+2)$. Now, by applying (A) to the covering "$X''' \to X^{''*}$" and (C) to the covering "$X^{'''*} \to X^{''*}$", we conclude that the Π_X-*equivariant* natural morphism

$$H^1(\Delta_{X''*}, \mathrm{New}^{\mathrm{tor}}_{S'''/S''*}(b+1/b+2)) \to H^1(\Delta_{X''*}, \mathrm{New}^{\mathrm{tor}}_{S''/S''*}(b+1/b+2))$$

[which maps the image of $\eta_{S'''}$ to the image of $\eta_{S''}$!] *factors* through a *"trace map"* as in (C) for the covering "$X^{'''*} \to X^{''*}$", hence in particular, through the endomorphism of $H^1(\Delta_{X''*}, \mathrm{New}^{\mathrm{tor}}_{S''/S''*}(b+1/b+2))$ [a module whose submodule of Π_X-invariants is *finite*, by Proposition 3.2, (i)] given by *multiplication by* $n^{'''*}$, in a Π_X-*equivariant* fashion. Thus, by taking n''' to be *"sufficiently large"*, we conclude that the image of $\eta_{S''}$ in $H^1(\Delta_{X''*}, \mathrm{New}^{\mathrm{tor}}_{S''/S''*}(b+1/b+2))$ is *zero*.

Now I *claim* that the image of $\eta_{S''}$ in

$$H^1(\Delta_{X'}, \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))$$

[obtained by applying the surjection

$$\mathrm{New}^{\mathrm{tor}}_{S''}(b+1/b+2)) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$$

of (A) applied to the covering "$X'' \to X'$"] is *zero*. Indeed, note that it follows immediately from the definitions that we have a *natural surjection* $\mathrm{New}^{\mathrm{tor}}_{S''/S''^*}(b+1/b+2) \twoheadrightarrow \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$ [induced, in effect, by the inclusion $\underline{\mathrm{Gal}}(X''/X''^*) \hookrightarrow \underline{\mathrm{Gal}}(X''/X')$]. Thus, since we have already shown that the image of $\eta_{S''}$ in the cohomology module $H^1(\Delta_{X''^*}, \mathrm{New}^{\mathrm{tor}}_{S''/S''^*}(b+1/b+2))$ is *zero*, it follows immediately that the image of $\eta_{S''}$ in $H^1(\Delta_{X''^*}, \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))$ is *zero*, hence that the image in question in the *claim* arises from a class

$$\begin{aligned}\in H^1(\underline{\mathrm{Gal}}(X''^*/X'), (\mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))^{\Delta_{X''^*}})\\ = H^1(\underline{\mathrm{Gal}}(X''^*/X'), \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)) = 0\end{aligned}$$

[where the last cohomology module vanishes since, by (A) applied to the covering "$X'' \to X'$", $\mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2)$ is an *induced* $\underline{\mathrm{Gal}}(X''^*/X')$-module]. This completes the proof of the *claim*.

Thus, in summary, we conclude that the image of $\eta_{S''}$ in the cohomology module $H^1(\Pi_X, \mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))$ arises from a unique class in

$$H^1(\Pi_X/\Delta_{X'}, (\mathrm{New}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))^{\Delta_{X'}}) \xrightarrow{\sim} H^1(\Pi_X/\Delta_{X'}, \underline{\mathrm{New}}^{\mathrm{tor}}_{S''/S'}(b+1/b+2))$$

which maps to the unique class in

$$H^1(\Pi_X/\Delta_{X'}, \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))$$

[a module which is *finite*, by Proposition 3.2, (i)] that gives rise to $\eta_{S'}$ via a morphism that *factors* through the endomorphism given by *multiplication by the order of* $c \cdot \underline{\mathrm{Gal}}(X''/X')$ [cf. (A), (B) applied to the covering "$X'' \to X'$"]. In particular, by taking n'' to be *"sufficiently large"*, we may conclude that $\eta_{S'} = 0$, as desired. That is to say:

> *This completes the proof that the two inclusions* $\Pi^{\leq b}_{U_S} \hookrightarrow \Pi^{\mathrm{TOR}\leq b+1}_{U_S}$ *differ by conjugation by an element of* $\mathrm{Ker}(\Pi^{\mathrm{TOR}\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\mathrm{LIE}\leq b+}_{U_S})$.

In particular, the proof of assertions (i), (ii) is complete.

Next, we consider assertion (iii). First, let us observe that when $b = 1$, assertion (iii) follows immediately from [the "pro-l version" of the argument applied to prove] Proposition 2.3, (i) [cf. the discussion preceding Proposition 3.4]. Thus, in the remainder of the proof of assertion (iii), we assume that $b \geq 2$. Note that since the elements of $\mathrm{Ker}(\Pi^{\leq b+1}_{U_S} \twoheadrightarrow \Pi^{\leq b}_{U_S})$ manifestly commute with the elements of $I^{\leq b+1}_{\overline{x}_*}$, it follows from conditions (a), (b), the fact that $b \geq 2$, and the assumption that $S = S_*$ is of *cardinality one* that β induces the *identity* on $\Pi^{\leq b+1}_{U_S}$[csp] [cf. the proof of assertion (ii) above]. Thus, to complete the proof of assertion (iii), it suffices to show that the *compatible system* of classes

$$\lambda_{S'} \in H^1(\Pi_X, \mathrm{New}_{S'}(b+1/b+2))$$

determined by β [cf. Proposition 3.4, (i); 3.1, (iv)] *vanishes*. Note that since $(\Delta^{(l)}_X)^{\mathrm{ab}}$ is of "weight ≤ 1", and $\mathrm{New}_{S'}(b+1/b+2)$ is of "weight $b+1 \geq 3$" [cf.

Proposition 3.2, (i)], it follows immediately from the Leray spectral sequence for $\Pi_X \twoheadrightarrow G_k$ that we have a *natural isomorphism*

$$H^1(G_k, (\mathrm{New}_{S'}(b+1/b+2))^{\Delta_X}) \xrightarrow{\sim} H^1(\Pi_X, \mathrm{New}_{S'}(b+1/b+2))$$

[where the superscript "Δ_X" denotes the Δ_X-invariants] and that the module $H^1(G_k, (\mathrm{New}_{S'}(b+1/b+2))^{\Delta_X})$ is *finite*. Thus, to show that the $\lambda_{S'} = 0$, it suffices to show that the inverse limit

$$\varprojlim_{X'} (\mathrm{New}_{S'}(b+1/b+2))^{\Delta_X}$$

[where X', S' are as described in the proof of assertions (i), (ii)] is *zero*. But this follows from *observation (B)* of the proof of assertions (i), (ii). This completes the proof of assertion (iii).

Next, we consider assertion (iv). In light of the definition of $\Pi_{U_S}^{\mathrm{TOR}\leq b+1}$, it suffices to show that any compatible system of D_X*-invariant* [relative to the *diagonal action* of D_X] classes

$$\kappa_{S'} \in \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)$$

[where X', S' are as described in the proof of assertions (i), (ii)] lies in the image of $I_{x_*}[U_S] \otimes (\mathbb{Q}_l/\mathbb{Z}_l)$ if $b = 1$ and vanishes if $b \geq 2$. [Here, we note that since $\mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2) = 0$ when b is *even*, we may assume without loss of generality that b is *odd*.] To do this, let X', X'', S', S'' be as in (A), (B). Now we would like to apply the *theory of the Appendix* [cf., especially, Theorem A.1] to the present situation. To do this, it is necessary to specify the data "(i), (ii), (iii), (vi), (vii), (viii), (ix), (x), (xi), (xii)" [cf. the discussion of the Appendix] to which this theory is to be applied.

We take the "d" of Theorem A.1 to be such that $2d = b+1$ [so the fact that b is *odd* implies that $d \geq 2$ whenever $b \geq 2$] and the prime number "l" of "(i)" to be the prime number l of the present discussion. We take the profinite group "Δ" of "(ii)" to be the quotient of the group Δ_X by the kernel of the quotient $(\Delta_X \supseteq) \Delta_{X'} \twoheadrightarrow \Delta^{\mathrm{ab}}_{X'} \twoheadrightarrow \Delta^{\mathrm{ab}}_{X'} \otimes \mathbb{Z}_l$; this group "$\Delta$" surjects onto $\Delta_X/\Delta_{X'}$, which we take to be the quotient group "G" of "(ii)", with kernel $\Delta^{\mathrm{ab}}_{X'} \otimes \mathbb{Z}_l$, which we take to be the subgroup "V" of "(ii)". Here, we recall that the condition of "(ii), (c)" concerning the *regular representation* follows immediately from [Milne], p. 187, Corollary 2.8 [cf. also [Milne], p. 187, Remark 2.9], in light of our assumption that X is proper hyperbolic, hence of *genus* ≥ 2. We take the profinite group "Γ" of "(ix)" to be the image $G_{k'} \subseteq G_k$ of $\Pi_{X'}$ in G_k [so "Γ" acts naturally on "Δ", "G", "H"]. Thus, "Δ_Γ" may be thought of as a *quotient* of $\Pi_X \times_{G_k} G_{k'}$, hence also as a *quotient* of $D_X \times_{G_k} G_{k'}$. Note that by consideration of *"weights"*, it follows that

$$(\mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2))^{G_{k'}}$$

is *finite*, hence *annihilated* by some finite power of l, which we take to be the number "N" of "(iii)". We take the covering $X'' \to X'$ of (A), (B) to be any $(S, \emptyset, \Sigma)$-admissible covering such that the resulting covering $X''_{\underline{k}} \to X'_{\underline{k}}$ is the covering determined by the resulting subgroup "$l^n \cdot V \subseteq \Delta$" [cf. the statement of Theorem A.1], so "J" may be identified with $\underline{\mathrm{Gal}}(X''/X')$. Next, we take the "$G$-torsor E_G" of "(vi)" to be S' and the "H-torsor E_H" of "(vii)" to be S''; thus, the natural surjection $S'' \twoheadrightarrow S'$ determines a surjection "$E_H \twoheadrightarrow E_G$" as in "(viii)". Note that S'' (respectively, S') may be thought of as a $\Delta_{U''_{S''}}$*-orbit* (respectively, $\Delta_{U'_{S'}}$*-orbit*) [via the action by conjugation] of the conjugacy class of subgroups of Δ_{U_S} determined by $I_{x_*}[U_S] \subseteq \Delta_{U_S}$. In particular, it follows that the *particular* member of this conjugacy class constituted by the subgroup $I_{x_*}[U_S] \subseteq \Delta_{U_S}$ determines a particular element $e_H \in E_H$ (respectively, $e_G \in E_G$) as in "(xi)". Moreover, the *diagonal action* of D_X — hence also of $D_X \times_{G_k} G_{k'} \subseteq D_X$ — on Δ_{U_S} determines an action of $D_X \times_{G_k} G_{k'} \subseteq D_X$ on E_H, E_G that *fixes* e_H, e_G, and [as is easily verified] *factors through the quotient* "Δ_Γ" of $D_X \times_{G_k} G_{k'} \twoheadrightarrow \Pi_X \times_{G_k} G_{k'}$; in particular, we obtain continuous actions of "Δ_Γ" on "E_G", "E_H" as in "(x)". Finally, we take the "Γ-module Λ" of "(xii)" to be the d-th tensor power of $M_X^{(l)} \otimes (\mathbb{Q}_l/\mathbb{Z}_l)$. This completes the *specification of the data* necessary to apply Theorem A.1.

Thus, by applying Theorem A.1 to the composite of the *second and third surjections* in the factorization of (A), we conclude that since $\kappa_{S''}$ is D_X-*invariant*, it follows that

$$\kappa_{S''} \in \mathrm{New}^{\mathrm{tor}}_{S''}(b + 1/b + 2)$$

maps to an element [i.e., $\kappa_{S'}$] of $N \cdot \mathrm{New}^{\mathrm{tor}}_{S'}(b+1/b+2)^{G_{k'}} = 0$ when $b \geq 2$ and to an element [i.e., $\kappa_{S'}$] in the image of $I_{x_*}[U_S] \otimes (\mathbb{Q}_l/\mathbb{Z}_l)$ when $b = 1$. This completes the proof of assertion (iv).

Finally, we consider assertion (v). It is immediate from the definitions that the various quotients in question are *cuspidally pro-l*. That these quotients are the *maximal cuspidally pro-l quotients* follows from the construction of $\Delta^{\leq \infty}_{U_S}$ and the easily verified fact that each $\Delta^{(l)}_{U'_{S'}}$ *injects* into $\mathrm{Lin}(\Delta^{(l)}_{U'_{S'}}(1/\infty))(\mathbb{Q}_l)$. Finally, the asserted *slimness* follows from the fact that the profinite groups in question may be written as inverse limits of profinite groups that admit normal open subgroups [with trivial centralizers] — namely, "$\Delta^{(l)}_{U'_{S'}}$", "$(\Pi^{(l)}_{U'_{S'}})^\dagger$", "$\Delta^{(l)}_{U_{X' \times X'}}$", "$(\Pi^{(l)}_{U_{X' \times X'}})^\dagger$" — which are *slim*, by Proposition 1.6, (i), (iii) [which implies that the quotients $\Delta^{(l)}_{U_{X' \times X'}} \twoheadrightarrow \Delta^{(l)}_{X'}$, $(\Pi^{(l)}_{U_{X' \times X'}})^\dagger \twoheadrightarrow (\Pi^{(l)}_{X'})^\dagger$, as well as the kernels of these quotients, are *slim*]. □

Remark 38. Proposition 3.6, (iii), may be regarded as a *"higher order, pro-l analogue"* of Proposition 2.3, (i).

Remark 39. It is important to note that if one *omits* [as was, *mistakenly*, done in an earlier version of this paper] the hypothesis that $S_0 = \emptyset$, then it *no longer holds* that the image of the trace map "$\mathrm{Tr}_{\mathcal{H}} : \mathcal{H}_{X''} \to \mathcal{H}_{X'}$" [appearing in the proof of Proposition 3.6, (i), (ii)] lies in $l^n \cdot \mathcal{H}_{X'}$. Indeed, this phenomenon may be understood by considering the trace map on first étale cohomology modules with $\mathbb{Z}_l$-coefficients associated to the *l^n-th power map* $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ on the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over k — a map which, as an easy computation reveals, is *surjective*.

We are now ready to prove the *main technical result* of the present §3:

Theorem 3.1. **(Reconstruction of Maximal Cuspidally Pro-*l* Extensions)** *Let X, Y be* **proper hyperbolic curves** *over a* **finite field***; denote the base fields of X, Y by k_X, k_Y, respectively. Suppose further that we have been given* **points** *$x_* \in X(k_X)$, $y_* \in Y(k_Y)$; write $S \stackrel{\mathrm{def}}{=} \{x_*\}$, $T \stackrel{\mathrm{def}}{=} \{y_*\}$ $U_S \stackrel{\mathrm{def}}{=} X \backslash S$, $V_T \stackrel{\mathrm{def}}{=} Y \backslash T$. Let Σ be a* **set of prime numbers** *that contains at least one prime number that is invertible in k_X, k_Y; thus, Σ determines various* **quotients** *Π_{U_S}, Π_X, $\Pi_{U_{X\times X}}$, $\Pi_{X\times X}$, Π_{V_T}, Π_Y, $\Pi_{U_{Y\times Y}}$, $\Pi_{Y\times Y}$ [cf. Proposition 1.6, (iii); the discussion preceding Proposition 1.5] of the* **étale fundamental groups** *of U_S, X, $U_{X\times X}$, $X \times X$, V_T, Y, $U_{Y\times Y}$, $Y \times Y$, respectively. Also, we write $\Pi_X \twoheadrightarrow G_{k_X}$, $\Pi_Y \twoheadrightarrow G_{k_Y}$ for the quotients determined by the respective* **absolute Galois groups** *of k_X, k_Y. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be a **Frobenius-preserving** *[hence also* **quasi-point-theoretic** *— cf. Remark 10]* **isomorphism** *of profinite groups that maps the decomposition group of x_* in Π_X [which is well-defined up to conjugation] to the decomposition group of y_* in Π_Y [which is well-defined up to conjugation]. Then for each prime $l \in \Sigma$ such that $l \neq p$, there exist* **commutative diagrams**

$$\begin{array}{ccc} \Pi^{\leq\infty}_{\overline{U}_S} & \xrightarrow{\alpha_\infty} & \Pi^{\leq\infty}_{\overline{V}_T} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow{\alpha} & \Pi_Y \end{array} \qquad \begin{array}{ccc} \Pi^{\leq\infty}_{\overline{U}_{X\times X}} & \xrightarrow{\alpha^\times_\infty} & \Pi^{\leq\infty}_{\overline{U}_{Y\times Y}} \\ \downarrow & & \downarrow \\ \Pi_{X\times X} & \xrightarrow{\alpha\times\alpha} & \Pi_{Y\times Y} \end{array}$$

— in which $\Pi_{U_S} \twoheadrightarrow \Pi^{\leq\infty}_{\overline{U}_S}$, $\Pi_{U_{X\times X}} \twoheadrightarrow \Pi^{\leq\infty}_{\overline{U}_{X\times X}}$, $\Pi_{V_T} \twoheadrightarrow \Pi^{\leq\infty}_{\overline{V}_T}$, $\Pi_{U_{Y\times Y}} \twoheadrightarrow \Pi^{\leq\infty}_{\overline{U}_{Y\times Y}}$ are the **maximal cuspidally pro-*l* quotients** *[cf. Proposition 3.6, (v)]; $\Pi_{X\times X} \cong \Pi_X \times_{G_{k_X}} \Pi_X$, $\Pi_{Y\times Y} \cong \Pi_Y \times_{G_{k_Y}} \Pi_Y$; the vertical arrows are the natural surjections; α_∞, $\alpha^\times_\infty$ are* **isomorphisms***, well-defined up to composition with a cuspidally inner automorphism, that are* **compatible***, relative to the natural surjections*

$$\Pi^{\leq\infty}_{\overline{U}_S} \twoheadrightarrow \Pi^{\text{c-ab},l}_{U_S}; \quad \Pi^{\leq\infty}_{\overline{U}_{X\times X}} \twoheadrightarrow \Pi^{\text{c-ab},l}_{U_{X\times X}}; \quad \Pi^{\leq\infty}_{\overline{V}_T} \twoheadrightarrow \Pi^{\text{c-ab},l}_{V_T}; \quad \Pi^{\leq\infty}_{\overline{U}_{Y\times Y}} \twoheadrightarrow \Pi^{\text{c-ab},l}_{U_{Y\times Y}}$$

*— where we use the superscript "*c-ab, l*" to denote the respective* **maximal cuspidally pro-l abelian quotients** *— with the isomorphisms*

$$\Pi^{\text{c-ab}}_{U_S} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{V_T}; \quad \Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{U_{Y\times Y}}$$

of Theorem 2.1*, (i); Theorem* 1.1*, (iii), respectively. Moreover, α_∞ (respectively, $\alpha^\times_\infty$) is* **compatible***, up cuspidally inner automorphisms, with the decomposition groups of x_*, y_* in $\Pi^{\leq\infty}_{U_S}$, $\Pi^{\leq\infty}_{V_T}$ (respectively, with the images of the decomposition groups D_X, D_Y in $\Pi^{\leq\infty}_{U_{X\times X}}$, $\Pi^{\leq\infty}_{U_{Y\times Y}}$). Finally, this condition of "compatibility with decomposition groups", together with the condition of making the above diagrams commute,* **uniquely determine** *the isomorphisms α_∞, $\alpha^\times_\infty$, up to composition with a cuspidally inner automorphism; in particular, $\alpha^\times_\infty$ is* **compatible***, up to composition with a cuspidally inner automorphism, with the automorphisms of $\Pi^{\leq\infty}_{U_{X\times X}}$, $\Pi^{\leq\infty}_{U_{Y\times Y}}$ given by* **switching the two factors**.

Proof. First, let us consider the isomorphism [i.e., more precisely: a *specific* member of the cuspidally inner equivalence class of isomorphisms]

$$\alpha^{\text{c-ab},l} : \Pi^{\text{c-ab},l}_{U_{X\times X}} \xrightarrow{\sim} \Pi^{\text{c-ab},l}_{U_{Y\times Y}}$$

arising from the isomorphism $\Pi^{\text{c-ab}}_{U_{X\times X}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{U_{Y\times Y}}$ of Theorem 1.1, (iii). Recall that since α is *Frobenius-preserving*, it is *quasi-point-theoretic* [cf. Remark 10], and that $\alpha^{\text{c-ab},l}$ is *compatible* with the images of D_X, D_Y, which we denote by $D^{(l)}_X$, $D^{(l)}_Y$. Thus, we may assume without loss of generality that our *choices of decomposition groups $D_{x_*}[U_S] \subseteq \Pi_{U_S}$, $D_{y_*}[V_T] \subseteq \Pi_{V_T}$*, as well as our *choices of splittings $G_{k_X} \hookrightarrow D_{x_*}[U_S]$, $G_{k_Y} \hookrightarrow D_{y_*}[V_T]$*, have images in $\Pi^{\text{c-ab},l}_{U_{X\times X}}$, $\Pi^{\text{c-ab},l}_{U_{Y\times Y}}$ that *correspond* via $\alpha^{\text{c-ab},l}$. In particular, it follows that $\alpha^{\text{c-ab},l}$ maps $\Pi^{\text{c-ab},l}_{U_S} \subseteq \Pi^{\text{c-ab},l}_{U_{X\times X}}$ isomorphically onto $\Pi^{\text{c-ab},l}_{V_T} \subseteq \Pi^{\text{c-ab},l}_{U_{Y\times Y}}$.

In the following argument, let us *identify* the "$\text{Lin}_{U_S}(1/\infty)$", "$\text{Lin}_X(1/\infty)$" portions of $\Delta^{\text{Lie}}_{U_S}$ with the [completions, relative to the natural filtration topology, of the] corresponding graded objects "$\text{Gr}_{\mathbb{Q}_l}(-)(1/\infty)$" via the *Galois invariant splittings* of Proposition 3.2, (ii), and similarly for V_T. Then, in light of our assumption that α is *Frobenius-preserving*, it follows immediately from the *naturality* of our constructions [cf., especially, Proposition 3.2, (iii)] that α induces, for each $\mathbb{Z} \ni b \geq 1$, *compatible isomorphisms*

$$\alpha^{\text{LIE}} : \Pi^{\text{LIE}}_{U_S} \xrightarrow{\sim} \Pi^{\text{LIE}}_{V_T}; \quad \alpha^{\text{LIE}\leq b} : \Pi^{\text{LIE}\leq b}_{U_S} \xrightarrow{\sim} \Pi^{\text{LIE}\leq b}_{V_T}$$

which are, moreover, compatible [with respect to the natural projections to Π_X, Π_Y] with the isomorphism α. Moreover, it follows from the *construction* of "$\Pi^{\text{LIE}\leq b}_{(-)}$" that the latter displayed isomorphism *maps $D^{\text{LIE}\leq b}_{x_*} \subseteq \Pi^{\text{LIE}\leq b}_{U_S}$ bijectively onto $D^{\text{LIE}\leq b}_{y_*} \subseteq \Pi^{\text{LIE}\leq b}_{V_T}$*, and that the resulting isomorphism $D^{\text{LIE}\leq b}_{x_*} \xrightarrow{\sim} D^{\text{LIE}\leq b}_{y_*}$ induces an *isomorphism*

$$D^{\leq b}_{x_*} \xrightarrow{\sim} D^{\leq b}_{y_*}$$

which is compatible [again by construction!] with the respective *Frobenius elements* "F_k" on either side.

Next, let us observe that since the isomorphism $\alpha^{\text{c-ab},l}$ induces an isomorphism $\Pi_{U_S}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab},l}$ that is compatible with the images of the *decompositions groups* $D_{x_*}[U_S]$, $D_{y_*}[V_T]$ and *Frobenius elements* in these decomposition groups, it follows immediately that for corresponding [i.e., via α] $(S, \emptyset, \Sigma)$-, $(T, \emptyset, \Sigma)$-admissible coverings $X' \to X$, $Y' \to Y$ [which induce coverings $U'_{S'} \to U_S$, $V_{T'} \to V_T$], $\alpha^{\text{c-ab},l}$ induces an isomorphism $\Delta_{U'_{S'}}^{\text{Lie}\leq 2} \xrightarrow{\sim} \Delta_{V_{T'}}^{\text{Lie}\leq 2}$ which is *compatible* with $\alpha^{\text{LIE}\leq 2}$. Moreover, although $\Delta_{U'_{S'}}^{\text{Lie}\leq 2}$, $\Delta_{V_{T'}}^{\text{Lie}\leq 2}$ are not center-free, the *semi-direct products* $\Delta_{U'_{S'}}^{\text{Lie}\leq 2} \rtimes H_X$, $\Delta_{V_{T'}}^{\text{Lie}\leq 2} \rtimes H_Y$ are easily seen to be *center-free* [cf. Proposition 1.6, (i)], for arbitrary open subgroups $H_X \subseteq G_{k_X}^{\dagger}$, $H_Y \subseteq G_{k_Y}^{\dagger}$ [where the daggers are as in Proposition 1.6, (i)] that correspond via α. Since $\Pi_{U_S}^{\text{LIE}\leq 2}$ (respectively, $\Pi_{V_T}^{\text{LIE}\leq 2}$) is an inverse limit of topological groups that admit normal closed subgroups of the form $\Delta_{U'_{S'}}^{\text{Lie}\leq 2} \rtimes H_X$ (respectively, $\Delta_{V_{T'}}^{\text{Lie}\leq 2} \rtimes H_Y$), we thus conclude [by applying the extension "$1 \to (-) \to \text{Aut}(-) \to \text{Out}(-) \to 1$" of §0 to these normal closed subgroups] that the isomorphism $\Pi_{U_S}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab},l}$ induced by $\alpha^{\text{c-ab},l}$ is *compatible* — relative to the natural inclusions

$$\Pi_{U_S}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{U_S}^{\leq 2} \hookrightarrow \Pi_{U_S}^{\text{LIE}\leq 2}; \quad \Pi_{V_T}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{V_T}^{\leq 2} \hookrightarrow \Pi_{V_T}^{\text{LIE}\leq 2}$$

[cf. the discussion preceding Proposition 3.4] — with $\alpha^{\text{LIE}\leq 2} : \Pi_{U_S}^{\text{LIE}\leq 2} \xrightarrow{\sim} \Pi_{V_T}^{\text{LIE}\leq 2}$.

In fact, since 3 is *odd*, it follows immediately from the definitions that the modules "$\text{New}_{S'}^{\mathbb{Q}}(3/4)$" *vanish*, hence [cf. Definition 3.3, (ii)] that we have an isomorphism $\Pi_{U_S}^{\text{LIE}\leq 3} \xrightarrow{\sim} \Pi_{U_S}^{\text{LIE}\leq 2+}$, which implies [cf. Proposition 3.4, (i)] that we have an isomorphism $\Pi_{U_S}^{\leq 3} \xrightarrow{\sim} \Pi_{U_S}^{\leq 2}$ [and similarly for V_T]. Thus, by Proposition 3.4, (iii), it follows that the isomorphism $\Pi_{U_S}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{V_T}^{\text{c-ab},l}$ induced by $\alpha^{\text{c-ab},l}$ is *compatible* — relative to the natural inclusions

$$\Pi_{U_S}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{U_S}^{\leq 3} \hookrightarrow \Pi_{U_S}^{\text{LIE}\leq 3}; \quad \Pi_{V_T}^{\text{c-ab},l} \xrightarrow{\sim} \Pi_{V_T}^{\leq 3} \hookrightarrow \Pi_{V_T}^{\text{LIE}\leq 3}$$

— with $\alpha^{\text{LIE}\leq 3} : \Pi_{U_S}^{\text{LIE}\leq 3} \xrightarrow{\sim} \Pi_{V_T}^{\text{LIE}\leq 3}$.

Next, let us observe that the *diagonal actions* of D_X, D_Y on $\Pi_{U_S}^{\text{LIE}}$, $\Pi_{V_T}^{\text{LIE}}$ clearly *factor* through $D_X^{(l)}$, $D_Y^{(l)}$ [hence determine *"diagonal actions"* of $D_X^{(l)}$, $D_Y^{(l)}$ on $\Pi_{U_S}^{\text{LIE}}$, $\Pi_{V_T}^{\text{LIE}}$]. Moreover, by what we have already shown concerning the *compatibility* of $\alpha^{\text{LIE}\leq 3}$ with $\alpha^{\text{c-ab},l}$ [cf. also the *compatibility* of $\alpha^{\text{c-ab},l}$ with $D_X^{(l)}$, $D_Y^{(l)}$] and the *compatibility* of $\alpha^{\text{c-ab},l}$ with the decomposition groups $D_{x_*}[U_S]$, $D_{y_*}[V_T]$, it follows [cf. Remarks 36, 37] that the conditions (a), (b), (c) of Proposition 3.5 are *compatible* with α^{LIE}, hence that α^{LIE} is *compatible with the diagonal actions of* $D_X^{(l)}$, $D_Y^{(l)}$ *on* $\Pi_{U_S}^{\text{LIE}}$, $\Pi_{V_T}^{\text{LIE}}$ [relative to the isomorphism $D_X^{(l)} \xrightarrow{\sim} D_Y^{(l)}$ induced by $\alpha^{\text{c-ab},l}$].

Now I *claim* that the isomorphism $\alpha^{\text{LIE}\leq b}$ maps $\Pi_{U_S}^{\leq b}$ bijectively onto $\Pi_{V_T}^{\leq b}$, thus inducing a *compatible inverse system* [parametrized by b] of *isomorphisms*

$$\alpha^{\leq b} : \Pi_{U_S}^{\leq b} \xrightarrow{\sim} \Pi_{V_T}^{\leq b}$$

that are compatible [with respect to the natural projections $\Pi_{U_S}^{\leq b} \twoheadrightarrow \Pi_X$, $\Pi_{V_T}^{\leq b} \twoheadrightarrow \Pi_Y$] with α. To verify this *claim*, we apply *induction on* b. The case $b = 1$ is vacuous; the case $b = 2$ follows from what we have already shown concerning the *compatibility* of $\alpha^{\mathrm{LIE}\leq 2}$ with $\alpha^{\text{c-ab},l}$. Thus, we assume that $b \geq 2$, and that the claim *has been verified* for "b" that are $\leq$ the b under consideration.

Now observe that by Propositions 3.4, (iii); 3.6, (ii), it follows that the isomorphism

$$\Pi_{U_S}^{\mathrm{LIE}\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\mathrm{LIE}\leq b+1}$$

maps $\Pi_{U_S}^{\leq b+1}$ bijectively onto a $\mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b+})$-conjugate of $\Pi_{V_T}^{\leq b+1}$. In particular, by *conjugating* by an appropriate element $\gamma \in \mathrm{Ker}(\Pi_{V_T}^{\mathrm{LIE}\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\mathrm{LIE}\leq b+})$, we obtain an *isomorphism*

$$\beta_{b+1} : \Pi_{U_S}^{\leq b+1} \xrightarrow{\sim} \Pi_{V_T}^{\leq b+1}$$

that is *compatible* with $\alpha^{\leq b}$ and, moreover, [since γ *commutes* with $I_{y_*}^{\leq b+1}$] *maps* $I_{x_*}^{\leq b+1}$ *bijectively onto* $I_{y_*}^{\leq b+1}$. Note that by Propositions 3.4, (i); 3.6, (iii), it follows that the choice of γ is *unique*, modulo $\mathrm{Ker}(\Pi_{V_T}^{\leq b+1} \twoheadrightarrow \Pi_{V_T}^{\leq b+})$. In particular, the image $\delta \in \Pi_{V_T}^{\mathrm{TOR}\leq b+1}$ of γ in $\Pi_{V_T}^{\mathrm{TOR}\leq b+1}$ is *uniquely determined*.

On the other hand, since α^{LIE} is *compatible with the diagonal actions of* $D_X^{(l)}$, $D_Y^{(l)}$ *on* $\Pi_{U_S}^{\mathrm{LIE}}$, $\Pi_{V_T}^{\mathrm{LIE}}$, it follows immediately, by *"transport of structure"*, that δ is *fixed* by the *diagonal action of* $D_Y^{(l)}$. But, by Proposition 3.6, (iv), this implies that $\delta = 0$. This completes the proof of the *claim*.

Thus, we obtain an isomorphism $\alpha_\infty : \Pi_{U_S}^{\leq\infty} \xrightarrow{\sim} \Pi_{V_T}^{\leq\infty}$ as in the statement of Theorem 3.1. Next, let us recall that $\Delta_{U_S}^{\leq\infty}$, $\Delta_{V_T}^{\leq\infty}$ are *slim* [cf. Proposition 3.6, (v)]. Thus, since this isomorphism α_∞ is *compatible with the diagonal actions of* $D_X^{(l)}$, $D_Y^{(l)}$, we may apply the isomorphism $\mathrm{Aut}(\Delta_{U_S}^{\leq\infty}) \xrightarrow{\sim} \mathrm{Aut}(\Delta_{V_T}^{\leq\infty})$ induced by α_∞ to obtain — i.e., by pulling back the extension

$$1 \to \Delta_{U_S}^{\leq\infty} \to \mathrm{Aut}(\Delta_{U_S}^{\leq\infty}) \to \mathrm{Out}(\Delta_{U_S}^{\leq\infty}) \to 1$$

[cf. §0] via the homomorphism

$$(D_X^{(l)} \twoheadrightarrow)\ \Pi_X \to \mathrm{Out}(\Delta_{U_S}^{\leq\infty})$$

arising from the diagonal action [and similarly for $\Delta_{V_T}^{\leq\infty}$] — an isomorphism $\alpha_\infty^\times : \Pi_{U_{X\times X}}^{\leq\infty} \xrightarrow{\sim} \Pi_{U_{Y\times Y}}^{\leq\infty}$ as in the statement of Theorem 3.1. Here, we note that the *"cuspidally inner indeterminacy"* of α_∞, $\alpha_\infty^\times$ that is referred to in the statement of Theorem 3.1 arises from the "cuspidally inner indeterminacy" in the *choice of corresponding decomposition groups* $D_{x_*}[U_S]$, $D_{y_*}[V_T]$ [more precisely: the images of these groups in $\Pi_{U_S}^{\leq\infty}$, $\Pi_{V_T}^{\leq\infty}$, as opposed to just in $\Pi_{U_S}^{\text{c-ab},l}$,

$\Pi_{V_T}^{\text{c-ab},l}$]. Finally, we observe that the asserted *uniqueness* follows immediately by considering *eigenspaces* relative to the Frobenius actions [cf. Proposition 3.2, (ii)], together with the construction of the isomorphism α^{LIE} [cf. also Propositions 1.10, (i); 2.3, (i)]. $\square$

Remark 40. The argument of the proof of Theorem 3.1 involving Proposition 3.6, (iv), may be regarded as a sort of *"higher order analogue"* of the argument applied in the proof of Theorem 1.1, (iii), involving Lemma 1.1; Proposition 1.8, (v).

Remark 41. At first glance, it may appear that the portion of Theorem 3.1 concerning $\alpha_\infty^\times$ may only be concluded when $X(k_X)$, $Y(k_Y)$ are *nonempty*. In fact, however, since $(\Pi_{U_X \times X}^{\leq\infty})^\dagger$, $(\Pi_{U_Y \times Y}^{\leq\infty})^\dagger$ are *slim* [cf. Proposition 3.6, (v)], it follows that the portion of Theorem 3.1 concerning $\alpha_\infty^\times$ may be concluded even without assuming that $X(k_X)$, $Y(k_Y)$ are *nonempty*, by applying Theorem 3.1 after passing to corresponding [via α] finite extensions of k_X, k_Y [cf. Remark 5].

Remark 42. It seems reasonable to expect that, when, say, $\Sigma = \{l\}$, the techniques applied in the proof of Theorem 3.1, together with the theory of [Mtm], should allow one to reconstruct the [geometrically pro-Σ] étale fundamental groups of the various *configuration spaces* [i.e., finite products of copies of X over k_X, with the various diagonals removed] "group-theoretically" from Π_X [under, say, an appropriate hypothesis of *"Frobenius-preservation"* as in Theorem 3.1]. This topic, however, lies beyond the scope of the present paper.

Remark 43. If the "cuspidalization of configuration spaces" [cf. Remark 42] can be achieved, then it seems likely that by applying an appropriate "specialization" operation, it should be possible to generalize Theorem 3.1 to the case where S, T are subsets of *arbitrary finite cardinality*.

Remark 44. One essential portion of the proof of Theorem 3.1 is the *Galois invariant splitting* of Proposition 3.2, (ii). Although it does not appear likely that such a splitting exists in the case of a *nonarchimedean local* base field [cf., e.g., the theory of [Mzk4]], it would be interesting to investigate the extent to which a result such as Theorem 3.1 may be generalized to the nonarchimedean local case, perhaps by making use of some sort of splitting such as the *Hodge-Tate decomposition*, or a splitting that arises via *crystalline* methods.

In the context of *absolute anabelian geometry over nonarchimedean local fields*, however, such p-adic Hodge-theoretic splittings might not be available, since the isomorphism class of the Galois module "$\mathbb{C}_p$" is not preserved by arbitrary automorphisms of the absolute Galois group of a nonarchimedean local field [cf. the theory of [Mzk3]].

The development of the theory underlying Theorem 3.1 was motivated by the following *important consequence*:

Corollary 3.1. **(Total Global Green-compatibility)** *In the situation of Theorem* 1.1*, (iii) [in the finite field case], suppose further that* $\Sigma^\dagger = \mathfrak{Primes}^\dagger$*, and that* X*,* Y *are* **Σ-separated** *[which implies that* α *is* **Frobenius-preserving** *and* **point-theoretic** *— cf. Remarks* 9*,* 10*]. Then the isomorphism* α *is* **totally globally Green-compatible**.

Proof. Indeed, we may apply Theorem 3.1 to the isomorphism α of Theorem 1.1, (iii), and arbitrary choices of sets of cardinality one $S = \{x_*\}$, $T = \{y_*\}$ that correspond via α. Let $l \in \Sigma^\dagger$. Then let us *observe* that the quotient $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$ satisfies the following property:

> If $\Pi_{U_S} \twoheadrightarrow Q$ is a finite quotient of Π_{U_S} such that for some quotient $Q \twoheadrightarrow Q'$ whose kernel has order a power of l, $\Pi_{U_S} \twoheadrightarrow Q'$ factors through $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$, then $\Pi_{U_S} \twoheadrightarrow Q$ also factors through $\Pi_{U_S} \twoheadrightarrow \Pi_{U_S}^{\leq\infty}$.

A similar statement holds for the quotient $\Pi_{V_T} \twoheadrightarrow \Pi_{V_T}^{\leq\infty}$. In light of this *observation*, together with our assumption that $\Sigma^\dagger = \mathfrak{Primes}^\dagger$ [which implies that α is *Frobenius-preserving*], it follows that the reasoning of [Tama], Corollary 2.10, Proposition 3.8 [cf. also Remark 10 of the present paper], may be applied to the isomorphism

$$\alpha_\infty : \Pi_{U_S}^{\leq\infty} \xrightarrow{\sim} \Pi_{V_T}^{\leq\infty}$$

of Theorem 3.1 to conclude that the isomorphism α_∞ maps the set of decomposition subgroups of the domain *bijectively onto* the set of decomposition subgroups of the codomain.

On the other hand, sorting through the definitions, the datum of the lifting of a decomposition group of Π_X, Π_Y corresponding to a point that does not belong to S, T to a [noncuspidal] decomposition group of the domain or codomain of α_∞ *determines*, by projection to $\Pi_{U_S}^{\text{c-ab},l}$, $\Pi_{V_T}^{\text{c-ab},l}$, the *$l$-adic portion of the Green's trivialization* associated to this point and the unique point of S or T. Since l is an *arbitrary* element of $\Sigma^\dagger = \mathfrak{Primes}^\dagger$, and the points x_*, y_* are *arbitrary* points that correspond via α, this shows that α is *globally Green-compatible.* That α is *totally globally Green-compatible* follows by applying this argument to the isomorphism induced by α between open subgroups of Π_X, Π_Y. □

Theorem 3.2. **(The Grothendieck Conjecture for Proper Hyperbolic Curves over Finite Fields)** *Let* X, Y *be* **proper hyperbolic curves** *over a* **finite field**; *denote the base fields of* X, Y *by* k_X, k_Y, *respectively. Write* Π_X, Π_Y *for the* **étale fundamental groups of** X, Y, *respectively. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an **isomorphism** *of profinite groups. Then* α *arises from a* **uniquely** *determined commutative diagram of* **schemes**

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\sim} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are **isomorphisms**; *the vertical arrows are the pro-finite étale universal coverings determined by the profinite groups* Π_X, Π_Y.

Proof. Theorem 3.2 follows formally from Corollaries 2.1, 3.1; Remarks 9, 10; Proposition 2.2, (ii). □

Appendix. Free Lie Algebras

In this Appendix, we discuss various elementary facts concerning *free Lie algebras* that are necessary in §3. In particular, we develop a sort of *"higher order analogue"* of the theory developed in Lemma 1.1.

Proposition A.1. **(Free Lie Algebras)** *Let* R *be a* **commutative ring with unity**; V *a* **finitely generated free** R*-module. Write* $\mathfrak{Lie}_R(V)$ *for the* **free Lie algebra** *over* R *associated to* V; *for* $\mathbb{Z} \ni b \geq 1$, *denote by* $\mathfrak{Lie}^b_R(V) \subseteq \mathfrak{Lie}_R(V)$ *the* R*-submodule generated by the "alternants of degree* b*" [cf. [Bour], Chapter II, §2.6]. Also, we shall denote by* $\mathcal{U}_R(V)$ *the* **enveloping algebra** *of* $\mathfrak{Lie}_R(V)$. *[Thus, we have a natural inclusion* $\mathfrak{Lie}_R(V) \hookrightarrow \mathcal{U}_R(V)$.*] Then:*

(i) Each $\mathfrak{Lie}^b_R(V)$ *is a* **finitely generated free** R*-module. Moreover, there is*

a natural isomorphism $V \xrightarrow{\sim} \mathfrak{Lie}^1_R(V)$.

(ii) Let $v \in V$ *be a nonzero element such that the quotient module* $V/R \cdot v$ *is* **free**. *Then the* **centralizer** *of* v *in* $\mathcal{U}_R(V)$ *is equal to the* R*-submodule of* $\mathcal{U}_R(V)$ *generated by the nonnegative powers of* v. *In particular, if* R *is a field of characteristic zero, then the* **centralizer** *of* v *in* $\mathfrak{Lie}_R(V)$ *is equal to* $R \cdot v$.

(iii) Suppose that the rank of V *over* R *is* ≥ 2. *Then the Lie algebra* $\mathfrak{Lie}_R(V)$ *is* **center-free**. *In particular, the* **adjoint representation** *of* $\mathfrak{Lie}_R(V)$ *is* **faithful**.

(iv) Let R' *be an* ***R*-algebra** *which is* **finitely generated** *and* **free** *as an* R*-module. Let* $\phi : R' \twoheadrightarrow R$ *be a surjection of* R*-algebras; suppose that* $V = V' \otimes_{R',\phi} R$*, for some finitely generated free* R'*-module* V' *[so we obtain a natural surjection* $V' \twoheadrightarrow V$ *compatible with* ϕ*]. Then the natural surjection* $V' \twoheadrightarrow V$ *induces a surjection of* R*-modules* $\mathfrak{Lie}^b_R(V') \twoheadrightarrow \mathfrak{Lie}^b_R(V)$ *that factors as a composite of natural surjections as follows:*

$$\mathfrak{Lie}^b_R(V') \twoheadrightarrow \mathfrak{Lie}^b_{R'}(V') \twoheadrightarrow \mathfrak{Lie}^b_R(V)$$

Here, the first arrow of this factorization is the arrow naturally induced by observing that every Lie algebra over R' *naturally determines a Lie algebra over* R*; the second arrow of this factorization is the arrow functorially induced by the natural* ϕ*-compatible surjection* $V' \twoheadrightarrow V$. *Finally, this second arrow induces an isomorphism* $\mathfrak{Lie}^b_{R'}(V') \otimes_{R',\phi} R \xrightarrow{\sim} \mathfrak{Lie}^b_R(V)$.

Proof. Assertion (i) follows immediately from [Bour], Chapter II, §2.11, Theorem 1, Corollary. Assertion (ii) follows from the well-known structure of the *enveloping algebra* $\mathcal{U}_R(V)$ [i.e., the natural isomorphism of $\mathcal{U}_R(V)$ with the *free associative algebra* determined by V over R; the fact that when R is a field of characteristic zero, the image of $\mathfrak{Lie}_R(V)$ in $\mathcal{U}_R(V)$ may be identified with the set of *primitive elements* — cf. [Bour], Chapter II, §3, Theorem 1, Corollaries 1,2], by considering the *effect on "words" of forming the commutator with* v — cf. the argument of [Mtm], Proposition 3.1 [which is given only in the case where R is a field of characteristic zero, but does not, in fact, make use of this assumption on R in an essential way]. Assertion (iii) follows immediately from assertion (ii) [by allowing the element "v" of assertion (ii) to range over the elements of an R-basis of V]. Assertion (iv) follows formally from the *universal property of a free Lie algebra*, together with the well-known *functoriality* of a free Lie algebra with respect to *tensor products* [cf. [Bour], Chapter II, §2.5, Proposition 3]. □

Next, let us suppose that we have been given data as follows:

(i) a *prime number* l;

(ii) a *profinite group* Δ that admits an *normal open subgroup* $V \subseteq \Delta$ such that the following conditions are satisfied: (a) V is *abelian* [so we shall regard V as a *module*]; (b) the topological module V is a *finitely generated free* R-*module*, where we write $R \stackrel{\text{def}}{=} \mathbb{Z}_l$; (c) the resulting action of the finite group $G \stackrel{\text{def}}{=} \Delta/V$ on V determines a G-module $V_{\mathbb{Q}_l} \stackrel{\text{def}}{=} V \otimes \mathbb{Q}_l$ that contains the *regular representation* of G;

(iii) a positive power N of l;

(iv) a collection of [not necessarily distinct!] elements $g_1, \ldots, g_d \in G$ [where $d \geq 1$ is an integer] of G at least one of which is *not equal to the identity element*.

Write

$$\zeta \stackrel{\text{def}}{=} \sum_{i=1}^{d} (1 - g_i) \in R[G]$$

[where $R[G]$ is the group ring of G with coefficients in R]. Then we have the following result:

Lemma A.1. **(Nontriviality of a Certain Operator)** *There exists an integer* $n \geq 1$ *such that the* **order** $|J_\zeta|$ *of the image*

$$J_\zeta \subseteq J$$

of the action of ζ *on [the finite group]* $J \stackrel{\text{def}}{=} V \otimes (\mathbb{Z}/l^n\mathbb{Z})$ *is* **divisible** *by* N.

Proof. Indeed, since the G-module $V_{\mathbb{Q}_l}$ contains the *regular representation* [cf. condition (ii), (c)], it follows that the image of the action of ζ on $V_{\mathbb{Q}_l}$ is a *nonzero* $\mathbb{Q}_l$-vector space, hence that the image of the action of ζ on the finitely generated free R-module V [cf. condition (ii), (b)] contains a rank one free R-module. Now Lemma A.1 follows immediately. $\square$

Next, let $J_\zeta \subseteq J$ be as in Lemma A.1; write $H \stackrel{\text{def}}{=} \Delta/(l^n \cdot V)$ [so $J \subseteq H$, $H/J = G$]. Also, let us assume that we have been given data as follows:

(v) a *collection of elements* $h_1, \ldots, h_d \in H$ that lift $g_1, \ldots, g_d \in G$;

(vi) a G-*torsor* E_G [whose G-action will be written as an *action from the left*];

(vii) an *H-torsor* E_H [whose H-action will be written as an *action from the left*];

(viii) a *surjection*

$$\epsilon : E_H \twoheadrightarrow E_G$$

that is *compatible* with the natural surjection $H \twoheadrightarrow G$;

(ix) a continuous action of a *profinite group* Γ on Δ that preserves the subgroup $V \subseteq \Delta$, hence determines a profinite group $\Delta_\Gamma \stackrel{\text{def}}{=} \Delta \rtimes \Gamma$ that acts continuously on G, H [in such a way that the restriction of this action to $\Delta \subseteq \Delta_\Gamma$ is the action of Δ on G, H by *conjugation*];

(x) *continuous actions* of Δ_Γ on E_G, E_H [which will be denoted via *superscripts*] that are *compatible* with the continuous actions of Δ_Γ on G, H, as well as with the surjection ϵ and, moreover, induce the *trivial* action of $\Gamma \subseteq \Delta_\Gamma$ on E_G [hence also on G];

(xi) an *element* [i.e., "basepoint"] $e_H \in E_H$, whose image via ϵ we denote by $e_G \in E_G$, such that e_H, e_G are *fixed* by the action of Δ_Γ on E_H, E_G.

Next, let us write

$$R_J \stackrel{\text{def}}{=} R[J]$$

for the group ring of J with coefficients in R. Thus, R_J is a *commutative R-algebra*, and we have a natural *augmentation homomorphism* $R_J \twoheadrightarrow R$ [which sends all of the elements of J to 1]. Moreover, if we write

$$\epsilon_M : M_H \stackrel{\text{def}}{=} R[E_H] \twoheadrightarrow M_G \stackrel{\text{def}}{=} R[E_G]$$

for the morphism of R_J-modules induced by ϵ on the respective free R-modules with bases given by the elements of E_H, E_G, then ϵ_M induces a *natural isomorphism* $M_H \otimes_{R_J} R \xrightarrow{\sim} M_G$. Thus, it follows from Proposition A.1, (iv), that, for $b \geq 1$ an integer, we have [in the notation of Proposition A.1] *natural surjections*

$$\mathfrak{Lie}^b_R(M_H) \twoheadrightarrow \mathfrak{Lie}^b_{R_J}(M_H) \twoheadrightarrow \mathfrak{Lie}^b_R(M_G)$$

the second of which determines a *natural isomorphism* $\mathfrak{Lie}^b_{R_J}(M_H) \otimes_{R_J} R \xrightarrow{\sim} \mathfrak{Lie}^b_R(M_G)$.

Now let

$$P(X_1, \ldots, X_d)$$

be an *"alternant monomial of degree d"* [i.e., a monomial element of $\mathfrak{Lie}^d_{\mathbb{Z}}(-)$ of the free $\mathbb{Z}$-module on the indeterminate symbols $X_1, \ldots, X_d$] in which each

X_i [for $i = 1, \ldots, d$] *appears precisely once*. Then $P(X_1, \ldots, X_d)$ determines an element

$$P(g_1 \cdot e_G, \ldots, g_i \cdot e_G, \ldots, g_d \cdot e_G)$$

of $\mathfrak{Lie}^d_R(M_G)$. Moreover, by allowing such $P(X_1, \ldots, X_d)$ and $g_1, \ldots, g_d$ to *vary* appropriately, we obtain a *Hall basis* [cf., e.g., [Bour], Chapter II, §2.11] of $\mathfrak{Lie}^d_R(M_G)$ [at least if $d \geq 2$; if $d = 1$, then one must also allow for the unique g_1 to be the identity element]. Similarly, by allowing such $P(X_1, \ldots, X_d)$ and $h_1, \ldots, h_d \in H$ to *vary* appropriately, we obtain a *Hall basis* [again, strictly speaking, if $d \geq 2$] of $\mathfrak{Lie}^d_{R_J}(M_H)$ of elements of the form $P(h_1 \cdot e_H, \ldots, h_d \cdot e_H)$.

Lemma A.2. (Relation of Superscript and Left Actions) *For any $v \in V \subseteq \Delta \subseteq \Delta_\Gamma$ that maps to $j \in J$, we have*

$$P(h_1 \cdot e_H, \ldots, h_i \cdot e_H, \ldots, h_d \cdot e_H)^v = \zeta(j) \cdot P(h_1 \cdot e_H, \ldots, h_i \cdot e_H, \ldots, h_d \cdot e_H)$$

in $\mathfrak{Lie}^d_{R_J}(M_H)$.

Proof. Indeed, we compute:

$$\begin{aligned} P(h_1 \cdot e_H, \ldots, h_i \cdot e_H, \ldots, h_d \cdot e_H)^v &= P(h_1^v \cdot e_H, \ldots, h_i^v \cdot e_H, \ldots, h_d^v \cdot e_H) \\ &= P(h_1^v \cdot h_1^{-1} \cdot h_1 \cdot e_H, \ldots, h_i^v \cdot h_i^{-1} \cdot h_i \cdot e_H, \ldots, h_d^v \cdot h_d^{-1} \cdot h_d \cdot e_H) \\ &= \Big(\prod_{i=1}^{d} [j, h_i]\Big) \cdot P(h_1 \cdot e_H, \ldots, h_i \cdot e_H, \ldots, h_d \cdot e_H) \\ &= \zeta(j) \cdot P(h_1 \cdot e_H, \ldots, h_i \cdot e_H, \ldots, h_d \cdot e_H) \end{aligned}$$

[where we apply the *R_J-module structure* of E_H and the fact that $e_H^v = e_H$ [cf. (xi)]]. $\square$

Next, let us assume that we have also been given the following data:

(xii) a *topological R-module* Λ equipped with a continuous action by Γ [which thus determines, via the natural surjection $\Delta_\Gamma \twoheadrightarrow \Gamma$, a continuous action by Δ_Γ on Λ].

Write:

$$V_\Gamma \stackrel{\text{def}}{=} V \rtimes \Gamma \subseteq \Delta_\Gamma;$$

$$F_J \stackrel{\text{def}}{=} J \cdot P(h_1 \cdot e_H, \ldots, h_d \cdot e_H) \subseteq \mathfrak{Lie}^d_{R_J}(M_H);$$

$$R[F_J] \stackrel{\text{def}}{=} R \cdot F_J = R_J \cdot P(h_1 \cdot e_H, \ldots, h_d \cdot e_H) \subseteq \mathfrak{Lie}^d_{R_J}(M_H);$$

$$\Lambda[F_J] \stackrel{\text{def}}{=} R[F_J] \otimes_R \Lambda \subseteq \mathfrak{Lie}^d_{R_J}(M_H) \otimes_R; \Lambda$$

$$F \stackrel{\text{def}}{=} P(g_1 \cdot e_G, \ldots, g_d \cdot e_G) \in \mathfrak{Lie}^d_R(M_G);$$

$$R[F] \stackrel{\text{def}}{=} R \cdot F \subseteq \mathfrak{Lie}^d_R(M_G); \quad \Lambda[F] \stackrel{\text{def}}{=} R[F] \otimes_R \Lambda \subseteq \mathfrak{Lie}^d_R(M_G) \otimes_R \Lambda$$

Thus, the natural surjection $\mathfrak{Lie}^d_{R_J}(M_H) \twoheadrightarrow \mathfrak{Lie}^d_R(M_G)$ determines [compatible] *natural surjections* $F_J \twoheadrightarrow \{F\}$, $R[F_J] \twoheadrightarrow R[F]$, $\Lambda[F_J] \twoheadrightarrow \Lambda[F]$. Also, we observe [cf. the fact that $\mathfrak{Lie}^d_{R_J}(M_H)$ is a *finitely generated free R_J-module*] that F_J is a *J-torsor* [relative to the action from the left], hence, in particular, a *finite set*.

Now observe that since V_Γ acts *trivially* on G, e_H [cf. (ix), (x), (xi)], it follows immediately that V_Γ *acts compatibly* on F_J, $R[F_J]$, $\Lambda[F_J]$, F, $R[F]$, $\Lambda[F]$, and that the natural action of V_Γ on $R[G]$ *preserves* ζ. In particular, it follows that V_Γ *preserves* $J_\zeta \subseteq J$, hence that V_Γ acts naturally on the *set of orbits*

$$(F_J \twoheadrightarrow)\ F_\zeta$$

of F_J with respect to the action of J_ζ; moreover, by Lemma A.2, it follows that this action of V_Γ on F_ζ *factors* through the quotient $V_\Gamma \twoheadrightarrow \Gamma$.

Now let us consider *invariants* with respect to the various *superscript* actions under consideration. Let us write

$$\text{Invar}(-,-)$$

for the set of *invariants* of the second argument in parentheses with respect to the *superscript* action of the group given by the first argument in parentheses. Then any element

$$\eta \in \text{Invar}(V_\Gamma, \Lambda[F_J])$$

may be regarded as a *Λ-valued function* on the set F_J which *descends* [cf. Lemma A.2] to a *Γ-invariant Λ-valued function on F_ζ*, i.e., an element $\eta_\zeta \in \text{Invar}(\Gamma, \Lambda[F_\zeta])$. Next, let us observe that [since η_ζ is Γ-invariant] the *sum* of the values $\in \Lambda$ of the Λ-valued function on F_ζ determined by η_ζ is a *Γ-invariant element* $\int \eta_\zeta \in \text{Invar}(\Gamma, \Lambda)$. Thus, the *sum*

$$\int \eta \in \Lambda$$

of the values $\in \Lambda$ of the Λ-valued function on F_J determined by η satisfies the relation

$$\int \eta = |J_\zeta| \cdot \int \eta_\zeta$$

in Λ. But the image of η in $\Lambda[F]$ is precisely the element $(\int \eta) \cdot F$. Thus, since, by Lemma A.1, $|J_\zeta|$ is *divisible* by N, we conclude the following:

Lemma A.3. **(Monomial-wise Computation of Invariants)** *The image*

$$\mathrm{Im}(\mathrm{Invar}(V_\Gamma, \Lambda[F_J])) \subseteq \Lambda[F]$$

of $\mathrm{Invar}(V_\Gamma, \Lambda[F_J]) \subseteq \Lambda[F_J]$ *in* $\Lambda[F]$ *lies in* $N \cdot \mathrm{Invar}(\Gamma, \Lambda[F])$.

Thus, by allowing $P(X_1, \ldots, X_d)$ and $h_1, \ldots, h_d \in H$ as in the above discussion to *vary* appropriately so as to obtain a *Hall basis* [again, strictly speaking, if $d \geq 2$] of $\mathfrak{Lie}^d_{R_J}(M_H)$ of elements of the form $P(h_1 \cdot e_H, \ldots, h_d \cdot e_H)$, we conclude the following:

Theorem A.1. **(Invariants of Free Lie Algebras)** *Let $d \geq 1$ be an integer. Suppose that we have been given data as in (i), (ii), (iii) above. Let $n \geq 1$ be an integer that satisfies the property of Lemma* A.1 *for all [of the finitely many] possible choices of data as in (iv) [relative to the given integer $d \geq 1$]; $J \stackrel{\mathrm{def}}{=} V/(l^n \cdot V) \subseteq H \stackrel{\mathrm{def}}{=} \Delta/(l^n \cdot V)$; $R_J \stackrel{\mathrm{def}}{=} R[J]$. Suppose that have also been given data as in (vi), (vii), (viii), (ix), (x), (xi), (xii) above; let $M_H \stackrel{\mathrm{def}}{=} R[E_H]$, $M_G \stackrel{\mathrm{def}}{=} R[E_G]$, $V_\Gamma \stackrel{\mathrm{def}}{=} V \rtimes \Gamma$ ($\subseteq \Delta_\Gamma$). Then the* **natural surjection**

$$\mathfrak{Lie}^d_{R_J}(M_H) \otimes_R \Lambda \twoheadrightarrow \mathfrak{Lie}^d_R(M_G) \otimes_R \Lambda$$

maps

$$\mathrm{Invar}(V_\Gamma, \mathfrak{Lie}^d_{R_J}(M_H) \otimes_R \Lambda)$$

into

$$N \cdot \mathrm{Invar}(V_\Gamma, \mathfrak{Lie}^d_R(M_G) \otimes_R \Lambda)$$

if $d \geq 2$. In a similar vein, the **natural surjection** $M_H \otimes_R \Lambda \twoheadrightarrow M_G \otimes_R \Lambda$ *maps* $\mathrm{Invar}(V_\Gamma, M_H \otimes_R \Lambda)$ *into* $N \cdot \mathrm{Invar}(V_\Gamma, M_G \otimes_R \Lambda) + \mathrm{Invar}(V_\Gamma, \Lambda) \cdot e_G \subseteq M_G \otimes_R \Lambda$.

References

[Bour] N. Bourbaki, *Lie Groups and Lie Algebras*, Springer Verlag (1989).

[DM] P. Deligne and D. Mumford, The Irreducibility of the Moduli Space of Curves of Given Genus, *IHES Publ. Math.* **36** (1969), pp. 75-109.

[FC] G. Faltings and C.-L. Chai, *Degenerations of Abelian Varieties*, Springer Verlag (1990).

[FK] E. Freitag and R. Kiehl, *Étale Cohomology and the Weil Conjecture*, Springer Verlag (1988).

[Kane] M. Kaneko, Certain automorphism groups of pro-l fundamental groups of punctured Riemann surfaces, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989), pp. 363-372.

[Knud] F. F. Knudsen, The Projectivity of the Moduli Space of Stable Curves, II, *Math. Scand.* **52** (1983), pp. 161-199.

[Mtm] M. Matsumoto, Galois representations on profinite braid groups on curves, *J. Reine Angew. Math.* **474** (1996), pp. 169-219.

[Milne] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series **33**, Princeton University Press (1980).

[Mzk1] S. Mochizuki, Correspondences on Hyperbolic Curves, *Journ. Pure Appl. Algebra* **131** (1998), pp. 227-244.

[Mzk2] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[Mzk3] S. Mochizuki, A Version of the Grothendieck Conjecture for p-adic Local Fields, *The International Journal of Math.* **8** (1997), pp. 499-506.

[Mzk4] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[Mzk5] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2003), pp. 77-122.

[Mzk6] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[Mzk7] S. Mochizuki, Topics Surrounding the Anabelian Geometry of Hyperbolic Curves, *Galois Groups and Fundamental Groups*, *Mathematical Sciences Research Institute Publications* **41**, Cambridge University Press (2003), pp. 119-165.

[Mzk8] S. Mochizuki, Galois Sections in Absolute Anabelian Geometry, *Nagoya Math. J.* **179** (2005), pp. 17-45.

[Mzk9] S. Mochizuki, *A Combinatorial Version of the Grothendieck Conjecture*, RIMS Preprint **1482** (December 2004); to appear in *Tohoku Math. J.*

[MB] L. Moret-Bailly, Métriques permises in *Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell*, edited by L. Szpiro, *Astérisque* **127**, Société Mathé-matique de France.

[Mumf] D. Mumford, *Abelian Varieties*, Oxford Univ. Press (1974).

[Naka] H. Nakamura, Galois Rigidity of Algebraic Mappings into some Hyperbolic Varieties, *Intern. J. Math.* **4** (1993), pp. 421-438.

[NTs] H. Nakamura and H. Tsunogai, Some finiteness theorems on Galois centralizers in pro-l mapping class groups, *J. reine angew. Math.* **441** (1993), pp. 115-144.

[Namba] M. Namba, *Branched coverings and algebraic functions*, Pitman Research Notes in Mathematics Series **161**, John Wiley & Sons (1987).

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften **323**, Springer Verlag (2000).

[Scott] W. R. Scott, *Group Theory*, Prentice-Hall (1964).

[Serre] J.-P. Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics **1500**, Springer Verlag (1992).

[Stix] J. Stix, A monodromy criterion for extending curves, *Int. Math. Res. Not.* **2005**, pp. 1787-1802.

[Tama] A. Tamagawa, The Grothendieck Conjecture for Affine Curves, *Compositio Math.* **109** (1997), pp. 135-194.

[Uchi] K. Uchida, Isomorphisms of Galois groups of algebraic function fields, *Ann. of Math.* **106** (1977), pp. 589-598.

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