COMMENTS ON "ABSOLUTE ANABELIAN CUSPIDALIZATIONS OF PROPER HYPERBOLIC CURVES"

Shinichi Mochizuki

February 2008

The argument given in Remark 1.18.2 [i.e., Remark 10 in the LaTeX Version], (iii), is somewhat sketchy and a bit misleading. This argument should read as follows:

(iii) Nevertheless, as was pointed out to the author by *A. Tamagawa*, even if X, Y are *not necessarily* Σ*-separated*, it is still possible to conclude, essentially from the arguments of [Tama], Corollary 2.10, Proposition 3.8, that:

Any *Frobenius-preserving* isomorphism α is *quasi-point-theoretic*.

Indeed, it suffices to give a *"group-theoretic" characterization* of the quasi-sections $D \subseteq \Pi_X$ which are decomposition groups of points $\in X^{\mathrm{cl}}$. Write

$$\widetilde{X} \to X$$

for the *pro-finite étale covering* corresponding to Π_X. If $E \subseteq \Pi_X$ is a closed subgroup whose image in G_{k_X} is *open*, then let us write k_E for the finite extension field of k_X determined by this image. If $J \subseteq \Pi_X$ is an open subgroup, then let us write $X_J \to X$ for the covering determined by J and $J_\Delta \stackrel{\mathrm{def}}{=} J \bigcap \Delta_X$. If $J \subseteq \Pi_X$ is an open subgroup such that J_Δ is a characteristic subgroup of Δ_X, then we shall say that J is *geometrically characteristic*. Now let $J \subseteq \Pi_X$ be a *geometrically characteristic open subgroup*. Let us refer to as a *descent-group for* J any open subgroup $H \subseteq \Pi_X$ such that $J \subseteq H$, $J_\Delta = H_\Delta$. Thus, a descent-group H for J may be thought of as an intermediate covering $X_J \to X_H \to X$ such that $X_H \times_{k_H} k_J \cong X_J$. Write

$$X_J(k_J)^{\text{fld-def}} \subseteq X_J(k_J)$$

for the subset of k_J-valued points of X_J that do *not* arise from points $\in X_H(k_H)$ for any descent-group $H \neq J$ for J — i.e., the k_J-valued points whose *field of definition* is k_J with respect to *all possible "descended forms" of* X_J. Thus, if $\widetilde{x}$ is a closed point of $\widetilde{X}$ that maps to $x \in X_J(k_J)$, and we write $D_{\widetilde{x}} \subseteq \Pi_X$ for the *stabilizer in* Π_X [i.e., "decomposition group"] of $\widetilde{x}$, then it is a *tautology* that x maps to a point $\in X_{H_x}(k_{H_x})$ for $H_x \stackrel{\mathrm{def}}{=} D_{\widetilde{x}} \cdot J_\Delta$ ($\supseteq J$) [so H_x forms a *descent-group for* J]; in particular, it follows immediately that:

Typeset by $\mathcal{AMS}$-TEX

$$x \in X_J(k_J)^{\text{fld-def}} \iff D_{\widetilde{x}} \subseteq J \iff H_x = J.$$

Now it follows immediately from this characterization of "fld-def" that if $J_1 \subseteq J_2 \subseteq \Pi_X$ are geometrically characteristic open subgroups such that $k_{J_1} = k_{J_2}$, then the natural map $X_{J_1}(k_{J_1}) \to X_{J_2}(k_{J_2})$ induces a map $X_{J_1}(k_{J_1})^{\text{fld-def}} \to X_{J_2}(k_{J_2})^{\text{fld-def}}$. Moreover, these considerations allow one to conclude [cf. the theory of [Tama]] that:

> A quasi-section $D \subseteq \Pi_X$ is a *decomposition group* of a point $\in X^{\text{cl}}$ if and and only if, for every geometrically characteristic open subgroup $J \subseteq \Pi_X$ such that $D \cdot J_\Delta = J$, it holds that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$.

Thus, to render this characterization of decomposition groups *"group-theoretic"*, it suffices to give a "group-theoretic" criterion for the condition that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$. In [Tama], the *Lefschetz trace formula* is applied to compute the cardinality of $X_J(k_J)$. On the other hand, if we use the notation "$|-|$" to denote the cardinality of a finite set, then one verifies immediately that

$$|X_J(k_J)| = \sum_H |X_H(k_H)^{\text{fld-def}}|$$

— where $H \supseteq J$ ranges over the *descent-groups* for J. In particular, by applying *induction* on $[\Pi_X : H]$, one concludes immediately from the above formula that $|X_J(k_J)^{\text{fld-def}}|$ may be computed from the $|X_H(k_H)|$, as H ranges over the *descent-groups* for J [while $|X_H(k_H)|$ may be computed, as in [Tama], from the *Lefschetz trace formula*]. This yields the desired "group-theoretic" characterization of the decomposition groups of Π_X.