The Mathematics of Mutually Alien Copies: from Gaussian Integrals to Inter-universal Teichmüller Theory

By

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Abstract

Inter-universal Teichmüller theory may be described as a construction of certain canonical deformations of the ring structure of a number field equipped with certain auxiliary data, which includes an elliptic curve over the number field and a prime number ≥ 5. In the present paper, we survey this theory by focusing on the rich analogies between this theory and the classical computation of the Gaussian integral. The main common features that underlie these analogies may be summarized as follows:

- the introduction of two mutually alien copies of the object of interest;
- the computation of the effect — i.e., on the two mutually alien copies of the object of interest — of two-dimensional changes of coordinates by considering the effect on infinitesimals;
- the passage from planar cartesian to polar coordinates and the resulting splitting, or decoupling, into radial — i.e., in more abstract valuation-theoretic terminology, “value group” — and angular — i.e., in more abstract valuation-theoretic terminology, “unit group” — portions;
- the straightforward evaluation of the radial portion by applying the quadraticity of the exponent of the Gaussian distribution;
- the straightforward evaluation of the angular portion by considering the metric geometry of the group of units determined by a suitable version of the natural logarithm function.

[Here, the intended sense of the descriptive “alien” is that of its original Latin root, i.e., a sense of abstract, tautological “otherness”.] After reviewing the classical computation of the Gaussian integral, we give a detailed survey of inter-universal Teichmüller theory by concentrating on the common features listed above. The paper concludes with a discussion of various historical aspects of the mathematics that appears in inter-universal Teichmüller theory.
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Introduction

In the present paper, we survey inter-universal Teichmüller theory by focusing on the rich analogies [cf. §3.8] between this theory and the classical computation of the Gaussian integral. Inter-universal Teichmüller theory concerns the construction of canonical deformations of the ring structure of a number field equipped with certain auxiliary data. The collection of data, i.e., consisting of the number field equipped with certain auxiliary data, to which inter-universal Teichmüller theory is applied is referred to as initial Θ-data [cf. §3.3, (i), for more details]. The principal components of a collection of initial Θ-data are

- the given number field,
- an elliptic curve over the number field, and
- a prime number \( l \geq 5 \).

The main applications of inter-universal Teichmüller theory to diophantine geometry [cf. §3.7, (iv), for more details] are obtained by applying the canonical deformation constructed for a specific collection of initial Θ-data to bound the height of the elliptic curve that appears in the initial Θ-data.

Let \( N \) be a fixed natural number \( > 1 \). Then the issue of bounding a given non-negative real number \( h \in \mathbb{R}_{\geq 0} \) may be understood as the issue of showing that \( N \cdot h \) is roughly equal to \( h \), i.e.,

\[
N \cdot h \sim h
\]

[cf. §2.3, §2.4]. When \( h \) is the height of an elliptic curve over a number field, this issue may be understood as the issue of showing that the height of the \([\text{in fact, in most cases, fictional]}\) “elliptic curve” whose \( q \)-parameters are the \( N \)-th powers “\( q^N \)” of the \( q \)-parameters “\( q \)” of the given elliptic curve is roughly equal to the height of the given elliptic curve, i.e., that, at least from the point of view of [global] heights,

\[
q^N \sim q
\]

[cf. §2.3, §2.4].

In order to verify the approximate relation \( q^N \sim q \), one begins by introducing two distinct — i.e., two “mutually alien” — copies of the conventional scheme
theory surrounding the given initial Θ-data. Here, the intended sense of the descriptive “alien” is that of its original Latin root, i.e., a sense of abstract, tautological “otherness”.

These two mutually alien copies of conventional scheme theory are glued together — by considering relatively weak underlying structures of the respective conventional scheme theories such as multiplicative monoids and profinite groups — in such a way that the “$q^N$” in one copy of scheme theory is identified with the “$q$” in the other copy of scheme theory. This gluing is referred to as the Θ-link. Thus, the “$q^N$” on the left-hand side of the Θ-link is glued to the “$q$” on the right-hand side of the Θ-link, i.e.,

\[ q_{\text{LHS}}^N \overset{\text{ident}}{=} q_{\text{RHS}} \]

[cf. §3.3, (vii), for more details]. Here, “$N$” is in fact taken not to be a fixed natural number, but rather a sort of symmetrized average over the values $j^2$, where $j = 1, \ldots, l^*$, and we write $l^* \overset{\text{def}}{=} (l - 1)/2$. Thus, the left-hand side of the above display

\[ \{q^j_{\text{LHS}}\}_j \]

bears a striking formal resemblance to the Gaussian distribution. One then verifies the desired approximate relation $q^N \approx q$ by computing

\[ \{q^j_{\text{LHS}}\}_j \]

— not in terms of $q_{\text{LHS}}$ [which is immediate from the definitions!], but rather — in terms of [the scheme theory surrounding]$q_{\text{RHS}}$

[which is a highly nontrivial matter!]. The conclusion of this computation may be summarized as follows:

up to relatively mild indeterminacies — i.e., “relatively small error terms” — $\{q^j_{\text{LHS}}\}_j$ may be “confused”, or “identified”, with $\{q^j_{\text{RHS}}\}_j$, that is to say,

\[ \{q^j_{\text{LHS}}\}_j \overset{\text{conf/ident}}{\overset{\text{def}}{=}} \{q^j_{\text{RHS}}\}_j \]

[cf. the discussion of §3.7, (i) especially, Fig. 3.19, as well as the discussion of §3.10, (ii), and §3.11, (iv), (v), for more details]. Once one is equipped with this “license” to confuse/identify $\{q^j_{\text{LHS}}\}_j$ with $\{q^j_{\text{RHS}}\}_j$, the derivation of the desired approximate relation

\[ \{q^j\}_j \overset{\text{approx}}{=\sim} q \]
and hence of the desired bounds on heights is an essentially formal matter [cf. §3.7, (ii), (iv); §3.11, (iv), (v)].

The starting point of the exposition of the present paper lies in the observation [cf. §3.8 for more details] that the main features of the theory underlying the computation just discussed of $\{q_j^2\}_{j}$ in terms of $q_{\text{RHS}}$ exhibit remarkable similarities — as is perhaps foreshadowed by the striking formal resemblance observed above to the Gaussian distribution — to the main features of the classical computation of the Gaussian integral, namely,

(1$^{\text{mf}}$) the introduction of two mutually alien copies of the object of interest [cf. §3.8, (1$^{\text{gau}}$), (2$^{\text{gau}}$)];

(2$^{\text{mf}}$) the computation of the effect — i.e., on the two mutually alien copies of the object of interest — of two-dimensional changes of coordinates by considering the effect on infinitesimals [cf. §3.8, (3$^{\text{gau}}$), (4$^{\text{gau}}$), (5$^{\text{gau}}$), (6$^{\text{gau}}$)];

(3$^{\text{mf}}$) the passage from planar cartesian to polar coordinates and the resulting splitting, or decoupling, into radial — i.e., in more abstract valuation-theoretic terminology, “value group” — and angular — i.e., in more abstract valuation-theoretic terminology, “unit group” — portions [cf. §3.8, (7$^{\text{gau}}$), (8$^{\text{gau}}$)];

(4$^{\text{mf}}$) the straightforward evaluation of the radial portion by applying the quadraticity of the exponent of the Gaussian distribution [cf. §3.8, (9$^{\text{gau}}$), (11$^{\text{gau}}$)];

(5$^{\text{mf}}$) the straightforward evaluation of the angular portion by considering the metric geometry of the group of units determined by a suitable version of the natural logarithm function [cf. §3.8, (10$^{\text{gau}}$), (11$^{\text{gau}}$)].

In passing, we mention that yet another brief overview of certain important aspects of inter-universal Teichmüller theory from a very elementary point of view may be found in §3.11.

The present paper begins, in §1, with a review of the classical computation of the Gaussian integral, by breaking down this familiar computation into steps in such a way as to facilitate the subsequent comparison with inter-universal Teichmüller theory. We then proceed, in §2, to discuss the portion of inter-universal Teichmüller theory that corresponds to (2$^{\text{mf}}$). The exposition of §2 was designed so as to be accessible to readers familiar with well-known portions of scheme theory and the theory of the étale fundamental group — i.e., at the level of [Harts] and [SGA1]. The various Examples that appear in this exposition of §2 include numerous well-defined and relatively straightforward mathematical assertions.
often without complete proofs. In particular, the reader may think of the task of supplying a complete proof for any of these assertions as a sort of “exercise” and hence of §2 itself as a sort of workbook with exercises.

At the level of papers, §2 is concerned mainly with the content of the “classical” paper [Uchi] of Uchida and the “preparatory papers” [FrdI], [FrdII], [GenEll], [AbsTopI], [AbsTopII], [AbsTopIII]. By contrast, the level of exposition of §3 is substantially less elementary than that of §2. In §3, we apply the conceptual infrastructure exposed in §2 to survey those aspects of inter-universal Teichmüller theory that correspond to (1\text{mf}), (3\text{mf}), (4\text{mf}), and (5\text{mf}), i.e., at the level of papers, to [EtTh], [IUTchI], [IUTchII], [IUTchIII], [IUTchIV]. Finally, in §4, we reflect on various historical aspects of the theory exposed in §2 and §3.

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§ 1. Review of the computation of the Gaussian integral

§ 1.1. Inter-universal Teichmüller theory via the Gaussian integral

The goal of the present paper is to pave the road, for the reader, from a state of complete ignorance of inter-universal Teichmüller theory to a state of general appreciation of the “game plan” of inter-universal Teichmüller theory by reconsidering the well-known computation of the Gaussian integral

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]

via polar coordinates from the point of view of a hypothetical high-school student who has studied one-variable calculus and polar coordinates, but has not yet had any exposure to multi-variable calculus. That is to say, we shall begin in the present §1
by reviewing this computation of the Gaussian integral by discussing how this computation might be explained to such a hypothetical high-school student. In subsequent §’s, we then proceed to discuss how various key steps in such an explanation to a hypothetical high-school student may be translated into the more sophisticated language of abstract arithmetic geometry in such a way as to yield a general outline of inter-universal Teichmüller theory based on the deep structural similarities between inter-universal Teichmüller theory and the computation of the Gaussian integral.

§ 1.2. Naive approach via changes of coordinates or partial integrations

In one-variable calculus, definite integrals that appear intractable at first glance are often reduced to much simpler definite integrals by performing suitable changes of coordinates or partial integrations. Thus:

Step 1: Our hypothetical high-school student might initially be tempted to perform a change of coordinates

\[ e^{-x^2} \sim u \]

and then [erroneously!] compute

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \cdot \int_{0}^{\infty} e^{-x^2} \, dx = - \int_{x=0}^{x=\infty} d(e^{-x^2}) = \int_{0}^{1} du = 1 \]

— only to realize shortly afterwards that this computation is in error, on account of the erroneous treatment of the infinitesimal “dx” when the change of coordinates was executed.

Step 2: This realization might then lead the student to attempt to repair the computation of Step 1 by considering various iterated partial integrations

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = - \int_{x=-\infty}^{x=\infty} \frac{1}{2x} d(e^{-x^2}) = \int_{x=-\infty}^{x=\infty} e^{-x^2} d\left(\frac{1}{2x}\right) = \ldots \]

— which, of course, lead nowhere.

§ 1.3. Introduction of identical but mutually alien copies

At this point, one might suggest to the hypothetical high-school student the idea of computing the Gaussian integral by first squaring the integral and then taking the square root of the value of the square of the integral. That is to say, in effect:

Step 3: One might suggest to the hypothetical high-school student that the Gaussian integral can in fact be computed by considering the product of two
identical — but mutually independent! — copies of the Gaussian integral
\[
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \cdot \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right)
\]
— i.e., as opposed to a single copy of the Gaussian integral.

Here, let us recall that our hypothetical high-school student was already in a mental state of extreme frustration as a result of the student’s intensive and heroic attempts in Step 2 which led only to an endless labyrinth of meaningless and increasingly complicated mathematical expressions. This experience left our hypothetical high-school student with the impression that the Gaussian integral was without question by far the most difficult integral that the student had ever encountered. In light of this experience, the suggestion of Step 3 evoked a reaction of intense indignation and distrust on the part of the student. That is to say, the idea that meaningful progress could be made in the computation of such an exceedingly difficult integral simply by considering two identical copies of the integral — i.e., as opposed to a single copy — struck the student as being utterly ludicrous.

Put another way, the suggestion of Step 3 was simply not the sort of suggestion that the student wanted to hear. Rather, the student was keenly interested in seeing some sort of clever partial integration or change of coordinates involving “\( \sin(-) \)”, “\( \cos(-) \)”, “\( \tan(-) \)”, “\( \exp(-) \)”, “\( \frac{1}{1+x^2} \)”, etc., i.e., of the sort that the student was used to seeing in familiar expositions of one-variable calculus.

§ 1.4. Integrals over two-dimensional Euclidean space

Only after quite substantial efforts at persuasion did our hypothetical high-school student reluctantly agree to proceed to the next step of the explanation:

**Step 4:** If one considers the “totality”, or “total space”, of the coordinates that appear in the product of two copies of the Gaussian integral of Step 3, then one can regard this product of integrals as a single integral
\[
\int_{\mathbb{R}^2} e^{-x^2} \cdot e^{-y^2} \, dx \, dy = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy
\]
over the Euclidean plane \( \mathbb{R}^2 \).

Of course, our hypothetical high-school student might have some trouble with Step 4 since it requires one to assimilate the notion of an integral over a space, i.e., the Euclidean plane \( \mathbb{R}^2 \), which is not an interval of the real line. This, however, may be explained by reviewing the essential philosophy behind the notion of the Riemann integral — a philosophy which should be familiar from one-variable calculus:
Step 5: One may think of integrals over more general spaces, i.e., such as the Euclidean plane $\mathbb{R}^2$, as computations

$$\text{net mass} = \lim \sum (\text{infinitesimals of zero mass})$$

of “net mass” by considering limits of sums of infinitesimals, i.e., such as “$dx\ dy$”, which one may think of as having “zero mass”.

§ 1.5. The effect on infinitesimals of changes of coordinates

Just as in one-variable calculus, computations of integrals over more general spaces can often be simplified by performing suitable changes of coordinates. Any [say, continuously differentiable] change of coordinates results in a new factor, given by the Jacobian, in the integrand. This factor constituted by the Jacobian, i.e., the determinant of a certain matrix of partial derivatives, may appear to be somewhat mysterious to our hypothetical high-school student, who is only familiar with changes of coordinates in one-variable calculus. On the other hand, the appearance of the Jacobian may be justified in a computational fashion as follows:

Step 6: Let $U, V \subseteq \mathbb{R}^2$ be open subsets of $\mathbb{R}^2$ and

$$U \ni (s, t) \mapsto (x, y) = (f(s, t), g(s, t)) \in V$$

a continuously differentiable change of coordinates such that the Jacobian

$$J \overset{\text{def}}{=} \det \begin{pmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial s} & \frac{\partial g}{\partial t} \end{pmatrix}$$

— which may be thought of as a continuous real-valued function on $U$ — is nonzero throughout $U$. Then for any continuous real-valued functions $\phi : U \to \mathbb{R}$, $\psi : V \to \mathbb{R}$ such that $\psi(f(s, t), g(s, t)) = \phi(s, t)$, the effect of the above change of coordinates on the integral of $\psi$ over $V$ may be computed as follows:

$$\int_V \psi \, dx \, dy = \int_U \phi \cdot J \, ds \, dt.$$

Step 7: In the situation of Step 6, the effect of the change of coordinates on the “infinitesimals” $dx\ dy$ and $ds\ dt$ may be understood as follows: First, one localizes to a sufficiently small open neighborhood of a point of $U$ over which the various partial derivatives of $f$ and $g$ are roughly constant, which implies that the change of coordinates determined by $f$ and $g$ is roughly linear. Then the effect of such a linear transformation on areas — i.e., in the language of Step 5, “masses” — of sufficiently small parallelograms is given by multiplying
by the **determinant** of the linear transformation. Indeed, to verify this, one observes that, after possible pre- and post-composition with a **rotation** [which clearly does not affect the computation of such areas], one may assume that one of the sides of the parallelogram under consideration is a **line segment on the s-axis whose left-hand endpoint is equal to the origin** \((0,0)\), and, moreover, that the linear transformation may be written as a **composite of toral dilations** and **unipotent** linear transformations of the form

\[
(s, t) \mapsto (a \cdot s, b \cdot t); \quad (s, t) \mapsto (s + c \cdot t, t)
\]

— where \(a, b, c \in \mathbb{R}\), and \(ab \neq 0\). On the other hand, in the case of such **“upper triangular”** linear transformations, the effect of the linear transformation on the area of the parallelogram under consideration is an easy computation at the level of high-school planar geometry.

§ 1.6. **Passage from planar cartesian to polar coordinates**

Once the **“innocuous” generalities** of Steps 5, 6, and 7 have been assimilated, one may proceed as follows:

**Step 8**: We apply Step 6 to the integral of Step 4, regarded as an integral over the **complement** \(\mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})\) of the negative \(x\)-axis in the Euclidean plane, and the **change of coordinates**

\[
\mathbb{R}_{>0} \times (-\pi, \pi) \ni (r, \theta) \mapsto (x, y) = (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})
\]

— where we write \(\mathbb{R}_{>0}\) for the set of positive real numbers and \((-\pi, \pi)\) for the open interval of real numbers between \(-\pi\) and \(\pi\).

**Step 9**: The change of coordinates of Step 8 allows one to compute as follows:

\[
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \cdot \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) = \int_{\mathbb{R}^2} e^{-x^2} \cdot e^{-y^2} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^2} e^{-x^2+y^2} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})} e^{-x^2+y^2} \, dx \, dy
\]

\[
= \int_{\mathbb{R}_{>0} \times (-\pi, \pi)} e^{-r^2} \, rdr \, d\theta
\]

\[
= \left( \int_0^{\infty} e^{-r^2} \cdot 2r \, dr \right) \cdot \left( \int_{-\pi}^{\pi} \frac{1}{2} \, d\theta \right)
\]
— where we observe that the final equality is notable in that it shows that, in the computation of the integral under consideration, the radial [i.e., “r”] and angular [i.e., “θ”] coordinates may be decoupled, i.e., that the integral under consideration may be written as a product of a radial integral and an angular integral.

**Step 10:** The radial integral of Step 9 may be evaluated

\[
\int_0^\infty e^{-r^2} \cdot 2r \, dr = \int_0^1 d(e^{-r^2}) = \int_0^1 du = 1
\]

by applying the change of coordinates

\[e^{-r^2} \sim u\]

that, in essence, appeared in the erroneous initial computation of Step 1!

**Step 11:** The angular integral of Step 9 may be evaluated as follows:

\[
\int_{-\pi}^{\pi} \frac{1}{2} \cdot d\theta = \pi
\]

Here, we note that, if one thinks of the Euclidean plane \(\mathbb{R}^2\) of Step 4 as the complex plane, i.e., if we write the change of coordinates of Step 8 in the form \(x + iy = r \cdot e^{i\theta}\), then, relative to the Euclidean coordinates \((x, y)\) of Step 4, the above evaluation of the angular integral may be regarded as arising from the change of coordinates given by considering the imaginary part of the natural logarithm

\[\log(r \cdot e^{i\theta}) = \log(r) + i\theta.\]

**Step 12:** Thus, in summary, we conclude that

\[
\left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \cdot \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right)
\]

\[
= \left( \int_0^\infty e^{-r^2} \cdot 2r \, dr \right) \cdot \left( \int_{-\pi}^{\pi} \frac{1}{2} \cdot d\theta \right) = \pi
\]

— i.e., that \(\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}\). Here, it is of interest to observe that, although in the approach to computing the Gaussian integral discussed above [i.e., starting in Step 3 and concluding in the present Step 12], the radial and angular integrals of Steps 10 and 11 arise quite naturally in the final computation of the present Step 12, if one just looks at the original Gaussian integral \(\int_{-\infty}^{\infty} e^{-x^2} \, dx\) on the real line from a naive point of view [cf. Steps 1 and 2],
it is essentially a hopeless task to identify “explicit portions” of this original Gaussian integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) on the real line that “correspond” precisely, in some sort of meaningful sense, to the radial and angular integrals of Steps 10 and 11.

§ 1.7. Justification of naive approach up to an “error factor”

Put another way, the content of the above discussion may be summarized as follows:

If one considers two identical — but mutually independent! — copies of the Gaussian integral, i.e., as opposed to a single copy, then the naively motivated coordinate transformation that gave rise to the erroneous computation of Step 1 may be “justified”, up to a suitable “error factor” \( \sqrt{\pi} \! \sqrt{\!} \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \!
situation of the ABC or Szpiro Conjectures, one may assume, without loss of generality, that, for any given finite set \( \Sigma \) of [archimedean and nonarchimedean] valuations of the rational number field \( \mathbb{Q} \),

the rational points under consideration lie, at each valuation of \( \Sigma \), inside some **compact subset** [i.e., of the set of rational points of the projective line minus three points over *some finite extension of the completion* of \( \mathbb{Q} \) at this valuation] satisfying certain properties.

In particular, when one computes the height of a rational point of the projective line minus three points as a suitable weighted sum of the valuations of the \( q \)-parameters of the corresponding elliptic curve, one may **ignore**, up to bounded discrepancies, contributions to the height that arise, say, from the **archimedean valuations** or from the nonarchimedean valuations that lie over some “exceptional” prime number such as 2.

**§ 2.2. Arithmetic degrees as global integrals**

As is well-known, the height of a rational point may be thought of as the **arithmetic degree** of a certain **arithmetic line bundle** over the field of definition of the rational point [cf. [Fsk], §1.3; [GenEll], §1]. Alternatively, from an id\('\)elic point of view, such arithmetic degrees of arithmetic line bundles over an NF may be thought of as logarithms of volumes — i.e., **“log-volumes”** — of certain **regions** inside the **ring of ad\('\)èles of the NF** [cf. [Fsk], §2.2; [AbsTopIII], Definition 5.9, (iii); [IUTchIII], Proposition 3.9, (iii)]. Relative to the point of view of the discussion of §1.4, such log-volumes may be thought of as **“net masses”**, that is to say,

as **“global masses”** [i.e., global log-volumes] that arise by summing up various **“local masses”** [i.e., local log-volumes], corresponding to the [archimedean and nonarchimedean] valuations of the NF under consideration.

This point of view of the discussion of §1.4 suggests further that such a global net mass should be regarded as some sort of

**integral** over an NF, that is to say, which arises by applying some sort of **mysterious “limit summation operation”** to some sort of **“zero mass infinitesimal”** object [i.e., corresponding to a differential form].

It is precisely this point of view that will be pursued in the discussion to follow via the following correspondences with terminology to be explained below:

\[
\begin{align*}
\text{zero mass objects} & \leftrightarrow \quad \text{“\text{\'{e}tale-like}” structures} \\
\text{positive/nonzero mass objects} & \leftrightarrow \quad \text{“\text{Frobenius-like}” structures}
\end{align*}
\]
§ 2.3. Bounding heights via global multiplicative subspaces

In the situation discussed in §2.1, one way to understand the problem of showing that the height \( h \in \mathbb{R} \) of a rational point is \textit{“small”} is as the problem of showing that, for some \textbf{fixed natural number} \( N > 1 \), the height \( h \) satisfies the \textbf{equation}

\[
N \cdot h \left( \overset{\text{def}}{=} h + h + \ldots + h \right) = h
\]

[which implies that \( h = 0! \)] — or, more generally, for a suitable \textit{“relatively small”} constant \( C \in \mathbb{R} \) [i.e., which is independent of the rational point under consideration], the \textbf{inequality}

\[
N \cdot h \leq h + C
\]

[which implies that \( h \leq \frac{1}{N-1} \cdot C! \)] — holds. Indeed, this is precisely the approach that is taken to bounding heights in the \textit{“tiny” special case} of the theory of [Falt1] that is given in the proof of [GenEll], Lemma 3.5. Here, we recall that the \textit{key assumption} in [GenEll], Lemma 3.5, that makes this sort of argument work is the assumption of the existence, for some prime number \( l \), of a certain kind of \textit{special rank one subspace} [i.e., a subspace whose \( \mathbb{F}_l \)-dimension is equal to 1] of the space of \( l \)-torsion points [i.e., a \( \mathbb{F}_l \)-vector space of dimension 2] of the elliptic curve under consideration. Such a rank one subspace is typically referred to in this context as a \textbf{global multiplicative subspace}, i.e., since it is a subspace defined over the \( \mathbb{Q} \) under consideration that \textit{coincides}, at each nonarchimedean valuation of the \( \mathbb{Q} \) at which the elliptic curve under consideration has potentially multiplicative reduction, with the rank one subspace of \( l \)-torsion points that arises, via the \textit{Tate uniformization}, from the [one-dimensional] space of \( l \)-torsion points of the \textit{multiplicative group} \( \mathbb{G}_m \). The quotient of the original given elliptic curve by such a \textbf{global multiplicative subspace} is an elliptic curve that is \textit{isogenous} to the original elliptic curve. Moreover,

the \textbf{q-parameters} of this isogenous elliptic curve are the \textbf{l-th powers} of the \( q \)-parameters of the original elliptic curve; thus, the \textbf{height} of this isogenous elliptic curve is [roughly, up to contributions of negligible order] \( l \) \textbf{times} the height of the original elliptic curve.

These properties of the isogenous elliptic curve allow one to \textbf{compute} the \textbf{height} of the \textit{isogenous elliptic curve} in terms of the \textbf{height} of the \textit{original elliptic curve} by calculating the effect of the isogeny relating the two elliptic curves on the respective \textbf{sheaves of differentials} and hence to conclude an \textit{inequality} \( “N \cdot h \leq h + C” \) of the desired type [for \( N = l \) — cf. the proof of [GenEll], Lemma 3.5, for more details]. At a more concrete level, this \textbf{computation} may be summarized as the \textbf{observation} that, by considering the effect of the isogeny under consideration on \textit{sheaves of differentials}, one may conclude that
multiplying heights by \( l \) — i.e., “raising \( q \)-parameters to the \( l \)-th power”

\[ q \mapsto q^l \]

— has the effect on logarithmic differential forms

\[ d\log(q) = \frac{dq}{q} \mapsto l \cdot d\log(q) \]

of multiplying by \( l \), i.e., at the level of heights, of adding terms of the order of \( \log(l) \), thus giving rise to inequalities that are roughly of the form “\( l \cdot h \leq h + \log(l) \)”.

On the other hand, in general,

such a global multiplicative subspace does not exist, and the issue of somehow “simulating” the existence of a global multiplicative subspace is one fundamental theme of inter-universal Teichmüller theory.

§ 2.4. Bounding heights via Frobenius morphisms on number fields

The simulation issue discussed in §2.3 is, in some sense, the fundamental reason for the construction of various types of “Hodge theaters” in [IUTchI] [cf. the discussion surrounding [IUTchI], Fig. 11.4; [IUTchI], Remark 4.3.1]. From the point of view of the present discussion, the fundamental additive and multiplicative symmetries that appear in the theory of \([\Theta^\pm_{\text{ell}}NP-]Hodge theaters\) [cf. §3.3, (v); §3.6, (i), below] and which correspond, respectively, to the additive and multiplicative structures of the ring \( \mathbb{F}_l \) [where \( l \) is the fixed prime number for which we consider \( l \)-torsion points], may be thought of as corresponding, respectively, to the symmetries in the equation

\[ N \cdot h \left( \overset{\text{def}}{=} h + h + \ldots + h \right) = h \]

of all the \( h \)'s [in the case of the additive symmetry] and of the \( h \)'s on the LHS [in the case of the multiplicative symmetry]. This portion of inter-universal Teichmüller theory is closely related to the analogy between inter-universal Teichmüller theory and the classical hyperbolic geometry of the upper half-plane. This analogy with the hyperbolic geometry of the upper half-plane is, in some sense, the central topic of [BogIUT] [cf. also §3.10, (vi); §4.1, (i); §4.3, (iii), of the present paper] and may be thought of as corresponding to the portion of inter-universal Teichmüller theory discussed in [IUTchI], [IUTchIII]. Since this aspect of inter-universal Teichmüller theory is already discussed in substantial detail in [BogIUT], we shall not discuss it in much detail in the present paper. On the other hand, another way of thinking about the above equation “\( N \cdot h = h \)” is as follows:

This equation may also be thought of as calling for the establishment of some sort of analogue for an \( NF \) of the Frobenius morphism in positive characteristic scheme theory, i.e., a Frobenius morphism that somehow “acts” naturally
on the entire situation [i.e., including the height $h$, as well as the $q$-parameters at nonarchimedean valuations of potentially multiplicative reduction, of a given elliptic curve over the NF] in such a way as to multiply arithmetic degrees [such as the height!] by $N$ and raise $q$-parameters to the $N$-th power — i.e.,

$$h \mapsto N \cdot h, \quad q \mapsto q^N$$

— and hence yield the equation “$N \cdot h = h$” [or inequality “$N \cdot h \leq h + C$”] via some sort of natural functoriality.

This point of view is also quite fundamental to inter-universal Teichmüller theory, and, in particular, to the analogy between inter-universal Teichmüller theory and the theory of the Gaussian integral, as reviewed in §1. These aspects of inter-universal Teichmüller theory are discussed in [IUTchII], [IUTchIII]. In the present paper, we shall concentrate mainly on the exposition of these aspects of inter-universal Teichmüller theory. Before proceeding, we remark that, ultimately, in inter-universal Teichmüller theory, we will, in effect, take “$N$” to be a sort of symmetrized average over the squares of the values $j = 1, 2, \ldots, l^*$, where $l^* \overset{\text{def}}{=} (l - 1)/2$, and $l$ is the prime number of §2.3. That is to say, whereas the [purely hypothetical!] naive analogue of the Frobenius morphism for an NF considered so far has the effect, on $q$-parameters of the elliptic curve under consideration at nonarchimedean valuations of potentially multiplicative reduction, of mapping $q \mapsto q^N$, the sort of assignment that we shall ultimately be interested in in inter-universal Teichmüller theory is an assignment [which is in fact typically written with the left- and right-hand sides reversed]

$$q \mapsto \{q^{j^2}\}_{j=1, \ldots, l^*}$$

— where $q$ denotes a $2l$-th root of the $q$-parameter $q$ — i.e., an assignment which, at least at a formal level, closely resembles a Gaussian distribution. Of course, such an assignment is not compatible with the ring structure of an NF, hence does not exist in the framework of conventional scheme theory. Thus, one way to understand inter-universal Teichmüller theory is as follows:

in some sense the fundamental theme of inter-universal Teichmüller theory consists of the development of a mechanism for computing the effect — e.g., on heights of elliptic curves [cf. the discussion of §2.3!] — of such non-scheme-theoretic “Gaussian Frobenius morphisms” on NF’s.

§ 2.5. Fundamental example of the derivative of a Frobenius lifting

In some sense, the most fundamental example of the sort of Frobenius action in the $p$-adic theory that one would like to somehow translate into the case of NF’s is the following [cf. [AbsTopII], Remark 2.6.2; [AbsTopIII], §15; [IUTchIII], Remark 3.12.4, (v)]:

$$h \mapsto N \cdot h, \quad q \mapsto q^N$$

— and hence yield the equation “$N \cdot h = h$” [or inequality “$N \cdot h \leq h + C$”] via some sort of natural functoriality.
Example 2.5.1. Frobenius liftings on smooth proper curves. Let $p$ be a prime number; $A$ the ring of Witt vectors of a perfect field $k$ of characteristic $p$; $X$ a smooth, proper curve over $A$ of genus $g_X \geq 2$; $\Phi : X \rightarrow X$ a Frobenius lifting, i.e., a morphism whose reduction modulo $p$ coincides with the Frobenius morphism in characteristic $p$. Thus, one verifies immediately that $\Phi$ necessarily lies over the Frobenius morphism on the ring of Witt vectors $A$. Write $\omega_{X_k}$ for the sheaf of differentials of $X_k \overset{def}{=} X \times_A k$ over $k$. Then the derivative of $\Phi$ yields, upon dividing by $p$, a morphism of line bundles

$$\Phi^*\omega_{X_k} \rightarrow \omega_{X_k}$$

which is easily verified to be generically injective. Thus, by taking global degrees of line bundles, we obtain an inequality

$$(p-1)(2g_X - 2) \leq 0$$

— hence, in particular, an inequality $g_X \leq 1$ — which may be thought of as being, in essence, a statement to the effect that $X$ cannot be hyperbolic. Note that, from the point of view discussed in §1.4, §1.5, §2.2, §2.3, §2.4, this inequality may be thought of as a computation of “global net masses”, i.e., global degrees of line bundles on $X_k$, via a computation of the effect of the “change of coordinates” $\Phi$ by considering the effect of this change of coordinates on “infinitesimals”, i.e., on the sheaf of differentials $\omega_{X_k}$.

§ 2.6. Positive characteristic model for mono-anabelian transport

One fundamental drawback of the computation discussed in Example 2.5.1 is that it involves the operation of differentiation on $X_k$, an operation which does not, at least in the immediate literal sense, have a natural analogue in the case of NF’s. This drawback does not exist in the following example, which treats certain subtle, but well-known aspects of anabelian geometry in positive characteristic and, moreover, may, in some sense, be regarded as the fundamental model, or prototype, for a quite substantial portion of inter-universal Teichmüller theory. In this example, Galois groups, or étale fundamental groups, in some sense play the role that is played by tangent bundles in the classical theory — a situation that is reminiscent of the approach of the [scheme-theoretic] Hodge-Arakelov theory of [HASurI], [HASurII], which is briefly reviewed in §2.14 below. One notion of central importance in this example — and indeed throughout inter-universal Teichmüller theory! — is the notion of a cyclotome, a term which is used to refer to an isomorphic copy of some quotient [by a closed submodule] of the familiar Galois module $\hat{\mathbb{Z}}(1)$, i.e., the “Tate twist” of
the trivial Galois module $\hat{\mathbb{Z}}$, or, alternatively, the rank one free $\hat{\mathbb{Z}}$-module equipped with the action determined by the **cyclotomic character**. Also, if $p$ is a **prime number**, then we shall write $\hat{\mathbb{Z}}_{\neq p}$ for the quotient $\hat{\mathbb{Z}}/\mathbb{Z}_p$.

**Example 2.6.1.** Mono-anabelian transport via the Frobenius morphism in positive characteristic.

(i) Let $p$ be a prime number; $k$ a finite field of characteristic $p$; $X$ a smooth, proper curve over $k$ of genus $g_X \geq 2$; $K$ the function field of $X$; $\overline{K}$ a separable closure of $K$. Write $\eta_X \overset{\text{def}}{=} \text{Spec}(K)$; $\overline{\eta}_X \overset{\text{def}}{=} \text{Spec}(\overline{K})$; $\overline{K} \subseteq \overline{K}$ for the algebraic closure of $k$ determined by $\overline{K}$; $\mu_{\overline{K}} \subseteq k$ for the group of roots of unity of $\overline{K}$; $\mu_{\overline{\mathbb{Z}}_{\neq p}} \overset{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\overline{K}})$; $G_K \overset{\text{def}}{=} \text{Gal}(\overline{K}/K)$; $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$; $\Pi_X$ for the quotient of $G_K$ determined by the maximal subextension of $\overline{K}$ that is unramified over $X$;

$$\Phi_X : X \to X, \quad \Phi_{\eta_X} : \eta_X \to \eta_X, \quad \Phi_{\overline{\eta}_X} : \overline{\eta}_X \to \overline{\eta}_X$$

for the respective Frobenius morphisms of $X$, $\eta_X$, $\overline{\eta}_X$. Thus, we have natural surjections $G_K \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$, and $\Pi_X$ may be thought of as [i.e., is naturally isomorphic to] the étale fundamental group of $X$ [for a suitable choice of basepoint]. Write $\Delta_X \overset{\text{def}}{=} \ker(\Pi_X \twoheadrightarrow G_k)$. Recall that it follows from elementary facts concerning separable and purely inseparable field extensions that [by considering $\Phi_{\overline{\eta}_X}$] $\Phi_{\eta_X}$ induces isomorphisms of Galois groups and étale fundamental groups

$$\Psi_X : \Pi_X \xrightarrow{\sim} \Pi_X, \quad \Psi_{\eta_X} : G_K \xrightarrow{\sim} G_K$$

— which is, in some sense, a quite remarkable fact since

the Frobenius morphisms $\Phi_X$, $\Phi_{\eta_X}$ themselves are morphisms "of degree $p > 1"$, hence, in particular, are by no means isomorphisms!

We refer to [IUTchIV], Example 3.6, for a more general version of this phenomenon.

(ii) Next, let us recall that it follows from the fundamental anabelian results of [Uchi] that there exists a purely group-theoretic functorial algorithm

$$G_K \mapsto \tilde{K}(G_K)_{\text{CFT}} \hookrightarrow G_K$$

— i.e., an algorithm whose input data is the abstract topological group $G_K$, whose functoriality is with respect to isomorphisms of topological groups, and whose output data is a field $\tilde{K}(G_K)_{\text{CFT}}$ equipped with a $G_K$-action. Moreover, if one allows oneself to apply the conventional interpretation of $G_K$ as a Galois group $\text{Gal}(\overline{K}/K)$, then there is a natural $G_K$-equivariant isomorphism

$$\rho : \tilde{K} \xrightarrow{\sim} \tilde{K}(G_K)_{\text{CFT}}$$
that arises from the reciprocity map of class field theory, applied to each of the finite subextensions of the extension $\widetilde{K}/K$. Since class field theory depends, in an essential way, on the field structure of $\widetilde{K}$ and $K$, it follows formally that, at least in an a priori sense, the construction of $\rho$ itself also depends, in an essential way, on the field structure of $\widetilde{K}$ and $K$. Moreover, the fact that the isomorphism $\widetilde{K}(G_K)^{\text{CFT}} \cong \widetilde{K}(G_K)^{\text{CFT}}$ and $\rho$ are [unlike $\Phi_{\eta_X}$ itself!] isomorphisms implies that the diagram

$$
\begin{array}{ccc}
\widetilde{K}(G_K)^{\text{CFT}} & \xrightarrow{\rho} & \widetilde{K}(G_K)^{\text{CFT}} \\
\Phi_{\eta_X}^* & \circlearrowleft & \Phi_{\eta_X}^* \\
K & \xrightarrow{\rho} & K
\end{array}
$$

fails to be commutative!

(iii) On the other hand, let us recall that consideration of the first Chern class of a line bundle of degree 1 on $X$ yields a natural isomorphism

$$
\lambda : \mu_{\widehat{Z}\neq p} \cong M_X \overset{\text{def}}{=} \text{Hom}_{\widehat{Z}\neq p}(H^2(\Delta_X, \widehat{Z}\neq p), \widehat{Z}\neq p)
$$

[cf., e.g., [Cusp], Proposition 1.2, (ii)]. Such a natural isomorphism between cyclotomes [i.e., such as $\mu_{\widehat{Z}\neq p}, M_X$] will be referred to as a cyclotomic rigidity isomorphism. Thus, if we let “$H$” range over the open subgroups of $G_K$, then, by composing this cyclotomic rigidity isomorphism [applied to the coefficients of “$H^1(\cdot)$”] with the Kummer morphism associated to the multiplicative group $(\widetilde{K}^H)^{\times}$ of the field $\widetilde{K}^H$ of $H$-invariants of $\widetilde{K}$, we obtain an embedding

$$
\kappa : \widetilde{K}^{\times} \hookrightarrow \lim_{\mapsto H} H^1(H, \mu_{\widehat{Z}\neq p}) \cong \lim_{\mapsto H} H^1(H, M_X)
$$

— whose construction depends only on the multiplicative monoid with $G_K$-action $\widetilde{K}^{\times}$ and the cyclotomic rigidity isomorphism $\lambda$. Note that the existence of the reconstruction algorithm $\widetilde{K}(\cdot)^{\text{CFT}}$ reviewed above implies that the kernels of the natural surjections $G_K \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$ may be reconstructed group-theoretically from the abstract topological group $G_K$. In particular, we conclude that $\lim_{\mapsto H} H^1(H, M_X)$ may be reconstructed group-theoretically from the abstract topological group $G_K$. Moreover, the anabelian theory of [Cusp] [cf., especially, [Cusp], Proposition 2.1; [Cusp], Theorem 2.1, (ii); [Cusp], Theorem 3.2] yields a purely group-theoretic functorial algorithm

$$
G_K \mapsto \widetilde{K}(G_K)^{\text{Kum}} \lhd G_K
$$

— i.e., an algorithm whose input data is the abstract topological group $G_K$, whose functoriality is with respect to isomorphisms of topological groups, and whose output data is a field $\widetilde{K}(G_K)^{\text{Kum}}$ equipped with a $G_K$-action which is constructed as the union
with \{0\} of the image of \(\kappa\). [In fact, the input data for this algorithm may be taken to be the abstract topological group \(\Pi_X\), but we shall not pursue this topic here.]

Thus, just as in the case of \(\widetilde{\mathcal{K}}(-)^\text{CFT}\), the fact that the isomorphism \(\widetilde{\mathcal{K}}(\Psi_{\eta X})^{\text{Kum}} : \widetilde{\mathcal{K}}(G_K)^{\text{Kum}} \sim \widetilde{\mathcal{K}}(G_K)^{\text{Kum}}\) and \(\kappa\) are [unlike \(\Phi_{\eta X}\) itself!] isomorphisms implies that the diagram

\[
\begin{array}{ccc}
\widetilde{\mathcal{K}}(G_K)^{\text{Kum}} & \xrightarrow{\Phi_{\eta X}^*} & \widetilde{\mathcal{K}}(G_K)^{\text{Kum}} \\
\uparrow & \sim \? & \uparrow \\
\bar{K} & \xrightarrow{\Phi_{\eta X}^*} & \bar{K}
\end{array}
\]

— where, by a slight abuse of notation, we write \(\kappa\) for the “formal union” of \(\kappa\) with \(\{0\}\) — fails to be commutative!

(iv) The \([a \ a \ p r i o r i]\) noncommutativity of the diagram of the final display of (iii) may be interpreted in two ways, as follows:

(a) If one starts with the assumption that this diagram is in fact commutative, then the fact that the Frobenius morphism \(\Phi_{\eta X}^*\) multiplies degrees of rational functions \(\in \bar{K}\) by \(p\), together with the fact that the vertical and upper horizontal arrows of the diagram are isomorphisms, imply [since the field \(\bar{K}\) is not perfect!] the erroneous conclusion that all degrees of rational functions \(\in \bar{K}\) are equal to zero! This sort of argument is formally similar to the argument “\(N \cdot h = h \implies h = 0\)” discussed in \S 2.3.

(b) One may regard the noncommutativity of this diagram as the problem of computing just how much “indeterminacy” one must allow in the objects and arrows that appear in the diagram in order to render the diagram commutative. From this point of view, one verifies immediately that a solution to this problem may be given by introducing “indeterminacies” as follows: One replaces

\[
\lambda \rightsquigarrow \lambda \cdot p^\mathbb{Z}
\]

the cyclotomic rigidity isomorphism \(\lambda\) by the orbit of \(\lambda\) with respect to composition with multiplication by arbitrary \(\mathbb{Z}\)-powers of \(p\), and one replaces

\[
\bar{K} \rightsquigarrow \bar{K}^{\text{pf}}, \quad \bar{K}(G_K)^{\text{Kum}} \rightsquigarrow (\bar{K}(G_K)^{\text{Kum}})^{\text{pf}}
\]

the fields \(\bar{K}, \bar{K}(G_K)^{\text{Kum}}\) by their perfections.

Here, we observe that interpretation (b) may be regarded as corresponding to the argument “\(N \cdot h \leq h + C \implies h \leq \frac{1}{N-1} \cdot C\)” discussed in \S 2.3. That is to say,

If, in the situation of (b), one can show that the indeterminacies necessary to render the diagram commutative are sufficiently mild, at least in the case
of the heights or q-parameters that one is interested in, then it is "reasonable to expect" that the resulting "contradiction in the style of interpretation (a)" between

* multiplying degrees by some integer [or rational number] > 1

and the fact that

* the vertical and upper horizontal arrows of the diagram are isomorphisms

should enable one to conclude that "N \cdot h \leq h + C" [and hence that "h \leq \frac{1}{N-1} \cdot C"].

This is precisely the approach that is in fact taken in inter-universal Teichmüller theory.

§ 2.7. The apparatus and terminology of mono-anabelian transport

Example 2.6.1 is exceptionally rich in structural similarities to inter-universal Teichmüller theory, which we proceed to explain in detail as follows. One way to understand these structural similarities is by considering the quite substantial portion of terminology of inter-universal Teichmüller theory that was, in essence, inspired by Example 2.6.1:

(i) **Links between “mutually alien” copies of scheme theory:** One central aspect of inter-universal Teichmüller theory is the study of certain "walls", or "filters" — which are often referred to as "links" — that separate two "mutually alien" copies of conventional scheme theory [cf. the discussions of [IUTchII], Remark 3.6.2; [IUTchIV], Remark 3.6.1]. The main example of such a link in inter-universal Teichmüller theory is constituted by [various versions of] the Θ-link. The log-link also plays an important role in inter-universal Teichmüller theory. The main motivating example for these links which play a central role in inter-universal Teichmüller theory is the [Frobenius morphism] $\Phi_{\eta_X}$ of Example 2.6.1. From the point of view of the discussion of §1.4, §1.5, §2.2, §2.3, §2.4, and §2.5, such a link corresponds to a change of coordinates.

(ii) **Frobenius-like objects:** The objects that appear on either side of a link and which are used in order to construct, or “set up”, the link, are referred to as "Frobenius-like". Put another way,

Frobenius-like objects are objects that, at least a priori, are only defined on one side of a link [i.e., either the domain or codomain], and, in particular, do not necessarily map isomorphically to corresponding objects on the opposite side of the link.
Thus, in Example 2.6.1, the “mutually alien” copies of $\bar{K}$ on either side of the $p$-power map $\Phi^*_\eta$ are Frobenius-like. Typically, Frobenius-like structures are characterized by the fact that they have positive/nonzero mass. That is to say, Frobenius-like structures represent the positive mass — i.e., such as degrees of rational functions in Example 2.6.1 or heights/degrees of arithmetic line bundles in the context of diophantine geometry — that one is ultimately interested in computing and, moreover, is, at least in an a priori sense, affected in a nontrivial way, e.g., multiplied by some factor $> 1$, by the link under consideration. From this point of view, Frobenius-like objects are characterized by the fact that the link under consideration gives rise to an “ordering”, or “asymmetry”, between Frobenius-like objects in the domain and codomain of the link under consideration [cf. the discussion of [FrdI], §I3, §I4].

(iii) Étale-like objects: By contrast, objects that appear on either side of a link that correspond to the “topology of some sort of underlying space” — such as the étale topology! — are referred to as “étale-like”. Typically, étale-like structures are mapped isomorphically — albeit via some indeterminate isomorphism! [cf. the discussion of §2.10 below] — to one another by the link under consideration. From this point of view, étale-like objects are characterized by the fact that the link under consideration gives rise to a “confusion”, or “symmetry”, between étale-like objects in the domain and codomain of the link under consideration [cf. the discussion of [FrdI], §I3, §I4]. Thus, in Example 2.6.1, the Galois groups/étale fundamental groups $G_K$, $\Pi_X$, which are mapped isomorphically to one another via $\Phi^*_\eta$, albeit via some “mysterious indeterminate isomorphism”, are étale-like. Objects that are algorithmically constructed from étale-like objects such as $G_K$ or $\Pi_X$ are also referred to as étale-like, so long as they are regarded as being equipped with the additional structure constituted by the algorithm applied to construct the object from some object such as $G_K$ or $\Pi_X$. Étale-like structures are regarded as having zero mass and are used as rigid containers for positive mass Frobenius-like objects, i.e., containers whose structure satisfies certain rigidity properties that typically arise from various anabelian properties and [as in Example 2.6.1!] allows one to compute the effect on positive mass Frobenius-like objects of the links, or “changes of coordinates”, under consideration.

(iv) Coric objects: In the context of consideration of some sort of link as in (i), coricity refers to the property of being invariant with respect to — i.e., the property of mapping isomorphically to a corresponding object on the opposite side of — the link under consideration. Thus, as discussed in (iii), étale-like objects, considered up to isomorphism, constitute a primary example of the notion of a coric object. On the other hand, [non-étale-like] Frobenius-like coric objects also arise naturally in various contexts. Indeed, in the situation of Example 2.6.1 [cf. the discussion of Example 2.6.1, (iv), (b)], not only étale-like objects such as $\Pi_X$, $G_K$, and $\bar{K}(G_K)^{Kum}$, but also
Frobenius-like objects such as the *perfections* $\tilde{K}^{pf}$ are coric.

(v) **The computational technique of mono-anabelian transport:** The technique discussed in (iii), i.e., of computing the effect on positive mass Frobenius-like objects of the links, or “changes of coordinates”, under consideration, by first applying some sort of Kummer isomorphism to pass from Frobenius-like to corresponding étale-like objects, then applying some sort of anabelian construction algorithm or rigidity property, and finally applying a suitable inverse Kummer isomorphism to pass from étale-like to corresponding Frobenius-like objects will be referred to as the technique of **mono-anabelian transport** [cf. Fig. 2.1 above]. In some sense, the **most fundamental prototype** for this technique is the situation described in Example 2.6.1, (iii) [cf. also the discussion of Example 2.6.1, (iv), (b)]. Here, the term “mono-anabelian” [cf. the discussion of [AbsTopIII], §I2] refers to the fact that the algorithm under consideration is an algorithm whose input data [typically] consists only of an abstract profinite group [i.e., that “just happens” to be isomorphic to a Galois group or étale fundamental group that arises from scheme theory!]. This term is used to distinguish from fully faithfulness results [i.e., to the effect that one has a natural bijection between certain types of morphisms of schemes and certain types of morphisms of profinite groups] of the sort that appear in various anabelian conjectures of Grothendieck. Such fully faithfulness results are referred to as “bi-anabelian”.

(vi) **Kummer-detachment indeterminacies versus étale-transport indeterminacies:** The first step that occurs in the procedure for mono-anabelian transport [cf. the discussion of (v)] is the passage, via some sort of Kummer isomorphism, from Frobenius-like objects to corresponding étale-like objects. This first step is referred to as **Kummer-detachment.** The indeterminacies that arise during this first step are referred to as **Kummer-detachment indeterminacies** [cf. [IUTchIII], Remark 1.5.4].
Such Kummer-detachment indeterminacies typically involve indeterminacies in the \textbf{cyclotomic rigidity isomorphism} that is applied, i.e., as in the situation discussed in Example 2.6.1, (iv), (b). On the other hand, in general, more complicated Kummer-detachment indeterminacies [i.e., that are not directly related to cyclotomic rigidity isomorphisms] can occur. By contrast, the indeterminacies that occur as a result of the fact that the étale-like structures under consideration may only be regarded as being known up to an \textit{indeterminate isomorphism} [cf. the discussion of (iii), as well as of §2.10 below] are referred to as \textbf{étale-transport indeterminacies} [cf. [IUTchIII], Remark 1.5.4].

(vii) \textbf{Arithmetic holomorphic structures versus mono-analytic structures:} A ring may be regarded as consisting of \textit{“two combinatorial dimensions”} — namely, the underlying \textbf{additive} and \textbf{multiplicative} structures of the ring — which are \textbf{intertwined} with one another in a rather complicated fashion [cf. the discussion of [AbsTopIII], §I3; [AbsTopIII], Remark 5.6.1]. In inter-universal Teichmüller theory, one is interested in \textbf{dismantling} this complicated intertwining structure by considering the underlying additive and multiplicative monoids associated to a ring \textbf{separately}. In this context, the \textbf{ring structure}, as well as other structures such as étale fundamental groups that are sufficiently rigid as to allow the algorithmic reconstruction of the ring structure, are referred to as \textbf{arithmetic holomorphic structures}. By contrast, structures that arise from dismantling the complicated intertwining inherent in a ring structure are referred to as \textbf{mono-analytic} — a term which may be thought of as a sort of arithmetic analogue of the notion of an underlying \textit{real analytic} structure in the context of complex holomorphic structures. From this point of view, the approach of Example 2.6.1, (ii), involving the \textbf{reciprocity} map of \textbf{class field theory} depends on the \textbf{arithmetic holomorphic structure} [i.e., the ring structure] of the field \(\tilde{K}\) in a quite essential and complicated way. By contrast, the \textbf{Kummer-theoretic} approach of Example 2.6.1, (iii), only depends on the \textbf{mono-analytic} structure constituted by the underlying \textbf{multiplicative monoid} of the field \(\tilde{K}\), together with the \textbf{cyclotomic rigidity isomorphism} \(\lambda\). Thus, although \(\lambda\) depends on the ring structure of the field \(\tilde{K}\), the Kummer-theoretic approach of Example 2.6.1, (iii), has the \textit{advantage}, from the point of view of dismantling the arithmetic holomorphic structure, of \textbf{isolating} the \textit{dependence of \(\kappa\) on the arithmetic holomorphic structure} of the field \(\tilde{K}\) in the “compact form” constituted by the \textbf{cyclotomic rigidity isomorphism} \(\lambda\).

§ 2.8. \textbf{Remark on the usage of certain terminology}

In the context of the discussion of §2.7, we remark that although terms such as \textit{“link”}, \textit{“Frobenius-like”}, \textit{“étale-like”}, \textit{“corie”}, \textit{“mono-anabelian transport”}, \textit{“Kummer-}
“detachment”, “cyclotomic rigidity isomorphism”, “mono-analytic”, and “arithmetic holomorphic structure” are well-defined in the various specific contexts in which they are applied, these terms do not admit general definitions that are applicable in all contexts.

In this sense, such terms are used in a way that is similar to the way in which terms such as “underlying” [cf., e.g., the “underlying topological space of a scheme”, the “underlying real analytic manifold of a complex manifold”] or “anabelian” are typically used in mathematical discussions. The term “multiradial”, which will be discussed in §3, is also used in this way. In this context, we remark that one aspect that complicates the use of the terms “Frobenius-like” and “étale-like” is the sort of curious process of evolution that these terms underwent as the author progressed from writing [FrdI], [FrdII] in 2005 to writing [IUTchI] in 2008 and finally to writing [IUTchII] and [IUTchIII] during the years 2009 - 2010. This “curious process of evolution” may be summarized as follows:

(1Fr/ét) Vague philosophical approach [i.e., “order-conscious” vs. “indifferent to order”]: The first stage in this evolutionary process consists of the vague philosophical characterizations of the notion of “Frobenius-like” via the term “order-conscious” and of the notion of “étale-like” via the phrase “indifferent to order”. These vague characterizations were motivated by the situation surrounding the monoids [i.e., in the case of “Frobenius-like”] that appear in the theory of Frobenioids [cf. [FrdI], [FrdII]] and the situation surrounding the Galois groups/arithmetic fundamental groups [i.e., in the case of “étale-like”] that appear in the base categories [“D”] of Frobenioids. This point of view is discussed in [FrdI], §I4, and is quoted and applied throughout [IUTchI].

(2Fr/ét) Characterization in the context of the log-theta-lattice: This point of view consists of the characterization of the notions of “Frobenius-like” and “étale-like” in the context of the specific links, i.e., the Θ- and log-links, that occur in the log-theta-lattice. This approach is developed throughout [IUTchII] [cf., especially, [IUTchII], Remark 3.6.2] and [IUTchIII] [cf., especially, [IUTchIII], Remark 1.5.4]. Related discussions may be found in [IUTchIV], Remarks 3.6.2, 3.6.3. At a purely technical/notational level, this approach may be understood as follows [cf. also the discussion in the final portion of §3.3, (vii), of the present paper]:

- Frobenius-like objects [or structures] are objects that, when embedded in the log-theta-lattice, are marked [via left-hand superscripts] by lattice coordinates “(n,m)” or, when not embedded in the log-theta-lattice, are marked [via left-hand superscripts] by daggers “†”/double daggers “‡”/asterisks “∗”.
étale-like objects [or structures] are objects that arise from [i.e., are often denoted as “functions (−)” of] the $\mathcal{D}/\mathcal{D}^\text{−}$-portions of the $\Theta^{\pm\text{ell}}\mathcal{NF}$-Hodge theaters that appear, or, when embedded in the log-theta-lattice, are marked [via left-hand superscripts] by vertically coric/bi-coric lattice coordinates “(n, o)”/“(o, o)”.

(3Fr/ét) **Abstract link-theoretic approach:** This is the the approach taken in §2.7 of the present paper. In this approach, the notions of “Frobenius-like” and “étale-like” are only defined in the context of a specific link. This approach arose, in discussions involving the author and Y. Hoshi, as a sort of abstraction/generalization of the situation that occurs in [IUTchII], [IUTchIII] in the case of the specific links, i.e., the $\Theta$- and log-links, that occur in the case of the log-theta-lattice [cf. (2Fr/ét)].

Thus, of these three approaches (1Fr/ét), (2Fr/ét), (3Fr/ét), the approach (3Fr/ét) is, in some sense, theoretically the most satisfying approach, especially from the point of view of considering possible generalizations of the theory of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV]. On the other hand, from the point of view of the more restricted goal of attaining a technically sound understanding of the content of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV], the purely technical/notational approach discussed in (2Fr/ét) is quite sufficient.

§ 2.9. **Mono-anabelian transport and the Kodaira-Spencer morphism**

The discussion of §2.6 and §2.7 may be summarized as follows: In some sense, the central theme of inter-universal Teichmüller theory is the computation via mono-anabelian transport — in the spirit of the discussion of Example 2.6.1, (iv), (b) — of the discrepancy between two [systems of] Kummer theories, that is to say, of the sort of indeterminacies that one must admit in order to render two systems of Kummer theories compatible — i.e., relative to the various gluings constituted by some link [cf. the Frobenius morphism $\Phi_{\eta_X}$ in Example 2.6.1; the discussion of §2.7, (i)] between the two systems of Kummer theories — with simultaneous execution, e.g., when one of the two systems of Kummer theories [cf. the objects in the lower right-hand corner of the diagram of §2.6, (iii), and Fig. 2.1] is held fixed.

In this context, it is of interest to observe that this approach of computing degrees of [“positive mass”] Frobenius-like objects by embedding them into rigid étale-like [“zero mass”] containers [cf. the discussion of §2.7, (iii), (v)] is formally similar to the classical definition of the Kodaira-Spencer morphism associated to a family of elliptic curves [cf. also the discussion of §3.1, (v), below]: Indeed, suppose [relative to the terminology of [Semi], §0] that $S^{\text{log}}$ is a smooth log curve over $\text{Spec}(\mathbb{C})$ [equipped with the trivial log structure], and that $E^{\text{log}} \to S^{\text{log}}$ is a stable log curve of type (1, 1)
[i.e., in essence, a family of elliptic curves whose origin is regarded as a “marked point”, and which is assumed to have stable reduction at the points of degeneration]. Write $\omega_{S^{\log}/C}$ for the sheaf of relative logarithmic differentials of $S^{\log} \to \text{Spec}(\mathbb{C})$; $(\mathcal{E}, \nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \omega_{S^{\log}/C})$ for the rank two vector bundle with logarithmic connection on $S^{\log}$ determined by the first logarithmic de Rham cohomology module of $E^{\log} \to S^{\log}$ and the logarithmic Gauss-Manin connection; $\omega_E \subseteq \mathcal{E}$ for the Hodge filtration on $\mathcal{E}$; $\tau_E$ for the $\mathcal{O}_S$-dual of $\omega_E$. Thus, we have a natural exact sequence $0 \to \omega_E \to \mathcal{E} \to \tau_E \to 0$. Then the Kodaira-Spencer morphism associated to the family $E^{\log} \to S^{\log}$ may be defined as the composite of morphisms in the diagram

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\nabla_{\mathcal{E}}} & \mathcal{E} \otimes_{\mathcal{O}_S} \omega_{S^{\log}/C} \\
\uparrow & & \downarrow \\
\omega_E & & \tau_E \otimes_{\mathcal{O}_S} \omega_{S^{\log}/C}
\end{array}
$$

— where the left-hand vertical arrow is the natural inclusion, and the right-hand vertical arrow is the natural surjection. The three arrows in this diagram may be regarded as corresponding to the three arrows in analogous positions in the diagrams of Example 2.6.1, (iii), and Fig. 2.1. That is to say, this diagram may be understood as a computation of the degree $\text{deg}(-)$ of the “positive mass” Frobenius-like object $\omega_E$ that yields an inequality $\text{deg}(\omega_E) \leq \text{deg}(\omega_{S^{\log}/C}) - \text{deg}(\omega_E)$, i.e.,

$$2 \cdot \text{deg}(\omega_E) \leq \text{deg}(\omega_{S^{\log}/C})$$

via a comparison of “alien copies” of the “positive mass” Frobenius-like object $\omega_E$ that lie on opposite sides of the “link” constituted by a deformation of the moduli/holomorphic structure of the family of elliptic curves under consideration. This comparison is performed by relating these Frobenius-like objects $\omega_E$ [or its dual] on either side of the deformation by means of the “zero mass” étale-like object $(\mathcal{E}, \nabla_{\mathcal{E}})$, i.e., which may be thought of as a local system on the open subscheme $U_S \subseteq S$ of the underlying scheme $S$ of $S^{\log}$ where the log structure of $S^{\log}$ is trivial. The tensor product with $\omega_{S^{\log}/C}$ — and the resulting appearance of the bound $\text{deg}(\omega_{S^{\log}/C})$ in the above inequality — may be understood as the indeterminacy that one must admit in order to achieve this comparison.

§ 2.10. Inter-universality: changes of universe as changes of coordinates

One fundamental aspect of the links [cf. the discussion of §2.7, (i)] — namely, the $\Theta$-link and log-link — that occur in inter-universal Teichmüller theory is their incompatibility with the ring structures of the rings and schemes that appear in their domains and codomains. In particular, when one considers the result of transporting an étale-like structure such as a Galois group [or étale fundamental group] across such
a link [cf. the discussion of §2.7, (iii)], one must abandon the interpretation of such a Galois group as a group of automorphisms of some ring or field structure [cf. [AbsTopIII], Remark 3.7.7, (i); [IUTchIV], Remarks 3.6.2, 3.6.3], i.e., one must regard such a Galois group as an abstract topological group that is not equipped with any of the "labelling structures" that arise from the relationship between the Galois group and various scheme-theoretic objects. It is precisely this state of affairs that results in

the quite central role played in inter-universal Teichmüller theory by results in

[mono-]anabelian geometry, i.e., by results concerned with reconstructing various scheme-theoretic structures from an abstract topological group that "just happens" to arise from scheme theory as a Galois group/étale fundamental group.

In this context, we remark that it is also this state of affairs that gave rise to the term "inter-universal": That is to say, the notion of a "universe", as well as the use of multiple universes within the discussion of a single set-up in arithmetic geometry, already occurs in the mathematics of the 1960’s, i.e., in the mathematics of Galois categories and étale topoi associated to schemes. On the other hand, in this mathematics of the Grothendieck school, typically one only considers relationships between universes — i.e., between labelling apparatuses for sets — that are induced by morphisms of schemes, i.e., in essence by ring homomorphisms. The most typical example of this sort of situation is the functor between Galois categories of étale coverings induced by a morphism of connected schemes. By contrast, the links that occur in inter-universal Teichmüller theory are constructed by partially dismantling the ring structures of the rings in their domains and codomains [cf. the discussion of §2.7, (vii)], hence necessarily result in

much more complicated relationships between the universes — i.e., between the labelling apparatuses for sets — that are adopted in the Galois categories that occur in the domains and codomains of these links, i.e., relationships that do not respect the various labelling apparatuses for sets that arise from correspondences between the Galois groups that appear and the respective ring/scheme theories that occur in the domains and codomains of the links.

That is to say, it is precisely this sort of situation that is referred to by the term "inter-universal". Put another way,

a change of universe may be thought of [cf. the discussion of §2.7, (i)] as a sort of abstract/combinatorial/arithmetic version of the classical notion of a "change of coordinates".

In this context, it is perhaps of interest to observe that, from a purely classical point of view, the notion of a [physical] "universe" was typically visualized as a copy of Euclidean three-space. Thus, from this classical point of view,
a “change of universe” literally corresponds to a “classical change of the coordinate system — i.e., the labelling apparatus — applied to label points in Euclidean three-space”!

Indeed, from an even more elementary point of view, perhaps the simplest example of the essential phenomenon under consideration here is the following purely combinatorial phenomenon: Consider the string of symbols

\[0\overline{1}0\]

— i.e., where “0” and “1” are to be understood as formal symbols. Then, from the point of view of the length two substring 01 on the left, the digit “1” of this substring may be specified by means of its “coordinate relative to this substring”, namely, as the symbol to the far right of the substring 01. In a similar vein, from the point of view of the length two substring 10 on the right, the digit “1” of this substring may be specified by means of its “coordinate relative to this substring”, namely, as the symbol to the far left of the substring 10. On the other hand,

neither of these specifications via “substring-based coordinate systems” is meaningful to the opposite length two substring; that is to say, only the solitary abstract symbol “1” is simultaneously meaningful, as a device for specifying the digit of interest, relative to both of the “substring-based coordinate systems”.

Finally, in passing, we note that this discussion applies, albeit in perhaps a somewhat trivial way, to the isomorphism of Galois groups \(\Psi_{\eta X} : G_K \cong G_K\) induced by the Frobenius morphism \(\Phi_{\eta X}\) in Example 2.6.1, (i): That is to say, from the point of view of classical ring theory, this isomorphism of Galois groups is easily seen to coincide with the identity automorphism of \(G_K\). On the other hand, if one takes the point of view that elements of various subquotients of \(G_K\) are equipped with labels that arise from the isomorphisms \(\rho\) or \(\kappa\) of Example 2.6.1, (ii), (iii), i.e., from the reciprocity map of class field theory or Kummer theory, then one must regard such labelling apparatuses as being incompatible with the Frobenius morphism \(\Phi_{\eta X}\). Thus, from this point of view, the isomorphism \(\Phi_{\eta X}\) must be regarded as a “mysterious, indeterminate isomorphism” [cf. the discussion of §2.7, (iii)].

§ 2.11. The two underlying combinatorial dimensions of a ring

Before proceeding, we pause to examine in more detail the two underlying combinatorial dimensions of a ring — i.e., constituted by addition and multiplication — is by means of semi-direct product
groups such as

\[ \mathbb{Z}_l \rtimes \mathbb{Z}_l^\times \text{ or } \mathbb{F}_l \rtimes \mathbb{F}_l^\times \]

— where \( l \) is a prime number; \( \mathbb{Z}_l \) denotes, by abuse of notation, the underlying additive profinite group of the ring “\( \mathbb{Z}_l \)” of \( l \)-adic integers; \( \mathbb{Z}_l^\times \) denotes the multiplicative profinite group of invertible \( l \)-adic integers; \( \mathbb{F}_l \), by abuse of notation, denotes the underlying additive group of the finite field “\( \mathbb{F}_l \)” of \( l \)-elements; \( \mathbb{F}_l^\times \) denotes the multiplicative group of the field \( \mathbb{F}_l \); \( \mathbb{Z}_l^\times \), \( \mathbb{F}_l^\times \) act on \( \mathbb{Z}_l \), \( \mathbb{F}_l \) via the ring structure of \( \mathbb{Z}_l \), \( \mathbb{F}_l \). Here, we note that both [the rings] \( \mathbb{Z}_l \) and \( \mathbb{F}_l \) are closely related to the fundamental ring \( \mathbb{Z} \). Indeed, \( \mathbb{Z} \) may be regarded as a dense subring of \( \mathbb{Z}_l \), while \( \mathbb{F}_l \) may be regarded as a “good finite discrete approximation” of \( \mathbb{Z} \) whenever \( l \) is “large” by comparison to the numbers of interest. Note, moreover, that if \( G_k \equiv \text{Gal}(\overline{k}/k) \) is the absolute Galois group of a mixed-characteristic local field [i.e., “MLF”] \( k \) of residue characteristic \( p \) for which \( \overline{k} \) is an algebraic closure, then the maximal tame quotient \( G_k \rightarrow G^\text{tm}_k \) is isomorphic to some open subgroup of the closed subgroup of the direct product

\[ \prod_{l' \neq p} \mathbb{Z}_{l'} \rtimes \mathbb{Z}_{l'}^\times \]

given by the inverse image via the quotient

\[ \left( \prod_{l' \neq p} \mathbb{Z}_{l'} \rtimes \mathbb{Z}_{l'}^\times \rightarrow \right) \prod_{l' \neq p} \mathbb{Z}_{l'}^\times \]

of the closed subgroup topologically generated by the image of \( p \) [cf. [NSW], Proposition 7.5.1]. Thus, if we assume that \( p \neq l \), then \( l \) may be thought of as one of the “\( l' \)’s” in the last two displays. In particular, from a purely cohomological point of view, the two combinatorial dimensions “\( \mathbb{Z}_l \)” and “\( \mathbb{Z}_l^\times \)” of the semi-direct product group \( \mathbb{Z}_l \rtimes \mathbb{Z}_l^\times \) — i.e., which correspond to the additive and multiplicative structures of the ring \( \mathbb{Z}_l \) — may be thought of as corresponding directly to the two \( l \)-cohomological dimensions [cf. [NSW], Theorem 7.1.8, (i)] of the profinite group \( G^\text{tm}_k \) or, equivalently [since \( l \neq p \)], of the profinite group \( G_k \). This suggests the point of view that the “restriction to \( l \)” should not be regarded as essential, i.e., that one should regard

the two underlying combinatorial dimensions of the ring \( k \) as corresponding to the two cohomological dimensions of its absolute Galois group \( G_k \),

and indeed, more generally, since the two cohomological dimensions of the absolute Galois group \( G_F \) of a [say, for simplicity, totally imaginary] number field \( F \) [cf. [NSW], Proposition 8.3.17] may be thought of, via the well-known classical theory of the Brauer group, as globalizations [cf. [NSW], Corollary 8.1.16; [NSW], Theorem 8.1.17] of the two cohomological dimensions of the absolute Galois groups of its [say, for simplicity, nonarchimedean] localizations, that one should regard
the two underlying combinatorial dimensions of a [totally imaginary] number field $F$ as corresponding to the two cohomological dimensions of its absolute Galois group $G_F$.

Moreover, in the case of the local field $k$, the two cohomological dimensions of $G_k$ may be thought of as arising [cf., e.g., the proof of [NSW], Theorem 7.1.8, (i)] from the one cohomological dimension of the maximal unramified quotient $G_k \to G_k^{unr} (\cong \hat{\mathbb{Z}})$ and the one cohomological dimension of the inertia subgroup $I_k \overset{\text{def}}{=} \text{Ker}(G_k \to G_k^{unr})$. Since $I_k$ and $G_k^{unr}$ may be thought of as corresponding, via local class field theory [cf. [NSW], Theorem 7.2.3], or, alternatively [i.e., "dually" — cf. [NSW], Theorem 7.2.6], via Kummer theory, to the subquotients of the multiplicative group $k^\times$ associated to $k$ given by the unit group $O_k^\times$ and the value group $k^\times/O_k^\times$ of $k$ [together with the corresponding subquotients associated to the various subextensions of $k$ in $\overline{k}$], we conclude that it is natural to regard

the two underlying combinatorial dimensions of the ring $k$, or, alternatively, the two cohomological dimensions of its absolute Galois group $G_k$, as corresponding to the natural exact sequence

$$1 \to O_k^\times \to k^\times \to k^\times/O_k^\times \to 1$$

— i.e., to the [non-split, i.e., at least when subject to the requirement of functoriality with respect to the operation of passing to finite extensions of $k$!] “decomposition” of $k^\times$ into its unit group $O_k^\times$ and value group $k^\times/O_k^\times$.

This situation is reminiscent of the [split!] decomposition of the multiplicative topological group $\mathbb{C}^\times$ associated to the field of complex numbers, i.e., which is equipped with a natural decomposition

$$\mathbb{C}^\times \cong S^1 \times \mathbb{R}_{>0}$$

as a direct product of its unit group $S^1$ and value group $\mathbb{R}_{>0}$ [i.e., the multiplicative group of positive real numbers].

§ 2.12. Mono-anabelian transport for mixed-characteristic local fields

The discussion of the two underlying combinatorial dimensions of a ring in §2.11 — especially, in the case of an MLF "$k$" — leads naturally, from the point of view of the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, to consideration of the following examples, which may be thought of as arithmetic analogues of the discussion in Step 7 of §1.5 of the effect of upper triangular linear transformations and rotations on "local masses". As one might expect from the discussion of §2.7, Kummer theory — i.e., applied to relate Frobenius-like structures.
to their étale-like counterparts — and cyclotomic rigidity isomorphisms play a central role in these examples. In the following examples, we use the notation of §2.11 for \( "k" \) and various objects related to \( k \); also, we shall write

\[
(\mathcal{O}_k^\times \subseteq \mathcal{O}_k^\geq \subseteq k^\times)
\]

for the topological multiplicative monoid of nonzero integral elements of \( k \),

\[
\mu_k \subseteq \mathcal{O}_k^\geq
\]

for the topological module of torsion elements of \( \mathcal{O}_k^\geq \), and

\[
\rho_k : k^\times \leftrightarrow (k^\times)^\wedge \sim \to G_k^{\text{ab}}
\]

for the composite of the embedding of \( k^\times \) into its profinite completion \((k^\times)^\wedge\) with the natural isomorphism [i.e., which arises from local class field theory] of \((k^\times)^\wedge\) with the abelianization \( G_k^{\text{ab}} \) of \( G_k \). Here, we recall, from local class field theory, that \( \rho_k \) is functorial with respect to passage to finite subextensions of \( k \) in \( \overline{k} \) and the Verlagerung homomorphism between abelianizations of open subgroups of \( G_k \). Also, we recall [cf., e.g., [AbsAnab], Proposition 1.2.1, (iii), (iv)] that the images \( \rho_k(\mu_k) \) \( \subseteq \rho_k(\mathcal{O}_k^\geq) \subseteq \rho_k(k^\times) \subseteq G_k^{\text{ab}} \) of \( \mu_k \), \( \mathcal{O}_k^\geq \), and \( k^\times \) via \( \rho_k \) may be constructed group-theoretically from the topological group \( G_k \). The notation introduced so far for various objects related to \( k \) will also be applied to finite subextensions of \( k \) in \( \overline{k} \), as well as [i.e., by passing to suitable inductive limits] to \( k \) itself. In particular, if we write

\[
\mu_{\overline{k}}(G_{k}) \subseteq \mathcal{O}_{\overline{k}}^\geq(G_{k}) \subseteq \overline{k}^\times(G_{k})
\]

for the respective inductive systems [or, by abuse of notation, when there is no fear of confusion, inductive limits], relative to the Verlagerung homomorphism between abelianizations of open subgroups of \( G_k \), of the [group-theoretically constructible!] submonoids \( \rho_k'(\mu_k') \subseteq \rho_k'(\mathcal{O}_k'^\geq) \subseteq \rho_k'(k'^\times) \subseteq G_{k'}^{\text{ab}} \) associated to the various open subgroups \( G_{k'} \subseteq G_k \) [i.e., where \( k' \) ranges over the finite subextensions of \( k \) in \( \overline{k} \)], then the various \( \rho_k' \) determine natural isomorphisms

\[
\rho_{\mu_{\overline{k}}} : \mu_{\overline{k}} \sim \to \mu_{\overline{k}}(G_{k}), \quad \rho_{\mathcal{O}_{\overline{k}}} : \mathcal{O}_{\overline{k}}^\geq \sim \to \mathcal{O}_{\overline{k}}^\geq(G_{k}), \quad \rho_{k^\times} : \overline{k}^\times \sim \to \overline{k}^\times(G_{k})
\]

of [multiplicative] \( G_{k'} \)-monoids. In the following, we shall also use the notation \( \mu_{\overline{k}}^{\mathcal{Z}} \overset{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\overline{k}}) \) and \( \mu_{\overline{k}}^{\mathcal{Z}}(G_{k}) \overset{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\overline{k}}(G_{k})) \).

**Example 2.12.1.** Nonarchimedean multiplicative monoids of local integers.

(i) In the following, we wish to regard the pair \( "G_k \curvearrowright \mathcal{O}_k^\geq" \) as an abstract ind-topological monoid \( "\mathcal{O}_k^\geq" \) [i.e., inductive system of topological monoids] equipped with
a continuous action by an abstract topological group \( "G_k" \). Thus, for instance, we may think of \( \bar{k}^\times \) as the groupification \((\mathcal{O}_F^\mathcal{p})_{\text{gp}}\) of the monoid \( \mathcal{O}_F^\mathcal{p} \), of \( \mu_{\bar{k}} \) as the subgroup of torsion elements of the monoid \( \mathcal{O}_F^\mathcal{p} \), and of \( \mathcal{O}_F^\mathcal{p} \subseteq k^\times \) as the result of considering the \( G_k \)-invariants \((\mathcal{O}_F^\mathcal{p})^{G_k} \subseteq (\bar{k}^\times)^{G_k} \) of the inclusion \( \mathcal{O}_F^\mathcal{p} \subseteq \bar{k}^\times \). Observe that, by considering the action of \( G_k \) on the various \( N \)-th roots, for \( N \) a positive integer, of elements of \( k^\times \), we obtain a natural Kummer map

\[
\kappa_k : k^\times \hookrightarrow H^1(G_k, \mu_{\bar{k}})
\]
— which may be composed with the natural isomorphism \( \rho_{\mu_{\bar{k}}} \) to obtain a natural embedding

\[
\kappa_k^{\text{Gal}} : k^\times \hookrightarrow H^1(G_k, \mu_{\bar{k}}(G_k))
\]
— where we note that the cohomology module in the codomain of this embedding may be constructed group-theoretically from the abstract topological group \( "G_k" \). On the other hand, it follows immediately from the definitions that \( \kappa_k \) may be constructed functorially from the abstract ind-topological monoid with continuous topological group action \( "G_k \lhd \mathcal{O}_F^\mathcal{p}" \).

(ii) In fact,

\( \rho_{\mu_{\bar{k}}} \) may also be constructed functorially from the abstract ind-topological monoid with continuous topological group action \( "G_k \lhd \mathcal{O}_F^\mathcal{p}" \).

Indeed, this follows formally from the fact that

there exists a canonical isomorphism \( \mathbb{Q}/\mathbb{Z} \cong H^2(G_k, \mu_{\bar{k}}) \) that may be constructed functorially from this data \( "G_k \lhd \mathcal{O}_F^\mathcal{p}" \).

— i.e., by applying this functorial construction to both the data \( "G_k \lhd \mathcal{O}_F^\mathcal{p}" \) and the data \( "G_k \lhd \mathcal{O}_F^\mathcal{p}(G_k)" \), and then observing that \( \rho_{\mu_{\bar{k}}} \) may be characterized as the unique isomorphism \( \mu_{\bar{k}} \cong \mu_{\bar{k}}(G_k) \) that is compatible with the isomorphisms \( \mathbb{Q}/\mathbb{Z} \cong H^2(G_k, \mu_{\bar{k}}) \) and \( \mathbb{Q}/\mathbb{Z} \cong H^2(G_k, \mu_{\bar{k}}(G_k)) \). To construct this canonical isomorphism [cf., e.g., the proof of [AbsAnab], Proposition 1.2.1, (vii); the statement and proof of [FrdII], Theorem 2.4, (ii); the statement of [AbsTopIII], Corollary 1.10, (i), (a); the statement of [AbsTopIII], Proposition 3.2, (i), for more details], we first observe that since \( \bar{k}^\times / \mu_{\bar{k}} \) is a \( \mathbb{Q} \)-vector space, it follows that we have natural isomorphisms \( H^2(G_k, \mu_{\bar{k}}) \cong H^2(G_k, \bar{k}^\times) \), \( H^2(I_k, \mu_{\bar{k}}) \cong H^2(I_k, \bar{k}^\times) \). Since, moreover, the inertia subgroup \( I_k \subseteq G_k \) is of cohomological dimension 1 [cf. the discussion of §2.11], we conclude that \( H^2(I_k, \bar{k}^\times) \cong H^2(I_k, \mu_{\bar{k}}) = 0 \). Next, let us recall that, by elementary Galois theory [i.e., “Hilbert’s Theorem 90”], one knows that \( H^1(I_k, \bar{k}^\times) = 0 \). Thus, we conclude from the Leray-Serre spectral sequence associated to the extension \( 1 \to I_k \to G_k \to G_k^{\text{unr}} \to 1 \) that,
if we write $k^{\text{unr}} \subseteq \overline{k}$ for the subfield of $k$-invariants of $\overline{k}$, then we have a natural isomorphism $H^2(G_k, \overline{k}^\times) \xrightarrow{\sim} H^2(G_k^{\text{unr}}, (k^{\text{unr}})^\times)$. On the other hand, the valuation map on $(k^{\text{unr}})^\times$ determines an isomorphism $H^2(G_k^{\text{unr}}, (k^{\text{unr}})^\times) \xrightarrow{\sim} H^2(G_k^{\text{unr}}, \mathbb{Z})$ [where again we apply “Hilbert’s Theorem 90”, this time to the residue field of $k$]. Moreover, by applying the isomorphism determined by the Frobenius element $G_k^{\text{unr}} \xrightarrow{\sim} \hat{\mathbb{Z}}$ [which is group-theoretically constructible — cf. [AbsAnab], Proposition 1.2.1, (iv)], together with the long exact sequence in Galois cohomology associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ [and the fact that $\mathbb{Q}$ is a $\mathbb{Q}$-vector space!], we obtain a natural isomorphism $H^2(G_k^{\text{unr}}, \mathbb{Z}) \xrightarrow{\sim} H^1(G_k^{\text{unr}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Thus, by taking the inverse of the composite of the various natural isomorphisms constructed so far [solely from the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}$!”], we obtain the desired canonical isomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \mu_\kappa)$.

(iii) The functorial construction given in (ii) — i.e., via well-known elementary techniques involving Brauer groups of the sort that appear in local class field theory — of the cyclotomic rigidity isomorphism $\rho_{\mu_\kappa}$ from the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}$” is perhaps the most fundamental case — at least in the context of the arithmetic of NF’s and MLF’s — of the phenomenon of cyclotomic rigidity. One formal consequence of the discussion of (i), (ii) is the fact that the operation of passing from the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}$” to the data “$G_k$” is fully faithful, i.e., one has a natural bijection

$$\text{Aut}(G_k \curvearrowright \mathcal{O}_k^\text{nr}) \xrightarrow{\sim} \text{Aut}(G_k)$$

— where the first “$\text{Aut}(-)$” denotes automorphisms of the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}$” consisting of an abstract ind-topological monoid with continuous topological group action; the second “$\text{Aut}(-)$” denotes automorphisms of the data “$G_k$” consisting of an abstract topological group [cf. [AbsTopIII], Proposition 3.2, (iv)]. Indeed, surjectivity follows formally from the functorial construction of the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}(G_k)$” from the abstract topological group $G_k$ [cf. the discussion at the beginning of the present §2.12]; injectivity follows formally from the fact that, as a consequence of the cyclotomic rigidity discussed in (ii), one has a functorial construction from the data “$G_k \curvearrowright \mathcal{O}_k^\text{nr}$” of the embedding $\kappa_k^\text{Gal} : k^\times \hookrightarrow H^1(G_k, \mu_\mathbb{Z}(G_k))$ [in fact applied in the case where “$k$” is replaced by arbitrary finite subextensions of $k$ in $\overline{k}$] into the container $H^1(G_k, \mu_\mathbb{Z}(G_k))$ [which may be constructed solely from the abstract topological group $G_k$]! Note that this situation may also be understood in terms of the general framework of mono-anabelian transport discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], by considering the commutative diagram

$$H^1(G_k, \mu_\mathbb{Z}(G_k)) \xrightarrow{\sim} H^1(G_k, \mu_\mathbb{Z}(G_k))$$

$$\downarrow \kappa_k^\text{Gal}|_{\mathcal{O}_k^\text{nr}} \quad \sim \quad \downarrow \kappa_k^\text{Gal}|_{\mathcal{O}_k^\text{nr}}$$

$$\mathcal{O}_k^\text{nr} \xrightarrow{\sim} \mathcal{O}_k^\text{nr}$$
— where the horizontal arrows are induced by some given automorphism of the data “$G_k \lhd \mathcal{O}_k^\times$”; the vertical arrows serve to embed the Frobenius-like data “$\mathcal{O}_k^\times$” into the étale-like container $H^1(G_k, \mu_k^\mathbb{Z}(G_k))$. Finally, we observe that the cyclotomic rigidity discussed in (ii) may be understood, relative to the exact sequence

$$1 \to \mathcal{O}_k^\times \to \mathbb{k}^\times \to \mathbb{k}^\times / \mathcal{O}_k^\times (\cong \mathbb{Q}) \to 1$$

— which, as was discussed in the final portion of §2.11, may be thought of as corresponding to the two underlying combinatorial dimensions of the ring $\mathbb{k}$ — as revolving around the rigidity of the two fundamental subquotients $\mu_k^\mathbb{Z} \subseteq \mathbb{k}^\times$ and $\mathbb{k}^\times \to \mathbb{k}^\times / \mathcal{O}_k^\times$ of $\mathbb{k}^\times$. When viewed in this light, the discussion of the present Example 2.12.1 may be thought of, relative to the analogy discussed in (ii), one may reason as follows. First, we observe that, by applying the functorial construction of (ii) in the case of the data “$G_k \lhd \mathcal{O}_k^\times (G_k)$”, one obtains a canonical isomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \mu_k^\mathbb{Z}(G_k))$. Since the cup product in group cohomology, together with this canonical isomorphism, determines a perfect duality [cf. [NSW], Theorem 7.2.6], one thus obtains a natural isomorphism $G_k^{ab} \xrightarrow{\sim} H^1(G_k, \mu_k^\mathbb{Z}(G_k))$. Write

$$\mathcal{O}_k^\times (G_k)^\text{Kum} \subseteq H^1(G_k, \mu_k^\mathbb{Z}(G_k))$$

for the image via this natural isomorphism of $\mathcal{O}_k^\times (G_k) \subseteq G_k^{ab}$ [i.e., the submodule of $G_k$-invariants of $\mathcal{O}_k^\times (G_k)$]. Thus, $\mathcal{O}_k^\times (G_k)^\text{Kum}$ may be constructed group-theoretically from the abstract topological group $G_k$. Next, let us recall the elementary fact that, relative to the natural inclusion $\mathbb{Q} \hookrightarrow \mathbb{Z} \otimes \mathbb{Q}$, we have an equality

$$\mathbb{Q}_{>0} \bigcap \hat{\mathbb{Z}}^\times = \{1\}$$

— where $\mathbb{Q}_{>0} \subseteq \mathbb{Q}$ denotes the multiplicative monoid of positive rational numbers. Now let us observe that it follows formally from this elementary fact — for instance, by considering the quotient $\mathcal{O}_k^\times (G_k) \to \mathcal{O}_k^\times (G_k)/\mathcal{O}_k^\times (G_k) (\cong \mathbb{N})$ by the submonoid of units $\mathcal{O}_k^\times (G_k) \subseteq \mathcal{O}_k^\times (G_k)$ — that the only element $\in \mathbb{Z}^\times$ that, relative to the natural action of $\hat{\mathbb{Z}}^\times$ on $H^1(G_k, \mu_k^\mathbb{Z}(G_k))$ [i.e., induced by the natural action of $\hat{\mathbb{Z}}^\times$ on $\mu_k^\mathbb{Z}(G_k)$], preserves the submonoid $\mathcal{O}_k^\times (G_k)$ is the identity element $1 \in \hat{\mathbb{Z}}^\times$. In particular, it follows that

the cyclotomic rigidity isomorphism $\rho_{\mu_k^\mathbb{Z}}$ may be characterized as the unique isomorphism $\mu_k^\mathbb{Z} \xrightarrow{\sim} \mu_k^\mathbb{Z}(G_k)$ that is compatible with the submonoids $\kappa_k(\mathcal{O}_k^\times) \subseteq H^1(G_k, \mu_k^\mathbb{Z})$ and $\mathcal{O}_k^\times (G_k)^\text{Kum} \subseteq H^1(G_k, \mu_k^\mathbb{Z}(G_k))$.
This characterization thus yields an alternative approach to the characterization of the cyclotomic rigidity isomorphism $\rho_{\mu_k}$ given in (ii) [cf. the discussion of [IUTchIII], Remark 2.3.3, (viii)]. On the other hand, there is a fundamental difference between this alternative approach and the approach of (ii): Indeed, one verifies immediately that the approach of (ii) is compatible with the profinite topology of $G_k$ in the sense that the construction of (ii) may be formulated as the result of applying a suitable limit operation to “finite versions” of this construction of (ii), i.e., versions in which “$G_k$” is replaced by the quotients of “$G_k$” by sufficiently small normal open subgroups of “$G_k$”, and “$O_k^\times$” is replaced by the submonoids of invariants with respect to such normal open subgroups. By contrast, the alternative approach just discussed is fundamentally incompatible with the profinite topology of $G_k$ in the sense that the crucial fact $\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$ — which may be thought of as a sort of discreteness property [cf. the discussion of [IUTchIII], Remark 3.12.1, (iii); [IUTchIV], Remark 2.3.3, (ii)] — may only be applied at the level of the full profinite group $G_k$ [i.e., at the level of Kummer classes with coefficients in some copy of $\hat{\mathbb{Z}}(1)$], not at the level of finite quotients of $G_k$ [i.e., at the level of Kummer classes with coefficients in some finite quotient of some copy of $\hat{\mathbb{Z}}(1)$]. Thus, in summary, although this alternative approach has the disadvantage of being incompatible with the profinite topology of $G_k$, various versions of this approach — i.e., involving constructions that depend, in an essential way, on the crucial fact $\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$ — will, nevertheless, play an important role in inter-universal Teichmüller theory [cf. the discussion of Example 2.13.1 below].

Example 2.12.2. Frobenius morphisms on nonarchimedean multiplicative monoids of local integers.

(i) One way to gain a further appreciation of the cyclotomic rigidity phenomenon discussed in Example 2.12.1 is to consider the pair “$G_k \curvearrowright O_k^\times”$, which again we regard as consisting of an abstract ind-topological monoid “$O_k^\times”” [i.e., inductive system of topological monoids] equipped with a continuous action by an abstract topological group “$G_k$”. Since $O_k^\times$ may be thought of as an inductive system/limit of profinite abelian groups, it follows immediately that there is a natural $G_k$-equivariant action of $\hat{\mathbb{Z}}^\times$ on the data “$G_k \curvearrowright O_k^\times””. Moreover, if $\alpha$ is an arbitrary automorphism of this data “$G_k \curvearrowright O_k^\times”” [i.e., regarded as an abstract ind-topological monoid equipped with a continuous action by an abstract topological group], then although it is not necessarily the case that $\alpha$ is compatible with the cyclotomic rigidity isomorphism $\rho_{\mu_k} : \mu_k \cong \mu_k(G_k)$, one verifies immediately [from the fact that, as an abstract abelian group, $\mu_k \cong \mathbb{Q}/\mathbb{Z}$, together with the elementary fact that $\text{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^\times$] that there always exists a unique element $\lambda \in \hat{\mathbb{Z}}^\times$ such that the automorphism $\lambda \cdot \alpha$ of the data “$G_k \curvearrowright O_k^\times”” is compatible with $\rho_{\mu_k}$. Thus, by arguing as in Example 2.12.1, (iii), one concludes that one has a natural
bijection

\[ \Aut(G_k \curvearrowright \mathcal{O}_k^\times) \cong \hat{\mathbb{Z}}^\times \times \Aut(G_k) \]

— where the first “Aut(−)” denotes automorphisms of the data “\(G_k \curvearrowright \mathcal{O}_k^\times\)” consisting of an abstract ind-topological monoid with continuous topological group action; the second “Aut(−)” denotes automorphisms of the data “\(G_k\)” consisting of an abstract topological group [cf. [AbsTopIII], Proposition 3.3, (ii); [FrdII], Remark 2.4.2]. Just as in the case of Example 2.12.1, (iii), this situation may also be understood in terms of the general framework of mono-anabelian transport discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], by considering the commutative diagram

\[
\begin{array}{ccc}
H^1(G_k, \mu_{\hat{\mathbb{Z}}_k}(G_k)) & \cong & H^1(G_k, \mu_{\hat{\mathbb{Z}}_k}(G_k)) \\
\uparrow^{\kappa_k^{\text{Gal}|_{\mathcal{O}_k^\times}}} & ? \cong ? & \uparrow^{\kappa_k^{\text{Gal}|_{\mathcal{O}_k^\times}}} \\
\mathcal{O}_k^\times & \cong & \mathcal{O}_k^\times
\end{array}
\]

— where the horizontal arrows are induced by some given automorphism of the data “\(G_k \curvearrowright \mathcal{O}_k^\times\)”; the vertical arrows serve to embed the Frobenius-like data “\(\mathcal{O}_k^\times\)” into the étale-like container \(H^1(G_k, \mu_{\hat{\mathbb{Z}}_k}(G_k))\); the diagram commutes [cf. “\(\cong ?\)”] up to the action of a suitable element \(\in \hat{\mathbb{Z}}^\times\).

(ii) Let \(\pi_k \in \mathcal{O}_k^\times\) be a uniformizer of \(\mathcal{O}_k\). Then one sort of intermediate type of data between the data “\(G_k \curvearrowright \mathcal{O}_k^\times\)” considered in (i) above and the data “\(G_k \curvearrowright \mathcal{O}_k^\boxtimes\)” considered in Example 2.12.1 is the data “\(G_k \curvearrowright \mathcal{O}_k^\times \cdot \mathcal{O}_k^\boxtimes (\subseteq \mathcal{O}_k^\boxtimes)\)”, which again we regard as consisting of an abstract ind-topological monoid “\(\mathcal{O}_k^\times \cdot \mathcal{O}_k^\boxtimes\)” [i.e., inductive system of topological monoids] equipped with a continuous action by an abstract topological group “\(G_k\)”. Here, we observe that \(\mathcal{O}_k^\times \cdot \mathcal{O}_k^\boxtimes = \mathcal{O}_k^\times \cdot \pi_k^N\). Let \(\mathbb{Z} \ni N \geq 2, \alpha \in \Aut(G_k \curvearrowright \mathcal{O}_k^\times)\). Then observe that \(N, \alpha\) determine — i.e., in the spirit of the discussion of §2.4 — a sort of Frobenius morphism \(\phi_{N,\alpha}\)

\[
\left(G_k \curvearrowright \mathcal{O}_k^\times \cdot \mathcal{O}_k^\boxtimes\right) \to \left(G_k \curvearrowright \mathcal{O}_k^\times \cdot \mathcal{O}_k^\boxtimes\right) \\
\pi_k \mapsto \pi_k^N
\]

that restricts to \(\alpha\) on the data “\(G_k \curvearrowright \mathcal{O}_k^\times\)”. From the point of view of the general framework of mono-anabelian transport discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], this sort of Frobenius morphism \(\phi_{N,\alpha}\) induces a commutative diagram

\[
\begin{array}{ccc}
H^1(G_k, \mu_{\hat{\mathbb{Z}}_k}(G_k)) & \cong & H^1(G_k, \mu_{\hat{\mathbb{Z}}_k}(G_k)) \\
\uparrow^{\kappa_k^{\text{Gal}|_{\mathcal{O}_k^\boxtimes}}} & ? \cong ? & \uparrow^{\kappa_k^{\text{Gal}|_{\mathcal{O}_k^\boxtimes}}} \\
\mathcal{O}_k^\boxtimes & \to & \mathcal{O}_k^\boxtimes
\end{array}
\]
— where the horizontal arrows are induced by $\phi_{N,\alpha}$; the vertical arrows serve to embed the Frobenius-like data “$\mathcal{O}_k^\times$” into the étale-like container $H^1(G_k, \mu_k^{\mathbb{Z}}(G_k))$; the diagram commutes [cf. “$\leadsto$”] up to the action of a suitable element $\in \mathbb{Z}$ on $\mathcal{O}_k^\times \subseteq \mathcal{O}_k^\mathbb{P}$ and a suitable element $\in \mathbb{N}$ [namely, $\mathbb{N} \subseteq \mathbb{N}$ on $\pi^N_k$]. Finally, we observe that the diagonal nature of the action of $\phi_{N,\alpha}$ on the unit group $\mathcal{O}_k^\times$ [via $\alpha$] and the value group $\pi^Z_k$ [by raising to the $N$-th power] portions of the ind-topological monoid $\mathcal{O}_k^\times \cdot \mathcal{O}_k^\mathbb{P}$ may be thought of, relative to the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, as corresponding to the discussion of the effect on “local masses” of the toral dilations that appeared in the discussion of Step 7 of §1.5.

**Example 2.12.3. Nonarchimedean logarithms.**

(i) The discussion of various simple cases of mono-anabelian transport in Examples 2.12.1, (iii); 2.12.2, (i), (ii), concentrated on the Kummer-theoretic aspects, i.e., in effect, on the Kummer-detachment indeterminacies [cf. §2.7, (vi)], or lack thereof, of the examples considered. On the other hand, another fundamental aspect of these examples [cf. the natural bijections of Examples 2.12.1, (iii); 2.12.2, (i)] is the étale-transport indeterminacies [cf. §2.7, (vi)] that occur as a result of the well-known existence of elements $\in \text{Aut}(G_k)$ that do not preserve the ring structure on $\mathcal{O}_k^\mathbb{P}$ (Gk)

— cf. [NSW], the Closing Remark preceding Theorem 12.2.7. By contrast, if $X$ is a hyperbolic curve of strictly Belyi type [cf. [AbsTopII], Definition 3.5] over $k$, and we write $\Pi_X$ for the étale fundamental group of $X$ [for a suitable choice of basepoint], then it follows from the theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9; [AbsTopIII], Remark 1.9.2; [AbsTopIII], Corollary 1.10], that

if one regards $G_k$ as a quotient $\Pi_X \rightarrow G_k$ of $\Pi_X$, then there exists a functorial algorithm for reconstructing this quotient $\Pi_X \rightarrow G_k$ of $\Pi_X$, together with the ring structure on [the union with \(\{0\}\) of] $\mathcal{O}_k^\mathbb{P}(G_k)$, from the abstract topological group $\Pi_X$.

Here, we recall that [it follows immediately from the definitions that] any connected finite étale covering of a once-punctured elliptic curve [i.e., an elliptic curve minus the origin] over $k$ that is defined over an NF is necessarily of strictly Belyi type.

(ii) Write $(\mathcal{O}_k^\times)^{\text{pf}}$ for the perfection [cf., e.g., [FrdII], §0] of the ind-topological monoid $\mathcal{O}_k^\times$. Thus, it follows immediately from the elementary theory of $p$-adic fields [cf., e.g., [Kobl], Chapter IV, §1, §2] that the $p$-adic logarithm determines a $G_k$-equivariant bijection

$$\log_k : (\mathcal{O}_k^\times)^{\text{pf}} \xrightarrow{\sim} k$$
with respect to which the operation of multiplication, which we shall often denote by the notation “\(\mathbb{E}\)”, in the domain corresponds to the operation of addition, which we shall often denote by the notation “\(\mathbb{H}\)”, in the codomain. This bijection fits into a diagram

\[
\ldots \quad \mathcal{O}_k^\mathcal{D} \supseteq \mathcal{O}_k^\mathcal{X} \quad \xrightarrow{\sim} \quad (\mathcal{O}_k^\mathcal{X})^{\text{pt}} \quad \xrightarrow{\kappa} \quad \overline{k} \supseteq \mathcal{O}_k^\mathcal{D} \quad \ldots
\]

— where the “\(\ldots\)” on the left and right denote the result of juxtaposing copies of the portion of the diagram “from \(\mathcal{O}_k^\mathcal{D}\) to \(\mathcal{O}_k^\mathcal{D}\)”, i.e., copies that are glued together along the initial/final instances of “\(\mathcal{O}_k^\mathcal{D}\)”. Here, we observe that the various objects that appear in this diagram may be regarded as being equipped with a natural action of \(\Pi_X\) [for \(X\) as in (i)], which acts via the natural quotient \(\Pi_X \to G_k\). To keep the notation simple, we shall denote the portion of the diagram “from \(\mathcal{O}_k^\mathcal{D}\) to \(\mathcal{O}_k^\mathcal{D}\)” by means of the notation \(\log: \mathcal{O}_k^\mathcal{D} \to \mathcal{O}_k^\mathcal{D}\). Thus, the diagram of the above display may be written

\[
\ldots \xrightarrow{\sim} \quad \Pi_X \quad \xrightarrow{\sim} \quad \Pi_X \quad \xrightarrow{\sim} \quad \Pi_X \quad \xrightarrow{\sim} \ldots
\]

\[
\ldots \xrightarrow{\log} \quad \mathcal{O}_k^\mathcal{D} \xrightarrow{\log} \quad \mathcal{O}_k^\mathcal{D} \xrightarrow{\log} \quad \mathcal{O}_k^\mathcal{D} \xrightarrow{\log} \ldots
\]

— i.e., regarded as a sequence of iterates of “\(\log\)”. Here, since the operation “\(\log\)” [i.e., which, in effect, converts “\(\mathbb{E}\)” into “\(\mathbb{H}\)”] is incompatible with the ring structures on [the union with \(\{0\}\) of] the copies of \(\mathcal{O}_k^\mathcal{D}\) in the domain and codomain of “\(\log\)”, we observe — in accordance with the discussion of §2.10! — that it is natural to regard the various copies of \(\mathcal{O}_k^\mathcal{D}\) as being equipped with distinct labels and the isomorphisms “\(\xrightarrow{\sim}\)” between different copies of \(\Pi_X\) as being indeterminate isomorphisms between distinct abstract topological groups. Such diagrams are studied in detail in [AbsTopIII], and, moreover, form the fundamental model for the \(\log\)-link of inter-universal Teichmüller theory [cf. §3.3, (ii), (vi), below], which is studied in detail in [IUTchIII].

(iii) It follows from the mono-anabelian theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9; [AbsTopIII], Corollary 1.10], that, if we regard \(G_k\) as a quotient of \(\Pi_X\), then the image, which we denote by \(\mathcal{O}_k^\mathcal{D}(\Pi_X) \subseteq H^1(G_k, \mu_k^\mathbb{Z}(G_k))\), of \(\mathcal{O}_k^\mathcal{D}\) via \(\kappa^\text{Gal}\) [cf. Example 2.12.1, (iii)] may be reconstructed — i.e., as a topological monoid equipped with a ring structure [on its union with \(\{0\}\)] — from the abstract topological group \(\Pi_X\). By applying this construction to arbitrary open subgroups of \(G_k\) and passing to inductive systems/limits, we thus obtain an ind-topological monoid \(\mathcal{O}_k^\mathcal{D}(\Pi_X)\) equipped with a natural continuous action by \(\Pi_X\) and a ring structure [on its union with \(\{0\}\)]. Thus, from the point of view of the general framework of mono-anabelian transport...
Shinichi Mochizuki discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], we obtain a diagram

\[ \mathcal{O}_k^\varphi(\Pi_X) \xrightarrow{\sim} \mathcal{O}_k^\varphi(\Pi_X) \]

\[ \cdots \xrightarrow{\text{Kum}} \xrightarrow{?} \xrightarrow{\text{Kum}} \cdots \]

\[ \mathcal{O}_k^\varphi \xrightarrow{\log} \mathcal{O}_k^\varphi \]

— where the upper horizontal arrow is induced by some indeterminate isomorphism \( \Pi_X \xrightarrow{\sim} \Pi_X \) [cf. the discussion of §2.10]; the lower horizontal arrow is the operation “\log” discussed in (ii); the vertical arrows are the “Kummer isomorphisms” determined by the various “\( \kappa_{\text{Gal}} \)” associated to open subgroups of \( G_k \); the “…” denote iterates of the square surrounding the “\( \sim ? \)”.

Thus, the vertical arrows of this diagram relate the various copies of Frobenius-like data \( \mathcal{O}_k^\varphi \) to the various copies of étale-like data \( \mathcal{O}_k^\varphi(\Pi_X) \), which are coric [cf. the discussion of §2.7, (iv)] with respect to the “link” constituted by the operation \( \log \).

(iv) The diagram of (iii) is, of course, far from being commutative [cf. the notation “\( \sim ? \)"], i.e., at least at the level of images of elements via the various composites of arrows in the diagram. On the other hand, if, instead of considering such images of elements via composites of arrows in the diagram, one considers regions [i.e., subsets] of \( \{0\} \) of the groupification of \( \mathcal{O}_k^\varphi \) or \( \mathcal{O}_k^\varphi(\Pi_X) \), then one verifies easily that the following observation holds:

Write

\[ \mathcal{I} \overset{\text{def}}{=} (2p)^{-1} \cdot \log_k(\mathcal{O}_k^\varphi) \subseteq k = \{0\} \cup (\mathcal{O}_k^\varphi)^{\text{sp}} \]

and \( \mathcal{I}(\Pi_X) \subseteq \{0\} \cup \mathcal{O}_k^\varphi(\Pi_X)^{\text{sp}} \) for the corresponding subset of the union with \( \{0\} \) of the groupification of \( \mathcal{O}_k^\varphi(\Pi_X) \). Then we have inclusions of “regions”

\[ \mathcal{O}_k^\varphi \subseteq \mathcal{I} \supseteq \log_k(\mathcal{O}_k^\varphi) \]

within \( \mathcal{I} \), as well as corresponding inclusions for \( \mathcal{I}(\Pi_X) \).

The compact “region” \( \mathcal{I} \), which is referred to as the log-shell, plays an important role in inter-universal Teichmüller theory. Note that one has both Frobenius-like [i.e., \( \mathcal{I} \)] and étale-like [i.e., \( \mathcal{I}(\Pi_X) \)] versions of the log-shell. Here, we observe that, from the point of view of the discussion of arithmetic holomorphic structures in §2.7, (vii), both of these versions are holomorphic in the sense that they depend, at least in an a priori sense, on “\( \log_k \)”, i.e., which is defined in terms of a power series that only makes sense if one is equipped with both “\( \boxplus \)” and “\( \boxtimes \)” [i.e., both the additive and multiplicative structures of the ring \( \bar{k} \)]. On the other hand, if one writes

\[ \mathcal{O}^{\times \mu} \]
for the quotient of “$\mathcal{O}^\times$” by its torsion subgroup [i.e., by the roots of unity], then $\log_k$ determines natural bijections of topological modules

$$\mathcal{O}_k^{\times,\mu} \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{I} \otimes \mathbb{Q}, \quad \mathcal{O}_k^{\times,\mu}(\Pi_X) \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{I}(\Pi_X) \otimes \mathbb{Q}$$

— i.e., within which the lattices $\mathcal{O}_k^{\times,\mu} \otimes (2p)^{-1}, \mathcal{O}_k^{\times,\mu}(\Pi_X) \otimes (2p)^{-1}$ correspond, respectively, to $\mathcal{I}, \mathcal{I}(\Pi_X)$. In particular, by applying these bijections,

we may think of the topological modules $\mathcal{I}, \mathcal{I}(\Pi_X)$ as objects constructed from the topological modules $\mathcal{O}_k^{\times,\mu}, \mathcal{O}_k^{\times,\mu}(G_k)$ [cf. the notation introduced at the beginning of the present §2.12; the notation of Example 2.12.1, (iv)], i.e., as objects constructed from mono-analytic structures [cf. the discussion of §2.7, (vii)].

That is to say, in addition to the holomorphic Frobenius-like and holomorphic étale-like versions of the log-shell discussed above, one may also consider mono-analytic Frobenius-like and mono-analytic étale-like versions of the log-shell. All four of these versions of the log-shell play an important role in inter-universal Teichmüller theory [cf. the discussion of §3.6, (iv), below; [IUTchIII], Definition 1.1, (i), (iv); [IUTchIII], Proposition 1.2, (v), (vi), (vii), (ix), (x); [IUTchIII], Remark 3.9.5, (vii), (Ob7); [IUTchIII], Remark 3.12.2, (iv), (v)]. Returning to the issue of the non-commutativity [i.e., “$\curvearrowright$?”] of the diagram of (iii), we observe the following:

the inclusions of “regions” discussed above may be interpreted as asserting that the holomorphic étale-like log shell $\mathcal{I}(\Pi_X)$ serves as a container for [i.e., as a “region” that contains] the images — i.e., of $\mathcal{O}_k^\times, \mathcal{O}_k^\times \subseteq \mathcal{O}_k^{\times,\mu}$, or, in the case of multiple iterates of $\log$, even smaller subsets of $\mathcal{O}_k^\times$ — via all possible composites of arrows of the diagram of (iii) [including the “…” on the left- and right-hand sides of the diagram!].

This property of the log-shell will be referred to as upper semi-commutativity [cf. [IUTchIII], Remark 1.2.2, (i), (iii)]. Thus, this property of upper semi-commutativity constitutes a sort of Kummer-detachment indeterminacy [cf. the discussion of §2.7, (vi)] and may be regarded as an answer to the question of computing the discrepancy between the two Kummer theories in the domain and codomain of the link “$\log$” [cf. the discussion at the beginning of §2.9]. Another important answer, in the context of inter-universal Teichmüller theory, to this computational question is given by the theory of log-volumes [i.e., where we use the term log-volume to refer to the natural logarithm of the volume $\in \mathbb{R}_{>0}$ of a region]:

There is a natural definition of the notion of the log-volume $\in \mathbb{R}$ of a region [i.e., compact open subset] of $k = \{0\} \cup (\mathcal{O}_k^\times)^{\text{gp}}$, which is normalized so that the
log-volume of $O_k$ is 0, while the log-volume of $p \cdot O_k$ is $-\log(p)$. This log-volume is compatible [in the evident sense] with passage between the four versions of log-shells discussed above, as well as with $\log$ in the sense that it assumes the same value $\in \mathbb{R}$ on regions that are mapped bijectively to one another via $\log_K$ [cf. [AbsTopIII], Proposition 5.7, (i); [IUTchIII], Proposition 1.2, (iii); [IUTchIII], Proposition 3.9, (i), (ii), (iv); [IUTchIII], Remark 3.9.4].

These properties of upper semi-commutativity and log-volume compatibility will be sufficient for the purposes of inter-universal Teichm"uller theory.

(v) Finally, we observe that since the operation $\log$ — which maps $\boxtimes \Rightarrow \boxplus$ and relates unit groups [cf. ($O_k^\times)^{pf}$] to value groups [i.e., nonzero non-units of $\overline{k}$] — may be thought of as an operation that "juggles", or "rotates", the two underlying combinatorial dimensions [cf. the discussion of §2.11] of the ring $\overline{k}$ [cf. [AbsTopIII], §I3], one may think of this operation $\log$, relative to the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, as corresponding to the discussion of the effect on "local masses" of the rotations that appeared in the discussion of Step 7 of §1.5.

§ 2.13. Mono-anabelian transport for monoids of rational functions

Let $k$ be either an MLF or an NF; $X$ a hyperbolic curve of strictly Belyi type [cf. [AbsTopII], Definition 3.5] over $k$; $\overline{K}_X$ an algebraic closure of the function field $K_X$ of $X$; $\overline{k} \subseteq \overline{K}_X$ the algebraic closure of $k$ determined by $\overline{K}_X$. Write $\mu_{\overline{k}} \subseteq \overline{k}$ for the group of roots of unity of $\overline{k}$; $\mu_{\overline{k}} \overset{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\overline{k}})$; $G_X \overset{\text{def}}{=} \text{Gal}(K_X/K_X)$; $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$; $G_X \rightarrow \Pi_X$ for the quotient of $G_X$ determined by the maximal subextension of $\overline{K}_X$ that is unramified over $X$ [so $\Pi_X$ may be thought of, for a suitable choice of basepoint, as the étale fundamental group of $X$]. Thus, when $k$ is an MLF, $X$ and $\Pi_X$ are as in Example 2.12.3, (i). Here, for simplicity, we assume further that $X$ is of genus $\geq 1$, and write $\Delta_X \overset{\text{def}}{=} \text{Ker}(\Pi_X \rightarrow G_k)$; $X^{\text{cp}}$ for the natural [smooth, proper] compactification of $X$; $\Delta_X \rightarrow \Delta_X^{\text{cp}}$ for the quotient of $\Delta_X$ by the cuspidal inertia groups of $\Delta_X$ [so $\Delta_X^{\text{cp}}$ may be naturally identified with the étale fundamental group, for a suitable basepoint, of $X^{\text{cp}} \times_k \overline{k}$];

$$M_X \overset{\text{def}}{=} \text{Hom}_{\hat{\mathbb{Z}}}(H^2(\Delta_X^{\text{cp}}, \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$$

[so $M_X (\cong \hat{\mathbb{Z}})$ is a cyclotome naturally associated to $\Delta_X$ — cf. [AbsTopIII], Proposition 1.4, (ii)]. Here, we recall that the quotient $\Delta_X \rightarrow \Delta_X^{\text{cp}}$, hence also the cyclotome $M_X$, may be constructed by means of a purely group-theoretic algorithm from the abstract topological group $\Pi_X$ [cf. [AbsTopI], Lemma 4.5, (v); [IUTchI], Remark 1.2.2, (ii)]. Now observe that
the Frobenius-like data that appears in the various examples [i.e., Examples 2.12.1, 2.12.2, 2.12.3] of mono-anabelian transport discussed in §2.12 only involve the two underlying combinatorial dimensions of [various portions of] the ind-topological monoid \("O_k^\approx\) of these examples.

That is to say, although, for instance, the étale-like data [i.e., \("\Pi_X\)" that appears in Example 2.12.3 involves the relative geometric dimension of \(X\) over \(k\) [i.e., in the case where \(k\) is an MLF], the Frobenius-like data [i.e., \("O_k^\approx\)"] that appears in these examples does not involve this geometric dimension of \(X\) over \(k\). On the other hand, in inter-universal Teichmüller theory, it will be of crucial importance [cf. the discussion of §3.4, §3.6, below; [IUTchIII], Remark 2.3.3] to consider such Frobenius-like data that involves the geometric dimension of \(X\) over \(k\) — i.e., in more concrete terms, to consider nonconstant rational functions on \(X\) — together with various evaluation operations that arise by evaluating such functions on various “special points” of \(X\). In fact, the fundamental importance of such evaluation operations may also be seen in the discussion of §2.14, (i), (ii), (iii), below. In the remainder of the present §2.13, we discuss what is perhaps the most fundamental example of cyclotomic rigidity and mono-anabelian transport for such geometric functions.

Example 2.13.1. Monoids of rational functions.

(i) In the following, we assume for simplicity that the field \(k\) is an NF. Recall that consideration of the first Chern class of a line bundle of degree 1 on \(X^{\mathrm{sp}}\) yields a natural isomorphism

\[
\lambda : \mu_k^\approx \cong M_X
\]

— cf., e.g., Example 2.6.1, (iii); [the evident NF version of] [Cusp], Proposition 1.2, (ii).

Next, observe that by considering the action of \(G_X\) on the various \(N\)-th roots, for \(N\) a positive integer, of elements of \(K_X^{\times} (= K_X \setminus \{0\}),\) we obtain a natural Kummer map

\[
\kappa_X : K_X^{\times} \hookrightarrow H^1(G_X, \mu_k^\approx)
\]

— which may be composed with the natural isomorphism \(\lambda\) to obtain a natural embedding

\[
\kappa_X^{\mathrm{Gal}} : K_X^{\times} \hookrightarrow H^1(G_X, M_X)
\]

— where we recall from the theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9] that the Galois group \(G_X\) [regarded up to inner automorphisms that arise from elements of \(\text{Ker}(G_X \twoheadrightarrow \Pi_X)\)], together with the cohomology module in the codomain of \(\kappa_X^{\mathrm{Gal}}\), the image of \(\kappa_X^{\mathrm{Gal}}\) in this cohomology module, and the field structure on the union \(K_X(\Pi_X)^{\mathrm{Kum}}\) of this image with \(\{0\},\) may be constructed group-theoretically from the abstract topological group \(\Pi_X\). Write \(G_X(\Pi_X)\) for “\(G_X\) regarded as an object
constructed in this way from $\Pi_X$; $$K_X(\Pi_X)^{\text{Kum}} \hookrightarrow G_X(\Pi_X)$$ for the inductive system/limit [which, by functoriality, is equipped with a natural action by $G_X(\Pi_X)$] of the result of applying this group-theoretic construction $\Pi_X \mapsto K_X(\Pi)^{\text{Kum}}$ to the various open subgroups of $G_X(\Pi_X)$.

(ii) Now let us regard the pair “$G_X \rightrightarrows K^\times_X$” as an abstract ind-monoid “$K^\times_X$” [i.e., inductive system of monoids] equipped with a continuous action by an abstract topological group “$G_X$” that arises, for some abstract quotient topological group “$G_X \twoheadrightarrow \Pi_X$”, as the topological group “$G_X(\Pi_X)$” of (i) [hence is only well-defined up to inner automorphisms that arise from elements of $\text{Ker}(G_X \twoheadrightarrow \Pi_X)$]. Thus, if we think of $\mu_\hat{k} \subseteq K^\times_X$ as the subgroup of torsion elements of the monoid $K^\times_X$, then, by considering the action of $G_X$ on the various $N$-th roots, for $N$ a positive integer, of elements of $K^\times_X$, we obtain the natural Kummer map $$\kappa_X : K^\times_X \rightarrow H^1(G_X, \mu_\hat{k})$$ discussed in (i). Moreover,

the $\{\pm 1\}$-orbit of the cyclotomic rigidity isomorphism $\lambda : \mu_\hat{k} \simeq M_X$ of (i) may be constructed functorially from the data “$G_X \rightrightarrows K^\times_X$” by applying the “alternative approach” discussed in Example 2.12.1, (iv), as follows [cf. [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi)]. Indeed, it follows formally from the elementary fact

$$\mathbb{Q}_{>0} \bigcap \hat{\mathbb{Z}}^\times = \{1\}$$

— for instance, by considering the various quotients $K^\times_X \twoheadrightarrow \mathbb{Z}$ determined by the discrete valuations of $K_X$ that arise from the closed points of $X_{\text{cp}}$, i.e., the quotients which, at the level of Kummer classes, are induced by restriction to the various cuspidal inertia groups [cf. the first display of [AbsTopIII], Proposition 1.6, (iii)] — that

the only isomorphisms $\mu_\hat{k} \simeq M_X$ that map the image of $\kappa_X$ into $K^\times_X(\Pi_X)^{\text{Kum}}$ (def $K_X(\Pi_X)^{\text{Kum}} \setminus \{0\}$) are the isomorphisms that belong to the $\{\pm 1\}$-orbit of $\lambda$.

As discussed in Example 2.12.1, (iv) [cf. also the discussion of [IUTchIII], Remark 2.3.3, (vii)], this approach to cyclotomic rigidity has the disadvantage of being incompatible with the profinite topology of $G_X$ [or $\Pi_X$].

(iii) We continue to use the notational conventions of (ii). Then observe that the functorial construction of the $\{\pm 1\}$-orbit of the cyclotomic rigidity isomorphism $\lambda$ given in (ii) may be interpreted in the fashion of Example 2.12.2, (i). That is to say, observe
that this functorial construction implies that if $\alpha$ is an arbitrary automorphism of the data $\mathcal{G}_X \curvearrowright \mathcal{K}_X \times X$, then either $\alpha$ or $-\alpha$ [i.e., the composite of $\alpha$ with the automorphism of the data $\mathcal{G}_X \curvearrowright \mathcal{K}_X \times X$ that raises elements of $\mathcal{K}_X$ to the power $-1$ and acts as the identity on $\mathcal{G}_X$] — but not both! — is compatible with $\lambda$, hence also with $\kappa_{\text{Gal}}$. In particular, by applying this observation to the various open subgroups of $\mathcal{G}_X$, one concludes that one has a natural bijection

$$\text{Aut}(\mathcal{G}_X \curvearrowright \mathcal{K}_X^x) \sim \{\pm 1\} \times \text{Aut}(\Pi_X)$$

— where the first “$\text{Aut}(-)$” denotes automorphisms of the data $\mathcal{G}_X \curvearrowright \mathcal{K}_X^x$ as described at the beginning of (ii); the second “$\text{Aut}(-)$” denotes automorphisms of the data “$\Pi_X$” consisting of an abstract topological group [cf. the discussion of [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi)]. Just as in the case of Example 2.12.2, (i), this situation may also be understood in terms of the general framework of mono-anabelian transport discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], by considering the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}_X^x(\Pi_X)_{\text{Kum}} & \sim & \mathcal{K}_X^x(\Pi_X)_{\text{Kum}} \\
\uparrow_{\text{Kum}} & \sim \uparrow_{\text{Kum}} & \\
\mathcal{K}_X^x & \sim & \mathcal{K}_X^x
\end{array}$$

— where $\mathcal{K}_X^x(\Pi_X)_{\text{Kum}} \overset{\text{def}}{=} \mathcal{K}_X(\Pi_X)_{\text{Kum}} \setminus \{0\}$; the horizontal arrows are induced by some given automorphism of the data “$\mathcal{G}_X \curvearrowright \mathcal{K}_X^x$”; the vertical arrows, which relate the Frobenius-like data $\mathcal{K}_X^x$ to the étale-like data $\mathcal{K}_X^x(\Pi_X)_{\text{Kum}}$, are the “Kummer isomorphisms” determined by the various “$\kappa_{\text{Gal}}$” associated to open subgroups of $\mathcal{G}_X$; the diagram commutes [cf. “$\sim$?”] up to the action of a suitable element $\in \{\pm 1\}$.

(iv) Finally, we pause to remark that one fundamental reason for the use of Kummer theory in inter-universal Teichmüller theory in the context of nonconstant rational functions [i.e., as in the discussion of the present Example 2.13.1] lies in the functoriality of Kummer theory with respect to the operation of evaluation of such functions at “special points” of $X$.

That is to say, [cf. the discussion of Example 2.6.1, (ii), (iii); §2.7, (vii)] although there exist many different versions — e.g., versions for “higher-dimensional fields” — of class field theory, these versions of class field theory do not satisfy such functoriality properties with respect to the operation of evaluation of functions at points [cf. the discussion of §2.14, §3.6, §4.2, below; [IUTchIV], Remark 2.3.3, (vi)].
§ 2.14. Finite discrete approximations of harmonic analysis

Finally, we conclude the present §2 by pausing to examine in a bit more detail the transition that was, in effect, made earlier in the present §2 in passing from derivatives [in the literal sense, as in the discussion of §2.5] to Galois groups/étale fundamental groups [i.e., as in the discussion of and subsequent to §2.6]. This transition is closely related to many of the ideas of the [scheme-theoretic] Hodge-Arakelov theory of [HASurI], [HASurII].

Example 2.14.1. Finite discrete approximation of differential calculus on the real line. We begin by recalling that the differential calculus of [say, infinitely differentiable] functions on the real line admits a finite discrete approximation, namely, by substituting

\[ \frac{df(x)}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} \sim f(X + 1) - f(X) \]

difference operators for derivatives [in the classical sense]. If \( d \) is a positive integer, then one verifies easily, by considering such difference operators in the case of polynomial functions of degree \( < d \) with coefficients \( \in \mathbb{Q} \), that evaluation at the elements \( 0, 1, \ldots, d - 1 \in \mathbb{Z} \subseteq \mathbb{Q} \) yields a natural isomorphism of \( \mathbb{Q} \)-vector spaces of dimension \( d \)

\[ \mathbb{Q}[X]^{<d} \left( \overset{\text{def}}{=} \bigoplus_{j=0}^{d-1} \mathbb{Q} \cdot X^j \right) \overset{\sim}{\rightarrow} \bigoplus_{0}^{d-1} \mathbb{Q} \]

— cf., e.g., the discussion of the well-known classical theory of Hilbert polynomials in [Harts], Chapter I, §7, for more details. In fact, it is not difficult to compute explicitly the “denominators” necessary to make this evaluation isomorphism into an isomorphism of finite free \( \mathbb{Z} \)-modules. This sort of “discrete function theory” [cf. also Example 2.14.2 below] may be regarded as the fundamental prototype for the various constructions of Hodge-Arakelov theory.

Example 2.14.2. Finite discrete approximation of Fourier analysis on the unit circle. In the spirit of the discussion of Example 2.14.1, we recall that classical function theory — i.e., in effect, Fourier analysis — on the unit circle \( S^1 \) admits a well-known finite discrete approximation: If \( d \) is a(n) [say, for simplicity] odd positive integer, so \( d^* \overset{\text{def}}{=} \frac{1}{2}(d - 1) \in \mathbb{Z} \), then one verifies easily that evaluation of polynomial functions of degree \( \in \{ -d^*, -d^* + 1, \ldots, -1, 0, 1, d^* - 1, d^* \} \) with coefficients \( \in \mathbb{Z} \) on the multiplicative group scheme \( \mathbb{G}_m^{\text{def}} = \text{Spec}(\mathbb{Z}[U, U^{-1}]) \) at [say, scheme-theoretic] points of the subscheme \( \mu_d \subseteq \mathbb{G}_m \) of \( d \)-torsion points of \( \mathbb{G}_m \) yields a natural isomorphism of finite free \( \mathbb{Z} \)-modules of rank \( d \)

\[ \bigoplus_{j=-d^*}^{d^*} \mathbb{Z} \cdot U^j \overset{\sim}{\rightarrow} \mathcal{O}_{\mu_d} \]
— where, by abuse of notation, we write $\mathcal{O}_{\mu_d}$ for the ring of global sections of the structure sheaf of the affine scheme $\mu_d$. If one base-changes via the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ into the field of complex numbers $\mathbb{C}$, then, when $d$ is sufficiently “large”, one may think of the \textit{totality of these $d$-torsion points}

$$\exp(2\pi i \cdot \frac{1}{d} \mathbb{Z}) = \left\{ \exp(2\pi i \cdot \frac{a}{d}), \exp(2\pi i \cdot \frac{1}{d}), \ldots, \exp(2\pi i \cdot \frac{(d-1)}{d}) \right\} \subseteq S^1$$

as a sort of \textbf{finite discrete approximation} of $S^1$ and hence, in particular, of \textit{adjacent pairs} of $d$-torsion points as “\textit{tangent vectors}” on $S^1$. That is to say, since [inverse systems of] such torsion points give rise to the \textit{étale fundamental group} of $\mathbb{G}_m \times \mathbb{Z} \mathbb{C}$, it is precisely this “\textit{picture}” of torsion points of $S^1$ that motivates the idea that

\textbf{Galois groups/étale fundamental groups} should be regarded as a sort of \textbf{arithmetic analogue} of the classical geometric notion of a \textit{tangent bundle}

— the discussion of §2.6. Moreover, if one regards $\mathbb{G}_m$ as the codomain of a \textit{nonzero function} [on some unspecified “space”], then this very classical “\textit{pictorial representation of a cyclotome}” [i.e., of torsion points of $S^1$] also explains, from a “pictorial point of view”, the importance of \textbf{cyclotomes} and \textbf{Kummer classes} in the discussion of §2.6, §2.7. That is to say,

\textbf{a Kummer class of a function}, which, so to speak, records the “\textit{arithmetic infinitesimal motion}” in $\mathbb{G}_m$ induced, via the function, by an “\textit{arithmetic infinitesimal motion}” in the space on which the function is defined may be thought of as a sort of “\textit{arithmetic logarithmic derivative}” of the function

— a point of view that is consistent with the usual point of view that the Kummer exact sequence in étale cohomology [i.e., which induces a connecting homomorphism in cohomology that computes the Kummer class of a function] should be thought of as a sort of “\textit{arithmetic logarithmic derivative}” of the function

Example 2.14.3. \textbf{Finite discrete approximation of harmonic analysis on complex tori.} Examples 2.14.1 and 2.14.2 admit a natural generalization to the case of \textbf{elliptic curves}. Indeed, let $E$ be an elliptic curve over a field $F$ of characteristic zero, $E^\dagger \to E$ the \textit{universal extension} of $E$, $\eta \in E(F)$ a [nontrivial] torsion point of order 2, $l \neq 2$ a prime number. Write $E[l] \subseteq E$ for the subscheme of \textit{l-torsion points}, $\mathcal{L} \overset{\text{def}}{=} \mathcal{O}_E(l \cdot [\eta])$ [where “$[\eta]$” denotes the effective divisor on $E$ determined by $\eta$]. Here, we recall that $E^\dagger \to E$ is an $A^1$-\textit{torsor} [so $E[l]$ may also be regarded as the subscheme $\subseteq E^\dagger$ of $l$-torsion points of $E^\dagger$]. In particular, it makes sense to speak of the sections $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{\leq l} \subseteq \Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})$ of $\mathcal{L}$ over $E^\dagger$ whose \textit{relative degree}, with respect to the
morphism $E^\dagger \to E$, is $< l$. Then the simplest version of the **fundamental theorem of Hodge-Arakelov theory** states that **evaluation** at the subscheme of $l$-torsion points $E[l] \subseteq E^\dagger$ yields a **natural isomorphism of $F$-vector spaces of dimension $l^2$**

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{< l} \sim \mathcal{L}|_{E[l]}$$

[cf. [HASurI], Theorem $A^\text{simple}$]. Moreover:

- When $F$ is an **NF**, this isomorphism is compatible, up to mild discrepancies, with natural integral structures on the LHS and RHS of the isomorphism at the nonarchimedean valuations of $F$ and with natural Hermitian metrics on the LHS and RHS of the isomorphism at the archimedean valuations of $F$.
- When $F$ is a complete discrete valuation field, and $E$ is a **Tate curve** over $F$, with special fiber isomorphic to $\mathbb{G}_m$, **Example 2.14.2** may be thought of as corresponding to the portion of the natural isomorphism of the above display that arises from the “special fiber of $E$”, while **Example 2.14.1** may be thought of as corresponding to the portion of the natural isomorphism of the above display that arises from the “special fiber of the relative dimension of the morphism $E^\dagger \to E$”.
- When $E$ is a **Tate curve**, the isomorphism, over $F$, of the above display may be interpreted as a result concerning the invertibility of the matrix determined by the values at the $l$-torsion points of certain **theta functions** associated to the Tate curve and their derivatives of order $< l$ [cf., §3.4, (iii), below; §3.6, (ii), below; [Fsk], §2.5; [EtTh], Proposition 1.4, for a review of the series representation of such theta functions].
- When $F$ is an **arbitrary field**, the isomorphism of the above display may be thought of as a sort of **discrete polynomial version** of the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, i.e., in the spirit of the discussion of Examples 2.14.1 and 2.14.2, above.
- When $F$ is an **NF**, the isomorphism of the above display may be thought of as a sort of **discrete globalized version** of the harmonic analysis involving “$\partial$, $\overset{\mathcal{O}}{\partial}$, Green’s functions, etc.” that appears at archimedean valuations in classical **Arakelov theory**. This is the reason for the appearance of the word “Arakelov” in the term “Hodge-Arakelov theory”. From this point of view, the computation of the discrepancy between natural integral structures/metrics on the LHS and RHS of the isomorphism of the above display may be thought of as a sort of computation of **analytic torsion** — a point of view that in some sense foreshadows the interpretation [cf. the discussion of §3.9, (iii), below] of inter-universal Teichmüller theory as the **computation** of a sort of **global arithmetic/Galois-theoretic** form of analytic torsion.
· The isomorphism of the above display may also be thought of as a sort of global ("function-theoretic"!) arithmetic version of the ("linear"!) comparison isomorphisms that occur in complex or \( p \)-adic Hodge theory. [That is to say, the LHS and RHS of the isomorphism of the above display correspond, respectively, to the "de Rham" and "étale" sides of comparison isomorphisms in \( p \)-adic Hodge theory.] This is the reason for the appearance of the word "Hodge" in the term "Hodge-Arakelov theory". This point of view gives rise to a natural definition for a sort of arithmetic version of the Kodaira-Spencer morphism discussed in §2.9, in which Galois groups play the role played by tangent bundles in the classical version of the Kodaira-Spencer morphism reviewed in §2.9 [cf. [HASurI], §1.4]. When \( E \) is a Tate curve over a complete discrete valuation field \( F \), this arithmetic Kodaira-Spencer morphism essentially coincides, when formulated properly, with the classical Kodaira-Spencer morphism reviewed in §2.9 [cf. [HASurII], Corollary 3.6].

· Relative to the point of view of "filtered crystals" [e.g., vector bundles equipped with a connection and filtration — cf. the data \((E, \nabla_E, \omega_E \subseteq E)\) of §2.9], the isomorphism of the above display may be thought of as a sort of discrete Galois-theoretic version of the "crystalline theta object" [cf. [HASurII], §2.3; the remainder of [HASurII], §2], i.e., in essence, the "nonlinear filtered crystal" constituted by the universal extension \( E^\dagger \) equipped with the ample line bundle \( L|_{E^\dagger} \), the natural crystal structure on \((E^\dagger, L|_{E^\dagger})\), and the "filtration" constituted by the morphism \( E^\dagger \to E \).

We refer to [HASurI], [HASurII], for more details concerning the ideas just discussed.

§ 3. Multiradiality: an abstract analogue of parallel transport

§ 3.1. The notion of multiradiality

So far, in §2, we have discussed various generalities concerning arithmetic changes of coordinates [cf. §2.10; the analogy discussed in §§2.2, §2.5, and §2.7 with the classical theory of §1.4 and §1.5], which are applied in effect to the two underlying combinatorial dimensions of a ring such as an MLF or an NF [cf. §2.7, (vii); §2.11; §2.12], and the approach to computing the effect of such arithmetic changes of coordinates — i.e., in the form of Kummer-detachment indeterminacies or étale-transport indeterminacies [cf. §2.7, (vi); §2.9] — by means of the technique of mono-anabelian transport [cf. §2.7, (v)]. By contrast, in the present §3, we turn to the issue of considering the particular arithmetic changes of coordinates that are of interest in the
Many aspects of these particular arithmetic changes of coordinates are highly reminiscent of the change of coordinates discussed in §1.6 from planar cartesian to polar coordinates. In some sense, the central notion that underlies the abstract combinatorial analogue, i.e., that is developed in inter-universal Teichmüller theory, of this change of coordinates from planar cartesian to polar coordinates is the notion of multiradiality.

(i) Types of mathematical objects: In the following discussion, we shall often speak of “types of mathematical objects”, i.e., such as groups, rings, topological spaces equipped with some additional structure, schemes, etc. This notion of a “type of mathematical object” is formalized in [IUTchIV], §3, by introducing the notion of a “species”. On the other hand, the details of this formalization are not so important for the following discussion of the notion of multiradiality. A “type of mathematical object” determines an associated category consisting of mathematical objects of this type — i.e., in a given universe, or model of set theory — and morphisms between such mathematical objects. On the other hand, in general, the structure of this associated category [i.e., as an abstract category!] contains considerably less information than the information that determines the “type of mathematical object” that one started with. For instance, if \( p \) is a prime number, then the “type of mathematical object” given by rings isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) [and ring homomorphisms] yields a category whose equivalence class as an abstract category is manifestly independent of the prime number \( p \).

(ii) Radial environments: A radial environment consists of a triple. The first member of this triple is a specific “type of mathematical object” that is referred to as radial data. The second member of this triple is a specific “type of mathematical object” that is referred to as coric data. The third member of this triple is a functorial algorithm that inputs radial data and outputs coric data; this algorithm is referred to as radial, while the resulting functor from the category of radial data to the category of coric data is referred to as the radial functor of the radial environment. We would like to think of the coric data as a sort of “underlying structure” of the “finer structure” constituted by the radial data and of the radial algorithm as an algorithm that forgets this “finer structure”, i.e., an algorithm that assigns to a collection of radial data the collection of underlying coric data of this given collection of radial data. We refer to [IUTchII], Example 1.7, (i), (ii), for more details.

(iii) Multiradiality and uniradiality: A radial environment is called multiradial if its associated radial functor is full. A radial environment is called uniradial if its associated radial functor is not full. One important consequence of the condition of multiradiality is the following switching property:

Consider the category of objects consisting of an ordered pair of collections of
radial data, together with an isomorphism between the associated collections of underlying coric data [and morphisms defined in the evident way]. Observe that this category admits a switching functor [from the category to itself] that assigns to an object of the category the object obtained by switching the two collections of radial data of the given object and replaces the isomorphism between associated collections of underlying coric data by the inverse to this isomorphism. Then multiradiality implies that the switching functor preserves the isomorphism classes of objects.

Indeed, one verifies immediately that multiradiality is in fact equivalent to the property that any object of the category discussed in the above display is, in fact, isomorphic to a “diagonal object”, i.e., an object given by considering an ordered pair of copies of a given collection of radial data, together with the identity isomorphism between the associated collections of underlying coric data — cf. the illustration of Fig. 3.1 below. We refer to [IUTchII], Example 1.7, (ii), (iii), for more details.

(iv) **Analogy with the Grothendieck definition of a connection:** Thus, in summary,

multiradiality concerns the issue of comparison between two collections of radial data that share a common collection of underlying coric data.

We shall often think of this sort of comparison as a comparison between two “holomorphic structures” that share a common “underlying real analytic structure” [cf. the examples discussed in §3.2 below]. Note that multiradiality may be thought of as a sort of abstract analogue of the notion of “parallel transport” or, alternatively, the Grothendieck definition of a connection [cf. the discussion of [IUTchII], Remark 1.7.1]. That is to say, given a scheme $X$ over a scheme $S$, the Grothendieck definition of a connection on an object $E$ over $X$ consists of an isomorphism between the fibers of $E$ at two distinct — but infinitesimally close! — points of $X$ that map to the same point of $S$. Thus, one may think of the fullness condition of multiradiality as the condition that there exist a sort of parallel transport isomorphism between two collections of radial data [i.e., corresponding to two “fibers”] that lifts a given isomorphism between collections of underlying coric data [i.e., corresponding to a path between the points over which the two fibers lie]. The indeterminacy in the choice of such a lifting may then be thought of, relative to this analogy with parallel transport, as a sort of “monodromy” associated to the multiradial environment.

(v) **The Kodaira-Spencer morphism via multiradiality:** The classical approach to proving the geometric version of the Szpiro Conjecture by means of the Kodaira-Spencer morphism was reviewed in §2.9. Here, we observe that this argument involving the Kodaira-Spencer morphism may be formulated in a way that
- renders explicit the analogy discussed in (iv) above between multiradiality and connections, and, moreover,
- renders explicit the relationship between this classical argument involving the Kodaira-Spencer morphism and the approach taken in inter-universal Teichmüller theory, that is to say, as a sort of “limiting case” or “degenerate version” of the argument [sketched in the Introduction to the present paper — cf. also the discussion of §2.3, §2.4] involving multiplication of the height “$h$” by a factor “$N$”, in the limit “$N \to \infty$” [in which case comparison between “$h$” and “$N \cdot h$”, or equivalently, between “$h$” and “$\frac{1}{N} \cdot h$”, becomes a comparison between “$h$” and “0”].

This formulation may be broken down into steps, as follows. Let $S^{\log}$ be as in §2.9, $\mathcal{L}$ a line bundle on $S$. Suppose that we are interested in bounding $\deg(\mathcal{L})$ [i.e., bounding the degree of $\mathcal{L}$ from above]. Then:

(1\textsuperscript{KS}) Write $p_1, p_2 : S \times S \to S$ for the natural projections from the direct product $S \times S$ to the first and second factors. Suppose that we are given an isomorphism

$$p_1^* \mathcal{L} \xrightarrow{\sim} p_2^* \mathcal{L}$$

of line bundles on $S \times S$ between the pull-backs of $\mathcal{L}$ via $p_1, p_2$. Then one verifies immediately, by restricting to various fibers of the direct product $S \times S$, that the existence of such an isomorphism implies that $\deg(\mathcal{L}) = 0$ [hence that $\deg(\mathcal{L})$ is bounded], as desired.

(2\textsuperscript{KS}) Write $S_\delta$ for the first infinitesimal neighborhood of the diagonal $(S \xrightarrow{\sim} ) \Delta_S \subseteq S \times S$. Suppose that we are given an isomorphism

$$p_1^* \mathcal{L}|_{S_\delta} \xrightarrow{\sim} p_2^* \mathcal{L}|_{S_\delta}$$

of line bundles on $S_\delta$ between the restrictions to $S_\delta$ of the two pull-backs via $p_1, p_2$ of $\mathcal{L}$. Since $S$ is proper [so any automorphism of the line bundle $\mathcal{L}$ on $S$ is
given by multiplication by a nonzero complex number], one verifies immediately that an isomorphism as in the above display may be thought of as a connection, in the sense of Grothendieck, on $L$ [cf. the discussion of (iv) above!]. In particular, since the base field $\mathbb{C}$ is of characteristic zero, we thus conclude again [from the elementary theory of de Rham-theoretic first Chern classes of line bundles on curves] that $\text{deg}(L) = 0$ [and hence that $\text{deg}(L)$ is bounded], as desired.

(3KS) Let $F$ be a rank two vector bundle that admits an exact sequence $0 \to L \to F \to L^{-1} \to 0$ of vector bundles on $S$. Thus, one may think of $F$ as a container for $L$. Write $S_\delta^{\log}$ for the first infinitesimal neighborhood of the “logarithmic diagonal” $(S_\delta^{\log} \cong \Delta_{S_\delta^{\log}} \subseteq S_\delta^{\log} \times S_\delta^{\log}$). Next, suppose that we are given an isomorphism

$$p_1^* L|_{S_\delta^{\log}} \cong p_2^* F|_{S_\delta^{\log}}$$

of vector bundles on [the underlying scheme of] $S_\delta^{\log}$ between the restrictions to $S_\delta^{\log}$ of the pull-backs via $p_1, p_2$ of $F$. [Thus, such an isomorphism arises, for instance, from a logarithmic connection on $F$.] Suppose, moreover, that this isomorphism has nilpotent monodromy, i.e., that the restriction of this isomorphism to each of the cusps of $S_\delta^{\log}$ differs from multiplication by a nonzero complex number by a nilpotent endomorphism of the fiber of $F$ at the cusp under consideration. Thus, we obtain two inclusions

$$p_1^* L|_{S_\delta^{\log}} \hookrightarrow p_1^* F|_{S_\delta^{\log}} \cong p_2^* F|_{S_\delta^{\log}} \hookleftarrow p_2^* L|_{S_\delta^{\log}}$$

[where the “$\cong$” is the isomorphism of the first display of the present (3KS)] of line bundles into a rank two vector bundle over $S_\delta^{\log}$; one verifies immediately that the images of these two inclusions coincide over the diagonal $\Delta_{S_\delta^{\log}} \subseteq S_\delta^{\log}$. That is to say,

the isomorphism $p_1^* F|_{S_\delta^{\log}} \cong p_2^* F|_{S_\delta^{\log}}$ allows one to use $F$ as a container for $L$ to compare the discrepancy between the two inclusions.

(4KS) Suppose that the images in $p_1^* F|_{S_\delta^{\log}} \cong p_2^* F|_{S_\delta^{\log}}$ of the two inclusions in the second display of (3KS) coincide. Then [since $S$ is proper — cf. the argument in (2KS)] the resulting isomorphism $p_1^* L|_{S_\delta^{\log}} \cong p_2^* L|_{S_\delta^{\log}}$ may be thought of as a logarithmic connection on $L$ with nilpotent monodromy, i.e., [since $L$ is of rank one!] a connection [without logarithmic poles!] on $L$. In particular, we are in the situation of (2KS), so we may conclude again that $\text{deg}(L) = 0$ [and hence that $\text{deg}(L)$ is bounded], as desired.
In general, of course, the images in $p_1^* \mathcal{F}|_{S_\delta^{\log}} \sim p_2^* \mathcal{F}|_{S_\delta^{\log}}$ of the two inclusions in the second display of (3KS) will not coincide. On the other hand, in this case [i.e., in which the images of the two inclusions do not coincide], one may consider [cf. the diagram in the second display of (3KS)] the composite

$$p_1^* \mathcal{L}|_{S_\delta^{\log}} \to p_1^* \mathcal{F}|_{S_\delta^{\log}} \sim p_2^* \mathcal{F}|_{S_\delta^{\log}} \to p_2^* \mathcal{L}^{-1}|_{S_\delta^{\log}}$$

[where the “$\to$” is the restriction to $S_\delta^{\log}$ of the given surjection $\mathcal{F} \to \mathcal{L}^{-1}$], whose restriction to $\Delta_{S_\delta^{\log}} \subseteq S_\delta^{\log}$ vanishes, hence determines a nonzero morphism of line bundles on $S$

$$\mathcal{L} \to \omega_{S_\delta^{\log}/\mathbb{C}} \otimes \mathcal{L}^{-1}$$

[where we recall that the ideal sheaf defining the closed [log] subscheme $\Delta_{S_\delta^{\log}} \subseteq S_\delta^{\log}$ may be naturally identified with the push-forward, via the natural inclusion $\Delta_{S_\delta^{\log}} \hookrightarrow S_\delta^{\log}$, of the sheaf of logarithmic differentials $\omega_{S_\delta^{\log}/\mathbb{C}}$]. Now one verifies immediately that, if one takes $\mathcal{F}$ to be the vector bundle “$\mathcal{E}$” of §2.9, equipped with the isomorphism as in the first display of (3KS) arising from the logarithmic connection $\nabla_{\mathcal{E}}$, and $\mathcal{L}$ to be the subbundle “$\omega_{\mathcal{E}} \subseteq \mathcal{E}$” of $\mathcal{E}$, then the nonzero morphism of the above display may be identified with the Kodaira-Spencer morphism $\omega_{\mathcal{E}} \to \tau_{\mathcal{E}} \otimes_{\mathcal{O}_S} \omega_{S_\delta^{\log}/\mathbb{C}}$ discussed in §2.9. Thus, in summary,

the Kodaira-Spencer morphism may be thought of as a measure of the discrepancy that arises when one fixes the “$\omega_{\mathcal{E}}$” on one factor of $S$ and compares it with the “$\omega_{\mathcal{E}}$” on a distinct, “alien” factor of $S$ by means of the common container “$\mathcal{E}$”, which is equipped with a connection “$\nabla_{\mathcal{E}}$” [i.e., an isomorphism as in the first display of (3KS)].

When formulated in this way, the Kodaira-Spencer morphism becomes manifestly analogous to the approach sketched in the Introduction to the present paper [cf. also the discussion of §2.3, §2.4] to bounding heights of elliptic curves [cf. the discussion of §3.7, (ii), (iv), below] by applying a suitable multiradiality property [cf. the discussion of §3.7, (i), below], i.e., [in the language of the Introduction] a “license to confuse”.

[Here, we note that, relative to the analogy with inter-universal Teichmüller theory, the situation that arises in (2KS), (4KS) corresponds to the [usual!] situation in which there actually exists a “global multiplicative subspace” — cf. the discussion of §2.3.] Finally, we remark in passing that the crystalline theta object referred to in the discussion of Example 2.14.3 may be thought of as a sort of intermediate stage between the situation discussed in (5KS) and the situation that is ultimately considered in inter-universal Teichmüller theory.
Fig. 3.2: The uniradiality of complex holomorphic structures

§ 3.2. Fundamental examples of multiradiality

The following examples may be thought of as fundamental prototypes of the phenomenon of multiradiality.

Example 3.2.1. Complex holomorphic structures on two-dimensional real vector spaces.

(i) Consider the radial environment in which the radial data is given by one-dimensional $\mathbb{C}$-vector spaces [and isomorphisms between such data], the coric data is given by two-dimensional $\mathbb{R}$-vector spaces [and isomorphisms between such data], and the radial algorithm assigns to a one-dimensional $\mathbb{C}$-vector space the associated underlying $\mathbb{R}$-vector space. Then one verifies immediately that this radial environment, shown in Fig. 3.2 above, is uniradial [cf. [Pano], Figs. 2.2, 2.3].

(ii) If $V$ is a two-dimensional real vector space, then write $\text{End}(V)$ for the $\mathbb{R}$-algebra of $\mathbb{R}$-linear endomorphisms of $V$ and $\text{GL}(V)$ for the group of invertible elements of $\text{End}(V)$. Observe that if $V$ is a two-dimensional real vector space, then a complex — i.e., “holomorphic” — structure on $V$ may be thought of as a homomorphism of $\mathbb{R}$-algebras $\mathbb{C} \rightarrow \text{End}(V)$. In particular, it makes sense to speak of a $\text{GL}$-orbit of complex structures on $V$, i.e., the set of $\text{GL}(V)$-conjugates of some such homomorphism. Now consider the radial environment in which a collection of radial data consists of a two-dimensional $\mathbb{R}$-vector space equipped with a $\text{GL}$-orbit of complex structures [and the morphisms between such data are taken to be the isomorphisms between such data], the coric data is the same as the coric data of (i), and the radial algorithm assigns to a two-dimensional $\mathbb{R}$-vector space equipped with a $\text{GL}$-orbit of complex structures the associated underlying $\mathbb{R}$-vector space. Then one verifies immediately that this radial environment, shown in Fig. 3.3 below, is [“tautologically”!] multiradial [cf. [Pano], Figs. 2.2, 2.3].

(iii) The examples of radial environments discussed in (i), (ii) are particularly of interest in the context of inter-universal Teichmüller theory in light of the relationship
between complex holomorphic structures as discussed in (i), (ii) and the geometry of the upper half-plane. That is to say, if, in the notation of (ii), we write \(GL(V) = GL^+(V) \coprod GL^-(V)\) for the decomposition of \(GL(V)\) determined by considering the sign of the determinant of an \(\mathbb{R}\)-linear automorphism of \(V\), then the space of moduli of complex holomorphic structures on \(V\) [i.e., the set of \(GL(V)\)-conjugates of a particular homomorphism of \(\mathbb{R}\)-algebras \(C \rightarrow \text{End}(V)\)] may be identified
\[
GL(V)/\mathbb{C}^\times \cong GL^+(V)/\mathbb{C}^\times \coprod GL^-(V)/\mathbb{C}^\times \cong \mathcal{H}^+ \coprod \mathcal{H}^-
\]
in a natural way with the disjoint union of the upper [i.e., \(\mathcal{H}^+\)] and lower [i.e., \(\mathcal{H}^-\)] half-planes. This observation is reminiscent of the deep connections between inter-universal Teichmüller theory and the hyperbolic geometry of the upper half-plane, as discussed in [BogIUT] [cf. also the discussion of §2.4, as well as of §3.10, (vi); §4.1, (i); §4.3, (iii), of the present paper]. This circle of ideas is also of interest, in the context of inter-universal Teichmüller theory, in the sense that it is reminiscent of the natural bijection
\[
\mathbb{C}^\times \setminus GL^+(V)/\mathbb{C}^\times \cong [0, 1) \ni \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto \frac{t}{t+1}
\]
[where \(\mathbb{R} \ni t \geq 1\)] between the space of double cosets on the left and the semi-closed interval \([0, 1)\) on the right, i.e., a bijection that is usually interpreted in classical complex Teichmüller theory as the map that assigns to a deformation of complex structure the dilation \(\in [0, 1)\) associated to this deformation [cf. [QuCnf], Proposition A.1, (ii)].

**Example 3.2.2.** Arithmetic fundamental groups of hyperbolic curves of strictly Belyi type over mixed-characteristic local fields.

(i) Let \(X \rightarrow \text{Spec}(k)\) and \(\Pi_X \rightarrow G_k\) be as in Example 2.12.3, (i). Consider the radial environment in which the radial data is given by topological groups \(\Pi\) that “just happen to be” abstractly isomorphic as topological groups to \(\Pi_X\) [and isomorphisms between topological groups], the coric data is given by topological groups \(G\) that “just
happen to be” abstractly isomorphic as topological groups to $G_k$ [and isomorphisms between topological groups], and the radial algorithm assigns to a topological group $\Pi$ the quotient group $\Pi \to G$ that corresponds to the [group-theoretically constructible! — cf. the discussion of Example 2.12.3, (i)] quotient $\Pi_X \to G_k$. Then one verifies immediately [cf. the two displays of Example 2.12.3, (i)!] that this radial environment, shown in Fig. 3.4 above, is uniradial.

(ii) In the remainder of the present (ii), we apply the notation “$\text{Aut}(\cdot)$” to denote the group of automorphisms of the topological group in parentheses. Consider the radial environment in which the radial data is given by triples

$$(\Pi, G, \alpha)$$

— where $\Pi$ is a topological group as in the radial data of (i), $G$ is a topological group as in the coric data of (i), and $\alpha$ is an $\text{Aut}(G)$-orbit of isomorphisms between $G$ and the quotient of $\Pi$ that corresponds to the [group-theoretically constructible!] quotient $\Pi_X \to G_k$ — [and isomorphisms between such triples], the coric data is the same as the coric data of (i), and the radial algorithm assigns to a triple $(\Pi, G, \alpha)$ the topological group $G$. Then one verifies immediately that this radial environment, shown in Fig. 3.5 above, is [“tautologically”!] multiradial [cf. [IUTchII], Example 1.8, (i)].
§ 3.3. The log-theta-lattice: $\Theta^{\pm \ell}F$-Hodge theaters, log-links, $\Theta$-links

The fundamental stage on which the constructions of inter-universal Teichmüller theory are performed is referred to as the log-theta-lattice.

(i) Initial $\Theta$-data: The log-theta-lattice is completely determined, up to isomorphism, once one fixes a collection of initial $\Theta$-data. Roughly speaking, this data consists of

- an elliptic curve $E_F$ over a number field $F$,
- an algebraic closure $\overline{F}$ of $F$,
- a prime number $l \geq 5$,
- a collection of valuations $V$ of a certain subfield $K \subseteq \overline{F}$, and
- a collection of valuations $V^\text{bad}_\text{mod}$ of a certain subfield $F^\text{mod} \subseteq \overline{F}$

that satisfy certain technical conditions — cf. [IUTchI], Definition 3.1, for more details. Here, we write $F^\text{mod} \subseteq F$ for the field of moduli of $E_F$, i.e., the field extension of $\mathbb{Q}$ obtained by adjoining the $j$-invariant of $E_F$; $K \subseteq \overline{F}$ for the extension field of $F$ generated by the fields of definition of the $l$-torsion points of $E_F$; $X_F \subseteq E_F$ for the once-punctured elliptic curve obtained by removing the origin from $E_F$; and $X_F \rightarrow C_F$ for the hyperbolic orbicurve obtained by forming the stack-theoretic quotient of $X_F$ by the natural action of $\{\pm 1\}$. Also, in the following, we shall write $V(-)$ for the set of all [nonarchimedean and archimedean] valuations of an NF “$(-)$” and append a subscripted element $\in V(-)$ to the NF to denote the completion of the NF at the element $\in V(-)$. We assume further that the following conditions are satisfied [cf. [IUTchI], Definition 3.1, for more details]:

- $F$ is Galois over $F^\text{mod}$ of degree prime to $l$ and contains the fields of definition of the 2-3-torsion points of $E_F$;
- the image of the natural inclusion $\text{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$ [well-defined up to composition with an inner automorphism] contains $SL_2(\mathbb{F}_l)$;
- $E_F$ has stable reduction at all of the nonarchimedean valuations of $F$;
- $C_K \overset{\text{def}}{=} C_F \times^F K$ is a $K$-core, i.e., does not admit a finite étale covering that is isomorphic to a finite étale covering of a Shimura curve [cf. [CanLift], Remarks 2.1.1, 2.1.2]; this condition implies that there exists a unique model $C^\text{mod}_{F^\text{mod}}$ of $C_F$ over $F^\text{mod}$ [cf. the discussion of [IUTchI], Remark 3.1.7, (i)];
- $V \subseteq V(K)$ is a subset such that the natural inclusion $F^\text{mod} \subseteq F \subseteq K$ induces a bijection $V \overset{\sim}{\rightarrow} V^\text{mod}$ between $V$ and the set $V^\text{mod} \overset{\text{def}}{=} V(F^\text{mod})$;
- $V^\text{bad}_\text{mod} \subseteq V^\text{mod}$ is a nonempty set of nonarchimedean valuations of odd residue characteristic over which $E_F$ has bad [i.e., multiplicative] reduction, that is to say, roughly speaking, the subset of the set of valuations where $E_F$ has bad
multiplicative reduction that will be “of interest” to us in the context of the constructions of inter-universal Teichmüller theory.

The above conditions in fact imply that $K$ is Galois over $F_{\text{mod}}$ [cf. [IUTchI], Remark 3.1.5]. We shall write

$$V_{\text{bad}}^{\text{mod}} \overset{\text{def}}{=} V_{\text{mod}} \times_{V_{\text{mod}}} V \subseteq V, \quad V_{\text{good}}^{\text{mod}} \overset{\text{def}}{=} V_{\text{mod}} \setminus V_{\text{bad}}^{\text{mod}}, \quad V^{\text{good}} \overset{\text{def}}{=} V \setminus V_{\text{bad}}$$

and apply the superscripts “non” and “arc” to $V$, $V_{\text{mod}}$, and $V(-)$ to denote the subsets of nonarchimedean and archimedean valuations, respectively. The data listed above determines, up to $K$-isomorphism [cf. [IUTchI], Remark 3.1.3], a finite étale covering $C_K \to C_K$ of degree $l$ such that the base-changed covering

$$X_K \overset{\text{def}}{=} C_K \times_{C_F} X_F \to X_K \overset{\text{def}}{=} X_F \times F K$$

arises from a rank one quotient $E_K[l] \to Q (\cong \mathbb{Z}/l\mathbb{Z})$ of the module $E_K[l]$ of $l$-torsion points of $E_K(K)$ [where we write $E_K \overset{\text{def}}{=} E_F \times F K$] which, at $v \in V_{\text{bad}}$, restricts to the quotient arising from coverings of the dual graph of the special fiber.

(ii) The log-theta-lattice: The log-theta-lattice, various versions of which are defined in [IUTchIII] [cf. [IUTchIII], Definitions 1.4; 3.8, (iii)], is a [highly noncommutative!] two-dimensional diagram that consists of three types of components, namely, •’s, ↑’s, and →’s [cf. the portion of Fig. 3.6 below that lies to the left of the “⊇”]. Each “•” in Fig. 3.6 represents a $\Theta^{\pm\text{ell}}NF$-Hodge theater, which may be thought of as a sort of miniature model of the conventional arithmetic geometry surrounding the given initial $\Theta$-data. Each vertical arrow “↑” in Fig. 3.6 represents a log-link, i.e., a certain type of gluing between various portions of the $\Theta^{\pm\text{ell}}NF$-Hodge theaters that constitute the domain and codomain of the arrow. Each horizontal arrow “→” in Fig. 3.6 represents a $\Theta$-link [various versions of which are defined in [IUTchI], [IUTchII], [IUTchIII]], i.e., another type of gluing between various portions of the $\Theta^{\pm\text{ell}}NF$-Hodge theaters that constitute the domain and codomain of the arrow. The portion of the log-theta-lattice that is ultimately actually used to prove the main results of inter-universal Teichmüller theory is shown in the portion of Fig. 3.6 — i.e., a sort of “infinite letter H” — that lies to the right of the “⊇”. On the other hand, the significance of considering the entire log-theta-lattice may be seen in the fact that — unlike the portion of Fig. 3.6 that lies to the right of the “⊇”! —

the [entire] log-theta-lattice is symmetric, up to unique isomorphism, with respect to arbitrary horizontal and vertical translations.

Various objects constructed from the •’s of the log-theta-lattice will be referred to as horizontally coric if they are invariant with respect to arbitrary horizontal transla-
tions, as \textbf{vertically coric} if they are \textit{invariant} with respect to arbitrary \textit{vertical} translations, and as \textbf{bi-coric} if they are \textit{both} horizontally and vertically coric. In this context, we observe that — unlike any \textit{finite} portion of a \textit{vertical line} of the log-theta-lattice! — each \textit{[infinite!] vertical line} of the log-theta-lattice is \textbf{symmetric}, up to \textit{unique isomorphism}, with respect to arbitrary \textit{vertical translations}.

As we shall see in §3.6, (iv), below, this is precisely why [cf. the portion of Fig. 3.6 that lies to the \textit{right} of the “⊇”] it will ultimately be necessary to work with the \textit{entire infinite vertical lines} of the log-theta-lattice [i.e., as opposed to with some finite portion of such a vertical line]. Finally, we remark that

the \textbf{two dimensions} of the log-theta-lattice may be thought of as corresponding to the \textbf{two underlying combinatorial dimensions} of a \textit{ring} [cf. the discussion of these two dimensions in the case of NF’s and MLF’s in §2.11], i.e., to \textbf{addition} and \textbf{multiplication}.

Indeed, the $\Theta$-\textbf{link} only involves the \textbf{multiplicative} structure of the rings that appear and, at an \textit{extremely rough level}, may be understood as corresponding to thinking of “numbers” as elements of the \textit{multiplicative monoid of positive integers}

$$\mathbb{N}_{\geq 1} \cong \bigoplus_{p} p^{\mathbb{N}}$$

— where $p$ ranges over the prime numbers, and $\mathbb{N}$ denotes the additive monoid of nonnegative integers — that is to say, as elements of an \textit{abstract monoid} that admits \textit{automorphisms} that \textit{switch distinct prime numbers} $p_1$, $p_2$, as well as \textit{endomorphisms} given by \textit{raising to the $N$-th power} [cf. the discussion of §2.4]. By contrast, the $\log$-\textbf{link} may be understood as corresponding to a \textit{link} between, or \textbf{rotation}/\textit{juggling} of, the
additive and multiplicative structures at the various completions of an NF that is obtained by means of the various natural logarithms defined on these completions [cf. the discussion of Example 2.12.3, (v)]. Here, we observe that the noncommutativity of the log-theta-lattice [which was mentioned at the beginning of the present (ii)] arises precisely from the fact that

the definition of the \( \Theta \)-link, which only involves the multiplicative structure of the rings that appear, is fundamentally incompatible with — i.e., only makes sense once one deactivates — the rotation/juggling of the additive and multiplicative structures that arises from the log-link.

In particular, the \( \Theta \)-link may only be defined if one distinguishes between the domain and codomain of the log-link, i.e., between distinct vertical coordinates in a single vertical line of the log-theta-lattice. Moreover,

this state of affairs, i.e., which requires one [in order to define the \( \Theta \)-link!] to distinguish the ring structures in the domain and codomain of the log-link [which are related in a non-ring-theoretic fashion to one another via the log-link!], makes it necessary to think of the [possibly tempered] arithmetic fundamental groups in the domain and codomain of the log-link as being related via indeterminate isomorphisms — i.e., as discussed in §2.10; Example 2.12.3, (ii) [cf. the discussion of [IUTchIII], Remark 1.2.2, (vi), (a); [IUTchIII], Remark 1.2.4, (i); [IUTchIV], Remark 3.6.3, (i)]. This situation may be understood by means of the analogy with the situation in complex Teichmüller theory:

one deforms one real dimension of the complex structure, while holding the other real dimension fixed — an operation that is only meaningful if these two distinct real dimensions are not subject to rotations, i.e., to indeterminacies with respect to the action of \( S^1 \subseteq \mathbb{C}^\times \) [cf. [IUTchI], Remark 3.9.3, (ii), (iii), (iv)].

Put another way, the portion of the log-theta-lattice that is “actually used” [cf. Fig. 3.6] exhibits substantial structural similarities to the natural bijection

\[
\mathbb{C}^\times \backslash GL^+(V) / \mathbb{C}^\times \xrightarrow{\sim} [0, 1)
\]

\[
\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto \frac{t-1}{t+1}
\]

[where \( \mathbb{R} \ni t \geq 1 \)] discussed in Example 3.2.1, (iii), that is to say:

the deformation of holomorphic structure \( \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right) \) may be thought of as corresponding to the single \( \Theta \)-link of this portion of the log-theta-lattice, while
the “$\mathbb{C}^\times$’s” on either side of the “$GL^+(V)$” may be thought of as corresponding, respectively, to the vertical lines of log-links — i.e., rotations of the arithmetic holomorphic structure! — on either side of the single $\Theta$-link.

In the context of this natural bijection discussed in Example 3.2.1, (iii), it is of interest to observe that this double coset space “$\mathbb{C}^\times \backslash GL^+(V) / \mathbb{C}^\times$” is also reminiscent of the double coset spaces associated to groups of matrices over $p$-adic fields that arise in the theory of Hecke correspondences. Alternatively, relative to the analogy with the two dimensions of an MLF, if the MLF under consideration is absolutely unramified, i.e., isomorphic to the quotient field of a ring of Witt vectors, then one may think of

- the log-link as corresponding to the Frobenius morphism in positive characteristic, i.e., to one of the two underlying combinatorial dimensions — namely, the slope zero dimension — of the MLF and of
- the $\Theta$-link as corresponding to the mixed characteristic extension structure of a ring of Witt vectors, i.e., to the transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$, which may be thought of as corresponding to the other of the two underlying combinatorial dimensions — namely, the positive slope dimension — of the MLF.

We refer to [IUTchI], §I4; [IUTchIII], Introduction; [IUTchIII], Remark 1.4.1, (iii); [IUTchIII], Remark 3.12.4; [Pano], §2, for more details concerning the numerous analogies between inter-universal Teichmüller theory and various aspects of the $p$-adic theory, such as the canonical liftings that play a central role in the $p$-adic Teichmüller theory of $[pOrd]$, $[pTch]$, $[pTchIn]$.

(iii) **The notion of a Frobenioid:** A Frobenioid is an abstract category whose abstract categorical structure may be thought of, roughly speaking, as encoding the theory of divisors and line bundles on various “coverings” — i.e., normalizations in various finite separable extensions of the function field — of a given normal integral scheme.

Here, the category of such “coverings” is referred to as the base category of the Frobenioid. All of the Frobenioids that play a [non-negligible] role in inter-universal Teichmüller theory are model Frobenioids [cf. FrdII, Theorem 5.2] whose base category corresponds to “some sort of” — that is to say, possibly tempered, in the sense of [André], §4; [Semi], Example 3.10 — arithmetic fundamental group [i.e., in the non-tempered case, the étale fundamental group of a normal integral scheme of finite type over some sort of “arithmetic field”]. In particular, all of the Frobenioids that play a [non-negligible] role in inter-universal Teichmüller theory are essentially equivalent to a collection of data as follows that satisfies certain properties:
· a topological group, i.e., the [possibly tempered] arithmetic fundamental group;
· for each open subgroup of the topological group, an abelian group, called the rational function monoid, i.e., since it is a category-theoretic abstraction of the multiplicative group of rational functions on the “covering” corresponding to the given open subgroup;
· for each open subgroup of the topological group, an abelian monoid, called the divisor monoid, i.e., since it is a category-theoretic abstraction of the monoid of Weil divisors on the “covering” corresponding to the given open subgroup.

In particular, such Frobenioids may be thought of as category-theoretic abstractions of various aspects of the multiplicative portion of the ring structure of a normal integral scheme. We refer to §3.5 below for more remarks on the use of Frobenioids in inter-universal Teichmüller theory.

(iv) $[\Theta^{\pm \text{ell}} \text{NF}^-]$Hodge theaters as “tautological solutions” to a purely combinatorial problem: The $\Theta^{\pm \text{ell}} \text{NF}^-\text{Hodge theater}$ associated to a given collection of initial $\Theta$-data as in (i) is a somewhat complicated system of Frobenioids [cf. [IUTchI], Definition 6.13, (i)]. The topological group data for these Frobenioids arises from various subquotients of the [possibly tempered] arithmetic fundamental groups of the hyperbolic orbicurves discussed in (i). The rational function monoid data for these Frobenioids arises from the multiplicative groups of nonzero elements of various finite extensions of the number field $F_{\text{mod}}$ of (i) or localizations [i.e., completions] of such NF’s at valuations lying over valuations $\in \mathcal{V}$. The divisor monoid data for these Frobenioids arises, in the case of NF’s, from the monoid of effective arithmetic divisors [in the sense of diophantine geometry — cf., e.g., [GenEll], §1; [FrdII], Example 6.3], possibly with real coefficients, and, in the case of localizations of NF’s, from the nonnegative portion of the value group of the associated valuation, possibly tensored over $\mathbb{Z}$ with $\mathbb{R}$. [In fact, at valuations in $\mathcal{V}_{\text{bad}}$, an additional type of Frobenioid, called a tempered Frobenioid, also appears — cf. the discussion of §3.4, (iv); §3.5, below.] For instance, in the case of localizations at valuations $\in \mathcal{V}_{\text{good}} \cap \mathcal{V}_{\text{non}}$, one Frobenioid that appears quite frequently in inter-universal Teichmüller theory consists of data that is essentially equivalent to the data

$$\text{“} \Pi_X \hookrightarrow \mathcal{O}_K^{\geq} \text{”}$$

considered in Example 2.12.3, (ii). [In the case of valuations $\in \mathcal{V}_{\text{bad}}$, “$\Pi_X$” is replaced by the corresponding tempered arithmetic fundamental group; in the case of valuations $\in \mathcal{V}_{\text{arc}} (\subseteq \mathcal{V}_{\text{good}})$, one applies the theory of [AbsTopIII], §2.] In general, the Frobenioids obtained by applying the operation of “passing to real coefficients” are referred to as realified Frobenioids [cf. [FrdII], Proposition 5.3]. The system of Frobenioids that
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constitutes a $\Theta^{\pm \ell}NF$-Hodge theater determines, by passing to the associated system of base categories, an apparatus that is referred to as a $D-\Theta^{\pm \ell}NF$-Hodge theater [cf. [IUTchI], Definition 6.13, (ii)]. The purpose of considering such systems of Frobenioids lies in the goal of

reassembling the distribution of primes in the number field $K$ [cf. the discussion of [IUTchI], Remark 4.3.1] in such a way as to render possible the construction of some sort of global version of the “Gaussian distribution”

$$\{q_j^{2}\}_{j=1,...,l^*}$$

discussed at the end of §2.4, i.e., which, a priori, is only defined at the valuations $\in \mathbb{V}(K)^{\text{non}}$ at which $E_K$ has bad multiplicative reduction [such as, for instance, the valuations $\in \mathbb{V}_{\text{bad}}$].

This “global version” amounts to the local and global value group portions of the data that appears in the domain portion of the $\Theta$-link [cf. (vii) below]. The reassembling, referred to above, of the distribution of primes in the number field $K$ was one of the fundamental motivating issues for the author in the development of the absolute monoid-anabelian geometry of [AbsTopIII], i.e., of a version of anabelian geometry that differs fundamentally from the well-known anabelian result of Neukirch-Uchida concerning absolute Galois groups of NF’s [cf., e.g., [NSW], Chapter XII, §2] in numerous ways, but, in particular, in] that its reconstruction of an NF does not depend on the distribution of primes in the NF [cf. the discussion of [IUTchI], Remarks 4.3.1, 4.3.2]. The problem, referred to above, of constructing a sort of “global Gaussian distribution” may in fact easily be seen to be

essentially equivalent to the “purely combinatorial” problem of constructing a “global multiplicative subspace” [cf. the discussion of §2.3], together with a “global canonical generator”, i.e., more precisely: a one-dimensional $\mathbb{F}_l$-subspace of the two-dimensional $\mathbb{F}_l$-vector space $E_K[l]$ of $l$-torsion points of the elliptic curve $E_K$, together with a generator, well-defined up to multiplication by $\pm 1$, of the quotient of $E_K[l]$ by this one-dimensional $\mathbb{F}_l$-subspace, such that this subspace and generator coincide, at the valuations $\in \mathbb{V}(K)$ that lie over valuations $\in \mathbb{V}_{\text{bad}}$, with a certain canonical such subspace and generator that arise from a generator [again, well-defined up to multiplication by $\pm 1$] of the Galois group [isomorphic to $\mathbb{Z}$] of the well-known infinite covering of a Tate curve.

Here, we note that such a “global canonical generator” determines a bijection, which is well-defined up to multiplication by $\pm 1$, of the quotient referred to above with the
underlying additive group of $\mathbb{F}_l$. In a word, the combinatorial structure of a $\Theta^{\pm\ell}NF$-Hodge theater furnishes a sort of “tautological solution” to the purely combinatorial problem referred to above by

simply ignoring the valuations $\in \mathcal{V}(K) \setminus \mathcal{V}$, for instance, by working only with Frobenioids — i.e., in effect, arithmetic divisors/line bundles — that arise from arithmetic divisors supported on the set of valuations $\mathcal{V}(K) \setminus \mathcal{V}$, i.e., as opposed to on the entire set $\mathcal{V}(K)$

[cf. the discussion of [IUTchI], Remark 4.3.1]. Here, we note that this sort of operation of discarding certain of the primes of an NF can only be performed if one forgets the additive structure of an NF [i.e., since a sum of elements of an NF that are invertible at a given nonempty set of primes is no longer necessarily invertible at those primes! — cf. [AbsTopIII], Remark 5.10.2, (iv)] and works only with multiplicative structures, e.g., with Frobenioids. Collections of local data — consisting, say, of local Frobenioids or local [possibly tempered] arithmetic fundamental groups — indexed by the elements of $\mathcal{V}$ are referred to as prime-strips [cf. Fig. 3.7 above; [IUTchI], Fig. I1.2, and the surrounding discussion]. In a word, prime-strips may be thought of as a sort of monoid- or Galois-theoretic version of the classical notion of adèles/idèles.

(v) The symmetries of a $\Theta^{\pm\ell}NF$-Hodge theater: Once the “tautological so-
“solution” furnished by the combinatorial structure of a $\Theta^{\pm}\text{ell}NF$-Hodge theater is applied, the quotient of $E_K[l]$ discussed in (iv) corresponds to the quotient “$Q$” of (i), i.e., in effect, to the set of cusps of the hyperbolic curve $\overline{X}_K$ of (i). One may then consider additive and multiplicative symmetries

$$F_l^{\times\pm} \overset{\text{def}}{=} F_l \times \{\pm 1\}, \quad F_l^* \overset{\text{def}}{=} F_l^{\times}/\{\pm 1\}$$

— where $F_l^{\times} \overset{\text{def}}{=} F_l \setminus \{0\}$, and $\pm 1$ acts on $F_l$ in the usual way — on the underlying sets

$$F_l = \{-l^*,\ldots,-1,0,1,\ldots,l^*\}, \quad F_l^* = \{1,\ldots,l^*\}$$

— where $l^* \overset{\text{def}}{=} (l - 1)/2$; the numbers listed in the above display are to be regarded modulo $l$; we think of $F_l$ as the quotient $Q$ [i.e., the set of cusps of $\overline{X}_K$ and its localizations at valuations $\in \mathcal{V}$] discussed above and of $F_l^*$ as a certain subquotient of $Q$. Here, we remark that this interpretation of the quotient $Q$ as a set of cusps of $\overline{X}_K$ induces a natural outer isomorphism of $F_l^{\times\pm}$ with the group of “geometric automorphisms”

$$\text{Aut}_K(\overline{X}_K)$$

[i.e., the group of $K$-automorphisms of the $K$-scheme $\overline{X}_K$], as well as a natural isomorphism of $F_l^*$ with a certain quotient of the image of the group of “arithmetic automorphisms”

$$\text{Aut}(\mathcal{C}_K) \hookrightarrow \text{Gal}(K/F_{\text{mod}})$$

[i.e., the group of automorphisms of the algebraic stack $\mathcal{C}_K$, which, as is easily verified, maps injectively into the Galois group $\text{Gal}(K/F_{\text{mod}})$]. The combinatorial structure of a $\Theta^{\pm}\text{ell}NF$-Hodge theater may then be summarized as

the system of Frobenioids obtained by localizing and gluing together various Frobenioids or [possibly tempered] arithmetic fundamental groups associated to $\overline{X}_K$ and $\mathcal{C}_K$ in the fashion prescribed by the combinatorial recipe

$$F_l^{\times\pm} \approx F_l \supseteq F_l^{\times} \to F_l^* \bowtie F_l^*$$

— cf. Fig. 3.8 below.

Here, we remark that, in Fig. 3.8:

- the squares with actions by $F_l^{\times\pm}$ and $F_l^*$ correspond to Frobenioids or arithmetic fundamental groups that arise from $\overline{X}_K$ and $\mathcal{C}_K$, respectively;
- each of the elements of $F_l$ or $F_l^*$ that appears in parentheses “(...)” corresponds to a single prime-strip;
- each portion enclosed in brackets “[...]” corresponds to a single prime-strip;
\[
\begin{bmatrix}
-l^* < \ldots < -1 < 0 \\
< 1 < \ldots < l^*
\end{bmatrix}
\quad \begin{bmatrix}
1 < \ldots \\
< l^*
\end{bmatrix}
\]

\[
\uparrow \quad \Rightarrow \text{glue!} \leftarrow \uparrow
\]

\[
\{\pm 1\} \quad \begin{bmatrix}
-l^* < \ldots < -1 < 0 \\
< 1 < \ldots < l^*
\end{bmatrix}
\quad \begin{bmatrix}
1 < \ldots \\
< l^*
\end{bmatrix}
\]

\[
\downarrow \quad \begin{bmatrix}
\pm \rightarrow \pm \\
\equiv \pm \downarrow
\end{bmatrix}
\quad \begin{bmatrix}
\star \rightarrow \star \\
\equiv \star \downarrow
\end{bmatrix}
\]

\ldots \text{cf. ordinary monodromy,} \quad \text{...cf. supersingular monodromy,}

\text{additive symmetries!} \quad \text{toral symmetries!}

Fig. 3.8: The combinatorial structure of a \(\Theta^{\pm \ell}NF\)-Hodge theater: a bookkeeping apparatus for \(l\)-torsion points

- the arrows “\(\uparrow\)” correspond to the relation of passing from the various \textit{individual elements} of \(F_l\) or \(F^*_l\) [i.e., one prime-strip for each individual element] to the \textit{entire set} \(F_l\) or \(F^*_l\) [i.e., one prime-strip for the entire set];
- the arrows “\(\downarrow\)” correspond to the relation of regarding the set of elements in parentheses “(…)” with \textbf{fixed labels} as the underlying set of a set equipped with an \textbf{action} by \(F^*_{\pm}\) or \(F^*_{l}\);
- the \textbf{gluing} is the gluing prescribed by the \textbf{surjection} \(F_l \supseteq F^*_l \rightarrow F^*_{l}\).

In this context, it is important to keep in mind that

the \(F^*_{l\pm}\) and \(F^*_{l}\)-symmetries of a \(\Theta^{\pm \ell}NF\)-Hodge theater play a \textit{fundamental role} in the \textit{Kummer-theoretic} aspects of inter-universal Teichmüller theory that are discussed in §3.6 below.

As remarked in §2.4, the \(F^*_{l\pm}\) and \(F^*_{l}\)-symmetries of a \(\Theta^{\pm \ell}NF\)-Hodge theater may be thought of as corresponding, respectively, to the \textbf{additive} and \textbf{multiplicative/toral symmetries} of the \textit{classical upper half-plane} [cf. [IUTchI], Remark 6.12.3, (iii); [BogIUT], for more details]. Alternatively, the \(F^*_{l\pm}\) and \(F^*_{l}\)-symmetries of a \(\Theta^{\pm \ell}NF\)-Hodge theater may be thought of as corresponding, respectively, to the \textbf{unipotent} ordinary and \textbf{toral} \textit{supersingular monodromy} — i.e., put another way, to the well-known structure of the \textit{\(p\)-Hecke correspondence} — that occurs in the well-known classical
$p$-adic theory surrounding the **moduli stack of elliptic curves** over the $p$-adic integers $\mathbb{Z}_p$ [cf. the discussion of [IUTchI], Remark 4.3.1; [IUTchII], Remark 4.11.4, (iii), (c)].

(vi) **log-links:** Each vertical arrow

$$
\bullet \xrightarrow{\log} \bullet
$$

of the log-theta-lattice relates the various *copies* of

“$\Pi_X \curvearrowright \mathcal{O}_K^\times$”

[cf. the discussion at the beginning of (iv)] that lie in the *prime-strips* of the domain $\Theta^{\pm \text{ell}}_{NF}$-Hodge theater “$\bullet$” of the log-link to the corresponding copy of “$\Pi_X \curvearrowright \mathcal{O}_K^\times$” that lies in a *prime-strip* of the codomain $\Theta^{\pm \text{ell}}_{NF}$-Hodge theater “$\bullet$” of the log-link in the fashion prescribed by the arrow “$\log$” of the diagram of Example 2.12.3, (iii) [with suitable modifications involving *tempered* arithmetic fundamental groups at the valuations $\in V_{\text{bad}}$ or the theory of [AbsTopIII], §2, at the valuations $\in V_{\text{arc}} (\subseteq V_{\text{good}})$]. In particular, the log-link may be thought of as “lying over” an **isomorphism** between the respective copies of “$\Pi_X$” which is **indeterminate** since [cf. the discussion of §2.10; the discussion at the end of Example 2.12.3, (ii)] the two copies of “$\Pi_X$” must be regarded as *distinct abstract topological groups*. Put another way, from the point of view of the discussion at the beginning of (iv),

the log-link induces an **indeterminate isomorphism** between the $D$-$\Theta^{\pm \text{ell}}_{NF}$-Hodge theaters associated to the $\Theta^{\pm \text{ell}}_{NF}$-Hodge theaters “$\bullet$” in the domain and codomain of the log-link, that is to say, these $D$-$\Theta^{\pm \text{ell}}_{NF}$-Hodge theaters associated to the “$\bullet$’s” of the log-theta-lattice are **vertically coric** [cf. [IUTchIII], Theorem 1.5, (i)].

Now recall from Example 2.12.3, (i), that each abstract topological group “$\Pi_X$” may be regarded as the *input data* for a **functorial algorithm** that allows one to reconstruct the base field [in this case an MLF] of the hyperbolic curve “$X$”. Put another way, from the point of view of the terminology discussed in §2.7, (vii), each copy of “$\Pi_X$” may be regarded as an **arithmetic holomorphic structure** on the quotient group “$\Pi_X \to G_k$” associated to $\Pi_X$ [cf. the discussion of Example 2.12.3, (i)]. Indeed, this is precisely the point of the analogy between the fundamental prototypical examples — i.e., Examples 3.2.1, 3.2.2 — of the phenomenon of multiradiance. The various “$X$’s” that occur in a $\Theta^{\pm \text{ell}}_{NF}$-Hodge theater are certain finite étale coverings of localizations of the hyperbolic curve $X_K$ at various valuations $\in \mathbb{V}$. These finite étale coverings are hyperbolic curves over $K_{\underline{v}}$ which are denoted $X_{\underline{v}}$ in the case of $\underline{v} \in \mathbb{V}_{\text{bad}}$ and $X_{\underline{v}}$ in
the case of $\nu \in \mathcal{V}^{\text{good}}$ [cf. [IUTchI], Definition 3.1, (e), (f)]. On the other hand, we recall from [AbsTopIII], Theorem 1.9 [cf. also [AbsTopIII], Remark 1.9.2], that this functorial algorithm may also be applied to the hyperbolic orbicurves $X_K, \mathcal{C}_K$, or $C_{F_{\text{mod}}}$, i.e., whose base fields are NF’s, in a fashion that is functorial [cf. the discussion of [IUTchI], Remarks 3.1.2, 4.3.2] with respect to passing to finite étale coverings, as well as with respect to localization at valuations of $\in \mathcal{V}$ [cf. also the theory of [AbsTopIII], §2, in the case of valuations $\in \mathcal{V}^{\text{arc}}$]. That is to say, in summary,

the various [possibly tempered, in the case of valuations $\in \mathcal{V}^{\text{bad}}$] arithmetic fundamental groups of finite étale coverings of $C_{F_{\text{mod}}}$ [such as $X_K, \mathcal{C}_K$, or $C_{F_{\text{mod}}}$ itself] and their localizations at valuations $\in \mathcal{V}$ that appear in a $\Theta^{\pm \text{ell}} \text{NF}$-Hodge theater may be regarded as abstract representations of the arithmetic holomorphic structure [i.e., ring structure — cf. the discussion of §2.7, (vii)] of the various base fields of these hyperbolic orbicurves.

Moreover, this state of affairs motivates the point of view that the various localizations and gluings that occur in the structure of a single $\Theta^{\pm \text{ell}} \text{NF}$-Hodge theater [cf. Fig. 3.8] or, as just described, in the structure of a log-link [i.e., a vertical arrow of the log-theta-lattice — cf. Fig. 3.6] may be thought of as arithmetic analytic continuations between various NF’s along the various gluings of prime-strips that occur [cf. the discussion of [IUTchI], Remarks 4.3.1, 4.3.2, 4.3.3, 5.1.4].

In this context, it is of interest to observe that, at a technical level, these arithmetic analytic continuations are achieved by applying the mono-anabelian theory of [AbsTopIII], §1 [or, in the case of archimedean valuations, the theory of [AbsTopIII], §2]. Moreover, this mono-anabelian theory of [AbsTopIII], §1, is, in essence, an elementary consequence of the theory of Belyi cuspidalizations developed in [AbsTopII], §3 [cf. [AbsTopIII], Remark 1.11.3]. Here, we recall that the term cuspidalization refers to a functorial algorithm in the arithmetic fundamental group of a hyperbolic curve for reconstructing the arithmetic fundamental group of some dense open subscheme of the hyperbolic curve. In particular, by considering Kummer classes of rational functions [cf. the discussion of Example 2.13.1, (i)],

**cuspidalization** may be thought of as a sort of “equivalence” between the function theory on a hyperbolic curve and the function theory on a dense open subscheme of the hyperbolic curve — a formulation that is very formally reminiscent of the classical notion of analytic continuation.

The Belyi cuspidalizations developed in [AbsTopII], §3, are achieved as a formal consequence of the elementary observation that
any “sufficiently small” dense open subscheme $U$ of the hyperbolic curve $P$ given by removing three points from the projective line may be regarded — via the use of a suitable Belyi map! — as a finite étale covering of $P$; in particular, the arithmetic fundamental group of $U$ may be recovered from the arithmetic fundamental group of $P$ by considering a suitable open subgroup of the arithmetic fundamental group of $P$ [cf. [AbsTopII], Example 3.6; [AbsTopII], Corollaries 3.7, 3.8, for more details].

This state of affairs is all the more fascinating in that the well-known construction of Belyi maps via an induction on the degree over $\mathbb{Q}$ of the ramification locus of certain rational maps between two projective lines is [cf. the discussion of [IUTchI], Remark 5.1.4] highly reminiscent of the well-known Schwarz lemma of elementary complex analysis, i.e., to the effect that the absolute value, relative to the respective Poincaré metrics, of the derivative at any point of a holomorphic map between copies of the unit disc is $\leq 1$.

(vii) $\Theta$-links: Various versions of the “$\Theta$-link” are defined in [IUTchI], [IUTchII], [IUTchIII] — cf. [IUTchI], Corollary 3.7, (i); [IUTchII], Corollary 4.10, (iii); [IUTchIII], Definition 3.8, (ii). In the present paper, we shall primarily be interested in the version of [IUTchIII], Definition 3.8, (ii). The versions of [IUTchII], Corollary 4.10, (iii), are partially simplified versions of the version that one is ultimately interested in [i.e., the version of [IUTchIII], Definition 3.8, (ii)], while the version of [IUTchI], Corollary 3.7, (i), is an even more drastically simplified version of these partially simplified versions. The $\Theta$-link may be understood, roughly speaking, as a realization of the version of the assignment “$q \mapsto q^N$” considered in the final portion of the discussion of §2.4, i.e., the assignment

$$“q \mapsto \{q^{j^2}\}_{j=1,\ldots,k^*}”$$

given by taking a sort of symmetrized average as “$N$” varies over the values $j^2$, for $j = 1, \ldots, k^*$. At a more technical level, the $\Theta$-link

$$ \bullet \xrightarrow{\Theta} \bullet $$

is a gluing between two $\Theta^{\pm \text{ell}} NF$-Hodge theaters “$\bullet$”, via an indeterminate [cf. the discussion of §2.10] isomorphism between certain gluing data arising from the domain $\Theta^{\pm \text{ell}} NF$-Hodge theater “$\bullet$” and certain gluing data arising from the codomain $\Theta^{\pm \text{ell}} NF$-Hodge theater “$\bullet$”. The gluing data that arises from the domain “$\bullet$” of the $\Theta$-link is as follows:

(a$^{\Theta}$) [Local unit group portion: Consider, in the notation of §2.11, §2.12 [cf., especially, Example 2.12.3, (iv)], the ind-topological monoid equipped with an action
by a topological group $G_k \curvearrowright \mathcal{O}_k^\times$, where we note that $\mathcal{O}_k^\times$ is a $\mathbb{Q}_p$-vector space.

Observe that for each open subgroup $H \subseteq G_k$, which determines a subfield $\mathbb{k}^H \subseteq \mathbb{k}$ of $H$-invariants of $\mathbb{k}$, the image of $\mathcal{O}_k^\times$ in $\mathcal{O}_k^\times$ determines an integral structure, or “lattice” [i.e., a finite free $\mathbb{Z}_p$-module], in the $\mathbb{Q}_p$-vector subspace of $H$-invariants of $\mathcal{O}_k^\times$. [Here, we note that the theory of the $p$-adic logarithm determines a natural isomorphism between this subspace and the $\mathbb{Q}_p$-vector space $\mathbb{k}$.] For each $v \in \mathbb{V}^\text{non}$, we take the unit group portion data at $v$, to be this data

$$(G_k \curvearrowright \mathcal{O}_k^\times, \{\mathcal{O}_k^\times \subseteq (\mathcal{O}_k^\times)^H\}_H)$$

— i.e., which we regard as an ind-topological monoid equipped with an action by a topological group, together with, for each open subgroup of the topological group, an integral structure — in the case $k \overset{\text{def}}{=} K_v$. When $k = K_v$, we shall write $G_v \overset{\text{def}}{=} G_k$.

An analogous construction may be performed for $v \in \mathbb{V}^\text{arc}$.

(b$^\Theta$) Local value group portion: At each $v \in \mathbb{V}^\text{bad}$, we take the local value group portion data at $v$ to be the formal monoid [abstractly isomorphic to the monoid $\mathbb{N}$] generated by

$$\{q_v^{2^j}\}_{j=1,...,l^*}$$

— i.e., where $q_v \in \mathcal{O}_{K_v}^\times$ is a $2l$-th root of the $q$-parameter $q_v$ of the elliptic curve $E_K$ at $v$; the data of the above display is regarded as a collection of elements of $\mathcal{O}_{K_v}^\times$ indexed by the elements of $\mathbb{F}_v^*$; each element of this collection is well-defined up to multiplication by a $2l$-th root of unity. An analogous, though somewhat more formal, construction may be performed for $v \in \mathbb{V}^\text{good}$.

(c$^\Theta$) Global value group portion: Observe that each of the local formal monoids [say, for simplicity, at $v \in \mathbb{V}^\text{bad}$] of (b$^\Theta$) may be realified. That is to say, the corresponding realified monoid is simply the monoid of $\mathbb{R}_{\geq 0}$-multiples [i.e., nonnegative real multiples] of the image of the given monoid $[\cong \mathbb{N}]$ inside the tensor product $\otimes_\mathbb{Z}\mathbb{R}$ of the groupification $[\cong \mathbb{Z}]$ of this given monoid. Note, moreover, that the product formula of elementary algebraic number theory yields a natural notion of “finite collections of elements of the groupifications $[\cong \mathbb{R}]$ of these realified monoids at $v \in \mathbb{V}$ whose sum $= 0$”. This data, consisting of a realified monoid at each $v \in \mathbb{V}$, together with a collection of “product formula relations”, determines a global realified Frobenioid. We take the global value group portion data to be this global realified Frobenioid.

The gluing data that arises from the codomain of the $\Theta$-link is as follows:
(a') **[Local] unit group portion:** For each $v \in \mathcal{V}$, we take the unit group portion data at $v$ to be the analogous data, i.e., this time constructed from the codomain $\bullet$ of the Θ-link, to the data of $(a^\Theta)$.

(b') **Local value group portion:** At each $v \in \mathcal{V}^{\text{bad}}$, we take the local value group portion data at $v$ to be the formal monoid [abstractly isomorphic to the monoid $\mathbb{N}$] generated by $q_v$ — i.e., where we apply the notational conventions of $(b^\Theta)$. An analogous, though somewhat more formal, construction may be performed for $v \in \mathcal{V}^{\text{good}}$.

(c') **Global value group portion:** We take the global value group portion data to be the global realified Frobenioid [cf. the data of $(c^\Theta)$] determined by the realifications of the local formal monoids of $(b^\Theta)$ at $v \in \mathcal{V}$, together with a naturally determined collection of “product formula relations”.

In fact, the above description is slightly inaccurate in a number of ways: for instance, in [IUTchI], [IUTchII], [IUTchIII], the data of $(a^\Theta)$, $(b^\Theta)$, $(c^\Theta)$, $(a^q)$, $(b^q)$, $(c^q)$ are constructed in a somewhat more intrinsic fashion directly from the various Frobenioids [and other data] that constitute the $\Theta^{\pm\text{ell}}N\mathcal{F}$-Hodge theater “•” under consideration. This sort of intrinsic construction exhibits, in a very natural fashion,

the ind-topological monoids $\mathcal{O}_K^{\times\mu}$ of $(a^\Theta)$ and $(a^q)$, the local formal monoids of $(b^\Theta)$ and $(b^q)$, and the global realified Frobenioids of $(c^\Theta)$ and $(c^q)$ as Frobenius-like objects.

By contrast,

the topological groups $G_k$ of $(a^\Theta)$ and $(a^q)$ are étale-like objects.

In this context, it is useful to note — cf. the discussion of the vertical coricity of $D^{\Theta^{\pm\text{ell}}}N\mathcal{F}$-Hodge theaters in (vi) — that

the unit group portion data of $(a^\Theta)$, $(a^q)$ is horizontally coric [cf. [IUTchIII], Theorem 1.5, (ii)], while the portion of this data constituted by the topological group $G_k$ is bi-coric [cf. [IUTchIII], Theorem 1.5, (iii)].

Indeed,

one way to think of Frobenius-like structures, in the context of the log-theta-lattice, is as structures that, at least $a$ priori, are confined to — i.e., at least $a$ priori, are only defined in — a fixed $\Theta^{\pm\text{ell}}N\mathcal{F}$-Hodge theater “•” of the log-theta-lattice.
Here, we note that since \([\text{just as in the case of the log-link — cf. the discussion in the final portion of (ii)!}]\) the \(\Theta\)-link is \textbf{fundamentally incompatible} with the ring structures in its domain and codomain, it is necessary to think of the \textbf{bi-coric} topological group \(\text{“} G_k \text{“} \) as being only well-defined up to some \textbf{indeterminate isomorphism} [cf. the discussion of \$2.10; \text{[IUTchIII]}, Remark 1.4.2, (i), (ii); \text{[IUTchIV]}, Remark 3.6.3, (i)].

Thus, in summary, the \(\Theta\)-link induces an \textbf{isomorphism} of the \textbf{unit group portion data} of \((a^{\Theta}), (a^{q}), \) on the one hand, and a \textbf{dilation}, by a factor given by a sort of \textbf{symmetrized average} of the \(j^2\), for \(j = 1, \ldots, l^*\), of the \textbf{local and global value group data} of \((b^{\Theta}), (c^{\Theta}), (b^{q}), (c^{q})\), on the other.

The object, well-defined up to isomorphism, of the \textbf{global realified Frobenioid} of \((c^{\Theta})\) determined by the unique collection of generators of the \textbf{local formal monoids} of \((b^{\Theta})\) at \(v \in \mathcal{V}^{\text{bad}}\) will be referred to as the \textbf{\(\Theta\)-pilot object} [cf. \text{[IUTchI]}, Definition 3.8, (i)]. In a similar vein, the object, well-defined up to isomorphism, of the \textbf{global realified Frobenioid} of \((c^{q})\) determined by the unique collection of generators of the \textbf{local formal monoids} of \((b^{q})\) at \(v \in \mathcal{V}^{\text{bad}}\) will be referred to as the \textbf{\(q\)-pilot object} [cf. \text{[IUTchI]}, Definition 3.8, (i)]. The \(\Theta\)- and \(q\)-pilot objects play a \textbf{central role} in the \textbf{main results} of \textbf{inter-universal Teichmüller theory}, which are the main topic of \$3.7 below.

\section*{3.4. Kummer theory and multiradial decouplings/cyclotomic rigidity}

The \textbf{first main result} of \textbf{inter-universal Teichmüller theory} [cf. \$3.7, (i)] consists of a \textbf{multiradial representation} of the \textbf{\(\Theta\)-pilot objects} discussed in \$3.3, (vii). Relative to the general discussion of \textbf{multiradiality} in \$3.1, this \textbf{multiradiality} may be understood as being with respect to the \textbf{radial algorithm}

\[(a^{\Theta}), (b^{\Theta}), (c^{\Theta}) \mapsto (a^{\Theta})\]

that associates to the \textbf{gluing data} in the \textbf{domain of the \(\Theta\)-link} the \textbf{horizontally coric unit group portion} of this data [cf. the discussion of \$3.3, (vii)]. The construction of this multiradial representation of \(\Theta\)-pilot objects consists of \textbf{two steps}. The \textbf{first step}, which we discuss in detail in the present \$3.4, is the construction of \textbf{multiradial cyclotomic rigidity} and \textbf{decoupling} algorithms for certain \textbf{special types of functions} on the hyperbolic curves under consideration. The \textbf{second step}, which we discuss in detail in \$3.6 below, concerns the \textbf{Galois evaluation} at certain special points — i.e., evaluation via \textbf{Galois sections of arithmetic fundamental groups} — of these functions to obtain certain \textbf{special values} that act on \textbf{processions} of \textbf{log-shells}.
(i) **The essential role of Kummer theory:** We begin with the fundamental observation that, despite the fact, for $\Box \in \{\Theta, q\}$, the construction of the data $(a^{\Box})$ (respectively, $(b^{\Box}); (c^{\Box})$) depends quite essentially on whether $\Box = \Theta$ or $\Box = q$, the indeterminate gluing isomorphism that constitutes a $\Theta$-link exists — i.e., the data $(a^{\Theta})$ (respectively, $(b^{\Theta}); (c^{\Theta})$) is indeed isomorphic to the data $(a^{q})$ (respectively, $(b^{q}); (c^{q})$) — precisely as a consequence of the fact that we regard the [ind-topological] monoids that occur in this data as *abstract [ind-topological] monoids* that are not equipped with the auxiliary data of how these [ind-topological] monoids “happen to be constructed”. That is to say, the inclusion of such auxiliary data would render the corresponding portions of data in the domain and codomain of the $\Theta$-link non-isomorphic! Such abstract [ind-topological] monoids are a sort of prototypical example of the notion of a Frobenius-like structure [cf. [IUTchIV], Example 3.6, (iii)]. A similar observation applies to the copies of “$\mathcal{O}_k^\circ$” that occur in the discussion of the log-link in §3.3, (vi). Thus, in summary, it is precisely by working with Frobenius-like structures such as abstract [ind-topological] monoids or abstract [global realified] Frobenoids that we are able to construct the non-ring/scheme-theoretic gluing isomorphisms [i.e., “non-ring/scheme-theoretic” in the sense that they do not arise from morphisms of rings/schemes!] of the log- and $\Theta$-links of §3.3, (vi), (vii) [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

By contrast, étale-like structures such as the “$G_k$’s” of $(a^{\Theta})$ and $(a^{q})$ [cf. §3.3, (vii)] or the “$\Pi_X$’s” of §3.3, (vi), will be used to compute various portions of the ring/scheme theory on the opposite side of a log- or $\Theta$-link via the technique of mono-anabelian transport, as discussed in §2.7, §2.9, i.e., by determining the sort of indeterminacies that one must admit in order to render the two systems of Kummer theories — which, we recall, are applied in order to relate Frobenius-like structures to corresponding étale-like structures — in the domain and codomain of the log- or $\Theta$-link compatible with simultaneous execution.

Here, we recall that Kummer classes are obtained, in essence, by considering cohomology classes that arise from the action of various Galois or arithmetic fundamental groups on the various roots of elements of an abstract monoid [cf. Examples 2.6.1, (iii); 2.12.1, (i); 2.13.1, (i)]. Thus, the key step in rendering such Kummer classes independent of any Frobenius-like structures lies in the algorithmic construction of a cyclotomic rigidity isomorphism between the group of torsion elements of the abstract monoid under consideration and some sort of étale-like cyclotome, i.e., that is constructed
directly from the Galois or arithmetic fundamental group under consideration [cf. the isomorphism “$\lambda$” of Example 2.6.1, (iii), (iv); the isomorphism “$\rho_{\mu_\tau}$” of Example 2.12.1, (i), (ii), (iv); the isomorphism “$\lambda$” of Example 2.13.1, (i), (ii)]. On the other hand, let us observe, relative to the multiradiality mentioned at the beginning of the present §3.4, that the coric data “$G_k \curvearrowright \mathcal{O}_k^{x_{\mu}}$” admits a [nontrivial] natural action by $\widehat{\mathbb{Z}}^\times$ $\mathcal{O}_k^{x_{\mu}} \curvearrowright \widehat{\mathbb{Z}}^\times$

[i.e., which is $G_k$-equivariant and compatible with the various integral structures that appear in the coric data] that lifts to a natural $\widehat{\mathbb{Z}}^\times$-action on $\mathcal{O}_k^{x}$ [cf. Example 2.12.2, (i)]. This $\widehat{\mathbb{Z}}^\times$-action induces a trivial $\widehat{\mathbb{Z}}^\times$-action on $\mathcal{O}_k$ [hence also on $\mu_\tau(G_k)$], but a nontrivial action of $\widehat{\mathbb{Z}}^\times$ on $\mu_\tau$. In particular, this $\widehat{\mathbb{Z}}^\times$-action on $\mathcal{O}_k^{x}$ is manifestly incompatible with the cyclotomic rigidity isomorphism $\rho_{\mu_\tau} : \mu_\tau \sim \mu_\tau(G_k)$ that was functorially constructed in Example 2.12.1, (ii), hence, in light of the functoriality of this construction, does not extend to an action of $\widehat{\mathbb{Z}}^\times$ on $\mathcal{O}_k^{\ominus}$. That is to say,

the naive approach just discussed to cyclotomic rigidity isomorphisms via the functorial construction of Example 2.12.1, (ii), is incompatible with the requirement of multiradiality, i.e., of the existence of liftings of arbitrary morphisms between collections of coric data.

This discussion motivates the following approach, which is fundamental to inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.2.1, (iii); [IUTchIII], Remark 2.2.2]:

in order to obtain multiradial cyclotomic rigidity isomorphisms for the local and global value group data $(b^\Theta)$ and $(c^\Theta)$, it is necessary to somehow decouple this data $(b^\Theta)$ and $(c^\Theta)$ from the unit group data of $(a^\Theta)$.

This decoupling is achieved in inter-universal Teichmüller theory by working with certain special types of functions, as described in (ii), (iii), (iv), below.

(ii) Multiradial decouplings/cyclotomic rigidity for $\kappa$-coric rational functions: The global realified Frobenioids of $(c^\Theta)$ may be interpreted as “realifications” of certain categories of $\ell^*$-tuples, indexed by $j = 1, \ldots, l^*$, of arithmetic line bundles on the number field $F_{\text{mod}}$. The ring structure — i.e., both the additive “$\oplus$” and multiplicative “$\otimes$” structures — of copies of this number field $F_{\text{mod}}$ is applied, ultimately, in inter-universal Teichmüller theory, in order to relate these global realified Frobenioids of $(c^\Theta)$, which are, in essence, a multiplicative notion, to the interpretation of
arithmetic line bundles in terms of **log-shells**, which are **modules**, i.e., whose group law is written **additively** [cf. [IUTchIII], Remarks 3.6.2, 3.10.1] — an interpretation with respect to which **global arithmetic degrees** correspond to **log-volumes** of certain regions inside the various log-shells at each $v \in \mathcal{V}$ [cf. the discussion of §2.2]. Thus, in summary,

the *essential structure of interest* that gives rise to the data of $(c^\Theta)$ consists of copies of the **number field** $F_{\text{mod}}$ indexed by $j = 1, \ldots, l^\ast$.

In particular, the **Kummer theory** [cf. the discussion of (i)!] concerning the data of $(c^\Theta)$ revolves around the Kummer theory of such copies of the number field $F_{\text{mod}}$. As discussed at the beginning of the present §3.4, elements of such copies of $F_{\text{mod}}$ will be constructed as **special values** at certain **special points** of certain **special types of functions**. Here, it is perhaps of interest to recall that

this approach to constructing elements of the **base field** [in this case, the number field $F_{\text{mod}}$] of a hyperbolic curve — i.e., by **evaluating Kummer classes** of **rational functions** on the hyperbolic curve at certain **special points** — is precisely the approach that is in fact applied in the **mono-anabelian reconstruction algorithms** discussed in [AbsTopIII], §1.

As discussed at the beginning of the present §3.4 [cf. also the discussion of (i)], the **first step** in the construction of multiradial representations of $\Theta$-pilot objects to be discussed in §3.7, (i), consists of formulating the **Kummer theory** of **suitable special types of rational functions** in such a way that we obtain **multiradial cyclotomic rigidity isomorphisms** that involve a **decoupling** of this Kummer theory for rational functions from the **unit group data** of $(a^\Theta)$. In the present case, i.e., which revolves around the construction of copies of $F_{\text{mod}}$,

the desired formulation of Kummer theory is achieved by considering a certain subset — called the **pseudo-monoid of $\kappa$-coric rational functions** [cf. [IUTchI], Remark 3.1.7, (i), (ii)] — of the group [i.e., multiplicative monoid] “$K_X^\kappa$” considered in Example 2.13.1, in the case where the hyperbolic curve “$X$” is taken to be the hyperbolic curve $\mathcal{X}_K$ of §3.3, (i) [cf. [IUTchI], Remark 3.1.2, (ii)].

Recall the hyperbolic orbicurve $C_{F_{\text{mod}}}$ discussed in §3.3, (i). Write $|C_{F_{\text{mod}}}|$ for the **coarse space** $|C_{F_{\text{mod}}}|$ associated to $C_{F_{\text{mod}}}$. Here, it is useful to recall the well-known fact that $|C_{F_{\text{mod}}}|$ is isomorphic to the **affine line over $L$**. We shall refer to the points of the compactification of $|C_{F_{\text{mod}}}|$ that arise from the 2-torsion points of the elliptic curve $E_F$ other than the origin as **strictly critical**. A **$\kappa$-coric rational function** is a rational function on $|C_{F_{\text{mod}}}|$ that

restricts to a **root of unity** at each **strictly critical point** of $|C_{F_{\text{mod}}}|$.
and, moreover, satisfies certain other [somewhat less essential] technical conditions [cf. [IUTchI], Remark 3.1.7, (i)]. Thus, a $\kappa$-coric rational function on $|C_{\text{mod}}|$ may also be regarded, by restriction, as a rational function on $X_K$. Although the $\kappa$-coric rational functions do not form a monoid [i.e., the product of two $\kappa$-coric rational functions is not necessarily a $\kappa$-coric rational function], it nevertheless holds that arbitrary positive powers of $\kappa$-coric rational functions are $\kappa$-coric. Moreover, every root of unity in $F_{\text{mod}}$ is $\kappa$-coric; a rational function on $|C_{\text{mod}}|$ is $\kappa$-coric if and only if some positive power of the rational function is $\kappa$-coric. One verifies immediately that [despite the fact that the $\kappa$-coric rational functions do not form a monoid] these elementary properties that are satisfied by $\kappa$-coric rational functions are sufficient for conducting Kummer theory with $\kappa$-coric rational functions. Then [cf. [IUTchI], Example 5.1, (v), for more details]:

- The desired decoupling of the pseudo-monoid of $\kappa$-coric rational functions from the unit group data of $(a^\theta)$ is achieved by means of the condition that evaluation at any of the strictly critical points — an operation that may be performed at the level of étale-like structures, i.e., by restricting Kummer classes to decomposition groups of points — yields a root of unity.
- The desired multiradial cyclotomic rigidity isomorphism is achieved by means of the technique discussed in Example 2.13.1, (ii) — i.e., involving the elementary fact

$$\mathbb{Q}_{>0} \bigcap \hat{\mathbb{Z}}^\times = \{1\}$$

— which is applied to the pseudo-monoid of $\kappa$-coric rational functions, i.e., as opposed to the entire multiplicative monoid $K_X^\times$.

As was mentioned in Example 2.13.1, (ii), this approach has the disadvantage of being incompatible with the profinite topology of the Galois or arithmetic fundamental groups involved [cf. the discussion of (iii) below; §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii)]. Also, we remark that although this approach only allows one to reconstruct the desired cyclotomic rigidity isomorphism up to multiplication by $\pm 1$, this will not yield any problems since we are, in fact, only interested in reconstructing copies of the entire multiplicative monoid $F_{\text{mod}}^\times$, which is closed under inversion [cf. the discussion of [IUTchIII], Remark 2.3.3, (vi); [IUTchIII], Remark 3.11.4].

(iii) Naive approach to cyclotomic rigidity for theta functions: Fix $\nu \in \mathcal{V}_{\text{bad}}$. Denote by means of a subscript $\underline{\nu}$ the result of base-changing objects over $K$ to $K_{\underline{\nu}}$. Thus, $X_{\underline{\nu}}$ is a “once-punctured Tate curve” over $K_{\underline{\nu}}$, hence determines a one-pointed stable curve of genus one $\mathcal{X}_{\underline{\nu}}$ over the ring of integers $\mathcal{O}_{K_{\underline{\nu}}}$ of $K_{\underline{\nu}}$ [where, for simplicity, we omit the notation for the single marked point, which arises from the cusp of $X_{\underline{\nu}}$]. In particular, the dual graph of the special fiber of $X_{\underline{\nu}}$ [i.e., more precisely: of $\mathcal{X}_{\underline{\nu}}$] is a “loop” [i.e., more precisely, consists of a single vertex and a single edge, both ends
Fig. 3.9: *Labels of irreducible components and orders of zeroes at cusps “*” and poles at irreducible components “□” of the theta function $\tilde{\Theta}_\mathfrak{p}$ on $\tilde{Y}_\mathfrak{p}$* of which abut to the single vertex. In particular, the universal covering [in the sense of classical algebraic topology!] of this dual graph determines [what is called] a tempered covering $Y_\mathfrak{p} \to X_\mathfrak{p}$ [i.e., at the level of models over $\mathcal{O}_{K_\mathfrak{p}}$], a tempered covering $\mathfrak{Y}_\mathfrak{p} \to \mathfrak{X}_\mathfrak{p}$ — cf. [André], §4; [Semi], Example 3.10], whose Galois group $\mathbb{Z} \overset{\text{def}}{=} \text{Gal}(Y_\mathfrak{p}/X_\mathfrak{p})$ is noncanonically isomorphic to $\mathbb{Z}$. Thus,

the special fiber of $Y_\mathfrak{p}$ [i.e., more precisely: of $\mathfrak{Y}_\mathfrak{p}$] consists of an infinite chain of copies of the “once-punctured/one-pointed projective line”, in which the “punctures/cusps” correspond to the points “1” of the copies of the projective line, and the point “$\infty$” of each such copy is glued to the point “0” of the adjacent copy [cf. the upper portion of Fig. 3.9 above; the discussion at the beginning of [EtTh], §1; [IUTchII], Proposition 2.1; [IUTchII], Remark 2.1.1].

If one fixes one such copy of the once-punctured projective line, together with an isomorphism $\mathbb{Z} \overset{\sim}{\to} \mathbb{Z}$, then the natural action of $\mathbb{Z}$ on $Y_\mathfrak{p}$ determines a natural bijection of the set of irreducible components of the special fiber of $Y_\mathfrak{p}$ — or, alternatively, of the set of cusps of $Y_\mathfrak{p}$ — with $\mathbb{Z}$ [cf. the “labels” of Fig. 3.9]. The “multiplication by 2” endomorphism of the elliptic curve $E_\mathfrak{p}$ [which may be thought of as the compactification of the affine hyperbolic curve $X_\mathfrak{p}$] determines, via base-change by $Y_\mathfrak{p} \to X_\mathfrak{p}$, a double covering $\tilde{Y}_\mathfrak{p} \to Y_\mathfrak{p}$ [i.e., at the level of models over $\mathcal{O}_{K_\mathfrak{p}}$], a double covering $\mathfrak{Y}_\mathfrak{p} \to \mathfrak{Y}_\mathfrak{p}$].

One verifies immediately that the set of irreducible components of the special fiber of $\tilde{Y}_\mathfrak{p}$ [i.e., more precisely: of $\mathfrak{Y}_\mathfrak{p}$] maps bijectively to the set of irreducible components of the special fiber of $Y_\mathfrak{p}$, while there exist precisely two cusps of $\tilde{Y}_\mathfrak{p}$ over each cusp of $Y_\mathfrak{p}$. The formal completion of $Y_\mathfrak{p}$ along the smooth locus [i.e., the complement of “0” and “$\infty$”] of the irreducible component of the special fiber labeled 0 is naturally isomorphic [in a fashion compatible with a choice of isomorphism $\mathbb{Z} \overset{\sim}{\to} \mathbb{Z}$] to a once-punctured copy of the multiplicative group “$\mathbb{G}_m$”. In particular, it makes sense to speak of the standard multiplicative coordinate “$U_\mathfrak{p}$” on this formal completion, as well as a square root [well-defined up to multiplication by $\pm 1$] “$\tilde{U}_\mathfrak{p}$” of $\mathfrak{U}_\mathfrak{p}$ on the base-change of this
formal completion by $\tilde{Y}_u \to Y_u$. The **theta function**

$$\tilde{\Theta}_u = \tilde{\Theta}_u(\tilde{U}_u) \overset{\text{def}}{=} q_u^{-\frac{k}{8}} \cdot \sum_{n \in \mathbb{Z}} \frac{1}{(-1)^n \cdot \tilde{U}_u^{2n+1}} \cdot \tilde{U}_u^{\frac{n+1}{2} \cdot \tilde{U}_u^{2n+1}}$$

may be thought of as a *meromorphic function* on $\tilde{Y}_u$ [cf. [EtTh], Proposition 1.4, and the preceding discussion], whose zeroes are precisely the *cusps*, with multiplicity 1, and whose *poles* are supported on the special fiber of $\tilde{Y}_u$, with multiplicity [relative to a square root $q_u^\frac{1}{2}$ of $q_u$] equal to $j^2$ at the irreducible component labeled $j$ [cf. Fig. 3.9]. In fact, in inter-universal Teichmüller theory, we shall mainly be interested in [a certain constant multiple of] the reciprocal of an $l$-th root of this theta function, namely,

$$\Theta_u \overset{\text{def}}{=} \left\{ \left( \sum_{m \in \mathbb{Z}} q_u^{\frac{1}{2} \cdot (m+\frac{1}{2})^2} \right)^{-1} \cdot \left( \sum_{n \in \mathbb{Z}} (-1)^n \cdot q_u^{\frac{1}{2} \cdot (\frac{n+1}{2})^2} \cdot U_n^{n+\frac{1}{2}} \right) \right\}^{-\frac{1}{l}}$$

— which may be thought of as a *meromorphic function* on $\tilde{Y}_u \overset{\text{def}}{=} \tilde{Y}_u \times_{X_u} \tilde{X}_u$ [cf. the notation of §3.3, (vi)] that is normalized by the condition that it assumes a value $\in \mu_{2l}$ [i.e., a $2l$-th root of unity] at the *points* of $\tilde{Y}_u$ that lie over the torsion points of $E_u$ of order precisely 4 and, moreover, meet the smooth locus of the irreducible component of the special fiber of $\tilde{Y}_u$ [i.e., more precisely: of $\tilde{Y}_u \overset{\text{def}}{=} \tilde{Y}_u \times_{X_u} \tilde{X}_u$, where $\tilde{X}_u$ and $\tilde{X}_u$ denote the respective normalizations of $X_u$ in $\tilde{X}_u$ and $\tilde{X}_u$] labeled 0 [i.e., the unique irreducible component of the special fiber of $\tilde{Y}_u$ that maps to the irreducible component of the special fiber of $\tilde{Y}_u$ labeled 0]. Such points of $\tilde{Y}_u$ are referred to as **zero-labeled evaluation points** [cf., e.g., [IUTchII], Corollary 2.6]. At a very rough level,

the approach to **multiradial decouplings/cyclotomic rigidity** taken in the case of the Kummer theory of special functions that surrounds the formal monoid of $(b^\Theta)$ may be understood as being “roughly similar” to the approach discussed in (ii) in the case of the global realified Frobenoids of $(c^\Theta)$, except that “$\kappa$-coric rational functions” are replaced by [normalized reciprocals of $l$-th roots of] **theta functions**.

[cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchIII], Remark 2.3.3]. That is to say,

- The desired **decoupling** [which is referred to in [EtTh], as “constant multiple rigidity”] of the [reciprocal of an $l$-th root of the] theta function from the unit group data of $(a^\Theta)$ is achieved by means of the condition that **evaluation** at any of the **zero-labeled evaluation points** — an operation that may be performed at the level of étale-like structures, i.e., by restricting Kummer classes to decomposition groups of points — yields a $2l$-th root of unity.
The desired **multiradial cyclotomic rigidity isomorphism** is, roughly speaking, achieved by means of the **"mod $N$ Kummer class version"**, for various positive integers $N$, of the technique discussed in Example 2.13.1, (ii) [cf., especially, the final display of Example 2.13.1, (ii)]: that is to say, such a “mod $N$ version” is possible — **without any $\{\pm 1\}$ indeterminacies!** — precisely as a consequence of the fact that the **order of each zero** of $\tilde{\Theta}_\nu$ at each cusp of $\tilde{Y}_\nu$ is **precisely one** [cf., e.g., the discussion of [IUTchIII], Remark 2.3.3, (vi)].

In this context, we remark that the decomposition groups of zero-labeled evaluation points may be reconstructed by applying the theory of **elliptic cuspidalizations** developed in [AbsTopII], §3. This theory proceeds in an essentially parallel fashion to the theory of Belyi cuspidalizations [cf. the discussion of §3.3, (vi)]. That is to say, **elliptic cuspidalizations** are achieved as a formal consequence of the **elementary observation** that the **dense open subscheme** of a once-punctured elliptic curve obtained by removing the $N$-torsion points, for $N$ a positive integer, may be regarded — via the use of the “**multiplication by $N$”** endomorphism of the elliptic curve! — as a **finite étale covering** of the given once-punctured elliptic curve; in particular, the arithmetic fundamental group of such a dense open subscheme may be recovered from the arithmetic fundamental group of the given once-punctured elliptic curve by considering a **suitable open subgroup** of the latter arithmetic fundamental group [cf. [AbsTopII], Example 3.2; [AbsTopII], Corollaries 3.3, 3.4, for more details].

Another important observation in this context [cf. also §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii)] is that the approach to **cyclotomic rigidity** described above involving **“mod $N$ Kummer classes”** is manifestly **compatible** with the **topology** of the Galois or tempered arithmetic fundamental groups involved. Since this “mod $N$ approach” depends, in an **essential way**, on the fact that the **order of the zero** of $\tilde{\Theta}_\nu$ at each cusp of $\tilde{Y}_\nu$ is **precisely one** [cf., the discussion of [IUTchIII], Remark 2.3.3, (vi)], it also serves to elucidate the importance of working with the **first power** of [reciprocals of $l$-th roots of the] theta function, i.e., as opposed to the $M$-th power, for $M \geq 2$ [cf. [IUTchII], Remark 3.6.4, (iii), (iv); [IUTchIII], Remark 2.1.1, (iv); [IUTchIII], Remark 2.3.3, (vii)]. On the other hand, the approach described thus far in the present (iii) has one **fundamental deficiency**, namely, the fact that the **orders of the poles** of $\tilde{\Theta}_\nu$ are not **compatible/symmetric** with respect to the action of $\mathbb{Z}_l$ on $Y_\nu$ [cf. Fig. 3.9] implies that the approach described thus far in the present (iii) to **multiradial cyclotomic rigidity** — i.e., involving, in effect, the **mod $N$ Kummer classes** of the [reciprocal of an $l$-th root of the] theta function — is **not compatible** with
the $\mathbb{Z}$-symmetries of $Y_{\underline{v}}$ [cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchIII], Remark 2.3.3, (iv)].

Here, we note that the quotient $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/l \cdot \mathbb{Z} = F_l$ may be identified with the subgroup $F_l \subseteq F_{\mathbb{Z}}$ in the discussion of symmetries of $\Theta^{\pm \text{ell}} NF$-Hodge theaters in §3.3, (v). Also, we remark that the mod $N$ Kummer class of the reciprocal of an $l$-th root of the theta function is indeed compatible with the $N \cdot l \cdot \mathbb{Z}$-symmetries of $Y_{\underline{v}}$. This means that, as one varies $N$, the obstruction to finding a coherent system of basepoints — i.e., a coherent notion of the “zero label” — of the resulting projective system lies in

$$\mathbb{R}^1 \lim_N N \cdot l \cdot \mathbb{Z} \cong \mathbb{R}^1 \lim_N N \cdot l \cdot \mathbb{Z} \cong l \cdot \hat{\mathbb{Z}}/l \cdot \mathbb{Z} \neq 0$$

[cf. the discussion of [EtTh], Remark 2.16.1]. Put another way, one may only construct a coherent system of basepoints if one is willing to replace $\mathbb{Z}$ by its profinite completion, i.e., to sacrifice the discrete nature of $\mathbb{Z}$ ($\cong \mathbb{Z}$). It is for this reason that the property of compatibility with, say, the $l \cdot \mathbb{Z}$-symmetries of $Y_{\underline{v}}$ is referred to, in [EtTh], as discrete rigidity. This sort of discrete rigidity plays an important role in inter-universal Teichmüller theory since a failure of discrete rigidity would obligate one to work with $\hat{\mathbb{Z}}$-multiples/powers of divisors, line bundles, or meromorphic functions — a state of affairs that would, for instance, obligate one to sacrifice the crucial notion of positivity/ampleness in discussions of divisors and line bundles [cf. the discussion of [IUTchIII], Remark 2.1.1, (v)].

(iv) Cyclotomic, discrete, and constant multiple rigidity for mono-theta environments: The incompatibility of the approach discussed in (iii) with $\mathbb{Z}$-symmetries [cf. the discussion at the end of (iii)] is remedied in the theory of [EtTh] by working with mono-theta environments, as follows: Write $\mathcal{L}_{\underline{X}_{\underline{v}}}$ for the ample line bundle of degree 1 on $\underline{X}_{\underline{v}}$ determined by the unique marked point of $\underline{X}_{\underline{v}}$ and

$$L_{\underline{X}_{\underline{v}}} \overset{\text{def}}{=} \mathcal{L}_{\underline{X}_{\underline{v}}}|_{\underline{X}_{\underline{v}}}, \quad \mathcal{L}_{\underline{Y}_{\underline{v}}} \overset{\text{def}}{=} \mathcal{L}_{\underline{X}_{\underline{v}}}|_{\underline{Y}_{\underline{v}}}, \quad L_{\underline{Y}_{\underline{v}}} \overset{\text{def}}{=} \mathcal{L}_{\underline{X}_{\underline{v}}}|_{\underline{Y}_{\underline{v}}}, \quad L_{\underline{Y}_{\underline{v}}} \overset{\text{def}}{=} \mathcal{L}_{\underline{X}_{\underline{v}}}|_{\underline{Y}_{\underline{v}}},$$

for the various pull-backs, or restrictions, of $\mathcal{L}_{\underline{X}_{\underline{v}}}$. Then the theta function $\tilde{\Theta}_{\underline{v}}$ on $\tilde{Y}_{\underline{v}}$ may be thought of as a ratio of two sections of the line bundle $\mathcal{L}_{\underline{v}}$ over $\tilde{Y}_{\underline{v}}$, which may be described as follows:

- The algebraic section of $\mathcal{L}_{\underline{v}}$ is the section [well-defined up to a $K_{\underline{v}}^X$-multiple] whose zero locus coincides with the locus of zeroes of $\tilde{\Theta}_{\underline{v}}$. The pair consisting of the line bundle $\mathcal{L}_{\underline{v}}$ and this algebraic section admits $\mathbb{Z}$-symmetries [cf. Fig. 3.9], i.e., automorphisms that lie over the automorphisms of $\mathbb{Z} = \text{Gal}(\underline{Y}_{\underline{v}}/\underline{X}_{\underline{v}})$.
- The theta section of $\mathcal{L}_{\underline{v}}$ is the section [well-defined up to a $K_{\underline{v}}^X$-multiple]
whose zero locus coincides with the \textit{locus of poles} of $\tilde{\Theta}_\Sigma$. One verifies immediately [cf. Fig. 3.9] that the \textit{theta section} is \textbf{not compatible} with the $\mathbb{Z}$-\textit{symmetries} of the \textit{algebraic section}.

The analogous operation, for this \textit{line bundle-theoretic data}, to considering various \textit{Kummer classes} of the \textit{theta function} is the operation of passing to the \textit{tempered arithmetic fundamental group} of the $\mathbb{G}_m$-torsor $L^\times_\Sigma$ associated to $L_\Sigma$ or to the \textit{morphisms} on tempered arithmetic fundamental groups induced by the \textit{algebraic and theta sections}. Here, “\textit{mod} $N$ \textit{Kummer classes}” correspond to considering the \textit{quotient} of the tempered arithmetic fundamental group of $L^\times_\Sigma$ that corresponds to coverings whose restriction to the “$\mathbb{G}_m$ \textit{fibers}” of the $\mathbb{G}_m$-torsor $L^\times_\Sigma$ is \textit{dominated} by the covering $\mathbb{G}_m \to \mathbb{G}_m$ given by raising to $N$-th power. Note that

\begin{itemize}
  \item \textbf{neither} the \textit{ratio} of the \textit{algebraic and theta sections} — i.e., the \textit{theta function}! — \textbf{nor} the \textit{pair} consisting of the \textit{algebraic and theta sections} is \textbf{compatible} with the $\mathbb{Z}$-\textit{symmetries} of $Y_\Sigma$.
\end{itemize}

On the other hand, it is not difficult to verify that the following triple of data is indeed \textbf{compatible, up to isomorphism}, with the $\mathbb{Z}$-\textit{symmetries} of $Y_\Sigma$:

\begin{enumerate}
  \item (\textit{a}$^{\mu - \Theta}$) the $\mathbb{G}_m$-torsor $L^\times_\Sigma$;
  \item (\textit{b}$^{\mu - \Theta}$) the \textit{group of automorphisms} of $L^\times_\Sigma$ generated by the $\mathbb{Z}$-\textit{symmetries} of $L^\times_\Sigma$ and the automorphisms determined by \textit{multiplication} by a constant $\in \mathbb{K}_\Sigma^\times$;
  \item (\textit{c}$^{\mu - \Theta}$) the \textit{theta section} of $\tilde{L}^\times_\Sigma \overset{\text{def}}{=} L^\times_\Sigma|_{\tilde{Y}_\Sigma}$.
\end{enumerate}

Indeed, the asserted \textit{compatibility} with the $\mathbb{Z}$-\textit{symmetries} of $Y_\Sigma$ is immediate for (\textit{a}$^{\mu - \Theta}$) and (\textit{b}$^{\mu - \Theta}$). On the other hand, with regard to (\textit{c}$^{\mu - \Theta}$), a direct calculation shows that application of a $\mathbb{Z}$-symmetry has the effect of multiplying the theta section by some \textit{meromorphic function} which is a product of integer powers of $\tilde{U}_\Sigma$ and $q_\Sigma^{\frac{1}{2}}$; moreover, a direct calculation shows that the \textit{group of automorphisms} of (\textit{b}$^{\mu - \Theta}$) is \textit{stabilized} by conjugation by the operation of multiplying by such a meromorphic function. That is to say, by applying such multiplication operations, we conclude that the \textit{triple of data} (\textit{a}$^{\mu - \Theta}$), (\textit{b}$^{\mu - \Theta}$), (\textit{c}$^{\mu - \Theta}$) is indeed \textbf{compatible, up to isomorphism}, with the $\mathbb{Z}$-\textit{symmetries} of $Y_\Sigma$, as desired [cf. [EtTh], Proposition 2.14, (ii), (iii), for more details]. This argument motivates the following definition [cf. the discussion of [IUTchIII], Remark 2.3.4]:

The \textbf{[mod $N$] mono-theta environment} is defined by considering the \textbf{[mod $N$] tempered arithmetic fundamental group} versions of the “\textit{l-th roots}” of the triple of data (\textit{a}$^{\mu - \Theta}$), (\textit{b}$^{\mu - \Theta}$), (\textit{c}$^{\mu - \Theta}$) discussed above [cf. [EtTh], Definition 2.13, (ii)].
[Indeed, the data \((a^{μ−Θ}), (b^{μ−Θ}), (c^{μ−Θ})\) correspond, respectively, to the data of [EtTh], Definition 2.13, (ii), (a), (b), (c).] In particular, the \textbf{functoriality} of the tempered arithmetic fundamental group [essentially — cf. [EtTh], Proposition 2.14, (ii), (iii), for more details] implies that a \textbf{mono-theta environment} admits \(l \cdot \mathbb{Z}\)-symmetries of the desired type, hence, in particular, that it satisfies the crucial property of \textbf{discrete rigidity} discussed in the final portion of (iii). Moreover, by forming suitable \textbf{commutators} in the group of automorphisms of \(L^\times X\), one may recover the desired \textbf{cyclotomic rigidity isomorphism} [cf. [EtTh], Corollary 2.19, (i); [IUTchII], Remark 1.1.1, for more details], i.e., that was discussed in a “rough form” in (iii), in a fashion that is

- \textbf{decoupled} from the unit group data of \((a^{Θ})\),
- manifestly \textbf{compatible} with the \textbf{topology} of the tempered arithmetic fundamental groups involved [since one works with “mod \(N\)” mono-theta environments!], and
- \textbf{compatible} with the \(\mathbb{F}_l^{\times \pm}\)-symmetries of \(Θ^{\pm \text{ell}} NF\)-Hodge theaters [cf. §3.3, (v); [IUTchII], Remark 1.1.1, (iv), (v)].

Indeed, these \textbf{multiradial decoupling/cyclotomic rigidity} properties of mono-theta environments are the main topic of [IUTchII], §1, and are summarized in [IUTchII], Corollaries 1.10, 1.12. Moreover, mono-theta environments have both \textbf{étale-like} and \textbf{Frobenius-like versions}, i.e., they may be constructed naturally [cf. [IUTchII], Proposition 1.2, (i), (ii)] either

- from the \textbf{tempered arithmetic fundamental group} [regarded as an \textit{abstract topological group}] of \(\mathbb{X}_v\), or
- from a certain “\textbf{tempered Frobenioid}”, i.e., a \textit{model Frobenioid} [cf. §3.3, (iii)] obtained by considering suitable \textit{divisors, line bundles, and meromorphic functions} on the various tempered coverings of \(\mathbb{X}_v\).

Finally, we close with the important \textbf{observation} that the various \textbf{rigidity} properties of mono-theta environments discussed above may be regarded as

essentially formal consequences of the \textbf{quadratic structure} of the \textbf{commutators} of the \textbf{theta groups} — or, equivalently, of the \textbf{curvature}, or \textbf{first Chern class} — associated to the line bundle \(L^\times X\).

[cf. the discussion of [IUTchII], Remark 1.1.1, (iv), (v); [IUTchIII], Remark 2.1.1]. This observation is of interest in that it shows that the theory of [EtTh] [or, indeed, a substantial portion of inter-universal Teichmüller theory!] yields an interesting \textbf{alternative interpretation} for the structure of \textbf{theta groups} to the classical \textbf{representation-theoretic} interpretation, i.e., involving irreducible representations of theta groups [cf. [IUTchIII], Remark 2.3.4, (iv)].
<table>
<thead>
<tr>
<th>Approach to cyclotomic rigidity</th>
<th>Applied to Kummer theory surrounding</th>
<th>Uni-/multi-radiality</th>
<th>Compatibility with profinite/tempered topologies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brauer groups/ local class field theory for “$G_k \curvearrowright \mathcal{O}_{\kappa}$”</td>
<td>$(a^\Theta)$</td>
<td>uniradial</td>
<td>compatible</td>
</tr>
<tr>
<td>mono-theta environments</td>
<td>$(b^\Theta)$</td>
<td>multiradial</td>
<td>compatible</td>
</tr>
<tr>
<td>$\kappa$-coric rational functions, via $\mathbb{Q}_{&gt;0} \cap \hat{\mathbb{Z}}^\times = {1}$</td>
<td>$(c^\Theta)$</td>
<td>multiradial</td>
<td>incompatible</td>
</tr>
</tbody>
</table>

Fig. 3.10: Three approaches to cyclotomic rigidity
(v) **Various approaches to cyclotomic rigidity:** The discussion in the present §3.4 of various properties of the three approaches to cyclotomic rigidity that appear in inter-universal Teichmüller theory is summarized in Fig. 3.10 above. The most naive approach, involving well-known properties from *local class field theory* applied to the data “$G_k \curvearrowright \mathcal{O}_k^\times$” [cf. Example 2.12.1, (ii), (iii), (iv)], is **compatible** with the **profinite topology** of the Galois or arithmetic fundamental groups involved, but suffers from the **fundamental defect** of being **uniradial**, i.e., of being “**un-decouplable**” from the unit group data of $(a^\Theta)$ [cf. the discussion of (i)]. By contrast, the approaches discussed in (ii) and (iv) involving $\kappa$-coric rational functions and **mono-theta environments** satisfy the crucial requirement of **multiradiality**, i.e., of being “**decouplable**” from the unit group data of $(a^\Theta)$. The approach via mono-theta environments also satisfies the important property of being **compatible** with the topology of the tempered arithmetic fundamental groups involved. By contrast, the approach via $\kappa$-coric rational functions is **not compatible** with the **profinite topology** of the Galois or arithmetic fundamental groups involved. This **incompatibility** in the case of the Kummer theory surrounding the **global** data of $(c^\Theta)$ will not, however, pose a problem, since compatibility with the **topologies** of the various Galois or [possibly tempered] arithmetic fundamental groups involved will only be of interest in the case of the Kummer theory surrounding the **local** data of $(a^\Theta)$ and $(b^\Theta)$ [cf. §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii), (viii)].

§ 3.5. Remarks on the use of Frobenioids

The **theory of Frobenioids** was developed in [FrdI], [FrdII] as a

solution to the problem of providing a **unified, intrinsic category-theoretic characterization** of various types of categories of **line bundles** and **divisors** that frequently appeared in the author’s research on the arithmetic of hyperbolic curves and, moreover, seemed, at least from a heuristic point of view, to be remarkably similar in structure.

These papers [FrdI], [FrdII] on Frobenioids were written in the spring of 2005, when the author only had a relatively rough, sketchy idea of how to formulate inter-universal Teichmüller theory. In particular,

anyone who reads these papers [FrdI], [FrdII] — or indeed, the Frobenioid-theoretic portion of [EtTh] — under the expectation that they were written as an **optimally efficient presentation of precisely those portions of the theory of Frobenioids that are actually used** in inter-universal Teichmüller theory will undoubtedly be disappointed.

In light of this state of affairs, it seems appropriate to pause at this point to make a few remarks on the use of Frobenioids in inter-universal Teichmüller theory. First of all,
at \( v \in V^{\text{arc}} \), the \( \text{["archimedean"] Frobenioids} \) that appear in inter-universal Teichmüller theory [cf. [IUTchI], Example 3.4] are essentially equivalent to the \textbf{topological monoid} \( \mathcal{O}_\mathbb{C} \) [i.e., the multiplicative topological monoid of nonzero complex numbers of norm \( \leq 1 \)] and hence \textit{may be ignored}.

On the other hand,

- at \( v \in V^{\text{non}} \), all of the \( \text{["nonarchimedean"] Frobenioids} \) that appear in inter-universal Teichmüller theory [cf., e.g., [IUTchI], Fig. II.2] — except for the \textit{tempered Frobenioids} mentioned in §3.4, (iv) — are essentially equivalent to \textit{either} the data [consisting of an abstract ind-topological monoid equipped with a continuous action by an abstract topological group]

\[ \text{"} G_k \curvearrowright \mathcal{O}_k^{\mathbb{D}} \text{"} \]

of Example 2.12.1, (i), \textit{or} the data [consisting of an abstract ind-topological monoid equipped with a continuous action by an abstract topological group]

\[ \text{"} \Pi_X \curvearrowright \mathcal{O}_k^{\mathbb{D}} \text{"} \]

of Example 2.12.3, (ii) [where \( \Pi_X \) is possibly replaced by the \textit{tempered arithmetic fundamental group} of \( X \)], \textit{or} the data obtained from one of these two types of data by replacing \( \mathcal{O}_k^{\mathbb{D}} \) by some \textit{subquotient} of \( \mathcal{O}_k^{\mathbb{D}} \) [as in Example 2.12.2, (i), (ii)].

Moreover, all of these “nonarchimedean” Frobenioids are \textbf{model Frobenioids} [cf. the discussion of §3.3, (iii)]. The \textit{only other types of Frobenioids} — all of which are \textbf{model Frobenioids} [cf. the discussion of §3.3, (iii)] — that appear in inter-universal Teichmüller theory are

- the \( \text{[possibly realified] global Frobenioids} \) associated to NF’s [cf. the discussion of §3.4, (ii)], which admit a \textit{simple elementary description} as categories of \textit{arithmetic line bundles} on NF’s [cf. [FrdI], Example 6.3; [IUTchIII], Example 3.6; [Fsk], §2.10, (i), (ii)];
- the \textbf{tempered Frobenioids} mentioned in §3.4, (iv).

Here, we note that

these last two examples — i.e., \textit{global Frobenioids} and \textit{tempered Frobenioids} — \textit{differ fundamentally} from the previous examples, which were essentially equivalent to an \textit{ind-topological monoid} that was, in some cases, equipped with a continuous action by a \textit{topological group}, in that their \textbf{Picard groups} [cf. [FrdI], Theorem 5.1] admit \textit{non-torsion elements}.
Indeed, *global Frobenioids* contain objects corresponding to arithmetic line bundles whose *arithmetic degree* is $\neq 0$, while *tempered Frobenioids* contain objects corresponding to line bundles for which *arbitrary positive tensor powers are nontrivial* such as [strictly speaking, the pull-back to $\mathcal{Y}_Y$ of] the line bundle “$L_\mathcal{Y}$” of §3.4, (iv). Finally, we remark that although the theory of tempered Frobenioids, which is developed in [EtTh], §3, §4, §5, is *somewhat complicated*, the only portions of these tempered Frobenioids that are *actually used* in inter-universal Teichmüller theory are the portions discussed in §3.4, (iii), (iv), i.e.,

(a$^{t-F}$) the *theta monoids* generated by local units [i.e., “$\mathcal{O}^\times$”] and nonnegative powers of roots of the [reciprocals of $l$-th roots of] theta functions that are constructed from tempered Frobenioids [cf. [IUTchI], Example 3.2; [IUTchII], Example 3.2, (i)];

(b$^{t-F}$) the *mono-theta environments* constructed from tempered Frobenioids [cf. [IUTchII], Proposition 1.2, (ii)], which are related to the monoids of (a$^{t-F}$), in that they *share the same submonoids of roots of unity*.

Indeed, étale-like versions of this “essential Frobenius-like data” of (a$^{t-F}$) and (b$^{t-F}$) are discussed in [IUTchII], Corollaries 1.10, 1.12; [IUTchIII], Theorem 2.2, (ii) [cf. the data “($a_\mathcal{Y}$), ($b_\mathcal{Y}$), ($c_\mathcal{Y}$), ($d_\mathcal{Y}$)” of loc. cit.]. Thus, from the point of view of studying inter-universal Teichmüller theory,

one may essentially *omit* the detailed study of [EtTh], §3, §4, §5, either by accepting the construction of the data (a$^{t-F}$) and (b$^{t-F}$) “on faith” or by regarding this data as data constructed from the scheme-theoretic objects discussed in [EtTh], §1, §2.

§ 3.6. Galois evaluation, labels, symmetries, and log-shells

In the present §3.6, we discuss the theory of Galois evaluation of the $\kappa$-coric rational functions and theta functions of §3.4, (ii), (iii). Here, we remark that the term “Galois evaluation” refers to the passage

\[
\text{abstract functions } \mapsto \text{values}
\]

by first passing from Frobenius-like — that is to say, in essence, [pseudo-]monoid-theoretic — versions of these functions [cf. the discussion of §3.4, (ii), (iii); §3.5] to étale-like versions of these functions via various forms of Kummer theory as discussed in §3.4, (ii), (iii), then evaluating these étale-like functions by restricting them to decomposition subgroups [that, say, arise from closed points of the curve under consideration] of the [possibly tempered] arithmetic fundamental group under consideration to obtain étale-like versions of the values of interest, and finally applying the
Kummer theory of the constant base field [i.e., as discussed in Example 2.12.1] to obtain Frobenius-like versions of the values of interest [cf. Fig. 3.11 below; [IUTchII], Remark 1.12.4]. In fact, it is essentially a tautology that the only way to construct an assignment “abstract functions \( \mapsto \) values” that is compatible with the operation of forming Kummer classes is precisely by applying [some variant of] this technique of Galois evaluation [cf. the discussion of [IUTchII], Remark 1.12.4]. Moreover, it is interesting in this context to observe [cf. the discussion of [IUTchII], Remark 1.12.4] that the well-known Section Conjecture of anabelian geometry — which, at least historically, was expected to be related to diophantine geometry [cf. the discussion of [IUTchI], §I5] — suggests strongly that, when one applies the technique of Galois evaluation, in fact, the only suitable subgroups of the [possibly tempered] arithmetic fundamental group under consideration for the operation of “evaluation” are precisely the decomposition subgroups that arise from the closed points of the curve under consideration!

From this point of view, it is also of interest to observe that, in the context of the evaluation of theta functions at torsion points [cf. (ii) below], it will be necessary to apply a certain “combinatorial version of the Section Conjecture” [cf. [IUTchI], Remark 2.5.1; the proof of [IUTchII], Corollary 2.4, (i)]. Finally, we remark that, in order to give a precise description of the Galois evaluation operations that are performed in inter-universal Teichmüller theory, it will be necessary to consider, in substantial detail,

- the labels of the points at which the functions are to be evaluated [i.e., the points that give rise to the decomposition subgroups mentioned above],
- the symmetries that act on these labels [cf. §3.3, (v)], and
- the log-shells that serve as containers for the values that are constructed.

(i) Passage to the étale-picture, combinatorial uni-/multiradiality of symmetries: Recall from the discussion of §3.3, (vi), that the \( D\Theta^{\pm\text{ell}}NF\)-Hodge theaters associated to the \( \Theta^{\pm\text{ell}}NF\)-Hodge theaters “\( \bullet \)” in the log-theta-lattice are vertically coric. That is to say, one may think of a \( D\Theta^{\pm\text{ell}}NF\)-Hodge theater, considered up to an indeterminate isomorphism, as an invariant of each vertical line of the log-theta-lattice. Moreover, the étale-like portion [i.e., the “\( G_\text{L}^{\ast}'s\)”] of the data of \( (a^g) \), or, equivalently, \( (a^q) \), of §3.3, (vii), again considered up to an indeterminate isomorphism, may be thought of as an object constructed from the \( D\Theta^{\pm\text{ell}}NF\)-Hodge theater associated to the \( \Theta^{\pm\text{ell}}NF\)-Hodge theater “\( \bullet \)” under consideration. In particular,

if one takes the radial data to consist of the \( D\Theta^{\pm\text{ell}}NF\)-Hodge theater associated to some vertical line of the log-theta-lattice [considered up to an indeterminate isomorphism!], the coric data to consist of the étale-like portion
<table>
<thead>
<tr>
<th>Frobenius-like version of functions</th>
<th>Kummer</th>
<th>étale-like version of functions</th>
</tr>
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<tbody>
<tr>
<td>evaluation</td>
<td>⇓</td>
<td>evaluation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frobenius-like version of values</th>
<th>Kummer$^{-1}$</th>
<th>étale-like version of values</th>
</tr>
</thead>
</table>

Fig. 3.11: The technique of Galois evaluation

[again considered up to an indeterminate isomorphism!] of the data of $(a^\Theta)$, or, equivalently, $(a^q)$, of §3.3, (vii), and the **radial algorithm** to be the assignment [i.e., “construction”] of the above discussion, then one obtains a radial environment, shown in Fig. 3.12 below, that is [“tautologically”!] **multiradial** [cf. [IUTchII], Corollary 4.11; [IUTchII], Fig. 4.3].

[Indeed, this multiradial environment may be thought of as being simply a *slightly more complicated version* of the multiradial environment of Example 3.2.2, (ii).] The diagram obtained by including, in the diagram of Fig. 3.12, not just two collections of radial data [that arise, say, from two adjacent vertical lines of the log-theta-lattice], but rather the collections of radial data that arise from *all of the vertical lines* of the log-theta-lattice is referred to as the **étale-picture** [cf. [IUTchII], Fig. 4.3]. Despite its tautological nature,

the **multiradiality** — i.e., **permutability** of $\mathcal{D}-\Theta^{\pm\text{ell}}NF$-Hodge theaters associated to **distinct vertical lines** of the log-theta-lattice — of the **étale-picture** is nonetheless *somewhat remarkable* since [prior to passage to the **étale-picture**!] the log-theta-lattice does **not** admit **symmetries** that **permute distinct vertical lines** of the log-theta-lattice.

Next, we consider the respective $\mathbb{F}^{\times\pm}_l$- **and** $\mathbb{F}^*_l$- **symmetries** of the constituent $\mathcal{D}-\Theta^{\pm\text{ell}}NF$-Hodge theaters [cf. §3.3, (v)]. In this context, it is useful to introduce symbols “$>$” and “$\geq$”:

- “$>$” denotes the **entire set** $\mathbb{F}_l$ that appears in the discussion of Fig. 3.8 in §3.3, (v), i.e., the notation “[...]]” in the upper left-hand corner of Fig. 3.8 [cf. [IUTchI], Fig. 6.5].
- “$\geq$” denotes the **entire set** $\mathbb{F}^*_l$ that appears in the discussion of Fig. 3.8 in
§3.3, (v), i.e., the notation “[...]” in the upper right-hand corner of Fig. 3.8 [cf. [IUTchI], Fig. 6.5].

- The gluing shown in Fig. 3.8 may be thought of as an assignment that sends

\[ 0, \succ \mapsto > \]

[cf. [IUTchI], Proposition 6.7; [IUTchI], Fig. 6.5; the discussion of [IUTchII], Remark 3.8.2, (ii); the symbol “△” of [IUTchII], Corollary 4.10, (i)].

- Ultimately, we shall be interested in computing weighted averages of log-volumes at the various labels in \( F_l \) or \( F_l^* \) (\( \subset |F_l| \overset{\text{def}}{=} F_l^* \cup \{0\} \)) [cf. [IUTchI], Remark 5.4.2; the computations of [IUTchIV], §1]. From this point of view, it is natural to think in terms of formal sums with rational coefficients

\[
[0], \; [\, |j| \,] \overset{\text{def}}{=} \frac{1}{2}([j] + [-j]), \\
[\succ] \overset{\text{def}}{=} \frac{1}{2}([0] + [1] + [-1] + \ldots + [l^*] + [-l^*]), \\
[>] \overset{\text{def}}{=} \frac{1}{2^{|l^*|}}([1] + [-1] + \ldots + [l^*] + [-l^*])
\]

— where the “\( j \)” and “...” indicate arguments that range within the positive integers between 1 and \( l^* = \frac{1}{2}(l - 1) \). Note that these assignments of formal sums are compatible with the gluing “0, \( \succ \mapsto > \)”, i.e., relative to which

\[ [0] \mapsto [>, \; [>] \mapsto [>, \; \frac{1}{(l^*+1)}([0] + [1] + \ldots + [|l^*|]) \mapsto [>] \]

— where we note that such relations may be easily verified by observing that the coefficients of “[\( j \)]” and “[\(-j \)]” always coincide and are independent of \( j \); thus, these relations may be verified by substituting a single indeterminate “\( w \)” for all of the symbols “[0]”, “[\( j \)]”, and “[\(-j \)]”.  

Fig. 3.12: The multiradiality of \( D-\Theta^{\pm \text{ell}}NF \)-Hodge theaters
If one extracts from the étale-picture the various $F_l^\ast$-symmetries of the respective $\mathcal{D}$-$\Theta^{\pm \text{ell}}$NF-Hodge theaters, then one obtains a diagram as in Fig. 3.13 above, i.e., a diagram of various distinct, independent $F_l^\ast$-actions that are "glued together at a common symbol 0" [cf. the discussion of [IUTchII], Remark 4.7.4; [IUTchII], Fig. 4.2]. This diagram may be thought of as a sort of combinatorial prototype for the phenomenon of multiradiality. On the other hand, if one extracts from the étale-picture the various $F_l^{\times \pm}$-symmetries of the respective $\mathcal{D}$-$\Theta^{\pm \text{ell}}$NF-Hodge theaters, then one obtains a diagram as in Fig. 3.14 above, i.e., a diagram of various mutually interfering $F_l^{\times \pm}$-actions that interfere with one another as a consequence of the fact that they are "glued together at a common symbol 0" [cf. the discussion of [IUTchII], Remark 4.7.4; [IUTchII], Fig. 4.1]. This diagram may be thought of as a sort of combinatorial prototype for the phenomenon of uniradiality. Finally, in this context, it is also of interest to observe that, if, in accordance with the point of view of the discussion of §2.14, one thinks of $F_l$ as a sort of finite discrete approximation of "$\mathbb{Z}$" [cf. [IUTchI], Remark 6.12.3, (i); [IUTchII], Remark 4.7.3, (i)], and one thinks "$\mathbb{Z}$" as the value group of the various completions [say, for simplicity, at $v \in \mathbb{V}^{\text{non}}$] of $K$, then

the $F_l^\ast$-symmetry corresponds to a symmetry that only involves the non-unit portions of these value groups at various $v \in \mathbb{V}^{\text{non}}$, while the $F_l^{\times \pm}$-symmetry
is a symmetry that involves a sort of “juggling” between local unit groups and local value groups.

This point of view is consistent with the fact [cf. (iii) below; Example 2.12.3, (v); §3.3, (ii), (vii); §3.4, (ii)] that the $\mathbb{F}_l^*$-symmetry is related only to the Kummer theory surrounding the global value group data $(c^\Theta)$, while [cf. (ii) below; Example 2.12.3, (v); §3.3, (ii), (vii); §3.4, (iii), (iv)] the $\mathbb{F}_l^{\pm}$-symmetry is related to both the [local] unit group data $(a^\Theta)$ and the local value group data $(b^\Theta)$, which are “juggled” about by the log-links of the log-theta-lattice.

(ii) **Theta values and local diagonals via the $\mathbb{F}_l^{\pm}$-symmetry:** Let $v \in \mathbb{V}^{\text{bad}}$. Write

\[ \Pi_{\pm} \subseteq \Pi_{\pm}^{\pm} \subseteq \Pi_{\pm}^{\text{cor}} \]

[cf. [IUTchII], Definition 2.3, (i)] for the inclusions of tempered arithmetic fundamental groups [for suitable choices of basepoints] determined by the finite étale coverings $X_{\mathbb{W}}^v \to X_{\mathbb{W}} \to G_{\mathbb{W}}$ [cf. §3.3, (i), (vi); the notational conventions discussed at the beginning of §3.4, (iii)]. These tempered arithmetic fundamental groups of hyperbolic orbicurves over $K_{\mathbb{W}}$ admit natural outer surjections to $G_{\mathbb{W}}$; write $\Delta_{\pm} \subseteq \Delta_{\pm}^\pm \subseteq \Delta_{\pm}^{\text{cor}}$ for the respective kernels of these surjections. In fact, $\Pi_{\pm}^\pm$ and $\Pi_{\pm}^{\text{cor}}$, together with the above inclusions, may be reconstructed functorially from the topological group $\Pi_{\mathbb{W}}$ [cf. [EtTh], Proposition 2.4]. The $\mathbb{F}_l^{\pm}$-symmetries of a $\Theta^{\text{ell}}NF$-Hodge theater [cf. §3.3, (v)] induce outer automorphisms of $\Pi_{\pm}^\pm$. Indeed, these outer automorphisms may be thought of as the outer automorphisms of $\Pi_{\mathbb{W}}^\pm$ induced by conjugation in $\Pi_{\mathbb{W}}^{\text{cor}}$ by the quotient group $\Pi_{\mathbb{W}}^{\text{cor}}/\Pi_{\mathbb{W}}^\pm$, which admits a natural outer isomorphism $\mathbb{F}_l^{\pm} \sim \to \Pi_{\mathbb{W}}^{\text{cor}}/\Pi_{\mathbb{W}}^\pm$ [cf. [IUTchII], Corollary 2.4, (iii)]. Moreover, since these outer automorphisms of $\Pi_{\mathbb{W}}^\pm$ arise from $K$-linear automorphisms of the hyperbolic curve $X_{K}$ [cf. the discussion of §3.3, (v); [IUTchII], Corollary 2.4, (iii)], let us observe that

the outer automorphisms of $\Pi_{\pm}^\pm$ under consideration may, in fact, be thought of as $\Delta_{\pm}^\pm$-outer automorphisms of $\Pi_{\mathbb{W}}^\pm$ [i.e., automorphisms defined up to composition with an inner automorphism induced by conjugation by an element of $\Delta_{\pm}^\pm$] induced by conjugation by elements of $\Delta_{\mathbb{W}}^{\text{cor}}$.

Next, let us recall from the discussion of §3.3, (v), concerning the $\mathbb{F}_l^{\pm}$-symmetry that elements of $\mathbb{F}_l$ may be thought of — up to $\mathbb{F}_l^{\pm}$-indeterminacies that may in fact, as a consequence of the structure of a $\Theta^{\text{ell}}NF$-Hodge theater, be synchronized in a fashion that is independent of the choice of $v \in \mathbb{V}^{\text{bad}}$ [cf. [IUTchI], Remark 6.12.4, (i), (ii), (iii)] — as labels of cusps of $X_{\mathbb{W}}$. Moreover, such cusps of $X_{\mathbb{W}}$ may be thought of, by applying a suitable functorial group-theoretic algorithm, as certain conjugacy classes of subgroups [i.e., cuspidal inertia subgroups] of $\Pi_{\mathbb{W}}^\pm$ [cf. [IUTchI], Definition 6.1, (iii)]. In particular, the above observation implies that, if we think of $G_{\mathbb{W}}$ as a quotient of one
of the tempered arithmetic fundamental groups $\Pi_\Sigma$, $\Pi_\Sigma^\pm$, $\Pi_\Sigma^{\text{cor}}$ discussed above, and we consider copies of this quotient $G_\Sigma$ equipped with labels $$(G_\Sigma)_t$$ where we think of $t \in F_l$ as a conjugacy class of cuspidal inertia subgroups of $\Pi_\Sigma^\pm$—then

the action of the $F_l^{\times\pm}$-symmetry [i.e., by conjugation in $\Pi_\Sigma^{\text{cor}}$] on these labeled quotients $\{(G_\Sigma)_t\}_{t \in F_l}$ induces symmetrizing isomorphisms between these labeled quotients that are free of any inner automorphism indeterminacies [cf. [IUTchII], Corollary 3.5, (i); [IUTchII], Remark 3.5.2, (iii); [IUTchII], Remark 4.5.3, (i)].

The existence of these symmetrizing isomorphisms is a phenomenon that is sometimes referred to as conjugate synchronization. Note that this sort of situation differs radically from the situation that arises for the isomorphisms induced by conjugation in $G_K := \text{Gal}(\mathcal{F}/K)$ between the various decomposition groups of $\nu$ [that is to say, copies of “$G_\nu$”], i.e., isomorphisms which are only well-defined up to composition with some indeterminate inner automorphism of the decomposition group under consideration [cf. the discussion of [IUTchII], Remark 2.5.2, (iii)]. Relative to the theme of “synchronizing”, another important role played by the $F_l^{\times\pm}$-symmetry is the role of synchronizing the $\pm$-indeterminacies that occur at the portions labeled by various valuations $\nu \in \mathbb{V}$ on the “left-hand side” [cf. Fig. 3.8] of a $\Theta_{\text{ell}}^\pm N\text{F-Hodge theater}$ [cf. [IUTchII], Remark 4.5.3, (iii)]. In this context, since tempered arithmetic fundamental groups, unlike conventional profinite étale fundamental groups, are only defined at a specific $\nu \in \mathbb{V}^{\text{bad}}$, one technical issue that arises, when one considers the task of relating the symmetrizing isomorphisms discussed above at different $\nu \in \mathbb{V}^{\text{bad}}$ [or, indeed, to the theory at valuations $\nu \in \mathbb{V}^{\text{good}}$] is the issue of comparing tempered and profinite conjugacy classes of various types of subgroups [i.e., such as cuspidal inertia groups] — an issue that is resolved [cf. the application of [IUTchI], Corollary 2.5, in the proof of [IUTchII], Corollary 2.4] by applying the theory of [Semi].

The symmetrizing isomorphisms discussed above may be applied not only to copies of the étale-like object $G_\Sigma$ but also to various Frobenius-like objects that are “closely related” to $G_\Sigma$ [cf. the pairs “$G_k \ltimes \mathcal{O}_k^\times$” of Example 2.12.1; [IUTchII], Corollary 3.6, (i)]. Moreover, an analogous theory of symmetrizing isomorphisms may be developed at valuations $\nu \in \mathbb{V}^{\text{good}}$ [cf. [IUTchII], Corollary 4.5, (iii); [IUTchII], Corollary 4.6, (iii)]. The graphs, or diagonals, of these symmetrizing isomorphisms at various valuations $\nu$
may be thought of as corresponding to the symbol “≻” discussed in (i), or indeed, after applying the gluing that appears in the structure of a \( \Theta^{\pm \text{ell}} N F \)-Hodge theater [cf. (i); §3.3, (v)], to the symbols “0”, “≻” [cf. IUTchII, Corollary 3.5, (iii); IUTchII, Corollary 3.6, (iii); IUTchII, Corollary 4.5, (iii); IUTchII, Corollary 4.6, (iii); IUTchII, Corollary 4.10, (i)]. Moreover,

the data labeled by these symbols “≻”, “0”, “≻”, form the data that is ultimately actually used in the horizontally coric unit group portion \((a^{\Theta}), (a^q)\) [cf. §3.3, (vii)] of the data in the codomain and domain of the \( \Theta \)-link [cf. IUTchIII, Theorem 1.5, (iii)].

The significance of this approach to constructing the data of \((a^{\Theta}), (a^q)\) lies in the fact that the “descent” [cf. IUTchIII, Remark 1.5.1, (i)] from the individual labels \( t \in \mathbb{F}_l \) to the symbols “≻” / “0” / “≻” that is effected by the various symmetrizing isomorphisms gives rise to horizontally coric data — i.e., data that is shared by the codomain and domain of the \( \Theta \)-link — that serves as a container [cf. the discussion of (iv) below] for the various theta values [well-defined up to multiplication by a 2l-th root of unity]

\[ q^{j^2} \]

— where we think of \( j = 1, \ldots, l^* \) as corresponding to an element of \( \mathbb{F}_l^* \) obtained by identifying two elements \( \pm t \in \mathbb{F}_l^* \subseteq \mathbb{F}_l \) — obtained by Galois evaluation [cf. IUTchII, Corollary 2.5; IUTchII, Remark 2.5.1; IUTchII, Corollary 3.5, (ii); IUTchII, Corollary 3.6, (ii)], i.e., by restricting the Kummer classes of the [reciprocals of \( l \)-th roots of] theta functions on \( \tilde{Y}_v \) discussed in §3.4, (iii) [cf. also the data “\((a^{l^*})\)” discussed in §3.5]

- first to the decomposition groups, denoted by the notation “▷”, in the open subgroup of \( \Pi_{\mathbb{F}} \) corresponding to \( \tilde{Y}_v \) determined [up to conjugation in \( \Pi_{\mathbb{F}} \) — cf. IUTchII, Proposition 2.2; IUTchII, Corollary 2.4] by the connected — i.e., so as not to give rise to distinct basepoints for distinct labels \( j = 1, \ldots, l^* \) [cf. the discussion of IUTchII, Remarks 2.6.1, 2.6.2, 2.6.3] — “line segment” of labels of irreducible components of the special fiber of \( \tilde{Y}_v \)

\[ \{-l^*, -l^* + 1, \ldots, -1, 0, 1, \ldots, l^* - 1, l^*\} \subseteq \mathbb{Z} \]

[cf. Fig. 3.9 and the surrounding discussion; IUTchII, Remark 2.1.1, (ii)]; and

- then to the decomposition groups associated to “evaluation points” — i.e., cusps translated by a zero-labeled evaluation point — labeled by \( \pm t \in \mathbb{F}_l^* \subseteq \mathbb{F}_l \).
Finally, we remark that the **Kummer theory** that relates the corresponding *étale-like* and *Frobenius-like* data that appears in the various **symmetrizing isomorphisms** just discussed only involves *local* data, i.e., the data of \((a^{\Theta})\) and \((b^{\Theta})\), hence [cf. the discussion of §3.4, (v); Fig. 3.10] is **compatible** with the **topologies** of the various tempered or profinite Galois or arithmetic fundamental groups involved. The **significance** of this **compatibility with topologies** lies in the fact it means that the **Kummer isomorphisms** that appear may be **computed** relative to some finite *étale covering* of the schemes involved, i.e., relative to a situation in which — unlike the situation that arises if one considers some sort of *projective limit* of multiplicative monoids associated to rings — the **ring structure** of the schemes involved is **still intact**. That is to say, since the \(\log\)-link is defined by applying the formal **power series** of the natural logarithm, an object that can **only be defined if both** the additive and the multiplicative structures of the [topological] rings involved are available,

this **computability** allows one to **compare** — hence to establish the **compatibility of** — the various **symmetrizing isomorphisms** just discussed in the **codomain and domain** of the \(\log\)-link [cf. [IUTchII], Remark 3.6.4, (i); [IUTchIII], Remark 1.3.2; the discussion of Step (vi) of the proof of [IUTchIII], Corollary 3.12].

This **compatibility** plays an important role in inter-universal Teichmüller theory.

(iii) **Number field values and global diagonals via the \(F^{\ast}\)-symmetry:** We begin by considering certain **field extensions** of the field \(F_{\text{mod}}\): write

- \(F_{\text{sol}} \subseteq F\) for the maximal solvable extension of \(F_{\text{mod}}\) in \(F\) [cf. [IUTchI], Definition 3.1, (b)];
- \(F(\mu_{l}, C_{F})\) for the field obtained by adjoining to the **function field** of \(C_{F}\) the \(l\)-th roots of unity [cf. the field \(F(\mu_{l}) \cdot L_{C}\) of [IUTchI], Remark 3.1.7, (iii)];
- \(F(\mu_{l}, \kappa\text{-sol})\) for the field obtained by adjoining to \(F(\mu_{l}, C_{F})\) arbitrary roots of \(F_{\text{sol}}\)-multiples of \(\kappa\text{-coric rational functions}\) [cf. §3.4, (ii)] in \(F(\mu_{l}, C_{F})\) [cf. the field \(F(\mu_{l}) \cdot L_{C}(\kappa\text{-sol})\) of [IUTchI], Remark 3.1.7, (iii)];
- \(F(C_{K})\) for the Galois closure over the field \(F(\mu_{l}, C_{F})\) of the **function field** of \(C_{K}\) [cf. the field \(L_{C}(C_{K})\) of [IUTchI], Remark 3.1.7, (iii)].

Then one verifies immediately, by applying the fact that the finite group \(SL_{2}(\mathbb{F}_{l})\) [where we recall from §3.3, (i), that \(l \geq 5\)] is **perfect**, that

\(F(\mu_{l}, \kappa\text{-sol})\) and \(F(C_{K})\) are **linearly disjoint** over \(F(\mu_{l}, C_{F})\) [cf. [IUTchI], Remark 3.1.7, (iii)].

It then follows, in an essentially **formal** way, from this **linear disjointness** [cf. [IUTchI],
Remark 3.1.7, (ii), (iii); [IUTchI], Example 5.1, (i), (v); [IUTchI], Remark 5.1.5; [IUTchII], Corollary 4.7, (i), (ii); [IUTchII], Corollary 4.8, (i), (ii)] that:

- the various elements in $F_{\text{mod}}$ or $F_{\text{sol}}$

may be obtained by Galois evaluation, i.e., by restricting the Kummer classes of the $\kappa$-coric rational functions discussed in §3.4, (ii), to the various decomposition groups that arise, respectively, from $F_{\text{mod}}$- or $F_{\text{sol}}$-rational points;

- this construction of $F_{\text{mod}}$ or $F_{\text{sol}}$ via Galois evaluation may be done in a fashion that is compatible with the labels $\in F_i^\pm$ and the $F_i^\pm$-symmetry that appear in a $\Theta^{\pm \text{ell}}$ $NF$-Hodge theater [cf. the discussion of §3.3, (v); the right-hand side of Fig. 3.8];

- in particular, this compatibility with labels and the $F_i^\pm$-symmetry induces symmetrizing isomorphisms between copies of $F_{\text{mod}}$ or $F_{\text{sol}}$ that determine graphs, or diagonals, which may be thought of as corresponding to the symbol “$>$” [cf. the discussion of (i), (ii)].

In this context, we note that this approach to constructing elements of $NF$’s by restricting Kummer classes of rational functions on hyperbolic curves to decomposition groups of points defined over an $NF$ is precisely the approach taken in the functorial algorithms of [AbsTopIII], Theorem 1.9 [cf., especially, [AbsTopIII], Theorem 1.9, (d)]. Also, we observe that, although much of the above discussion runs in a somewhat parallel fashion to the discussion in (ii) of the $F_i^{\pm}$-symmetry and the construction of theta values via Galois evaluation, there are important differences, as well, between the theta and $NF$ cases [cf. [IUTchIII], Remark 2.3.3]:

- First of all, the symmetrizing isomorphisms/diagonals associated to the $F_i^\pm$-symmetry are not compatible with the symmetrizing isomorphisms/diagonals associated to the $F_i^{\pm}$-symmetry, except on the respective restrictions of these two collections of symmetrizing isomorphisms to copies of $F_{\text{mod}}$ [cf. [IUTchII], Remark 4.7.2]. Moreover, for various technical reasons related to conjugate synchronization, it is of fundamental importance in the theory to isolate the $F_i^\pm$-symmetry from the $F_i^{\pm}$-symmetry [cf. the discussion of [IUTchII], Remarks 2.6.2, 4.7.3, 4.7.5, 4.7.6].

- Unlike the Kummer theory applied in the theta case, the Kummer theory applied in the $NF$ case is not compatible with the topologies of the various profinite Galois or arithmetic fundamental groups that appear [cf. the discussion of §3.4, (v)]. On the other hand, this will not cause any problems since
there is no issue, in the NF case, of applying formal power series such as the power series of the natural logarithm [cf. the final portion of the discussion of (ii); [IUTchIII], Remark 2.3.3, (vii), (viii)].

- In the Galois evaluation applied in the theta case, one is concerned with constructing, at a level where the arithmetic holomorphic structure [i.e., the ring structure] is still intact, theta values that depend, in an essential way, on the label “j”. By contrast, in the Galois evaluation applied in the NF case, one only constructs, at such a level where the arithmetic holomorphic structure is still intact, the totality of [various copies of] the multiplicative monoid $F_{\text{mod}}^\times$ associated to the number field $F_{\text{mod}}$: that is to say, a dependence on the label “j” only appears at the level of the mono-analytic structures constituted by the global realified Frobenioids of $(e^\Theta)$, i.e., in the form of a sort of ratio, or weight, “$j^2$” [cf. the fourth display of [IUTchII], Corollary 4.5, (v)] between the arithmetic degrees at the label “j” and the arithmetic degrees at the label “1” [cf. [IUTchIII], Remark 2.3.3, (iii); [IUTchIII], Remark 3.11.4, (i)].

In the context of this final difference between the theta and NF cases, it is perhaps of interest to observe that a similar sort of “weighted copy” $(F_{\text{mod}}^\times)^{j^2}$ $(\subseteq F_{\text{mod}}^\times)$ of $F_{\text{mod}}^\times$ is not possible at the level of arithmetic holomorphic structures [that is to say, in the sense that it is not compatible with the additive interpretation of line bundles, i.e., in terms of modules] since this “weighted copy” $(F_{\text{mod}}^\times)^{j^2}$ $(\subseteq F_{\text{mod}}^\times)$ is not closed under addition [cf. the discussion of the final portion of §3.3, (iv); [AbsTopIII], Remark 5.10.2, (iv)].

(iv) Actions on log-shells: At this point, the reader may have noticed two apparent shortcomings [which are, in fact, closely related!] in the theory of Galois evaluation developed thus far in the present §3.6:

- Unlike the case with the Kummer theory of $\kappa$-coric rational functions and theta functions/mono-theta environments discussed in §3.4, the discussion of Galois evaluation given above does not mention any multiradiality properties.
- Ultimately, one is interested in relating [the Kummer theory surrounding] Frobenius-like structures — such as the theta values and copies of NF’s that arise from Galois evaluation — in the domain of the $\Theta$-link to [the Kummer theory surrounding] Frobenius-like structures in the codomain of the $\Theta$-link, i.e., in accordance with the discussion of the technique of mono-anabelian transport in §2.7, §2.9. On the other hand, since the theta values and copies of NF’s that arise from Galois evaluation are not [necessarily] local units at the various $z \in \mathbb{V}$, it is by no means clear how to relate this Galois evaluation output data to the codomain of the $\Theta$-link using the horizontally coric portion — i.e., the [local] unit group portion $(a^\Theta)$ and $(a^q)$ — of the $\Theta$-link.
In fact, these two shortcomings are closed related: That is to say, the existence of the\textit{ obstruction} discussed in §3.4, (i), to the approach to \textit{Kummer theory} taken in Example 2.12.1, (ii), that arises from the \textit{natural action} “$\mathcal{O}_k^\times \times \hat{\mathbb{Z}}^\times$” implies — in light of the \textit{nontrivial extension structure} that exists between the \textit{value groups} and \textit{units} of finite subextensions of “$k$” in “$\overline{k}$” [cf. the discussion in the final portion of Example 2.12.1, (iii)] — that, at least in any sort of \textit{a priori} or \textit{natural} sense,

the \textbf{output data} — i.e., \textit{theta values} and \textit{copies of NF’s} — of the \textit{Galois evaluation} operations discussed in (ii), (iii) above [i.e., which lies in various copies of “$k$”] is \textbf{by no means multiradial} [cf. [IUTchII], Remark 2.9.1, (iii); [IUTchII], Remark 3.4.1, (ii); [IUTchII], Remark 3.7.1; [IUTchIII], Remark 2.2.1, (iv)].

One of the \textbf{fundamental ideas} of inter-universal Teichmüller theory is that

one may apply the theory of the \textit{log-link} and \textit{log-shells} to obtain a \textbf{solution} to these closely related shortcomings.

More precisely, from the point of view of the \textit{log-theta-lattice}, the \textit{log-link} from the lattice point $(n, m - 1)$ to the lattice point $(n, m)$ [where $n, m \in \mathbb{Z}$] allows one to construct [cf. the notation of Example 2.12.3, (iv)] a \textit{holomorphic Frobenius-like log-shell} “$\mathcal{I}$” at $(n, m)$ from the “$\mathcal{O}_k^\times \times \hat{\mathbb{Z}}^\times$” at $(n, m - 1)$. Thus,

the \textbf{output data} — i.e., \textit{theta values} and \textit{copies of NF’s} — of the \textit{Galois evaluation} operations discussed in (ii), (iii) above at $(n, m)$ \textbf{acts naturally} on the “$\mathcal{I} \otimes \mathbb{Q}$” [i.e., the copy of “$k$”] at $(n, m)$ that arises from this \textit{log-link} from $(n, m - 1)$ to $(n, m)$ [cf. [IUTchIII], Proposition 3.3, (i); [IUTchIII], Proposition 3.4, (ii); [IUTchIII], Definition 3.8, (ii)].

On the other hand, this gives rise to a \textbf{fundamental dilemma}:

Since the construction of the \textit{log-link} — i.e., at a more concrete level, the \textit{formal power series} of the \textit{natural logarithm} — can only be defined if \textit{both} the additive \textit{and} the multiplicative structures of the [topological] rings involved are available, the \textit{log-link} and hence, in particular, the construction of \textit{log-shells} just discussed, at, say, the lattice point $(n, m)$, are \textbf{meaningless}, at least in any \textit{a priori} sense, from the point of view of the lattice point $(n + 1, m)$, i.e., the \textit{codomain} of the \textit{Θ-link} from $(n, m)$ to $(n + 1, m)$ [cf. [IUTchIII], Remark 3.11.3; Steps (iii) and (iv) of the proof of [IUTchIII], Corollary 3.12].

Another of the \textbf{fundamental ideas} of inter-universal Teichmüller theory is the following [cf. the discussion in §3.3, (ii), of the \textit{symmetry} of the \textit{[infinite!] vertical lines} of the log-theta-lattice with respect to arbitrary \textbf{vertical translations}]:
By considering structures that are invariant with respect to vertical shifts of the log-theta-lattice — i.e., vertically coric structures such as holomorphic/mono-analytic étale-like log-shells that serve as containers for Frobenius-like objects such as holomorphic Frobenius-like log-shells [cf. Fig. 3.15 below] or notions of vertical invariance such as upper semi-commutativity or log-volume compatibility [cf. Example 2.12.3, (iv)] — and then transporting these invariant structures to the opposite side of the Θ-link by means of mono-analytic Frobenius-like log-shells [i.e., which may be constructed directly from the data “$O^X_{\kappa}$” that appears in the horizontally coric data $(a^\Theta), (a^g)$ of the Θ-link], one may construct multiradial containers for the output data — i.e., theta values and copies of NF’s — of the Galois evaluation operations discussed above.

Fig. 3.15: A vertical line of the log-theta-lattice [shown horizontally]: holomorphic Frobenius-like structures “•” at each lattice point related, via various Kummer isomorphisms [i.e., vertical or diagonal arrows], to vertically coric holomorphic étale-like structures “◦”

Here, we observe that each of the four types of log-shells discussed in Example 2.12.3, (iv), plays an indispensable role in the theory [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob7); [IUTchIII], Remark 3.12.2, (iv), (v)]:

- the holomorphic Frobenius-like log-shells satisfy the property [unlike their mono-analytic counterparts!] that the log-link — whose construction requires the use of the topological ring structure on these log-shells! — may be applied to them, as well as the property [unlike their étale-like counterparts!] that they belong to a fixed vertical position of a vertical line of the log-theta-lattice, hence are meaningful even in the absence of the log-link and, in particular, may be related directly to the Θ-link;
- the holomorphic étale-like log-shells allow one [unlike their Frobenius-like counterparts!] to relate holomorphic Frobenius-like log-shells at different vertical positions of a vertical line of the log-theta-lattice to one another in a fashion [unlike their mono-analytic counterparts!] that takes into account the log-link [whose construction requires the use of the topological ring structure on these log-shells!];
- the mono-analytic Frobenius-like log-shells satisfy the property [unlike their holomorphic counterparts!] that they may be constructed directly from
the data “$O^\times_{\mathbb{K}}$” that appears in the horizontally coric data $(a^\Theta), (a^q)$ of the $\Theta$-link, as well as the property [unlike their étale-like counterparts!] that they belong to a fixed vertical position of a vertical line of the log-theta-lattice, hence are meaningful even in the absence of the log-link and, in particular, may be related directly to the $\Theta$-link [cf. the discussion in the final portion of §3.3, (ii)];

- the mono-analytic étale-like log-shells satisfy the property [unlike their holomorphic counterparts!] that they may be constructed directly from the data “$G_k$” that appears in the horizontally coric data $(a^\Theta), (a^q)$ of the $\Theta$-link, as well as the property [unlike their Frobenius-like counterparts, when taken alone!] that they may be used, in conjunction with their Frobenius-like counterparts, to relate mono-analytic Kummer theory [i.e., Kummer isomorphisms between mono-analytic Frobenius-like/étale-like log-shells] to holomorphic Kummer theory [i.e., Kummer isomorphisms between holomorphic Frobenius-like/étale-like log-shells].

In particular, the significance of working with mono-analytic Frobenius-like log-shells may be understood as follows. If, instead of working with mono-analytic Frobenius-like log-shells, one simply passes from holomorphic Frobenius-like log-shells at arbitrary vertical positions [in a single vertical line of the log-theta-lattice] to holomorphic étale-like log-shells [via Kummer isomorphisms] and then to mono-analytic étale-like log-shells [by forgetting various structures] — i.e.,

\[
\begin{array}{ccc}
\text{holomorphic} & \sim & \text{holomorphic} \\
\text{Frobenius-like} & \sim & \text{étale-like} \\
\text{log-shells} & \sim & \text{log-shells} \\
\end{array}
\]

— then the relationship between the mono-analytic étale-like log-shells [whose vertical position is indeterminate!] and the various holomorphic Frobenius-like log-shells [in the vertical line under consideration] is subject simultaneously to indeterminacies arising from both the log-link [i.e., “(Ind3)” — cf. the discussion of §3.7, (i), below] and the $\Theta$-link [i.e., “(Ind1)” — cf. the discussion of §3.7, (i), below]. Relative to the analogy discussed in the final portion of §3.3, (ii), between the log-theta-lattice and “$\mathbb{C}^\times \setminus GL^+(V)/\mathbb{C}^\times$”, such simultaneous indeterminacies correspond to indeterminacies with respect to the action of the subgroup of $GL^+(V)$ generated by $\mathbb{C}^\times$ and “$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$”. By contrast, by stipulating that the passage from holomorphic étale-like log-shells to mono-analytic étale-like log-shells be executed in conjunction with the Kummer isomorphisms [implicit in the data that is glued together in definition of the $\Theta$-link] with corresponding Frobenius-like log-shells at the vertical position “0” — i.e.,
Holomorphic Frobenius-like log-shells $\sim$ Holomorphic étale-like log-shells $\sim$ Mono-analytic étale-like log-shells

$(\text{Ind}3) \sim \uparrow \sim \uparrow \sim (\text{Ind}1,2)$

Holomorphic Frobenius-like log-shells at 0 $\sim$ Mono-analytic Frobenius-like log-shells at 0

[where the vertical arrows denote the respective Kummer isomorphisms] — one obtains a “decoupling” of the log-link/Θ-link indeterminacies, i.e.,

- a partially rigid relationship between the holomorphic/mono-analytic étale-like log-shells and the holomorphic/mono-analytic Frobenius-like log-shells at the vertical position 0 [in the vertical line under consideration] that is subject only to indeterminacies arising from the Θ-link [i.e., “(Ind1), (Ind2)” — cf. the discussion of §3.7, (i), below], together with
- a partially rigid relationship between the holomorphic Frobenius-like log-shells at the vertical position 0 and the various holomorphic Frobenius-like log-shells at arbitrary vertical positions [in the vertical line under consideration], i.e., via holomorphic étale-like log-shells, that is subject only to indeterminacies arising from the log-link [i.e., “(Ind3)” — cf. the discussion of §3.7, (i), below].

Relative to the analogy discussed in the final portion of §3.3, (ii), between the log-theta-lattice and “$\mathbb{C}^x\backslash GL^+(V)/\mathbb{C}^x$”, this “decoupling” of indeterminacies corresponds to an indeterminacy with respect to an action of $\mathbb{C}^x$ from the left, together with a distinct action of “$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$” from the right.

(v) **Processions:** In order to achieve the multiradiality discussed in (iv) for Galois evaluation output data, one further technique must be introduced, namely, the use of processions [cf. [IUTch1], Definition 4.10], which serve as a sort of mono-analytic substitute for the various labels in $\mathbb{F}_l^*, |\mathbb{F}_l| (= \mathbb{F}_l^* \cup \{0\})$, or $\mathbb{F}_l$ discussed in (i). That is to say, since these labels are closely related to the various cuspidal inertia subgroups of the geometric fundamental groups “$\Delta$” of the hyperbolic orbicurves involved [cf. the
discussion of (i), (ii); §2.13; §3.3, (v)], it follows that these labels are not horizontally coric [i.e., not directly visible to the opposite side of the Θ-link — cf. the discussion of [IUTchIV], Remark 3.6.3, (ii)] and indeed do not even admit, at least in any a priori sense, any natural multiradial formulation. The approach taken in inter-universal Teichmüller theory to dealing with this state of affairs [cf. [IUTchI], Proposition 6.9] is to consider the diagram of inclusions of finite sets

\[
S^\pm_1 \hookrightarrow S^\pm_{1+1=2} \hookrightarrow \ldots \hookrightarrow S^\pm_{j+1} \hookrightarrow \ldots \hookrightarrow S^\pm_{1+l^*} = l^*.
\]

— where we write \(S^\pm_{j+1} \overset{\text{def}}{=} \{0, 1, \ldots, j\}\), for \(j = 0, \ldots, l^*\), and we think of each of these finite sets as being subject to arbitrary permutation automorphisms. That is to say, we think of

the set \(S^\pm_{j+1}\) as a container for the labels \(0, 1, \ldots, j\), and of the label “\(j\)” as “some” element of this container set, i.e., for each \(j\), there is an indeterminacy of \(j + 1\) possibilities for the element of this container set that corresponds to \(j\).

Here, we note in passing that this sort of indeterminacy is substantially milder than the indeterminacies that occur if one considers each \(j\) only as “some” element of \(S^\pm_{l^*}\), in which case every \(j\) is subject to an indeterminacy of \(l\) possibilities — cf. [IUTchI], Proposition 6.9, (i), (ii). One then regards

each such container set as an index set for a collection — which is referred to as a “capsule” [cf. [IUTchI], §0] — of copies of some sort of étale-like mono-analytic prime-strip.

An étale-like mono-analytic prime-strip is, roughly speaking, a collection of copies of data “\(G_k\)” indexed by \(v \in V\) [cf. [IUTchI], Fig. I1.2, and the surrounding discussion; [IUTchI], Definition 4.1, (iii)]. Now each étale-like mono-analytic prime-strip in a capsule, as described above, gives rise, at each \(v \in V\), to

mono-analytic étale-like log-shells, on which the Galois evaluation output data acts, in the fashion described in (iv), up to various indeterminacies that arise from the passage from holomorphic Frobenius-like log-shells to mono-analytic étale-like log-shells [cf. Figs. 3.16, 3.17 below, where each “/±” denotes an étale-like mono-analytic prime-strip].

These indeterminacies will be discussed in more detail in §3.7, (i), below. In fact, ultimately, from the point of various log-volume computations, it is more natural to consider the Galois evaluation output data as acting, up to various indeterminacies, on certain tensor products of the various log-shells indexed by a particular container set \(S^\pm_{j+1}\). Such tensor products are referred to as tensor packets [cf. [IUTchIII], Propositions 3.1, 3.2].
In the following, we outline the statements of and relationships between the main results [cf. [IUTchIII], Theorem 3.11; [IUTchIII], Corollary 3.12; [IUTchIV], Theorem 1.10; [IUTchIV], Corollaries 2.2, 2.3] of inter-universal Teichmüller theory.

(i) Multiradial representation of the $\Theta$-pilot object up to mild indeterminacies: The content of the discussion of §3.4, §3.5, and §3.6 may be summarized as follows [cf. [IUTchIII], Theorem A; [IUTchIII], Theorem 3.11]:

the data in the domain $(a^\Theta)$, $(b^\Theta)$, and $(c^\Theta)$ [cf. §3.3, (vii)] of the $\Theta$-link may be expressed in a fashion that is multiradial, when considered up to certain indeterminacies (Ind1), (Ind2), (Ind3) [cf. the discussion below], with respect to the radial algorithm

$$(a^\Theta), (b^\Theta), (c^\Theta) \mapsto (a^\Theta)$$

[cf. the discussion at the beginning of §3.4] by regarding this [Frobenius-like!] data $(a^\Theta)$, $(b^\Theta)$, and $(c^\Theta)$ [up to the indeterminacies (Ind1), (Ind2), (Ind3)] as data [cf. Fig. 3.18 below] that is constructed by

- first applying the Kummer theory and multiradial decouplings/cyclotomic rigidity of $\kappa$-coric rational functions in the case of $(c^\Theta)$ [cf. §3.4, (ii), (v)] and of theta functions/mono-theta environments in the case of $(b^\Theta)$ [cf. §3.4, (iii), (iv), (v); §3.5]; and
- then applying the theory of Galois evaluation, log-shells, and processions, together with symmetrizing isomorphisms at $l$-torsion points via the $F_{\text{mod}}^{x+}$-symmetry, in the case of $(b^\Theta)$ [cf. §3.6, (i), (ii), (iv), (v)], and at decomposition groups corresponding to $F_{\text{mod}}^{-}/F_{\text{sol}}$-rational points via the $F_{\text{mod}}^{-}$-symmetry, in the case of $(c^\Theta)$ [cf. §3.6, (i), (iii), (iv), (v)].
Here, we recall from §3.6, (iv), (v), that the data \((b^\Theta)\) and \((c^\Theta)\) act on processes of tensor packets that arise from the mono-analytic étale-like log-shells constructed from the data \((a^\Theta)\). The indeterminacies \((\text{Ind1}), (\text{Ind2}), (\text{Ind3})\) referred to above act on these log-shells and may be described as follows:

**Ind1** These indeterminacies are the \textbf{étale-transport} indeterminacies [cf. §2.7, (vi); Example 2.12.3, (i)] that occur as a result of the automorphisms [which, as was discussed in Example 2.12.3, (i), do not, in general, preserve the ring structure] of the various “\(G_k\)’s” that appear in the data \((a^\Theta)\).

**Ind2** These indeterminacies are the \textbf{Kummer-detachment} indeterminacies [cf. §2.7, (vi)] that occur as a result of the identification of, or confusion between, mono-analytic Frobenius-like and mono-analytic étale-like log-shells [cf. the discussion of §3.6, (iv)]. At a more concrete level, these indeterminacies arise from the action of the group of “isometries” — which is often denoted

\[\text{Ism}(-)\]

[cf. [IUTChII], Example 1.8, (iv)] — of the data \(\mathcal{O}_K^{\times \mu}\) [which we regard as equipped with a system of integral structures, or lattices] of \((a^\Theta)\), i.e., the \textbf{compact topological group} of \(G_k\)-equivariant automorphisms of the ind-topological module \(\mathcal{O}_K^{\times \mu}\) that, for each open subgroup \(H \subseteq G_k\), preserve the lattice \(\mathcal{O}_K^{\times \mu} \subseteq (\mathcal{O}_K^{\times \mu})^H\).

**Ind3** These indeterminacies are the \textbf{Kummer-detachment} indeterminacies [cf. §2.7, (vi); §3.6, (iv)] of the \textbf{log-Kummer correspondence} [cf. [IUTChIII], Remark 3.12.2, (iv), (v)], i.e., the system of log-links and Kummer isomorphisms of a particular vertical line of the log-theta-lattice [cf. Fig. 3.15].

In addition to these “explicitly visible” indeterminacies, there are also “invisible indeterminacies” [cf. [IUTChIII], Remark 3.11.4] that in fact arise, but may be ignored in the above description, essentially as a formal consequence of the way in which the various objects that appear are defined:
The various theta values and copies of \( F_{\text{mod}}^x \) that occur as Frobenius-like Galois evaluation output data at various vertical positions of the log-Kummer correspondence [cf. the discussion of \( \S 3.6, \text{(iv), (v)} \)] satisfy an important “non-interference” property [cf. \[IUTchIII\], Proposition 3.5, (ii), (c); \[IUTchIII\], Proposition 3.10, (ii)]: namely, the intersection of such output data with the product of the local units [i.e., “\( \mathcal{O}^x \)’”] at those elements of \( \mathcal{V} \) at which the output data in question occurs consists only of roots of unity. As a result, the only “possible confusion”, or “indeterminacy”, that occurs as a consequence of possibly applying iterates of the log-link to the various local units consists of a possible multiplication by a root of unity. On the other hand, since the theta values and copies of \( F_{\text{mod}}^x \) that occur as Frobenius-like Galois evaluation output data are defined in such a way as to be stable under the action by multiplication by such roots of unity, this indeterminacy may, in fact, be ignored [cf. the discussion of \[IUTchIII\], Remark 3.11.4, (i)].

The indeterminacy of possible multiplication by \( \pm 1 \) in the cyclotomic rigidity isomorphism that is applied in the Kummer theory of \( \kappa \)-coric rational functions [cf. the final portion of the discussion of \( \S 3.4, \text{(ii)} \)] may be ignored since the global Frobenioids related to the data \( (c^\Theta) \), i.e., that arise from copies of \( F_{\text{mod}} \), only require the use of the totality of [copies of] the multiplicative monoid \( F_{\text{mod}}^x \), which is stabilized by the operation of inversion [cf. the discussion of \[IUTchIII\], Remark 3.11.4, (i)].

At this point, it is useful to recall [cf. the discussion at the beginning of \( \S 3.4, \text{(i)} \)] that it was possible to define, in \( \S 3.3, \text{(vii)} \), the gluing isomorphisms that constitute the \( \Theta \)-link between the domain data \( (a^\Theta), (b^\Theta), (c^\Theta) \) and the codomain data \( (a^\varphi), (b^\varphi), (c^\varphi) \) precisely because we worked with various abstract monoids or global realified Frobenioids, i.e., as opposed to the “conventional scheme-like representations” of this data \( (a^\Theta), (b^\Theta), (c^\Theta) \) in terms of theta values and copies of NF’s. In particular, one way to interpret the multiradial representation discussed above [cf. the discussion of “simultaneous execution” at the beginning of \( \S 2.9 \) and \( \S 3.4, \text{(i)} \); the discussion of the properties “IPL”, “SHE”, “APT”, “HIS” in \[IUTchIII\], Remark 3.11.1] is as follows:

This multiradial representation may be understood as the [somewhat surprising!] assertion that not only the domain data \( (a^\Theta), (b^\Theta), (c^\Theta) \), but also the codomain data \( (a^\varphi), (b^\varphi), (c^\varphi) \) — or, indeed,

any collection of data

[i.e., not just the codomain data \( (a^\varphi), (b^\varphi), (c^\varphi) \)!]

that is isomorphic to the domain data \( (a^\Theta), (b^\Theta), (c^\Theta) \)

[cf., e.g., the discussion concerning “\( q^\lambda \)” in the second to last display of \( \S 3.8 \) below] — can, when regarded up to suitable indeterminacies, be represented
via the “conventional scheme-like representation” of \((a^\Theta),\ (b^\Theta),\ (c^\Theta)\)
in a fashion that is compatible with the original “conventional scheme-like representation” of the given collection of data [i.e., such as the “conventional scheme-like representation” of the data \((a^q),\ (b^q),\ (c^q)\)].

If we specialize this interpretation to the case of the data \((a^q),\ (b^q),\ (c^q)\), then we obtain the following [again somewhat surprising!] interpretation of the multiradial representation discussed above [cf. the discussion of §2.4]:

If one takes a symmetrized average over \(N = 1^2, 2^2, \ldots, j^2, \ldots, (l^\star)^2\), and one works up to suitable indeterminacies, then the arithmetic line bundle determined by “\(q\)”

[i.e., the \(2l\)-th roots of the \(q\)-parameter at the valuations \(\in \mathcal{V}^{\text{bad}}\), or, alternatively, the “\(q\)-pilot object”] may be identified — i.e., from the point of view of performing any sort of computation that takes into account the “suitable indeterminacies” — with the arithmetic line bundle determined by “\(q^N\)”

[i.e., the “\(\Theta\)-pilot object”] obtained by raising the arithmetic line bundle determined by “\(q\)” to the \(N\)-th tensor power

[cf. Fig. 3.19 above, where “LHS” and “RHS” denote, respectively, the left-hand and right-hand sides, i.e., the domain and codomain, of the \(\Theta\)-link].

(ii) **Log-volume estimates:** The interpretation discussed in the final portion of (i) leads naturally to an estimate of the arithmetic degree of the \(q\)-pilot object [cf. [IUTchIII], Theorem B; [IUTchIII], Corollary 3.12], as follows [cf. Steps (x), (xi) of the proof of [IUTchIII], Corollary 3.12; [IUTchIII], Fig. 3.8]:

---

Fig. 3.19: The *gluing*, or tautological identification “=" of the \(\Theta\)-link from the point of view of the *multiradial representation*
One starts with the Frobenius-like version of the \( q \)-pilot object — i.e., “\( q_{\text{RHS}}^{\Theta} \)” — on the RHS of the \( \Theta \)-link. All subsequent computations are to be understood as computations that are performed relative to the fixed arithmetic holomorphic structure of this RHS of the \( \Theta \)-link.

The isomorphism class determined by this \( q \)-pilot object in the global realified Frobenioid of \((c^q)\) [cf. §3.3, (vii)] is sent, via the \( \Theta \)-link, to the isomorphism class determined by the \( \Theta \)-pilot object — i.e., “\( \{q_{\text{LHS}}^{j}\}_j \)” — in the global realified Frobenioid of \((c^q)\) [cf. §3.3, (vii)].

One then applies the multiradial representation discussed in (i) [cf. Fig. 3.19].

One observes that the log-volume, suitably normalized, on the log-shells that occur in this multiradial representation is invariant with respect to the indeterminacies (Ind1) and (Ind2), as well as with respect to the invisible indeterminacies, discussed in (i).

On the other hand, the upper semi-commutativity indeterminacy (Ind3) — i.e., “commutativity” at the level of inclusions of regions in log-shells [cf. the discussion of Example 2.12.3, (iv)] — may be understood as asserting that the log-volume of the multiradial representation of the \( \Theta \)-pilot object must be interpreted as an upper bound.

The multiradial representation of the \( \Theta \)-pilot object “\( \{q_{\text{LHS}}^{j}\}_j \)” can only be compared to the isomorphism class determined by the original \( q \)-pilot object “\( q_{\text{RHS}}^{\Theta} \)” in the global realified Frobenioid of \((c^q)\), i.e., not to the specific arithmetic line bundle given by a copy of the trivial arithmetic line bundle with fixed local trivializations “1” multiplied by 2\( l \)-th roots of \( q \)-parameters [cf. IUTchIII], Remarks 3.9.5, (vii), (viii), (ix), (x); Remark 3.12.2, (v)].

In particular, in order to perform such a comparison between the multiradial representation of the \( \Theta \)-pilot object “\( \{q_{\text{LHS}}^{j}\}_j \)” and the isomorphism class determined by the original \( q \)-pilot object “\( q_{\text{RHS}}^{\Theta} \)” in the global realified Frobenioid of \((c^q)\), it is necessary to make the output data of the multiradial representation into a collection of “\( \mathcal{O}_k \)-modules” [where we use the notation “\( \mathcal{O}_k \)” to denote the various completions of the ring of integers of \( K \) at, for simplicity, the valuations \( \in \mathcal{V}^{\text{non}} \)], i.e., relative to the arithmetic holomorphic structure of the RHS of the \( \Theta \)-link!

Such “\( \mathcal{O}_k \)-modules” are obtained by, essentially [cf. IUTchIII], Remark 3.9.5, for more details], forming the “\( \mathcal{O}_k \)-modules generated by” the various tensor packets of log-shells [cf. the discussion of §3.6, (iv), (v)] that appear in the multiradial
representation, i.e., which, a priori, are [up to a factor given by a suitable power of “p”] just topological modules

\[ \log(O^\times_k) \otimes \log(O^\times_k) \otimes \ldots \otimes \log(O^\times_k) \]

— that is to say, tensor products of \( j + 1 \) copies of \( \log(O^\times_k) \) at the portion of the multiradial representation labeled by \( j \). This operation yields a slightly enlargement of the multiradial representation, which is referred to as the **holomorphic hull** [cf. [IUTchIII], Corollary 3.12; [IUTchIII], Remark 3.9.5] of the multiradial representation.

(9\textsuperscript{est}) The **log-volume**, when applied to the original \( q \)-pilot object \( \Theta \)-pilot object \( \{ q^{j^2} \}_{\text{LHS}} \) may be interpreted as the arithmetic degree of these objects [cf. §2.2; [IUTchIII], Remark 1.5.2, (i), (iii); [IUTchIII], Proposition 3.9, (iii); [IUTchIII], Remark 3.10.1, (iii)].

(10\textsuperscript{est}) In particular, any **upper bound** on the log-volume of the holomorphic hull of the multiradial representation of the \( \Theta \)-pilot object \( \{ q^{j^2} \}_{\text{LHS}} \) may be interpreted [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (viii), (ix); [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.2, (i), (ii)] as an upper bound on the log-volume of the original [Frobenius-like] \( q \)-pilot object \( \Theta \)-pilot object \( \{ q^{j^2} \}_{\text{RHS}} \).

(11\textsuperscript{est}) This comparison of log-volumes was obtained by considering the images of various Frobenius-like objects in the étale-like tensor packets of log-shells of the multiradial representation. In particular, one must apply the log-Kummer correspondence on both the LHS [in the case of the \( \Theta \)-pilot object \( \{ q^{j^2} \}_{\text{LHS}} \)] and the RHS [in the case of the original [Frobenius-like] \( q \)-pilot object \( \{ q^{j^2} \}_{\text{RHS}} \)] of the \( \Theta \)-link. On the other hand, this does not affect the resulting inequality, in light of the compatibility of log-volumes with the arrows of the log-Kummer correspondence [cf. Example 2.12.3, (iv); §3.6, (iv), as well as the discussion of [IUTchIII], Remark 3.9.5, (vii), (viii), (ix); [IUTchIII], Remark 3.12.2, (iv), (v)].

(12\textsuperscript{est}) Thus, in summary, one obtains an **inequality** of log-volumes [cf. [IUTchIII], Theorem B; [IUTchIII], Corollary 3.12]

\[
\log-\text{vol.} \left( \begin{array}{c}
\Theta\text{-pilot object up to mild indeterminacies,} \\
i.e., (\text{Ind1}), (\text{Ind2}), (\text{Ind3}), \\
\text{plus formation of} \\
\text{holomorphic hull}
\end{array} \right) \geq \log-\text{vol.} \left( q\text{-pilot object} \right) (\approx 0)
\]
— where the log-volume of the \( q \)-pilot object on the right-hand side of the inequality is negative and of negligible absolute value by comparison to the terms of interest [to be discussed in more detail in (iv) below] on the left-hand side of the inequality.

(iii) **Comparison with a result of Stewart-Yu:** Recall that in [StYu], an inequality is obtained which may be thought of as a sort of “weak version of the ABC Conjecture”, i.e., which is, roughly speaking, weaker than the inequality of the usual ABC Conjecture in that it contains an undesired exponential operation “\( \exp(-) \)” in its upper bound. This sort of deviation from the inequality of the usual ABC Conjecture is of interest from the point of view of the “vertical shift” discussed in §3.6, (iv), which, on the one hand, gives rise to the indeterminacy (Ind3) [cf. the discussion of (i); (ii), (5est)] and, on the other hand, arises from the fact that the horizontally coric portion of the data related by the \( \Theta \)-link differs from the sort of data in which one is ultimately interested precisely by a single iterate of the log-link, i.e., a single vertical shift in the log-theta-lattice.

(iv) **Computation of log-volumes:** Let us return to the discussion of (ii). It remains to compute the left-hand side of the inequality of (ii), (12est), in more elementary terms. This is done in [IUTchIV], Theorem 1.10. This computation yields a rather strong version of the Szpiro Conjecture inequality, in the case of elliptic curves over NF’s that admit initial \( \Theta \)-data [cf. §3.3, (i)] that satisfies certain technical conditions. The existence of such initial \( \Theta \)-data that satisfies certain technical conditions is then verified in [IUTchIV], Corollary 2.2, (ii), for elliptic curves over NF’s that satisfy certain technical conditions by applying the techniques of [GenEll], §3, §4. Here, we remark in passing that the prime number \( l \) that appears in this initial \( \Theta \)-data constructed in [IUTchIV], Corollary 2.2, (ii), is roughly of the order of the square root of the height of the elliptic curve under consideration [cf. [IUTchIV], Corollary 2.2, (ii), (C1)]. This initial \( \Theta \)-data yields a version of the Szpiro Conjecture inequality [cf. [IUTchIV], Corollary 2.2, (ii), (iii)], which, although somewhat weaker and less effective than the inequality of [IUTchIV], Theorem 1.10, is still rather strong in the sense that it implies that, if we restrict, for simplicity, to the case of elliptic curves over \( \mathbb{Q} \), then

the “\( \epsilon \) terms” that appear in the Szpiro Conjecture inequality concerning the height \( h \) may be bounded above by terms of the order of

\[
h^{1/2} \cdot \log(h)
\]

— i.e., at least in the case of elliptic curves over \( \mathbb{Q} \) whose moduli are “compactly bounded”, in the sense that the moduli lie inside given fixed compact subsets of the sets of rational points of the moduli of elliptic curves over \( \mathbb{R} \) and \( \mathbb{Q}_2 \).
[cf. [IUTchIV], Remark 1.10.5, (ii), (iii); [IUTchIV], Remark 2.2.1, (i), (ii)]. Here, we recall from these Remarks in [IUTchIV] [cf. also [Mss]; [vFr], §2] that

This \( \frac{1}{2} \) in the exponent of \( h \) is of interest in light of the existence of sequences of \( \text{"abc sums"} \) for which this \( \frac{1}{2} \) is asymptotically attained, i.e., as a bound from below, but only if one works with \( \text{abc sums} \) that correspond to elliptic curves whose moduli are not necessarily compactly bounded.

This prompts the following question:

\textit{Can one construct sequences of \( \text{abc sums} \) with similar asymptotic behavior, but which correspond to elliptic curves whose moduli are indeed compactly bounded?}

At the time of writing, it appears that no definitive answer to this question is known, although there does exist some preliminary work in this direction [cf. [Wada]]. In this context, it is also of interest to recall [cf. the discussion of [IUTchIV], Remark 2.2.1; [vFr], §2] that this \( \frac{1}{2} \) is highly reminiscent of the \( \frac{1}{2} \) that appears in the Riemann hypothesis. So far, in the above discussion, we have restricted ourselves to versions of the Szpiro Conjecture inequality for elliptic curves over NF’s that satisfy various technical conditions. On the other hand,

by applying the theory of noncritical Belyi maps [cf. the discussion in the final portion of §2.1; [GenEll], Theorem 2.1; [IUTchIV], Corollary 2.3; [IUTchIV], Theorem A] — which may be thought of as a sort of arithmetic version of analytic continuation [cf. the discussion of §3.3, (vi); [IUTchI], Remark 5.1.4; [IUTchIV], Remark 2.2.4, (iii)] — one may derive the inequalities of the Vojta/Szpiro/ABC Conjectures in their usual form.

We refer to [Fsk], §1.3, §2.12, for another — and, in certain respects, more detailed — discussion of these aspects of inter-universal Teichmüller theory. Although a detailed discussion of the somewhat technical, but entirely elementary computation of the left-hand side of the inequality of (ii), \( (12^{\text{est}}) \), lies beyond the scope of the present paper, we conclude the present (iv) with a summary of the very simple computation of the leading term of the left-hand side of the inequality of (ii), \( (12^{\text{est}}) \):

- First, one notes [cf. [IUTchIV], Proposition 1.2, (i); [IUTchIV], Proposition 1.3, (i); the second to last display of Step (v) of the proof of [IUTchIV], Theorem 1.10] that, if one ignores [since we are only interested in computing the leading term!] the archimedean valuations of \( K \), as well as the nonarchimedean valuations of \( K \) whose ramification index over \( \mathbb{Q} \) is “large” [i.e., is \( \geq \) the cardinality of the set of nonzero elements of the residue field of the corresponding prime of
then the resulting log-volume [suitably normalized] of the tensor packet of log-shells corresponding to the label \( j \in \{1, 2, \ldots, l^*\} \) is

\[
(j + 1) \cdot \log(\text{deg})
\]

— where we write \( \log(\text{deg}) \) for the arithmetic degree [suitably normalized, so as to be invariant with respect to finite extensions — cf. [IUTchIV], Definition 1.9, (i)] of the arithmetic divisor determined by the different ideal of the number field \( K \) over \( \mathbb{Q} \).

* On the other hand, the effect — i.e., on the tensor packet of log-shells corresponding to the label \( j \) — of multiplying by the theta value \( q_j^2 \) [cf. Figs. 3.16, 3.18] at \( \mathfrak{v} \in \mathcal{V}^\text{bad} \) [cf. the second to last display of Step (v) of the proof of [IUTchIV], Theorem 1.10] is given by

\[
- \frac{j^2}{2l} \cdot \log(q)
\]

— where we write \( \log(q) \) for the arithmetic degree [again suitably normalized] of the arithmetic divisor determined by the \( q \)-parameters of the elliptic curve \( E \) over \( F \) at the valuations that lie over valuations \( \in \mathcal{V}^\text{mod} \).

* Thus, the leading term of the log-volume of the left-hand side of the inequality of (ii), (12est), is given by

\[
\sum_{j=1}^{l^*} (j + 1) \cdot \log(\text{deg}) - \frac{j^2}{2l} \cdot \log(q) \approx \frac{1}{2} \cdot (\frac{l}{2})^2 \cdot \log(\text{deg}) - \frac{1}{3 \cdot 2l} \cdot (\frac{l}{2})^3 \cdot \log(q)
\]

\[
= \frac{l^2}{8} \cdot \log(\text{deg}) - \frac{l^2}{48} \cdot \log(q)
\]

\[
= \frac{l^2}{48} \{6 \cdot \log(\text{deg}) - \log(q)\}
\]

— where the notation \( \approx \) denotes a possible omission of terms that do not affect the leading term. That is to say, one obtains a “large positive constant” \( \frac{l^2}{48} \) times precisely the quantity — i.e.,

\[
6 \cdot \log(\text{deg}) - \log(q)
\]

— that one wishes to bound from below in order to conclude [a suitable version of] the Szpiro Conjecture inequality.

As discussed in [IUTchIV], Remark 1.10.1 [cf. also the discussion of §3.9, (i), (ii), below], this “computation of the leading term”, which was originally motivated by the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII], was in fact known to the author around the year 2000. Put another way, one of the primary motivations for the development of inter-universal Teichmüller theory was precisely the problem of establishing a suitable framework, or geometry, in which this computation could be performed.
§ 3.8. Comparison with the Gaussian integral

At this point, it is of interest to compare and contrast the theory of "arithmetic changes of coordinates" [i.e., §2] and multiradial representations [i.e., the present §3] discussed so far in the present paper with the classical computation of the Gaussian integral, as discussed in §1. In the following, the various "Steps" refer to the "Steps" in the computation of the Gaussian integral, as reviewed in §1.

(1\textsuperscript{gau}) The naive change of coordinates $e^{-x^2} \sim u$ of Step 1 [cf. also Step 2] is formally reminiscent [cf. [IUTchII], Remark 1.12.5, (ii)] of the assignment
\[
\{q^j\}_{j=1,...,l} \mapsto q
\]
that appears in the definition of the $\Theta$-link [cf. §2.4; §3.3, (vii), (b$\Theta$), (b$\vartheta$)].

(2\textsuperscript{gau}) The introduction of two "mutually alien" copies of the Gaussian integral in Step 3 may be thought of as corresponding to the appearance of the two $\Theta^{\pm}\text{ell}\, NF$-Hodge theaters "\bullet" in the domain and codomain of the $\Theta$-link [cf. §2.7, (i); the two vertical lines in the right-hand portion of Fig. 3.6], which may be thought of as representing two "mutually alien" copies of the conventional scheme theory surrounding the elliptic curve under consideration, i.e., the elliptic curve that appears in the initial $\Theta$-data of §3.3, (i).

(3\textsuperscript{gau}) The two dimensions of the Euclidean space $\mathbb{R}^2$ that appears in Step 4 may be thought of as corresponding to the two dimensions of the log-theta-lattice [cf. §3.3, (ii)], which are closely related to the two underlying combinatorial dimensions of a ring. Here, we recall from §2.11 that these two underlying combinatorial dimensions of a ring may be understood quite explicitly in the case of $NF$’s, $MLF$’s, or the field of complex numbers.

(4\textsuperscript{gau}) The point of view discussed in Step 5 that integrals may be thought of as computations of net masses as limits of sums of infinitesimals of zero mass may be understood as consisting of two aspects: First of all, the summation of local contributions that occurs in an integral may be regarded as corresponding to the use of prime-strips [i.e., local data indexed by elements of $\mathbb{V}$ — cf. the discussion of §3.3, (iv)] and the computation of heights in terms of log-volumes, as discussed in §2.2. On the other hand, the limit aspect of an integral, which involves the consideration of some sort of notion of infinitesimals [i.e., "zero mass" objects], may be thought of as corresponding to the fundamental dichotomy between Frobenius-like and étale-like objects [cf. §2.2; §2.7, (ii), (iii); §2.8; §2.9; §3.3, (iii); §3.4, (i); §3.5].

(5\textsuperscript{gau}) The generalities concerning the effect on integrals of changes of coordinates on local patches of the Euclidean plane in Step 6 may be thought of as corresponding
<table>
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<td>$\Theta$-link ${q^2}_{j=1,\ldots,t*} \mapsto q$</td>
</tr>
<tr>
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<td>two dimensions of the Euclidean plane $\mathbb{R}^2$</td>
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<tr>
<td>unipotent linear transformations, toral dilations, and rotations</td>
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</tr>
</tbody>
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Fig. 3.20.1: Comparison between *inter-universal Teichmüller theory* and the classical computation of the *Gaussian integral*
to the generalities on the *computational technique* of *mono-anabelian transport* as a mechanism for computing the effect of “*arithmetic changes of coordinates*” [cf. the discussion of §2.7, §2.8, §2.9, §2.10, §2.11]. This technique is motivated by the *concrete examples* given in §2.5, §2.6 of *changes of coordinates* related to *positive characteristic Frobenius morphisms*, as well as by the discussion of examples of *finite discrete approximations* of harmonic analysis given in §2.14.

(6*ga*) The fundamental examples given in Step 7 of *linear changes of coordinates* in the Euclidean plane — i.e., *unipotent* linear transformations, *toral dilations*, and *rotations* — may be thought of as corresponding to the fundamental examples of *arithmetic changes of coordinates* in the case of MLF’s that were discussed in §2.12 [cf. also the discussion of §2.3, §2.4, and §2.13; §3.3, (vi)].

(7*ga*) The passage from *planar cartesian* to *polar coordinates* discussed in Step 8 may be understood as a sort of *rotation*, or *continuous deformation*, between the *two mutually alien copies* of the Gaussian integral introduced in Step 3, i.e., which correspond to the $x$- and $y$-axes. Thus, this passage to polar coordinates may be regarded as corresponding [cf. the discussion at the beginning of §3.1; §3.1, (iv); §3.2; the discussion surrounding Figs. 3.1, 3.3, 3.5, 3.6, 3.12, 3.13, 3.19; the discussion of [IUTchII], Remark 1.12.5] to the “deformation”, or “*parallel transport*”, between distinct collections of *radial data* that appears in the definition of the notion of *multiradiality* and, in particular, to the passage from the *log-theta-lattice* to the *étale-picture* [cf. §3.6, (i)] and, ultimately, to the *multiradial representation* of §3.7, (i).

(8*ga*) The fundamental *decoupling* into *radial* and *angular* coordinates discussed in Step 9 may be understood as corresponding to the discussion in §3.4 of *Kummer theory* for *special types* of *functions* via *multiradial decouplings/cyclotomic rigidity* [cf. also the discussion of *unit group* and *value group* portions in §2.11; §3.3, (vii)].

(9*ga*) The efficacy of the *change of coordinates* that renders possible the evaluation of the *radial integral* in Step 10 may be understood as an *essentially formal* consequence of the *quadratic* nature of the *exponent* that appears in the Gaussian distribution. This *fundamental aspect* of the computation of the Gaussian integral may be regarded as corresponding to the fact that the *rigidity* properties of *mono-theta environments* that underlie the *multiradial decouplings/cyclotomic rigidity* discussed in §3.4, (iii), (iv) are, in essence, *formal consequences* [cf. the discussion in the final portion of §3.4, (iv)] of the *quadratic* structure of the *commutators* of the *theta groups* associated to the ample line bundles that appear in the theory [cf. the discussion of [IUTchIII], Remark 2.1.1; [IUTchIV], Remark
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</tr>
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<td>decoupling into radial and angular coordinates</td>
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<tr>
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<tr>
<td>naive change of coordinates &quot;justifiable&quot; up to a suitable &quot;error factor&quot; ( \sqrt{\pi} ) arising from the square root of the angular integral over the complex units</td>
<td>naive approach to bounding heights via &quot;Gaussian Frobenius morphisms&quot; on NF’s &quot;justifiable&quot; up to a suitable &quot;log-different error factor&quot; arising from the indeterminacies (Ind1), (Ind2), (Ind3) acting on log-shells</td>
</tr>
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Fig. 3.20.2: Comparison between inter-universal Teichmüller theory and the classical computation of the Gaussian integral
2.2.2; the discussion of the **functional equation of the theta function** in [Pano], §3.

In particular, the evaluation of the **radial integral** in Step 10 corresponds to the portion of inter-universal Teichmüller theory that relates to the [local] **value group portion** \((b^\Theta)\) of the \(\Theta\)-link [cf. \((1^{\text{gau}})\)].

\((10^{\text{gau}})\) The **angular integral** of Step 11 is an integral over the **unit group** of the field of complex numbers that is evaluated by executing the **change of coordinates** determined by the **imaginary part** of the **natural logarithm**. This important aspect of the computation of the Gaussian integral may be regarded as corresponding to the theory of **Galois evaluation** and **log-shells** exposed in §3.6 — cf., especially, the theory involving **iterates** of the \(\text{log-link}\) discussed in §3.6, (iv); §3.7, (i). In this context, it is of interest to recall [cf. Example 2.12.3, (v); §3.3, (ii), (vi)] that the \(\text{log-link}\) may be understood as a sort of **arithmetic rotation**, or **juggling**, of the **two underlying combinatorial dimensions of a ring** that essentially concerns the [local] **unit group portion**, i.e., \((a^\Theta), (a^q)\), of the \(\Theta\)-link [cf. §3.3, (vii); §3.6, (iv)].

\((11^{\text{gau}})\) The **final computation** of the Gaussian integral in Step 12 may be summarized [cf. §1.7] as asserting that the **naive change of coordinates** of \((1^{\text{gau}})\) may in fact be “justified”, provided that one allows for a suitable “**error factor**” given by the **square root** of the **angular integral** of \((10^{\text{gau}})\). This conclusion may be understood as corresponding to the **computation**, discussed in §3.7, (ii), (iii), (iv), of the **left-hand side** of the **inequality** of §3.7, (ii), \((12^{\text{est}})\). This computation may be summarized as asserting that one obtains a **bound** on the **height** of the elliptic curve under consideration whose **leading term** is the “**log-different** \(\log(\delta^K)\)” [cf. the final portion of §3.7, (iv)] that one expects from [a suitable version of] the **Szpiro Conjecture**. Put another way, this computation may be summarized as asserting that

the **naive approach** outlined in §2.3, §2.4 to **bounding heights** via “**Gaussian Frobenius morphisms**” on NF’s may in fact be “justified”, provided that one allows for a suitable “**error factor**” that arises from the **indeterminacies** (Ind1), (Ind2), (Ind3) acting on the **log-shells** — i.e., the [local] **unit group portion** \((a^\Theta), (a^q)\) of the \(\Theta\)-link [cf. the discussion of \((10^{\text{gau}})\)] — that appear in the **multiradial representation** discussed in §3.7, (i) [cf. also the discussion of [IUTchIV], Remark 2.2.2].

Finally, in this context, we observe [cf. the final portion of Step 12] that, just as in the case of the computation of the Gaussian integral, it is **essentially a hopeless task** to **identify “explicit portions”** of the original Gaussian integral \(\int_{-\infty}^{\infty} e^{-x^2} \, dx\) on the real line that “**correspond**” precisely, in some sort of meaningful sense, to
the radial and angular integrals of Steps 10 and 11, it is essentially a hopeless task to

trace, in some sort of explicit or readily computable fashion, the way in which the value group portions \((b^\Theta), (c^\Theta)\) in the domain of the \(\Theta\)-link [cf. \((g^{\text{gau}}})]\) appear within the multiradial representation via tensor packets of log-shells [cf. \((10^{\text{gau}}})]\) expressed in objects arising from the codomain of the \(\Theta\)-link

[cf. the discussion of “APT” in [IUTchIII], Remark 3.11.1, (iv)].

The various observations discussed in the present §3.8 are summarized in Figs. 3.20.1, 3.20.2, above. Finally, with regard to \((1^{\text{gau}}})\), we note that

the left-hand side “\(\{q_j^2\}_{j=1,...,l^*}\)” of the assignment discussed in \((1^{\text{gau}}})\) cannot be replaced by “\(q^\lambda\)” for \(1 \neq \lambda \in \mathbb{Q}_{>0}\)

or by

“\(\{q^{N-j^2}\}_{j=1,...,l^*}\)” for \(2 \leq N \in \mathbb{N}\).

Indeed, this property of the left-hand side of the assignment discussed in \((1^{\text{gau}}})\) is, in the case of “\(q^\lambda\)” a consequence of the

- the lack [i.e., in the case of “\(q^\lambda\)”] of a theory of multiradial decouplings/cyclotomic rigidity of the sort [cf. the discussion of §3.4, (iii), (iv)] that exists in the case of theta functions and mono-theta environments

[cf. the discussion of [IUTchIII], Remark 2.2.2, (i), (ii), (iii)] and, in the case of “\(\{q^{N-j^2}\}\)_{j=1,...,l^*}”, a consequence of the

- special role [cf. the discussion of §3.4, (iii)] played by the first power of [reciprocals of \(l\)-th roots of the] theta function;
- the condition [cf. the discussion at the beginning of §3.6] that the assignment “abstract functions \(\mapsto\) values” that occurs in the passage from theta functions to theta values be obtained by applying the technique of Galois evaluation

[cf. [IUTchII], Remark 3.6.4, (iii), (iv); [IUTchIII], Remark 2.1.1, (iv); the discussion of the final portion of Step (xii) of the proof of [IUTchIII], Corollary 3.12]. Moreover, the negation of this property of the left-hand side of the assignment discussed in \((1^{\text{gau}}})\) would imply a stronger version of the Szpiro Conjecture inequality that is in fact false [cf. [IUTchIV], Remark 2.3.2, (ii)]. By contrast,
the right-hand side “$q$” of the assignment discussed in $(1^{\text{gau}})$ can be replaced by “$q^\lambda$” for $1 \neq \lambda \in \mathbb{Q}_{>0}$, without any substantive effect on the theory; moreover, doing so does not result in any substantive improvement in the estimates discussed in §3.7, (ii), (iii), (iv)

[cf. [IUTchIII], Remark 3.12.1, (ii)]. In this context, it is of interest to observe that:

This sort of qualitative difference between the left- and right-hand sides of the assignment $\{q^j\}_{j=1}^{\lambda} \mapsto q$ is reminiscent of the qualitative difference — e.g., the presence or absence of the exponential! — between the left- and right-hand sides of the naive change of coordinates $e^{-x^2} \mapsto u$.

§ 3.9. Relation to scheme-theoretic Hodge-Arakelov theory

In the present §3.9, we pause to reconsider the theory of multiradial representations developed in the present §3 from the point of view of the scheme-theoretic Hodge-Arakelov theory discussed in Example 2.14.3 — a theory which, as discussed in the final portion of §3.7, (iv), played a central role in motivating the development of inter-universal Teichmüller theory.

(i) Hodge filtrations and theta trivializations: We begin by examining the natural isomorphism of $F$-vector spaces of dimension $l^2$

$$\Gamma(E^t, L|_{E^t})^<t \xrightarrow{\sim} L|_{E[l]}$$

that constitutes the fundamental theorem of Hodge-Arakelov theory discussed in Example 2.14.3 in a bit more detail [cf., e.g., the discussion surrounding [Pano], Theorem 1.1] in the case where $F$ is an NF. First of all, we observe that, although both the domain and codomain of this isomorphism are $F$-vector spaces of dimension $l^2$, by considering the natural action of suitable theta groups on the domain and codomain and applying the well-known theory of irreducible representations of theta groups, one may conclude that, up to “uninteresting redundancies”, this isomorphism may in fact be [“essentially”] regarded as an isomorphism between $F$-vector spaces of dimension $l$. The left-hand side of this isomorphism of $l$-dimensional $F$-vector spaces admits a natural Hodge filtration that arises by considering the subspaces of relative degree $< t$, for $t = 0, \ldots, l - 1$. Moreover, one verifies easily that, if we write $\omega_E$ for the cotangent space of $E$ at the origin of $E$, and $\tau_E$ for the dual of $\omega_E$, then the adjacent subquotients of this Hodge filtration are the 1-dimensional $F$-vector spaces

$$\tau_E^\otimes t$$

for $t = 0, \ldots, l - 1$, tensored with some fixed 1-dimensional $F$-vector space [i.e., which is independent of $t$], which we shall ignore since its arithmetic degree [i.e., when regarded
as being equipped with natural integral structures at the nonarchimedean valuations of $F$ and natural Hermitian structures at the archimedean valuations of $F$] is sufficiently small that its omission does not affect the computation of the leading terms of interest. On the other hand, the right-hand side of the isomorphism under consideration admits a natural theta trivialization [i.e., a natural isomorphism with the $F$-vector space $F^\oplus l$ given by the direct sum of $l$ copies of the $F$-vector space $F$, which we think of as being labeled by $t = 0, \ldots, l - 1$, which is compatible — up to contributions that are sufficiently small as to have no effect on the computation of the leading terms of interest — with the various natural integral structures and natural Hermitian metrics involved, except at the valuations where $E$ has potentially multiplicative reduction, where one must adjust the natural integral structure [i.e., the integral structure determined by the ring of integers $O_F$ of $F$] on the copy of $F$ in $F^\oplus l$ labeled $t \in \{0, \ldots, l - 1\}$ by a factor of

$$q^{i^2/4}$$

— where the notation “$q$” denotes a $2l$-th root of the $q$-parameter of $E$ at the valuation under consideration. Next, let us observe that [again up to contributions that are sufficiently small as to have no effect on the computation of the leading terms of interest] one may replace the label $t \in \{0, \ldots, l - 1\}$ by a label $j \in \{1, \ldots, l^*\}$, where we think of $t$ as $\approx 2j$. Write $\Omega^\log_E \overset{\text{def}}{=} \omega_E^\otimes 2$. Then the 1-dimensional $F$-vector spaces — i.e., which we think of as arithmetic line bundles by equipping these 1-dimensional $F$-vector spaces with natural integral structures and natural Hermitian metrics — corresponding to the label $j \in \{1, \ldots, l^*\}$ on the left- and right-hand sides of the natural isomorphism of $l$-dimensional $F$-vector spaces determined by the fundamental theorem of Hodge-Arakelov theory assume the form

$$\{ (\Omega^\log_E) \otimes j \} \vee, \qquad q^{i^2} \cdot F$$

— where the notation “$\vee$” denotes the dual. Put another way, if we tensor the dual of the left-hand side contribution at $j \in \{1, \ldots, l^*\}$ with the right-hand side contribution at $j \in \{1, \ldots, l^*\}$, then we obtain the conclusion that the natural isomorphism under consideration may be thought of “at a very rough level” — i.e., by replacing the Hodge filtration with its semi-simplification, etc. — as a sort of global section of some sort of weighted average over $j$ of the [arithmetic line bundles corresponding to the] 1-dimensional $F$-vector spaces

$$q^{i^2} \cdot (\Omega^\log_E) \otimes j$$

— where $j$ ranges over the elements of $\{1, \ldots, l^*\}$.

In fact, the above discussion may be translated into purely geometric terms by working with the tautological one-dimensional semi-abelian scheme over the natural
compactification of the moduli stack of elliptic curves [over, say, a field of characteristic zero]. Then “$\Omega_{E}^{\log}$” may be thought of as the line bundle of logarithmic differentials on this compactified moduli stack. Moreover, one can compute global degrees “$\deg(-)$”

$$\sum_{j=1}^{l^*} \deg((\Omega_{E}^{\log})^\otimes j) - \deg(\frac{i^2}{2l} \cdot [\infty]) = \sum_{j=1}^{l^*} j \cdot \deg(\Omega_{E}^{\log}) - \frac{i^2}{2l} \cdot \deg([\infty])$$

$$\approx \frac{1}{2} \cdot (\frac{1}{2})^2 \cdot \deg(\Omega_{E}^{\log}) - \frac{1}{32} \cdot (\frac{1}{2})^3 \cdot \deg([\infty])$$

$$= \frac{i^2}{8} \cdot \deg(\Omega_{E}^{\log}) - \frac{i^2}{48} \cdot \deg([\infty])$$

$$= \frac{i^2}{48} \{ \deg((\Omega_{E}^{\log})^\otimes 6) - \deg([\infty]) \}$$

— where the notation “$\approx$” denotes a possible omission of terms that do not affect the leading term; “$[\infty]$” denotes the effective divisor on the compactified moduli stack under consideration determined by the point at infinity, i.e., the scheme-theoretic zero locus of the $q$-parameter; by abuse of notation, we use the same notation for “compactified moduli stack versions” of the corresponding objects introduced in the discussion of elliptic curves over NF’s. That is to say, in summary,

the determinant of the natural isomorphism that appears in the fundamental theorem of Hodge-Arakelov theory is simply [an invertible constant multiple of] some positive tensor power of the well-known discriminant modular form of weight 12, i.e., a global section of $\omega_{E}^{\otimes 12} = (\Omega_{E}^{\log})^\otimes 6$ whose unique zero is a zero of order 1 at the point at infinity of the compactified moduli stack of elliptic curves

[cf. [Pano], §1; the discussion of the final portion of [HASurI], §1.2, for more details].

(ii) Comparison with inter-universal Teichmüller theory: First, we begin with the observation that, relative to the classical analogy between NF’s and one-dimensional functions fields [over some field of constants], it is natural to think of

- log-shells as localized absolute arithmetic analogues of the notion of the sheaf of logarithmic differentials.

Indeed, this point of view is supported by the fact that the log-shell associated to a finite extension of $\mathbb{Q}_p$ [for some prime number $p$] whose absolute ramification index is $\leq p - 2$ coincides with the dual fractional ideal to the different ideal of the given finite extension of $\mathbb{Q}_p$ [cf. [IUTchIV], Proposition 1.2, (i); [IUTchIV], Proposition 1.3, (i)]. Thus, it is natural to regard

- the [arithmetic line bundle corresponding to the] 1-dimensional $F$-vector space $q^{i^2} \cdot (\Omega_{E}^{\log})^\otimes j$
— where $j$ ranges over the elements of $\{1, \ldots, l^*\}$ — of the discussion of (i) as a sort of scheme-theoretic analogue, or precursor, of the portion labeled by $j$ of the multiradial representation discussed in §3.7, (i) [cf. also the explicit display of §3.7, (ii), (8$^{\text{est}}$)];

- the resulting computation of global degrees “deg(−)” given in (i) as a sort of scheme-theoretic analogue, or precursor, of the computation of the leading term of the log-volume of the left-hand side of the inequality of §3.7, (ii), (12$^{\text{est}}$) [cf. the final portion of §3.7, (iv)].

Indeed, this was precisely the point of view of the author around the year 2000 that motivated the author to develop inter-universal Teichmüller theory.

(iii) **Analytic torsion interpretation:** In conventional Arakelov theory for varieties over NF’s, analytic torsion refers to a metric invariant, at the archimedean valuations of an NF, that measures the way in which the space of global [holomorphic/algebraic] sections of a line bundle — which is regarded, by means of various considerations in harmonic analysis, as a subspace of the Hilbert space of $L^2$-class sections of the line bundle — is embedded inside this ambient Hilbert space of $L^2$-class sections.

Since this ambient Hilbert space of $L^2$-class sections may be regarded as a topological invariant, i.e., which is unaffected by deformations of the holomorphic moduli of the variety under consideration, the notion of analytic torsion may be understood as a measure of the way in which the subspace constituted by the space of global algebraic sections — which depends, in a quite essential fashion, on the holomorphic moduli of the variety under consideration — is embedded inside this topological invariant.

When formulated in this way,

the notion of analytic torsion becomes highly reminiscent of the computational technique of mono-analytic transport [cf. the discussion of §2.7; §2.9; §3.1, (v)] and, in particular, of the use of log-shells to construct the “multiradial containers” [cf. the discussion of §3.6, (iv)] for the various arithmetic holomorphic structures that appear in the multiradial representation discussed in §3.7, (i).

Indeed, from this point of view, scheme-theoretic Hodge-Arakelov theory may be understood as a sort of intermediate step — i.e., a finite discrete approximation, in the spirit of the discussion of §2.14, which is, moreover, [unlike the classical notion of analytic torsion!] defined over NF’s — between the classical notion of analytic torsion and inter-universal Teichmüller theory. Put another way,
• the natural isomorphism that appears in the fundamental theorem of scheme-theoretic Hodge-Arakelov theory may be understood as a sort of polynomial-theoretic discretization of the theory surrounding the classical notion of analytic torsion, while
• inter-universal Teichmüller theory may be understood as a sort of global Galois-theoretic version over NF’s of the theory surrounding the classical notion of analytic torsion

[cf. the discussion of [IUTchIV], Remark 1.10.4].

§ 3.10. The technique of tripodal transport

In the present §3.10, we re-examine inter-universal Teichmüller theory once again, this time from the point of view of the technique of tripodal transport. Various versions of this technique may also be seen in previous work of the author concerning

• p-adic Teichmüller theory,
• scheme-theoretic Hodge-Arakelov theory, and
• combinatorial anabelian geometry.

The proof given by

• Bogomolov [cf. [ABKP], [Zh], [BogIUT], as well as the discussion of §4.3, (iii), below] of the geometric version of the Szpiro Conjecture over the complex numbers

may also be re-interpreted from the point of view of this technique.

(i) The notion of tripodal transport: The general notion of tripodal transport may be summarized as follows [cf. also Fig. 3.21 below]:

(1\textsuperscript{trp}) One starts with a “nontrivial property” of interest [i.e., that one wishes to verify!] associated to some sort of given arithmetic holomorphic structure — such as a hyperbolic curve or a number field [cf. the discussion of §2.7, (vii)].

(2\textsuperscript{trp}) One observes that this nontrivial property of interest [i.e., associated to the given arithmetic holomorphic structure] may be derived by combining a “relatively trivial” property, again associated to the given arithmetic holomorphic structure, with some sort of alternative property of interest.

(3\textsuperscript{trp}) One establishes some sort of parallel transport mechanism — which is typically not compatible with the given arithmetic [i.e., scheme-/ring-theoretic!] holomorphic structure — that allows one to reduce the issue of verifying the alternative
property of interest for the given arithmetic holomorphic structure to a “corresponding version” in the case of the tripod [i.e., the projective line minus three points] of this alternative property of interest.

\(4^{\text{trp}}\) One verifies the alternative property of interest in the case of the tripod.

\(5^{\text{trp}}\) By combining \((1^{\text{trp}}), (2^{\text{trp}}), (3^{\text{trp}}), (4^{\text{trp}})\), one concludes that the original nontrivial property of interest associated to the given arithmetic holomorphic structure does indeed hold, as desired.

Here, we note that the steps \((3^{\text{trp}}), (4^{\text{trp}})\) are often very closely related, and, indeed, at times, it is difficult to isolate these two steps from one another. This sort of argument might strike some readers at first glance as “mysterious” or “astonishing” in the sense that ultimately, one is able to conclude the original nontrivial property of interest [cf. \((1^{\text{trp}})\)] associated to the given arithmetic holomorphic structure [cf. \((5^{\text{trp}})\)] despite that fact that the nontrivial content of the argument centers around the arithmetic surrounding the tripod [cf. \((3^{\text{trp}}), (4^{\text{trp}})\)], in sharp contrast to the fact that the argument only requires the use of a “relatively trivial” observation concerning the given arithmetic holomorphic structure [cf. \((2^{\text{trp}})\)].

![Tripodal transport](image)

**Fig. 3.21: Tripodal transport**

Perhaps it is most natural to regard this sense of “mysteriousness” or “astonishment” as a reflection of the potency of the parallel transport mechanism [cf. \((3^{\text{trp}})\)] that is employed. This “potency” is, in many of the examples discussed below, derived as a consequence of various rigidity properties, such as anabelian properties. Such rigidity properties may only be derived by
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applying the mechanism of parallel transport via rigidity properties — not to relatively simple “types of mathematical objects” such as vector spaces or modules, as is typically the case in classical instances of parallel transport! — but rather to complicated mathematical objects [cf. the discussion of [IUTchIV], Remark 3.3.2], such as the sort of Galois groups/étale fundamental groups that occur in anabelian geometry, i.e., mathematical objects whose intrinsic structure is sufficiently rich to allow one to establish rigidity properties that are sufficiently “potent” to compensate for the “loss of structure” that arises from sacrificing compatibility with classical scheme-/ring-theoretic structures.

Here, we note that it is necessary to sacrifice compatibility with classical scheme-/ring-theoretic structures precisely because such structures typically constitute a fundamental obstruction to relating the arithmetic surrounding the given arithmetic holomorphic structure to the arithmetic surrounding the tripod. A typical example of this sort of “fundamental obstruction” may be seen by considering, for instance, the case of two [scheme-theoretically! non-isomorphic proper hyperbolic curves over an algebraically closed field of characteristic zero, which, nonetheless, have isomorphic étale fundamental groups. This point of view, i.e., of overcoming the sort of “fundamental obstruction” to parallel transport that arises from imposing the restriction of working within a fixed scheme/ring theory, is closely related to the introduction of the notions of Frobenius-like and étale-like structures — cf. the discussion of §2.7, (ii), (iii); §2.8.

(ii) Inter-universal Teichmüller theory via tripodal transport: We begin our discussion by observing that, when viewed from the point of view of the notion of tripodal transport, inter-universal Teichmüller theory may be recapitulated as follows:

The fundamental log volume estimate \((12^{est})\) [cf. \((1^{trp})\)] is obtained in the argument discussed in §3.7, (ii) [cf. \((5^{trp})\)], by combining [cf. \((9^{est}), (10^{est}), (11^{est})\)] a relatively simple argument [cf. \((2^{trp})\)] carried out in the arithmetic holomorphic structure of the RHS of the \(\Theta\)-link [cf. \((1^{est}), (7^{est})\)], involving relatively simple operations such as the formation of the holomorphic hull [cf. \((6^{est}), (7^{est}), (8^{est})\)], with the parallel transport mechanism [cf. \((3^{trp})\), as well as the discussion of §3.1, (iv), (v)] furnished by the multiradial representation [cf. \((2^{est}), (3^{est}), (4^{est}), (5^{est})\)], which is established by considering various properties of objects [cf. §3.4, §3.6], such as the theta function on the Tate curve [cf. §3.4, (iii), (iv); Fig. 3.9], on the LHS of the \(\Theta\)-link [cf. \((4^{trp})\)].

Here, we recall from the discussion of §3.4, (iii), (iv); Fig. 3.9, that the theory surrounding the theta function on the Tate curve may be thought
of as a sort of function-theoretic representation of the $p$-adic arithmetic geometry of a copy of the tripod for which the cusps “0” and “∞” are subject to the involution symmetry that permutes these two cusps and leaves the cusp “1” fixed.

Also, we recall from the discussion of §3.7, (i) [cf. also the discussion of the properties “IPL”, “SHE”, “APT”, “HIS” in [IUTchIII], Remark 3.11.1] that the parallel transport mechanism furnished by the multiradial representation revolves around the following central property:

the algorithm that yields the multiradial representation converts any collection of input data [i.e., not just the codomain data $(a^q), (b^q), (c^q)$ of the Θ-link!] that is isomorphic to the domain data $(a^Θ), (b^Θ), (c^Θ)$ of the Θ-link — i.e., in somewhat more technical terminology [cf. [IUTchII], Definition 4.9, (viii)], any $\mathcal{F}^{\min} \times _{\mu} \text{-prime-strip}$ — into output data that is expressed in terms of the arithmetic holomorphic structure of the input data, i.e., of the codomain of the Θ-link.

Finally, at a more technical level, we recall from §3.3, (vi); §3.4, (ii); §3.4, (iii), (iv), that this parallel transport mechanism is established by applying

- the theory of the étale theta function developed in [EtTh];
- the theory of [local and global] mono-anabelian reconstruction developed in [AbsTopII], [AbsTopIII].

Here, it is of interest to observe that both the theory of elliptic cuspidalization, which plays an important role in [EtTh], and the theory of Belyi cuspidalization, which plays an important role in [AbsTopII], [AbsTopIII], may be regarded as essentially formal consequences of the fundamental anabelian results obtained in [pGC]. The rigidity properties developed in [EtTh] also depend, in a fundamental way, on the interpretation [i.e., as rigidity properties of the desired type!] given in [EtTh] of the theta symmetries of the theta function on the Tate curve.

(iii) $p$-adic Teichmüller theory via tripodal transport: When viewed from the point of view of the notion of tripodal transport, a substantial portion of the $p$-adic Teichmüller theory of $[p\text{Ord}], [p\text{Tch}], [p\text{TchIn}]$ may be summarized as follows:

One constructs a theory of canonical indigenous bundles, canonical Frobenius liftings, and associated canonical Galois representations into $PGL_2(-)$ [for a suitable “(-)” — cf. $[p\text{TchIn}]$, Theorems 1.2, 1.4, for more details] for quite general $p$-adic hyperbolic curves [cf. $(1_{\text{trp}}), (2_{\text{trp}}), (5_{\text{trp}})]$ by establishing a parallel transport mechanism [cf. $(5_{\text{trp}})$] that allows one to transport
similar canonical objects associated to the *tautological family of elliptic curves* over the *tripod* [cf. (4\textsuperscript{trp})].

Here, we recall that, prior to \([p\text{Ord}]\), the existence of such canonical objects associated to a \(p\)-adic hyperbolic curve was only known in the case of *Shimura curves*, i.e., such as the *tripod*. From the point of view of the notion of *tripodal transport*, it is also of interest to observe that:

The notion of an **ordinary Frobenius lifting** [cf. \([p\text{TchIn}], \text{Theorem } 1.3\)], which plays a *central role* in \([p\text{Ord}],[p\text{Tch}],[p\text{TchIn}]\), may be understood as a sort of *\(p\)-adic generalization* of the most fundamental example of a *Frobenius lifting*, namely, the endomorphism

\[
T \mapsto T^p
\]

[where \(T\) denotes the standard coordinate on the projective line] of the *tripod* over a \(p\)-adic field. This endomorphism is *equivariant* with respect to the *symmetry* of the tripod which permutes the cusps “0” and “\(\infty\)” and leaves the cusp “1” fixed.

At a more technical level, we recall that the *parallel transport mechanism* employed in \(p\)-adic Teichmüller theory revolves around the following two fundamental technical tools:

· the fact that the natural morphism from the *moduli stack of nilcurves* [i.e., pointed stable curves equipped with an indigenous bundle whose \(p\)-curvature is square nilpotent] to the corresponding moduli stack of pointed stable curves is a *finite, flat*, and *local complete intersection morphism* of degree \(p\) to the power of the dimension of these moduli stacks [cf. \([p\text{TchIn}], \text{Theorem } 1.1\)];

· various *strong rigidity properties*, with respect to *deformation*, that hold precisely over the *ordinary locus* of the *moduli stack of nilcurves*, i.e., the *étale locus* of the natural morphism from the moduli stack of nilcurves to the corresponding moduli stack of pointed stable curves.

In this context, it is of interest to observe, considering the fundamental role played by such notions as *differentials* and *curvature* in the *classical differential-geometric version of parallel transport*, that both of these fundamental technical tools rely on various subtle properties of the *\(p\)-curvature* and *Frobenius actions on differentials*. This relationship with *differentials* is also interesting from the point of view of the fundamental role played by the theory of \([p\text{GC}]\) in the discussion of \([\text{EtTh}],[\text{AbsTopII}],[\text{AbsTopIII}]\) in (ii), since *differentials*, treated from a *\(p\)-adic Hodge-theoretic* point of view, play a fundamental role in \([p\text{GC}]\). Finally, we observe that although anabelian results do not play *any role* in the parallel transport mechanism of *\(p\)-adic Teichmüller*
theory, it is interesting to note that $p$-adic Teichmüller theory has an important application to absolute anabelian geometry [cf. [CanLift], §3, as well as the discussion of [IUTchI], §4; [IUTchII], Remark 4.11.4, (iii)].

(iv) **Scheme-theoretic Hodge-Arakelov theory via tripodal transport:**

When viewed from the point of view of the notion of tripodal transport, the fundamental theorem of Hodge-Arakelov theory, i.e., the natural isomorphism reviewed at the beginning of §3.9, (i) [cf. also Example 2.14.3; [HASurI]; [HASurII]], may be understood as follows:

One verifies [cf. the discussion of [HASurI], §1.1] that the natural morphism obtained by evaluating sections of an ample line bundle over the universal vectorial extension of an elliptic curve at torsion points [cf. the discussion at the beginning of Example 2.14.3] is indeed an isomorphism [cf. (1_{trp}), (5_{trp})] by verifying that it is an isomorphism in the case of Tate curves by means of an explicit computation involving derivatives of theta functions [cf. (4_{trp})] and then proceeding to parallel transport this isomorphism in the case of Tate curves to the entire compactified moduli stack of elliptic curves in characteristic 0 by means of an explicit computation [the leading term portion of which is reviewed in §3.9, (i)] of the degrees of the vector bundles on this compactified moduli stack that constitute the domain and codomain of the natural morphism under consideration [cf. (2_{trp}), (3_{trp})].

Here, we recall from the discussion of (ii) above; §3.4, (iii), (iv); Fig. 3.9, that

the theory surrounding the theta function on the Tate curve may be thought of as a sort of function-theoretic representation of the [not necessarily $p$-adic arithmetic geometry of a copy of the tripod for which the cusps “0” and “$\infty$” are subject to the involution symmetry that permutes these two cusps and leaves the cusp “1” fixed.

In this context, it is also perhaps of interest to recall that there is an alternative approach to the parallel transport mechanism discussed above [i.e., computing degrees of vector bundles on the compactified moduli stack of elliptic curves], namely, the parallel transport mechanism applied in the proof of [HASurII], Theorem 4.3, which exploits various special properties of the Frobenius and Verschiebung morphisms in positive characteristic. Finally, we observe that although the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII] is not directly related, in a logical sense, to anabelian geometry, it nevertheless played a central role, as was discussed in detail in §3.9, in motivating the development of inter-universal Teichmüller theory, which may be understood as a sort of reformulation of the essential content of the scheme-theoretic
Hodge-Arakelov theory of [HASurI], [HASurII] via techniques based on anabelian geometry.

(v) Combinatorial anabelian geometry via tripodal transport: Let $F$ be a number field, $\mathcal{F}$ an algebraic closure of $F$, $X$ a hyperbolic curve over $F$, $n \geq 1$ an integer. Write $X_n$ for the $n$-th configuration space of $X$ [cf., e.g., [MT], Definition 2.1, (i)]; $\Pi_n$ for the étale fundamental group of $X_n \times_F \mathcal{F}$ [for a suitable choice of basepoint]; $\Pi \overset{\text{def}}{=} \Pi_1$; $\Pi^{\text{trpd}}$ for “$\Pi$” in the case where $X$ is the tripod [i.e., the projective line minus three points]; $G_F \overset{\text{def}}{=} \text{Gal}(\mathcal{F}/F)$; $\text{Out}^{\text{FC}}(\Pi_n)$ for the group of outer automorphisms of $\Pi_n$ satisfying certain technical conditions [i.e., “FC”] involving the fiberwise subgroups and cuspidal inertia subgroups [cf. [CombCusp], Definition 1.1, (ii), for more details]. Thus, it follows from the definition of “$\text{Out}^{\text{FC}}$” that the natural projection $X_{n+1} \rightarrow X_n$ given by forgetting the $(n+1)$-st factor determines a homomorphism

$$\phi_{n+1} : \text{Out}^{\text{FC}}(\Pi_{n+1}) \rightarrow \text{Out}^{\text{FC}}(\Pi_n)$$

[cf. the situation discussed in [NodNon], Theorem B]; the natural action of $G_F$ on $X_n \times_F \mathcal{F}$ determines an outer Galois representation

$$\rho_n : G_F \rightarrow \text{Out}^{\text{FC}}(\Pi_n)$$

[cf. the situation discussed in [NodNon], Theorem C]. Write $\rho \overset{\text{def}}{=} \rho_1$, $\rho^{\text{trpd}}$ for “$\rho$” in the case where $X$ is the tripod. Then

the proof of the injectivity [cf. [NodNon], Theorem C] of

$$\rho : G_F \rightarrow \text{Out}^{\text{FC}}(\Pi)$$

given in [NodNon] is perhaps the most transparent/prototypical example of the phenomenon of tripodal transport.

Indeed, this proof may be summarized as follows:

One makes the [“relatively trivial”! — cf. (2 trp)] observation that $\rho$ admits a factorization

$$\rho = \phi_2 \circ \phi_3 \circ \rho_3 : G_F \rightarrow \text{Out}^{\text{FC}}(\Pi_3) \rightarrow \text{Out}^{\text{FC}}(\Pi_2) \rightarrow \text{Out}^{\text{FC}}(\Pi)$$

— which allows one to reduce [cf. (2 trp)] the verification of the desired injectivity of $\rho$ [cf. (1 trp), (5 trp)] to the verification of the injectivity of $\phi_2 \overset{\text{def}}{=} \phi_2 \circ \phi_3$ [cf. (3 trp), (4 trp)] and $\rho_3$ [cf. (4 trp)]. Then:

- One observes that the injectivity of $\phi_2 \overset{\text{def}}{=} \phi_2 \circ \phi_3$ depends only on the type “$(g, r)$” [i.e., the genus and number of punctures] of $X$, hence may
be verified in the case of — i.e., may be “parallel transported” [cf. (3<sup<trp></sup>)] to the case of — a totally degenerate pointed stable curve, i.e., a pointed curve obtained by gluing together some collection of tripods along the various cusps of the tripods [cf. (4<sup<trp></sup>)]. On the other hand, in the case of such a totally degenerate pointed stable curve, the desired injectivity [i.e., of the analogue of “$\phi_{23}$”] may be verified by applying the purely combinatorial/group-theoretic techniques of combinatorial anabelian geometry developed in [CbCusp], [NodNon] [cf. [NodNon], Theorem B].

One verifies the injectivity of $\rho_3$ by applying a certain natural homomorphism called the tripod homomorphism

$$\tau : \text{Out}^{FC}(\Pi_3) \to \text{Out}^{FC}(\Pi^{tpd})$$

[cf. [CbTpII], Theorem C, (ii)], which satisfies the property that $\rho^{tpd} = \tau \circ \rho_3 : G_F \to \text{Out}^{FC}(\Pi_3) \to \text{Out}^{FC}(\Pi^{tpd})$ and hence allows one to conclude the injectivity of $\rho_3$ from the well-known injectivity result of Belyi to the effect that $\rho^{tpd}$ is injective [cf. (4<sup<trp></sup>)].

Here, it is interesting to note, especially in light of the discussion of anabelian results and differentials in the final portions of (i), (ii), (iii), the central role played by combinatorial anabelian geometry — i.e., in particular, various combinatorial versions of the Grothendieck Conjecture such as [NodNon], Theorem A — in the parallel transport mechanism discussed above. Such combinatorial versions of the Grothendieck Conjecture concern group-theoretic characterizations of the decomposition of a pointed stable curve into various irreducible components glued together along the nodes of the curve. This sort of decomposition may be interpreted as a sort of discrete version of the notion of a differential, i.e., which may be thought of as a decomposition of a ring/scheme structure into infinitesimals. Finally, we emphasize that this proof of the injectivity of $\rho$ is a particularly striking example of the phenomenon of tripodal transport, in the sense that the issue of relating the injectivity of $\rho$ for an arbitrary $X$ to the injectivity of $\rho^{tpd}$, i.e., in the case of the tripod, seems, a priori, to be entirely intractable, at least so long as one restricts oneself to morphisms between schemes [cf. the discussion in the final portion of (i)].

(vi) **Tripodal transport and Bogomolov’s proof:** Often, as in the examples discussed in (ii), (iii), (iv), above, the tripod that appears in instances of the phenomenon of tripodal transport is a tripod in which the cusps “0” and “∞” play a distinguished, but symmetric role, which is somewhat different from the role played by the cusp “1”. When considered from this point of view, the tripod may thought of as the underlying scheme of the group scheme $\mathbb{G}_m$ [with its origin removed], hence, in
particular, as a sort of algebraic version of the topological circle \( S^1 \). If one thinks of the tripod in this way, i.e., as corresponding to \( S^1 \), then the proof given by Bogomolov [cf. [ABKP], [Zh], [BogIUT], as well as the discussion of §4.3, (iii), below] of the geometric version of the Szpiro Conjecture over the complex numbers may also be understood as an instance, albeit in a somewhat generalized sense, of the technique of tripodal transport.

To explain further, we introduce notation as follows:

- Write \( \text{Aut}_\pi(\mathbb{R}) \) for the group of self-homeomorphisms \( \mathbb{R} \sim \mathbb{R} \) that commute with translation by \( \pi \in \mathbb{R} \). Thus, if we think of \( S^1 \) as the quotient \( \mathbb{R}/(2\pi \cdot \mathbb{Z}) \), then \( \text{Aut}_\pi(\mathbb{R}) \) may be understood as the group of self-homeomorphisms of \( \mathbb{R} \) that lift elements of the group \( \text{Aut}_+(S^1) \) of orientation-preserving self-homeomorphisms of \( S^1 \) that commute with multiplication by \(-1\) on \( S^1 \). In particular, we have a natural exact sequence \( 1 \to 2\pi \cdot \mathbb{Z} \to \text{Aut}_\pi(\mathbb{R}) \to \text{Aut}_+(S^1) \to 1 \).
- Write \( \text{Aut}_\pi(\mathbb{R}_{\geq 0}) \) for the group of self-homeomorphisms \( \mathbb{R}_{\geq 0} \sim \mathbb{R}_{\geq 0} \) that stabilize and restrict to the identity on the subset \( \pi \cdot \mathbb{N} \subseteq \mathbb{R}_{\geq 0} \).
- Write \( \mathbb{R}_{|\pi|} \) for the set of \( \text{Aut}_\pi(\mathbb{R}_{\geq 0}) \)-orbits of \( \mathbb{R}_{\geq 0} \) [relative to the natural action of \( \text{Aut}_\pi(\mathbb{R}_{\geq 0}) \) on \( \mathbb{R}_{\geq 0} \)]. Thus,

\[
\mathbb{R}_{|\pi|} = \left( \bigcup_{n \in \mathbb{N}} \left\{ n \cdot \pi \right\} \right) \cup \left( \bigcup_{m \in \mathbb{N}} \left\{ (m \cdot \pi, (m + 1) \cdot \pi) \right\} \right)
\]

— where we use the notation “\([-\]” to denote the element in \( \mathbb{R}_{|\pi|} \) determined by an element or nonempty subset of \( \text{Aut}_\pi(\mathbb{R}_{\geq 0}) \)-orbit; we use the notation “\((-, -)\)” to denote an open interval in \( \mathbb{R}_{\geq 0} \); we observe that the natural order relation on \( \mathbb{R}_{\geq 0} \) induces a natural order relation on \( \mathbb{R}_{|\pi|} \).
- Write \( \delta^{\text{sup}} : \text{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{|\pi|} \) for the map that assigns to \( \alpha \in \text{Aut}_\pi(\mathbb{R}) \) the element \( \sup(\delta(\alpha)) \in \mathbb{R}_{|\pi|} \), where we observe that

\[
\delta(\alpha) \overset{\text{def}}{=} \left\{ \left\lfloor \alpha(x) - x \right\rfloor \mid x \in \mathbb{R} \right\} \subseteq \mathbb{R}_{|\pi|}
\]

is a finite subset [cf. the definition of \( \text{Aut}_\pi(\mathbb{R}) \)] of \( \mathbb{R}_{|\pi|} \), and that [as is easily verified, by observing that for any \( \beta \in \text{Aut}_\pi(\mathbb{R}) \) and \( x, y \in \mathbb{R} \) such that \( x \leq y \), there exists a \( \gamma \in \text{Aut}_\pi(\mathbb{R}_{\geq 0}) \) such that \( \beta(y) - \beta(x) = \beta((y-x)+x) - \beta(x) = \gamma(y-x) \) the assignments \( \delta(-), \delta^{\text{sup}}(-) \) are \( \text{Aut}_\pi(\mathbb{R}) \)-conjugacy invariant.
- Write \( SL_2(\mathbb{R})^\sim \) for universal covering of \( SL_2(\mathbb{R}) \). Thus, we have a natural central extension of topological groups \( 1 \to \mathbb{Z} \to SL_2(\mathbb{R})^\sim \to SL_2(\mathbb{R}) \to 1 \). By composing the natural embedding \( S^1 \to \mathbb{R}^{2x} \overset{\text{def}}{=} \mathbb{R}^2 \setminus \{(0,0)\} \) with the natural projection \( \mathbb{R}^{2x} \to \mathbb{R}^{2z} \overset{\text{def}}{=} \mathbb{R}^{2x}/\mathbb{R}_{>0} \), we obtain a natural homeomorphism \( S^1 \sim \mathbb{R}^{2z} \), hence [by considering the natural action of \( SL_2(\mathbb{R}) \) on \( \mathbb{R}^{2x}, \mathbb{R}^{2z} \)] natural actions

\[
SL_2(\mathbb{R}) \sim S^1; \quad SL_2(\mathbb{R})^\sim \sim \mathbb{R}
\]
[where we think of $\mathbb{R}$ as the universal covering of $S^1 = \mathbb{R}/(2\pi \cdot \mathbb{Z})$, the latter of which determines a natural injective homomorphism $SL_2(\mathbb{R}) \sim \hookrightarrow Aut_\pi(\mathbb{R})$]

[which, at times, we shall use to think of $SL_2(\mathbb{R}) \sim$ as a subgroup of $Aut_\pi(\mathbb{R})$. We may assume without loss of generality that the generator “1” of $Z \hookrightarrow SL_2(\mathbb{R}) \sim$ was chosen so as to act on $\mathbb{R}$ in the positive direction.

Write $SL_2(\mathbb{Z}) \sim \overset{\text{def}}{=} SL_2(\mathbb{R}) \sim \times_{SL_2(\mathbb{R})} SL_2(\mathbb{Z}).$ Thus, we have a natural central extension of discrete groups $1 \to Z \to SL_2(\mathbb{Z}) \sim \to SL_2(\mathbb{Z}) \to 1$. One shows easily [e.g., by considering the discriminant modular form, as in [BogIUT]] that the abelianization of $SL_2(\mathbb{Z}) \sim$ is isomorphic to $\mathbb{Z}$, and hence that there exists a unique surjective homomorphism $\chi : SL_2(\mathbb{Z}) \sim \to \mathbb{Z}$

that maps positive elements of $Z \hookrightarrow SL_2(\mathbb{Z}) \sim$ to positive elements of $\mathbb{Z}$.]

In some sense, the fundamental phenomenon that underlies Bogomolov’s proof is the following elementary fact:

Whereas the $SL_2(\mathbb{Z})$-conjugacy classes of the unipotent elements $\tau^m \overset{\text{def}}{=} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$

\textbf{differ} for different positive integers $m$, the $SL_2(\mathbb{R})$-conjugacy classes of these elements \textbf{coincide} for different positive integers $m$.]

In the context of Bogomolov’s proof, if one thinks of $SL_2(\mathbb{Z})$ as the topological fundamental group of the moduli stack of elliptic curves over the complex numbers, then such unipotent elements arise as the images in $SL_2(\mathbb{Z})$ — via the [outer] homomorphism induced on topological fundamental groups by the classifying morphism associated to a family of one-dimensional complex tori over a hyperbolic Riemann surface $S$ of finite type — of the natural generators of cuspidal inertia groups of the topological fundamental group of $S$. In this situation, the positive integer $m$ then corresponds to the \textbf{valuation} of the $q$-parameter at a cusp of $S$. Next, we recall [cf., e.g., [BogIUT], (B1)] that unipotent elements of $SL_2(\mathbb{R})$ admit \textbf{canonical liftings} to $SL_2(\mathbb{R}) \sim$. In particular, it makes sense to apply both $\delta^{\sup}$ and $\chi$ to the canonical lifting $\tilde{\tau}^m \in SL_2(\mathbb{Z}) \sim$ of $\tau^m$. Since $\chi$ is a \textbf{homomorphism}, we have $\chi(\tilde{\tau}^m) = m$
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[cf., e.g., [BogIUT], (B3)]. On the other hand, since $\delta^{\sup}(-)$ is $\text{Aut}_\pi(\mathbb{R})$- [hence, in particular, $SL_2(\mathbb{R})^\sim$] conjugacy invariant, we have

$$\delta^{\sup}(\tau^m) < [\pi]$$

[cf., e.g., [BogIUT], (B1)] for arbitrary $m$. It is precisely by applying both $\chi$ and $\delta^{\sup}(-)$ to a certain natural relation [arising from the image in $SL_2(\mathbb{Z})$ of the “usual defining relation” of the topological fundamental group of $S$] between elements $\in SL_2(\mathbb{Z})$ lifted to $SL_2(\mathbb{Z})^\sim$ that one is able to derive the geometric version of the Szpiro inequality, that is to say, to bound the height of the given family of one-dimensional complex tori — i.e., more concretely, in essence, the sum of the “$m$’s” arising from the various cusps of $S$ [cf., e.g., [BogIUT], (B4)] — by a number that depends only on the genus and number of cusps of $S$ and not on the “$m$’s” themselves [cf., e.g., [BogIUT], (B2), (B5)].

From the point of view of the technique of tripodal transport, one may summarize this argument as follows:

one bounds the height [i.e., essentially, the sum of the “$m$’s”] of the given family of one-dimensional complex tori [cf. (5 trp)] — which is a reflection of the holomorphic moduli of this family [cf. (1 trp)] — by combining a “relatively trivial” [cf. (2 trp)] object $\chi$ arising from the holomorphic structure of the moduli stack of elliptic curves over the complex numbers [i.e., from the discriminant modular form] with the parallel transport mechanism [cf. (3 trp)] given by passing from the “holomorphic” $SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})^\sim$ to the “real analytic” $SL_2(\mathbb{R})$, $SL_2(\mathbb{R})^\sim$, i.e., in essence, by passing to the $\text{Aut}_+(S^1)$-invariant geometry of $S^1$, as reflected in the $\text{Aut}_\pi(\mathbb{R})$-conjugacy invariant map $\delta^{\sup} : \text{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{[\pi]}$ [cf. (4 trp)].

From the point of view of the analogy [cf. the discussion of (ii) above; [BogIUT]] between Bogomolov’s proof and inter-universal Teichmüller theory, we observe that:

· The canonical lifts discussed above of unipotent elements $\in SL_2(\mathbb{Z})$ to $SL_2(\mathbb{Z})^\sim$ correspond to the theory of the étale theta function [i.e., [EtTh]] in inter-universal Teichmüller theory.

· The $\text{Aut}_+(S^1)$-invariant geometry of $S^1$, as reflected in the $\text{Aut}_\pi(\mathbb{R})$-conjugacy invariant map $\delta^{\sup} : \text{Aut}_\pi(\mathbb{R}) \to \mathbb{R}_{[\pi]}$, corresponds to the theory of mono-analytic log-shells and related log-volume estimates [cf. (12 est); §3.7, (iv); [IUTchIV], §1, §2] in inter-universal Teichmüller theory. In particular, $\text{Aut}_+(S^1)/\text{Aut}_\pi(\mathbb{R})$-indeterminacies in Bogomolov’s proof — in which both the additive [i.e., corresponding to unipotent subgroups of $SL_2(\mathbb{R})$] and multiplicative [i.e., corresponding to toral, or equivalently, compact subgroups of $SL_2(\mathbb{R})$] dimensions of $SL_2(\mathbb{R})$ are “confused” within the single dimension of $S^1$ — correspond to the indeterminacies (Ind1), (Ind2), (Ind3) of
inter-universal Teichmüller theory.

- The role played by $SL_2(\mathbb{Z}), SL_2(\mathbb{Z})^\sim, \chi$ corresponds to the role played by the fixed arithmetic holomorphic structure of the RHS of the $\Theta$-link [cf. $(1^{\text{est}}), (6^{\text{est}}), (7^{\text{est}}), (8^{\text{est}}), (9^{\text{est}}), (10^{\text{est}}), (11^{\text{est}})]$ in the argument of §3.7, (ii). By contrast, the role played by $SL_2(\mathbb{R}), SL_2(\mathbb{R})^\sim, \delta^{\text{sup}}(-)$ corresponds to the role played by the multiradial representation [cf. $(2^{\text{est}}), (3^{\text{est}}), (4^{\text{est}}), (5^{\text{est}})]$ in the argument of §3.7, (ii).

In particular, one has natural correspondences

\[
SL_2(\mathbb{R}), SL_2(\mathbb{R})^\sim, \delta^{\text{sup}}(-) \leftrightarrow \text{[IUTchIII], Theorem 3.11};
\]

\[
SL_2(\mathbb{Z}), SL_2(\mathbb{Z})^\sim, \chi \leftrightarrow \text{[IUTchIII], Corollary 3.12 (\Leftarrow \text{Theorem 3.11})}
\]

— i.e., where, more precisely, the RHS of the latter correspondence is to be understood as referring to the derivation of [IUTchIII], Corollary 3.12, from [IUTchIII], Theorem 3.11. These last two correspondences are particularly interesting in light of the well-documented historical fact that the theory/estimates in Bogomolov’s proof related to $SL_2(\mathbb{R}), SL_2(\mathbb{R})^\sim, \delta^{\text{sup}}(-)$ were apparently already known to Milnor in the 1950’s [cf. [MlWd]], while the idea of combining these estimates with the theory surrounding $SL_2(\mathbb{Z}), SL_2(\mathbb{Z})^\sim, \chi$ appears to have been unknown until the work of Bogomolov around the year 2000 [cf. [ABKP]]. Moreover, these last two correspondences — and, indeed, the entire analogy between Bogomolov’s proof and inter-universal Teichmüller theory — are also of interest in the following sense:

Bogomolov’s proof only involves working with elements $\in SL_2(\mathbb{R}), SL_2(\mathbb{R})^\sim$ that arise from topological fundamental groups, hence may be applied not only to algebraic/holomorphic families of elliptic curves, but also to arbitrary topological families of one-dimensional complex tori that satisfy suitable conditions at the points of degeneration, i.e., “bad reduction”.

This aspect of Bogomolov’s proof is reminiscent of the fact that the initial $\Theta$-data of inter-universal Teichmüller theory [cf. §3.3, (i)] essentially only involves data that arises from various arithmetic fundamental groups associated to an elliptic curve over a number field. In particular, this aspect of Bogomolov’s proof suggests strongly that perhaps, in the future, some version of inter-universal Teichmüller theory could be developed in which the initial $\Theta$-data of the current version of inter-universal Teichmüller theory is replaced by some collection of topological groups that satisfies conditions analogous to the conditions satisfied by the collection of arithmetic fundamental groups that appear in the initial $\Theta$-data of the current version of inter-universal Teichmüller theory, but that does not necessarily arise, in a literal sense, from an elliptic curve over a number field.
§ 3.11. Mathematical analysis of elementary conceptual discomfort

We conclude our exposition, in the present §3, of the main ideas of inter-universal Teichmüller theory by returning to our discussion of the point of view of a hypothetical high-school student, in the style of §1. Often the sort of deep conceptual discomfort that such a hypothetical high-school student might experience when attempting to understand various elementary ideas in mathematics may be analyzed and elucidated more constructively when viewed from the more sophisticated point of view of a professional mathematician. Moreover, this sort of approach to mathematical analysis of conceptual discomfort may be applied to the analysis of the discomfort that some mathematicians appear to have experienced when studying various central ideas of inter-universal Teichmüller theory, such as the log- and Θ-links.

(i) Proof by mathematical induction: Some high-school students encounter substantial discomfort in accepting the notion of proof by mathematical induction, for instance, in the case of proofs of facts such as the following:

Example 3.11.1. Sum of squares of consecutive integers. For any positive integer \( n \), it holds that

\[
\sum_{j=1}^{n} j^2 = \frac{1}{6}n(2n+1)(n+1).
\]

Such discomfort is at times expressed by assertions to the effect that they cannot believe that it is not possible to simply give some sort of more direct argument that applies to all positive integers at once — i.e., without resorting to such indirect and “non-intuitive” devices of reasoning as the induction hypothesis — in the style of proofs of facts such as the following:

Example 3.11.2. Square of a sum. For any positive integers \( n \) and \( m \), it holds that

\[
(n+m)^2 = n^2 + 2nm + m^2.
\]

On the other hand, from the more sophisticated point of view of a professional mathematician, the situation surrounding the usual proofs of these facts in Examples 3.11.1, 3.11.2 may be understood as follows:

The fact in Example 3.11.2 in fact holds for arbitrary elements “\( n \)” and “\( m \)” in an arbitrary commutative ring and hence, in particular, is best understood as a consequence of the axioms of a commutative ring. By contrast, the fact in Example 3.11.1 is a fact that depends on the structure of the particular ring \( \mathbb{Z} \), or, essentially equivalently, on the structure of the particular monoid \( \mathbb{N} \). In particular, it is natural that any proof of the fact in Example 3.11.1
should depend, in an essential way, on the \textit{definition} of \([\mathbb{Z} \text{ or }] \mathbb{N}\). On the other hand, the logical structure of an argument by \textbf{mathematical induction} is, in essence, simply a rephrasing of the \textbf{very definition} of \(\mathbb{N}\).

In light of this state of affairs, although it seems to be rather difficult to formulate and prove, in a rigorous way, the assertion that there does not exist a proof of the fact in Example 3.11.1 that does not essentially involve mathematical induction, at least from the standard point of view of mathematics at, say, the undergraduate or graduate level, it does not seem natural or reasonable to expect the existence of a proof of the fact in Example 3.11.1 that does not essentially involve mathematical induction.

(ii) \textbf{Identification of the domain and codomain of the logarithm:} A somewhat different situation from the situation discussed in (i) may be seen in the case of the notion of a \textit{logarithm}. Some high-school students encounter substantial discomfort in accepting the notion of a logarithm on the grounds that a number in the \textbf{exponent} of an expression such as

\[ a^b \]

[where, say, \(a, b \in \mathbb{R}_{>0}\)], i.e., “\(b\)”, seems to have a \textbf{fundamentally different meaning} from a number \textbf{not} in the exponent, i.e., “\(a\)”. In light of this “fundamental difference in meaning” between numbers in and not in the exponent, a function such as the \textit{logarithm}, i.e., which “converts” [cf. such relations as \(\log(a^b) = b \cdot \log(a)\)] numbers in the exponent into numbers not in the exponent, seems, from the point of view of such students, to be \textbf{infinitely mysterious} or intractable in nature. From the point of view of a professional mathematician, this sort of “fundamental difference in meaning” between numbers in and not in the exponent may be understood as the difference between iteration of the monoid operation in the underlying \textit{multiplicative} and \textit{additive monoids} of the topological field \(\mathbb{R}\), i.e., as the difference between the \textit{multiplicative} and \textit{additive structures} of the topological field \(\mathbb{R}\). The [natural] logarithm on positive real numbers may then be understood as a certain \textit{natural isomorphism}

\[ \log : \mathbb{R}_{>0} \sim \rightarrow \mathbb{R} \]

between the underlying \textbf{positive} \textit{multiplicative} and \textit{additive monoids} of the topological field \(\mathbb{R}\). Thus, the substantial discomfort that some high-school students encounter in accepting the notion of a \textit{logarithm} may be understood as

a difficulty in accepting the \textbf{identification} of the underlying additive monoid of the topological field \(\mathbb{R}\) that contains the multiplicative monoid \(\mathbb{R}_{>0}\) in the \textbf{domain} of log with the underlying additive monoid of the topological field \(\mathbb{R}\) that appears in the \textbf{codomain} of log on the grounds that the map \(\log\) is \textbf{not}
compatible with [i.e., does not arise from a ring homomorphism between] the ring structures in its domain and codomain.

Here, we recall that this identification is typically “taken for granted” or “regarded as not requiring any justification” in discussions concerning the natural logarithm on positive real numbers. A similar identification that is “taken for granted” or “regarded as not requiring any justification” may be seen in typical discussions concerning the \( p \)-adic logarithm, as well as in the closely related identification, in the context of \( p \)-adic Hodge theory, of the copy of “\( \mathbb{Z}_p \)” lying inside the base field of a \( p \)-adic variety with the copy of “\( \mathbb{Z}_p \)” that acts on certain types of \( \text{étale} \) local systems on the variety [cf. the discussion of [EtTh], Remark 2.16.2; the discussion in the final portion of [Pano], §3; the discussion of “mysterious tensor products” in [BogIUT]]. By contrast, in the case of the log-link in inter-universal Teichmüller theory, it is of crucial importance, as discussed in the latter portion of §3.3, (ii), to distinguish the domain and codomain of the log-link, since confusion of the domain and codomain of the log-link — i.e., confusion of the multiplicative and additive structures that occur in the domain of the \( \Theta \)-link — would yield a situation in which the \( \Theta \)-link is not well-defined.

In particular, interestingly enough, although the substantial discomfort that some high-school students experience when studying the logarithm is inconsistent with the point of view typically taken in discussions of the natural logarithm on positive real numbers or the \( p \)-adic logarithm in the context of \( p \)-adic Hodge theory, this substantial discomfort of some high-school students is, somewhat remarkably, consistent with the situation surrounding the log-link in the context of the log-theta-lattice in inter-universal Teichmüller theory.

(iii) **Conceptual content of the ABC inequality:** Yet another kind of situation — which resembles, in certain aspects, the situation discussed in (i), but is related, in other aspects, to the situation discussed in (ii) — may be seen in elementary discussions of the ABC inequality for rational integers [i.e., an immediate consequence of the Szpiro Conjecture inequality discussed in §3.7, (iv)]. This most fundamental version of the ABC inequality may be stated as follows:

There exists a positive real number \( \lambda \) such that for all triples \((a, b, c)\) of relatively prime positive integers satisfying \( a + b = c \), it holds that

\[
abc \leq \left( \prod_{p|abc} p \right)^\lambda.
\]

Here, we observe that whereas the left-hand side “LHS” of this inequality is a quantity that measures the size — i.e., from a more advanced point of view, the height — of
the triple \((a, b, c)\) relative to the **additive structure** of the additive monoid \(\mathbb{N}\), the **right-hand side** \(\text{"RHS"}\) of this inequality is a quantity that arises from thinking of the triple \((a, b, c)\) in terms of the **multiplicative monoid** \(\mathbb{N}_{\geq 1}\) modulo the **quotient relation** \(p \sim p^n\) [for \(n\) a positive integer] that **identifies primes with arbitrary positive powers of primes**. Note that since the **multiplicative structure** of \(\mathbb{N}_{\geq 1}\) may be derived immediately from the **additive structure** of \(\mathbb{N}\) — e.g., by thinking of \(a \cdot b\) as the sum \(a + \cdots + a\) of \(b\) copies of \(a\) — one may also think of the LHS of the above inequality as a measure of the size of the triple \((a, b, c)\) relative to the **ring structure** of \(\mathbb{Z}\) [i.e., which involves both the **additive** and **multiplicative** structures of \(\mathbb{Z}\)]. In particular, one may understand, from a more **conceptual point of view**, the **"trivial inequality"** in the **opposite direction**, i.e.,

\[
abc \geq \prod_{p \mid abc} p
\]

as a reflection of the **elementary observation** that the **multiplicative monoid** \(\mathbb{N}_{\geq 1}\) considered modulo \(\sim\) may be **"easily derived from"** — or, in other words, is **"dominated/controlled by"** — the **additive structure/ring structure** of \(\mathbb{Z}\):

\[
\left(\text{additive structure/ring structure of } \mathbb{Z}\right) \succ \left(\text{multiplicative monoid } \mathbb{N}_{\geq 1} \text{ modulo } \sim\right).
\]

By contrast, the fundamental form of the **ABC inequality** recalled above may be understood, at a more **conceptual level**, as the assertion that, up to a certain **"indeterminacy"** [corresponding to **"\(\lambda\)"**], the **multiplicative monoid** \(\mathbb{N}_{\geq 1}\) considered modulo \(\sim\) is **sufficiently potent** as to **dominate/control** the **additive structure/ring structure** of \(\mathbb{Z}\):

\[
\left(\text{additive structure/ring structure of } \mathbb{Z}\right) \prec \left(\text{multiplicative monoid } \mathbb{N}_{\geq 1} \text{ modulo } \sim\right)
\]

— a somewhat **startling** assertion, since, at least from an **a priori** point of view, passing from, say, the ring \(\mathbb{Z}\) to the multiplicative monoid \(\mathbb{N}_{\geq 1}\) modulo \(\sim\) appears to involve quite a **substantial loss of data/structure**. On the other hand, this **conceptual interpretation** of the ABC inequality is remarkably reminiscent of the **\(\Theta\)-link** [cf. the discussion of the **log-link in (ii)**] and **multiradial representation** of inter-universal Teichmüller theory, which in effect assert [cf. the discussion of §3.7, (i)] that

the **multiplicative** monoids/Frobenioids, together with Galois actions, that appear in the data glued together via the **\(\Theta\)-link** are **sufficiently potent** as to **dominate/control** up to certain indeterminacies — via the **multiradial representation** of the **\(\Theta\)-pilot** — the **ring structure/ arithmetic holomorphic structure** in the domain of the **\(\Theta\)-link**, i.e.,
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ring structure that gives to rise to Θ-pilot
≺ (multiplicative monoids/Frobenioids and Galois actions in Θ-link).

That is to say, in summary,

the conceptual interpretation of the ABC inequality discussed above is already sufficiently rich as to strongly suggest numerous characteristic features of inter-universal Teichmüller theory, i.e., such as the multiradial representation, up to suitable indeterminacies, of various distinct ring structures related by the much weaker [a priori] data — i.e., multiplicative monoids/Frobenioids and Galois actions — that appears in the Θ-link.

In particular, from the point of view of this conceptual interpretation of the ABC inequality, such characteristic features of inter-universal Teichmüller theory are quite natural and indeed appear remarkably close to being “inevitable” in some suitable sense [cf. the discussion of (i)]. Finally, we observe that this interplay, with regard to “domination/control”, between rigid ring structures [which determine the “height”] and weaker structures with indeterminacies [for which primes are identified with their positive powers] via the multiradial representation is remarkably reminiscent of the discussion in §3.10, (vi), of the interplay, in the context of Bogomolov’s proof of the geometric version of the Szpiro Conjecture, between the rigid $SL_2(\mathbb{Z})$ [where conjugacy classes of unipotent elements determine the “height”] and the less rigid $SL_2(\mathbb{R})$ [where conjugacy classes of unipotent elements identify arbitrary positive powers of such elements] via “$\delta^{\sup}(-)$”.

(iv) Logical AND vs. logical OR, multiple copies, and multiradiality: As discussed at the beginning of §3.3, (ii), each lattice point in the log-theta-lattice [i.e., each “•” in Fig. 3.6] represents a $\Theta^{\pm\text{ell}}NF$-Hodge theater, which may be thought of as a sort of miniature model of the conventional scheme/ring theory surrounding the given initial Θ-data. In the following discussion, we shall use the notation “$(-)R$”, where $(-) \in \{\dag, \dagger\}$, to denote a particular such model of conventional scheme/ring theory; we shall write “$*$” for some $\mathcal{F}^{\uparrow\cdot\uparrow\cdot\uparrow\cdot\uparrow\cdot\uparrow\cdot\uparrow\cdot\uparrow\cdot\uparrow\cdot\text{prime-strip}}$ [i.e., some collection of data] that is isomorphic either to the domain data $(a^\Theta)$, $(b^\Theta)$, $(c^\Theta)$ or, equivalently, to the codomain data $(a^\theta)$, $(b^\theta)$, $(c^\theta)$ of the Θ-link — cf. the discussion of the latter portion of §3.7, (i); the discussion of §3.10, (ii); [IUTchII], Definition 4.9, (viii)], regarded up to isomorphism. Thus, the Θ-link may be thought of as consisting of the assignments

$$* \mapsto \uparrow q^N \in \dagger R; \quad * \mapsto \dagger q \in \uparrow R$$

— where $N = 1, 2, \ldots, j^2, \ldots, (l^*)^2$; “$\uparrow q^N$” denotes the domain data $(a^\Theta)$, $(b^\Theta)$, $(c^\Theta)$ of the Θ-link, which belongs [i.e., “$\in$”] to the model of conventional scheme/ring theory
in the domain of the Θ-link; “‡q” denotes the codomain data \((a^q), (b^q), (c^q)\) of the Θ-link, which belongs [i.e., “∈”] to the model of conventional scheme/ring theory “‡R” in the codomain of the Θ-link. In this context, we observe that one fundamental — but entirely elementary! — issue that arises when considering these two assignments “‡ \(\mapsto \‡q^N\), “‡ \(\mapsto \‡q\)” is the issue of what precisely is the logical relationship between these two assignments that constitute the Θ-link?

In a word, the answer to this question — which underlies, in an essential way, the entire logical structure of inter-universal Teichmüller theory — is as follows:

the Θ-link is to be understood — as a matter of definition! — as a construction with respect to which these two assignments are simultaneously valid, that is to say, from the point of view of symbolic logical relators,

\[
(\ast \mapsto \‡q^N) \land (\ast \mapsto \‡q)
\]

— i.e., where “\(\land\)” denotes the logical relator “AND” [cf. the discussion of the “distinct labels approach”, “\(\land\)” in [IUTchIII], Remark 3.11.1, (vii); [IUTchIII], Remark 3.12.2, (ii), \((c^{itw}), (f^{itw})\)].

Here, we observe that:

(1\textsuperscript{and}) If one forgets the distinct labels “‡”, “‡”, then the resulting collections of data \(q^N \in \mathbb{R}, q \in \mathbb{R}\) are different. In particular, if one deletes the distinct labels “‡”, “‡”, then the crucial logical relator “\(\land\)” no longer holds [i.e., leads immediately to a contradiction!]. That is to say, it is precisely by distinguishing the two copies “‡R”, “‡R” that one obtains a well-defined construction of the Θ-link, as described above. This situation is reminiscent of the discussion of distinct copies in §1.3.

(2\textsuperscript{and}) One way to understand the notion of multiradiality in the case of the multiradial representation of the Θ-pilot is as the [highly nontrivial!] property of an algorithm that allows one to maintain the validity of this crucial logical relator “\(\land\)” throughout the execution of the algorithm — cf. the discussion of “simultaneous execution/meaningfulness” in §2.9; §3.4, (i); §3.7, (i), as well as the discussion of the properties “IPL”, “SHE”, “APT”, “HIS” in [IUTchIII], Remark 3.11.1. This point of view is reminiscent of the single vector bundle “\(p^*_1\mathcal{F}|_{S^\log_k}\overset{\sim}{\rightarrow} p^*_2\mathcal{F}|_{S^\log_k}\)” of §3.1, (v), (3\textsuperscript{KS}) [i.e., which serves simultaneously as a pull-back via \(p_1\) and as a pull-back via \(p_2\)], as well as of the discussion of §1.4, §1.5, §1.6 [i.e., of integration on \(\mathbb{R}^2\), as opposed to \(\mathbb{R}\)].
It is precisely by applying this interpretation [cf. (2^and)] of multiradiality — i.e., maintenance of the validity of this crucial logical relator “∧” throughout the execution of the multiradial algorithm — that one may conclude [cf. §3.7, (ii), (10^est), (11^est), (12^est)], in an essentially formal fashion, that the multiradial representation of the Θ-pilot, regarded up to suitable indeterminacies, is — simultaneously [cf. “∧”!] — a representation of the original q-pilot in the codomain of the Θ-link.

Put another way, one fundamental cause of certain frequently [and, at times, somewhat vociferously!] articulated misunderstandings of inter-universal Teichmüller theory is precisely

the misunderstanding that the Θ-link is to be understood — as a matter of definition! — as a construction with respect to which the two assignments “∗↦→‡qN”, “∗↦→†q” are not necessarily required to be simultaneously valid, that is to say, from the point of view of symbolic logical relators,

\[(∗↦→‡qN) ∨ (∗↦→†q)\]

— i.e., where “∨” denotes the logical relator “OR”.

Here, we observe that, if one takes the point of view of this misunderstanding, then:

(1^or) The logical relator “∨” remains valid even if one forgets the distinct labels “‡”, “†”. In particular, the use of distinct copies throughout inter-universal Teichmüller theory seems entirely superfluous — cf. the discussion of distinct copies in §1.3.

(2^or) The multiradial representation of the Θ-pilot — whose nontriviality lies precisely in the maintenance of the validity of the crucial logical relator “∧” throughout the execution of the multiradial algorithm! [cf. (2^and), (3^and)] — appears to be valid [which is not surprising since, in general, \(∧ ⇒ ∨！\)], but entirely devoid of any interesting content.

(3^or) As a result of the point of view of (2^or), the conclusion [cf. §3.7, (ii), (10^est), (11^est), (12^est)], in an essentially formal fashion, that the multiradial representation of the Θ-pilot, regarded up to suitable indeterminacies, is — simultaneously [i.e., “∧”!] — a representation of the original q-pilot in the codomain of the Θ-link appears somewhat abrupt, mysterious, or entirely unjustified.

Thus, in summary, confusion, in the context of the Θ-link and the multiradial representation of the Θ-pilot, between the logical relators “∧” and “∨” — which is, in essence, an entirely elementary issue [cf. the discussion of §1.3, as well as of [IUTchIII], Remark 3.11.1, (vii); [IUTchIII], Remark 3.12.2, (ii), (e^toy), (f^toy)] — has
the potential to lead to very deep repercussions with regard to understanding the essential logical structure of inter-universal Teichmüller theory. Finally, we conclude the present discussion by observing that one way to approach the task of understanding these aspects of the essential logical structure of inter-universal Teichmüller theory is by considering the following elementary combinatorial and numerical models:

Example 3.11.3. Elementary combinatorial model of “∧ vs. “∨”, multiple copies, and multiradiality. The ideas discussed in the present §3.11, (iv), may be summarized/expressed simply, in terms of elementary combinatorics, as in Fig. 3.22 below. Here, “∗” corresponds, in the above discussion, to “*”, i.e., to some abstract $F^{±}×μ$-prime-strip; each of the boxes in the upper left- and right-hand corners of the diagram corresponds, in the above discussion, to “$(-)R$”, where $(-) ∈ \{\†, \‡\}$, i.e., to some abstract $Θ^{±}ellNF$-Hodge theater; each of these boxes is equipped with two distinct substructures “$(-)0$”, “$(-)1$” [which may be thought of as corresponding, in the above discussion, respectively, to $(-)q$, $(-)q^N$] such that ∗ is glued [cf. the horizontal arrows emanating from either side of ∗] to $^\dagger 1$, $^\dagger 0$; the lower box in the center is to be understood as a copy of either of the two upper boxes on the left and right whose relationship to ∗ is, by definition, indeterminate, i.e., ∗ corresponds either to $^0 0$ or to $^0 1$; the diagonal arrows on the left and right then correspond to the operation of forgetting the datum of which of “$(-)0$”, “$(-)1$” is glued to ∗. Thus, the central portion of Fig. 3.22, delimited by dotted lines on either side, is to be thought of as containing objects that are, by definition, neutral/symmetric with respect to the portion marked with a “‡” on the left and the portion marked with a “†” on the right. Here, the gluings of ∗ in the upper portion of the diagram are to be understood as being [by definition!] simultaneously valid, i.e.,

$$(∗↦→^\dagger 1) \land (∗↦→^\dagger 0)$$

— a situation that is consistent/well-defined precisely because the two labels “†” and “‡” are regarded as being distinct [cf. (1 and)]. In particular, this upper “∧” portion of the diagram may be regarded as a sort of tautological, or initial, multiradial algorithm [cf. (2 and)], which is essentially equivalent to the “distinct labels approach” discussed in [IUTchIII], Remark 3.11.1, (vii). Then the operation of passing, via the diagonal arrows, from the upper “∧” portion of the diagram to the lower central box — i.e., to

$$(∗↦→^0 1) \lor (∗↦→^0 0)$$

— may be understood as corresponding to the “forced identification approach” discussed in [IUTchIII], Remark 3.11.1, (vii), or [alternatively and essentially equivalently!], from the point of view of the above discussion, as corresponding to the passage from “∧” to “∨” [where we recall that, in general, $∧ \implies ∨$], i.e., to

$$(∗↦→^\dagger 1) \lor (∗↦→^\dagger 0)$$
Here, we observe that this “∨” approach may also be regarded as a sort of “trivial multiradial algorithm” [cf. (2\textsuperscript{nd}), (2\textsuperscript{or})], i.e., in the sense, that, in general, it holds that $A \lor B$ is equivalent to $(A \lor B) \land (A \lor B)$.

![Diagram](image)

**Fig. 3.22:** Elementary combinatorial model of the Θ-link

**Example 3.11.4.** Elementary numerical model of “∧ vs. “∨”, multiple copies, and multiradiality. Alternatively, the ideas exposed in the present §3.11, (iv), may be summarized/expressed in terms of elementary numerical manipulations, as follows. First of all, the overall general logical flow of inter-universal Teichmüller theory — i.e., starting from the definition of the Θ-link, proceeding to the multiradial representation of the Θ-pilot [cf. [IUTchIII], Theorem 3.11], and finally, culminating in a final numerical estimate [cf. [IUTchIII], Corollary 3.12] — may be represented by means of real numbers $A, B \in \mathbb{R}_{>0}$ and $\epsilon, N \in \mathbb{R}$ such that $0 \leq \epsilon < 1$ in the following way:

- **Θ-link:**
  
  \[ (N \overset{\text{def}}{=} -2B) \land (N \overset{\text{def}}{=} -A); \]

- **multiradial representation of the Θ-pilot:**
  
  \[ (N = -2A + \epsilon) \land (N = -A); \]

- **final numerical estimate:**
  
  \[-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A < 1.\]

Thus, the Θ-link [cf. (1\textsuperscript{st})] and multiradial representation of the Θ-pilot [cf. (2\textsuperscript{nd})] are meaningful/nontrivial precisely because of the logical relator “∧”, whose use obligates one, in the definition of the Θ-link, to consider *a priori distinct* real numbers $A, B$;
the passage from the multiradial representation of the $\Theta$-pilot to the final numerical estimate is immediate/straightforward/logically transparent [cf. (3\textsuperscript{and})]. By contrast, if one replaces “∧” by “∨”, then our elementary numerical model of the logical structure of inter-universal Teichmüller theory takes the following form:

- **“∨” version of $\Theta$-link:**
  $$(N \overset{\text{def}}{=} -2B) \lor (N \overset{\text{def}}{=} -A) \quad [\text{cf. } (N \overset{\text{def}}{=} -2A) \lor (N \overset{\text{def}}{=} -A)];$$

- **“∨” version of multiradial representation of the $\Theta$-pilot:**
  $$(N = -2A + \epsilon) \lor (N = -A);$$

- **final numerical estimate:**
  $$-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A < 1.$$

That is to say, the use of distinct real numbers $A, B$ in the definition of the “∨” version of $\Theta$-link seems entirely superfluous [cf. (1\textsuperscript{or})]. This motivates one to identify $A$ and $B$ — i.e., to suppose “for the sake of simplicity” that $A = B$ — which then has the effect of rendering the definition of the original “∧” version of the $\Theta$-link invalid/self-contradictory [cf. (1\textsuperscript{and}), (1\textsuperscript{or})]. Once one identifies $A$ and $B$, i.e., once one supposes “for the sake of simplicity” that $A = B$, the passage from the “∨” version of $\Theta$-link to the [resulting “∨” version of the] multiradial representation of the $\Theta$-pilot seems entirely meaningless/devoid of any interesting content [cf. (2\textsuperscript{or})]. The passage from the [resulting meaningless “∨” version of the] multiradial representation of the $\Theta$-pilot to the final numerical estimate then seems abrupt/mysterious/entirely unjustified, i.e., put another way, looks as if

one erroneously replaced the “∨” in the meaningless “∨” version of the multiradial representation of the $\Theta$-pilot by an “∧” without any mathematical justification whatsoever [cf. (3\textsuperscript{or})].

This is precisely the pernicious chain of misunderstandings that has given rise to a substantial amount of unnecessary confusion concerning inter-universal Teichmüller theory.

(v) **Closed loops and two-dimensionality:** In the context of the discussion of (iv), it is useful to observe that one way to think of the construction algorithm of the multiradial representation of the $\Theta$-pilot is as follows [cf. the discussion of [IUTchIII], Remark 3.9.5, (viii)]:

- \[ \text{(v) Closed loops and two-dimensionality: In the context of the discussion of (iv), it is useful to observe that one way to think of the construction algorithm of the multiradial representation of the } \Theta\text{-pilot is as follows [cf. the discussion of [IUTchIII], Remark 3.9.5, (viii)]:} \]
(1) This construction may be thought of as a construction of a certain subquotient of the portion of the log-theta-lattice on the right-hand side of Fig. 3.6 [i.e., consisting of two vertical lines of log-links joined by a single horizontal Θ-link] that restricts to the identity on the vertical line of log-links that contains the codomain of the Θ-link.

(2) Alternatively, this subquotient may be thought of as a sort of projection of the “Θ-intertwining” [i.e., the structure on an abstract $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip as the $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip arising from the Θ-pilot object appearing in the domain of the Θ-link]

— up to suitable indeterminacies — onto

the “q-intertwining” [i.e., the structure on an abstract $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip as the $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip arising from the q-pilot object appearing in the codomain of the Θ-link].

Here, we observe further [cf. the discussion of §3.10, (ii)] that this algorithm converts any collection of input data — i.e., any $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip — into output data that is expressed in terms of the arithmetic holomorphic structure of the input data $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip, hence makes it possible to construct, in effect,

(3) a single $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip that is simultaneously equipped with both the q-intertwining and the Θ-intertwining, regarded up to suitable indeterminacies [cf. discussion of the crucial logical relator “∧” in (iv) above; the discussion of the final portion of [IUTchIII], Remark 3.9.5, (ix)].

Put another way, this algorithm allows one to

(4) construct a closed loop — i.e., from a given input data [q-intertwined!] $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip back to the given input data [q-intertwined!] $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip — whose output data consists of the Θ-intertwining [up to suitable indeterminacies] on this single given input data [q-intertwined!] $F\overset{\Theta}{\Rightarrow}×\mu$-prime-strip [cf. the discussion of the final portion of [IUTchIII], Remark 3.9.5, (ix)].

Here, we note that it is precisely this closed nature of the loop that allows one to derive [cf. §3.7, (ii), (10est), (11est), (12est)] nontrivial consequences [cf. the discussion of (3and), (3or) in (iv)] from the multiradial representation of the Θ-pilot. That is to say, one entirely elementary/“general nonsense” observation that may be made in this context is the following:
If, by contrast, the algorithm only yielded [not a closed loop, but rather] an “open path” — i.e., from one [“input”] type of mathematical object to some distinct/non-comparable [“output”] type of mathematical object — then one could only conclude from the algorithm some sort of relationship between the structure of the input object and the structure of the output object; it would not, however, be possible to conclude anything about the intrinsic structure of either the input or the output objects.

In this context, it is also important to note the crucial role played by the notion/definition [cf. [IUTchII], Definition 4.9, (viii)] of an \(\mathcal{F}^+\times\mu\)-prime-strip, i.e., by the particular sort of data that appears in an \(\mathcal{F}^+\times\mu\)-prime-strip. That is to say:

The data contained in an \(\mathcal{F}^+\times\mu\)-prime-strip is, on the one hand, sufficiently strong to suffice as input data for the construction algorithm of the multiradial representation of the \(\Theta\)-pilot, but, on the other hand, sufficiently weak so as to yield isomorphic collections of data [hence, in particular, to allow one to define the \(\Theta\)-link!] from the data arising from the \(q\)-pilot and \(\Theta\)-pilot objects.

Indeed, it is precisely this simultaneous sufficient strength/weakness that makes it possible to construct a single \(\mathcal{F}^+\times\mu\)-prime-strip that is simultaneously equipped with both the \(q\)-intertwining and the \(\Theta\)-intertwining, regarded up to suitable indeterminacies [cf. (3\textsuperscript{cl})]. Here, we note that, at a more concrete level:

The crucial sufficient strength/weakness properties discussed in (6\textsuperscript{cl}) may be understood as a consequence of the fact that an \(\mathcal{F}^+\times\mu\)-prime-strip is comprised of both unit group and value group portions — i.e., of portions corresponding to the “two arithmetic/combinatorial dimensions” of the discussion of §2.11 — but comprised in such a way that these two arithmetic/combinatorial dimensions are independent of one another, i.e., not [at least in any a priori sense!] subject to any ring structure/intertwining such as the \(q\)- or \(\Theta\)-intertwinings.

Moreover, as discussed in [IUTchIII], Remark 3.9.5, (vii), (Ob7); [IUTchIII], Remark 3.9.5, (ix), (x):

The crucial sufficient strength/weakness properties discussed in (6\textsuperscript{cl}), (7\textsuperscript{cl}) would fail to hold if various portions of the collection of data that constitutes an \(\mathcal{F}^+\times\mu\)-prime-strip are omitted.

For instance [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob7); [IUTchIII], Remark 3.9.5, (ix), (x)]:

If, in the argument of §3.7, (ii), one omits the formation of the holomorphic hull [cf. §3.7, (ii), (8\textsuperscript{ext})], then the resulting argument amounts, in essence, to
an attempt to establish a closed loop as in (3\textsuperscript{clid}), (4\textsuperscript{clid}), in a situation in which \( F^{\text{clid}} \times \mu \)-prime-strips are replaced by some alternative type of mathematical object that does not satisfy [cf. (8\textsuperscript{clid})] the crucial sufficient strength/weakness properties discussed in (6\textsuperscript{clid}), (7\textsuperscript{clid}), hence does not give rise to such a closed loop or, in particular, to the nontrivial conclusions arising from a closed loop [cf. (5\textsuperscript{clid})].

\textsection 4. Historical comparisons and analogies

\textsection 4.1. Numerous connections to classical theories

Many discussions of inter-universal Teichmüller theory exhibit a tendency to emphasize the novelty of many of the ideas and notions that constitute the theory. On the other hand, another important aspect of many of these ideas and notions of inter-universal Teichmüller theory is their quite substantial relationship to numerous classical theories. One notable consequence of this latter aspect of inter-universal Teichmüller theory is the following:

one obstacle that often hampers the progress of mathematicians in their study of inter-universal Teichmüller theory is a lack of familiarity with such classical theories, many of which date back to the 1960’s or 1970’s [or even earlier]!

(i) Contrast with classical numerical computations: We begin our discussion by recalling the famous computation in the late nineteenth century by William Shanks of \( \pi \) to 707 places, which was later found, with the advent of digital computing devices in the twentieth century, to be correct only up to 527 places!

The work that went into this sort of computation may strike some mathematicians as being reminiscent, in a certain sense, of the sheer number of pages of the various papers — i.e., such as [Semi], [FrdI], [FrdIII], [EtTh], [GenEll], [AbsTopIII], [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] — that one must study in order to achieve a thorough understanding of inter-universal Teichmüller theory. In fact, however, inter-universal Teichmüller theory differs quite fundamentally from the computation of Shanks in that, as was discussed throughout the present paper, and especially in \( \textsection 3.8 \), \( \textsection 3.9 \), \( \textsection 3.10 \), \( \textsection 3.11 \), the central ideas of inter-universal Teichmüller theory are rather compact and conceptual in nature and revolve around the issue of comparison, by means of the notions of mono-anabelian transport and multiradiality, of mutually alien copies of miniature models of conventional scheme theory in a fashion that exhibits remarkable similarities to the
compact and conceptual nature of the classical computation of the Gaussian integral by means of the introduction of two “mutually alien copies” of this integral.

One way of briefly summarizing these remarkable similarities [i.e., which are discussed in more detail in §3.8, as well as in the Introduction to the present paper] is as follows:

- the “theta portion” — i.e., the Θ-link — of the log-theta-lattice of inter-universal Teichmüller theory may be thought of as a sort of statement of the main computational problem of inter-universal Teichmüller theory and may be understood as corresponding to the various Gaussians that appear in the classical computation of the Gaussian integral, while
- the “log portion” — i.e., the log-link — of the log-theta-lattice of inter-universal Teichmüller theory may be thought of as a sort of solution of the main computational problem of inter-universal Teichmüller theory and may be understood as corresponding to the angular portion of the representation via polar coordinates of the [square of the] Gaussian integral.

In this context, it is of interest to recall the remarkable similarities of certain aspects of inter-universal Teichmüller theory to the theory surrounding the functional equation of the theta function — i.e., “Jacobi’s identity” [cf. the discussion of the final portion of [Pano], §3; the discussion preceding [Pano], Theorem 4.1] — which may be thought of as a sort of function-theoretic version of the computation of the Gaussian integral that may be obtained, roughly speaking, by interpreting this computation of the Gaussian integral in the context of the hyperbolic geometry of the upper half-plane. As discussed in §2.4; Example 3.2.1, (iii); §3.3, (v), the analogy between certain aspects of inter-universal Teichmüller theory and the hyperbolic geometry of the upper half-plane has not been exposed in the present paper in much detail since it has already been exposed in substantial detail in [BogIUT] [cf. also §3.10, (vi); §4.3, (iii), of the present paper]. On the other hand, it is of interest to recall that classically,

Jacobi’s identity was often appreciated as a relatively compact and conceptual way to achieve a startling improvement in computational accuracy in the context of explicit numerical calculations of values of the theta function [cf. the discussion preceding [Pano], Theorem 4.1].

(ii) Explicit examples of connections to classical theories: Next, we review various explicit examples of connections between inter-universal Teichmüller theory, as exposed thus far in the present paper, and various classical theories:

(1cls) Recall from the discussion of §2.10 that the notion of a “universe”, as well as the use of multiple universes within the discussion of a single set-up in arithmetic
geometry, already occurs in the mathematics of the 1960’s, i.e., in the mathematics of *Galois categories* and *étale topoi* associated to schemes [cf. [SGA1], [SGA4]].

(2\textsuperscript{cls}) One important aspect of the appearance of *universes* in the theory of *Galois categories* is the *inner automorphism indeterminacies* that occur when one relates Galois categories associated to distinct schemes via a morphism between such schemes [cf. [SGA1], Exposé V, §5, §6, §7]. These indeterminacies may be regarded as *distant ancestors*, or *prototypes*, of the more *drastic* indeterminacies — cf., e.g., the indeterminacies (Ind1), (Ind2), (Ind3) discussed in §3.7, (i) — that occur in inter-universal Teichmüller theory.

(3\textsuperscript{cls}) The theory of *Tate* developed in the 1960’s [cf. [Serre], Chapter III, Appendix] concerning *Hodge-Tate representations* plays a *fundamental role* in the theory of \([\mathbb{Q}_p]_{GC}\) [cf. also [AbsTopI], §3]. This theory of \([\mathbb{Q}_p]_{GC}\) and [AbsTopI], §3, may be regarded as a *precursor* of the theory of *log-shells* developed in [AbsTopIII], §3, §4, §5.

(4\textsuperscript{cls}) The approach of *Faltings* to *p*-adic *Hodge theory* via the technique of *almost étale extensions* [cf. [Falt2]] plays a *central role* in the *p*-adic *anabelian geometry* developed in \([p]_{GC}\). Here, we recall that this theory of \([p]_{GC}\) constitutes the *crucial technical tool* that underlies the *Belyi* and *elliptic cuspidalizations* of [AbsTopII], §3, which, in turn, play a quite essential role in the theory of [EtTh], [AbsTopIII], hence, in particular, in inter-universal Teichmüller theory.

(5\textsuperscript{cls}) The work of *Tate* in the 1960’s concerning *theta functions* on uniformizations of *Tate curves* [cf. [Mumf2], §5] plays a *fundamental role* in [EtTh], hence also in inter-universal Teichmüller theory.

(6\textsuperscript{cls}) The *scheme-theoretic Hodge-Arakelov theory* discussed in Example 2.14.3 and §3.9 may be regarded as a *natural extension* of [the portion concerning *elliptic curves* of] Mumford’s theory of *algebraic theta functions* [cf. [Mumf1]].

(7\textsuperscript{cls}) The *invariance of the étale site*, up to isomorphism, with respect to the *Frobenius morphism in positive characteristic* [cf. the discussion of Example 2.6.1, (i)] was well-known [cf. [SGA1], Exposé IX, Théorème 4.10] to the Grothendieck school in the 1960’s. As discussed in §2.6, §2.7, this phenomenon, taken together with the *fundamental work* of Uchida [cf. [Uchi]] in the 1970’s concerning the *anabelian geometry* of one-dimensional function fields over a finite field, may be regarded as the *fundamental prototype* for the apparatus of *mono-anabelian transport* — and, in particular, for the terms “*Frobenius-like*” and “*étale-like*” — which plays a *central role* in inter-universal Teichmüller theory.
The use of abstract [commutative] monoids, e.g., in the theory of Frobenioids [cf. §3.3, (iii); §3.5], which plays a fundamental role in inter-universal Teichmüller theory, was motivated by the use of such monoids in the theory of log schemes [cf. [Kato1], [Kato2]], which, in turn, was motivated by the use of such monoids in the classical theory of toric varieties developed in the 1970’s [cf. [KKMS]].

Recall from the discussion of §3.1, (iv), (v), that the notion of multiradiality, which plays a fundamental role in inter-universal Teichmüller theory, may be regarded as a sort of abstract combinatorial analogue of the Grothendieck definition of a connection, i.e., which plays a central role in the classical theory of the crystalline site and dates back to the 1960’s [cf. [GrCrs]].

(iii) Monoids and Galois theory: With regard to (ii), (8cls), we observe that the important role played by abstract [commutative] monoids in inter-universal Teichmüller theory is reminiscent of the way in which such monoids are used by many mathematicians in research related to “geometry over $\mathbb{F}_1$” [i.e., the fictitious “field with one element”]. On the other hand, the way in which such monoids are used in inter-universal Teichmüller theory differs fundamentally from the way in which such monoids are used in conventional research on geometry over $\mathbb{F}_1$ in the following respect:

- in inter-universal Teichmüller theory, various anabelian and Kummer-theoretic aspects of Galois or arithmetic fundamental groups that act on such monoids play a fundamental role [cf. the discussion of mono-anabelian transport in §2.7, §2.9];
- by contrast, at least to the author’s knowledge at the time of writing, research on geometry over $\mathbb{F}_1$ does not involve, in any sort of essential way, such anabelian or Kummer-theoretic aspects of Galois or arithmetic fundamental groups acting on monoids.

Indeed, this fundamental difference between inter-universal Teichmüller theory and conventional research on geometry over $\mathbb{F}_1$ might give rise to various interesting questions and hence stimulate further research. Finally, in this context, it is perhaps of interest to note that although there is no specific mathematical object in inter-universal Teichmüller theory that may be said to correspond to “$\mathbb{F}_1$”, in some sense the notion of a “field with one element” may, at a more conceptual level, be thought of as corresponding to the notion of coricity/cores/coric objects — that is to say, objects that are invariant with respect to [i.e., “lie under, in a unique way”] various operations such as links [cf. the discussion of §2.7, (iv)] — a notion which is indeed central to inter-universal Teichmüller theory.
(iv) **Techniques to avoid stacks and 2-categories:** Some mathematicians appear to have a strong aversion to the use of such notions as “categories of categories” or algebraic stacks — i.e., notions that obligate one to work with 2-categories — in arithmetic geometry. Here, we observe that the substantive mathematical phenomenon that obligates one, in such situations, to work with 2-categories is essentially the same phenomenon as the phenomenon constituted by the inner automorphism indeterminacies discussed in (ii), (2\textsuperscript{cls}). On the other hand, in inter-universal Teichmüller theory, various “general nonsense” techniques are applied that allow one, in such situations, to work with categories [i.e., as opposed to 2-categories] and thus avoid the cumbersome complications that arise from working with 2-categories:

- In inter-universal Teichmüller theory, one typically works with slim categories such as Galois categories that arise from slim profinite groups [i.e., profinite groups for which the centralizer of every open subgroup is trivial] or temp-slim tempered groups [cf. [Semi], Remark 3.4.1]. The use of slim categories allows one, in effect, to think of “categories of categories” as categories [i.e., rather than 2-categories]. Indeed, in the case of slim profinite groups, this point of view is precisely the point of view that underlies the theory of [GeoAnbd]. Generalities concerning slim categories may be found in [FrDI], Appendix.
- Another important “general nonsense” technique that is used in inter-universal Teichmüller theory to keep track explicitly of the various types of indeterminacies that occur is the notion of a poly-morphism [cf. [IUTchI], §0], i.e., a [possibly empty] subset of the set of arrows between two objects in a category. Thus, there is a natural way to compose two poly-morphisms [i.e., that consist of composable arrows] to obtain a new poly-morphism. Consideration of such composites of poly-morphisms allows one to trace how various indeterminacies interact with one another.

In this context, it is perhaps useful to observe that, from a more classical point of view, the inner automorphism indeterminacies discussed in (ii), (2\textsuperscript{cls}), correspond to the indeterminacy in the choice of a basepoint of a [say, connected, locally contractible] topological space.

That is to say, in anabelian geometry, working with

**slim anabelioids as opposed to slim profinite groups**

corresponds, in essence, to working, in classical topology, with

**topological spaces as opposed to pointed topological spaces.**
Since many natural maps between topological spaces — i.e., such as localization maps! — are not compatible with choices of distinguished points, it is often more natural, in many discussions of classical topology, to make use [not only of the notion of a “pointed topological space”, but also] of the notion of a “topological space” [i.e., that is not equipped with the choice of a distinguished point!]. It is precisely for this reason that in many discussions — i.e., such as those that occur in inter-universal Teichmüller theory, for instance, in the case of localizations at various primes of an NF! [cf. the discussion of §3.3, (iv), (v), (vi)] — involving the geometry of categories, it is much more natural and less cumbersome to work with slim categories such as slim anabelioids [i.e., as opposed to profinite groups].

(v) Notational complexity and mutually alien copies: Some readers of the papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] have expressed bafflement at the degree of complexity of the notation — i.e., by comparison to the degree of complexity of notation that is typical in conventional papers on arithmetic geometry — that appears in these papers. This complexity of notation may be understood as a natural consequence of

- the need to distinguish between objects that belong to distinct copies, i.e., distinct “miniature models”, of conventional scheme theory [cf., e.g., the labels “n,m” for the various lattice points “” in the log-theta-lattice, as discussed in §2.8, (2Fr/´et)]; §3.3, (ii); §3.6, (iv); [IUTchIII], Definition 3.8, (iii)], together with
- the need to distinguish between distinct objects — such as distinct cyclotomes related by nontrivial cyclotomic rigidity isomorphisms [cf., e.g., the discussion of §2.6, §2.12, §2.13, §3.4, as well as the discussion of §4.2, (i), below] — within a single miniature model of conventional scheme theory that are related to one another via structures that are “taken for granted” in conventional discussions of arithmetic geometry, but whose precise specification is in fact highly nontrivial in the context of situations where one considers multiple miniature models of conventional scheme theory.

Put another way, this complexity of notation may be regarded as an inevitable consequence of the central role played in inter-universal Teichmüller theory by “mutually alien copies/multiple miniature models” of conventional scheme theory and the resulting inter-universality issues that arise [cf. the discussion of §2.7, (i), (ii); §2.10; §3.8]. In particular, this complexity of notation is by no means superfluous.

§ 4.2. Contrasting aspects of class field theory and Kummer theory

We begin our discussion by observing that the role played by local class field theory [cf. the discussion of §2.11; §2.12, especially Example 2.12.1, (ii), (iii); §3.4, (v)]
in inter-universal Teichmüller theory is, in many respects, not particularly prominent, while global class field theory [for NF’s] is entirely absent from inter-universal Teichmüller theory. This situation for class field theory contrasts sharply with the very central role played by Kummer theory in inter-universal Teichmüller theory [cf. the discussion of mono-anabelian transport in §2.7, §2.9!]. In fact, this state of affairs is both natural and indeed somewhat inevitable for a number of reasons, which we pause to survey in the discussion to follow [cf. also Fig. 4.1 below].

(i) Strong functoriality properties and the central role of cyclotomic rigidity in Kummer theory: Perhaps the most conspicuous difference between class field theory and Kummer theory is the fact that, whereas

- class field theory may only be formulated for a certain special class of arithmetic fields [a class which in fact includes the function fields of all the integral schemes that appear in inter-universal Teichmüller theory], e.g., for global fields [i.e., fields that are finitely generated over an NF or a finite field] or certain types of completions or localizations of such global fields,
- Kummer theory, by contrast, may be formulated, by using the Kummer exact sequence in étale cohomology [cf., e.g., the discussion at the beginning of [Cusp], §2], for [essentially] arbitrary types of schemes and even for abstract monoids [cf. [FrdII], Definition 2.1, (ii)] that satisfy relatively weak conditions and do not necessarily arise from the multiplicative structure of a commutative ring.

A closely related difference between class field theory and Kummer theory is the fact that, whereas

- class field theory only satisfies very limited functoriality properties, i.e., for finite separable field extensions and certain types of localization operations associated to a valuation,
- Kummer theory satisfies very strong functoriality properties, for [essentially] arbitrary morphisms between [essentially] arbitrary schemes or between abstract monoids that satisfy suitable, relatively weak conditions.

These properties of Kummer theory make

Kummer theory much more suitable for use in anabelian geometry, where it is natural to consider morphisms between arithmetic fundamental groups that correspond to quite general morphisms between quite general schemes, i.e., where by “quite general”, we mean by comparison to the restrictions that arise if one attempts to apply class field theory.

Perhaps the most fundamental example of this sort of situation [i.e., that is of interest
in anabelian geometry, but to which class field theory cannot, at least in any immediate way, be applied] is the situation that arises if one considers the operation of

**evaluation of various types of functions** on, say, a curve, at a closed **point**

of the curve

[cf. the discussion of (ii) below; Example 2.13.1, (iv); §2.14; §3.6; [IUTchIV], Remark 2.3.3, (vi)]. On the other hand, one highly nontrivial and quite delicate aspect of Kummer theory that does not appear in class field theory is the issue of

establishing **cyclo
tomic rigidity isomorphisms** between cyclotomes constructed from the various rings, monoids, Galois groups, or arithmetic fundamental groups that appear in a particular situation.

Various **examples** of such isomorphisms between cyclotomes may be seen in the theory discussed in [PrfGC], [the discussion preceding] Lemma 9.1; [AbsAnab], Lemma 2.5; [Cusp], Proposition 1.2, (ii); [FrdII], Theorem 2.4, (ii); [EtTh], Corollary 2.19, (i); [AbsTopIII], Corollary 1.10, (ii); [Cusp], Remarks 3.2.1, 3.2.2 [cf. also Example 2.12.1, (ii); Example 2.13.1, (ii); §3.4, (ii), (iii), (iv), (v), of the present paper]. All of these examples concern “Kummer-faithful” situations [cf. [AbsTopIII], Definition 1.5], i.e., situations in which

the **Kummer map** on the multiplicative monoid [e.g., which arises from the multiplicative structure of a ring] of interest is injective.

Then the cyclotomic rigidity issues that arise typically involve the cyclotomes obtained by considering the **torsion subgroups** of such multiplicative monoids. This sort of situation **contrasts sharply** with the sort of highly “non-Kummer-faithful” situation considered in [PopBog], i.e., where one works with function fields over algebraic closures of finite fields [cf. [PopBog], Theorem I]. That is to say, in the sort of situation considered in [PopBog], the Kummer map vanishes on the roots of unity of the base field, and the Kummer theory that is applied [cf. [PopBog], §5.2] does not revolve around the issue of establishing cyclotomic rigidity isomorphisms. In particular, in the context of this sort of application of Kummer theory, it is natural to think of the image of the Kummer map as a sort of projective space, i.e., a quotient by the action of multiplication by nonzero elements of the base field. Thus, in summary, the relationship just discussed between “Kummer-faithful Kummer theory” and “non-Kummer-faithful Kummer theory” may be thought of as the difference between

“injective Kummer theory” and “projective Kummer theory”.

(ii) **The functoriality of Kummer theory with respect to evaluation of special functions at torsion points:** As mentioned in (i), the operation of evaluation of various types of [special] functions on, say, a curve, at various types of
of the curve plays a fundamental role in the Kummer theory that is applied in inter-universal Teichmüller theory [cf. the discussion of Example 2.13.1, (iv); §2.14; §3.6; [IUTchIV], Remark 2.3.3, (vi)]. This contrasts sharply with the fact that class field theory may only be related to the operation of evaluation of special functions at special points in very restricted classical cases, namely, the theory of exponential functions in the case of \( \mathbb{Q} \) or modular and elliptic functions in the case of imaginary quadratic fields.

Indeed, the goal of generalizing the theory that exists in these very restricted cases to the case of arbitrary NF’s is precisely the content of Kronecker’s Jugendtraum, or Hilbert’s twelfth problem [cf. the discussion of [IUTchIV], Remark 2.3.3, (vii)]. Indeed, in light of this state of affairs, one is tempted to regard inter-universal Teichmüller theory as a sort of “realization/solution” of the “version” of Kronecker’s Jugendtraum, or Hilbert’s twelfth problem, that one obtains if one replaces class field theory by Kummer theory!

(iii) The arithmetic holomorphicity of global class field theory versus the mono-analyticity of Kummer theory: Another important aspect of the fundamental differences between class field theory and Kummer theory that were highlighted in the discussion of (i) is the following:

- whereas the essential content of class field theory reflects various delicate arithmetic properties that are closed related to the arithmetic holomorphic structure of the very restricted types of arithmetic fields to which it may be applied,
- the very general and strongly functorial nature of Kummer theory makes Kummer theory more suited to treating the sorts of mono-analytic structures that arise in inter-universal Teichmüller theory [cf. the discussion of §2.7, (vii)]. Indeed, at a very naive level, this phenomenon may be seen in the difference between the “input data” for class field theory and Kummer theory, i.e., very restricted arithmetic fields in the case of class field theory versus very general types of abstract multiplicative monoids in the case of Kummer theory [cf. the discussion of (i)]. Another important instance of this phenomenon may be seen in the fact that whereas

- the global reciprocity law, which plays a central role in class field theory for NF’s, involves a nontrivial “intertwining” relationship, for any prime number \( l \), between the local unit determined by \( l \) at nonarchimedean valuations


<table>
<thead>
<tr>
<th><strong>Class field theory</strong></th>
<th><strong>Kummer theory</strong></th>
</tr>
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<tbody>
<tr>
<td>may be formulated only for special arithmetic fields</td>
<td>may be formulated for very general abstract multiplicative monoids</td>
</tr>
<tr>
<td>satisfies only very limited functoriality properties</td>
<td>satisfies very strong functoriality properties</td>
</tr>
<tr>
<td>limited range of applicability to anabelian geometry</td>
<td>wide range of applicability to anabelian geometry</td>
</tr>
<tr>
<td>cyclotomic rigidity isomorphisms are irrelevant</td>
<td>cyclotomic rigidity isomorphisms play a central role</td>
</tr>
<tr>
<td>no known compatibility with evaluation of functions at points</td>
<td>compatible with evaluation of functions at points</td>
</tr>
<tr>
<td>closely related to the arithmetic holomorphic structure of very restricted types of arithmetic fields</td>
<td>applicable to mono-analytic structures such as abstract multiplicative monoids</td>
</tr>
<tr>
<td>incompatible with local unit group/value group decouplings</td>
<td>compatible with local unit group/value group decouplings</td>
</tr>
<tr>
<td>related to global Dirichlet density of primes</td>
<td>naturally applied in conjunction with Prime Number Theorem</td>
</tr>
<tr>
<td>verification, involving cyclotomic extensions, of the global reciprocity law</td>
<td>global cyclotomic rigidity algorithms via ( \mathbb{Q}_{&gt;0} \cap \hat{\mathbb{Z}}^x = {1} )</td>
</tr>
</tbody>
</table>

Fig. 4.1: Comparison between class field theory and Kummer theory
of residue characteristic \( \neq l \) and the nonzero element of the value group determined by \( l \) at nonarchimedean valuations of residue characteristic \( l \),

- the compatibility of the Kummer theory applied in inter-universal Teichmüller theory with various “splittings”/“decouplings” between the local unit group and value group portions of this Kummer theory plays a central role in inter-universal Teichmüller theory

— cf. the discussion of §3.4; §3.8, \((8^{gaau})\); [IUTchIV], Remark 2.3.3, (v). A closely related fact is the fact that such local unit group/value group splittings are incompatible with [the multiplicative version of] Hilbert’s Theorem 90, which plays a central role in class field theory: that is to say, in the notation of Examples 2.12.1, 2.12.2, one verifies immediately that whereas \( H^1(G_k, \bar{k}^\times) = 0 \),

\[
H^1(G_k, \mu^\infty_k) \neq 0, \quad H^1(G_k, \mathcal{O}_k^\times) \neq 0, \quad H^1(G_k, \mu_k^\infty \cdot \pi^Q_k) \neq 0
\]

— where we write \( \mu^\infty_k \cdot \pi^Q_k \subseteq \bar{k}^\times \) for the subgroup of elements for which some positive power \( \in \pi^Q_k \). Finally, we recall from the discussion of [IUTchIV], Remark 2.3.3, (i), (ii), (iv), that a sort of analytic number theory version of this phenomenon may be seen in the fact that whereas

- class field theory is closely related — especially if one takes the point of view of early approaches to class field theory such as the approach attributed to Weber — to the “coherent aggregations” of primes that appear in discussions of the Dirichlet density of primes, e.g., in the context of the Tchebotarev density theorem,

- the Kummer-theoretic approach of inter-universal Teichmüller theory gives rise to the multiradial representation discussed in §3.7, (i), which leads to log-volume estimates [cf. the discussion of §3.7, (ii), (iv); the application of [IUTchIV], Proposition 1.6, and [IUTchIV], Proposition 2.1, (ii), in the explicit calculations of [IUTchIV], §1, §2] that involve, in an essential way, the Prime Number Theorem, i.e., which, so to speak, counts primes “one by one”, in effect “deactivating the coherent aggregations of primes” that appear in discussions of the Dirichlet density of primes.

(iv) Global reciprocity law versus global cyclotomic rigidity: Finally, we recall from the discussion of [“(b-4)” in] [IUTchIV], Remark 2.3.3, (i), (ii), that

- the use of cyclotomic extensions in classical approaches to verifying the global reciprocity law in class field theory for NF’s, i.e., to verifying that, in effect, the reciprocity map vanishes on idèles that arise from elements of the NF under consideration,
may be thought of as corresponding to

• the approach taken in inter-universal Teichmüller theory to constructing cyclotomic rigidity isomorphisms for the Kummer theory related to NF’s [cf. §3.4, (ii), (v)], i.e., in effect, by applying the elementary fact that \( \mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\} \) [cf. the discussion of the latter portion of [IUTchIII], Remark 3.12.1, (iii)].

Indeed, both of these phenomena concern the fact that some version of the product formula — that is to say, which, \textit{a priori} [or from a more naive, elementary point of view], is only known to hold for the Frobenius-like multiplicative monoids that arise from NF’s — in fact holds [i.e., in the form of the global reciprocity law or the elementary fact that \( \mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\} \)] at the level of \( \text{étale-like} \) profinite Galois groups.

§ 4.3. Arithmetic and geometric versions of the Mordell Conjecture

(i) Rough qualitative connections with Faltings’ proof of the Mordell Conjecture: First, we begin by observing [cf. [IUTchIV], Remark 2.3.3, (i), (ii), for more details] that there are numerous rough, qualitative correspondences — some of which are closely related to the topics discussed in §4.1 and §4.2 — between various components of the proof of the Mordell Conjecture given in [Falt1] and inter-universal Teichmüller theory:

\( (1^\text{st}) \) Various well-known aspects of classical algebraic number theory related to the “geometry of numbers”, such as the theory of heights and the Hermite-Minkowski theorem, are applied in [Falt1]. Similar aspects of classical algebraic number theory may be seen in the “non-interference” property [i.e., the fact that the only nonzero elements of an NF that are integral at all nonarchimedean and archimedean valuations of the NF are the roots of unity] for copies of the number field \( F_{\text{mod}} \) discussed in §3.7, (i), as well as in the use of global realified Frobenioids associated to NF’s [i.e., which are essentially an abstract category-theoretic version of the classical notions of arithmetic degrees and heights].

\( (2^\text{nd}) \) Global class field theory for NF’s, as well as the closely related notion of Dirichlet density of primes, plays an important role in [Falt1]. These aspects of [Falt1] are compared and contrasted in substantial detail in the discussion of §4.2 with the Kummer theory that plays a central role in inter-universal Teichmüller theory.

\( (3^\text{rd}) \) The theory of Hodge-Tate decompositions of \( p \)-adic Tate modules of abelian varieties over MLF’s plays an important role both in [Falt1] and, as discussed in §4.1, (ii), (4\text{cls}), in inter-universal Teichmüller theory.
(4th) The computations, applied in [Falt1], of the ramification that occurs in the theory surrounding finite flat group schemes bear a rough resemblance to the ramification computations involving log-shells in [AbsTopIII]; [IUTchIV], Propositions 1.1, 1.2, 1.3, 1.4.

(5th) The hidden endomorphisms [cf. the discussion of [AbsTopII], Introduction] that underlie the theory of Belyi and elliptic cuspidalizations [cf. the discussion of §3.3, (vi); §3.4, (iii)], which play an important role in inter-universal Teichmüller theory, as well as the theory of noncritical Belyi maps that is applied [cf. the discussion of §3.7, (iv)], via [GenEll], §2, in [IUTchIV], §2, may be thought of as a sort of analogue for hyperbolic curves of the theory of isogenies and Tate modules of abelian varieties that plays a central role in [Falt1].

(6th) The important role played by polarizations of abelian varieties in [Falt1] may be compared to the quite central role played by commutators of theta groups in the theory of rigidity properties of mono-theta environments, and hence in inter-universal Teichmüller theory as a whole [cf. the discussion of §3.4, (iv); §3.8, (9^{\text{gau}}); §4.1, (ii), (5^{\text{cls}})].

(7th) The logarithmic geometry of toroidal compactifications, which plays an important role in [Falt1], may be compared to the logarithmic geometry of special fibers of stable curves. The latter instance of logarithmic geometry is the starting point for the combinatorial anabelian geometry of tempered fundamental groups developed in [Semi], which plays an important role throughout inter-universal Teichmüller theory.

(ii) Arithmetic holomorphicity versus mono-analyticity/multiradiality:
One way to summarize the discussion of (i), as well as a substantial portion of the discussion of §4.2 [cf. [IUTchIV], Remark 2.3.3, (iii)], is as follows:

inter-universal Teichmüller theory may be understood, to a substantial extent, as a sort of hyperbolic, mono-analytic/multiradial analogue of the abelian, arithmetic holomorphic theory of [Falt1].

Indeed, this is precisely the point of view of the discussion of §4.2, (iii), concerning the relationship between the essentially arithmetic holomorphic nature of global class field theory and the essentially mono-analytic nature of Kummer theory. If one takes the point of view [cf. the discussion of §2.3, §2.4, §2.5, §2.6; Examples 2.14.2, 2.14.3] that Galois or arithmetic fundamental groups should be thought of as “arithmetic tangent bundles”, then the point of view of the present discussion may be formulated in the following way [cf. the discussion of the final portion of [IUTchI], §12]: Many results in
the conventional framework of arithmetic geometry that concern Galois or arithmetic fundamental groups may be understood as results to the effect that some sort of

\[ H^0(\text{arithmetic tangent bundle}) \]

does indeed coincide with some sort of very small collection of scheme-theoretic — i.e., arithmetic holomorphic — auto-/endo-morphisms. Indeed, examples of this sort of phenomenon include

(1\textsuperscript{hol}) the version of the Tate Conjecture proven in [Falt1];

(2\textsuperscript{hol}) various bi-anabelian results [cf. the discussion of §2.7, (v)] — i.e., fully faithfulness results in the style of various versions of the “Grothendieck Conjecture” — in anabelian geometry;

(3\textsuperscript{hol}) the “tiny” special case of the theory of [Falt1] discussed in §2.3 to the effect that “Frobenius endomorphisms of NF’s” of the desired type [i.e., that yield bounds on heights — cf. the discussion of §2.4!] cannot exist, i.e., so long as one restricts oneself to working within the framework of conventional scheme theory;

(4\textsuperscript{hol}) the results of [Wiles] concerning Galois representations [cf. the discussion of [IUTchI], §I5], which may be summarized as asserting, in essence, that, roughly speaking, nontrivial deformations of Galois representations that satisfy suitable natural conditions do not exist.

All of the results just stated assert some sort of

“arithmetic holomorphic nonexistence”

[up to a very small number of exceptions], hence lie in a fundamentally different direction from the content of inter-universal Teichmüller theory, which, in effect, concerns the construction — or

“non-arithmetic holomorphic existence”

— of a “Frobenius endomorphism of an NF”, by working outside the framework of conventional scheme theory, i.e., by considering suitable mono-analytic/multiradial deformations of the arithmetic holomorphic structure, that is to say, at the level of suggestive notation, by considering

\[ H^1(\text{arithmetic tangent bundle}) \].

(iii) Comparison with the metric proofs of Parshin and Bogomolov in the complex case: Parshin [cf. [Par]] and Bogomolov [cf. [ABKP], [Zh]] have given proofs of geometric versions over the complex numbers of the Mordell and Szpiro Conjectures, respectively [cf. the discussion of [IUTchIV], Remarks 2.3.4, 2.3.5]. Parshin’s proof of the geometric version of the Mordell Conjecture is discussed
in detail in [IUTchIV], Remark 2.3.5, while Bogomolov’s proof of the geometric version of the Szpiro Conjecture is discussed in detail in [BogIUT] [cf. also §3.10, (vi), of the present paper]. The relationships of these two proofs to one another, as well as to the arithmetic theory, may be summarized as follows:

(1\textsuperscript{PB}) Both proofs revolve around the consideration of metric estimates of the displacements that arise from various natural actions of elements of the [usual topological] fundamental groups that appear [cf. the discussion at the beginning of [IUTchIV], Remark 2.3.5].

(2\textsuperscript{PB}) Both Parshin’s and Bogomolov’s proofs concern the metric geometry of the complex spaces that appear. On the other hand, these two proofs differ fundamentally in that whereas the metric geometry that appears in Parshin’s proof concerns the holomorphic geometry that arises from the Kobayashi distance — i.e., in effect, the Schwarz lemma of elementary complex analysis — the metric geometry that appears in Bogomolov’s proof concerns the real analytic hyperbolic geometry of the upper half-plane [cf. [IUTchIV], Remark 2.3.5, (PB1)].

(3\textsuperscript{PB}) The difference observed in (2\textsuperscript{PB}) is interesting in that it corresponds precisely to the difference discussed in (i) above between the proof of the arithmetic Mordell Conjecture [for NF’s!] in [Falt1] and inter-universal Teichmüller theory [cf. the discussion of [IUTchIV], Remark 2.3.5, (i)].

(4\textsuperscript{PB}) Parshin’s proof concerns, as one might expect from the statement of the Mordell Conjecture, rough, qualitative estimates. This state of affairs contrasts sharply, again as one might expect from the statement of the Szpiro Conjecture, with Bogomolov’s proof, which concerns effective, quantitative estimates [cf. the discussion of [IUTchIV], Remark 2.3.5, (ii)].

(5\textsuperscript{PB}) The appearance of the Kobayashi distance — i.e., in essence, the Schwarz lemma of elementary complex analysis — in (2\textsuperscript{PB}) is of interest in light of the point of view discussed in §3.3, (vi); §3.7, (iv), concerning the correspondence between the use of Belyi maps in inter-universal Teichmüller theory, i.e., in the context of Belyi cuspidalizations or height estimates, as a sort of means of arithmetic analytic continuation, and the classical complex theory surrounding the Schwarz lemma [cf. the discussion of [IUTchIV], Remark 2.3.5, (iii)].

§ 4.4. Atavistic resemblance in the development of mathematics

(i) Questioning strictly linear models of evolution: Progress in mathematics is often portrayed as a strictly linear affair — a process in which old theories or
ideas are rendered essentially obsolete, and hence forgotten, as soon as the essential content of those theories or ideas is “suitably extracted/absorbed” and formulated in a more modern form, which then becomes known as the state of the art. The historical development of mathematics is then envisioned as a sort of towering edifice that is subject to a perpetual appending of higher and higher floors, as new states of the art are discovered. On the other hand, it is often overlooked that there is in fact no intrinsic justification for this sort of strictly linear model of evolution. Put another way, there is

no rigorous justification for excluding the possibility that a particular approach to mathematical research that happens to be embraced without doubt by a particular community of mathematicians as the path forward in this sort of strictly linear evolutionary model may in fact be nothing more than a dramatic “wrong turn”, i.e., a sort of unproductive march into a meaningless cul de sac.

Indeed, Grothendieck’s original idea that anabelian geometry could shed light on diophantine geometry [cf. the discussion at the beginning of [IUTchI], §15] suggests precisely this sort of skepticism concerning the linear evolutionary model that arose in the 1960’s to the effect that progress in arithmetic geometry was best understood as a sort of

strictly linear march toward the goal of realizing the theory of motives, i.e., a sort of idealized version of the notion of a Weil cohomology.

In more recent years, another major “linear evolutionary model” that has arisen, partly as a result of the influence of the work of Wiles [cf. [Wiles]] concerning Galois representations, asserts that progress in arithmetic geometry is best understood as a sort of

strictly linear march toward the goal of realizing the representation-theoretic approach to arithmetic geometry constituted by the Langlands program.

As discussed in §4.3, (ii); [IUTchI], §15,

(a^{app}) the “mono-anabelian” approach to arithmetic geometry constituted by inter-universal Teichmüller theory

diffs fundamentally not only from

(b^{app}) the motive-/cohomology-theoretic and representation-theoretic approaches to arithmetic geometry just discussed — both of which may be characterized as “abelian”!
but also from

\((c^{\text{app}})\) the “bi-anabelian” approach involving the \textit{section conjecture} that was apparently originally envisioned by Grothendieck.

Here, we recall, moreover, the point of view of the \textit{dichotomy} discussed in §4.3, (ii), concerning \textbf{arithmetic holomorphicity} and \textbf{mono-analyticity/multiradiality}, i.e., to the effect that the \textit{difference} between \((a^{\text{app}})\), on the one hand, and \textit{both} \((b^{\text{app}})\) and \((c^{\text{app}})\), on the other, may be understood [if, for the sake of brevity, one applies the term “holomorphic” as an abbreviation of the term “arithmetically holomorphic”] as the difference between

\textbf{non-holomorphic existence} and \textbf{holomorphic nonexistence}.

Another way to understand, at a \textit{very rough level}, the \textit{difference} between \((a^{\text{app}})\) and \((b^{\text{app}})\) is as a reflection of the \textit{deep structural differences} between

[discrete or profinite] \textbf{free groups} and \textbf{matrix groups} [with discrete or profinite coefficients]

— cf. the discussion of [IUTchI], §I5. Finally, at a \textit{much more elementary level}, we note that the theory of \textbf{Galois groups} — which may be thought of as a mechanism that allows one to pass from \textit{field theory} to \textit{group theory} — plays a \textbf{fundamental role} in \((a^{\text{app}})\), \((b^{\text{app}})\), and \((c^{\text{app}})\). From this point of view, the difference between \((a^{\text{app}})\), on the one hand, and \((b^{\text{app}})\) [and, to a slightly lesser extent, \((c^{\text{app}})\)], on the other, may be understood as the difference between the \textbf{“inequalities”}

\textbf{group theory} \(\gg\) \textbf{field theory} and

\textbf{field theory} \(\gg\) \textbf{group theory}

— i.e., the issue of whether one regards [abstract] \textit{group theory} as the \textbf{central object of interest}, while \textit{field theory} [which we understand as including \textit{vector spaces} over fields, hence also \textit{representation theory}] is relegated to playing only a \textit{subordinate role}, or \textit{vice versa}. In this context, it is perhaps of interest to note that \textit{common central features} that appear in both \textit{inter-universal Teichmüller theory} and the \textit{work of Wiles} [cf. [Wiles]] concerning \textit{Galois representations} — i.e., in both \((a^{\text{app}})\) and \((b^{\text{app}})\) — include not only

\begin{itemize}
\item the central use of \textbf{Galois groups} [as discussed above], but also
\item the central use of \textbf{function theory on the upper half-plane}, i.e., \textbf{theta functions} in the case of inter-universal Teichmüller theory and \textbf{modular forms} in the case of [Wiles].
\end{itemize}
On the other hand, just as in the case of Galois groups discussed above, the approaches taken in inter-universal Teichmüller theory and [Wiles] to using *function theory on the upper half-plane* — i.e., *theta functions* versus *modular forms* — differ quite substantially.

(ii) **Examples of atavistic development:** An alternative point of view to the sort of *strictly linear evolutionary model* discussed in (i) is the point of view that progress in mathematics is best understood as a much more complicated *family tree*, i.e., not as a tree that consists solely of a *single trunk without branches* that continues to grow upward in a strictly linear manner, but rather as

a *much more complicated organism*, whose growth is sustained by an *intricate mechanism of interaction* among a vast *multitude of branches*, some of which sprout *not from branches of relatively recent vintage*, but rather from *much older, more ancestral branches* of the organism that were entirely irrelevant to the recent growth of the organism.

In the context of the present paper, it is of interest to note that this point of view, i.e., of

*substantially different multiple evolutionary branches* that sprout from a *single common ancestral branch*, is reminiscent of the notion of “*mutually alien copies*”, which forms a *central theme* of the present paper [cf. the discussion of §2.7, (i), (ii); §3.8].

Phenomena that support this point of view of an *“atavistic model of mathematical development”* may be seen in many of the examples discussed in §§4.1, §4.2, and §4.3 such as the following:

(1$^{\text{atv}}$) The very elementary construction of *Belyi maps* in the early 1980’s, or indeed *noncritical Belyi maps* in [NCBelyi], could easily have been discovered in the late nineteenth century [cf. §4.3, (i), (5$^{\text{flt}}$); §4.3, (iii), (5$^{\text{PB}}$)].

(2$^{\text{atv}}$) The application of Belyi maps to *Belyi cuspidalization* [cf. [AbsTopII], §3] could easily have been discovered in the mid-1990’s [cf. also (1$^{\text{atv}}$)].

(3$^{\text{atv}}$) The application of noncritical Belyi maps to *height estimates* in [GenEll], §2, could easily have been discovered in the mid-1980’s [cf. also (1$^{\text{atv}}$)].

(4$^{\text{atv}}$) The *Galois-theoretic* interpretation of the *Gaussian integral* or *Jacobi’s identity* furnished by inter-universal Teichmüller theory [cf. the discussion of §3.8; the discussion at the end of §3.9, (iii); the discussion of the final portion of §4.1, (i)] could easily have been discovered much earlier than in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV].
The interpretation of changes of universe in the context of non-ring-theoretic "arithmetic changes of coordinates" as in the discussion of §2.10 is entirely elementary and could easily have been discovered in the 1960's [cf. §4.1, (ii), (1\text{cls}), (2\text{cls})].

The use of Hodge-Tate representations as in [Q_pGC] or [AbsTopI], §3, could easily have been discovered in the 1960's [cf. §4.1, (ii), (3\text{cls})].

The use of Hodge-Tate decompositions as in [pGC] could easily have been discovered in the 1980's [cf. §4.1, (ii), (4\text{cls})].

The anabelian approach to theta functions on Tate curves taken in [EtTh] [cf. §3.4, (iii), (iv)] could easily have been discovered in the mid-1990's [cf. §4.1, (ii), (5\text{cls})].

The non-representation-theoretic use of the structure of theta groups in the theory of [EtTh] [cf. the discussion at the end of §3.4, (iv)] could easily have been discovered in the 1980's [cf. §4.1, (ii), (5\text{cls}), (6\text{cls})].

Scheme-theoretic Hodge-Arakelov theory, which may be regarded as a natural extension of the [the portion concerning elliptic curves of] Mumford's theory of algebraic theta functions, could easily have been discovered in the late 1960's [cf. §4.1, (ii), (6\text{cls})].

The technique of mono-anabelian transport in the context of positive characteristic anabelian geometry, i.e., in the style of Example 2.6.1, could easily have been discovered in the 1980's [cf. §4.1, (ii), (7\text{cls})].

The use of monoids as in the theory of Frobenioids could easily have been discovered in the mid-1990's [cf. §4.1, (ii), (8\text{cls})].

The notion of multiradiality is entirely elementary and could easily have been discovered in the late 1960's [cf. §4.1, (ii), (9\text{cls})].

The point of view of taking a Kummer-theoretic approach to Kronecker's Jugendtraum, i.e., as discussed in §4.2, (ii), could easily have been discovered much earlier than in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV].

In this context, we note that the atavistic model of mathematical development just discussed also suggests the possibility that the theory of Frobenioids — which, as was discussed in §3.5, was originally developed for reasons that were [related to, but, strictly speaking] independent of inter-universal Teichmüller theory, and is, in fact, only used in inter-universal Teichmüller theory in a relatively weak sense — may give rise, at some distant future date, to further developments of interest that are not directly related to inter-universal Teichmüller theory.
(iii) **Escaping from the cage of deterministic models of mathematical development:** The adoption of **strictly linear evolutionary models** of progress in mathematics of the sort discussed in (i) tends to be **highly attractive** to many mathematicians in light of the **intoxicating simplicity** of such strictly linear evolutionary models, by comparison to the **more complicated point of view** discussed in (ii). This **intoxicating simplicity** also makes such strictly linear evolutionary models — together with **strictly linear numerical evaluation devices** such as the “number of papers published”, the “number of citations of published papers”, or other like-minded **narrowly defined data formats** that have been concocted for **measuring progress in mathematics** — **highly enticing** to administrators who are charged with the tasks of **evaluating, hiring, or promoting** mathematicians. Moreover, this state of affairs that regulates the collection of individuals who are granted the license and resources necessary to actively engage in mathematical research tends to have the effect, over the long term, of **stifling efforts** by young researchers to conduct **long-term mathematical research** in directions that **substantially diverge** from the **strictly linear evolutionary models** that have been adopted, thus making it exceedingly difficult for **new “unanticipated” evolutionary branches** in the development of mathematics to sprout. Put another way,

**inappropriately narrowly defined strictly linear evolutionary models** of progress in mathematics exhibit a **strong and unfortunate** tendency in the profession of mathematics as it is currently practiced to become something of a **self-fulfilling prophecy** — a “prophecy” that is often zealously rationalized by dubious bouts of **circular reasoning**.

In particular, the issue of

**escaping from the cage** of such **narrowly defined deterministic models** of mathematical development stands out as an issue of **crucial strategic importance** from the point of view of charting a **sound, sustainable course** in the future development of the field of mathematics, i.e., a course that **cherishes the privilege to foster genuinely novel and unforeseen evolutionary branches** in its development.

**Bibliography**


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