

COMMENTS ON “ABSOLUTE ANABELIAN CUSPIDALIZATIONS OF PROPER HYPERBOLIC CURVES”

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(1.) The argument given in Remark 1.18.2 [i.e., Remark 10 in the LaTeX Version], (iii), is somewhat sketchy and a bit misleading. This argument should read as follows:

(iii) Nevertheless, as was pointed out to the author by A. Tamagawa, even if $X, Y$ are not necessarily $\Sigma$-separated, it is still possible to conclude, essentially from the arguments of [Tama], Corollary 2.10, Proposition 3.8, that:

Any Frobenius-preserving isomorphism $\alpha$ is quasi-point-theoretic.

Indeed, it suffices to give a “group-theoretic” characterization of the quasi-sections $D \subseteq \Pi X$ which are decomposition groups of points $\in X^\text{cl}$. Suppose, for simplicity, that $X$ is proper. Write

$$\tilde{X} \to X$$

for the pro-finite étale covering corresponding to $\Pi X$. If $E \subseteq \Pi X$ is a closed subgroup whose image in $G_{kX}$ is open, then let us write $k_E$ for the finite extension field of $k_X$ determined by this image. If $J \subseteq \Pi X$ is an open subgroup, then let us write $X_J \to X$ for the covering determined by $J$ and $J_\Delta \overset{\text{def}}{=} J \cap \Delta_X$. If $J \subseteq \Pi X$ is an open subgroup such that $J_\Delta$ is a characteristic subgroup of $\Delta_X$, then we shall say that $J$ is geometrically characteristic. Now let $J \subseteq \Pi X$ be a geometrically characteristic open subgroup. Let us refer to as a descent-group for $J$ any open subgroup $H \subseteq \Pi X$ such that $J \subseteq H$, $J_\Delta = H_\Delta$. Thus, a descent-group $H$ for $J$ may be thought of as an intermediate covering $X_J \to X_H \to X$ such that $X_H \times_{k_H} k_J \cong X_J$. Write

$$X_J(k_J)^\text{fld-def} \subseteq X_J(k_J)$$

for the subset of $k_J$-valued points of $X_J$ that do not arise from points $\in X_H(k_H)$ for any descent-group $H \neq J$ for $J$ — i.e., the $k_J$-valued points whose field of definition is $k_J$ with respect to all possible “descended forms” of $X_J$. Thus, if $\tilde{x}$ is a closed point of $\tilde{X}$ that maps to $x \in X_J(k_J)$, and we write $D_{\tilde{x}} \subseteq \Pi X$ for the stabilizer in $\Pi X$ [i.e., “decomposition group”] of $\tilde{x}$, then it is a tautology that $x$ maps to a point $\in X_{H_x}(k_{H_x})$ for $H_x \overset{\text{def}}{=} D_{\tilde{x}} \cdot J_\Delta \supseteq J$ [so $H_x$ forms a descent-group for $J$]; in particular, it follows immediately that:
Now it follows immediately from this characterization of “fld-def” that if \( J_1 \subseteq J_2 \subseteq \Pi_X \) are geometrically characteristic open subgroups such that \( k_{J_1} = k_{J_2} \), then the natural map \( X_{J_1}(k_{J_1}) \to X_{J_2}(k_{J_2}) \) induces a map \( X_{J_1}(k_{J_1})^{\text{fld-def}} \to X_{J_2}(k_{J_2})^{\text{fld-def}} \). Moreover, these considerations allow one to conclude [cf. the theory of [Tama]] that:

A quasi-section \( D \subseteq \Pi_X \) is a decomposition group of a point \( \in X^{\text{cl}} \) if and only if, for every geometrically characteristic open subgroup \( J \subseteq \Pi_X \) such that \( D \cdot J_\Delta = J \), it holds that \( X_{J}(k_{J})^{\text{fld-def}} \neq \emptyset \).

Thus, to render this characterization of decomposition groups “group-theoretic”, it suffices to give a “group-theoretic” criterion for the condition that \( X_{J}(k_{J})^{\text{fld-def}} \neq \emptyset \).

In [Tama], the Lefschetz trace formula is applied to compute the cardinality of \( X_{J}(k_{J}) \). On the other hand, if we use the notation “\(|-|\)” to denote the cardinality of a finite set, then one verifies immediately that

\[
|X_{J}(k_{J})| = \sum_{H} |X_{H}(k_{H})^{\text{fld-def}}|
\]

— where \( H \supseteq J \) ranges over the descent-groups for \( J \). In particular, by applying induction on \([\Pi_X : J]\), one concludes immediately from the above formula that \([X_{J}(k_{J})^{\text{fld-def}}]\) may be computed from the \([X_{H}(k_{H})]\), as \( H \) ranges over the descent-groups for \( J \) [while \(|X_{H}(k_{H})|\) may be computed, as in [Tama], from the Lefschetz trace formula]. This yields the desired “group-theoretic” characterization of the decomposition groups of \( \Pi_X \).

(2.) In the proof of Theorem 1.16, (iii) [i.e., Theorem 1.1, (iii), in the LaTeX version], the phrase “\( Z'_X \to X, Z'_Y \to Y \) are diagonal coverings” that appears at the beginning of this proof should read “\( Z'_X \to X \times X, Z'_Y \to Y \times Y \) are diagonal coverings”.

(3.) In the proof of Proposition 2.2, (i), it is asserted that one has a “natural isomorphism \( H^1(k, \Delta_X^{ab}) \cong J(k) \wedge (\supseteq J(k)) \)” . In fact, however, at least from an a priori point of view, one only has a natural injection \( (J(k) \subseteq) J(k) \wedge \hookrightarrow H^1(k, \Delta_X^{ab}) \). This does not have any effect on the proof, however, since the proof only requires the use of this natural injection.

(4.) In the proof of Proposition 1.6, (i), the phrase “fact that \( G_k^\dagger \) acts faithfully on \( M_X \) via the cyclotomic character” should read “fact that the image in Aut(\( M_X \)) of the action of \( G_k^\dagger \) on \( M_X \) via the cyclotomic character is infinite”.

\[ x \in X_{J}(k_{J})^{\text{fld-def}} \iff D_x \subseteq J \iff H_x = J. \]