THE MATHEMATICS OF MUTUALLY ALIEN COPIES: FROM GAUSSIAN INTEGRALS TO INTER-UNIVERSAL TEICHMÜLLER THEORY

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February 2018

Abstract. **Inter-universal Teichmüller theory** may be described as a construction of certain

canonical deformations of the **ring structure** of a **number field**

equipped with certain auxiliary data, which includes an **elliptic curve** over the number field and a **prime number** ≥ 5. In the present paper, we survey this theory by focusing on the *rich analogies* between this theory and the classical computation of the **Gaussian integral**. The main **common features** that underlie these analogies may be summarized as follows:

- the introduction of **two mutually alien copies** of the object of interest;
- the computation of the effect — i.e., on the two mutually alien copies of the object of interest — of **two-dimensional changes of coordinates** by considering the effect on **infinitesimals**;
- the passage from **planar cartesian** to **polar coordinates** and the resulting **splitting**, or **decoupling**, into **radial** — i.e., in more abstract valuation-theoretic terminology, **"value group"** — and **angular** — i.e., in more abstract valuation-theoretic terminology, **"unit group"** — portions;
- the straightforward evaluation of the **radial portion** by applying the **quadraticity** of the exponent of the Gaussian distribution;
- the straightforward evaluation of the **angular portion** by considering the *metric geometry* of the *group of units* determined by a suitable version of the natural **logarithm** function.

[Here, the intended sense of the descriptive **"alien"** is that of its original Latin root, i.e., a sense of **abstract, tautological "otherness"**.] After reviewing the classical computation of the Gaussian integral, we give a detailed survey of inter-universal Teichmüller theory by concentrating on the *common features* listed above. The paper concludes with a discussion of various **historical aspects** of the mathematics that appears in inter-universal Teichmüller theory.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Introduction

In the present paper, we survey **inter-universal Teichmüller theory** by focusing on the *rich analogies* [cf. §3.8] between this theory and the classical computation of the **Gaussian integral**. Inter-universal Teichmüller theory concerns the construction of

canonical deformations of the **ring structure** of a **number field**

equipped with certain auxiliary data. The collection of data, i.e., consisting of the number field equipped with certain auxiliary data, to which inter-universal Teichmüller theory is applied is referred to as **initial Θ-data** [cf. §3.3, (i), for more details]. The principal components of a collection of initial Θ-data are

· the given **number field**,
· an **elliptic curve** over the number field, and
· a **prime number** $l \geq 5$.

The **main applications** of inter-universal Teichmüller theory to **diophantine geometry** [cf. §3.7, (iv), for more details] are obtained by applying the *canonical deformation* constructed for a specific collection of *initial* Θ*-data* to **bound** the **height** of the elliptic curve that appears in the initial Θ-data.

Let N be a *fixed natural number* > 1. Then the issue of *bounding a given nonnegative real number* $h \in \mathbb{R}_{\geq 0}$ may be understood as the issue of showing that $N \cdot h$ is **roughly equal** to h, i.e.,

$$N \cdot h \quad \text{``}\approx\text{''} \quad h$$

[cf. §2.3, §2.4]. When h is the **height** of an *elliptic curve* over a number field, this issue may be understood as the issue of showing that the *height* of the [in fact, in most cases, *fictional*!] *"elliptic curve"* whose *q-parameters* are the ***N*-th powers** "q^N" of the ***q*-parameters** "q" of the given elliptic curve is **roughly equal** to the height of the given elliptic curve, i.e., that, at least from point of view of [global] *heights*,

$$q^N \quad \text{``}\approx\text{''} \quad q$$

[cf. §2.3, §2.4].

In order to verify the *approximate relation* q^N "$\approx$" q, one begins by introducing **two distinct** — i.e., two **"mutually alien"** — **copies** of the **conventional scheme theory** surrounding the given initial Θ-data. Here, the intended sense of the descriptive *"alien"* is that of its original Latin root, i.e., a sense of

abstract, tautological "otherness".

These two mutually alien copies of conventional scheme theory are **glued together** — by considering relatively weak underlying structures of the respective conventional scheme theories such as **multiplicative monoids** and **profinite groups** — in such a way that the "q^N" in *one copy of scheme theory* is **identified** with the "q" in the *other copy of scheme theory*. This gluing is referred to as the **Θ-link**. Thus, the "q^N" on the *left-hand side* of the Θ-link is *glued* to the "q" on the *right-hand side* of the Θ-link, i.e.,

$$q_{\mathrm{LHS}}^N \quad \text{``}=\text{''} \quad q_{\mathrm{RHS}}$$

[cf. §3.3, (vii), for more details]. Here, "N" is in fact taken *not* to be a *fixed natural number*, but rather a sort of **symmetrized average** over the values j^2, where $j = 1, \ldots, l^*$, and we write $l^* \stackrel{\text{def}}{=} (l-1)/2$. Thus, the *left-hand side* of the above display

$$\{q_{\mathrm{LHS}}^{j^2}\}_j$$

bears a *striking formal resemblance* to the **Gaussian distribution**. One then verifies the desired *approximate relation* q^N "$\approx$" q by **computing**

$$\{q_{\mathrm{LHS}}^{j^2}\}_j$$

— **not** in terms of q_{LHS} [which is immediate from the definitions!], but rather — **in terms of** [the scheme theory surrounding]

$$q_{\mathrm{RHS}}$$

[which is a *highly nontrivial* matter!]. The *conclusion* of this *computation* may be summarized as follows:

> up to **relatively mild indeterminacies** — i.e., *"relatively small error terms"* — $\{q_{\mathrm{LHS}}^{j^2}\}_j$ may be **"confused"**, or **"identified"**, with $\{q_{\mathrm{RHS}}^{j^2}\}_j$, that is to say,
>
> $$\begin{array}{ccc} \{q_{\mathrm{LHS}}^{j^2}\}_j & \overset{!!}{\leftrightarrows} & \{q_{\mathrm{RHS}}^{j^2}\}_j \\ (\text{“}=\text{”} \quad q_{\mathrm{RHS}}) & & \end{array}$$

[cf. the discussion of §3.7, (i), especially, Fig. 3.19, for more details]. Once one is equipped with this **"license"** to *confuse/identify* $\{q_{\mathrm{LHS}}^{j^2}\}_j$ with $\{q_{\mathrm{RHS}}^{j^2}\}_j$, the derivation of the desired *approximate relation*

$$\{q^{j^2}\}_j \quad \text{“}\approx\text{”} \quad q$$

and hence of the desired **bounds** on **heights** is an *essentially formal matter* [cf. §3.7, (ii), (iv)].

The starting point of the exposition of the present paper lies in the *observation* [cf. §3.8 for more details] that the *main features* of the theory underlying the *computation* just discussed of $\{q_{\mathrm{LHS}}^{j^2}\}_j$ in terms of q_{RHS} exhibit **remarkable similarities** — as is perhaps foreshadowed by the *striking formal resemblance* observed above to the *Gaussian distribution* — to the **main features** of the classical computation of the **Gaussian integral**, namely,

(1^{mf}) the introduction of **two mutually alien copies** of the object of interest [cf. §3.8, (1^{gau}), (2^{gau})];

(2^{mf}) the computation of the effect — i.e., on the two mutually alien copies of the object of interest — of **two-dimensional changes of coordinates** by considering the effect on **infinitesimals** [cf. §3.8, (3^{gau}), (4^{gau}), (5^{gau}), (6^{gau})];

(3^{mf}) the passage from **planar cartesian** to **polar coordinates** and the resulting **splitting**, or **decoupling**, into **radial** — i.e., in more abstract valuation-theoretic terminology, **"value group"** — and **angular** — i.e., in more abstract valuation-theoretic terminology, **"unit group"** — portions [cf. §3.8, (7^{gau}), (8^{gau})];

(4^{mf}) the straightforward evaluation of the **radial portion** by applying the **quadraticity** of the exponent of the Gaussian distribution [cf. §3.8, (9^{gau}), (11^{gau})];

(5^{mf}) the straightforward evaluation of the **angular portion** by considering the *metric geometry* of the *group of units* determined by a suitable version of the natural **logarithm** function [cf. §3.8, (10^{gau}), (11^{gau})].

The present paper begins, in §1, with a **review** of the classical computation of the **Gaussian integral**, by breaking down this familiar computation into *steps* in such a way as to facilitate the subsequent *comparison* with inter-universal Teichmüller theory. We then proceed, in §2, to discuss the portion of inter-universal Teichmüller theory that corresponds to (2^{mf}). The exposition of §2 was designed so as to be accessible to readers familiar with well-known portions of **scheme theory** and the theory of the **étale fundamental group** — i.e., at the level of [Harts] and [SGA1]. The various Examples that appear in this exposition of §2 include numerous

well-defined and **relatively straightforward mathematical assertions**

often without complete proofs. In particular, the reader may think of the task of supplying a complete proof for any of these assertions as a sort of *"exercise"* and hence of §2 itself as a sort of

workbook with exercises.

At the level of papers, §2 is concerned mainly with the content of the *"classical"* paper [Uchi] of Uchida and the *"preparatory papers"* [FrdI], [FrdII], [GenEll], [AbsTopI], [AbsTopII], [AbsTopIII]. By contrast, the level of exposition of §3 is substantially **less elementary** than that of §2. In §3, we apply the **conceptual infrastructure** exposed in §2 to survey those aspects of inter-universal Teichmüller theory that correspond to (1^{mf}), (3^{mf}), (4^{mf}), and (5^{mf}), i.e., at the level of papers, to [EtTh], [IUTchI], [IUTchII], [IUTchIII], [IUTchIV]. Finally, in §4, we reflect on various **historical aspects** of the theory exposed in §2 and §3.

Acknowledgements:

The author wishes to express his appreciation for the stimulating comments that he has received from numerous mathematicians concerning the theory exposed in the present paper and, especially, his deep gratitude to *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, and *Ivan Fesenko* for the very active and devoted role that they played both in discussing this theory with the author and in disseminating it to others. In particular, the author would like to thank *Yuichiro Hoshi* for introducing the notion of *mono-anabelian transport* as a means of formulating a technique that is frequently applied throughout the theory. This notion plays a central role in the expository approach adopted in the present paper.

Section 1: Review of the computation of the Gaussian integral

§**1.1.** **<u>Inter-universal Teichmüller theory via the Gaussian integral</u>:** The goal of the present paper is to pave the road, for the reader, from a state of *complete ignorance* of inter-universal Teichmüller theory to a state of *general appreciation* of the **"game plan"** of **inter-universal Teichmüller theory** by reconsidering the well-known *computation* of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx \;=\; \sqrt{\pi}$$

via *polar coordinates* from the point of view of a hypothetical high-school student who has studied one-variable calculus and polar coordinates, but has not yet had any exposure to **multi-variable calculus**. That is to say, we shall begin in the present §1 by reviewing this computation of the Gaussian integral by discussing how this computation might be explained to such a hypothetical high-school student. In subsequent §'s, we then proceed to discuss how various **key steps** in such an explanation to a hypothetical high-school student may be translated into the more sophisticated language of **abstract arithmetic geometry** in such a way as to yield a **general outline** of inter-universal Teichmüller theory based on the *deep structural similarities* between inter-universal Teichmüller theory and the computation of the Gaussian integral.

§**1.2.** **<u>Naive approach via changes of coordinates or partial integrations</u>:** In one-variable calculus, definite integrals that appear intractable at first glance are often reduced to much simpler definite integrals by performing suitable *changes of coordinates* or *partial integrations*. Thus:

> **<u>Step 1</u>**: Our hypothetical high-school student might initially be tempted to perform a **change of coordinates**
>
> $$e^{-x^2} \rightsquigarrow u$$
>
> and then [erroneously!] compute
>
> $$\int_{-\infty}^{\infty} e^{-x^2}\ dx \ = \ 2 \cdot \int_{0}^{\infty} e^{-x^2}\ dx \ = \ -\int_{x=0}^{x=\infty} d(e^{-x^2}) \ = \ \int_{0}^{1} du \ = \ 1$$
>
> — only to realize shortly afterwards that this computation is in error, on account of the **erroneous treatment of the infinitesimal "dx"** when the change of coordinates was executed.
>
> **<u>Step 2</u>**: This realization might then lead the student to attempt to repair the computation of Step 1 by considering various iterated *partial integrations*
>
> $$\int_{-\infty}^{\infty} e^{-x^2}\ dx \ = \ -\int_{x=-\infty}^{x=\infty} \frac{1}{2x} d(e^{-x^2}) \ = \ \int_{x=-\infty}^{x=\infty} e^{-x^2} d\Big(\frac{1}{2x}\Big) \ = \ \dots$$
>
> — which, of course, *lead nowhere*.

§**1.3.** **<u>Introduction of identical but mutually alien copies</u>:** At this point, one might suggest to the hypothetical high-school student the idea of computing the Gaussian integral by first **squaring** the integral and then taking the **square root** of the value of the square of the integral. That is to say, in effect:

> **<u>Step 3</u>**: One might suggest to the hypothetical high-school student that the Gaussian integral can in fact be computed by considering the product of **two identical — but mutually independent! — copies** of the **Gaussian integral**
>
> $$\Big(\int_{-\infty}^{\infty} e^{-x^2}\ dx\Big) \cdot \Big(\int_{-\infty}^{\infty} e^{-y^2}\ dy\Big)$$
>
> — i.e., as opposed to a *single copy* of the Gaussian integral.

Here, let us recall that our hypothetical high-school student was already in a mental state of *extreme frustration* as a result of the student's *intensive and heroic attempts* in Step 2 which led only to an **endless labyrinth** of meaningless and increasingly complicated mathematical expressions. This experience left our hypothetical high-school student with the impression that the Gaussian integral was without question **by far the most difficult integral** that the student had ever encountered. In light of this experience, the suggestion of Step 3 evoked a reaction of intense **indignation** and **distrust** on the part of the student. That is to say,

> the idea that meaningful progress could be made in the computation of such an exceedingly difficult integral simply by considering *two identical copies* of the integral — i.e., as opposed to a *single copy* — struck the student as being utterly **ludicrous**.

Put another way, the suggestion of Step 3 was simply *not* the sort of suggestion that the student *wanted to hear*. Rather, the student was keenly interested in seeing some sort of **clever partial integration** or **change of coordinates** involving "sin(−)", "cos(−)", "tan(−)", "exp(−)", "$\frac{1}{1+x^2}$", etc., i.e., of the sort that the student was used to seeing in familiar expositions of **one-variable** calculus.

§**1.4.** **Integrals over two-dimensional Euclidean space:** Only after quite substantial efforts at persuasion did our hypothetical high-school student reluctantly agree to proceed to the next step of the explanation:

> **Step 4**: If one considers the **"totality"**, or **"total space"**, of the coordinates that appear in the product of two copies of the Gaussian integral of Step 3, then one can regard this product of integrals as a **single integral**
>
> $$\int_{\mathbb{R}^2} e^{-x^2} \cdot e^{-y^2} \, dx \, dy \;=\; \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy$$
>
> over the **Euclidean plane** $\mathbb{R}^2$.

Of course, our hypothetical high-school student might have some trouble with Step 4 since it requires one to assimilate the notion of an **integral** over a space, i.e., the *Euclidean plane* $\mathbb{R}^2$, which is *not an interval of the real line*. This, however, may be explained by reviewing the essential philosophy behind the notion of the **Riemann integral** — a philosophy which should be familiar from **one-variable calculus**:

> **Step 5**: One may think of integrals over more general spaces, i.e., such as the Euclidean plane $\mathbb{R}^2$, as computations
>
> $$\textit{net mass} \;=\; \lim \sum \; (\text{infinitesimals of } \textit{zero mass})$$
>
> of **"net mass"** by considering *limits of sums of* **infinitesimals**, i.e., such as "*dx dy*", which one may think of as having **"zero mass"**.

§**1.5.** **The effect on infinitesimals of changes of coordinates:** Just as in one-variable calculus, computations of integrals over more general spaces can often be *simplified* by performing suitable *changes of coordinates*. Any [say, continuously differentiable] change of coordinates results in a *new factor*, given by the **Jacobian**, in the integrand. This factor constituted by the Jacobian, i.e., the determinant of a certain *matrix of partial derivatives*, may appear to be somewhat *mysterious* to our hypothetical high-school student, who is only familiar with changes of coodinates in one-variable calculus. On the other hand, the appearance of the Jacobian may be justified in a *computational fashion* as follows:

Step 6: Let $U, V \subseteq \mathbb{R}^2$ be open subsets of $\mathbb{R}^2$ and

$$U \ni (s,t) \mapsto (x,y) = (f(s,t), g(s,t)) \in V$$

a *continuously differentiable* **change of coordinates** such that the **Jacobian**

$$J \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial s} & \frac{\partial g}{\partial t} \end{pmatrix}$$

— which may be thought of as a continuous real-valued function on U — is *nonzero* throughout U. Then for any continuous real-valued functions $\phi : U \to \mathbb{R}$, $\psi : V \to \mathbb{R}$ such that $\psi(f(s,t), g(s,t)) = \phi(s,t)$, the effect of the above change of coordinates on the integral of ψ over V may be computed as follows:

$$\int_V \psi \ dx \ dy \ = \ \int_U \phi \cdot J \ ds \ dt.$$

Step 7: In the situation of Step 6, the effect of the change of coordinates on the **"infinitesimals"** $dx\ dy$ and $ds\ dt$ may be understood as follows: First, one **localizes** to a sufficiently small open neighborhood of a point of U over which the various partial derivatives of f and g are roughly *constant*, which implies that the change of coordinates determined by f and g is roughly *linear*. Then the effect of such a linear transformation on **areas** — i.e., in the language of Step 5, **"masses"** — of sufficiently small **parallelograms** is given by **multiplying** by the **determinant** of the linear transformation. Indeed, to verify this, one observes that, after possible pre- and post-composition with a **rotation** [which clearly does not affect the computation of such areas], one may assume that one of the sides of the parallelogram under consideration is a *line segment on the s-axis whose left-hand endpoint is equal to the origin* $(0,0)$, and, moreover, that the linear transformation may be written as a *composite* of **toral dilations** and **unipotent** linear transformations of the form

$$(s,t) \mapsto (a \cdot s, b \cdot t); \qquad (s,t) \mapsto (s + c \cdot t, t)$$

— where $a, b, c \in \mathbb{R}$, and $ab \neq 0$. On the other hand, in the case of such **"upper triangular"** linear transformations, the effect of the linear transformation on the area of the parallelogram under consideration is an easy computation at the level of high-school planar geometry.

§**1.6.** **Passage from planar cartesian to polar coordinates:** Once the *"innocuous" generalities* of Steps 5, 6, and 7 have been assimilated, one may proceed as follows:

__Step 8__: We *apply Step 6* to the *integral of Step 4*, regarded as an integral over the *complement* $\mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})$ of the negative x-axis in the Euclidean plane, to the *change of coordinates*

$$\mathbb{R}_{>0} \times (-\pi, \pi) \ \ni \ (r, \theta) \ \mapsto (x, y) = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})$$

— where we write $\mathbb{R}_{>0}$ for the set of positive real numbers and $(-\pi, \pi)$ for the open interval of real numbers between $-\pi$ and π.

__Step 9__: The change of coordinates of Step 8 allows one to compute as follows:

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \ &= \ \int_{\mathbb{R}^2} e^{-x^2} \cdot e^{-y^2}\ dx\ dy \\ &= \ \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dx\ dy \\ &= \ \int_{\mathbb{R}^2 \setminus (\mathbb{R}_{\leq 0} \times \{0\})} e^{-(x^2+y^2)}\ dx\ dy\end{aligned}$$

$$\begin{aligned}&= \ \int_{\mathbb{R}_{>0} \times (-\pi, \pi)} e^{-r^2}\ rdr\ d\theta \\ &= \ \left(\int_0^{\infty} e^{-r^2} \cdot 2rdr\right) \cdot \left(\int_{-\pi}^{\pi} \tfrac{1}{2} \cdot d\theta\right)\end{aligned}$$

— where we observe that the *final equality* is notable in that it shows that, in the computation of the integral under consideration, the **radial** [i.e., "r"] and **angular** [i.e., "θ"] coordinates may be **decoupled**, i.e., that the integral under consideration may be written as a product of a *radial integral* and an *angular integral*.

__Step 10__: The **radial integral** of Step 9 may be evaluated

$$\int_0^{\infty} e^{-r^2} \cdot 2rdr \ = \ \int_0^1 d(e^{-r^2}) \ = \ \int_0^1 du \ = \ 1$$

by applying the **change of coordinates**

$$e^{-r^2} \ \rightsquigarrow \ u$$

that, in essence, appeared in the *erroneous* initial computation of Step 1!

__Step 11__: The **angular integral** of Step 9 may be evaluated as follows:

$$\int_{-\pi}^{\pi} \tfrac{1}{2} \cdot d\theta \ = \ \pi$$

Here, we note that, if one thinks of the Euclidean plane $\mathbb{R}^2$ of Step 4 as the *complex plane*, i.e., if we write the change of coordinates of Step 8 in the form $x + iy = r \cdot e^{i\theta}$, then, relative to the Euclidean coordinates (x, y) of Step 4, the above evaluation of the angular integral may be regarded as arising from the **change of coordinates** given by considering the *imaginary part* of the **natural logarithm**

$$\log(r \cdot e^{i\theta}) \; = \; \log(r) \; + \; i\theta.$$

<u>**Step 12**</u>: Thus, in summary, we conclude that

$$\begin{aligned} \Big(\int_{-\infty}^{\infty} e^{-x^2} \; dx \Big)^2 \; &= \; \Big(\int_{-\infty}^{\infty} e^{-x^2} \; dx \Big) \cdot \Big(\int_{-\infty}^{\infty} e^{-y^2} \; dy \Big) \\ &= \; \Big(\int_{0}^{\infty} e^{-r^2} \cdot 2r dr \Big) \cdot \Big(\int_{-\pi}^{\pi} \tfrac{1}{2} \cdot d\theta \Big) \; = \; \pi \end{aligned}$$

— i.e., that $\int_{-\infty}^{\infty} e^{-x^2} \; dx \; = \; \sqrt{\pi}$.

§**1.7.** <u>**Justification of naive approach up to an "error factor"**</u>**:** Put another way, the content of the above discussion may be summarized as follows:

> If one considers **two identical — but mutually independent! — copies** of the **Gaussian integral**, i.e., as opposed to a *single copy*, then the **naively motivated coordinate transformation** that gave rise to the erroneous computation of Step 1 may be **"justified"**, up to a suitable **"error factor"** $\sqrt{\pi}$!

In this context, it is of interest to note that the technique applied in the above discussion for evaluating the integral of the *Gaussian distribution* "e^{-x^2}" **cannot**, in essence, **be applied** to integrals of **functions other than the Gaussian distribution**. Indeed, this essentially unique relationship between the technique of the above discussion and the Gaussian distribution may be understood as being, in essence, a consequence of the fact that the exponential function determines an *isomorphism of Lie groups* between the "additive Lie group" of real numbers and the "multiplicative Lie group" of positive real numbers. We refer to [Bell], [Dawson] for more details.

Section 2: Changes of universe as arithmetic changes of coordinates

§**2.1.** <u>**The issue of bounding heights: the ABC and Szpiro Conjectures**</u>**:** In diophantine geometry, i.e., more specifically, the diophantine geometry of **rational points** of an **algebraic curve** over a **number field** [i.e., an **"NF"**], one is typically concerned with the issue of **bounding heights** of such rational points. A brief exposition of various *conjectures* related to this issue of bounding heights of rational points may be found in [Fsk], §1.3. In this context, the case where the algebraic curve under consideration is the *projective line minus three points* corresponds most directly to the so-called **ABC** and — by thinking of this projective line as the *"λ-line"* that appears in discussions of the **Legendre form** of the Weierstrass equation for an elliptic curve — **Szpiro Conjectures**. In this case, the height of a rational point may be thought of as a suitable weighted sum of the **valuations** of the ***q*-parameters** of the elliptic curve determined by the rational point at the nonarchimedean primes of potentially multiplicative reduction [cf. the discussion at the end of [Fsk], §2.2; [GenEll], Proposition 3.4]. Here, it is also useful to recall [cf. [GenEll], Theorem 2.1] that, in the situation of the ABC or Szpiro Conjectures, one may assume, without loss of generality, that, for any given finite set Σ of [archimedean and nonarchimedean] valuations of the rational number field $\mathbb{Q}$,

> the rational points under consideration lie, at each valuation of Σ, inside some **compact subset** [i.e., of the set of rational points of the projective line minus three points over *some finite extension of the completion* of $\mathbb{Q}$ at this valuation] satisfying certain properties.

In particular, when one computes the height of a rational point of the projective line minus three points as a suitable weighted sum of the valuations of the q-parameters of the corresponding elliptic curve, one may **ignore**, up to bounded discrepancies, contributions to the height that arise, say, from the **archimedean valuations** or from the nonarchimedean valuations that lie over some "exceptional" prime number such as 2.

§**2.2.** **Arithmetic degrees as global integrals:** As is well-known, the height of a rational point may be thought of as the **arithmetic degree** of a certain **arithmetic line bundle** over the field of definition of the rational point [cf. [Fsk], §1.3; [GenEll], §1]. Alternatively, from an **idèlic** point of view, such arithmetic degrees of arithmetic line bundles over an NF may be thought of as logarithms of volumes — i.e., **"log-volumes"** — of certain **regions** inside the *ring of adèles of the NF* [cf. [Fsk], §2.2; [AbsTopIII], Definition 5.9, (iii); [IUTchIII], Proposition 3.9, (iii)]. Relative to the point of view of the discussion of §1.4, such log-volumes may be thought of as **"net masses"**, that is to say,

> as **"global masses"** [i.e., global log-volumes] that arise by summing up various **"local masses"** [i.e., local log-volumes], corresponding to the [archimedean and nonarchimedean] valuations of the NF under consideration.

This point of view of the discussion of §1.4 suggests further that such a global net mass should be regarded as some sort of

> **integral** over an NF, that is to say, which arises by applying some sort of **mysterious "limit summation operation"** to some sort of **"zero mass infinitesimal"** object [i.e., corresponding a differential form].

It is precisely this point of view that will be pursued in the discussion to follow via the following correspondences with terminology to be explained below:

$$\begin{array}{rcl} \textbf{zero mass}\text{ objects} & \longleftrightarrow & \textbf{``étale-like''}\text{ structures} \\ \textbf{positive/nonzero mass}\text{ objects} & \longleftrightarrow & \textbf{``Frobenius-like''}\text{ structures} \end{array}$$

§**2.3.** **Bounding heights via global multiplicative subspaces:** In the situation discussed in §2.1, one way to understand the problem of showing that the **height** $h \in \mathbb{R}$ of a rational point is **"small"** is as the problem of showing that, for some **fixed natural number** $N > 1$, the height h satisfies the **equation**

$$N \cdot h \left(\stackrel{\text{def}}{=} h + h + \ldots + h \right) = h$$

[which implies that $h = 0$!] — or, more generally, for a suitable *"relatively small"* constant $C \in \mathbb{R}$ [i.e., which is independent of the rational point under consideration], the **inequality**

$$N \cdot h \ \leq \ h + C$$

[which implies that $h \leq \frac{1}{N-1} \cdot C$!] — holds. Indeed, this is precisely the approach that is taken to *bounding heights* in the *"tiny" special case* of the theory of [Falt1] that is given in the proof of [GenEll], Lemma 3.5. Here, we recall that the *key assumption* in [GenEll], Lemma 3.5, that makes this sort of argument work is the assumption of the existence, for some prime number l, of a certain kind of *special rank one subspace* [i.e., a subspace whose $\mathbb{F}_l$-dimension is equal to 1] of the space of *l-torsion points* [i.e., a $\mathbb{F}_l$-vector space of dimension 2] of the elliptic curve under consideration. Such a rank one subspace is typically referred to in this context as a **global multiplicative subspace**, i.e., since it is a subspace defined over the NF under consideration that *coincides*, at each nonarchimedean valuation of the NF at which the elliptic curve under consideration has potentially multiplicative reduction, with the rank one subspace of l-torsion points that arises, via the *Tate uniformization*, from the [one-dimensional] space of l-torsion points of the *multiplicative group* $\mathbb{G}_\mathrm{m}$. The quotient of the original given elliptic curve by such a *global multiplicative subspace* is an elliptic curve that is **isogenous** to the original elliptic curve. Moreover,

> the ***q*-parameters** of this isogenous elliptic curve are the ***l*-th powers** of the q-parameters of the original elliptic curve; thus, the **height** of this isogenous elliptic curve is [roughly, up to contributions of negligible order] ***l* times** the height of the original elliptic curve.

These properties of the isogenous elliptic curve allow one to **compute** the *height* of the *isogenous elliptic curve* in terms of the *height* of the *original elliptic curve* by calculating the effect of the isogeny relating the two elliptic curves on the respective **sheaves of differentials** and hence to conclude an *inequality* "$N \cdot h \ \leq \ h + C$" of the desired type [for $N = l$ — cf. the proof of [GenEll], Lemma 3.5, for more details]. At a more concrete level, this *computation* may be summarized as the **observation** that, by considering the effect of the isogeny under consideration on *sheaves of differentials*, one may conclude that

> *"multiplying heights by l* — i.e., *"raising **q-parameters** to the **l-th power**"*
>
> $$q \ \mapsto \ q^l$$
>
> — has the effect on **logarithmic differential forms**
>
> $$d\log(q) = \frac{dq}{q} \ \mapsto \ l \cdot d\log(q)$$
>
> of *multiplying by l*, i.e., at the level of **heights**, of **adding** terms of the order of $\log(l)$, thus giving rise to *inequalities* that are roughly of the form "$l \cdot h \ \leq \ h + \log(l)$".

On the other hand, in general,

> such a global multiplicative subspace *does not exist*, and the issue of somehow **"simulating"** *the existence of a global multiplicative subspace* is one **fundamental theme** of inter-universal Teichmüller theory.

§**2.4.** **Bounding heights via Frobenius morphisms on number fields:** The simulation issue discussed in §2.3 is, in some sense, the fundamental reason for the **construction** of various types of **"Hodge theaters"** in [IUTchI] [cf. the discussion surrounding [IUTchI], Fig. I1.4; [IUTchI], Remark 4.3.1]. From the point of view of the present discussion, the fundamental **additive** and **multiplicative symmetries** that appear in the theory of *[*$\Theta^{\pm \mathrm{ell}}$*NF-]Hodge theaters* [cf. §3.3, (v); §3.6, (i), below] and which correspond, respectively, to the *additive* and *multiplicative structures* of the ring $\mathbb{F}_l$ [where l is the *fixed* prime number for which we consider *l-torsion points*], may be thought of as corresponding, respectively, to the **symmetries** in the equation

$$N \cdot h \left(\stackrel{\mathrm{def}}{=} \ h + h + \ldots + h \right) \ = h$$

of *all the h's* [in the case of the additive symmetry] and of *the h's on the LHS* [in the case of the multiplicative symmetry]. This portion of inter-universal Teichmüller theory is closely related to the analogy between inter-universal Teichmüller theory and the **classical hyperbolic geometry** of the **upper half-plane**. This analogy with the hyperbolic geometry of the upper half-plane is, in some sense, the **central topic** of [BogIUT] and may be thought of as corresponding to the portion of inter-universal Teichmüller theory discussed in [IUTchI], [IUTchIII]. Since this aspect of inter-universal Teichmüller theory is already discussed in substantial detail in [BogIUT], we shall not discuss it in much detail in the present paper. On the other hand, another way of thinking about the above equation "$N \cdot h = h$" is as follows:

> This equation may also be thought of as calling for the establishment of some sort of analogue for an **NF** of the **Frobenius morphism** in positive characteristic scheme theory, i.e., a Frobenius morphism that somehow *"acts" naturally on the entire situation* [i.e., including the height h, as well as the q-parameters at nonarchimedean valuations of potentially multiplicative reduction, of a given elliptic curve over the NF] in such a way as to *multiply arithmetic degrees* [such as the height!] by N and *raise q-parameters* to the N-th power — i.e.,
>
> $$h \ \mapsto \ N \cdot h, \quad q \ \mapsto \ q^N$$
>
> — and hence yield the equation "$N \cdot h = h$" [or inequality "$N \cdot h \leq h + C$"] via some sort of *natural functoriality.*

This point of view is also quite **fundamental** to inter-universal Teichmüller theory, and, in particular, to the analogy between inter-universal Teichmüller theory and the theory of the **Gaussian integral**, as reviewed in §1. These aspects of inter-universal Teichmüller theory are discussed in [IUTchII], [IUTchIII]. In the present paper, we shall concentrate mainly on the exposition of these aspects of inter-universal Teichmüller theory. Before proceeding, we remark that, ultimately, in inter-universal Teichmüller theory, we will, in effect, take "N" to be a sort of **symmetrized average** over the **squares** of the values $j = 1, 2, \ldots, l^*$, where $l^* \stackrel{\mathrm{def}}{=} (l-1)/2$, and l is the prime number of §2.3. That is to say, whereas the [purely hypothetical!] **naive analogue** of the **Frobenius morphism** for an NF considered so far has the effect, on q-parameters of the elliptic curve under consideration at nonarchimedean valuations of potentially multiplicative reduction, of mapping $q \ \mapsto \ q^N$, the sort of assignment that we shall ultimately be interested in inter-universal Teichmüller theory is an assignment [which is in fact typically written with the *left-* and *right-* hand sides *reversed*]

$$\underline{\underline{q}} \ \mapsto \ \{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*}$$

— where $\underline{\underline{q}}$ denotes a $2l$*-th root* of the q-parameter q — i.e., an assignment which, at least at a formal level, closely resembles a **Gaussian distribution**. Of course, such an assignment is **not compatible** with the **ring structure** of an NF, hence does not exist in the **framework** of **conventional scheme theory**. Thus, one way to understand inter-universal Teichmüller theory is as follows:

> in some sense the *fundamental theme* of inter-universal Teichmüller theory consists of the development of a mechanism for **computing** the effect — e.g., on *heights of elliptic curves* [cf. the discussion of §2.3!] — of such **non-scheme-theoretic "Gaussian Frobenius morphisms"** on NF's.

§**2.5.** **Fundamental example of the derivative of a Frobenius lifting:** In some sense, the most fundamental example of the sort of **Frobenius action** in the p-adic theory [cf. the discussion of §2.5] that one would like to somehow **translate** into the case of **NF's** is the following [cf. [AbsTopII], Remark 2.6.2; [AbsTopIII], §I5; [IUTchIII], Remark 3.12.4, (v)]:

Example 2.5.1: Frobenius liftings on smooth proper curves. Let p be a prime number; A the ring of Witt vectors of a perfect field k of characteristic p; X a *smooth, proper curve* over A of genus $g_X \geq 2$; $\Phi : X \rightarrow X$ a *Frobenius lifting*, i.e., a morphism whose reduction modulo p coincides with the *Frobenius morphism* in characteristic p. Thus, one verifies immediately that Φ necessarily lies over the Frobenius morphism on the ring of Witt vectors A. Write ω_{X_k} for the sheaf of differentials of $X_k \stackrel{\text{def}}{=} X \times_A k$ over k. Then the **derivative** of Φ yields, upon dividing by p, a morphism of line bundles

$$\Phi^*\omega_{X_k} \ \rightarrow \ \omega_{X_k}$$

which is easily verified to be *generically injective*. Thus, by taking *global degrees of line bundles*, we obtain an **inequality**

$$(p-1)(2g_X - 2) \ \leq \ 0$$

— hence, in particular, an inequality $g_X \leq 1$ — which may be thought of as being, in essence, a statement to the effect that X is **cannot be hyperbolic**. Note that, from the point of view discussed in §1.4, §1.5, §2.2, §2.3, §2.4, this inequality may be thought of as

> a computation of **"global net masses"**, i.e., *global degrees of line bundles* on X_k, via a computation of the effect of the **"change of coordinates"** Φ by considering the effect of this change of coordinates on **"infinitesimals"**, i.e., on the **sheaf of differentials** ω_{X_k}.

§**2.6.** **Positive characteristic model for mono-anabelian transport:** One *fundamental drawback* of the computation discussed in Example 2.5.1 is that it involves the operation of *differentiation* on X_k, an operation which does not, at least in the immediate literal sense, have a natural analogue in the case of *NF's*. This drawback does not exist in the following example, which treats certain **subtle**, but **well-known** aspects of **anabelian geometry** in **positive characteristic** and, moreover, may, in some sense, be regarded as the **fundamental model**, or **prototype**, for a quite substantial portion of inter-universal Teichmüller theory. In this example, **Galois groups**, or **étale fundamental groups**, in some sense play the role that is played by **tangent bundles** in the classical theory — a situation that is reminiscent of the approach of the [scheme-theoretic] **Hodge-Arakelov theory** of [HASurI], [HASurII], which is briefly reviewed in §2.14 below. One notion of *central importance* in this example — and indeed throughout inter-universal Teichmüller theory! — is the notion of a **cyclotome**, a term which is used to refer to an *isomorphic copy* of some quotient [by a closed submodule] of the familiar Galois module "$\widehat{\mathbb{Z}}(1)$", i.e., the **"Tate twist"** of the trivial Galois module "$\widehat{\mathbb{Z}}$", or, alternatively, the rank one free $\widehat{\mathbb{Z}}$-module equipped with the action determined by the **cyclotomic character**. Also, if p is a *prime number*, then we shall write $\widehat{\mathbb{Z}}_{\neq p}$ for the quotient $\widehat{\mathbb{Z}}/\mathbb{Z}_p$.

<u>Example 2.6.1</u>: **Mono-anabelian transport via the Frobenius morphism in positive characteristic.**

(i) Let p be a prime number; k a finite field of characteristic p; X a smooth, proper curve over k of genus $g_X \geq 2$; K the function field of X; $\widetilde{K}$ a *separable closure* of K. Write $\eta_X \stackrel{\text{def}}{=} \mathrm{Spec}(K)$; $\widetilde{\eta}_X \stackrel{\text{def}}{=} \mathrm{Spec}(\widetilde{K})$; $\overline{k} \subseteq \widetilde{K}$ for the algebraic closure of k determined by $\widetilde{K}$; $\boldsymbol{\mu}_{\overline{k}} \subseteq \overline{k}$ for the group of *roots of unity* of $\overline{k}$; $\boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}_{\neq p}} \stackrel{\text{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\overline{k}})$; $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{K}/K)$; $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X for the quotient of G_K determined by the maximal subextension of $\widetilde{K}$ that is *unramified* over X;

$$\Phi_X : X \to X, \quad \Phi_{\eta_X} : \eta_X \to \eta_X, \quad \Phi_{\widetilde{\eta}_X} : \widetilde{\eta}_X \to \widetilde{\eta}_X$$

for the respective *Frobenius morphisms* of X, η_X, $\widetilde{\eta}_X$. Thus, we have natural surjections $G_K \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$, and Π_X may be thought of as [i.e., is *naturally isomorphic* to] the *étale fundamental group* of X [for a suitable choice of basepoint]. Write $\Delta_X \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_X \twoheadrightarrow G_k)$. Recall that it follows from elementary facts concerning *separable* and *purely inseparable* field extensions that [by considering $\Phi_{\widetilde{\eta}_X}$] Φ_{η_X} induces **isomorphisms** of **Galois groups** and **étale fundamental groups**

$$\Psi_X : \Pi_X \xrightarrow{\sim} \Pi_X, \quad \Psi_{\eta_X} : G_K \xrightarrow{\sim} G_K$$

— which is, in some sense, a *quite remarkable fact* since

> the **Frobenius morphisms** Φ_X, Φ_{η_X} themselves are morphisms *"of degree $p > 1$"*, hence, in particular, are **by no means isomorphisms**!

We refer to [IUTchIV], Example 3.6, for a more general version of this phenomenon.

(ii) Next, let us recall that it follows from the *fundamental anabelian* results of [Uchi] that there exists a **purely group-theoretic functorial algorithm**

$$G_K \;\mapsto\; \widetilde{K}(G_K)^{\mathrm{CFT}} \curvearrowleft G_K$$

— i.e., an algorithm whose *input data* is the *abstract topological group* G_K, whose **functoriality** is with respect to isomorphisms of topological groups, and whose *output data* is a field $\widetilde{K}(G_K)^{\mathrm{CFT}}$ equipped with a G_K-action. Moreover, if one allows oneself to apply the *conventional interpretation* of G_K as a *Galois group* $\mathrm{Gal}(\widetilde{K}/K)$, then there is a *natural* G_K*-equivariant isomorphism*

$$\rho : \widetilde{K} \ \stackrel{\sim}{\to} \ \widetilde{K}(G_K)^{\mathrm{CFT}}$$

that arises from the **reciprocity map** of **class field theory**, applied to each of the finite subextensions of the extension $\widetilde{K}/K$. Since class field theory depends, in an essential way, on the *field structure* of $\widetilde{K}$ and K, it follows formally that, at least in an *a priori* sense, the construction of ρ itself also depends, in an essential way, on the **field structure** of $\widetilde{K}$ and K. Moreover, the fact that the isomorphism $\widetilde{K}(\Psi_{\eta_X})^{\mathrm{CFT}} : \widetilde{K}(G_K)^{\mathrm{CFT}} \stackrel{\sim}{\to} \widetilde{K}(G_K)^{\mathrm{CFT}}$ and ρ are [unlike Φ_{η_X} itself!] *isomorphisms* implies that the diagram

$$\begin{array}{ccc}
\widetilde{K}(G_K)^{\mathrm{CFT}} & \stackrel{\widetilde{K}(\Psi_{\eta_X})^{\mathrm{CFT}}}{\longleftarrow} & \widetilde{K}(G_K)^{\mathrm{CFT}} \\
\uparrow \rho & \curvearrowright ? & \uparrow \rho \\
\widetilde{K} & \stackrel{\Phi^*_{\eta_X}}{\longrightarrow} & \widetilde{K}
\end{array}$$

fails to be commutative!

(iii) On the other hand, let us recall that consideration of the *first Chern class* of a line bundle of degree 1 on X yields a *natural isomorphism*

$$\lambda : \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}_{\neq p}} \ \stackrel{\sim}{\to} \ M_X \ \stackrel{\mathrm{def}}{=} \ \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\neq p}}(H^2(\Delta_X, \widehat{\mathbb{Z}}_{\neq p}), \widehat{\mathbb{Z}}_{\neq p})$$

[cf., e.g., [Cusp], Proposition 1.2, (ii)]. Such a natural isomorphism between cyclotomes [i.e., such as $\boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}_{\neq p}}$, M_X] will be referred to as a *cyclotomic rigidity isomorphism*. Thus, if we let "H" range over the open subgroups of G_K, then, by *composing* this cyclotomic rigidity isomorphism [applied to the coefficients of "$H^1(-)$"] with the **Kummer morphism** associated to the multiplicative group $(\widetilde{K}^H)^\times$ of the field $\widetilde{K}^H$ of H-invariants of $\widetilde{K}$, we obtain an embedding

$$\kappa : \widetilde{K}^\times \ \hookrightarrow \ \varinjlim_H H^1(H, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}_{\neq p}}) \ \stackrel{\sim}{\to} \ \varinjlim_H H^1(H, M_X)$$

— whose construction depends only on the **multiplicative monoid** with G_K-action $\widetilde{K}^\times$ and the **cyclotomic rigidity isomorphism** λ. Note that the existence of the reconstruction algorithm $\widetilde{K}(-)^{\mathrm{CFT}}$ reviewed above implies that the kernels of the natural surjections $G_K \twoheadrightarrow \Pi_X \twoheadrightarrow G_k$ may be reconstructed group-theoretically from the abstract topological group G_K. In particular, we conclude that $\varinjlim_H H^1(H, M_X)$ may be reconstructed group-theoretically from the abstract topological group G_K. Moreover, the *anabelian* theory of [Cusp] [cf., especially, [Cusp], Proposition 2.1; [Cusp], Theorem 2.1, (ii); [Cusp], Theorem 3.2] yields a **purely group-theoretic functorial algorithm**

$$G_K \ \mapsto \ \widetilde{K}(G_K)^{\mathrm{Kum}} \curvearrowleft G_K$$

— i.e., an algorithm whose *input data* is the *abstract topological group* G_K, whose **functoriality** is with respect to isomorphisms of topological groups, and whose *output data* is a field $\widetilde{K}(G_K)^{\mathrm{Kum}}$ equipped with a G_K-action which is constructed as the union with $\{0\}$ of the image of κ. [In fact, the input data for this algorithm may be taken to be the abstract topological group Π_X, but we shall not pursue this topic here.] Thus, just as in the case of "$\widetilde{K}(-)^{\mathrm{CFT}}$", the fact that the isomorphism $\widetilde{K}(\Psi_{\eta_X})^{\mathrm{Kum}} : \widetilde{K}(G_K)^{\mathrm{Kum}} \xrightarrow{\sim} \widetilde{K}(G_K)^{\mathrm{Kum}}$ and κ are [unlike Φ_{η_X} itself!] *isomorphisms* implies that the diagram

$$\begin{array}{ccc} \widetilde{K}(G_K)^{\mathrm{Kum}} & \xleftarrow{\widetilde{K}(\Psi_{\eta_X})^{\mathrm{Kum}}} & \widetilde{K}(G_K)^{\mathrm{Kum}} \\ \uparrow \kappa & \curvearrowright ? & \uparrow \kappa \\ \widetilde{K} & \xrightarrow{\Phi^*_{\eta_X}} & \widetilde{K} \end{array}$$

— where, by a slight abuse of notation, we write "κ" for the "formal union" of κ with "$\{0\}$" — **fails to be commutative**!

(iv) The [*a priori*] *noncommutativity* of the diagram of the final display of (iii) may be interpreted in two ways, as follows:

(a) If one starts with the assumption that this diagram *is* in fact *commutative*, then the fact that the Frobenius morphism $\Phi^*_{\eta_X}$ *multiplies degrees* of rational functions $\in \widetilde{K}$ by p, together with the fact that the vertical and upper horizontal arrows of the diagram are *isomorphisms*, imply [since the field $\widetilde{K}$ is *not perfect*!] the *erroneous conclusion* that all degrees of rational functions $\in \overline{K}$ are *equal to zero*! This sort of argument is **formally similar** to the argument "$N \cdot h = h \implies h = 0$" discussed in §2.3.

(b) One may regard the noncommutativity of this diagram as the **problem** of **computing** just how much **"indeterminacy"** one must allow in the objects and arrows that appear in the diagram in order to render the diagram **commutative**. From this point of view, one verifies immediately that a **solution** to this problem may be given by introducing *"indeterminacies"* as follows: One *replaces*

$$\lambda \rightsquigarrow \lambda \cdot p^{\mathbb{Z}}$$

the **cyclotomic rigidity isomorphism** λ by the **orbit** of λ with respect to composition with multiplication by arbitrary $\mathbb{Z}$-powers of p, and one *replaces*

$$\widetilde{K} \rightsquigarrow \widetilde{K}^{\mathrm{pf}}, \quad \widetilde{K}(G_K)^{\mathrm{Kum}} \rightsquigarrow (\widetilde{K}(G_K)^{\mathrm{Kum}})^{\mathrm{pf}}$$

the fields $\widetilde{K}$, $\widetilde{K}(G_K)^{\mathrm{Kum}}$ by their **perfections**.

Here, we observe that intepretation (b) may be regarded as corresponding to the argument "$N \cdot h \leq h + C \implies h \leq \frac{1}{N-1} \cdot C$" discussed in §2.3. That is to say,

If, in the situation of (b), one can show that the **indeterminacies** necessary to render the diagram **commutative** are **sufficiently mild**, at least in the case of the *heights* or *q-parameters* that one is interested in, then it is *"reasonable to expect"* that the resulting *"contradiction in the style of interpretation (a)"* between

multiplying degrees by some integer [or rational number] > 1

and the fact that

the vertical and upper horizontal arrows of the diagram are isomorphisms

should enable one to conclude that "$N \cdot h \leq h + C$" [and hence that "$h \leq \frac{1}{N-1} \cdot C$"].

This is precisely the approach that is in fact taken in inter-universal Teichmüller theory.

§**2.7.** **<u>The apparatus and terminology of mono-anabelian transport</u>:** Example 2.6.1 is *exceptionally rich* in **structural similarities** to inter-universal Teichmüller theory, which we proceed to explain in detail as follows. One way to understand these structural similarities is by considering the quite substantial portion of **terminology** of inter-universal Teichmüller theory that was, in essence, **inspired** by Example 2.6.1:

(i) **<u>Links between "mutually alien" copies of scheme theory</u>:** One central aspect of inter-universal Teichmüller theory is the study of certain **"walls"**, or **"filters"** — which are often referred to as **"links"** — that separate two **"mutually alien" copies** of **conventional scheme theory** [cf. the discussions of [IUTchII], Remark 3.6.2; [IUTchIV], Remark 3.6.1]. The main example of such a link in inter-universal Teichmüller theory is constituted by [various versions of] the **Θ-link**. The **$\mathfrak{log}$-link** also plays an important role in inter-universal Teichmüller theory. The main motivating example for these links which play a central role in inter-universal Teichmüller theory is the **Frobenius morphism** Φ_{η_X} of Example 2.6.1. From the point of view of the discussion of §1.4, §1.5, §2.2, §2.3, §2.4, and §2.5, such a link corresponds to a **change of coordinates**.

(ii) **<u>Frobenius-like objects</u>:** The objects that appear on either side of a link and which are used in order to construct, or "set up", the link, are referred to as **"Frobenius-like"**. Put another way,

> **Frobenius-like objects** are objects that, at least *a priori*, are *only* **defined on one side** *of a link* [i.e., either the domain or codomain], and, in particular, *do* **not** *necessarily map* **isomorphically** *to corresponding objects on the opposite side of the link.*

Thus, in Example 2.6.1, the "mutually alien" copies of $\widetilde{K}$ on either side of the p-power map $\Phi^*_{\eta_X}$ are *Frobenius-like*. Typically, Frobenius-like structures are characterized by the fact that they have **positive/nonzero mass**. That is to say, Frobenius-like structures represent the positive mass — i.e., such as *degrees of rational functions* in Example 2.6.1 or *heights/degrees of arithmetic line bundles* in the context of diophantine geometry — that one is *ultimately interested in computing* and, moreover, is, at least in an *a priori* sense, *affected in a nontrivial way*, e.g., *multiplied by some factor* > 1, by the link under consideration. From this point of view, Frobenius-like objects are characterized by the fact that the link under consideration gives rise to an **"ordering"**, or **"asymmetry"**, between Frobenius-like objects in the domain and codomain of the link under consideration [cf. the discussion of [FrdI], §I3, §I4].

(iii) **<u>Étale-like objects</u>:** By contrast, objects that appear on either side of a link that correspond to some sort of *"underlying topology"* — such as the *étale topology*! — are referred to as **"étale-like"**. Typically, étale-like structures are mapped *isomorphically* — albeit via some *indeterminate isomorphism*! [cf. the discussion of §2.10 below] — to one another by the link under consideration. From this point of view, étale-like objects are characterized by the fact that the link under consideration gives rise to a **"confusion"**, or **"symmetry"**, between étale-like objects in the domain and codomain of the link under consideration [cf. the discussion of [FrdI], §I3, §I4]. Thus, in Example 2.6.1, the Galois groups/étale fundamental groups G_K, Π_X, which are mapped *isomorphically* to one another via Φ_{η_X}, albeit via some *"mysterious indeterminate isomorphism"*, are *étale-like*. Objects that are **algorithmically constructed from étale-like objects** such as G_K or Π_X are also referred to as *étale-like*, so long as they are regarded as being equipped with the additional structure constituted by the algorithm applied to construct the object from some object such as G_K or Π_X. Étale-like structures are regarded as having **zero mass** and are used as **rigid containers** for positive mass Frobenius-like objects, i.e., containers whose rigidity typically arises from various **anabelian** properties and [as in Example 2.6.1!] allows one to **compute** *the effect on positive mass Frobenius-like objects of the links, or "changes of coordinates", under consideration.*

(iv) **<u>Coric objects</u>:** In the context of consideration of some sort of link as in (i), **coricity** refers to the property of being *invariant* with respect to — i.e., the property of **mapping isomorphically** to a corresponding object on the **opposite side** of — the link under consideration. Thus, as discussed in (iii), **étale-like** objects, considered *up to isomorphism*, constitute a *primary* example of the notion of a **coric** object. On the other hand, *[non-étale-like] Frobenius-like coric objects* also arise naturally in various contexts. Indeed, in the situation of Example 2.6.1 [cf. the discussion of Example 2.6.1, (iv), (b)], not only *étale-like* objects such as Π_X, G_K, and $\widetilde{K}(G_K)^{\mathrm{Kum}}$, but also *Frobenius-like* objects such as the *perfections* $\widetilde{K}^{\mathrm{pf}}$ are *coric.*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{étale-like}\\ \text{object}\end{array}} \arrow[r, "\overset{\text{mono-anabelian rigidity}}{\curvearrowright}", "\sim"'] & \boxed{\begin{array}{c}\text{étale-like}\\ \text{object}\end{array}} \arrow[d, Rightarrow, "\text{Kummer}"', "\text{isom.}^{-1}"] \\
\boxed{\begin{array}{c}\text{Frobenius-like}\\ \text{object}\end{array}} \arrow[u, Rightarrow, "\text{Kummer}", "\text{isom.}"'] \arrow[r, rightsquigarrow, "\text{link}"] & \boxed{\begin{array}{c}\text{Frobenius-like}\\ \text{object}\end{array}}
\end{tikzcd}$$

Fig. 2.1: Mono-anabelian transport

(v) <u>**The computational technique of mono-anabelian transport:**</u> The technique discussed in (iii), i.e., of

> **computing** the effect on **positive mass Frobenius-like objects** of the **links**, or *"changes of coordinates"*, under consideration, by first applying some sort of **Kummer isomorphism** to pass from Frobenius-like to corresponding **étale-like objects**, then applying some sort of **anabelian** construction algorithm or rigidity property, and finally applying a suitable **inverse Kummer isomorphism** to pass from étale-like to corresponding Frobenius-like objects

will be referred to as the technique of **mono-anabelian transport** [cf. Fig. 2.1 above]. In some sense, the *most fundamental* **prototype** for this technique is the situation described in Example 2.6.1, (iii) [cf. also the discussion of Example 2.6.1, (iv), (b)]. Here, the term *"mono-anabelian"* [cf. the discussion of [AbsTopIII], §I2] refers to the fact that the algorithm under consideration is an algorithm whose *input data* [typically] consists only of an *abstract profinite group* [i.e., that "just happens" to be isomorphic to a Galois group or étale fundamental group that arises from scheme theory!]. This term is used to distinguish from *fully faithfulness* results [i.e., to the effect that one has a natural bijection between certain types of morphisms of schemes and certain types of morphisms of profinite groups] of the sort that appear in various anabelian conjectures of Grothendieck. Such fully faithfulness results are referred to as *"bi-anabelian"*.

(vi) <u>**Kummer-detachment indeterminacies versus étale-transport indeterminacies:**</u> The *first step* that occurs in the procedure for *mono-anabelian transport* [cf. the discussion of (v)] is the passage, via some sort of *Kummer isomorphism*, from *Frobenius-like* objects to corresponding *étale-like* objects. This first step is referred to as the *Kummer-detachment*. The indeterminacies that arise during this first step are referred to as **Kummer-detachment indeterminacies** [cf. [IUTchIII], Remark 1.5.4]. Such Kummer-detachment indeterminacies typically involve indeterminacies in the **cyclotomic rigidity isomorphism** that is applied, i.e., as in the situation discussed in Example 2.6.1, (iv), (b). On the other hand, in general, more complicated Kummer-detachment indeterminacies [i.e., that are not directly related to cyclotomic rigidity isomorphisms] can occur. By contrast, the indeterminacies that occur as a result of the fact that the étale-like structures under consideration may only be regarded as being known up to an *indeterminate isomorphism* [cf. the discussion of (iii), as well as of §2.10 below] are referred to as **étale-transport indeterminacies** [cf. [IUTchIII], Remark 1.5.4].

(vii) <u>**Arithmetic holomorphic structures versus mono-analytic structures:**</u> A ring may be regarded as consisting of **"two combinatorial dimensions"** — namely, the underlying **additive** and **multiplicative** structures of the ring — which are **intertwined** with one another in a rather complicated fashion [cf. the discussion of [AbsTopIII], §I3; [AbsTopIII], Remark 5.6.1]. In inter-universal Teichmüller theory, one is interested in **dismantling** this complicated intertwining structure by considering the underlying additive and multiplicative monoids associated to a ring **separately**. In this context, the **ring structure**, as well as other structures such as étale fundamental groups that are sufficiently rigid as to allow the algorithmic reconstruction of the ring structure, are referred to as **arithmetic holomorphic structures**. By contrast, structures that arise from dismantling the complicated intertwining inherent in a ring structure are referred to as **mono-analytic** — a term which may be thought of as a sort of arithmetic analogue of the notion of an underlying *real analytic* structure in the context of complex holomorphic structures. From this point of view, the approach of Example 2.6.1, (ii), involving the **reciprocity** map of **class field theory** depends on the **arithmetic holomorphic structure** [i.e., the ring structure] of the field $\widetilde{K}$ in a quite essential and complicated way. By contrast, the **Kummer-theoretic** approach of Example 2.6.1, (iii), only depends on the **mono-analytic** structure constituted by the underlying **multiplicative monoid** of the field $\widetilde{K}$, together with the **cyclotomic rigidity isomorphism** λ. Thus, although λ depends on the ring structure of the field $\widetilde{K}$, the Kummer-theoretic approach of Example 2.6.1, (iii), has the *advantage*, from the point of view of dismantling the arithmetic holomorphic structure, of

> **isolating** the *dependence of* κ *on the arithmetic holomorphic structure* of the field $\widetilde{K}$ in the "compact form" constituted by the *cyclotomic rigidity isomorphism* λ.

§**2.8.** <u>**Remark on the usage of certain terminology**</u>**:** In the context of the discussion of §2.7, we remark that although terms such as *"link"*, *"Frobenius-like"*, *"étale-like"*, *"coric"*, *"mono-anabelian transport"*, *"Kummer-detachment"*, *"cyclotomic rigidity isomorphism"*, *"mono-analytic"*, and *"arithmetic holomorphic structure"* are

> **well-defined** in the various **specific contexts** in which they are applied, these terms do **not** admit **general definitions** that are applicable in *all contexts*.

In this sense, such terms are used in a way that is similar to the way in which terms such as *"underlying"* [cf., e.g., the "underlying topological space of a scheme", the "underlying real analytic manifold of a complex manifold"] or *"anabelian"* are typically used in mathematical discussions. The term *"multiradial"*, which will be discussed in §3, is also used in this way. In this context, we remark that one aspect that complicates the use of the terms *"Frobenius-like"* and *"étale-like"* is the sort of **curious process of evolution** that these terms underwent as the author progressed from writing [FrdI], [FrdII] in 2005 to writing [IUTchI] in 2008 and finally to writing [IUTchII] and [IUTchIII] during the years 2009 - 2010. This *"curious process of evolution"* may be summarized as follows:

($1^{\text{Fr-ét}}$) **Vague philosophical approach [i.e., "order-conscious" vs. "indifferent to order"]**: The *first stage* in this evolutionary process consists of the *vague philosophical characterizations* of the notion of *"Frobenius-like"* via the term *"order-conscious"* and of the notion of *"étale-like"* via the phrase *"indifferent to order"*. These vague characterizations were motivated by the situation surrounding the **monoids** [i.e., in the case of "Frobenius-like"] that appear in the theory of Frobenioids [cf. [FrdI], [FrdII]] and the situation surrounding the **Galois groups/arithmetic fundamental groups** [i.e., in the case of "étale-like"] that appear in the *base categories* ["$\mathcal{D}$"] of Frobenoids. This point of view is discussed in [FrdI], §I4, and is quoted and applied throughout [IUTchI].

($2^{\text{Fr-ét}}$) **Characterization in the context of the log-theta-lattice**: This point of view consists of the characterization of the notions of *"Frobenius-like"* and *"étale-like"* in the context of the **specific links**, i.e., the Θ- and $\mathfrak{log}$-links, that occur in the log-theta-lattice. This approach is developed throughout [IUTchII] [cf., especially, [IUTchII], Remark 3.6.2] and [IUTchIII] [cf., especially, [IUTchIII], Remark 1.5.4]. Related discussions may be found in [IUTchIV], Remarks 3.6.2, 3.6.3. At a *purely technical/notational level*, this approach may be understood as follows [cf. also the discussion in the final portion of §3.3, (vii), of the present paper]:

- **Frobenius-like** objects [or structures] are objects that, when embedded in the *log-theta-lattice*, are marked [via *left-hand superscripts*] by *lattice coordinates "(n, m)"* or, when not embedded in the *log-theta-lattice*, are marked [via *left-hand superscripts*] by *daggers "$\dagger$"/double daggers "$\ddagger$"/asterisks "$*$"*.
- **étale-like** objects [or structures] are objects that arise from [i.e., are often denoted as *"functions $(-)$"* of] the *$\mathcal{D}/\mathcal{D}^{\vdash}$-portions* of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters that appear, or, when embedded in the *log-theta-lattice*, are marked [via *left-hand superscripts*] by *vertically coric/bi-coric lattice coordinates "$(n, \circ)$"/"$(\circ, \circ)$"*.

($3^{\text{Fr-ét}}$) **Abstract link-theoretic approach**: This is the the approach taken in §2.7 of the present paper. In this approach, the notions of *"Frobenius-like"* and *"étale-like"* are **only defined** *in the context of a* **specific link**. This approach arose, in discussions involving the author and *Y. Hoshi*, as a sort of **abstraction/generalization** of the situation that occurs in [IUTchII], [IUTchIII] in the case of the *specific links*, i.e., the Θ- and $\mathfrak{log}$-links, that occur in the case of the *log-theta-lattice* [cf. ($2^{\text{Fr-ét}}$)].

Thus, of these three approaches ($1^{\text{Fr-ét}}$), ($2^{\text{Fr-ét}}$), ($3^{\text{Fr-ét}}$), the approach ($3^{\text{Fr-ét}}$) is, in some sense, *theoretically the most satisfying approach*, especially from the point of view of considering *possible generalizations* of the theory of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV]. On the other hand, from the point of view of the more restricted goal of *attaining a technically sound understanding* of the content of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV], the *purely technical/notational* approach discussed in ($2^{\text{Fr-ét}}$) is *quite sufficient*.

§**2.9.** <u>**Mono-anabelian transport and the Kodaira-Spencer morphism**</u>**:** The discussion of §2.6 and §2.7 be summarized as follows: In some sense, the central theme of inter-universal Teichmüller theory is the **computation** via **mono-anabelian transport** — in the *spirit of the discussion of Example 2.6.1, (iv), (b)* — of the **discrepancy** between **two** [systems of] **Kummer theories**, that is to say,

> of the sort of **indeterminacies** that one must admit in order to render *two systems of Kummer theories* **compatible** — i.e., relative to the various **gluings** constituted by some **link** [cf. the *Frobenius morphism* Φ_{η_X} in Example 2.6.1; the discussion of §2.7, (i)] between the two systems of Kummer theories — with **simultaneous execution**, e.g., when **one** of the two systems of Kummer theories [cf. the objects in the *lower right-hand corner* of the diagram of §2.6, (iii), and Fig. 2.1] is held **fixed**.

In this context, it is of interest to observe that this approach of computing degrees of ["positive mass"] **Frobenius-like** objects by *embedding* them into **rigid étale-like** ["zero mass"] **containers** [cf. the discussion of §2.7, (iii), (v)] is *formally similar* to the classical definition of the **Kodaira-Spencer morphism** associated to a *family of elliptic curves*: Indeed, suppose [relative to the terminology of [Semi], §0] that $S^{\log}$ is a *smooth log curve* over $\mathrm{Spec}(\mathbb{C})$ [equipped with the trivial log structure], and that $E^{\log} \to S^{\log}$ is a *stable log curve* of *type* $(1, 1)$ [i.e., in essence, a family of elliptic curves whose origin is regarded as a "marked point", and which is assumed to have *stable reduction* at the points of degeneration]. Write $\omega_{S^{\log}/\mathbb{C}}$ for the sheaf of relative logarithmic differentials of $S^{\log} \to \mathrm{Spec}(\mathbb{C})$; $(\mathcal{E}, \nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \omega_{S^{\log}/\mathbb{C}})$ for the rank two vector bundle with logarithmic connection on $S^{\log}$ determined by the *first logarithmic de Rham cohomology module* of $E^{\log} \to S^{\log}$ and the *logarithmic Gauss-Manin connection*; $\omega_E \subseteq \mathcal{E}$ for the *Hodge filtration* on $\mathcal{E}$; τ_E for the $\mathcal{O}_S$-dual of ω_E. Thus, we have a natural exact sequence $0 \to \omega_E \to \mathcal{E} \to \tau_E \to 0$. Then the *Kodaira-Spencer morphism* associated to the family $E^{\log} \to S^{\log}$ may be defined as the composite of morphisms in the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla_{\mathcal{E}}} & \mathcal{E} \otimes_{\mathcal{O}_S} \omega_{S^{\log}/\mathbb{C}} \\ \uparrow & & \downarrow \\ \omega_E & & \tau_E \otimes_{\mathcal{O}_S} \omega_{S^{\log}/\mathbb{C}} \end{array}$$

— where the left-hand vertical arrow is the natural inclusion, and the right-hand vertical arrow is the natural surjection. The three arrows in this diagram may be regarded as *corresponding to the three arrows in analogous positions* in the diagrams of Example 2.6.1, (iii), and Fig. 2.1. That is to say, this diagram may be understood as a *computation* of the degree $\deg(-)$ of the "positive mass" Frobenius-like object ω_E that yields an *inequality* $\deg(\omega_E) \ \leq \ \deg(\omega_{S^{\log}/\mathbb{C}}) - \deg(\omega_E)$, i.e.,

$$2 \cdot \deg(\omega_E) \ \leq \ \deg(\omega_{S^{\log}/\mathbb{C}})$$

via a **comparison** of **"alien copies"** of the "positive mass" **Frobenius-like** object ω_E that lie on opposite sides of the **"link"** constituted by a **deformation** of the **moduli/holomorphic structure** of the family of elliptic curves under consideration. This comparison is performed by relating these Frobenius-like objects ω_E [or its dual] on either side of the deformation by means of the "zero mass" **étale-like** object $(\mathcal{E}, \nabla_{\mathcal{E}})$, i.e., which may be thought of as a *local system* on the open subscheme $U_S \subseteq S$ of the underlying scheme S of $S^{\log}$ where the log structure of $S^{\log}$ is trivial. The tensor product with $\omega_{S^{\log}/\mathbb{C}}$ — and the resulting appearance of the bound $\deg(\omega_{S^{\log}/\mathbb{C}})$ in the above inequality — may be understood as the **indeterminacy** that one must admit in order to achieve this comparison.

§**2.10.** **Inter-universality: changes of universe as changes of coordinates:** One fundamental aspect of the *links* [cf. the discussion of §2.7, (i)] — namely, the **Θ-link** and **𝔩𝔬𝔤-link** — that occur in inter-universal Teichmüller theory is their **incompatibility** with the **ring structures** of the rings and schemes that appear in their domains and codomains. In particular, when one considers the result of *transporting* an **étale-like** structure such as a Galois group [or étale fundamental group] across such a link [cf. the discussion of §2.7, (iii)], one must **abandon** the interpretation of such a Galois group as a group of **automorphisms** of some **ring** [or field] structure [cf. [AbsTopIII], Remark 3.7.7, (i); [IUTchIV], Remarks 3.6.2, 3.6.3], i.e., one must regard such a Galois group as an **abstract topological group** that is **not** equipped with any of the "labelling structures" that arise from the relationship between the Galois group and various scheme-theoretic objects. It is precisely this state of affairs that results in

> the quite *central role* played in inter-universal Teichmüller theory by results in **[mono-]anabelian geometry**, i.e., by results concerned with *reconstructing* various scheme-theoretic structures from an *abstract topological group* that "just happens" to arise from scheme theory as a Galois group/étale fundamental group.

In this context, we remark that it is also this state of affairs that gave rise to the term **"inter-universal"**: That is to say, the notion of a *"universe"*, as well as the use of *multiple universes* within the discussion of a single set-up in arithmetic geometry, already occurs in the mathematics of the 1960's, i.e., in the mathematics of Galois categories and étale topoi associated to schemes. On the other hand, in this mathematics of the Grothendieck school, typically one only considers *relationships between universes* — i.e., between *labelling apparatuses for sets* — that are *induced by morphisms of schemes*, i.e., in essence by *ring homomorphisms*. The most typical example of this sort of situation is the *functor between Galois categories of étale coverings induced by a morphism of connected schemes*. By contrast, the **links** that occur in inter-universal Teichmüller theory are constructed by *partially dismantling the ring structures* of the rings in their domains and codomains [cf. the discussion of §2.7, (vii)], hence necessarily result in

> *much more complicated* **relationships between the universes** — i.e., between the *labelling apparatuses for sets* — that are adopted in the Galois categories that occur in the domains and codomains of these links, i.e., *relationships* that do **not respect** the various labelling apparatuses for sets that arise from correspondences between the Galois groups that appear and the respective **ring/scheme theories** that occur in the domains and codomains of the links.

That is to say, it is precisely this sort of situation that is referred to by the term **"inter-universal"**. Put another way,

> a **change of universe** may be thought of [cf. the discussion of §2.7, (i)] as a sort of **abstract/combinatorial/arithmetic** version of the classical notion of a **"change of coordinates"**.

In this context, it is perhaps of interest to observe that, from a purely classical point of view, the notion of a [physical] *"universe"* was typically visualized as a copy of *Euclidean three-space*. Thus, from this classical point of view,

> a *"change of universe"* literally corresponds to a "classical change of the *coodinate system* — i.e., the *labelling apparatus* — applied to label points in Euclidean three-space"!

Indeed, from an *even more elementary* point of view, perhaps the simplest example of the essential phenomenon under consideration here is the following **purely combinatorial phenomenon**: Consider the **string of symbols**

010

— i.e., where "0" and "1" are to be understood as formal symbols. Then, from the point of view of the length two *substring* 01 on the *left*, the digit "1" of this substring may be specified by means of its "coordinate relative to this substring", namely, as the *symbol to the far right* of the substring 01. In a similar vein, from the point of view of the length two *substring* 10 on the *right*, the digit "1" of this substring may be specified by means of its "coordinate relative to this substring", namely, as the *symbol to the far left* of the substring 10. On the other hand,

> **neither** of these specifications via **"substring-based coordinate systems"** is **meaningful** to the **opposite** length two substring; that is to say, only the solitary **abstract symbol "1"** is **simultaneously meaningful**, as a device for specifying the digit of interest, relative to **both** of the "substring-based coordinate systems".

Finally, in passing, we note that this discussion applies, albeit in perhaps a *somewhat trivial* way, to the *isomorphism of Galois groups* $\Psi_{\eta_X} : G_K \xrightarrow{\sim} G_K$ induced by the Frobenius morphism Φ_{η_X} in Example 2.6.1, (i): That is to say, from the point of view of classical ring theory, this isomorphism of Galois groups is easily seen to coincide with the *identity automorphism* of G_K. On the other hand, if one takes the point of view that elements of various subquotients of G_K are equipped with *labels* that arise from the isomorphisms ρ or κ of Example 2.6.1, (ii), (iii), i.e., from the *reciprocity* map of *class field theory* or *Kummer theory*, then one must regard such labelling apparatuses are being **incompatible** with the **Frobenius morphism** Φ_{η_X}. Thus, from this point of view, the isomorphism Φ_{η_X} must be regarded as a **"mysterious, indeterminate isomorphism"** [cf. the discussion of §2.7, (iii)].

§**2.11.** **The two underlying combinatorial dimensions of a ring:** Before proceeding, we pause to examine in more detail the **two underlying combinatorial dimensions** of a **ring** discussed in §2.7, (vii) [cf. also [AbsTopIII], §I3]. One way of expressing these two underlying combinatorial dimensions of a ring — i.e., constituted by **addition** and **multiplication** — is by means of *semi-direct product groups* such as

$$\mathbb{Z}_l \rtimes \mathbb{Z}_l^\times \quad \text{or} \quad \mathbb{F}_l \rtimes \mathbb{F}_l^\times$$

— where l is a prime number; $\mathbb{Z}_l$ denotes, by abuse of notation, the underlying additive profinite group of the ring "$\mathbb{Z}_l$" of l-adic integers; $\mathbb{Z}_l^\times$ denotes the multiplicative profinite group of invertible l-adic integers; $\mathbb{F}_l$, by abuse of notation, denotes the underlying additive group of the finite field "$\mathbb{F}_l$" of l elements; $\mathbb{F}_l^\times$ denotes the multiplicative group of the field $\mathbb{F}_l$; $\mathbb{Z}_l^\times$, $\mathbb{F}_l^\times$ act on $\mathbb{Z}_l$, $\mathbb{F}_l$ via the *ring structure* of $\mathbb{Z}_l$, $\mathbb{F}_l$. Here, we note that both [the rings] $\mathbb{Z}_l$ and $\mathbb{F}_l$ are closely related to the *fundamental ring* $\mathbb{Z}$. Indeed, $\mathbb{Z}$ may be regarded as a dense subring of $\mathbb{Z}_l$, while $\mathbb{F}_l$ may be regarded as a **"good finite discrete approximation"** of $\mathbb{Z}$ whenever l is *"large"* by comparison to the numbers of interest. Note, moreover, that if $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ is the **absolute Galois group** of a **mixed-characteristic local field** [i.e., **"MLF"**] k of residue characteristic p for which $\overline{k}$ is an *algebraic closure*, then the *maximal tame quotient* $G_k \twoheadrightarrow G_k^{\text{tm}}$ is isomorphic to some open subgroup of the closed subgroup of the direct product

$$\prod_{l' \neq p} \mathbb{Z}_{l'} \rtimes \mathbb{Z}_{l'}^{\times}$$

given by the inverse image via the quotient

$$\Big(\prod_{l' \neq p} \mathbb{Z}_{l'} \rtimes \mathbb{Z}_{l'}^{\times} \twoheadrightarrow \Big) \prod_{l' \neq p} \mathbb{Z}_{l'}^{\times}$$

of the closed subgroup topologically generated by the image of p [cf. [NSW], Proposition 7.5.1]. Thus, if we assume that $p \neq l$, then l may be thought of as one of the "l''s" in the last two displays. In particular, from a purely *cohomological point of view*, the *two combinatorial dimensions* "$\mathbb{Z}_l$" and "$\mathbb{Z}_l^{\times}$" of the semi-direct product group $\mathbb{Z}_l \rtimes \mathbb{Z}_l^{\times}$ — i.e., which correspond to the *additive* and *multiplicative* structures of the ring $\mathbb{Z}_l$ — may be thought of as corresponding directly to the *two l-cohomological dimensions* [cf. [NSW], Theorem 7.1.8, (i)] of the profinite group G_k^{tm} or, equivalently [since $l \neq p$], of the profinite group G_k. This suggests the point of view that the "restriction to l" should not be regarded as essential, i.e., that one should regard

> the **two underlying combinatorial dimensions** of the **ring** k as corresponding to the **two cohomological dimensions** of its **absolute Galois group** G_k,

and indeed, more generally, since the two cohomological dimensions of the absolute Galois G_F of a [say, for simplicity, totally imaginary] number field F [cf. [NSW], Proposition 8.3.17] may be thought of, via the well-known classical theory of the *Brauer group*, as *globalizations* [cf. [NSW], Corollary 8.1.16; [NSW], Theorem 8.1.17] of the two cohomological dimensions of the absolute Galois groups of its [say, for simplicity, *nonarchimedean*] *localizations*, that one should regard

> the **two underlying combinatorial dimensions** of a [totally imaginary] **number field** F as corresponding to the **two cohomological dimensions** of its **absolute Galois group** G_F.

Moreover, in the case of the local field k, the *two cohomological dimensions* of G_k may be thought of as arising [cf., e.g., the proof of [NSW], Theorem 7.1.8, (i)] from the *one cohomological dimension* of the *maximal unramified quotient* $G_k \twoheadrightarrow G_k^{\mathrm{unr}}$ ($\cong \widehat{\mathbb{Z}}$) and the *one cohomological dimension* of the *inertia subgroup* $I_k \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G_k \twoheadrightarrow G_k^{\mathrm{unr}})$. Since I_k and G_k^{unr} may be thought of as corresponding, via **local class field theory** [cf. [NSW], Theorem 7.2.3], or, alternatively [i.e., *"dually"* — cf. [NSW], Theorem 7.2.6], via **Kummer theory**, to the subquotients of the multiplicative group $k^{\times}$ associated to k given by the *unit group* $\mathcal{O}_k^{\times}$ and the *value group* $k^{\times}/\mathcal{O}_k^{\times}$ of k [together with the corresponding subquotients associated to the various subextensions of k in $\overline{k}$], we conclude that it is natural to regard

> the *two underlying combinatorial dimensions* of the *ring* k, or, alternatively, the *two cohomological dimensions* of its *absolute Galois group* G_k, as corresponding to the natural exact sequence
>
> $$1 \to \mathcal{O}_k^{\times} \to k^{\times} \to k^{\times}/\mathcal{O}_k^{\times} \to 1$$

— i.e., to the [*non-split*, i.e., at least when subject to the requirement of *functoriality* with respect to the operation of passing to finite extensions of k!] "decomposition" of $k^\times$ into its **unit group** $\mathcal{O}_k^\times$ and **value group** $k^\times/\mathcal{O}_k^\times$.

This situation is reminiscent of the [split!] decomposition of the multiplicative topological group $\mathbb{C}^\times$ associated to the field of **complex numbers**, i.e., which is equipped with a natural decomposition

$$\mathbb{C}^\times \stackrel{\sim}{\to} \mathbb{S}^1 \times \mathbb{R}_{>0}$$

as a direct product of its **unit group** $\mathbb{S}^1$ and **value group** $\mathbb{R}_{>0}$ [i.e., the multiplicative group of positive real numbers].

§2.12. **Mono-anabelian transport for mixed-characteristic local fields:** The discussion of the *two underlying combinatorial dimensions* of a *ring* in §2.11 — especially, in the case of an *MLF* "k"— leads naturally, from the point of view of the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, to consideration of the following examples, which may be thought of as *arithmetic analogues* of the discussion in Step 7 of §1.5 of the effect of **upper triangular** linear transformations and **rotations** on **"local masses"**. As one might expect from the discussion of §2.7, **Kummer theory** — i.e., applied to relate **Frobenius-like** structures to their **étale-like** counterparts — and **cyclotomic rigidity isomorphisms** play a central role in these examples. In the following examples, we use the notation of §2.11 for "k" and various objects related to k; also, we shall write

$$(\mathcal{O}_k^\times \subseteq) \qquad \mathcal{O}_k^\rhd \qquad (\subseteq k^\times)$$

for the *topological multiplicative monoid of nonzero integral elements* of k,

$$\boldsymbol{\mu}_k \subseteq \mathcal{O}_k^\rhd$$

for the *topological module of torsion elements* of $\mathcal{O}_k^\rhd$, and

$$\rho_k : k^\times \hookrightarrow (k^\times)^\wedge \stackrel{\sim}{\to} G_k^{\mathrm{ab}}$$

for the composite of the *embedding* of $k^\times$ into its profinite completion $(k^\times)^\wedge$ with the *natural isomorphism* [i.e., which arises from *local class field theory*] of $(k^\times)^\wedge$ with the *abelianization* G_k^{ab} of G_k. Here, we recall, from local class field theory, that ρ_k is *functorial* with respect to passage to *finite subextensions* of k in $\overline{k}$ and the *Verlagerung* homomorphism between abelianizations of open subgroups of G_k. Also, we recall [cf., e.g., [AbsAnab], Proposition 1.2.1, (iii), (iv)] that the *images* $\rho_k(\boldsymbol{\mu}_k) \subseteq \rho_k(\mathcal{O}_k^\rhd) \subseteq \rho_k(k^\times) \subseteq G_k^{\mathrm{ab}}$ of $\boldsymbol{\mu}_k$, $\mathcal{O}_k^\rhd$, and $k^\times$ via ρ_k may be **constructed group-theoretically** from the topological group G_k. The notation introduced so far for various objects related to k will also be applied to finite subextensions of k in $\overline{k}$, as well as [i.e., by passing to suitable *inductive limits*] to $\overline{k}$ itself. In particular, if we write

$$\boldsymbol{\mu}_{\overline{k}}(G_k) \subseteq \mathcal{O}_{\overline{k}}^\rhd(G_k) \subseteq \overline{k}^\times(G_k)$$

for the respective *inductive systems* [or, by abuse of notation, when there is no fear of confusion, *inductive limits*], relative to the *Verlagerung* homomorphism between abelianizations of open subgroups of G_k, of the [group-theoretically constructible!] *submonoids* $\rho_{k'}(\boldsymbol{\mu}_{k'}) \subseteq \rho_{k'}(\mathcal{O}^{\triangleright}_{k'}) \subseteq \rho_{k'}((k')^{\times}) \subseteq G^{\mathrm{ab}}_{k'}$ associated to the various open subgroups $G_{k'} \subseteq G_k$ [i.e., where k' ranges over the finite subextensions of k in $\overline{k}$], then the various $\rho_{k'}$ determine *natural isomorphisms*

$$\rho_{\boldsymbol{\mu}_{\overline{k}}} : \boldsymbol{\mu}_{\overline{k}} \xrightarrow{\sim} \boldsymbol{\mu}_{\overline{k}}(G_k), \qquad \rho_{\mathcal{O}^{\triangleright}_{\overline{k}}} : \mathcal{O}^{\triangleright}_{\overline{k}} \xrightarrow{\sim} \mathcal{O}^{\triangleright}_{\overline{k}}(G_k), \qquad \rho_{\overline{k}^{\times}} : \overline{k}^{\times} \xrightarrow{\sim} \overline{k}^{\times}(G_k)$$

of [multiplicative] G_k-monoids. In the following, we shall also use the notation $\boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\overline{k}})$ and $\boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\overline{k}}(G_k))$.

<u>Example 2.12.1</u>**: Nonarchimedean multiplicative monoids of local integers.**

(i) In the following, we wish to regard the pair "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" as an *abstract ind-topological monoid* "$\mathcal{O}^{\triangleright}_{\overline{k}}$" [i.e., inductive system of topological monoids] *equipped with a continuous action* by an *abstract topological group* "G_k". Thus, for instance, we may think of $\overline{k}^{\times}$ as the *groupification* $(\mathcal{O}^{\triangleright}_{\overline{k}})^{\mathrm{gp}}$ of the monoid $\mathcal{O}^{\triangleright}_{\overline{k}}$, of $\boldsymbol{\mu}_{\overline{k}}$ as the subgroup of torsion elements of monoid $\mathcal{O}^{\triangleright}_{\overline{k}}$, and of $\mathcal{O}^{\triangleright}_{k} \subseteq k^{\times}$ as the result of considering the G_k*-invariants* $(\mathcal{O}^{\triangleright}_{\overline{k}})^{G_k} \subseteq (\overline{k}^{\times})^{G_k}$ of the inclusion $\mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \overline{k}^{\times}$. Observe that, by considering the action of G_k on the various N*-th roots*, for N a positive integer, of elements of $k^{\times}$, we obtain a natural **Kummer map**

$$\kappa_k : k^{\times} \hookrightarrow H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}})$$

— which may be *composed* with the natural isomorphism $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ to obtain a *natural embedding*

$$\kappa^{\mathrm{Gal}}_k : k^{\times} \hookrightarrow H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k))$$

— where we note that the cohomology module in the codomain of this embedding may be **constructed group-theoretically** from the abstract topological group "G_k". On the other hand, it follows immediately from the definitions that κ_k may be **constructed functorially** from the abstract ind-topological monoid with continuous topological group action "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$".

(ii) In fact,

> $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ may also be **constructed functorially** from the abstract ind-topological monoid with continuous topological group action "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$".

Indeed, this follows *formally* from the fact that

> there exists a **canonical isomorphism** $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \boldsymbol{\mu}_{\overline{k}})$ that may be **constructed functorially** from this data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$".

— i.e., by applying this functorial construction to *both* the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" *and* the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}(G_k)$" and then observing that $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ may be *characterized* as the *unique isomorphism* $\boldsymbol{\mu}_{\overline{k}} \xrightarrow{\sim} \boldsymbol{\mu}_{\overline{k}}(G_k)$ that is *compatible* with the isomorphisms

$\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \boldsymbol{\mu}_{\overline{k}})$ and $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \boldsymbol{\mu}_{\overline{k}}(G_k))$. To *construct* this canonical isomorphism [cf., e.g., the proof of [AbsAnab], Proposition 1.2.1, (vii); the statement and proof of [FrdII], Theorem 2.4, (ii); the statement of [AbsTopIII], Corollary 1.10, (i), (a); the statement of [AbsTopIII], Proposition 3.2, (i), for more details], we first observe that since $\overline{k}^{\times}/\boldsymbol{\mu}_{\overline{k}}$ is a $\mathbb{Q}$-*vector space*, it follows that we have *natural isomorphisms* $H^2(G_k, \boldsymbol{\mu}_{\overline{k}}) \xrightarrow{\sim} H^2(G_k, \overline{k}^{\times})$, $H^2(I_k, \boldsymbol{\mu}_{\overline{k}}) \xrightarrow{\sim} H^2(I_k, \overline{k}^{\times})$. Since, moreover, the *inertia subgroup* $I_k \subseteq G_k$ is of *cohomological dimension* 1 [cf. the discussion of §2.11], we conclude that $H^2(I_k, \overline{k}^{\times}) \xrightarrow{\sim} H^2(I_k, \boldsymbol{\mu}_{\overline{k}}) = 0$. Next, let us recall that, by elementary Galois theory [i.e., *"Hilbert's Theorem 90"*], one knows that $H^1(I_k, \overline{k}^{\times}) = 0$. Thus, we conclude from the *Leray-Serre spectral sequence* associated to the extension $1 \to I_k \to G_k \to G_k^{\mathrm{unr}} \to 1$ that, if we write $k^{\mathrm{unr}} \subseteq \overline{k}$ for the subfield of I_k-invariants of $\overline{k}$, then we have a *natural isomorphism* $H^2(G_k, \overline{k}^{\times}) \xrightarrow{\sim} H^2(G_k^{\mathrm{unr}}, (k^{\mathrm{unr}})^{\times})$. On the other hand, the *valuation map* on $(k^{\mathrm{unr}})^{\times}$ determines an *isomorphism* $H^2(G_k^{\mathrm{unr}}, (k^{\mathrm{unr}})^{\times}) \xrightarrow{\sim} H^2(G_k^{\mathrm{unr}}, \mathbb{Z})$ [where again we apply *"Hilbert's Theorem 90"*, this time to the residue field of k]. Moreover, by applying the isomorphism determined by the *Frobenius element* $G_k^{\mathrm{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}$ [which is *group-theoretically constructible* — cf. [AbsAnab], Proposition 1.2.1, (iv)], together with the long exact sequence in Galois cohomology associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ [and the fact that $\mathbb{Q}$ is a $\mathbb{Q}$-*vector space*!], we obtain a *natural isomorphism* $H^2(G_k^{\mathrm{unr}}, \mathbb{Z}) \xrightarrow{\sim} H^1(G_k^{\mathrm{unr}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Thus, by taking the *inverse* of the *composite* of the various *natural isomorphisms* constructed so far [solely from the *data* "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$"!], we obtain the desired *canonical isomorphism* $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \boldsymbol{\mu}_{\overline{k}})$.

(iii) The *functorial construction* given in (ii) — i.e., via well-known elementary techniques involving **Brauer groups** of the sort that appear in **local class field theory** — of the **cyclotomic rigidity isomorphism** $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ from the *data* "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" is perhaps the *most fundamental case* — at least in the context of the arithmetic of *NF's* and *MLF's* — of the phenomenon of **cyclotomic rigidity**. One *formal consequence* of the discussion of (i), (ii) is the fact that the operation of passing from the *data* "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" to the *data* "$G_k$" is *fully faithful*, i.e., one has a *natural bijection*

$$\mathrm{Aut}(G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}) \quad \xrightarrow{\sim} \quad \mathrm{Aut}(G_k)$$

— where the first "Aut(−)" denotes automorphisms of the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" consisting of an *abstract ind-topological monoid with continuous topological group action*; the second "Aut(−)" denotes automorphisms of the data "G_k" consisting of an *abstract topological group* [cf. [AbsTopIII], Proposition 3.2, (iv)]. Indeed, *surjectivity* follows formally from the *functorial construction* of the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}(G_k)$" from the abstract topological group G_k [cf. the discussion at the beginning of the present §2.12]; *injectivity* follows formally from the fact that, as a consequence of the *cyclotomic rigidity* discussed in (ii), one has a *functorial construction* from the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" of the **embedding** $\kappa_k^{\mathrm{Gal}} : k^{\times} \hookrightarrow H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k))$ [in fact applied in the case where "k" is replaced by *arbitrary finite subextensions* of k in $\overline{k}$] into the **container** $H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k))$ [which may be constructed *solely* from the *abstract topological group* G_k!]. Note that this situation may also be understood in terms of the general framework of **mono-anabelian transport** discussed in §2.7,

(v) [cf. also Example 2.6.1, (iii), (iv)], by considering the *commutative diagram*

$$\begin{array}{ccc} H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k)) & \xrightarrow{\sim} & H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k)) \\ \uparrow \kappa_k^{\mathrm{Gal}}|_{\mathcal{O}_{\overline{k}}^{\rhd}} & \curvearrowright & \uparrow \kappa_k^{\mathrm{Gal}}|_{\mathcal{O}_{\overline{k}}^{\rhd}} \\ \mathcal{O}_{\overline{k}}^{\rhd} & \xrightarrow{\sim} & \mathcal{O}_{\overline{k}}^{\rhd} \end{array}$$

— where the horizontal arrows are induced by *some given automorphism* of the data "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\rhd}$"; the vertical arrows serve to embed the **Frobenius-like** data "$\mathcal{O}_{\overline{k}}^{\rhd}$" into the **étale-like container** $H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$. Finally, we observe that the *cyclotomic rigidity* discussed in (ii) may be understood, relative to the exact sequence

$$1 \to \mathcal{O}_{\overline{k}}^{\times} \to \overline{k}^{\times} \to \overline{k}^{\times}/\mathcal{O}_{\overline{k}}^{\times} \ (\cong \mathbb{Q}) \to 1$$

— which, as was discussed in the final portion of §2.11, may be thought of as corresponding to the *two underlying combinatorial dimensions* of the *ring* $\overline{k}$ — as revolving around the **rigidity** of the *two fundamental subquotients* $\boldsymbol{\mu}_{\overline{k}} \subseteq \overline{k}^{\times}$ and $\overline{k}^{\times} \twoheadrightarrow \overline{k}^{\times}/\mathcal{O}_{\overline{k}}^{\times}$ of $\overline{k}^{\times}$. When viewed in this light, the discussion of the present Example 2.12.1 may be thought of, relative to the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, as corresponding to the discussion of the effect on **"local masses"** of the **unipotent** linear transformations that appeared in the discussion of Step 7 of §1.5.

(iv) At this point, it is perhaps of interest to observe that there is an *alternative approach* to constructing the cyclotomic rigidity isomorphism $\rho_{\boldsymbol{\mu}_{\overline{k}}}$. That is to say, instead of reasoning as in (ii), one may reason as follows. First, we observe that, by applying the *functorial construction* of (ii) in the case of the data "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\rhd}(G_k)$", one obtains a *canonical isomorphism* $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$. Since the *cup product* in group cohomology, together with this canonical isomorphism, determines a *perfect duality* [cf. [NSW], Theorem 7.2.6], one thus obtains a *natural isomorphism* $G_k^{\mathrm{ab}} \xrightarrow{\sim} H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$. Write

$$\mathcal{O}_{\overline{k}}^{\rhd}(G_k)^{\mathrm{Kum}} \quad \subseteq \quad H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$$

for the image via this natural isomorphism of $\mathcal{O}_k^{\rhd}(G_k) \subseteq G_k^{\mathrm{ab}}$ [i.e., the submodule of G_k-invariants of $\mathcal{O}_{\overline{k}}^{\rhd}(G_k)$]. Thus, $\mathcal{O}_k^{\rhd}(G_k)^{\mathrm{Kum}}$ may be *constructed group-theoretically* from the *abstract topological group* G_k. Next, let us recall the elementary fact that, relative to the natural inclusion $\mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, we have an equality

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

— where $\mathbb{Q}_{>0} \subseteq \mathbb{Q}$ denotes the multiplicative monoid of positive rational numbers. Now let us observe that it follows formally from this elementary fact — for instance, by considering the *quotient* $\mathcal{O}_k^{\rhd}(G_k) \twoheadrightarrow \mathcal{O}_k^{\rhd}(G_k)/\mathcal{O}_k^{\times}(G_k)$ ($\cong \mathbb{N}$) — that the *only element* $\in \widehat{\mathbb{Z}}^{\times}$ that, relative to the *natural action* of $\widehat{\mathbb{Z}}^{\times}$ on $H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$ [i.e., induced by the natural action of $\widehat{\mathbb{Z}}^\times$ on $\boldsymbol{\mu}_{\overline{k}}(G_k)$], *preserves* the submonoid $\mathcal{O}_{\overline{k}}^{\triangleright}(G_k)$ is the *identity element* $1 \in \widehat{\mathbb{Z}}^\times$. In particular, it follows that

> the cyclotomic rigidity isomorphism $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ may be *characterized* as the *unique isomorphism* $\boldsymbol{\mu}_{\overline{k}} \xrightarrow{\sim} \boldsymbol{\mu}_{\overline{k}}(G_k)$ that is *compatible* with the submonoids $\kappa_k(\mathcal{O}_{\overline{k}}^{\triangleright}) \subseteq H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}})$ and $\mathcal{O}_{\overline{k}}^{\triangleright}(G_k)^{\mathrm{Kum}} \subseteq H^1(G_k, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}}(G_k))$.

This characterization thus yields an *alternative approach* to the characterization of the cyclotomic rigidity isomorphism $\rho_{\boldsymbol{\mu}_{\overline{k}}}$ given in (ii) [cf. the discussion of [IUTchIII], Remark 2.3.3, (viii)]. On the other hand, there is a **fundamental difference** between this alternative approach and the approach of (ii): Indeed, one verifies immediately that the approach of (ii) is **compatible** with the **profinite topology** of G_k in the sense that the construction of (ii) may be formulated as the result of applying a *suitable limit operation* to *"finite versions"* of this construction of (ii), i.e., versions in which "G_k" is replaced by the *quotients* of "G_k" by *sufficiently small normal open subgroups* of "G_k", and "$\mathcal{O}_{\overline{k}}^{\triangleright}$" is replaced by the *submonoids of invariants* with respect to such normal open subgroups. By contrast, the alternative approach just discussed is **fundamentally incompatible** with the profinite topology of G_k in the sense that the *crucial fact* $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$ — which may be thought of as a sort of **discreteness** property [cf. the discussion of [IUTchIII], Remark 3.12.1, (iii); [IUTchIV], Remark 2.3.3, (ii)] — may only be applied at the level of the *full profinite group* G_k [i.e., at the level of *Kummer classes* with coefficients in some copy of $\widehat{\mathbb{Z}}(1)$], *not* at the level of finite quotients of G_k [i.e., at the level of *Kummer classes* with coefficients in some *finite quotient* of some copy of $\widehat{\mathbb{Z}}(1)$]. Thus, in summary, although this alternative approach has the **disadvantage** of being **incompatible** with the **profinite topology** of G_k, various versions of this approach — i.e., involving constructions that depend, in an essential way, on the *crucial fact* $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$ — will, nevertheless, play an *important role* in inter-universal Teichmüller theory [cf. the discussion of Example 2.13.1 below].

<u>Example 2.12.2</u>: **Frobenius morphisms on nonarchimedean multiplicative monoids of local integers.**

(i) One way to gain a further appreciation of the *cyclotomic rigidity* phenomenon discussed in Example 2.12.1 is to consider the pair "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$", which again we regard as consisting of an *abstract ind-topological monoid* "$\mathcal{O}_{\overline{k}}^{\times}$" [i.e., inductive system of topological monoids] *equipped with a continuous action* by an *abstract topological group* "G_k". Since $\mathcal{O}_{\overline{k}}^{\times}$ may be thought of as an inductive system/limit of *profinite abelian groups*, it follows immediately that there is a *natural G_k-equivariant action* of $\widehat{\mathbb{Z}}^\times$ on the data "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$". Moreover, if α is an *arbitrary automorphism* of this data "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$" [i.e., regarded as an abstract ind-topological monoid equipped with a continuous action by an abstract topological group], then although it is *not* necessarily the case that α is *compatible* with the *cyclotomic rigidity isomorphism* $\rho_{\boldsymbol{\mu}_{\overline{k}}} : \boldsymbol{\mu}_{\overline{k}} \xrightarrow{\sim} \boldsymbol{\mu}_{\overline{k}}(G_k)$, one verifies immediately [from the fact that, as an abstract abelian group, $\boldsymbol{\mu}_{\overline{k}} \cong \mathbb{Q}/\mathbb{Z}$, together with the elementary fact that $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^\times$] that there always exists a *unique element* $\lambda \in \widehat{\mathbb{Z}}^\times$ such that the automorphism $\lambda \cdot \alpha$ of the data "$G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$" is *compatible* with $\rho_{\boldsymbol{\mu}_{\overline{k}}}$. Thus, by arguing as in Example 2.12.1, (iii), one concludes that one has a *natural bijection*

$$\mathrm{Aut}(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}}) \quad \stackrel{\sim}{\to} \quad \widehat{\mathbb{Z}}^{\times} \times \mathrm{Aut}(G_k)$$

— where the first "Aut($-$)" denotes automorphisms of the data "$G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$" consisting of an *abstract ind-topological monoid with continuous topological group action*; the second "Aut($-$)" denotes automorphisms of the data "G_k" consisting of an *abstract topological group* [cf. [AbsTopIII], Proposition 3.3, (ii); [FrdII], Remark 2.4.2]. Just as in the case of Example 2.12.1, (iii), this situation may also be understood in terms of the general framework of **mono-anabelian transport** discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], by considering the *commutative diagram*

$$\begin{array}{ccc} H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k)) & \stackrel{\sim}{\longrightarrow} & H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k)) \\ \uparrow \kappa^{\mathrm{Gal}}_k|_{\mathcal{O}^{\times}_k} & \curvearrowright ? & \uparrow \kappa^{\mathrm{Gal}}_k|_{\mathcal{O}^{\times}_k} \\ \mathcal{O}^{\times}_k & \stackrel{\sim}{\longrightarrow} & \mathcal{O}^{\times}_k \end{array}$$

— where the horizontal arrows are induced by *some given automorphism* of the data "$G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$"; the vertical arrows serve to embed the **Frobenius-like** data "$\mathcal{O}^{\times}_k$" into the **étale-like container** $H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k))$; the diagram *commutes* [cf. "$\curvearrowright$?"] *up to the action of a suitable element* $\in \widehat{\mathbb{Z}}^{\times}$.

(ii) Let $\pi_k \in \mathcal{O}^{\triangleright}_k$ be a *uniformizer* of $\mathcal{O}_k$. Then one sort of intermediate type of data between the data "$G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$" considered in (i) above and the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" considered in Example 2.12.1 is the data "$G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}} \cdot \mathcal{O}^{\triangleright}_k$ ($\subseteq \mathcal{O}^{\triangleright}_{\overline{k}}$)", which again we regard as consisting of an *abstract ind-topological monoid* "$\mathcal{O}^{\times}_{\overline{k}}$" [i.e., inductive system of topological monoids] *equipped with a continuous action* by an *abstract topological group* "G_k". Here, we observe that $\mathcal{O}^{\times}_{\overline{k}} \cdot \mathcal{O}^{\triangleright}_k = \mathcal{O}^{\times}_{\overline{k}} \cdot \pi^{\mathbb{N}}_k$. Let $\mathbb{Z} \ni N \geq 2$, $\alpha \in \mathrm{Aut}(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}})$. Then observe that N, α determine — i.e., in the spirit of the discussion of §2.4 — a sort of **Frobenius morphism** $\phi_{N,\alpha}$

$$\begin{array}{ccc} \left(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}} \cdot \mathcal{O}^{\triangleright}_k\right) & \to & \left(G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}} \cdot \mathcal{O}^{\triangleright}_k\right) \\ \pi_k & \mapsto & \pi^N_k \end{array}$$

that restricts to α on the data "$G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$". From the point of view of the general framework of **mono-anabelian transport** discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], this sort of Frobenius morphism $\phi_{N,\alpha}$ induces a *commutative diagram*

$$\begin{array}{ccc} H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k)) & \stackrel{\sim}{\longrightarrow} & H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k)) \\ \uparrow \kappa^{\mathrm{Gal}}_k|_{\mathcal{O}^{\triangleright}_k} & \curvearrowright ? & \uparrow \kappa^{\mathrm{Gal}}_k|_{\mathcal{O}^{\triangleright}_k} \\ \mathcal{O}^{\triangleright}_k & \longrightarrow & \mathcal{O}^{\triangleright}_k \end{array}$$

— where the horizontal arrows are induced by $\phi_{N,\alpha}$; the vertical arrows serve to embed the **Frobenius-like** data "$\mathcal{O}^{\triangleright}_k$" into the **étale-like container** $H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k))$;

the diagram *commutes* [cf. "$\curvearrowright$?"] *up to the action of a suitable element* $\in \widehat{\mathbb{Z}}^\times$ *on* $\mathcal{O}_k^\times \subseteq \mathcal{O}_k^\rhd$ *and a suitable element* $\in \mathbb{N}$ [namely, $N \in \mathbb{N}$] *on* $\pi_k^{\mathbb{N}}$. Finally, we observe that the **diagonal** nature of the action of $\phi_{N,\alpha}$ on the **unit group** $\mathcal{O}_{\overline{k}}^\times$ [via α] and the **value group** $\pi_k^{\mathbb{Z}}$ [by raising to the N-th power] portions of the ind-topological monoid $\mathcal{O}_{\overline{k}}^\times \cdot \mathcal{O}_k^\rhd$ may be thought of, relative to the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, as corresponding to the discussion of the effect on **"local masses"** of the **toral dilations** that appeared in the discussion of Step 7 of §1.5.

__Example 2.12.3__: **Nonarchimedean logarithms.**

(i) The discussion of various simple cases of *mono-anabelian transport* in Examples 2.12.1, (iii); 2.12.2, (i), (ii), concentrated on the *Kummer-theoretic aspects*, i.e., in effect, on the *Kummer-detachment indeterminacies* [cf. §2.7, (vi)], or lack thereof, of the examples considered. On the other hand, another fundamental aspect of these examples [cf. the *natural bijections* of Examples 2.12.1, (iii); 2.12.2, (i)] is the **étale-transport indeterminacies** [cf. §2.7, (vi)] that occur as a result of the well-known

> **existence** of elements $\in \mathrm{Aut}(G_k)$ that do **not** preserve the **ring structure** on [the union with $\{0\}$ of] $\mathcal{O}_{\overline{k}}^\rhd(G_k)$

— cf. [NSW], the Closing Remark preceding Theorem 12.2.7. By contrast, if X is a **hyperbolic curve of strictly Belyi type** [cf. [AbsTopII], Definition 3.5] over k, and we write Π_X for the *étale fundamental group* of X [for a suitable choice of basepoint], then it follows from the theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9; [AbsTopIII], Remark 1.9.2; [AbsTopIII], Corollary 1.10], that

> if one *regards* G_k as a **quotient** $\Pi_X \twoheadrightarrow G_k$ of Π_X, then there exists a **functorial algorithm** for reconstructing this quotient $\Pi \twoheadrightarrow G_k$ of Π_X, together with the **ring structure** on [the union with $\{0\}$ of] $\mathcal{O}_{\overline{k}}^\rhd(G_k)$, from the *abstract topological group* Π_X.

Here, we recall that [it follows immediately from the definitions that] any connected finite étale covering of a **once-punctured elliptic curve** [i.e., an elliptic curve minus the origin] over k that is **defined over an NF** is necessarily *of strictly Belyi type*.

(ii) Write $(\mathcal{O}_{\overline{k}}^\times)^{\mathrm{pf}}$ for the *perfection* [cf., e.g., [FrdI], §0] of the ind-topological monoid $\mathcal{O}_{\overline{k}}^\times$. Thus, it follows immediately from the elementary theory of p-adic fields [cf., e.g., [Kobl], Chapter IV, §1, §2] that the ***p*-adic logarithm** determines a *G_k-equivariant bijection*

$$\log_{\overline{k}} : (\mathcal{O}_{\overline{k}}^\times)^{\mathrm{pf}} \quad \xrightarrow{\sim} \quad \overline{k}$$

with respect to which the operation of **multiplication**, which we shall often denote by the notation "⊠", in the domain corresponds to the operation of **addition**, which we shall often denote by the notation "⊞", in the codomain. This bijection fits into a *diagram*

$$\ldots \quad \mathcal{O}_{\overline{k}}^\rhd \quad \supseteq \quad \mathcal{O}_{\overline{k}}^\times \quad \twoheadrightarrow \quad (\mathcal{O}_{\overline{k}}^\times)^{\mathrm{pf}} \quad \xrightarrow{\sim} \quad \overline{k} \quad \supseteq \quad \mathcal{O}_{\overline{k}}^\rhd \quad \ldots$$

— where the "…" on the left and right denote the result of *juxtaposing* copies of the portion of the diagram "from $\mathcal{O}^{\triangleright}_{\overline{k}}$ to $\mathcal{O}^{\triangleright}_{\overline{k}}$", i.e., copies that are *glued together* along initial/final instances of "$\mathcal{O}^{\triangleright}_{\overline{k}}$". Here, we observe that the various objects that appear in this diagram may be regarded as being equipped with a **natural action** of Π_X [for X as in (i)], which acts via the natural quotient $\Pi_X \twoheadrightarrow G_k$. To keep the notation simple, we shall denote the portion of the diagram "from $\mathcal{O}^{\triangleright}_{\overline{k}}$ to $\mathcal{O}^{\triangleright}_{\overline{k}}$" by means of the notation $\mathfrak{log} : \mathcal{O}^{\triangleright}_{\overline{k}} \to \mathcal{O}^{\triangleright}_{\overline{k}}$. Thus, the diagram of the above display may be written

$$
\begin{array}{ccccccccc}
\dots & \stackrel{\sim}{\to} & \Pi_X & \stackrel{\sim}{\to} & \Pi_X & \stackrel{\sim}{\to} & \Pi_X & \stackrel{\sim}{\to} & \dots \\
 & & \curvearrowright & & \curvearrowright & & \curvearrowright & & \\
\dots & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{O}^{\triangleright}_{\overline{k}} & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{O}^{\triangleright}_{\overline{k}} & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{O}^{\triangleright}_{\overline{k}} & \stackrel{\mathfrak{log}}{\longrightarrow} & \dots
\end{array}
$$

— i.e., regarded as a sequence of *iterates of "$\mathfrak{log}$"*. Here, since the operation "$\mathfrak{log}$" [i.e., which, in effect, converts "$\boxtimes$" into "$\boxplus$"] is **incompatible** with the **ring structures** on [the union with $\{0\}$ of] the copies of $\mathcal{O}^{\triangleright}_{\overline{k}}$ in the domain and codomain of "$\mathfrak{log}$", we observe — in accordance with the discussion of §2.10! — that it is natural to regard the various copies of $\mathcal{O}^{\triangleright}_{\overline{k}}$ as being equipped with *distinct labels* and the *isomorphisms* " $\stackrel{\sim}{\to}$ " between different copies of Π_X as being **indeterminate isomorphisms** between *distinct abstract topological groups*. Such diagrams are studied in detail in [AbsTopIII], and, moreover, form the *fundamental model* for the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory [cf. §3.3, (ii), (vi), below], which is studied in detail in [IUTchIII].

(iii) It follows from the **mono-anabelian** theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9; [AbsTopIII], Corollary 1.10], that, if we regard G_k as a *quotient* of Π_X, then the *image*, which we denote by $\mathcal{O}^{\triangleright}_{k}(\Pi_X)$ $(\subseteq H^1(G_k, \boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}}(G_k)))$, of $\mathcal{O}^{\triangleright}_{k}$ via $\kappa^{\mathrm{Gal}}_{k}$ [cf. Example 2.12.1, (iii)] may be *reconstructed* — i.e., as a *topological monoid* equipped with a *ring structure* [on its union with $\{0\}$] — from the *abstract topological group* Π_X. By applying this construction to *arbitrary open subgroups* of G_k and passing to inductive systems/limits, we thus obtain an *ind-topological monoid* $\mathcal{O}^{\triangleright}_{\overline{k}}(\Pi_X)$ equipped with a *natural continuous action by* Π_X and a *ring structure* [on its union with $\{0\}$]. Thus, from the point of view of the general framework of **mono-anabelian transport** discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], we obtain a *diagram*

$$
\begin{array}{ccccc}
 & \mathcal{O}^{\triangleright}_{\overline{k}}(\Pi_X) & \stackrel{\sim}{\longrightarrow} & \mathcal{O}^{\triangleright}_{\overline{k}}(\Pi_X) & \\
\dots & \uparrow \scriptstyle{\mathrm{Kum}} & \curvearrowright ? & \uparrow \scriptstyle{\mathrm{Kum}} & \dots \\
 & \mathcal{O}^{\triangleright}_{\overline{k}} & \stackrel{\mathfrak{log}}{\longrightarrow} & \mathcal{O}^{\triangleright}_{\overline{k}} &
\end{array}
$$

— where the upper horizontal arrow is induced by some **indeterminate isomorphism** $\Pi_X \stackrel{\sim}{\to} \Pi_X$ [cf. the discussion of §2.10]; the lower horizontal arrow is the operation "$\mathfrak{log}$" discussed in (ii); the vertical arrows are the **"Kummer isomorphisms"** determined by the various "$\kappa^{\mathrm{Gal}}_{k}$" associated to open subgroups of G_k; the "…" denote iterates of the square surrounding the "$\curvearrowright$?". Thus, the vertical arrows of this diagram relate the various copies of **Frobenius-like** data $\mathcal{O}^{\triangleright}_{\overline{k}}$ to the various copies of **étale-like** data $\mathcal{O}^{\triangleright}_{\overline{k}}(\Pi_X)$, which are **coric** [cf. the discussion of §2.7, (iv)] with respect to the **"link"** constituted by the operation $\mathfrak{log}$.

(iv) The diagram of (iii) is, of course, **far from being commutative** [cf. the notation "$\curvearrowright$?"], i.e., at least at the level of *images of elements* via the various composites of arrows in the diagram. On the other hand, if, instead of considering such images of elements via composites of arrows in the diagram, one considers **regions** [i.e., subsets] of [the union with $\{0\}$ of the *groupification* of] $\mathcal{O}^{\triangleright}_{k}$ $(\subseteq \mathcal{O}^{\triangleright}_{\overline{k}})$ or $\mathcal{O}^{\triangleright}_{k}(\Pi_X)$ $(\subseteq \mathcal{O}^{\triangleright}_{\overline{k}}(\Pi_X))$, then one verifies easily that the following *observation* holds:

> Write
> $$\mathcal{I} \stackrel{\mathrm{def}}{=} (2p)^{-1} \cdot \log_{\overline{k}}(\mathcal{O}^{\times}_{k}) \subseteq k = \{0\} \cup (\mathcal{O}^{\triangleright}_{k})^{\mathrm{gp}}$$
> and $\mathcal{I}(\Pi_X) \subseteq \{0\} \cup \mathcal{O}^{\triangleright}_{k}(\Pi_X)^{\mathrm{gp}}$ for the corresponding subset of the union with $\{0\}$ of the groupification of $\mathcal{O}^{\triangleright}_{k}(\Pi_X)$. Then we have **inclusions of "regions"**
> $$\mathcal{O}^{\triangleright}_{k} \subseteq \mathcal{I} \supseteq \log_{\overline{k}}(\mathcal{O}^{\times}_{k})$$
> within $\mathcal{I}$, as well as corresponding inclusions for $\mathcal{I}(\Pi_X)$.

The compact "region" $\mathcal{I}$, which is referred to as the **log-shell**, plays an important role in inter-universal Teichmüller theory. Note that one has both **Frobenius-like** [i.e., $\mathcal{I}$] and **étale-like** [i.e., $\mathcal{I}(\Pi_X)$] versions of the log-shell. Here, we observe that, from the point of view of the discussion of **arithmetic holomorphic structures** in §2.7, (vii), both of these versions are **holomorphic** in the sense that they depend, at least in an *a priori* sense, on "$\log_{\overline{k}}$", i.e., which is defined in terms of a *power series* that only makes sense if one is equipped with *both* "$\boxplus$" *and* "$\boxtimes$" [i.e., both the additive and multiplicative structures of the ring $\overline{k}$]. On the other hand, if one writes
$$\text{“}\mathcal{O}^{\times\boldsymbol{\mu}}\text{”}$$
for the *quotient* of "$\mathcal{O}^{\times}$" by its *torsion subgroup* [i.e., by the roots of unity], then $\log_{\overline{k}}$ determines *natural bijections of topological modules*
$$\mathcal{O}^{\times\boldsymbol{\mu}}_{k} \otimes \mathbb{Q} \stackrel{\sim}{\to} \mathcal{I} \otimes \mathbb{Q}, \quad \mathcal{O}^{\times\boldsymbol{\mu}}_{k}(\Pi_X) \otimes \mathbb{Q} \stackrel{\sim}{\to} \mathcal{I}(\Pi_X) \otimes \mathbb{Q}$$
— i.e., within which the lattices $\mathcal{O}^{\times\boldsymbol{\mu}}_{k} \otimes (2p)^{-1}$, $\mathcal{O}^{\times\boldsymbol{\mu}}_{k}(\Pi_X) \otimes (2p)^{-1}$ correspond, respectively, to $\mathcal{I}$, $\mathcal{I}(\Pi_X)$. In particular, by applying these *bijections*,

> we may think of the *topological modules* $\mathcal{I}$, $\mathcal{I}(\Pi_X)$ as objects constructed from the *topological modules* $\mathcal{O}^{\times\boldsymbol{\mu}}_{k}$, $\mathcal{O}^{\times\boldsymbol{\mu}}_{k}(G_k)$ [cf. the notation introduced at the beginning of the present §2.12; the notation of Example 2.12.1, (iv)], i.e., as objects constructed from *mono-analytic* structures [cf. the discussion of §2.7, (vii)].

That is to say, in addition to the **holomorphic Frobenius-like** and **holomorphic étale-like** versions of the *log-shell* discussed above, we also have **mono-analytic Frobenius-like** and **mono-analytic étale-like** versions of the **log-shell**. All four of these versions of the log-shell play an important role in inter-universal Teichmüller theory [cf. the discussion of §3.6, (iv), below; [IUTchIII], Definition 1.1, (i), (iv); [IUTchIII], Proposition 1.2, (v), (vi), (viii), (ix), (x); [IUTchIII], Remark 3.12.2, (iv), (v)]. Returning to the issue of the *noncommutativity* [i.e., "$\curvearrowright$?"] of the *diagram* of (iii), we observe the following:

> the *inclusions of "regions"* discussed above may be *interpreted* as asserting that the *holomorphic étale-like log shell* $\mathcal{I}(\Pi_X)$ serves as a **container** for [i.e., as a "region" that contains] the **images** — i.e., of $\mathcal{O}^{\triangleright}_{\overline{k}}$, $\mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}}$, or, in the case of multiple iterates of $\mathfrak{log}$, *even smaller* subsets of $\mathcal{O}^{\times}_{\overline{k}}$ — via **all possible composites** of arrows of the *diagram* of (iii) [including the "..." on the left- and right-hand sides of the diagram!].

This property of the log-shell is referred to as **upper semi-commutativity** [cf. [IUTchIII], Remark 1.2.2, (i), (iii)]. Thus, this property of upper semi-commutativity constitutes a sort of **Kummer-detachment indeterminacy** [cf. the discussion of §2.7, (vi)] and may be regarded as an *answer* to the question of **computing** the **discrepancy** between the **two Kummer theories** in the domain and codomain of the **link** "$\mathfrak{log}$" [cf. the discussion at the beginning of §2.9]. Another important *answer*, in the context of inter-universal Teichmüller theory, to this computational question is given by the theory of **log-volumes** [i.e., where we use the term *log-volume* to refer to the *natural logarithm* of the *volume* $\in \mathbb{R}_{>0}$ of a region]:

> There is a *natural definition* of the notion of the *log-volume* $\in \mathbb{R}$ of a *region* [i.e., compact open subset] of $k = \{0\} \cup (\mathcal{O}^{\triangleright}_{k})^{\mathrm{gp}}$, which is *normalized* so that the log-volume of $\mathcal{O}_k$ is 0, while the log-volume of $p \cdot \mathcal{O}_k$ is $-\log(p)$. This log-volume is **compatible** [in the evident sense] with passage between the *four versions* of *log-shells* discussed above, as well as with $\mathfrak{log}$ in the sense that it assumes the *same value* $\in \mathbb{R}$ on regions that are mapped *bijectively* to one another via $\log_{\overline{k}}$ [cf. [AbsTopIII], Proposition 5.7, (i); [IUTchIII], Proposition 1.2, (iii); [IUTchIII], Proposition 3.9, (i), (ii), (iv)].

These properties of **upper semi-commutativity** and **log-volume compatibility** will be sufficient for the purposes of inter-universal Teichmüller theory.

(v) Finally, we observe that since the operation $\mathfrak{log}$ — which maps

$$\boxtimes \rightsquigarrow \boxplus$$

and relates **unit groups** [cf. $(\mathcal{O}^{\times}_{\overline{k}})^{\mathrm{pf}}$] to **value groups** [i.e., nonzero non-units of $\overline{k}$] — may be thought of as an operation that **"juggles"**, or **"rotates"**, the **two underlying combinatorial dimensions** [cf. the discussion of §2.11] of the ring $\overline{k}$ [cf. [AbsTopIII], §I3], one may think of this operation $\mathfrak{log}$, relative to the analogy discussed in §2.2, §2.3, §2.4, §2.5, and §2.7 with the classical theory of §1.4 and §1.5, as corresponding to the discussion of the effect on **"local masses"** of the **rotations** that appeared in the discussion of Step 7 of §1.5.

§**2.13.** **Mono-anabelian transport for monoids of rational functions:** Let k be *either* an **MLF** *or* an **NF**; X a **hyperbolic curve of strictly Belyi type** [cf. [AbsTopII], Definition 3.5] over k; $\overline{K}_X$ an *algebraic closure* of the *function field* K_X of X; $\overline{k} \subseteq \overline{K}_X$ the *algebraic closure* of k determined by $\overline{K}_X$. Write $\boldsymbol{\mu}_{\overline{k}} \subseteq \overline{k}$ for the group of *roots of unity* of $\overline{k}$; $\boldsymbol{\mu}^{\widehat{\mathbb{Z}}}_{\overline{k}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{\mu}_{\overline{k}})$; $G_X \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}_X/K_X)$;

$G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$; $G_X \twoheadrightarrow \Pi_X$ for the quotient of G_X determined by the maximal subextension of $\overline{K}_X$ that is *unramified* over X [so Π_X may be thought of, for a suitable choice of basepoint, as the *étale fundamental group* of X]. Thus, when k is an MLF, X and Π_X are as in Example 2.12.3, (i). Here, for simplicity, we assume further that X is *of genus* ≥ 1, and write $\Delta_X \stackrel{\text{def}}{=} \text{Ker}(\Pi_X \twoheadrightarrow G_k)$; X^{cp} for the natural [smooth, proper] *compactification* of X; $\Delta_X \twoheadrightarrow \Delta_X^{\text{cp}}$ for the quotient of Δ_X by the *cuspidal inertia groups* of Δ_X [so Δ_X^{cp} may be naturally identified with the étale fundamental group, for a suitable basepoint, of $X^{\text{cp}} \times_k \overline{k}$];

$$M_X \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathbb{Z}}}(H^2(\Delta_X^{\text{cp}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$$

[so M_X ($\cong \widehat{\mathbb{Z}}$) is a *cyclotome* naturally associated to Δ_X — cf. [AbsTopIII], Proposition 1.4, (ii)]. Here, we recall that the quotient $\Delta_X \twoheadrightarrow \Delta_X^{\text{cp}}$, hence also the cyclotome M_X, may be constructed by means of a *purely group-theoretic algorithm* from the *abstract topological group* Π_X [cf. [AbsTopI], Lemma 4.5, (v); [IUTchI], Remark 1.2.2, (ii)]. Now *observe* that

> the **Frobenius-like** data that appears in the various examples [i.e., Examples 2.12.1, 2.12.2, 2.12.3] of **mono-anabelian transport** discussed in §2.12 only involve the **two underlying combinatorial dimensions** of [various portions of] the ind-topological monoid "$\mathcal{O}^{\rhd}_{\overline{k}}$" of these examples.

That is to say, although, for instance, the **étale-like** data [i.e., "Π_X"] that appears in Example 2.12.3 involves the relative **geometric dimension** of X over k [i.e., in the case where k is an MLF], the Frobenius-like data [i.e., "$\mathcal{O}^{\rhd}_{\overline{k}}$"] that appears in these examples does **not** involve this geometric dimension of X over k. On the other hand, in inter-universal Teichmüller theory, it will be of *crucial importance* [cf. the discussion of §3.4, §3.6, below; [IUTchIII], Remark 2.3.3] to consider such Frobenius-like data that involves the geometric dimension of X over k — i.e., in more concrete terms, to consider **nonconstant rational functions** on X — together with various **evaluation** operations that arise by evaluating such functions on various **"special points"** of X [or coverings of X]. In fact, the fundamental importance of such evaluation operations may also be seen in the discussion of §2.14, (i), (ii), (iii), below. In the remainder of the present §2.13, we discuss what is perhaps the *most fundamental example* of **cyclotomic rigidity** and **mono-anabelian transport** for such geometric functions.

<u>Example 2.13.1</u>: Monoids of rational functions.

(i) In the following, we assume for simplicity that the field k is an **NF**. Recall that consideration of the *first Chern class* of a line bundle of degree 1 on X^{cp} yields a *natural isomorphism*

$$\lambda : \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}} \stackrel{\sim}{\to} M_X$$

— cf., e.g., Example 2.6.1, (iii); [the evident NF version of] [Cusp], Proposition 1.2, (ii). Next, observe that by considering the action of G_X on the various *N-th roots*, for N a positive integer, of elements of $K_X^{\times}$ ($\stackrel{\text{def}}{=} K_X \setminus \{0\}$), we obtain a natural **Kummer map**

$$\kappa_X : K_X^{\times} \hookrightarrow H^1(G_X, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}})$$

— which may be *composed* with the natural isomorphism λ to obtain a *natural embedding*

$$\kappa_X^{\mathrm{Gal}} : K_X^\times \hookrightarrow H^1(G_X, M_X)$$

— where we recall from the theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9] that the *Galois group* G_X [regarded up to inner automorphisms that arise from elements of $\mathrm{Ker}(G_X \twoheadrightarrow \Pi_X)$], together with the *cohomology module* in the codomain of κ_X^{Gal}, the *image* of κ_X^{Gal} in this cohomology module, and the *field structure* on the *union* $K_X(\Pi_X)^{\mathrm{Kum}}$ of this image with $\{0\}$, may be **constructed group-theoretically** from the abstract topological group Π_X. Write $G_X(\Pi_X)$ for "G_X regarded as an object constructed in this way from Π_X";

$$\overline{K}_X(\Pi_X)^{\mathrm{Kum}} \curvearrowleft G_X(\Pi_X)$$

for the inductive system/limit [which, by *functoriality*, is equipped with a natural action by $G_X(\Pi_X)$] of the result of applying this group-theoretic construction $\Pi_X \mapsto K_X(\Pi)^{\mathrm{Kum}}$ to the various open subgroups of $G_X(\Pi_X)$.

(ii) Now let us regard the pair "$G_X \curvearrowright \overline{K}_X^\times$" as an *abstract ind-monoid* "$\overline{K}_X^\times$" [i.e., inductive system of monoids] *equipped with a continuous action* by an *abstract topological group* "G_X" that arises, for some *abstract quotient topological group* "$G_X \twoheadrightarrow \Pi_X$", as the topological group "$G_X(\Pi_X)$" of (i) [hence is only well-defined up to inner automorphisms that arise from elements of $\mathrm{Ker}(G_X \twoheadrightarrow \Pi_X)$]. Thus, if we think of $\boldsymbol{\mu}_{\overline{k}} \subseteq \overline{K}_X^\times$ as the *subgroup of torsion elements* of the monoid $\overline{K}_X^\times$, then, by considering the action of G_X on the various N*-th roots*, for N a positive integer, of elements of $K_X^\times$, we obtain the natural **Kummer map**

$$\kappa_X : K_X^\times \hookrightarrow H^1(G_X, \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}})$$

discussed in (i). Moreover,

> the $\{\pm 1\}$**-orbit** of the **cyclotomic rigidity isomorphism** $\lambda : \boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}} \xrightarrow{\sim} M_X$ of (i) may be **constructed functorially** from the data "$G_X \curvearrowright \overline{K}_X^\times$"

by applying the *"alternative approach"* discussed in Example 2.12.1, (iv), as follows [cf. [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi)]. Indeed, it follows formally from the elementary fact

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$$

— for instance, by considering the various *quotients* $K_X^\times \twoheadrightarrow \mathbb{Z}$ determined by the *discrete valuations* of K_X that arise from the *closed points* of X^{cp}, i.e., the *quotients* which, at the level of *Kummer classes*, are induced by restriction to the various *cuspidal inertia groups* [cf. the first display of [AbsTopIII], Proposition 1.6, (iii)] — that

> the *only isomorphisms* $\boldsymbol{\mu}_{\overline{k}}^{\widehat{\mathbb{Z}}} \xrightarrow{\sim} M_X$ that *map the image of* κ_X *into* $K_X^\times(\Pi_X)^{\mathrm{Kum}}$ ($\stackrel{\mathrm{def}}{=} K_X(\Pi_X)^{\mathrm{Kum}} \setminus \{0\}$) are the isomorphisms that belong to the $\{\pm 1\}$*-orbit of* λ.

As discussed in Example 2.12.1, (iv) [cf. also the discussion of [IUTchIII], Remark 2.3.3, (vii)], this approach to cyclotomic rigidity has the **disadvantage** of being **incompatible** with the **profinite topology** of G_X [or Π_X].

(iii) We continue to use the notational conventions of (ii). Then observe that the *functorial construction of the* $\{\pm 1\}$*-orbit of the cyclotomic rigidity isomorphism* λ given in (ii) may be interpreted in the fashion of Example 2.12.2, (i). That is to say, *observe* that this functorial construction implies that if α is an *arbitrary automorphism* of the data "$G_X \curvearrowright \overline{K}_X^\times$", then *either* α *or* $-\alpha$ [i.e., the composite of α with the automorphism of the data "$G_X \curvearrowright \overline{K}_X^\times$" that raises elements of $\overline{K}_X^\times$ to the power -1 and acts as the identity on G_X] — but *not both*! — is *compatible* with λ, hence also with κ_X^{Gal}. In particular, by applying this *observation* to the various open subgroups of G_X, one concludes that one has a *natural bijection*

$$\mathrm{Aut}(G_X \curvearrowright \overline{K}_X^\times) \quad \stackrel{\sim}{\to} \quad \{\pm 1\} \times \mathrm{Aut}(\Pi_X)$$

— where the first "Aut(−)" denotes automorphisms of the data "$G_X \curvearrowright \overline{K}_X^\times$" as described at the beginning of (ii); the second "Aut(−)" denotes automorphisms of the data "Π_X" consisting of an *abstract topological group* [cf. the discussion of [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi)]. Just as in the case of Example 2.12.2, (i), this situation may also be understood in terms of the general framework of **mono-anabelian transport** discussed in §2.7, (v) [cf. also Example 2.6.1, (iii), (iv)], by considering the *commutative diagram*

$$\begin{array}{ccc} \overline{K}_X^\times(\Pi_X)^{\mathrm{Kum}} & \stackrel{\sim}{\longrightarrow} & \overline{K}_X^\times(\Pi_X)^{\mathrm{Kum}} \\ \Big\uparrow \mathrm{Kum} & \curvearrowright ? & \Big\uparrow \mathrm{Kum} \\ \overline{K}_X^\times & \stackrel{\sim}{\longrightarrow} & \overline{K}_X^\times \end{array}$$

— where $\overline{K}_X^\times(\Pi_X)^{\mathrm{Kum}} \stackrel{\mathrm{def}}{=} \overline{K}_X(\Pi_X)^{\mathrm{Kum}} \setminus \{0\}$; the horizontal arrows are induced by *some given automorphism* of the data "$G_X \curvearrowright \overline{K}_X^\times$"; the vertical arrows, which relate the **Frobenius-like** data $\overline{K}_X^\times$ to the **étale-like** data $\overline{K}_X^\times(\Pi_X)^{\mathrm{Kum}}$, are the **"Kummer isomorphisms"** determined by the various "κ_X^{Gal}" associated to open subgroups of G_X; the diagram *commutes* [cf. "$\curvearrowright$?"] *up to the action of a suitable element* $\in \{\pm 1\}$.

(iv) Finally, we pause to remark that one **fundamental reason** for the use of **Kummer theory** in inter-universal Teichmüller theory in the context of **nonconstant rational functions** [i.e., as in the discussion of the present Example 2.13.1] lies in

> the **functoriality** of Kummer theory with respect to the operation of **evaluation** of such functions at **"special points"** of X.

That is to say, [cf. the discussion of Example 2.6.1, (ii), (iii); §2.7, (vii)] although there exist many different *versions* — e.g., versions for *"higher-dimensional fields"* — of **class field theory**, these versions of class field theory do **not satisfy** such functionality properties with respect to the operation of evaluation of functions at points [cf. the discussion of §2.14, §3.6, §4.2, below; [IUTchIV], Remark 2.3.3, (vi), (vii)].

§**2.14.** **Finite discrete approximations of harmonic analysis:** Finally, we conclude the present §2 by pausing to examine in a bit more detail the *transition* that was, in effect, made earlier in the present §2 in passing from **derivatives** [in the literal sense, as in the discussion of §2.5] to **Galois groups/étale fundamental groups** [i.e., as in the discussion of and subsequent to §2.6]. This transition is closely related to many of the ideas of the [scheme-theoretic] **Hodge-Arakelov theory** of [HASurI], [HASurII].

Example 2.14.1: Finite discrete approximation of differential calculus on the real line. We begin by recalling that the **differential calculus** of [say, infinitely differentiable] functions on the **real line** admits a **finite discrete approximation**, namely, by substituting

$$\tfrac{df(x)}{dx} \;=\; \lim_{\delta\to 0} \tfrac{f(x+\delta)-f(x)}{\delta} \quad \rightsquigarrow \quad f(X+1)-f(X)$$

difference operators for *derivatives* [in the classical sense]. If d is a positive integer, then one verifies easily, by considering such difference operators in the case of *polynomial functions of degree* $< d$ with *coefficients* $\in \mathbb{Q}$, that **evaluation** at the elements $0, 1, \ldots, d-1 \in \mathbb{Z} \subseteq \mathbb{Q}$ yields a *natural isomorphism of* $\mathbb{Q}$*-vector spaces of dimension* d

$$\mathbb{Q}[X]^{<d} \;\; \Big(\stackrel{\mathrm{def}}{=} \bigoplus_{j=0}^{d-1} \mathbb{Q}\cdot X^j\Big) \quad \stackrel{\sim}{\to} \quad \bigoplus_{0}^{d-1} \mathbb{Q}$$

— cf., e.g., the discussion of the well-known classical theory of *Hilbert polynomials* in [Harts], Chapter I, §7, for more details. In fact, it is not difficult to compute explicitly the *"denominators"* necessary to make this evaluation isomorphism into an isomorphism of *finite free* $\mathbb{Z}$*-modules*. This sort of *"discrete function theory"* [cf. also Example 2.14.2 below] may be regarded as the **fundamental prototype** for the various constructions of Hodge-Arakelov theory.

Example 2.14.2: Finite discrete approximation of Fourier analysis on the unit circle. In the spirit of the discussion of Example 2.14.1, we recall that classical function theory — i.e., in effect, **Fourier analysis** — on the **unit circle** $\mathbb{S}^1$ admits a well-known **finite discrete approximation**: If d is a(n) [say, for simplicity] *odd* positive integer, so $d^* \stackrel{\mathrm{def}}{=} \frac{1}{2}(d-1) \in \mathbb{Z}$, then one verifies easily that **evaluation** of *polynomial functions of degree* $\in \{-d^*, -d^*+1, \ldots, -1, 0, 1, d^*-1, d^*\}$ with *coefficients* $\in \mathbb{Z}$ on the *multiplicative group scheme* $\mathbb{G}_{\mathrm{m}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{Z}[U, U^{-1}])$ at [say, scheme-theoretic] points of the subscheme $\boldsymbol{\mu}_d \subseteq \mathbb{G}_{\mathrm{m}}$ of ***d*-torsion points** of $\mathbb{G}_{\mathrm{m}}$ yields a *natural isomorphism of finite free* $\mathbb{Z}$*-modules of rank* d

$$\bigoplus_{j=-d^*}^{d^*} \mathbb{Z}\cdot U^j \quad \stackrel{\sim}{\to} \quad \mathcal{O}_{\boldsymbol{\mu}_d}$$

— where, by abuse of notation, we write $\mathcal{O}_{\boldsymbol{\mu}_d}$ for the ring of global sections of the structure sheaf of the affine scheme $\boldsymbol{\mu}_d$. If one base-changes via the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ into the field of complex numbers $\mathbb{C}$, then, when d is sufficiently *"large"*, one may think of the *totality of these* d*-torsion points*

$$\exp(2\pi i\cdot \tfrac{1}{d}\mathbb{Z}) \;=\; \Big\{\exp(2\pi i\cdot \tfrac{0}{d}), \exp(2\pi i\cdot \tfrac{1}{d}), \ldots, \exp(2\pi i\cdot \tfrac{(d-1)}{d})\Big\} \;\subseteq\; \mathbb{S}^1$$

as a sort of **finite discrete approximation** of $\mathbb{S}^1$ and hence, in particular, of *adjacent pairs* of d-torsion points as **"tangent vectors"** on $\mathbb{S}^1$. That is to say, since [inverse systems of] such torsion points give rise to the *étale fundamental group* of $\mathbb{G}_m \times_{\mathbb{Z}} \mathbb{C}$, it is precisely this **"picture"** of torsion points of $\mathbb{S}^1$ that motivates the idea that

> **Galois groups/étale fundamental groups** should be regarded as a sort of **arithmetic analogue** of the classical geometric notion of a **tangent bundle**

— the discussion of §2.6. Moreover, if one regards $\mathbb{G}_m$ as the codomain of a *nonzero function* [on some unspecified "space"], then this very classical *"pictorial representation of a* **cyclotome***"* [i.e., of torsion points of $\mathbb{S}^1$] also explains, from a "pictorial point of view", the importance of **cyclotomes** and **Kummer classes** in the discussion of §2.6, §2.7. That is to say,

> a **Kummer class of a function**, which, so to speak, records the *"arithmetic infinitesimal motion"* in $\mathbb{G}_m$ induced, via the function, by an *"arithmetic infinitesimal motion"* in the space on which the function is defined may be thought of as a sort of **"arithmetic logarithmic derivative"** of the function

— a point of view that is consistent with the usual point of view that the Kummer exact sequence in étale cohomology [i.e., which induces a connecting homomorphism in cohomology that computes the Kummer class of a function] should be thought of as a sort of arithmetic analogue of the *exponential exact sequence* $1 \to 2\pi i \cdot \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1$ that appears in the theory of sheaf cohomology of sheaves of holomorphic functions on a complex space.

<u>Example 2.14.3</u>: **Finite discrete approximation of harmonic analysis on complex tori.** Examples 2.14.1 and 2.14.2 admit a natural generalization to the case of **elliptic curves**. Indeed, let E be an elliptic curve over a field F of characteristic zero, $E^\dagger \to E$ the *universal extension* of E, $\eta \in E(F)$ a [nontrivial] torsion point of order 2, $l \neq 2$ a prime number. Write $E[l] \subseteq E$ for the subscheme of *l-torsion points*, $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(l \cdot [\eta])$ [where "$[\eta]$" denotes the effective divisor on E determined by η]. Here, we recall that $E^\dagger \to E$ is an $\mathbb{A}^1$*-torsor* [so $E[l]$ may also be regarded as the subscheme $\subseteq E^\dagger$ of l-torsion points of $E^\dagger$]. In particular, it makes sense to speak of the sections $\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{<l} \subseteq \Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})$ of $\mathcal{L}$ over $E^\dagger$ whose *relative degree*, with respect to the morphism $E^\dagger \to E$, is $< l$. Then the simplest version of the **fundamental theorem of Hodge-Arakelov theory** states that **evaluation** at the subscheme of ***l*-torsion points** $E[l] \subseteq E^\dagger$ yields a *natural isomorphism of F-vector spaces of dimension l^2*

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{<l} \quad \stackrel{\sim}{\to} \quad \mathcal{L}|_{E[l]}$$

[cf. [HASurI], Theorem $\mathrm{A}^{\mathrm{simple}}$]. Moreover:

- When F is an **NF**, this isomorphism is compatible, up to mild discrepancies, with **natural integral structures** on the LHS and RHS of the isomorphism at the nonarchimedean valuations of F and with **natural Hermitian metrics** on the LHS and RHS of the isomorphism at the archimedean valuations of F.

· When F is a complete discrete valuation field, and E is a **Tate curve** over F, with special fiber isomorphic to $\mathbb{G}_{\mathrm{m}}$, **Example 2.14.2** may be thought of as corresponding to the portion of the natural isomorphism of the above display that arises from the **"special fiber of** E**"**, while **Example 2.14.1** may be thought of as corresponding to the portion of the natural isomorphism of the above display that arises from the **"special fiber of the relative dimension of the morphism** $E^{\dagger} \to E$**"**.

· When E is a **Tate curve**, the isomorphism, over F, of the above display may be interpreted as a result concerning the *invertibility of the matrix* determined by the values at the l-torsion points of certain **theta functions** associated to the Tate curve and their **derivatives of order** $< l$ [cf., §3.6 below; [Fsk], §2.5; [EtTh], Proposition 1.4, for a review of the series representation of such theta functions].

· When F is an *arbitrary field*, the isomorphism of the above display may be thought of as a sort of **discrete polynomial version** of the **Gaussian integral** $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, i.e., in the spirit of the discussion of Examples 2.14.1 and 2.14.2, above.

· When F is an **NF**, the isomorphism of the above display may be thought of as a sort of **discrete globalized version** of the **harmonic analysis** involving *"∂, $\overline{\partial}$, Green's functions, etc."* that appears at *archimedean valuations* in classical **Arakelov theory**. This is the reason for the appearance of the word "Arakelov" in the term "Hodge-Arakelov theory". From this point of view, the computation of the *discrepancy* between *natural integral structures/metrics* on the LHS and RHS of the isomorphism of the above display may be thought of as a sort of computation of **analytic torsion** — a point of view that in some sense foreshadows the *interpretation* [cf. the discussion of §3.9, (iii), below] of inter-universal Teichmüller theory as the **computation** of a sort of **global arithmetic/Galois-theoretic** form of **analytic torsion**.

· The isomorphism of the above display may also be thought of as a sort of **global arithmetic version** of the **comparison isomorphisms** that occur in complex or p-adic **Hodge theory**. [That is to say, the LHS and RHS of the isomorphism of the above display correspond, respectively, to the *"de Rham"* and *"étale"* sides of comparison isomorphisms in p-adic Hodge theory.] This is the reason for the appearance of the word "Hodge" in the term "Hodge-Arakelov theory". This point of view gives rise to a natural definition for a sort of **arithmetic** version of the **Kodaira-Spencer morphism** discussed in §2.9, in which **Galois groups** play the role played by **tangent bundles** in the classical version of the Kodaira-Spencer morphism reviewed in §2.9 [cf. [HASurI], §1.4]. When E is a *Tate curve* over a complete discrete valuation field F, this arithmetic Kodaira-Spencer morphism essentially *coincides*, when formulated properly, with the classical Kodaira-Spencer morphism reviewed in §2.9 [cf. [HASurII], §3].

· Relative to the point of view of *"filtered crystals"* [e.g., vector bundles equipped with a connection and filtration — cf. the data $(\mathcal{E}, \nabla_{\mathcal{E}}, \omega_E \subseteq \mathcal{E})$ of §2.9], the isomorphism of the above display may be thought of as a sort of **discrete Galois-theoretic version** of the **"crystalline theta object"** [cf. [HASurII], §2], i.e., the **"nonlinear filtered crystal"** constituted by the *universal extension* $E^\dagger$ equipped with the *ample line bundle* $\mathcal{L}|_{E^\dagger}$, the *natural structure of crystal* on $(E^\dagger, \mathcal{L}|_{E^\dagger})$, and the *"filtration"* constituted by the morphism $E^\dagger \to E$.

We refer to [HASurI], [HASurII], for more details concerning the ideas just discussed.

Section 3: Multiradiality: an abstract analogue of parallel transport

§**3.1.** <u>**The notion of multiradiality**</u>**:** So far, in §2, we have discussed various *generalities* concerning **arithmetic changes of coordinates** [cf. §2.10; the analogy discussed in §2.2, §2.5, and §2.7 with the classical theory of §1.4 and §1.5], which are applied in effect to the **two underlying combinatorial dimensions** of a ring such as an MLF or an NF [cf. §2.7, (vii); §2.11; §2.12], and the approach to computing the effect of such arithmetic changes of coordinates — i.e., in the form of **Kummer-detachment indeterminacies** or **étale-transport indeterminacies** [cf. §2.7, (vi); §2.9] — by means of the technique of **mono-anabelian transport** [cf. §2.7, (v)]. By contrast, in the present §3, we turn to the issue of considering the **particular** *arithmetic changes of coordinates* that are of interest in the context of inter-universal Teichmüller theory [cf. the discussion of §2.1, §2.3, §2.4]. Many aspects of these *particular* arithmetic changes of coordinates are highly reminiscent of the change of coordinates discussed in §1.6 from **planar cartesian** to **polar** coordinates. In some sense, the central notion that underlies the *abstract combinatorial analogue*, i.e., that is developed in inter-universal Teichmüller theory, of this change of coordinates from *planar cartesian* to *polar* coordinates is the notion of **multiradiality**.

(i) <u>**Types of mathematical objects**</u>**:** In the following discussion, we shall often speak of **"types of mathematical objects"**, i.e., such as groups, rings, topological spaces equipped with some additional structure, schemes, etc. This notion of a "type of mathematical object" is *formalized* in [IUTchIV], §3, by introducing the notion of a *"species"*. On the other hand, the details of this formalization are not so important for the following discussion of the notion of multiradiality. A "type of mathematical object" determines an associated **category** consisting of mathematical objects of this type — i.e., in a given *universe*, or *model of set theory* — and morphisms between such mathematical objects. On the other hand, in general, the structure of this associated category [i.e., as an *abstract category*!] contains *considerably less information* than the information that determines the "type of mathematical object" that one started with. For instance, if p is a prime number, then the "type of mathematical object" given by *rings isomorphic to* $\mathbb{Z}/p\mathbb{Z}$ [and ring homomorphisms] yields a *category* whose equivalence class as an *abstract category* is manifestly independent of the prime number p.

(ii) <u>**Radial environments**</u>: A **radial environment** consists of a **triple**. The first member of this triple is a specific "type of mathematical object" that is referred to as **radial data**. The second member of this triple is a specific "type of mathematical object" that is referred to as **coric data**. The third member of this triple is a **functorial algorithm** that *inputs radial data* and *outputs coric* *data*; this algorithm is referred to as **radial**, while the resulting *functor* from the category of radial data to the category of coric data is referred to as the **radial functor** of the radial environment. We would like to think of the *coric data* as a sort of *"underlying structure"* of the *"finer structure"* constituted by the *radial data* and of the *radial algorithm* as an algorithm that *forgets* this "finer structure", i.e., an algorithm that assigns to a collection of radial data the *collection of underlying coric data* of this given collection of radial data. We refer to [IUTchII], Example 1.7, (i), (ii), for more details.

(iii) **Multiradiality and uniradiality:** A radial environment is called **multiradial** if its associated radial functor is *full*. A radial environment is called **uniradial** if its associated radial functor is *not full*. One important consequence of the condition of *multiradiality* is the following **switching property**:

> Consider the category of *objects* consisting of an *ordered pair* of collections of radial data, together with an *isomorphism* between the associated collections of *underlying coric data* [and *morphisms* defined in the evident way]. Observe that this category admits a **switching functor** [from the category to itself] that assigns to an object of the category the object obtained by *switching* the two collections of radial data of the given object and replaces the isomorphism between associated collections of underlying coric data by the *inverse* to this isomorphism. Then **multiradiality** implies that *the switching functor preserves the isomorphism classes of objects*.

Indeed, one verifies immediately that multiradiality is in fact *equivalent* to the property that any object of the category discussed in the above display is, in fact, *isomorphic* to a *"diagonal object"*, i.e., an object given by considering an *ordered pair of copies* of a given collection of radial data, together with the *identity isomorphism* between the associated collections of underlying coric data — cf. the illustration of Fig. 3.1 below. We refer to [IUTchII], Example 1.7, (ii), (iii), for more details.

(iv) **Analogy with the Grothendieck definition of a connection:** Thus, in summary,

> **multiradiality** concerns the issue of **comparison** between **two collections of radial data** that share a **common collection of underlying coric data**.

We shall often think of this sort of comparison as a comparison between **two "holomorphic structures"** that share a **common "underlying real analytic structure"** [cf. the examples discussed in §3.2 below]. Note that multiradiality may be thought of as a sort of **abstract analogue** of the notion of **"parallel transport"** or, alternatively, the **Grothendieck definition of a connection** [cf. the discussion of [IUTchII], Remark 1.7.1]. That is to say, given a scheme X over a scheme S, the Grothendieck definition of a connection on an object $\mathcal{E}$ over X consists of an *isomorphism* between the *fibers* of $\mathcal{E}$ at *two distinct* — but *infinitesimally close*! — *points of* X that map to the *same point of* S. Thus, one may think of the *fullness* condition of *multiradiality* as the condition that there exist a sort of *parallel transport isomorphism* between *two collections of radial data* [i.e., corresponding to *two "fibers"*] that *lifts* a given *isomorphism* between collections of underlying coric data [i.e., corresponding to a *path* between the points over which the two fibers lie]. The **indeterminacy** in the *choice* of such a lifting may then be thought of, relative to this analogy with parallel transport, as a sort of **"monodromy"** associated to the multiradial environment.

$$\begin{tikzcd}
\boxed{\begin{matrix}\text{radial}\\ \text{data}\end{matrix}} \arrow[dr, "\substack{\text{radial}\\ \text{algorithm}}"'] & \overset{\text{permutable?}}{\leftrightarrows} & \boxed{\begin{matrix}\text{radial}\\ \text{data}\end{matrix}} \arrow[dl, "\substack{\text{radial}\\ \text{algorithm}}"] \\
& \boxed{\begin{matrix}\text{coric}\\ \text{data}\end{matrix}} &
\end{tikzcd}$$

Fig. 3.1: Multiradiality vs. uniradiality

(v) <u>**The Kodaira-Spencer morphism via multiradiality:**</u> The classical approach to proving the *geometric version of the Szpiro Conjecture* by means of the *Kodaira-Spencer morphism* was reviewed in §2.9. Here, we observe that this argument involving the Kodaira-Spencer morphism may be *formulated* in a way that

- *renders explicit* the analogy discussed in (iv) above between **multiradiality** and **connections**
- *renders explicit* the relationship between this *classical argument* involving the Kodaira-Spencer morphism and the approach taken in *inter-universal Teichmüller theory*, that is to say, as a sort of *"limiting case" or "degenerate version"* of the argument [sketched in the Introduction to the present paper — cf. also the discussion of §2.3, §2.4] involving **multiplication** of the **height "h"** by a **factor "N"**, in the **limit** "$N \to \infty$" [in which case comparison between "h" and "$N \cdot h$", or equivalently, between "h" and "$\frac{1}{N} \cdot h$", becomes a comparison between "h" and "0"].

This *formulation* may be broken down into *steps*, as follows. Let $S^{\log}$ be as in §2.9, $\mathcal{L}$ a line bundle on S. Suppose that we are interesting in **bounding** $\deg(\mathcal{L})$ [i.e., bounding the degree of $\mathcal{L}$ from above]. Then:

(1^{KS}) Write $p_1, p_2 : S \times S \to S$ for the natural projections from the *direct product* $S \times S$ to the first and second factors. Suppose that we are given an *isomorphism*

$$p_1^* \mathcal{L} \xrightarrow{\sim} p_2^* \mathcal{L}$$

of line bundles on $S \times S$ between the pull-backs of $\mathcal{L}$ via p_1, p_2. Then one verifies immediately, by restricting to various *fibers* of the direct product $S \times S$, that the existence of such an *isomorphism* implies that $\deg(\mathcal{L}) = 0$ [hence that $\deg(\mathcal{L})$ is *bounded*], as desired.

(2^{KS}) Write S_δ for the *first infinitesimal neighborhood* of the *diagonal* ($S \xrightarrow{\sim}$) $\Delta_S \subseteq S \times S$. Suppose that we are given an *isomorphism*

$$p_1^* \mathcal{L}|_{S_\delta} \xrightarrow{\sim} p_2^* \mathcal{L}|_{S_\delta}$$

of line bundles on S_δ between the restrictions to S_δ of the two pull-backs via p_1, p_2 of $\mathcal{L}$. Since S is *proper* [so any automorphism of the line bundle $\mathcal{L}$ on S is given by multiplication by a nonzero complex number], one verifies immediately that an isomorphism as in the above display may be thought of as a **connection**, in the sense of Grothendieck, on $\mathcal{L}$ [cf. the discussion of (iv) above!]. In particular, since the base field $\mathbb{C}$ is of *characteristic zero*, we thus conclude again [from the elementary theory of de Rham-theoretic first Chern classes of line bundles on curves] that $\deg(\mathcal{L}) = 0$ [and hence that $\deg(\mathcal{L})$ is *bounded*], as desired.

(3^{KS}) Let $\mathcal{F}$ be a *rank two vector bundle* that admits an exact sequence $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{L}^{-1} \to 0$ of vector bundles on S. Thus, one may think of $\mathcal{F}$ as a *container* for $\mathcal{L}$. Write $S^{\log}_\delta$ for the *first infinitesimal neighborhood* of the *"logarithmic diagonal"* ($S^{\log} \xrightarrow{\sim}$) $\Delta_{S^{\log}} \subseteq S^{\log} \times S^{\log}$. Next, suppose that we are given an *isomorphism*

$$p_1^*\mathcal{F}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S^{\log}_\delta}$$

of vector bundles on [the underlying scheme of] $S^{\log}_\delta$ between the restrictions to $S^{\log}_\delta$ of the pull-backs via p_1, p_2 of $\mathcal{F}$. [Thus, such an *isomorphism* arises, for instance, from a *logarithmic connection* on $\mathcal{F}$.] Suppose, moreover, that this *isomorphism* has *nilpotent monodromy*, i.e., that the restriction of this *isomorphism* to each of the cusps of $S^{\log}$ differs from multiplication by a nonzero complex number by a *nilpotent endomorphism* of the fiber of $\mathcal{F}$ at the cusp under consideration. Thus, we obtain *two inclusions*

$$p_1^*\mathcal{L}|_{S^{\log}_\delta} \hookrightarrow p_1^*\mathcal{F}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S^{\log}_\delta} \hookleftarrow p_2^*\mathcal{L}|_{S^{\log}_\delta}$$

[where the " $\xrightarrow{\sim}$ " is the *isomorphism* of the first display of the present (3^{KS})] of line bundles into a rank two vector bundle over $S^{\log}_\delta$; one verifies immediately that the images of these two inclusions *coincide* over the diagonal $\Delta_{S^{\log}} \subseteq S^{\log}_\delta$. That is to say,

> the **isomorphism** $p_1^*\mathcal{F}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S^{\log}_\delta}$ allows one to use $\mathcal{F}$ as a **container** for $\mathcal{L}$ to **compare the discrepancy** between the two [restrictions to $S^{\log}_\delta$ of] pull-backs $p_1^*\mathcal{L}|_{S^{\log}_\delta}$, $p_1^*\mathcal{L}|_{S^{\log}_\delta}$.

(4^{KS}) Suppose that the images in $p_1^*\mathcal{F}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S^{\log}_\delta}$ of the *two inclusions* in the second display of (3^{KS}) *coincide*. Then [since S is *proper* — cf. the argument in (2^{KS})] the resulting isomorphism $p_1^*\mathcal{L}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{L}|_{S^{\log}_\delta}$ may be thought of as a *logarithmic connection* on $\mathcal{L}$ with *nilpotent monodromy*, i.e., [since $\mathcal{L}$ is of *rank one*!] a *connection* [without logarithmic poles!] on $\mathcal{L}$. In particular, we are in the situation of (2^{KS}), so we may conclude again that $\deg(\mathcal{L}) = 0$ [and hence that $\deg(\mathcal{L})$ is *bounded*], as desired.

(5^{KS}) In general, of course, the images in $p_1^*\mathcal{F}|_{S^{\log}_\delta} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S^{\log}_\delta}$ of the *two inclusions* in the second display of (3^{KS}) *will* **not** *coincide*. On the other hand, in this case [i.e., in which the images of the two inclusions do *not* coincide], one may consider [cf. the diagram in the second display of (3^{KS})] the *composite*

$$p_1^*\mathcal{L}|_{S_\delta^{\log}} \hookrightarrow p_1^*\mathcal{F}|_{S_\delta^{\log}} \xrightarrow{\sim} p_2^*\mathcal{F}|_{S_\delta^{\log}} \twoheadrightarrow p_2^*\mathcal{L}^{-1}|_{S_\delta^{\log}}$$

[where the "$\twoheadrightarrow$" is the restriction to $S_\delta^{\log}$ of the given surjection $\mathcal{F} \twoheadrightarrow \mathcal{L}^{-1}$], whose restriction to $\Delta_{S^{\log}} \subseteq S_\delta^{\log}$ *vanishes*, hence determines a *nonzero morphism* of line bundles on S

$$\mathcal{L} \rightarrow \omega_{S^{\log}/\mathbb{C}} \otimes \mathcal{L}^{-1}$$

[where we recall that the ideal sheaf defining the closed [log] subscheme $\Delta_{S^{\log}} \subseteq S_\delta^{\log}$ may be naturally identified with the push-forward, via the natural inclusion $\Delta_{S^{\log}} \hookrightarrow S_\delta^{\log}$, of the sheaf of logarithmic differentials $\omega_{S^{\log}/\mathbb{C}}$]. Now one verifies immediately that, if one takes $\mathcal{F}$ to be the vector bundle "$\mathcal{E}$" of §2.9, equipped with the isomorphism as in the first display of (3^{KS}) arising from the logarithmic connection $\nabla_{\mathcal{E}}$, and $\mathcal{L}$ to be the subbundle "$\omega_E \subseteq \mathcal{E}$" of §2.9, then the *nonzero morphism* of the above display may be identified with the *Kodaira-Spencer morphism* $\omega_E \rightarrow \tau_E \otimes_{\mathcal{O}_S} \omega_{S^{\log}/\mathbb{C}}$ discussed in §2.9. Thus, in summary,

> the **Kodaira-Spencer morphism** may be thought of as a measure of the **discrepancy** that arises when one **fixes** the "ω_E" on **one factor** of S and **compares** it with the "ω_E" on a **distinct, "alien" factor** of S by means of the **common container** "$\mathcal{E}$", which is equipped with a **connection** "$\nabla_{\mathcal{E}}$" [i.e., an isomorphism as in the first display of (3^{KS})].

When *formulated* in this way, the Kodaira-Spencer morphism becomes manifestly analogous to the approach sketched in the Introduction to the present paper [cf. also the discussion of §2.3, §2.4] to **bounding heights** of elliptic curves [cf. the discussion of §3.7, (ii), (iv), below] by applying a suitable **multiradiality** property [cf. the discussion of §3.7, (i), below], i.e., [in the language of the Introduction] a *"license to confuse"*.

[Here, we note that, relative to the analogy with inter-universal Teichmüller theory, the situation that arises in (2^{KS}), (4^{KS}) corresponds to the [unusual!] situation in which there actually exists a *"global multiplicative subspace"* — cf. the discussion of §2.3.] Finally, we remark in passing that the **crystalline theta object** referred to in the discussion of Example 2.14.3 may be thought of as a sort of *intermediate stage* between the situation discussed in (5^{KS}) and the situation that is ultimately considered in inter-universal Teichmüller theory.

§**3.2.** <u>**Fundamental examples of multiradiality**</u>**:** The following examples may be thought of as **fundamental prototypes** of the phenomenon of **multiradiality**.

<u>**Example 3.2.1**</u>**: Complex holomorphic structures on two-dimensional real vector spaces.**

(i) Consider the *radial environment* in which the **radial data** is given by *one-dimensional* $\mathbb{C}$-*vector spaces* [and isomorphisms between such data], the **coric data** is given by *two-dimensional* $\mathbb{R}$-*vector spaces* [and isomorphisms between such data], and the **radial algorithm** assigns to a one-dimensional $\mathbb{C}$-vector space the associated *underlying* $\mathbb{R}$-*vector space*. Then one verifies immediately that this radial environment, shown in Fig. 3.2 below, is **uniradial** [cf. [Pano], Figs. 2.2, 2.3].

$$
\begin{tikzcd}
\boxed{\mathbb{C} \overset{\sim}{\rightarrow} \mathbb{R}^2} \arrow[dr, "\text{radial algorithm}"'] & \not\cong & \boxed{\mathbb{C} \overset{\sim}{\rightarrow} \mathbb{R}^2} \arrow[dl, "\text{radial algorithm}"] \\
& \boxed{\mathbb{R}^2 \curvearrowleft GL_2(\mathbb{R})} &
\end{tikzcd}
$$

Fig. 3.2: The uniradiality of complex holomorphic structures

(ii) If V is a two-dimensional real vector space, then write $\mathrm{End}(V)$ for the $\mathbb{R}$-algebra of $\mathbb{R}$-linear endomorphisms of V and $GL(V)$ for the group of invertible elements of $\mathrm{End}(V)$. Observe that if V is a two-dimensional real vector space, then a *complex* — i.e., *"holomorphic"* — structure on V may be thought of as a *homomorphism of* $\mathbb{R}$-*algebras* $\mathbb{C} \to \mathrm{End}(V)$. In particular, it makes sense to speak of a **GL-orbit of complex structures** on V, i.e., the set of $GL(V)$-conjugates of some such homomorphism. Now consider the *radial environment* in which a collection of **radial data** consists of a *two-dimensional* $\mathbb{R}$-*vector space equipped with a* GL-*orbit of complex structures* [and the morphisms between such data are taken to be the isomorphisms between such data], the **coric data** is the same as the coric data of (i), and the **radial algorithm** assigns to a two-dimensional $\mathbb{R}$-vector space equipped with a GL-orbit of complex structures the associated *underlying* $\mathbb{R}$-*vector space*. Then one verifies immediately that this radial environment, shown in Fig. 3.3 below, is [**"tautologically"**!] **multiradial** [cf. [Pano], Figs. 2.2, 2.3].

$$
\begin{tikzcd}
\boxed{\mathbb{C} \overset{\sim}{\rightarrow} \mathbb{R}^2 \curvearrowleft GL_2(\mathbb{R})} \arrow[dr, "\text{radial algorithm}"'] & \leftrightarrows & \boxed{\mathbb{C} \overset{\sim}{\rightarrow} \mathbb{R}^2 \curvearrowleft GL_2(\mathbb{R})} \arrow[dl, "\text{radial algorithm}"] \\
& \boxed{\mathbb{R}^2 \curvearrowleft GL_2(\mathbb{R})} &
\end{tikzcd}
$$

Fig. 3.3: The multiradiality of $GL_2(\mathbb{R})$-orbits of complex holomorphic structures

(iii) The examples of radial environments discussed in (i), (ii) are particularly of interest in the context of inter-universal Teichmüller theory in light of the relationship between *complex holomorphic structures* as discussed in (i), (ii) and the *geometry of the upper half-plane*. That is to say, if, in the notation of (ii), we write $GL(V) = GL^+(V) \coprod GL^-(V)$ for the decomposition of $GL(V)$ determined by considering the *sign* of the determinant of an $\mathbb{R}$-linear automorphism of V, then the space of **moduli of complex holomorphic structures** on V [i.e., the set of $GL(V)$-conjugates of a particular homomorphism of $\mathbb{R}$-algebras $\mathbb{C} \hookrightarrow \mathrm{End}(V)$] may be *identified*

$$GL(V)/\mathbb{C}^\times \quad \xrightarrow{\sim} \quad GL^+(V)/\mathbb{C}^\times \quad \coprod \quad GL^-(V)/\mathbb{C}^\times \quad \xrightarrow{\sim} \quad \mathfrak{H}^+ \quad \coprod \quad \mathfrak{H}^-$$

in a natural way with the *disjoint union* of the *upper* [i.e., $\mathfrak{H}^+$] and *lower* [i.e., $\mathfrak{H}^-$] *half-planes*. This observation is reminiscent of the deep connections between inter-universal Teichmüller theory and the **hyperbolic geometry** of the **upper half-plane**, as discussed in [BogIUT] [cf. also the discussion of §2.4, as well as of §4.1, (i), below, of the present paper]. This circle of ideas is also of interest, in the context of inter-universal Teichmüller theory, in the sense that it is reminiscent of the *natural bijection*

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL^+(V)/\mathbb{C}^\times & \xrightarrow{\sim} & [0,1) \\ \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) & \mapsto & \frac{t-1}{t+1} \end{array}$$

[where $\mathbb{R} \ni t \geq 1$] between the space of double cosets on the left and the semi-closed interval $[0,1)$ on the right, i.e., a bijection that is usually interpreted in classical complex Teichmüller theory as the map that assigns to a **deformation of complex structure** the **dilation** $\in [0,1)$ associated to this deformation [cf. [QuCnf], Proposition A.1, (ii)].

Example 3.2.2: Arithmetic fundamental groups of hyperbolic curves of strictly Belyi type over mixed-characteristic local fields.

(i) Let $X \to \mathrm{Spec}(k)$ and $\Pi_X \twoheadrightarrow G_k$ be as in Example 2.12.3, (i). Consider the *radial environment* in which the **radial data** is given by *topological groups* Π that "just happen to be" *abstractly isomorphic* as topological groups to Π_X [and isomorphisms between topological groups], the **coric data** is given by *topological groups* G that "just happen to be" *abstractly isomorphic* as topological groups to G_k [and isomorphisms between topological groups], and the **radial algorithm** assigns to a topological group Π the *quotient group* $\Pi \twoheadrightarrow G$ that corresponds to the [*group-theoretically constructible!* — cf. the discussion of Example 2.12.3, (i)] quotient $\Pi_X \twoheadrightarrow G_k$. Then one verifies immediately [cf. the two displays of Example 2.12.3, (i)!] that this radial environment, shown in Fig. 3.4 below, is **uniradial**.

$$\begin{tikzcd} \boxed{\Pi \twoheadrightarrow G} \arrow[dr, "\substack{\text{radial}\\ \text{algorithm}}"'] & \not\cong & \boxed{\Pi \twoheadrightarrow G} \arrow[dl, "\substack{\text{radial}\\ \text{algorithm}}"] \\ & \boxed{\begin{array}{l} \curvearrowleft \mathrm{Aut}(G) \\ G \end{array}} & \end{tikzcd}$$

Fig. 3.4: The uniradiality of local arithmetic holomorphic structures

(ii) In the remainder of the present (ii), we apply the notation "Aut(−)" to denote the group of automorphisms of the topological group in parentheses. Consider the *radial environment* in which the **radial data** is given by *triples*

$$(\Pi, G, \alpha)$$

— where Π is a topological group as in the radial data of (i), G is a topological group as in the coric data of (i), and α is an $\mathrm{Aut}(G)$-*orbit of isomorphisms* between G and the quotient of Π that corresponds to the [*group-theoretically constructible!*] quotient $\Pi_X \twoheadrightarrow G_k$ — [and isomorphisms between such triples], the **coric data** is the same as the coric data of (i), and the **radial algorithm** assigns to a triple (Π, G, α) the *topological group* G. Then one verifies immediately that this radial environment, shown in Fig. 3.5 below, is [**"tautologically"**!] **multiradial** [cf. [IUTchII], Example 1.8, (i)].

$$
\begin{array}{ccc}
\boxed{\begin{array}{l} \qquad \curvearrowleft \ \mathrm{Aut}(G) \\ \Pi \ \twoheadrightarrow \ G \end{array}} & \leftrightarrows & \boxed{\begin{array}{l} \qquad \curvearrowleft \ \mathrm{Aut}(G) \\ \Pi \ \twoheadrightarrow \ G \end{array}} \\
\text{radial algorithm} \ \searrow & & \swarrow \ \text{radial algorithm} \\
& \boxed{\begin{array}{l} \curvearrowleft \ \mathrm{Aut}(G) \\ G \end{array}} &
\end{array}
$$

Fig. 3.5: The multiradiality of $\mathrm{Aut}(G)$-orbits of local arithmetic holomorphic structures

§3.3. <u>The log-theta-lattice: $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, $\mathfrak{log}$-links, Θ-links</u>: The *fundamental stage* on which the constructions of inter-universal Teichmüller theory are performed is referred to as the **log-theta-lattice**.

(i) **<u>Initial Θ-data</u>:** The *log-theta-lattice* is completely determined, up to isomorphism, once one fixes a collection of **initial Θ-data**. Roughly speaking, this data consists of

- an **elliptic curve** E_F over a **number field** F,
- an **algebraic closure** $\overline{F}$ of F,
- a **prime number** $l \geq 5$,
- a *collection of* **valuations** $\underline{\mathbb{V}}$ of a certain subfield $K \subseteq \overline{F}$, and
- a *collection of* **valuations** $\underline{\mathbb{V}}^{\mathrm{bad}}_{\mathrm{mod}}$ of a certain subfield $F_{\mathrm{mod}} \subseteq \overline{F}$

that satisfy certain technical conditions — cf. [IUTchI], Definition 3.1, for more details. Here, we write $F_{\mathrm{mod}} \subseteq F$ for the **field of moduli** of E_F, i.e., the field extension of $\mathbb{Q}$ obtained by adjoining the ***j*-invariant** of E_F; $K \subseteq \overline{F}$ for the extension field of F generated by the fields of definition of the ***l*-torsion points** of E_F; $X_F \subseteq E_F$ for the **once-punctured elliptic curve** obtained by removing the origin from E_F; and $X_F \to C_F$ for the *hyperbolic orbicurve* obtained by forming the *stack-theoretic quotient* of X_F by the natural action of $\{\pm 1\}$. Also, in the following, we shall write $\mathbb{V}(-)$ for the set of *all [nonarchimedean and archimedean]* **valuations** of an NF "$(-)$" and append a *subscripted* element $\in \mathbb{V}(-)$ to the NF to denote the *completion* of the NF at the element $\in \mathbb{V}(-)$. We assume further that the following *conditions* are satisfied [cf. [IUTchI], Definition 3.1, for more details]:

- F is **Galois** over F_{mod} of degree **prime to *l*** and contains the fields of definition of the **2·3-torsion points** of E_F;
- the image of the *natural inclusion* $\mathrm{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$ [well-defined up to composition with an inner automorphism] **contains** $SL_2(\mathbb{F}_l)$;

- E_F has **stable reduction** at all of the nonarchimedean valuations of F;
- $C_K \stackrel{\text{def}}{=} C_F \times_F K$ is a K-**core**, i.e., does not admit a finite étale covering that is isomorphic to a finite étale covering of a **Shimura curve** [cf. [CanLift], Remarks 2.1.1, 2.1.2]; this condition implies that there exists a *unique* model $C_{F_{\text{mod}}}$ of C_F over F_{mod} [cf. the discussion of [IUTchI], Remark 3.1.7, (i)];
- $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$ is a subset such that the natural inclusion $F_{\text{mod}} \subseteq F \subseteq K$ induces a **bijection** $\underline{\mathbb{V}} \stackrel{\sim}{\to} \mathbb{V}_{\text{mod}}$ between $\underline{\mathbb{V}}$ and the set $\mathbb{V}_{\text{mod}} \stackrel{\text{def}}{=} \mathbb{V}(F_{\text{mod}})$;
- $\mathbb{V}^{\text{bad}}_{\text{mod}} \subseteq \mathbb{V}_{\text{mod}}$ is a **nonempty set** of nonarchimedean valuations of *odd* residue characteristic over which E_F has **bad [i.e., multiplicative]** reduction, that is to say, roughly speaking, the *subset* of the set of valuations where E_F has bad multiplicative reduction that will be **"of interest"** to us in the context of the constructions of inter-universal Teichmüller theory.

The above conditions in fact imply that K is *Galois* over F_{mod} [cf. [IUTchI], Remark 3.1.5]. We shall write

$$\underline{\mathbb{V}}^{\text{bad}} \stackrel{\text{def}}{=} \mathbb{V}^{\text{bad}}_{\text{mod}} \times_{\mathbb{V}_{\text{mod}}} \underline{\mathbb{V}} \subseteq \underline{\mathbb{V}}, \quad \mathbb{V}^{\text{good}}_{\text{mod}} \stackrel{\text{def}}{=} \mathbb{V}_{\text{mod}} \setminus \mathbb{V}^{\text{bad}}_{\text{mod}}, \quad \underline{\mathbb{V}}^{\text{good}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\text{bad}}$$

and apply the superscripts "non" and "arc" to $\underline{\mathbb{V}}$, $\mathbb{V}_{\text{mod}}$, and $\mathbb{V}(-)$ to denote the subsets of *nonarchimedean* and *archimedean* valuations, respectively. The data listed above determines, up to K-isomorphism [cf. [IUTchI], Remark 3.1.3], a *finite étale covering* $\underline{C}_K \to C_K$ of degree l such that the base-changed covering

$$\underline{X}_K \stackrel{\text{def}}{=} \underline{C}_K \times_{C_F} X_F \to X_K \stackrel{\text{def}}{=} X_F \times_F K$$

arises from a *rank one quotient* $E_K[l] \twoheadrightarrow Q$ $(\cong \mathbb{Z}/l\mathbb{Z})$ of the module $E_K[l]$ of l-torsion points of $E_K(K)$ [where we write $E_K \stackrel{\text{def}}{=} E_F \times_F K$] which, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, restricts to the quotient arising from *coverings of the dual graph of the special fiber*.

(ii) **The log-theta-lattice:** The **log-theta-lattice**, various versions of which are defined in [IUTchIII] [cf. [IUTchIII], Definitions 1.4; 3.8, (iii)], is a **[highly noncommutative!] two-dimensional diagram** that consists of *three types of components*, namely, $\bullet$'s, $\uparrow$'s, and $\to$'s [cf. the portion of Fig. 3.6 below that lies to the *left* of the "$\supseteq$"]. Each "$\bullet$" in Fig. 3.6 represents a $\Theta^{\pm\text{ell}}$**NF-Hodge theater**, which may be thought of as a sort of *miniature model* of the conventional arithmetic geometry surrounding the given initial Θ-data. Each vertical arrow "$\uparrow$" in Fig. 3.6 represents a **log-link**, i.e., a certain type of **gluing** between various portions of the $\Theta^{\pm\text{ell}}$NF-Hodge theaters that constitute the domain and codomain of the arrow. Each horizontal arrow "$\to$" in Fig. 3.6 represents a **Θ-link** [various versions of which are defined in [IUTchI], [IUTchII], [IUTchIII]], i.e., another type of **gluing** between various portions of the $\Theta^{\pm\text{ell}}$NF-Hodge theaters that constitute the domain and codomain of the arrow. The portion of the log-theta-lattice that is ultimately *actually used* to prove the main results of inter-universal Teichmüller theory is shown in the portion of Fig. 3.6 — i.e., a sort of *"infinite letter H"* — that lies to the *right* of the "$\supseteq$". On the other hand, the significance of considering the *entire log-theta-lattice* may be seen in the fact that — unlike the portion of Fig. 3.6 that lies to the *right* of the "$\supseteq$"! —

> the *[entire] log-theta-lattice* is **symmetric**, up to *unique isomorphism*, with respect to arbitrary **horizontal** and **vertical translations**.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}
\quad \supseteq \quad
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & \xrightarrow{\Theta} & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\vdots & & \vdots
\end{array}
$$

Fig. 3.6: The entire *log-theta-lattice* and the portion that is *actually used*

Various objects constructed from the •'s of the log-theta-lattice will be referred to as **horizontally coric** if they are *invariant* with respect to arbitrary *horizontal* translations, as **vertically coric** if they are *invariant* with respect to arbitrary *vertical* translations, and as **bi-coric** if they are *both* horizontally and vertically coric. In this context, we observe that — unlike any *finite* portion of a *vertical line* of the log-theta-lattice! —

> each *[infinite!] vertical line* of the log-theta-lattice is **symmetric**, up to *unique isomorphism*, with respect to arbitrary **vertical translations**.

As we shall see in §3.6, (iv), below, this is precisely why [cf. the portion of Fig. 3.6 that lies to the *right* of the "$\supseteq$"] it will ultimately be necessary to work with the *entire infinite vertical lines* of the log-theta-lattice [i.e., as opposed to with some finite portion of such a vertical line]. Finally, we remark that

> the **two dimensions** of the log-theta-lattice may be thought of as corresponding to the **two underlying combinatorial dimensions** of a **ring** [cf. the discussion of these two dimensions in the case of NF's and MLF's in §2.11], i.e., to **addition** and **multiplication**.

Indeed, the **Θ-link** only involves the **multiplicative** structure of the rings that appear and, at an *extremely rough level*, may be understood as corresponding to thinking of *"numbers"* as elements of the *multiplicative monoid of positive integers*

$$\mathbb{N}_{\geq 1} \;\cong\; \bigoplus_{p} p^{\mathbb{N}}$$

— where p ranges over the prime numbers, and $\mathbb{N}$ denotes the additive monoid of nonnegative integers — that is to say, as elements of an *abstract monoid* that admits *automorphisms* that *switch distinct prime numbers* p_1, p_2, as well as *endomorphisms* given by *raising to the N-th power* [cf. the discussion of §2.4]. By contrast, the **$\mathfrak{log}$-link** may be understood as corresponding to a *link* between, or **rotation/juggling** of, the **additive** and **multiplicative** structures at the various completions of an NF that is obtained by means of the various *natural logarithms* defined on these completions [cf. the discussion of Example 2.12.3, (v)]. Here, we observe that the **noncommutativity** of the log-theta-lattice [which was mentioned at the beginning of the present (ii)] arises precisely from the fact that

> the definition of the **Θ-link**, which only involves the **multiplicative** structure of the rings that appear, is **fundamentally incompatible** with — i.e., only makes sense once one **deactivates** — the **rotation/juggling** of the **additive** and **multiplicative** structures that arises from the **$\mathfrak{log}$-link**.

In particular, the Θ-*link* may *only be defined* if one *distinguishes* between the *domain* and *codomain* of the $\mathfrak{log}$-*link*, i.e., between *distinct vertical coordinates* in a single vertical line of the log-theta-lattice. Moreover,

> this state of affairs, i.e., which requires one [in order to define the Θ-link!] to **distinguish** the **ring structures** in the domain and codomain of the $\mathfrak{log}$-link [which are related in a **non-ring-theoretic** fashion to one another via the $\mathfrak{log}$-link!], makes it necessary to think of the [possibly tempered] *arithmetic fundamental groups* in the domain and codomain of the $\mathfrak{log}$-link as being related via **indeterminate isomorphisms**

— i.e., as discussed in §2.10; Example 2.12.3, (ii) [cf. the discussion of [IUTchIII], Remark 1.2.2, (vi), (a); [IUTchIII], Remark 1.2.4, (i); [IUTchIV], Remark 3.6.3, (i)]. This situation may be understood by means of the analogy with the situation in **complex Teichmüller theory**:

> one **deforms** *one real dimension* of the complex structure, while *holding the other real dimension* **fixed** — an operation that is *only meaningful* if these two distinct real dimensions are **not subject to rotations**, i.e., to *indeterminacies* with respect to the action of $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ [cf. [IUTchI], Remark 3.9.3, (ii), (iii), (iv)].

Put another way, the portion of the *log-theta-lattice* that is *"actually used"* [cf. Fig. 3.6] exhibits *substantial structural similarities* to the *natural bijection*

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL^+(V)/\mathbb{C}^\times & \stackrel{\sim}{\to} & [0,1) \\ \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right) & \mapsto & \frac{t-1}{t+1} \end{array}$$

[where $\mathbb{R} \ni t \geq 1$] discussed in Example 3.2.1, (iii), that is to say:

> the **deformation of holomorphic structure** "$\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$" may be thought of as corresponding to the **single Θ-link** of this portion of the log-theta-lattice, while the "$\mathbb{C}^\times$'s" on either side of the "$GL^+(V)$" may be thought of as corresponding, respectively, to the vertical lines of **$\mathfrak{log}$-links** — i.e., **rotations** of the **arithmetic holomorphic structure**! — on either side of the single Θ-link.

In the context of this *natural bijection* discussed in Example 3.2.1, (iii), it is of interest to observe that this double coset space "$\mathbb{C}^\times \backslash GL^+(V)/\mathbb{C}^\times$" is also *reminiscent* of the *double coset spaces* associated to groups of matrices over p-adic fields that arise in the theory of **Hecke correspondences**. Alternatively, relative to the analogy with the **two dimensions** of an MLF, if the MLF under consideration is *absolutely unramified*, i.e., isomorphic to the quotient field of a ring of **Witt vectors**, then one may think of

- the **$\mathfrak{log}$-link** as corresponding to the **Frobenius morphism** in positive characteristic, i.e., to *one* of the *two underlying combinatorial dimensions* — namely, the *slope zero* dimension — of the MLF and of
- the **Θ-link** as corresponding to the **mixed characteristic extension structure** of a ring of Witt vectors, i.e., to the *transition from* $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ *to* $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$, which may be thought of as corresponding to the *other* of the *two underlying combinatorial dimensions* — namely, the *positive slope* dimension — of the MLF.

We refer to [IUTchI], §I4; [IUTchIII], Introduction; [IUTchIII], Remark 1.4.1, (iii); [IUTchIII], Remark 3.12.4; [Pano], §2, for more details concerning the numerous analogies between inter-universal Teichmüller theory and various aspects of the ***p*-adic theory**, such as the **canonical liftings** that play a central role in the ***p*-adic Teichmüller theory** of [pOrd], [pTch].

(iii) **The notion of a Frobenioid:** A **Frobenioid** is an *abstract category* whose abstract categorical structure may be thought of, roughly speaking, as encoding

> the theory of **divisors** and **line bundles** on various **"coverings"** — i.e., normalizations in various finite separable extensions of the function field — of a given normal integral scheme.

Here, the category of such "coverings" is referred to as the **base category** of the Frobenioid. All of the Frobenioids that play a [non-negligible] role in inter-universal Teichmüller theory are **model Frobenioids** [cf. [FrdI], Theorem 5.2] whose **base category** corresponds to "some sort of" — that is to say, possibly **tempered**, in the sense of [André], §4; [Semi], Example 3.10 — **arithmetic fundamental group** [i.e., in the non-tempered case, the *étale fundamental group* of a normal integral scheme of finite type over some sort of *"arithmetic field"*]. In particular, all of the Frobenioids that play a [non-negligible] role in inter-universal Teichmüller theory are *essentially equivalent* to a collection of data as follows that satisfies certain properties:

- a **topological group**, i.e., the [possibly tempered] arithmetic fundamental group;
- for each *open subgroup* of the topological group, an abelian group, called the **rational function monoid**, i.e., since it is a category-theoretic abstraction of the multiplicative group of *rational functions* on the "covering" corresponding to the given open subgroup;
- for each *open subgroup* of the topological group, an abelian monoid, called the **divisor monoid**, i.e., since it is a category-theoretic abstraction of the monoid of *Weil divisors* on the "covering" corresponding to the given open subgroup.

In particular, such Frobenioids may be thought of as category-theoretic abstractions of various aspects of the **multiplicative** portion of the ring structure of a normal integral scheme. We refer to §3.5 below for more remarks on the use of Frobenioids in inter-universal Teichmüller theory.

(iv) **<u>$\Theta^{\pm\mathrm{ell}}$NF-]Hodge theaters as "tautological solutions" to a purely combinatorial problem</u>:** The $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** associated to a given collection of *initial* Θ*-data* as in (i) is a somewhat complicated **system** of **Frobenioids** [cf. [IUTchI], Definition 6.13, (i)]. The *topological group* data for these Frobenioids arises from various subquotients of the [possibly tempered] arithmetic fundamental groups of the hyperbolic orbicurves discussed in (i). The *rational function monoid* data for these Frobenioids arises from the multiplicative groups of nonzero elements of various finite extensions of the *number field* F_{mod} of (i) or localizations [i.e., completions] of such NF's at valuations lying over valuations $\in \underline{\mathbb{V}}$. The *divisor monoid* data for these Frobenioids arises, in the case of NF's, from the *monoid of effective arithmetic divisors* [in the sense of diophantine geometry — cf., e.g., [GenEll], §1; [FrdI], Example 6.3], possibly with *real coefficients*, and, in the case of localizations of NF's, from the *nonnegative portion* of the *value group* of the associated valuation, possibly tensored over $\mathbb{Z}$ with $\mathbb{R}$. [In fact, at valuations in $\underline{\mathbb{V}}^{\mathrm{bad}}$, an additional type of Frobenioid, called a *tempered Frobenioid*, also appears — cf. the discussion of §3.4, (iv); §3.5, below.] For instance, in the case of localizations at valuations $\in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$, one Frobenioid that appears quite frequently in inter-universal Teichmüller theory consists of data that is essentially equivalent to the data

$$\text{“}\Pi_X \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}\text{”}$$

considered in Example 2.12.3, (ii). [In the case of valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$, "$\Pi_X$" is replaced by the corresponding *tempered* arithmetic fundamental group; in the case of valuations $\in \underline{\mathbb{V}}^{\mathrm{arc}}$ ($\subseteq \underline{\mathbb{V}}^{\mathrm{good}}$), one applies the theory of [AbsTopIII], §2.] In general, the Frobenioids obtained by applying the operation of *"passing to real coefficients"* are referred to as **realified Frobenioids** [cf. [FrdI], Proposition 5.3]. The system of Frobenioids that constitutes a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater determines, by passing to the associated *system of base categories*, an apparatus that is referred to as a $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** [cf. [IUTchI], Definition 6.13, (ii)]. The purpose of considering such systems of Frobenioids lies in the goal of

> **reassembling** the **distribution of primes** in the number field K [cf. the discussion of [IUTchI], Remark 4.3.1] in such a way as to render possible the construction of some sort of **global version** of the **"Gaussian distribution"**
>
> $$\{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*}$$
>
> discussed at the end of §2.4, i.e., which, *a priori*, is only *defined* at the valuations $\in \mathbb{V}(K)^{\mathrm{non}}$ at which E_K has *bad multiplicative reduction* [such as, for instance, the valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$].

This "global version" amounts to the local and global *value group portions* of the data that appears in the *domain* portion of the Θ**-link** [cf. (vii) below]. The *reassembling*, referred to above, of the *distribution of primes* in the number field K was *one of the fundamental motivating issues* for the author in the development of the **absolute mono-anabelian geometry** of [AbsTopIII], i.e., of a version of anabelian geometry that *differs fundamentally* from the well-known anabelian result of **Neukirch-Uchida** concerning absolute Galois groups of NF's [cf., e.g., [NSW], Chapter XII, §2] in [numerous ways, but, in particular, in] that its reconstruction of an NF does **not depend** on the **distribution of primes** in the NF [cf. the discussion of [IUTchI], Remarks 4.3.1, 4.3.2]. The problem, referred to above, of constructing a sort of **"global Gaussian distribution"** may in fact easily be seen to be

> essentially equivalent to the **"purely combinatorial"** problem of constructing a **"global multiplicative subspace"** [cf. the discussion of §2.3], together with a **"global canonical generator"**, i.e., more precisely: a one-dimensional $\mathbb{F}_l$*-subspace* of the two-dimensional $\mathbb{F}_l$-vector space $E_K[l]$ of l-torsion points of the elliptic curve E_K, together with a *generator*, well-defined up to multiplication by ± 1, of the *quotient* of $E_K[l]$ by this one-dimensional $\mathbb{F}_l$-subspace, such that this subspace and generator *coincide*, at the valuations $\in \mathbb{V}(K)$ that lie over valuations $\in \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, with a certain *canonical* such subspace and generator that arise from a generator [again, well-defined up to multiplication by ± 1] of the *Galois group* [isomorphic to $\mathbb{Z}$] of the well-known *infinite covering* of a *Tate curve.*

Here, we note that such a *"global canonical generator"* determines a *bijection*, which is well-defined up to multiplication by ± 1, of the *quotient* referred to above with the *underlying additive group* of $\mathbb{F}_l$. In a word, the *combinatorial structure* of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater furnishes a sort of **"tautological solution"** to the *purely combinatorial problem* referred to above by

> **simply ignoring** the valuations $\in \mathbb{V}(K) \backslash \underline{\mathbb{V}}$, for instance, by working only with **Frobenioids** — i.e., in effect, **arithmetic divisors/line bundles** — that arise from arithmetic divisors **supported** on the set of valuations $\underline{\mathbb{V}} \quad (\subseteq \mathbb{V}(K))$, i.e., as opposed to on the entire set $\mathbb{V}(K)$

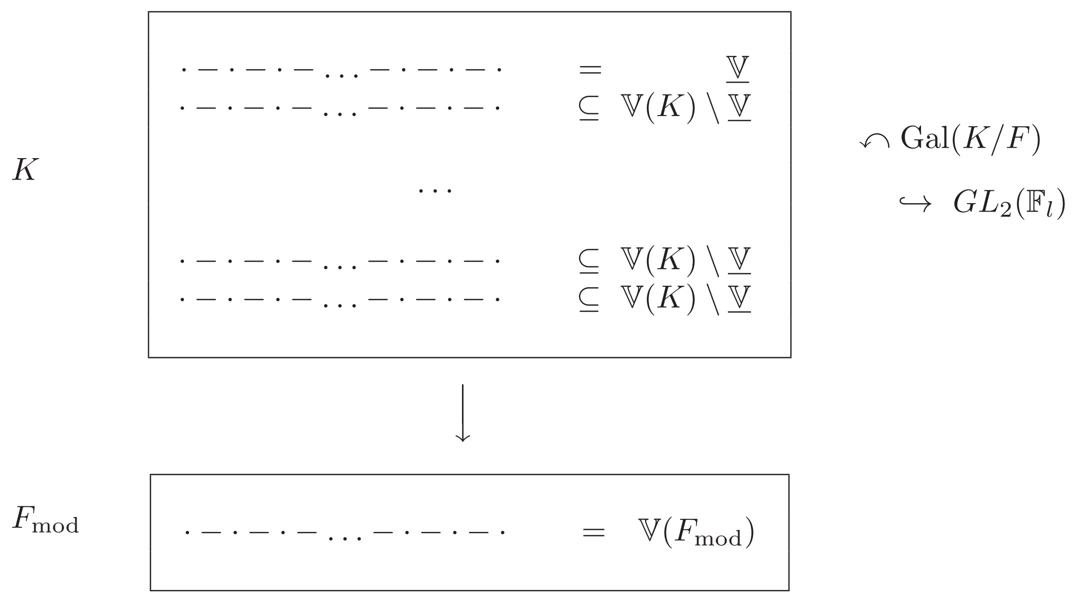

Fig. 3.7: Prime-strips as "sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F_{\mathrm{mod}})$

[cf. the discussion of [IUTchI], Remark 4.3.1]. Here, we note that this sort of operation of *discarding* certain of the primes of an NF can only be performed if one **forgets** the **additive structure** of an NF [i.e., since a sum of elements of an NF that are *invertible* at a given nonempty set of primes is *no longer necessarily invertible* at those primes! — cf. [AbsTopIII], Remark 5.10.2, (iv)] and works only with **multiplicative structures**, e.g., with Frobenioids. Collections of *local data* — consisting, say, of *local Frobenioids* or *local [possibly tempered] arithmetic fundamental groups* — indexed by the elements of $\underline{\mathbb{V}}$ are referred to as **prime-strips** [cf. Fig. 3.7 above; [IUTchI], Fig. I1.2, and the surrounding discussion]. In a word, prime-strips may be thought of as a sort of **monoid-** or **Galois-theoretic** version of the classical notion of **adèles/idèles**.

(v) **<u>The symmetries of a $\Theta^{\pm\text{ell}}$NF-Hodge theater:</u>** Once the *"tautological solution"* furnished by the combinatorial structure of a $\Theta^{\pm\text{ell}}$NF-Hodge theater is applied, the *quotient* of $E_K[l]$ discussed in (iv) corresponds to the quotient "Q" of (i), i.e., in effect, to the **set of cusps** of the hyperbolic curve $\underline{X}_K$ of (i). One may then consider **additive** and **multiplicative symmetries**

$$\mathbb{F}_l^{\rtimes\pm} \stackrel{\text{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}, \quad \mathbb{F}_l^{\circledast} \stackrel{\text{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}$$

— where $\mathbb{F}_l^{\times} \stackrel{\text{def}}{=} \mathbb{F}_l \setminus \{0\}$, and ± 1 acts on $\mathbb{F}_l$ in the usual way — on the *underlying sets*

$$\mathbb{F}_l = \{-l^{\circledast}, \ldots, -1, 0, 1, \ldots, l^{\circledast}\}, \quad \mathbb{F}_l^{\circledast} = \{1, \ldots, l^{\circledast}\}$$

— where $l^{\circledast} \stackrel{\text{def}}{=} (l-1)/2$; the numbers listed in the above display are to be regarded *modulo* l; we think of $\mathbb{F}_l$ as the *quotient* Q [i.e., the *set of cusps* of $\underline{X}_K$ and its localizations at valuations $\in \underline{\mathbb{V}}$] discussed above and of $\mathbb{F}_l^{\circledast}$ as a certain *subquotient* of Q. Here, we remark that this *interpretation* of the quotient Q as a *set of cusps* of $\underline{X}_K$ induces a *natural outer isomorphism* of $\mathbb{F}_l^{\rtimes\pm}$ with the group of **"geometric automorphisms"**

$$\mathrm{Aut}_K(\underline{X}_K)$$

[i.e., the group of K-automorphisms of the K-scheme $\underline{X}_K$], as well as a *natural isomorphism* of $\mathbb{F}_l^{\circledast}$ with a certain quotient of the image of the group of **"arithmetic automorphisms"**

$$\mathrm{Aut}(\underline{C}_K) \hookrightarrow \mathrm{Gal}(K/F_{\text{mod}})$$

[i.e., the group of automorphisms of the algebraic stack $\underline{C}_K$, which, as is easily verified, maps *injectively* into the Galois group $\mathrm{Gal}(K/F_{\text{mod}})$]. The **combinatorial structure** of a $\Theta^{\pm\text{ell}}$NF-Hodge theater may then be summarized as

> the system of Frobenioids obtained by *localizing* and *gluing together* various *Frobenioids* or *[possibly tempered] arithmetic fundamental groups* associated to $\underline{X}_K$ and $\underline{C}_K$ in the fashion prescribed by the **combinatorial recipe**
>
> $$\mathbb{F}_l^{\rtimes\pm} \curvearrowright \mathbb{F}_l \supseteq \mathbb{F}_l^{\times} \twoheadrightarrow \mathbb{F}_l^{\circledast} \curvearrowleft \mathbb{F}_l^{\circledast}$$
>
> — cf. Fig. 3.8 below.

$$
\begin{array}{ccccc}
 & \left[\begin{array}{l} -l^{\circledast} \; < \; \dots \; < \; -1 \; < 0 \\ \qquad\;\; < \; 1 \; < \; \dots \; < \; l^{\circledast} \end{array}\right] & & \left[\begin{array}{l} 1 \; < \; \dots \\ \quad < \; l^{\circledast} \end{array}\right] \\
 & \Uparrow & \Rightarrow \;\; \textbf{glue!} \;\; \Leftarrow & \Uparrow \\
\overset{\{\pm 1\}}{\curvearrowright} & \left(\begin{array}{l} -l^{\circledast} \; < \; \dots \; < \; -1 \; < 0 \\ \qquad\;\; < \; 1 \; < \; \dots \; < \; l^{\circledast} \end{array}\right) & & \left(\begin{array}{l} 1 \; < \; \dots \\ \quad < \; l^{\circledast} \end{array}\right) \\
 & \Downarrow & & \Downarrow \\
 & \begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} & & \begin{array}{ccc} \circledast & \rightarrow & \circledast \\ \uparrow & \overset{\mathbb{F}_l^{\circledast}}{\curvearrowright} & \downarrow \\ \circledast & \leftarrow & \circledast \end{array}
\end{array}
$$

. . . cf. **ordinary monodromy**, **additive symmetries**!

. . . cf. **supersingular monodromy**, **toral symmetries**!

Fig. 3.8: The *combinatorial structure* of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater: a bookkeeping apparatus for *l-torsion points*

Here, we remark that, in Fig. 3.8:

- the **squares** with actions by $\mathbb{F}_l^{\rtimes\pm}$ and $\mathbb{F}_l^{\circledast}$ correspond to Frobenioids or arithmetic fundamental groups that arise from $\underline{X}_K$ and $\underline{C}_K$, respectively;
- *each of the elements* of $\mathbb{F}_l$ or $\mathbb{F}_l^{\circledast}$ that appears in *parentheses* "(. . .)" corresponds to a **single prime-strip**;
- *each portion enclosed in brackets* "[. . .]" corresponds to a **single prime-strip**;
- the arrows "$\Uparrow$" correspond to the relation of passing from the various **individual elements** of $\mathbb{F}_l$ or $\mathbb{F}_l^{\circledast}$ [i.e., one prime-strip for each individual element] to the **entire set** $\mathbb{F}_l$ or $\mathbb{F}_l^{\circledast}$ [i.e., one prime-strip for the entire set];
- the arrows "$\Downarrow$" correspond to the relation of regarding the set of elements in parentheses "(. . .)" with **fixed labels** as the underlying set of a set equipped with an **action** by $\mathbb{F}_l^{\rtimes\pm}$ or $\mathbb{F}_l^{\circledast}$;
- the **gluing** is the gluing prescribed by the **surjection** $\mathbb{F}_l \supseteq \mathbb{F}_l^{\times} \twoheadrightarrow \mathbb{F}_l^{\circledast}$.

In this context, it is important to keep in mind that

> the $\mathbb{F}_l^{\rtimes\pm}$**- and** $\mathbb{F}_l^{\circledast}$**-symmetries** of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater play a *fundamental role* in the **Kummer-theoretic** aspects of inter-universal Teichmüller theory that are discussed in §3.6 below.

As remarked in §2.4, the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater may be thought of as corresponding, respectively, to the **additive** and **multiplicative/toral symmetries** of the *classical upper half-plane* [cf. [IUTchI], Remark 6.12.3, (iii); [BogIUT], for more details]. Alternatively, the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater may be thought of as corresponding, respectively, to the **[unipotent] ordinary** and **[toral] supersingular monodromy**

— i.e., put another way, to the well-known structure of the ***p*-Hecke correspondence** — that occurs in the well-known classical p-adic theory surrounding the **moduli stack of elliptic curves** over the p-adic integers $\mathbb{Z}_p$ [cf. the discussion of [IUTchI], Remark 4.3.1; [IUTchII], Remark 4.11.4, (iii), (c)].

(vi) **$\mathfrak{log}$-links:** Each vertical arrow

$$\begin{array}{c} \bullet \\ \uparrow \mathfrak{log} \\ \bullet \end{array}$$

of the log-theta-lattice relates the various *copies* of

$$\text{“}\Pi_X \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}\text{”}$$

[cf. the discussion at the beginning of (iv)] that lie in the *prime-strips* of the *domain* $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater “$\bullet$” of the $\mathfrak{log}$-*link* to the *corresponding copy* of “$\Pi_X \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$” that lies in a *prime-strip* of the *codomain* $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater “$\bullet$” of the $\mathfrak{log}$-*link* in the fashion prescribed by the arrow “$\mathfrak{log}$” of the diagram of Example 2.12.3, (iii) [with suitable modifications involving *tempered* arithmetic fundamental groups at the valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$ or the theory of [AbsTopIII], §2, at the valuations $\in \underline{\mathbb{V}}^{\mathrm{arc}}$ ($\subseteq \underline{\mathbb{V}}^{\mathrm{good}}$)]. In particular, the $\mathfrak{log}$-link may be thought of as “lying over” an **isomorphism** between the respective copies of “Π_X” which is **indeterminate** since [cf. the discussion of §2.10; the discussion at the end of Example 2.12.3, (ii)] the two copies of “Π_X” must be regarded as *distinct abstract topological groups*. Put another way, from the point of view of the discussion at the beginning of (iv),

> the $\mathfrak{log}$-link induces an **indeterminate isomorphism** between the **$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** associated to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters “$\bullet$” in the *domain* and *codomain* of the $\mathfrak{log}$-link, that is to say, these $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters associated to the $\bullet$’s of the log-theta-lattice are **vertically coric** [cf. [IUTchIII], Theorem 1.5, (i)].

Now recall from Example 2.12.3, (i), that each *abstract topological group* “Π_X” may be regarded as the *input data* for a **functorial algorithm** that allows one to reconstruct the **base field** [in this case an MLF] of the hyperbolic curve “X”. Put another way, from the point of view of the terminology discussed in §2.7, (vii), each copy of “Π_X” may be regarded as an **arithmetic holomorphic structure** on the quotient group “$\Pi_X \twoheadrightarrow G_k$” associated to Π_X [cf. the discussion of Example 2.12.3, (i)]. Indeed, this is precisely the point of the *analogy* between the *fundamental prototypical examples* — i.e., Examples 3.2.1, 3.2.2 — of the phenomenon of *multiradiality*. The various “X’s” that occur in a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater are certain *finite étale coverings* of localizations of the hyperbolic curve $\underline{X}_K$ at various valuations $\in \underline{\mathbb{V}}$. These finite étale coverings are hyperbolic curves over $K_{\underline{v}}$ which are denoted $\underline{\underline{X}}_{\underline{v}}$ in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and $\underline{X}_{\rightarrow\underline{v}}$ in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ [cf. [IUTchI], Definition 3.1, (e), (f)]. On the other hand, we recall from [AbsTopIII], Theorem 1.9 [cf. also [AbsTopIII], Remark 1.9.2], that this **functorial algorithm** may also be applied to the hyperbolic orbicurves $\underline{X}_K$, $\underline{C}_K$, or $C_{F_{\mathrm{mod}}}$, i.e., whose *base fields* are **NF’s**, in a fashion that is **functorial** [cf. the discussion of [IUTchI], Remarks 3.1.2, 4.3.2] with respect to passing to **finite étale coverings**, as well as with respect to **localization** at valuations of $\in \underline{\mathbb{V}}$ [cf. also the theory of [AbsTopIII], §2, in the case of valuations $\in \underline{\mathbb{V}}^{\mathrm{arc}}$]. That is to say, in summary,

> the various [possibly *tempered*, in the case of valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$] **arithmetic fundamental groups** of finite étale coverings of $C_{F_{\mathrm{mod}}}$ [such as $\underline{X}_K$, $\underline{C}_K$, or $C_{F_{\mathrm{mod}}}$ itself] and their localizations at valuations $\in \underline{\mathbb{V}}$ that appear in a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater may be regarded as *abstract representations* of the **arithmetic holomorphic structure** [i.e., *ring structure* — cf. the discussion of §2.7, (vii)] of the various *base fields* of these hyperbolic orbicurves.

Moreover, this state of affairs motivates the point of view that

> the various **localizations** and **gluings** that occur in the structure of a **single** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** [cf. Fig. 3.8] or, as just described, in the structure of a $\mathfrak{log}$**-link** [i.e., a vertical arrow of the log-theta-lattice — cf. Fig. 3.6] may be thought of as **arithmetic analytic continuations** between various **NF's** along the various gluings of **prime-strips** that occur [cf. the discussion of [IUTchI], Remarks 4.3.1, 4.3.2, 4.3.3, 5.1.4].

In this context, it is of interest to observe that, at a technical level, these *arithmetic analytic continuations* are achieved by applying the *mono-anabelian theory* of [AbsTopIII], §1 [or, in the case of *archimedean* valuations, the theory of [AbsTopIII], §2]. Moreover, this mono-anabelian theory of [AbsTopIII], §1, is, in essence, an elementary consequence of the theory of **Belyi cuspidalizations** developed in [AbsTopII], §3 [cf. [AbsTopIII], Remark 1.11.3]. Here, we recall that the term *cuspidalization* refers to a functorial algorithm in the arithmetic fundamental group of a hyperbolic curve for reconstructing the arithmetic fundamental group of some *dense open subscheme* of the hyperbolic curve. In particular, by considering *Kummer classes of rational functions* [cf. the discussion of Example 2.13.1, (i)],

> **cuspidalization** may be thought of as a sort of **"equivalence"** between the function theory on a hyperbolic curve and the function theory on a *dense open subscheme* of the hyperbolic curve — a formulation that is very *formally reminiscent* of the *classical notion of* **analytic continuation**.

The **Belyi cuspidalizations** developed in [AbsTopII], §3, are achieved as a formal consequence of the *elementary observation* that

> any "sufficiently small" **dense open subscheme** U of the hyperbolic curve P given by removing three points from the projective line may be regarded — via the use of a suitable **Belyi map**! — as a **finite étale covering** of P; in particular, the arithmetic fundamental group of U may be recovered from the arithmetic fundamental group of P by considering a *suitable open subgroup* of the arithmetic fundamental group of P [cf. [AbsTopII], Example 3.6; [AbsTopII], Corollaries 3.7, 3.8, for more details].

This state of affairs is all the more fascinating in that the well-known *construction of Belyi maps* via an induction on the *degree over* $\mathbb{Q}$ *of the ramification locus* of certain rational maps between two projective lines is [cf. the discussion of [IUTchI], Remark 5.1.4] *highly reminiscent* of the well-known **Schwarz lemma** of elementary complex analysis, i.e., to the effect that the absolute value, relative to the respective

Poincaré metrics, of the derivative at any point of a holomorphic map between copies of the unit disc is ≤ 1.

(vii) <u>**Θ-links:**</u> Various versions of the **"Θ-link"** are defined in [IUTchI], [IUTchII], [IUTchIII] — cf. [IUTchI], Corollary 3.7, (i); [IUTchII], Corollary 4.10, (iii); [IUTchIII], Definition 3.8, (ii). In the present paper, we shall primarily be interested in the version of [IUTchIII], Definition 3.8, (ii); the versions of [IUTchII], Corollary 4.10, (iii), are partially simplified versions of the version that one is ultimately interested in [i.e., the version of [IUTchIII], Definition 3.8, (ii)], while the version of [IUTchI], Corollary 3.7, (i), is an even more drastically simplified version of these partially simplified versions. The Θ-link may be understood, roughly speaking, as a realization of the version of the *assignment "$q \mapsto q^N$"* considered in the final portion of the discussion of §2.4, i.e., the *assignment*

$$\text{“}\underline{\underline{q}} \mapsto \{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*}\text{”}$$

given by taking a sort of *symmetrized average* as "N" varies over the values j^2, for $j = 1, \ldots, l^*$. At a more technical level, the Θ-link

$$\bullet \ \stackrel{\Theta}{\longrightarrow} \ \bullet$$

is a **gluing** between two $\Theta^{\pm\text{ell}}$NF-Hodge theaters "•", via an **indeterminate** [cf. the discussion of §2.10] **isomorphism** between certain *gluing data* arising from the *domain* $\Theta^{\pm\text{ell}}$NF-Hodge theater "•" and certain *gluing data* arising from the *codomain* $\Theta^{\pm\text{ell}}$NF-Hodge theater "•". The **gluing data** that arises from the **domain** "•" of the Θ-link is as follows:

(a^Θ) <u>**[Local] unit group portion:**</u> Consider, in the notation of §2.11, §2.12 [cf., especially, Example 2.12.3, (iv)], the *ind-topological monoid* equipped with an *action by a topological group* $G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$, where we note that $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$ is a $\mathbb{Q}_p$*-vector space*. Observe that for each open subgroup $H \subseteq G_k$, which determines a subfield $\overline{k}^H \subseteq \overline{k}$ of H-invariants of $\overline{k}$, the image of $\mathcal{O}^{\times}_{\overline{k}^H}$ in $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$ determines an *integral structure*, or *"lattice"* [i.e., a finite free $\mathbb{Z}_p$-module], in the $\mathbb{Q}_p$-vector subspace of H-invariants $(\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}})^H$ of $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$. [Here, we note that the theory of the *p-adic logarithm* determines a *natural isomorphism* between this subspace and the $\mathbb{Q}_p$-vector $\overline{k}^H$.] For each $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, we take the *unit group portion data at* $\underline{v}$, to be this data

$$(G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}, \ \{\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}^H} \subseteq (\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}})^H\}_H)$$

— i.e., which we regard as an *ind-topological monoid equipped with an action by a topological group*, together with, *for each open subgroup of the topological group, an integral structure* — in the case $k \stackrel{\text{def}}{=} K_{\underline{v}}$. When $k = K_{\underline{v}}$, we shall write $G_{\underline{v}} \stackrel{\text{def}}{=} G_k$. An analogous construction may be performed for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.

(b^Θ) <u>**Local value group portion:**</u> At each $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, we take the *local value group portion data at* $\underline{v}$ to be the *formal monoid* [abstractly isomorphic to the monoid $\mathbb{N}$] generated by

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\ldots,l^*}$$

— i.e., where $\underline{\underline{q}}_{\underline{v}} \in \mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ is a $2l$-th root of the *q-parameter* $q_{\underline{v}}$ of the elliptic curve E_K at $\underline{v}$; the data of the above display is regarded as a collection of elements of $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ indexed by the elements of $\mathbb{F}_l^{\circledast}$; each element of this collection is well-defined up to multiplication by a *2l-th root of unity*. An analogous, though somewhat *more formal*, construction may be performed for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$.

(c^{Θ}) **Global value group portion:** Observe that each of the *local formal monoids* [say, for simplicity, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$] of ($b^{\Theta}$) may be *realified*. That is to say that, the corresponding *realified monoid* is simply the monoid of $\mathbb{R}_{\geq 0}$-multiples [i.e., nonnegative real multiples] of the image of the given monoid [$\cong \mathbb{N}$] inside the tensor product $\otimes_{\mathbb{Z}}\mathbb{R}$ of the *groupification* [$\cong \mathbb{Z}$] of this given monoid. Note, moreover, that the *product formula* of elementary algebraic number theory yields a natural notion of "*finite collections* of elements of the *groupifications* [$\cong \mathbb{R}$] of these realified monoids at $\underline{v} \in \underline{\mathbb{V}}$ whose *sum* $= 0$". This data, consisting of a realified monoid at each $\underline{v} \in \underline{\mathbb{V}}$, together with a collection of "product formula relations", determines a **global realified Frobenioid**. We take the *global value group portion data* to be this global realified Frobenioid.

The **gluing data** that arises from the **codomain** of the Θ-link is as follows:

(a^q) **[Local] unit group portion:** For each $\underline{v} \in \underline{\mathbb{V}}$, we take the *unit group portion data at* $\underline{v}$ to be the *analogous data*, i.e., this time constructed from the *codomain* "•" of the Θ-link, to the data of (a^{Θ}).

(b^q) **Local value group portion:** At each $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, we take the *local value group portion data at* $\underline{v}$ to be the *formal monoid* [abstractly isomorphic to the monoid $\mathbb{N}$] generated by

$$\underline{\underline{q}}_{\underline{v}}$$

— i.e., where we apply the notational conventions of (b^{Θ}). An analogous, though somewhat *more formal*, construction may be performed for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$.

(c^q) **Global value group portion:** We take the *global value group portion data* to be the *global realified Frobenioid* [cf. the data of (c^{Θ})] determined by the *realifications* of the *local formal monoids* of (b^q) at $\underline{v} \in \underline{\mathbb{V}}$, together with a naturally determined collection of "product formula relations".

In fact, the above description is *slightly inaccurate* in a number of ways: for instance, in [IUTchI], [IUTchII], [IUTchIII], the data of (a^{Θ}), (b^{Θ}), (c^{Θ}), (a^q), (b^q), (c^q) are constructed in a somewhat more **intrinsic** fashion directly from the various **Frobenioids** [and other data] that constitute the $\Theta^{\pm\text{ell}}$NF-Hodge theater "•" under consideration. This sort of *intrinsic* construction exhibits, in a very natural fashion,

the *ind-topological monoids* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$" of ($a^{\Theta}$) and ($a^q$), the *local formal monoids* of (b^{Θ}) and (b^q), and the *global realified Frobenioids* of (c^{Θ}) and (c^q) as **Frobenius-like** objects.

By contrast,

> the *topological groups* "G_k" of (a^Θ) and (a^q) are **étale-like** objects.

In this context, it is useful to note — cf. the discussion of the **vertical coricity** of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters in (vi) — that

> the **unit group portion data** of (a^Θ), (a^q) is **horizontally coric** [cf. [IUTchIII], Theorem 1.5, (ii)], while the portion of this data constituted by the **topological group** "G_k" is **bi-coric** [cf. [IUTchIII], Theorem 1.5, (iii)].

Indeed,

> one way to think of **Frobenius-like** structures, in the context of the log-theta-lattice, is as structures that, at least *a priori*, are **confined to** — i.e., at least *a priori*, are **only defined in** — a **fixed** $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater "$\bullet$" of the log-theta-lattice.

Here, we note that

> since [just as in the case of the $\mathfrak{log}$-link — cf. the discussion in the final portion of (ii)!] the Θ-link is **fundamentally incompatible** with the **ring structures** in its domain and codomain, it is necessary to think of the **bi-coric** topological group "G_k" as being only well-defined up to some **indeterminate isomorphism** [cf. the discussion of §2.10; [IUTchIII], Remark 1.4.2, (i), (ii); [IUTchIV], Remark 3.6.3, (i)].

Thus, in summary, the

> **Θ-link** induces an **isomorphism** of the **unit group portion data** of (a^Θ), (a^q), on the one hand, and a **dilation**, by a *factor* given by a sort of *symmetrized average* of the j^2, for $j = 1, \ldots, l^*$, of the **local** and **global value group data** of (b^Θ), (c^Θ), (b^q), (c^q), on the other.

The object, well-defined up to *isomorphism*, of the *global realified Frobenioid* of (c^Θ) determined by the unique collection of generators of the *local formal monoids* of (b^Θ) at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ will be referred to as the **Θ-pilot object** [cf. [IUTchI], Definition 3.8, (i)]. In a similar vein, the object, well-defined up to *isomorphism*, of the *global realified Frobenioid* of (c^q) determined by the unique collection of generators of the *local formal monoids* of (b^q) at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ will be referred to as the ***q*-pilot object** [cf. [IUTchI], Definition 3.8, (i)]. The Θ- and q-pilot objects play a *central role* in the *main results* of inter-universal Teichmüller theory, which are the main topic of §3.7 below.

§**3.4.** **<u>Kummer theory and multiradial decouplings/cyclotomic rigidity</u>:** The *first main result* of inter-universal Teichmüller theory [cf. §3.7, (i)] consists of a **multiradial representation** of the **Θ-pilot objects** discussed in §3.3, (vii). Relative to the general discussion of multiradiality in §3.1, this multiradiality may be understood as being with respect to the *radial algorithm*

$$(a^\Theta),\ (b^\Theta),\ (c^\Theta) \quad \mapsto \quad (a^\Theta)$$

that associates to the *gluing data* in the *domain* of the Θ-*link* the *horizontally coric unit group portion* of this data [cf. the discussion of §3.3, (vii)]. The construction of this multiradial representation of Θ-pilot objects consists of *two steps*. The *first step*, which we discuss in detail in the present §3.4, is the construction of **multiradial cyclotomic rigidity** and **decoupling** algorithms for certain **special types of functions** on the hyperbolic curves under consideration. The *second step*, which we discuss in detail in §3.6 below, concerns the **Galois evaluation** at certain special points — i.e., evaluation via *Galois sections* of *arithmetic fundamental groups* — of these functions to obtain certain **special values** that act on **processions** of **log-shells**.

(i) **The essential role of Kummer theory:** We begin with the *fundamental observation* that, despite the fact, for $\Box \in \{\Theta, q\}$, the construction of the data $(a^{\Box})$ (respectively, $(b^{\Box})$; $(c^{\Box})$) *depends quite essentially* on whether $\Box = \Theta$ or $\Box = q$, the **indeterminate gluing isomorphism** that constitutes a Θ-*link* **exists** — i.e., the data (a^{Θ}) (respectively, (b^{Θ}); (c^{Θ})) is indeed *isomorphic* to the data (a^{q}) (respectively, (b^{q}); (c^{q})) — precisely as a consequence of the fact that we regard the [ind-topological] monoids that occur in this data as **abstract [ind-topological] monoids** that are **not** equipped with the *auxiliary data* of how these [ind-topological] monoids *"happen to be constructed"*. That is to say, the inclusion of such auxiliary data would render the corresponding portions of data in the *domain* and *codomain* of the Θ-link *non-isomorphic*! Such *abstract [ind-topological] monoids* are a sort of prototypical example of the notion of a *Frobenius-like* structure [cf. [IUTchIV], Example 3.6, (iii)]. A similar observation applies to the copies of "$\mathcal{O}^{\rhd}_{\overline{k}}$" that occur in the discussion of the $\mathfrak{log}$-*link* in §3.3, (vi). Thus, in summary,

> it is precisely by working with **Frobenius-like structures** such as abstract [ind-topological] monoids or abstract [global realified] Frobenioids that we are able to construct the **non-ring/scheme-theoretic gluing isomorphisms** [i.e., "non-ring/scheme-theoretic" in the sense that they do not arise from *morphisms of rings/schemes*!] of the **$\mathfrak{log}$- and Θ-links** of §3.3, (vi), (vii) [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

By contrast,

> **étale-like structures** such as the "G_k's" of (a^{Θ}) and (a^{q}) [cf. §3.3, (vii)] or the "Π_X's" of §3.3, (vi), will be used to **compute** various portions of the ring/scheme theory on the *opposite side* of a $\mathfrak{log}$- or Θ-link via the technique of **mono-anabelian transport**, as discussed in §2.7, §2.9, i.e., by determining the sort of **indeterminacies** that one must admit in order to render the two systems of **Kummer theories** — which, we recall, are applied in order to relate **Frobenius-like structures** to corresponding **étale-like structures** — in the *domain* and *codomain* of the $\mathfrak{log}$- or Θ-link **compatible** with **simultaneous execution**.

Here, we recall that *Kummer classes* are obtained, in essence, by considering cohomology classes that arise from the action of various Galois or arithmetic fundamental groups on the various *roots* of elements of an *abstract monoid* [cf. Examples 2.6.1, (iii); 2.12.1, (i); 2.13.1, (i)]. Thus, the key step in rendering such Kummer classes *independent* of any *Frobenius-like structures* lies in the *algorithmic construction* of a **cyclotomic rigidity isomorphism** between the group of torsion elements of the abstract monoid under consideration and some sort of **étale-like** cyclotome, i.e., that is constructed directly from the Galois or arithmetic fundamental group under consideration [cf. the isomorphism "λ" of Example 2.6.1, (iii), (iv); the isomorphism "$\rho_{\boldsymbol{\mu}_{\overline{k}}}$" of Example 2.12.1, (i), (ii), (iv); the isomorphism "λ" of Example 2.13.1, (i), (ii)]. On the other hand, let us observe, relative to the **multiradiality** mentioned at the beginning of the present §3.4, that the *coric data* "$G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$" admits a [nontrivial!] **natural action** by $\widehat{\mathbb{Z}}^{\times}$

$$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$$

[i.e., which is G_k*-equivariant* and compatible with the various *integral structures* that appear in the coric data] that *lifts* to a natural $\widehat{\mathbb{Z}}^{\times}$-action on $\mathcal{O}^{\times}_{\overline{k}}$ [cf. Example 2.12.2, (i)]. This $\widehat{\mathbb{Z}}^{\times}$-action induces a *trivial* action of $\widehat{\mathbb{Z}}^{\times}$ on G_k [hence also on $\boldsymbol{\mu}_{\overline{k}}(G_k)$], but a *nontrivial* action of $\widehat{\mathbb{Z}}^{\times}$ on $\boldsymbol{\mu}_{\overline{k}}$. In particular, this $\widehat{\mathbb{Z}}^{\times}$-action on $\mathcal{O}^{\times}_{\overline{k}}$ is **manifestly incompatible** with the **cyclotomic rigidity isomorphism**

$$\rho_{\boldsymbol{\mu}_{\overline{k}}} : \boldsymbol{\mu}_{\overline{k}} \stackrel{\sim}{\to} \boldsymbol{\mu}_{\overline{k}}(G_k)$$

that was *functorially constructed* in Example 2.12.1, (ii), hence, in light of the *functoriality* of this construction, does *not extend* to an action of $\widehat{\mathbb{Z}}^{\times}$ on $\mathcal{O}^{\rhd}_{\overline{k}}$. That is to say,

> the naive approach just discussed to **cyclotomic rigidity isomorphisms** via the functorial construction of Example 2.12.1, (ii), is **incompatible** with the requirement of **multiradiality**, i.e., of the existence of *liftings* of arbitrary morphisms between collections of *coric data.*

This discussion motivates the following approach, which is *fundamental* to inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.2.1, (iii); [IUTchIII], Remark 2.2.2]:

> in order to obtain **multiradial cyclotomic rigidity isomorphisms** for the *local and global value group data* (b^{Θ}) and (c^{Θ}), it is necessary to somehow **decouple** this data (b^{Θ}) and (c^{Θ}) from the *unit group data* of (a^{Θ}).

This *decoupling* is achieved in inter-universal Teichmüller theory by working with certain *special types of functions*, as described in (ii), (iii), (iv), below.

(ii) **Multiradial decouplings/cyclotomic rigidity for κ-coric rational functions:** The *global realified Frobenioids* of (c^{Θ}) may be interpreted as *"realifications"* of certain *categories of [l^*-tuples, indexed by $j = 1, \ldots, l^*$, of] arithmetic line bundles* on the number field F_{mod}. The **ring structure** — i.e., both the *additive* "$\boxplus$" and *multiplicative* "$\boxtimes$" structures — of copies of this number field F_{mod} is applied, ultimately, in inter-universal Teichmüller theory, in order to relate these **global realified Frobenioids** of (c^{Θ}), which are, in essence, a **multiplicative** notion, to the interpretation of arithmetic line bundles in terms of **log-shells**, which are *modules*, i.e., whose group law is written **additively** [cf. [IUTchIII], Remarks 3.6.2, 3.10.1] — an interpretation with respect to which *global arithmetic degrees* correspond to *log-volumes* of certain regions inside the various log-shells at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of §2.2]. Thus, in summary,

> the *essential structure of interest* that gives rise to the data of (c^{Θ}) consists of copies of the **number field** F_{mod} indexed by $j = 1, \ldots, l^{*}$.

In particular, the **Kummer theory** [cf. the discussion of (i)!] concerning the data of (c^{Θ}) revolves around the Kummer theory of such copies of the number field F_{mod}. As discussed at the beginning of the present §3.4, elements of such copies of F_{mod} will be constructed as *special values* at certain *special points* of certain *special types of functions*. Here, it is perhaps of interest to recall that

> this approach to constructing elements of the *base field* [in this case, the number field F_{mod}] of a hyperbolic curve — i.e., by **evaluating Kummer classes** of **rational functions** on the hyperbolic curve at certain **special points** — is precisely the approach that is in fact applied in the **mono-anabelian reconstruction algorithms** discussed in [AbsTopIII], §1.

As discussed at the beginning of the present §3.4 [cf. also the discussion of (i)], the *first step* in the construction of multiradial representations of Θ-pilot objects to be discussed in §3.7, (i), consists of formulating the **Kummer theory** of *suitable special types of rational functions* in such a way that we obtain **multiradial cyclotomic rigidity isomorphisms** that involve a **decoupling** of this Kummer theory for rational functions from the *unit group data* of (a^{Θ}). In the present case, i.e., which revolves around the construction of copies of F_{mod},

> the desired formulation of Kummer theory is achieved by considering a certain subset — called the **pseudo-monoid of κ-coric rational functions** [cf. [IUTchI], Remark 3.1.7, (i), (ii)] — of the group [i.e., multiplicative monoid] "$K_X^{\times}$" considered in Example 2.13.1, in the case where the hyperbolic curve "X" is taken to be the hyperbolic curve $\underline{X}_K$ of §3.3, (i) [cf. [IUTchI], Remark 3.1.2, (ii)].

Recall the hyperbolic orbicurve $C_{F_{\mathrm{mod}}}$ discussed in §3.3, (i). Write $|C_{F_{\mathrm{mod}}}|$ for the *coarse space* $|C_{F_{\mathrm{mod}}}|$ associated to $C_{F_{\mathrm{mod}}}$. Here, it is useful to recall the well-known fact that $|C_{F_{\mathrm{mod}}}|$ is isomorphic to the *affine line* over L. We shall refer to the points of the compactification of $|C_{F_{\mathrm{mod}}}|$ that arise from the 2-*torsion points* of the elliptic curve E_F other than the origin as *strictly critical.* A **κ-coric rational function** is a rational function on $|C_{F_{\mathrm{mod}}}|$ that

> **restricts** to a **root of unity** at each **strictly critical point** of $|C_{F_{\mathrm{mod}}}|$

and, moreover, satisfies certain other [somewhat less essential] technical conditions [cf. [IUTchI], Remark 3.1.7, (i)]. Thus, a κ-coric rational function on $|C_{F_{\mathrm{mod}}}|$ may also be regarded, by restriction, as a rational function on $\underline{X}_K$. Although the κ-coric rational functions do not form a *monoid* [i.e., the product of two κ-coric rational functions is not necessarily a κ-coric rational function], it nevertheless holds that *arbitrary positive powers* of κ-coric rational functions are κ-coric. Moreover, every root of unity in F_{mod} is κ-coric; a rational function on $|C_{F_{\mathrm{mod}}}|$ is κ-coric if and only if some positive power of the rational function is κ-coric. One verifies immediately that [despite the fact that the κ-coric rational functions do not form a *monoid*] these elementary properties that *are* satisfied by κ-coric rational functions are *sufficient* for conducting **Kummer theory** with κ-coric rational functions. Then [cf. [IUTchI], Example 5.1, (v), for more details]:

· The desired **decoupling** of the pseudo-monoid of κ-coric rational functions from the *unit group data* of (a^{Θ}) is achieved by means of the condition that **evaluation** at any of the **strictly critical points** — an operation that may be performed at the level of **étale-like structures**, i.e., by **restricting Kummer classes** to decomposition groups of points — yields a root of unity.

· The desired **multiradial cyclotomic rigidity isomorphism** is achieved by means of the technique discussed in Example 2.13.1, (ii) — i.e., involving the elementary fact

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

— which is applied to the *pseudo-monoid of κ-coric rational functions*, i.e., as opposed to the entire multiplicative monoid $K_X^{\times}$.

As was mentioned in Example 2.13.1, (ii), this approach has the **disadvantage** of being **incompatible** with the **profinite topology** of the Galois or arithmetic fundamental groups involved [cf. the discussion of (iii) below; §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii)]. Also, we remark that although this approach only allows one to reconstruct the desired cyclotomic rigidity isomorphism up to *multiplication by* ± 1, this will not yield any problems since we are, in fact, only interested in reconstructing copies of the *entire multiplicative monoid* $F_{\mathrm{mod}}^{\times}$, which is *closed* under inversion [cf. the discussion of [IUTchIII], Remark 2.3.3, (vi); [IUTchIII], Remark 3.11.4].

(iii) **Naive approach to cyclotomic rigidity for theta functions:** Fix $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Denote by means of a *subscript* $\underline{v}$ the result of *base-changing* objects over K to $K_{\underline{v}}$. Thus, $X_{\underline{v}}$ is a *"once-punctured Tate curve"* over $K_{\underline{v}}$, hence determines a *one-pointed stable curve of genus one* $\mathfrak{X}_{\underline{v}}$ over the ring of integers $\mathcal{O}_{K_{\underline{v}}}$ of $K_{\underline{v}}$ [where, for simplicity, we omit the notation for the *single marked point*, which arises from the *cusp* of $X_{\underline{v}}$]. In particular, the *dual graph* of the special fiber of $X_{\underline{v}}$ [i.e., more precisely: of $\mathfrak{X}_{\underline{v}}$] is a *"loop"* [i.e., more precisely, consists of a *single vertex* and a *single edge*, both ends of which abut to the single vertex]. In particular, the *universal covering* [in the sense of classical algebraic topology!] of this dual graph determines [what is called] a **tempered covering** $Y_{\underline{v}} \to X_{\underline{v}}$ [i.e., at the level of models over $\mathcal{O}_{K_{\underline{v}}}$, a tempered covering $\mathfrak{Y}_{\underline{v}} \to \mathfrak{X}_{\underline{v}}$ — cf. [André], §4; [Semi], Example 3.10], whose *Galois group* $\underline{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(Y_{\underline{v}}/X_{\underline{v}})$ is *noncanonically isomorphic* to $\mathbb{Z}$. Thus,

the special fiber of $Y_{\underline{v}}$ [i.e., more precisely: of $\mathfrak{Y}_{\underline{v}}$] consists of an *infinite chain* of copies of the "once-punctured/one-pointed projective line", in which the *"punctures/cusps"* correspond to the points "1" of the copies of the projective line, and the point "∞" of *each such copy* is glued to the point "0" of the *adjacent* copy [cf. the upper portion of Fig. 3.9 below; the discussion at the beginning of [EtTh], §1; [IUTchII], Proposition 2.1;

[IUTchII], Remark 2.1.1].

labels:	-2	-1	0	1	2	
$\ldots$	$\boxed{0 \quad * \quad \infty}$	$\boxed{0 \quad * \quad \infty}$	$\boxed{0 \quad * \quad \infty}$	$\boxed{0 \quad * \quad \infty}$	$\boxed{0 \quad * \quad \infty}$	$\ldots$
orders of zeroes:	1	1	1	1	1	
poles:	2^2	1^2	0	1^2	2^2	

Fig. 3.9: *Labels* of irreducible components and *orders* of *zeroes* at *cusps* "$*$" and *poles* at *irreducible components* "$\Box$" of the *theta function* $\ddot{\Theta}_{\underline{v}}$ on $\ddot{Y}_{\underline{v}}$

If one *fixes* one such copy of the once-punctured projective line, together with an isomorphism $\underline{\mathbb{Z}} \overset{\sim}{\to} \mathbb{Z}$, then the natural action of $\underline{\mathbb{Z}}$ on $Y_{\underline{v}}$ determines a *natural bijection* of the set of *irreducible components of the special fiber* of $Y_{\underline{v}}$ — or, alternatively, of the set of *cusps* of $Y_{\underline{v}}$ — with $\mathbb{Z}$ [cf. the **"labels"** of Fig. 3.9]. The *"multiplication by* 2*"* endomorphism of the *elliptic curve* $E_{\underline{v}}$ [which may be thought of as the compactification of the *affine hyperbolic curve* $X_{\underline{v}}$] determines, via base-change by $Y_{\underline{v}} \to X_{\underline{v}}$, a *double covering* $\ddot{Y}_{\underline{v}} \to Y_{\underline{v}}$ [i.e., at the level of models over $\mathcal{O}_{K_{\underline{v}}}$, a double covering $\ddot{\mathfrak{Y}}_{\underline{v}} \to \mathfrak{Y}_{\underline{v}}$]. One verifies immediately that the set of irreducible components of the special fiber of $\ddot{Y}_{\underline{v}}$ [i.e., more precisely: of $\ddot{\mathfrak{Y}}_{\underline{v}}$] maps *bijectively* to the set of irreducible components of the special fiber of $Y_{\underline{v}}$, while there exist *precisely two cusps* of $\ddot{Y}_{\underline{v}}$ over each cusp of $Y_{\underline{v}}$. The *formal completion* of $Y_{\underline{v}}$ along the smooth locus [i.e., the complement of "0" and "∞"] of the irreducible component of the special fiber labeled 0 is *naturally isomorphic* [in a fashion compatible with a choice of isomorphism $\underline{\mathbb{Z}} \overset{\sim}{\to} \mathbb{Z}$] to a once-punctured copy of the multiplicative group "$\mathbb{G}_m$". In particular, it makes sense to speak of the *standard multiplicative coordinate* "$U_{\underline{v}}$" on this *formal completion*, as well as a square root [well-defined up to multiplication by ± 1] "$\ddot{U}_{\underline{v}}$" of $U_{\underline{v}}$ on the base-change of this *formal completion* by $\ddot{Y}_{\underline{v}} \to Y_{\underline{v}}$. The **theta function**

$$\ddot{\Theta}_{\underline{v}} \;=\; \ddot{\Theta}_{\underline{v}}(\ddot{U}_{\underline{v}}) \;\overset{\mathrm{def}}{=}\; q_{\underline{v}}^{-\frac{1}{8}} \cdot \textstyle\sum_{n\in\mathbb{Z}} (-1)^n \cdot q_{\underline{v}}^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{U}_{\underline{v}}^{2n+1}$$

may be thought of as a *meromorphic function* on $\ddot{Y}_{\underline{v}}$ [cf. [EtTh], Proposition 1.4, and the preceding discussion], whose *zeroes* are precisely the *cusps*, with *multiplicity* 1, and whose *poles* are supported on the *special fiber* of $\ddot{Y}_{\underline{v}}$, with *multiplicity* [relative to a square root $q_{\underline{v}}^{\frac{1}{2}}$ of $q_{\underline{v}}$] equal to j^2 at the irreducible component labeled j [cf. Fig. 3.9]. In fact, in inter-universal Teichmüller theory, we shall mainly be interested in [a certain constant multiple of] the *reciprocal of an l-th root* of this theta function, namely,

$$\underline{\underline{\Theta}}_{\underline{v}} \;\overset{\mathrm{def}}{=}\; \left\{ \left(\sqrt{-1}\cdot \sum_{m\in\mathbb{Z}} q_{\underline{v}}^{\frac{1}{2}(m+\frac{1}{2})^2}\right)^{-1} \cdot \left(\sum_{n\in\mathbb{Z}} (-1)^n \cdot q_{\underline{v}}^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot U_{\underline{v}}^{n+\frac{1}{2}}\right)\right\}^{-\frac{1}{l}}$$

— which may be thought of as a *meromorphic function* on $\ddot{\underline{\underline{Y}}}_v \stackrel{\mathrm{def}}{=} \ddot{Y}_{\underline{v}} \times_{\underline{X}_{\underline{v}}} \underline{\underline{X}}_v$ that is *normalized* by the condition that it assumes a value $\in \boldsymbol{\mu}_{2l}$ [i.e., a $2l$-th root of unity] at the *points of* $\ddot{\underline{\underline{Y}}}_v$ that lie over the torsion points of $E_{\underline{v}}$ of *order precisely* 4 and, moreover, meets the smooth locus of the *irreducible component* of the special fiber of $\ddot{\underline{\underline{Y}}}_v$ [i.e., more precisely: of $\ddot{\underline{\underline{\mathfrak{Y}}}}_v \stackrel{\mathrm{def}}{=} \ddot{\mathfrak{Y}}_{\underline{v}} \times_{\underline{\mathfrak{X}}_{\underline{v}}} \underline{\underline{\mathfrak{X}}}_v$, where $\underline{\mathfrak{X}}_{\underline{v}}$ and $\underline{\underline{\mathfrak{X}}}_v$ denote the respective *normalizations* of $\mathfrak{X}_{\underline{v}}$ in $\underline{X}_{\underline{v}}$ and $\underline{\underline{X}}_v$] *labeled* 0 [i.e., the unique irreducible component of the special fiber of $\ddot{\underline{\underline{Y}}}_v$ that maps to the irreducible component of the special fiber of $\ddot{Y}_{\underline{v}}$ labeled 0]. Such points of $\ddot{\underline{\underline{Y}}}_v$ are referred to as **zero-labeled evaluation points** [cf., e.g., [IUTchII], Corollary 2.6]. At a very rough level,

> the approach to **multiradial decouplings/cyclotomic rigidity** taken in the case of the **Kummer theory** of special functions that surrounds the formal monoid of (b^{Θ}) may be understood as being "roughly similar" to the approach discussed in (ii) in the case of the global realified Frobenioids of (c^{Θ}), except that "κ-coric rational functions" are replaced by [normalized reciprocals of l-th roots of] **theta functions**.

[cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchIII], Remark 2.3.3]. That is to say,

- The desired **decoupling** [which is referred to in [EtTh], as *"constant multiple rigidity"*] of the [reciprocal of an l-th root of the] theta function from the *unit group data* of (a^{Θ}) is achieved by means of the condition that **evaluation** at any of the **zero-labeled evaluation points** — an operation that may be performed at the level of **étale-like structures**, i.e., by **restricting Kummer classes** to decomposition groups of points — yields a $2l$-th root of unity.
- The desired **multiradial cyclotomic rigidity isomorphism** is, roughly speaking, achieved by means of the **"mod N Kummer class version"**, for various positive integers N, of the technique discussed in Example 2.13.1, (ii) [cf., especially, the final display of Example 2.13.1, (ii)]: that is to say, such a "mod N version" is possible – *without any* $\{\pm 1\}$ *indeterminacies!* — *precisely* as a consequence of the fact that the **order of each zero** of $\ddot{\Theta}_{\underline{v}}$ at each cusp of $\ddot{Y}_{\underline{v}}$ is **precisely one** [cf., e.g., the discussion of [IUTchIII], Remark 2.3.3, (vi)].

In this context, we remark that the decomposition groups of *zero-labeled evaluation points* may be reconstructed by applying the theory of **elliptic cuspidalizations** developed in [AbsTopII], §3. This theory proceeds in an essentially parallel fashion to the theory of Belyi cuspidalizations [cf. the discussion of §3.3, (vi)]. That is to say, *elliptic cuspidalizations* are achieved as a formal consequence of the *elementary observation* that

> the **dense open subscheme** of a once-punctured elliptic curve obtained by removing the *N-torsion points*, for N a positive integer, may be regarded — via the use of the *"multiplication by N"* endomorphism of the elliptic curve! — as a **finite étale covering** of the given once-punctured elliptic curve; in particular, the arithmetic fundamental group of such a dense open subscheme may be recovered from the arithmetic fundamental

group of the given once-punctured elliptic curve by considering a *suitable open subgroup* of the latter arithmetic fundamental group [cf. [AbsTopII], Example 3.2; [AbsTopII], Corollaries 3.3, 3.4, for more details].

Another important observation in this context [cf. also §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii)] is that the approach to *cyclotomic rigidity* described above involving **"mod N Kummer classes"** is manifestly **compatible** with the **topology** of the Galois or tempered arithmetic fundamental groups involved. Since this "mod N approach" depends, in an *essential way*, on the fact that the *order of the zero* of $\ddot{\Theta}_{\underline{v}}$ at each cusp of $\ddot{Y}_{\underline{v}}$ is *precisely one* [cf., the discussion of [IUTchIII], Remark 2.3.3, (vi)], it also serves to elucidate the importance of working with the **first power** of [reciprocals of l-th roots of the] theta function, i.e., as opposed to the *M-th power*, for $M \geq 2$ [cf. [IUTchII], Remark 3.6.4, (iii), (iv); [IUTchIII], Remark 2.3.3, (vii)]. On the other hand, the approach described thus far in the present (iii) has one **fundamental deficiency**, namely, the fact that the *orders of the poles* of $\ddot{\Theta}_{\underline{v}}$ are **not compatible/symmetric** with respect to the action of $\underline{\mathbb{Z}}$ on $Y_{\underline{v}}$ [cf. Fig. 3.9] implies that

> the approach described thus far in the present (iii) to **multiradial cyclotomic rigidity** — i.e., involving, in effect, the **mod N Kummer classes** of the [reciprocal of an l-th root of the] **theta function** — is **not compatible** with the **$\underline{\mathbb{Z}}$-symmetries** of $Y_{\underline{v}}$ [cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchIII], Remark 2.3.3, (iv)].

Here, we note that the quotient $\underline{\mathbb{Z}} \xrightarrow{\sim} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/l \cdot \mathbb{Z} = \mathbb{F}_l$ may be *identified* with the subgroup $\mathbb{F}_l \subseteq \mathbb{F}_l^{\rtimes\pm}$ in the discussion of *symmetries of* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* in §3.3, (v). Also, we remark that the mod N Kummer class of the reciprocal of an l-th root of the theta function is **indeed compatible** with the **$N \cdot l \cdot \underline{\mathbb{Z}}$-symmetries** of $Y_{\underline{v}}$. This means that, *as one varies N*, the *obstruction* to finding a *coherent system of basepoints* — i.e., a coherent notion of the **"zero label"** — of the resulting projective system lies in

$$\mathbb{R}^1 \varprojlim_N N \cdot l \cdot \underline{\mathbb{Z}} \quad \xrightarrow{\sim} \quad \mathbb{R}^1 \varprojlim_N N \cdot l \cdot \mathbb{Z} \quad \xrightarrow{\sim} \quad l \cdot \widehat{\mathbb{Z}}/l \cdot \mathbb{Z} \quad \neq \quad 0$$

[cf. the discussion of [EtTh], Remark 2.16.1]. Put another way, one may only construct a coherent system of basepoints if one is willing to **replace** $\underline{\mathbb{Z}}$ by its *profinite completion*, i.e., to **sacrifice** the **discrete** nature of $\underline{\mathbb{Z}}$ ($\cong \mathbb{Z}$). It is for this reason that the property of *compatibility* with, say, the *$l \cdot \underline{\mathbb{Z}}$-symmetries* of $Y_{\underline{v}}$ is referred to, in [EtTh], as **discrete rigidity**. This sort of discrete rigidity plays an important role in inter-universal Teichmüller theory since a failure of discrete rigidity would obligate one to work with **$\widehat{\mathbb{Z}}$-multiples/powers** of divisors, line bundles, or meromorphic functions — a state of affairs that would, for instance, obligate one to sacrifice the crucial notion of **positivity/ampleness** in discussions of divisors and line bundles [cf. the discussion of [IUTchIII], Remark 2.1.1, (v)].

(iv) **<u>Cyclotomic, discrete, and constant multiple rigidity for mono-theta environments</u>:** The *incompatibility* of the approach discussed in (iii) with *$\underline{\mathbb{Z}}$-symmetries* [cf. the discussion at the end of (iii)] is remedied in the theory of [EtTh] by working with **mono-theta environments**, as follows: Write $\mathfrak{L}_{\mathfrak{X}_{\underline{v}}}$ for the ample line bundle of degree 1 on $\mathfrak{X}_{\underline{v}}$ determined by the *unique marked point* of $\mathfrak{X}_{\underline{v}}$ and

$$\mathfrak{L}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathfrak{L}_{\mathfrak{X}_{\underline{v}}}|_{\mathfrak{Y}_{\underline{v}}}, \quad \ddot{\mathfrak{L}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathfrak{L}_{\mathfrak{X}_{\underline{v}}}|_{\ddot{\mathfrak{Y}}_{\underline{v}}},$$
$$L_{X_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathfrak{L}_{\mathfrak{X}_{\underline{v}}}|_{X_{\underline{v}}}, \quad L_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathfrak{L}_{\underline{v}}|_{Y_{\underline{v}}}, \quad \ddot{L}_{\underline{v}} \stackrel{\mathrm{def}}{=} \ddot{\mathfrak{L}}_{\underline{v}}|_{\ddot{Y}_{\underline{v}}}$$

for the various pull-backs, or restrictions, of $\mathfrak{L}_{\mathfrak{X}_{\underline{v}}}$. Then the **theta function** $\ddot{\Theta}_{\underline{v}}$ on $\ddot{Y}_{\underline{v}}$ may be thought of as a **ratio** of *two sections* of the line bundle $\ddot{\mathfrak{L}}_{\underline{v}}$ over $\ddot{\mathfrak{Y}}_{\underline{v}}$, which may be described as follows:

- The **algebraic section** of $\ddot{\mathfrak{L}}_{\underline{v}}$ is the section [well-defined up to a $K_{\underline{v}}^\times$-multiple] whose zero locus coincides with the *locus of zeroes* of $\ddot{\Theta}_{\underline{v}}$. The pair consisting of the line bundle $\mathfrak{L}_{\underline{v}}$ and this *algebraic section* admits $\underline{\mathbb{Z}}$**-symmetries** [cf. Fig. 3.9], i.e., automorphisms that lie over the automorphisms of $\underline{\mathbb{Z}} = \mathrm{Gal}(Y_{\underline{v}}/X_{\underline{v}})$.
- The **theta section** of $\ddot{\mathfrak{L}}_{\underline{v}}$ is the section [well-defined up to a $K_{\underline{v}}^\times$-multiple] whose zero locus coincides with the *locus of poles* of $\ddot{\Theta}_{\underline{v}}$. One verifies immediately [cf. Fig. 3.9] that the *theta section* is **not compatible** with the $\underline{\mathbb{Z}}$**-symmetries** of the *algebraic section*.

The analogous operation, for this *line bundle-theoretic data*, to considering various *Kummer classes* of the *theta function* is the operation of passing to the **tempered arithmetic fundamental group** of the $\mathbb{G}_\mathrm{m}$-*torsor* $L_{\underline{v}}^\times$ associated to $L_{\underline{v}}$ or to the *morphisms* on tempered arithmetic fundamental groups induced by the *algebraic* and *theta sections*. Here, "mod N Kummer classes" correspond to considering the *quotient* of the tempered arithmetic fundamental group of $L_{\underline{v}}^\times$ that corresponds to coverings whose restriction to the *"$\mathbb{G}_\mathrm{m}$ fibers"* of the $\mathbb{G}_\mathrm{m}$-torsor $L_{\underline{v}}^\times$ is *dominated* by the covering $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ given by *raising to N-th power*. Note that

> **neither** the **ratio** of the *algebraic* and *theta sections* — i.e., the **theta function**! — **nor** the **pair** consisting of the *algebraic* and *theta sections* is **compatible** with the $\underline{\mathbb{Z}}$**-symmetries** of $Y_{\underline{v}}$.

On the other hand, it is not difficult to verify that the following triple of data is indeed **compatible, up to isomorphism,** with the $\underline{\mathbb{Z}}$**-symmetries** of $Y_{\underline{v}}$:

($a^{\mu\text{-}\Theta}$) the $\mathbb{G}_\mathrm{m}$-*torsor* $L_{\underline{v}}^\times$;

($b^{\mu\text{-}\Theta}$) the *group of automorphisms* of $L_{\underline{v}}^\times$ generated by the $\underline{\mathbb{Z}}$-*symmetries* of $L_{\underline{v}}^\times$ and the automorphisms determined by *multiplication* by a *constant* $\in K_{\underline{v}}^\times$;

($c^{\mu\text{-}\Theta}$) the *theta section* of $\ddot{L}_{\underline{v}}^\times \stackrel{\mathrm{def}}{=} L_{\underline{v}}^\times|_{\ddot{Y}_{\underline{v}}}$.

Indeed, the asserted *compatibility* with the $\underline{\mathbb{Z}}$**-symmetries** of $Y_{\underline{v}}$ is immediate for ($a^{\mu\text{-}\Theta}$) and ($b^{\mu\text{-}\Theta}$). On the other hand, with regard to ($c^{\mu\text{-}\Theta}$), a direct calculation shows that application of a $\underline{\mathbb{Z}}$-symmetry has the effect of multiplying the theta section by *some meromorphic function* which is a product of *integer powers* of $\ddot{U}_{\underline{v}}$ and $q_{\underline{v}}^{\frac{1}{2}}$; moreover, a direct calculation shows that the *group of automorphisms* of ($b^{\mu\text{-}\Theta}$) is *stabilized* by conjugation by the operation of multiplying by such a meromorphic function. That is to say, by applying such multiplication operations, we conclude that the *triple of data* ($a^{\mu\text{-}\Theta}$), ($b^{\mu\text{-}\Theta}$), ($c^{\mu\text{-}\Theta}$) is indeed *compatible, up* *to isomorphism,* with the $\underline{\mathbb{Z}}$**-symmetries** of $Y_{\underline{v}}$, as desired [cf. [EtTh], Proposition 2.14, (ii), (iii), for more details]. This argument motivates the following definition [cf. the discussion of [IUTchIII], Remark 2.3.4]:

> The [mod N] **mono-theta environment** is defined by considering the [mod N] *tempered arithmetic fundamental group* versions of the *"l-th roots"* of the triple of data $(a^{\mu\text{-}\Theta})$, $(b^{\mu\text{-}\Theta})$, $(c^{\mu\text{-}\Theta})$ discussed above [cf. [EtTh], Definition 2.13, (ii)].

[Indeed, the data $(a^{\mu\text{-}\Theta})$, $(b^{\mu\text{-}\Theta})$, $(c^{\mu\text{-}\Theta})$ correspond, respectively, to the data of Definition 2.13, (ii), (a), (b), (c).] In particular, the **functoriality** of the tempered arithmetic fundamental group [essentially — cf. [EtTh], Proposition 2.14, (ii), (iii), for more details] implies that a **mono-theta environment** admits $l \cdot \underline{\mathbb{Z}}$**-symmetries** of the desired type, hence, in particular, that it satisfies the crucial property of **discrete rigidity** discussed in the final portion of (iii). Moreover, by forming suitable **commutators** in the group of automorphisms of $L^{\times}_{\underline{v}}$, one may recover the desired **cyclotomic rigidity isomorphism** [cf. [EtTh], Corollary 2.19, (i); [IUTchII], Remark 1.1.1, for more details], i.e., that was discussed in a *"rough form"* in (iii), in a fashion that is

- · **decoupled** from the unit group data of (a^{Θ}),
- · manifestly **compatible** with the **topology** of the tempered arithmetic fundamental groups involved [since one works with *"mod N"* mono-theta environments!], and
- · **compatible** with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** of $\Theta^{\pm\text{ell}}$NF-Hodge theaters [cf. §3.3, (v); [IUTchII], Remark 1.1.1, (iv), (v)].

Indeed, these **multiradial decoupling/cyclotomic rigidity** properties of mono-theta environments are the main topic of [IUTchII], §1, and are summarized in [IUTchII], Corollaries 1.10, 1.12. Moreover, mono-theta environments have both **étale-like** and **Frobenius-like versions**, i.e., they may be constructed naturally [cf. [IUTchII], Proposition 1.2, (i), (ii)] *either*

- · from the **tempered arithmetic fundamental group** [regarded as an *abstract topological group*!] of $\underline{\underline{X}}_{\underline{v}}$, *or*
- · from a certain **"tempered Frobenioid"**, i.e., a *model Frobenioid* [cf. §3.3, (iii)] obtained by considering suitable *divisors*, *line bundles*, and *meromorphic functions* on the various tempered coverings of $\underline{\underline{X}}_{\underline{v}}$.

Finally, we close with the important *observation* that the various **rigidity** properties of mono-theta environments discussed above may be regarded as

> essentially formal consequences of the **quadratic structure** of the **commutators** of the **theta groups** — or, equivalently, of the **curvature**, or **first Chern class** — associated to the line bundle $L_{X_{\underline{v}}}$

[cf. the discussion of [IUTchII], Remark 1.1.1, (iv), (v); [IUTchIII], Remark 2.1.1]. This observation is of interest in that it shows that the theory of [EtTh] [or, indeed, a substantial portion of inter-universal Teichmüller theory!] yields an interesting **alternative interpretation** for the structure of **theta groups** to the classical **representation-theoretic** interpretation, i.e., involving irreducible representations of theta groups [cf. [IUTchIII], Remark 2.3.4, (iv)].

(v) **Various approaches to cyclotomic rigidity:** The discussion in the present §3.4 of various properties of the *three approaches to cyclotomic rigidity* that appear in inter-universal Teichmüller theory is summarized in Fig. 3.10 below. The most naive approach, involving well-known properties from *local class field theory* applied to the data "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$", is **compatible** with the **profinite topology** of the Galois or arithmetic fundamental groups involved, but suffers from the *fundamental defect* of being **uniradial**, i.e., of being **"un-decouplable"** from the *unit group data* of (a^{Θ}) [cf. the discussion of (i)]. By contrast, the approaches discussed in (ii) and (iv) involving **κ-coric rational functions** and **mono-theta environments** satisfy the crucial requirement of **multiradiality**, i.e., of being **"decouplable"** from the *unit group data* of (a^{Θ}). The approach via mono-theta environments also satisfies the important property of being **compatible** with the **topology** of the tempered arithmetic fundamental groups involved. By contrast, the approach via κ-coric rational functions is **not compatible** with the *profinite topology* of the Galois or arithmetic fundamental groups involved. This *incompatibility* in the case of the Kummer theory surrounding the **global** data of (c^{Θ}) will not, however, pose a problem, since *compatibility* with the *topologies* of the various Galois or [possibly tempered] arithmetic fundamental groups involved will only be of interest in the case of the Kummer theory surrounding the **local** data of (a^{Θ}) and (b^{Θ}) [cf. §3.6, (ii), below; [IUTchIII], Remark 2.3.3, (vii), (viii)].

Approach to cyclotomic rigidity	*Applied to Kummer theory surrounding*	*Uni-/multi-radiality*	*Compatibility with profinite/ tempered topologies*
Brauer groups/ local class field theory for "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$"	(a^{Θ})	**uniradial**	**compatible**
mono-theta environ-ments	(b^{Θ})	**multiradial**	**compatible**
κ-coric rational functions, via $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$	(c^{Θ})	**multiradial**	**incompatible**

Fig. 3.10: Three approaches to cyclotomic rigidity

§3.5. Remarks on the use of Frobenioids: The *theory of Frobenioids* was developed in [FrdI], [FrdII] as a

> *solution* to the problem of providing a **unified, intrinsic category-theoretic characterization** of various types of categories of *line bundles* and *divisors* that frequently appeared in the author's research on the

arithmetic of hyperbolic curves and, moreover, seemed, at least from a *heuristic* point of view, to be *remarkably similar in structure*.

These papers [FrdI], [FrdII] on Frobenioids were written in the spring of 2005, when the author only had a relatively rough, sketchy idea of how to formulate inter-universal Teichmüller theory. In particular,

> anyone who reads these papers [FrdI], [FrdII] — or indeed, the Frobenioid-theoretic portion of [EtTh] — under the expectation that they were written as an *optimally efficient presentation of precisely those portions of the theory of Frobenioids that are actually used* in inter-universal Teichmüller theory will undoubtedly be disappointed.

In light of this state of affairs, it seems appropriate to pause at this point to make a few remarks on the use of Frobenioids in inter-universal Teichmüller theory. First of all,

· at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, the **[“archimedean”] Frobenioids** that appear in inter-universal Teichmüller theory [cf. [IUTchI], Example 3.4] are essentially equivalent to the **topological monoid** $\mathcal{O}^{\rhd}_{\mathbb{C}}$ [i.e., the multiplicative topological monoid of nonzero complex numbers of norm ≤ 1] and hence *may be ignored*.

On the other hand,

· at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, all of the **[“nonarchimedean”] Frobenioids** that appear in inter-universal Teichmüller theory [cf., e.g., [IUTchI], Fig. I1.2] — except for the *tempered Frobenioids* mentioned in §3.4, (iv)! — are essentially equivalent to *either* the data [consisting of an abstract ind-topological monoid equipped with a continuous action by an abstract topological group]

$$\text{“}G_k \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}\text{”}$$

of Example 2.12.1, (i), *or* the data [consisting of an abstract ind-topological monoid equipped with a continuous action by an abstract topological group]

$$\text{“}\Pi_X \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}\text{”}$$

of Example 2.12.3, (ii) [where Π_X is possibly replaced by the *tempered arithmetic fundamental group* of X], *or* the data obtained from one of these two types of data by *replacing* “$\mathcal{O}^{\rhd}_{\overline{k}}$” by some *subquotient* of “$\mathcal{O}^{\rhd}_{\overline{k}}$” [as in Example 2.12.2, (i), (ii)].

Moreover, all of these “nonarchimedean” Frobenioids are **model Frobenioids** [cf. the discussion of §3.3, (iii)]. The *only other types of Frobenioids* — all of which are **model Frobenioids** [cf. the discussion of §3.3, (iii)] — that appear in inter-universal Teichmüller theory are

· the **[possibly realified] global Frobenioids** associated to NF’s [cf. the discussion of §3.4, (ii)], which admit a *simple elementary description* as categories of *arithmetic line bundles* on NF’s [cf. [FrdI], Example 6.3; [IUTchIII], Example 3.6; [Fsk], §2.10, (i), (ii)];

· the **tempered Frobenioids** mentioned in §3.4, (iv).

Here, we note that

> these last two examples — i.e., *global Frobenioids* and *tempered Frobenioids* — *differ fundamentally* from the previous examples, which were essentially equivalent to an *ind-topological monoid* that was, in some cases, equipped with a continuous action by a *topological group*, in that their **Picard groups** [cf. [FrdI], Theorem 5.1] admit **non-torsion elements**.

Indeed, *global Frobenioids* contain objects corresponding to arithmetic line bundles whose *arithmetic degree* is $\neq 0$, while *tempered Frobenioids* contain objects corresponding to line bundles for which *arbitrary positive tensor powers* are *nontrivial* such as [strictly speaking, the pull-back to $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ of] the line bundle "$L_{\underline{v}}$" of §3.4, (iv). Finally, we remark that although the theory of tempered Frobenioids, which is developed in [EtTh], §3, §4, §5, is *somewhat complicated*, the only portions of these tempered Frobenioids that are *actually used* in inter-universal Teichmüller theory are the portions discussed in §3.4, (iii), (iv), i.e.,

($a^{\text{t-F}}$) the **"theta monoids"** generated by **local units** [i.e., "$\mathcal{O}^{\times}$"] and nonnegative powers of roots of the [reciprocals of l-th roots of] **theta functions** that are constructed from tempered Frobenioids [cf. [IUTchI], Example 3.2; [IUTchII], Example 3.2, (i)];

($b^{\text{t-F}}$) the **mono-theta environments** constructed from tempered Frobenioids [cf. [IUTchII], Proposition 1.2, (ii)], which are related to the monoids of ($a^{\text{t-F}}$), in that they *share the* **same** *submonoids of roots of unity.*

Indeed, *étale-like versions* of this *"essential Frobenius-like data"* of ($a^{\text{t-F}}$) and ($b^{\text{t-F}}$) are discussed in [IUTchII], Corollaries 1.10, 1.12; [IUTchIII], Theorem 2.2, (ii) [cf. the data "$(a_{\underline{v}})$, $(b_{\underline{v}})$, $(c_{\underline{v}})$, $(d_{\underline{v}})$" of *loc. cit.*]. Thus, from the point of view of studying inter-universal Teichmüller theory,

> one may essentially *omit* the detailed study of [EtTh], §3, §4, §5, either by *accepting the construction of the data ($a^{\text{t-F}}$) and ($b^{\text{t-F}}$) "on faith"* or by *regarding this data as data constructed from the scheme-theoretic objects discussed in [EtTh], §1, §2.*

§**3.6.** **Galois evaluation, labels, symmetries, and log-shells:** In the present §3.6, we discuss the theory of **Galois evaluation** of the **κ-coric rational functions** and **theta functions** of §3.4, (ii), (iii). Here, we remark that the term *"Galois evaluation"* refers to the passage

$$\textbf{abstract functions} \quad \mapsto \quad \textbf{values}$$

by first passing from **Frobenius-like** — that is to say, in essence, **[pseudo-]monoid-theoretic** — versions of these functions [cf. the discussion of §3.4, (ii), (iii); §3.5] to **étale-like** versions of these functions via various forms of **Kummer theory** as discussed in §3.4, (ii), (iii), then **evaluating** these étale-like functions by **restricting** them to *decomposition subgroups* [that, say, arise from *closed points* of the *curve* under consideration] of the *[possibly tempered] arithmetic fundamental group* under consideration to obtain **étale-like** versions of the **values** of interest, and finally applying the **Kummer theory** of the *constant base field* [i.e., as discussed in Example 2.12.1] to obtain **Frobenius-like** versions of the values of interest [cf. Fig. 3.11 below; [IUTchII], Remark 1.12.4]. In fact, it is essentially a **tautology** that the *only way* to construct an *assignment* "abstract functions $\mapsto$ values" that is **compatible** with the operation of forming **Kummer classes** is precisely by applying [some variant of] this technique of Galois evaluation [cf. the discussion of [IUTchII], Remark 1.12.4]. Moreover, it is interesting in this context to observe [cf. the discussion of [IUTchII], Remark 1.12.4] that the well-known **Section Conjecture** of anabelian geometry — which, at least historically, was expected to be related to diophantine geometry [cf. the discussion of [IUTchI], §I5] — suggests strongly that, when one applies the technique of Galois evaluation, in fact,

> the **only suitable subgroups** of the [possibly tempered] arithmetic fundamental group under consideration for the operation of **"evaluation"** are precisely the **decomposition subgroups** that arise from the closed points of the curve under consideration!

From this point of view, it is also of interest to observe that, in the context of the *evaluation of theta functions at torsion points* [cf. (ii) below], it will be necessary to apply a certain **"combinatorial version of the Section Conjecture"** [cf. [IUTchI], Remark 2.5.1; the proof of [IUTchII], Corollary 2.4, (i)]. Finally, we remark that, in order to give a precise description of the Galois evaluation operations that are performed in inter-universal Teichmüller theory, it will be necessary to consider, in substantial detail,

- · the **labels** of the points at which the functions are to be evaluated [i.e., the points that give rise to the *decomposition subgroups* mentioned above],
- · the **symmetries** that act on these labels [cf. §3.3, (v)], and
- · the **log-shells** that serve as **containers** for the values that are constructed.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\textit{Frobenius-like}\\ \textit{version of}\\ \textbf{functions}\end{array}} \arrow[r, Rightarrow, "\textbf{Kummer}"] \arrow[d, dotted, "\textbf{evalu-}"', "\textbf{ation}"] \arrow[dr, phantom, "\curvearrowright"] & \boxed{\begin{array}{c}\textit{étale-like}\\ \textit{version of}\\ \textbf{functions}\end{array}} \arrow[d, Rightarrow, "\textbf{evalu-}"', "\textbf{ation}"] \\
\boxed{\begin{array}{c}\textit{Frobenius-like}\\ \textit{version of}\\ \textbf{values}\end{array}} & \boxed{\begin{array}{c}\textit{étale-like}\\ \textit{version of}\\ \textbf{values}\end{array}} \arrow[l, Rightarrow, "\textbf{Kummer}^{-1}"']
\end{tikzcd}$$

Fig. 3.11: The technique of Galois evaluation

(i) <u>**Passage to the étale-picture, combinatorial uni-/multiradiality of symmetries:**</u> Recall from the discussion of §3.3, (vi), that the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters associated to the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters "$\bullet$" in the log-theta-lattice are **vertically coric**. That is to say, one may think of a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater*, considered up to an *indeterminate isomorphism*, as an **invariant** of each **vertical line** of the log-theta-lattice. Moreover, the *étale-like portion* [i.e., the "$G_{\underline{v}}$'s"] of the data of (a^{Θ}), or, equivalently, (a^{q}), of §3.3, (vii), again considered up to an *indeterminate isomorphism*, may be thought of as an object *constructed from* *the* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* associated to the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater "$\bullet$" under consideration. In particular,

> if one takes the **radial data** to consist of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* associated to *some vertical line* of the log-theta-lattice [considered up to an *indeterminate isomorphism*!], the **coric data** to consist of the étale-like portion [again considered up to an *indeterminate isomorphism*!] of the data of (a^{Θ}), or, equivalently, (a^{q}), of §3.3, (vii), and the **radial algorithm** to be the assignment [i.e., *"construction"*] of the above discussion, then one obtains a radial environment, shown in Fig. 3.12 below, that is **["tautologically"!] multiradial** [cf. [IUTchII], Corollary 4.11; [IUTchII], Fig. 4.3].

[Indeed, this multiradial environment may be thought of as being simply a *slightly more complicated version* of the multiradial environment of Example 3.2.2, (ii).] The diagram obtained by including, in the diagram of Fig. 3.12, not just two collections of radial data [that arise, say, from two adjacent vertical lines of the log-theta-lattice], but rather the collections of radial data that arise from *all of the vertical lines* of the log-theta-lattice is referred to as the **étale-picture** [cf. [IUTchII], Fig. 4.3]. Despite its *tautological nature*,

> the **multiradiality** — i.e., **permutability** of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters associated to **distinct vertical lines** of the log-theta-lattice — of the **étale-picture** is nonetheless *somewhat remarkable* since [prior to passage to the étale-picture!] the log-theta-lattice does **not** admit **symmetries** that *permute distinct vertical lines* of the log-theta-lattice.

$$\begin{tikzcd}[column sep=small]
\boxed{\begin{array}{c}\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\text{NF-Hodge}\\ \text{theater in}\\ \text{the } \textit{domain}\\ \text{of the } \Theta\text{-link}\end{array}} \arrow[dr, "\substack{\text{radial}\\ \text{algorithm}}"'] & \leftrightarrows & \boxed{\begin{array}{c}\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\text{NF-Hodge}\\ \text{theater in}\\ \text{the } \textit{codomain}\\ \text{of the } \Theta\text{-link}\end{array}} \arrow[dl, "\substack{\text{radial}\\ \text{algorithm}}"] \\
& \boxed{\begin{array}{c}\text{étale-like}\\ \text{portion "}G_{\underline{v}}\text{"}\\ \text{of } (a^{\Theta}),\ (a^{q})\end{array}} &
\end{tikzcd}$$

Fig. 3.12: The multiradiality of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters

Next, we consider the respective $\mathbb{F}_l^{\rtimes \pm}$**- and** $\mathbb{F}_l^{\circledast}$**-symmetries** of the constituent $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters [cf. §3.3, (v)]. In this context, it is useful to introduce **symbols** "$\succ$" and "$>$":

- "$\succ$" denotes the **entire set** $\mathbb{F}_l$ that appears in the discussion of Fig. 3.8 in §3.3, (v), i.e., the notation "[...]" in the *upper left-hand corner* of Fig. 3.8 [cf. [IUTchI], Fig. 6.5].
- "$>$" denotes the **entire set** $\mathbb{F}_l^{\circledast}$ that appears in the discussion of Fig. 3.8 in §3.3, (v), i.e., the notation "[...]" in the *upper right-hand corner* of Fig. 3.8 [cf. [IUTchI], Fig. 6.5].

· The **gluing** shown in Fig. 3.8 may be thought of as an *assignment* that sends

$$0,\ \succ \quad \mapsto \quad >$$

[cf. [IUTchI], Proposition 6.7; [IUTchI], Fig. 6.5; the discussion of [IUTchII], Remark 3.8.2, (ii); the symbol "$\triangle$" of [IUTchII], Corollary 4.10, (i)].

· Ultimately, we shall be interested in computing *weighted averages* of *log-volumes* at the various *labels* in $\mathbb{F}_l$ or $\mathbb{F}_l^{\divideontimes}$ ($\subseteq |\mathbb{F}_l| \stackrel{\text{def}}{=} \mathbb{F}_l^{\divideontimes} \cup \{0\}$) [cf. [IUTchI], Remark 5.4.2; the computations of [IUTchIV], §1]. From this point of view, it is natural to think in terms of *formal sums* with rational coefficients

$$[0], \quad [\ |j|\ ] \stackrel{\text{def}}{=} \tfrac{1}{2}([j] + [-j]),$$
$$[\succ] \stackrel{\text{def}}{=} \tfrac{1}{l}([0] + [1] + [-1] + \ldots + [l^{\divideontimes}] + [-l^{\divideontimes}]),$$
$$[>] \stackrel{\text{def}}{=} \tfrac{1}{l^{\divideontimes}}([\ |1|\ ] + \ldots + [\ |l^{\divideontimes}|\ ]) \ = \ \tfrac{1}{2l^{\divideontimes}}([1] + [-1] + \ldots + [l^{\divideontimes}] + [-l^{\divideontimes}])$$

— where the "j" and "$\ldots$" indicate arguments that range within the positive integers between 1 and $l^{\divideontimes} = \frac{1}{2}(l-1)$. Note that these assignments of formal sums are *compatible* with the *gluing* "$0,\ \succ \mapsto >$", i.e., relative to which

$$[0] \ \mapsto \ [>], \quad [\succ] \ \mapsto \ [>], \quad \tfrac{1}{(l^{\divideontimes}+1)}([0] + [\ |1|\ ] + \ldots + [\ |l^{\divideontimes}|\ ]) \ \mapsto \ [>]$$

— where we note that such relations may be easily verified by observing that the coefficients of "$[j]$" and "$[-j]$" always *coincide* and are *independent of* j; thus, these relations may be verified by *substituting a single indeterminate* "w" for all of the symbols "$[0]$", "$[j]$", and "$[-j]$".

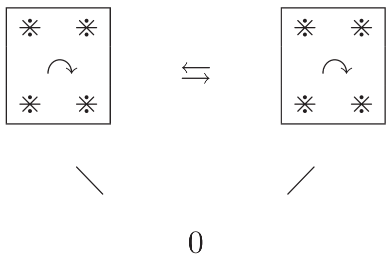

Fig. 3.13: Combinatorial *multiradiality* of $\mathbb{F}_l^{\divideontimes}$-symmetries

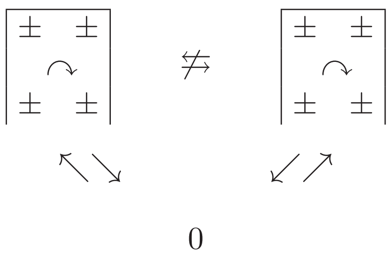

Fig. 3.14: Combinatorial *uniradiality* of $\mathbb{F}_l^{\rtimes\pm}$-symmetries

If one *extracts from the étale-picture* the various $\mathbb{F}_l^{\divideontimes}$**-symmetries** of the respective $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters, then one obtains a diagram as in Fig. 3.13 above, i.e., a diagram of **various distinct, independent** $\mathbb{F}_l^{\divideontimes}$**-actions** that are *"glued together* *at a common symbol* 0*"* [cf. the discussion of [IUTchII], Remark 4.7.4; [IUTchII], Fig. 4.2]. This diagram may be thought of as a sort of **combinatorial prototype** for the phenomenon of **multiradiality**. On the other hand, if one extracts from the *étale-picture* the various $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** of the respective $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters, then one obtains a diagram as in Fig. 3.14 above, i.e., a diagram of **various mutually interfering** $\mathbb{F}_l^{\rtimes\pm}$**-actions** that interfere with one another as a consequence of the fact that they are *"glued together at a common symbol* 0*"* [cf. the discussion of [IUTchII], Remark 4.7.4; [IUTchII], Fig. 4.1]. This diagram may be thought of as a sort of **combinatorial prototype** for the phenomenon of **uniradiality**. Finally, in this context, it is also of interest to observe that, if, in accordance with the point of view of the discussion of §2.14,

> one thinks of $\mathbb{F}_l$ as a sort of **finite discrete approximation** of "$\mathbb{Z}$" [cf. [IUTchI], Remark 6.12.3, (i); [IUTchII], Remark 4.7.3, (i)],

and one thinks "$\mathbb{Z}$" as the **value group** of the various completions [say, for simplicity, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] of K, then

> the $\mathbb{F}_l^{\circledast}$**-symmetry** corresponds to a symmetry that only involves the **non-unit portions** of these value groups at various $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, while the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** is a symmetry that involves a sort of **"juggling"** between local unit groups and local value groups.

This point of view is consistent with the fact [cf. (iii) below; Example 2.12.3, (v); §3.3, (ii), (vii); §3.4, (ii)] that the $\mathbb{F}_l^{\circledast}$-symmetry is related *only* to the Kummer theory surrounding the **global value group data** (c^{Θ}), while [cf. (ii) below; Example 2.12.3, (v); §3.3, (ii), (vii); §3.4, (iii), (iv)] the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is related to both the **[local] unit group data** (a^{Θ}) and the **local value group data** (b^{Θ}), which are **"juggled"** about by the $\mathfrak{log}$-links of the log-theta-lattice.

(ii) <u>**Theta values and local diagonals via the** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry:**</u> Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Write

$$\Pi_{\underline{v}} \quad \subseteq \quad \Pi_{\underline{v}}^{\pm} \quad \subseteq \quad \Pi_{\underline{v}}^{\mathrm{cor}}$$

[cf. [IUTchII], Definition 2.3, (i)] for the inclusions of *tempered arithmetic fundamental groups* [for suitable choices of basepoints] determined by the finite étale coverings $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to C_{\underline{v}}$ [cf. the notational conventions discussed at the beginning of §3.4, (iii)]. These tempered arithmetic fundamental groups of hyperbolic orbicurves over $K_{\underline{v}}$ admit *natural outer surjections* to $G_{\underline{v}}$; write $\Delta_{\underline{v}} \subseteq \Delta_{\underline{v}}^{\pm} \subseteq \Delta_{\underline{v}}^{\mathrm{cor}}$ for the respective *kernels* of these surjections. In fact, $\Pi_{\underline{v}}^{\pm}$ and $\Pi_{\underline{v}}^{\mathrm{cor}}$, together with the above inclusions, may be *reconstructed functorially* from the *topological group* $\Pi_{\underline{v}}$ [cf. [EtTh], Proposition 2.4]. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. §3.3, (v)] induce *outer automorphisms* of $\Pi_{\underline{v}}^{\pm}$. Indeed, these outer automorphisms may be thought of as the outer automorphisms of $\Pi_{\underline{v}}^{\pm}$ induced by *conjugation* in $\Pi_{\underline{v}}^{\mathrm{cor}}$ by the quotient group $\Pi_{\underline{v}}^{\mathrm{cor}}/\Pi_{\underline{v}}^{\pm}$, which admits a *natural outer isomorphism* $\mathbb{F}_l^{\rtimes\pm} \stackrel{\sim}{\to} \Pi_{\underline{v}}^{\mathrm{cor}}/\Pi_{\underline{v}}^{\pm}$ [cf. [IUTchII], Corollary 2.4, (iii)]. Moreover, since these outer automorphisms of $\Pi_{\underline{v}}^{\pm}$ arise from *K-linear automorphisms* of the hyperbolic curve $\underline{X}_K$ [cf. the discussion of §3.3, (v)], let us *observe* that

> the outer automorphisms of $\Pi_{\underline{v}}^{\pm}$ under consideration may, in fact, be thought of as $\Delta_{\underline{v}}^{\pm}$**-outer automorphisms** of $\Pi_{\underline{v}}^{\pm}$ [i.e., automorphisms

defined up to composition with an inner automorphism induced by conjugation by an element of $\Delta^{\pm}_{\underline{v}}$].

Next, let us recall from the discussion of §3.3, (v), concerning the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** that elements of $\mathbb{F}_l$ may be thought of — up to $\mathbb{F}_l^{\rtimes\pm}$**-indeterminacies** that may in fact, as a consequence of the *structure* of a $\Theta^{\pm\text{ell}}$*NF-Hodge theater*, be **synchronized** in a fashion that is **independent** of the choice of $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [cf. [IUTchI], Remark 6.12.4, (i), (ii), (iii)] — as **labels** of **cusps** of $\underline{X}_{\underline{v}}$. Moreover, such cusps of $\underline{X}_{\underline{v}}$ may be thought of, by applying a suitable *functorial group-theoretic algorithm*, as certain conjugacy classes of subgroups [i.e., cuspidal inertia subgroups] of $\Pi^{\pm}_{\underline{v}}$ [cf. [IUTchI], Definition 6.1, (iii)]. In particular, the above *observation* implies that, if we think of $G_{\underline{v}}$ as a *quotient* of one of the tempered arithmetic fundamental groups $\Pi_{\underline{v}}$, $\Pi^{\pm}_{\underline{v}}$, $\Pi^{\text{cor}}_{\underline{v}}$ discussed above, and we consider *copies* of this quotient $G_{\underline{v}}$ equipped with *labels*

$$(G_{\underline{v}})_t$$

— where we think of $t \in \mathbb{F}_l$ as a conjugacy class of cuspidal inertia subgroups of $\Pi^{\pm}_{\underline{v}}$ — then

> the action of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry [i.e., by conjugation in $\Pi^{\text{cor}}_{\underline{v}}$] on these *labeled quotients* $\{(G_{\underline{v}})_t\}_{t\in\mathbb{F}_l}$ induces **symmetrizing isomorphisms** between these labeled quotients that are **free** of any **inner automorphism indeterminacies** [cf. [IUTchII], Corollary 3.5, (i); [IUTchII], Remark 3.5.2, (iii); [IUTchII], Remark 4.5.3, (i)].

The existence of these symmetrizing isomorphisms is a phenomenon that is sometimes referred to as **conjugate synchronization**. Note that this sort of situation *differs radically* from the situation that arises for the *isomorphisms* induced by conjugation in $G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/K)$ between the various *decomposition groups* of $\underline{v}$ [that is to say, copies of "$G_{\underline{v}}$"], i.e., isomorphisms which are *only well-defined* up to composition with some **indeterminate inner automorphism** of the decomposition group under consideration [cf. the discussion of [IUTchII], Remark 2.5.2, (iii)]. In this context, since *tempered arithmetic fundamental groups*, unlike conventional *profinite* étale fundamental groups, are only defined at a *specific* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, one *technical issue* that arises, when one considers the task of relating the *symmetrizing isomorphisms* discussed above at *different* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [or, indeed, to the theory at valuations $\in \underline{\mathbb{V}}^{\text{good}}$] is

> the issue of *comparing* **tempered** *and* **profinite** *conjugacy classes* of various types of subgroups [i.e., such as cuspidal inertia groups] — an issue that is resolved [cf. the application of [IUTchI], Corollary 2.5, in the proof of [IUTchII], Corollary 2.4] by applying the theory of [Semi].

The *symmetrizing isomorphisms* discussed above may be applied *not only* to copies of the *étale-like* object $G_{\underline{v}}$ but also to various *Frobenius-like* objects that are *"closely related"* to $G_{\underline{v}}$ [cf. the pairs "$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$" of Example 2.12.1; [IUTchII], Corollary 3.6, (i)]. Moreover, an analogous theory of symmetrizing isomorphisms may be developed at valuations $\in \underline{\mathbb{V}}^{\text{good}}$ [cf. [IUTchII], Corollary 4.5, (iii); [IUTchII], Corollary 4.6, (iii)]. The **graphs**, or **diagonals**, of these symmetrizing isomorphisms at various valuations $\in \underline{\mathbb{V}}$ may be thought of as corresponding to the **symbol** "$\succ$"

discussed in (i), or indeed, after applying the **gluing** that appears in the structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater [cf. (i); §3.3, (v)], to the **symbols** "0", ">" [cf. [IUTchII], Corollary 3.5, (iii); [IUTchII], Corollary 3.6, (iii); [IUTchII], Corollary 4.5, (iii); [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.10, (i)]. Moreover,

> the data labeled by these symbols "$\succ$", "0", ">", form the data that is ultimately *actually used* in the **horizontally coric unit group portion** (a^Θ), (a^q) [cf. §3.3, (vii)] of the data in the *codomain* and *domain* of the Θ-link [cf. [IUTchIII], Theorem 1.5, (iii)].

The *significance* of this approach to constructing the data of (a^Θ), (a^q) lies in the fact that the **"descent"** [cf. [IUTchIII], Remark 1.5.1, (i)] from the **individual labels** $t \in \mathbb{F}_l$ to the **symbols** "$\succ$"/"0"/">" that is effected by the various **symmetrizing isomorphisms** gives rise to **horizontally coric** data — i.e., data that is **shared** by the *codomain* and *domain* of the Θ-link — that serves as a **container** [cf. the discussion of (iv) below] for the various **theta values** [well-defined up to *multiplication by a* $2l$*-th root of unity*]

$$\underline{\underline{q}}^{j^2}$$

— where we think of $j = 1, \ldots, l^*$ as corresponding to an element of $\mathbb{F}_l^*$ obtained by identifying two elements $\pm t \in \mathbb{F}_l^\times \subseteq \mathbb{F}_l$ — obtained by **Galois evaluation** [cf. [IUTchII], Corollary 2.5; [IUTchII], Remark 2.5.1; [IUTchII], Corollary 3.5, (ii); [IUTchII], Corollary 3.6, (ii)], i.e., by restricting the **Kummer classes** of the [reciprocals of l-th roots of] **theta functions** on $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ discussed in §3.4, (iii) [cf. also the data "$(a^{\text{t-F}})$" discussed in §3.5]

· *first* to the **decomposition groups**, denoted by the notation "$\ddot{\blacktriangleright}$", in the open subgroup of $\Pi_{\underline{v}}$ corresponding to $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ determined [up to conjugation in $\Pi_{\underline{v}}$ — cf. [IUTchII], Proposition 2.2; [IUTchII], Corollary 2.4] by the **connected** — i.e., so as not to give rise to *distinct basepoints* for *distinct labels* $j = 1, \ldots, l^*$ [cf. the discussion of [IUTchII], Remarks 2.6.1, 2.6.2, 2.6.3] — **"line segment"** of **labels** of irreducible components of the special fiber of $\ddot{Y}_{\underline{v}}$

$$\{-l^*, -l^*+1, \ldots, -1, 0, 1, \ldots, l^*-1, l^*\} \quad \subseteq \quad \mathbb{Z}$$

[cf. Fig. 3.9 and the surrounding discussion; [IUTchII], Remark 2.1.1, (ii)]; and

· *then* to the **decomposition groups** associated to **"evaluation points"** — i.e., *cusps translated by a zero-labeled evaluation point* — labeled by $\pm t \in \mathbb{F}_l^\times \subseteq \mathbb{F}_l$.

Finally, we remark that the **Kummer theory** that relates the corresponding *étale-like* and *Frobenius-like* data that appears in the various **symmetrizing isomorphisms** just discussed only involves **local** data, i.e., the data of (a^Θ) and (b^Θ), hence [cf. the discussion of §3.4, (v); Fig. 3.10] is **compatible** with the **topologies** of the various tempered or profinite Galois or arithmetic fundamental groups involved. The *significance* of this *compatibility with topologies* lies in the fact it means that the *Kummer isomorphisms* that appear may be **computed** *relative to some finite étale covering* of the schemes involved, i.e., relative to a situation in which — unlike the situation that arises if one considers some sort of **projective limit** of multiplicative monoids associated to rings — the **ring structure** of the schemes involved is **still intact**. That is to say, since the $\mathfrak{log}$-*link* is defined by applying the formal **power series** of the natural logarithm, an object that can *only be defined* if *both* the additive *and* the multiplicative structures of the [topological] rings involved are available,

> this **computability** allows one to **compare** — hence to establish the **compatibility** of — the various **symmetrizing isomorphisms** just discussed in the *codomain* and *domain* of the $\mathfrak{log}$**-link** [cf. [IUTchII], Remark 3.6.4, (i); [IUTchIII], Remark 1.3.2; the discussion of Step (vi) of the proof of [IUTchIII], Corollary 3.12].

This *compatibility* plays an important role in inter-universal Teichmüller theory.

(iii) **Number field values and global diagonals via the $\mathbb{F}_l^{\circledast}$-symmetry:** We begin by considering certain *field extensions* of the field F_{mod}: write

- $F_{\text{sol}} \subseteq \overline{F}$ for the *maximal solvable extension* of F_{mod} in $\overline{F}$ [cf. [IUTchI], Definition 3.1, (b)];
- $F(\boldsymbol{\mu}_l, C_F)$ for the field obtained by adjoining to the *function field* of C_F the l-th roots of unity [cf. the field "$F(\boldsymbol{\mu}_l) \cdot L_C$" of [IUTchI], Remark 3.1.7, (iii)];
- $F(\boldsymbol{\mu}_l, \kappa\text{-sol})$ for the field obtained by adjoining to $F(\boldsymbol{\mu}_l, C_F)$ *arbitrary roots* of F_{sol}-*multiples* of κ-*coric rational functions* [cf. §3.4, (ii)] in $F(\boldsymbol{\mu}_l, C_F)$ [cf. the field "$F(\boldsymbol{\mu}_l) \cdot L_C(\kappa\text{-sol})$" of [IUTchI], Remark 3.1.7, (iii)];
- $F(\underline{C}_K)$ for the *Galois closure* over the field $F(\boldsymbol{\mu}_l, C_F)$ of the *function field* of $\underline{C}_K$ [cf. the field "$L_C(\underline{C}_K)$" of [IUTchI], Remark 3.1.7, (iii)].

Then one verifies immediately, by applying the fact that the finite group $SL_2(\mathbb{F}_l)$ [where we recall from §3.3, (i), that $l \geq 5$] is *perfect*, that

> $F(\boldsymbol{\mu}_l, \kappa\text{-sol})$ and $F(\underline{C}_K)$ are **linearly disjoint** over $F(\boldsymbol{\mu}_l, C_F)$ [cf. [IUTchI], Remark 3.1.7, (iii)].

It then follows, in an essentially *formal* way, from this *linear disjointness* [cf. [IUTchI], Remark 3.1.7, (ii), (iii); [IUTchI], Example 5.1, (i), (v); [IUTchI], Remark 5.1.5; [IUTchII], Corollary 4.7, (i), (ii); [IUTchII], Corollary 4.8, (i), (ii)] that:

- the various **elements** in
$$F_{\text{mod}} \text{ or } F_{\text{sol}}$$
may be obtained by **Galois evaluation**, i.e., by restricting the **Kummer classes** of the κ**-coric rational functions** discussed in §3.4, (ii), to the various **decomposition groups** that arise, respectively, from F_{mod}- or F_{sol}-rational points;
- this construction of F_{mod} or F_{sol} via Galois evaluation may be done in a fashion that is **compatible** with the **labels** $\in \mathbb{F}_l^{\circledast}$ and the $\mathbb{F}_l^{\circledast}$**-symmetry** that appear in a $\Theta^{\pm\text{ell}}$**NF-Hodge theater** [cf. the discussion of §3.3, (v); the *right-hand side* of Fig. 3.8];

· in particular, this *compatibility* with *labels* and the $\mathbb{F}_l^{\circledast}$*-symmetry* induces **symmetrizing isomorphisms** between copies of F_{mod} or F_{sol} that determine **graphs**, or **diagonals**, which may be thought of as corresponding to the **symbol** "$>$" [cf. the discussion of (i), (ii)].

In this context, we note that this approach to constructing *elements of NF's* by *restricting Kummer classes* of *rational functions on hyperbolic curves* to *decomposition groups of points defined over an NF* is precisely the approach taken in the **functorial algorithms** of [AbsTopIII], Theorem 1.9 [cf., especially, [AbsTopIII], Theorem 1.9, (d)]. Also, we observe that, although much of the above discussion runs in a somewhat *parallel* fashion to the discussion in (ii) of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** and the construction of **theta values** via **Galois evaluation**, there are important **differences**, as well, between the **theta** and **NF** cases [cf. [IUTchIII], Remark 2.3.3]:

· First of all, the **symmetrizing isomorphisms/diagonals** associated to the $\mathbb{F}_l^{\circledast}$-symmetry are **not** compatible with the symmetrizing isomorphisms/diagonals associated to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, *except* on the respective restrictions of these two collections of symmetrizing isomorphisms to **copies of** F_{mod} [cf. [IUTchII], Remark 4.7.2].

· Unlike the *Kummer theory* applied in the *theta case*, the **Kummer theory** applied in the *NF case* is **not** compatible with the **topologies** of the various profinite Galois or arithmetic fundamental groups that appear [cf. the discussion of §3.4, (v)]. On the other hand, this will not cause any problems since there is **no issue**, in the *NF case*, of applying formal power series such as the power series of the *natural logarithm* [cf. the final portion of the discussion of (ii); [IUTchIII], Remark 2.3.3, (vii), (viii)].

· In the Galois evaluation applied in the *theta case*, one is concerned with constructing, at a level where the **arithmetic holomorphic structure** [i.e., the *ring structure*] is **still intact**, **theta values** that **depend**, in an essential way, on the **label** "j". By contrast, in the Galois evaluation applied in the *NF case*, one only constructs, at such a level where the arithmetic holomorphic structure is still intact, the **totality** of [various copies of] the multiplicative monoid $F_{\mathrm{mod}}^{\times}$ associated to the number field F_{mod}: that is to say, a **dependence** on the **label** "j" only appears at the level of the **mono-analytic structures** constituted by the **global realified Frobenioids** of (c^{Θ}), i.e., in the form of a sort of *ratio*, or *weight*, "j^2" [cf. the fourth display of [IUTchII], Corollary 4.5, (v)] between the *arithmetic degrees* at the label "j" and the *arithmetic degrees* at the label "1" [cf. [IUTchIII], Remark 2.3.3, (iii); [IUTchIII], Remark 3.11.4, (i)].

In the context of this final *difference* between the *theta* and *NF cases*, it is perhaps of interest to observe that a similar sort of *"weighted copy"* $(F_{\mathrm{mod}}^{\times})^{j^2}$ $(\subseteq F_{\mathrm{mod}}^{\times})$ of $F_{\mathrm{mod}}^{\times}$ is **not possible** at the level of **arithmetic holomorphic structures** [that is to say, in the sense that it is *not compatible* with the *additive* interpretation of line bundles, i.e., in terms of *modules*] since this "weighted copy" $(F_{\mathrm{mod}}^{\times})^{j^2}$ $(\subseteq F_{\mathrm{mod}}^{\times})$ is **not closed under addition** [cf. the discussion of the final portion of §3.3, (iv); [AbsTopIII], Remark 5.10.2, (iv)].

(iv) **Actions on log-shells:** At this point, the reader may have noticed *two* *apparent shortcomings* [which are, in fact, closely related!] in the theory of **Galois evaluation** developed thus far in the present §3.6:

- Unlike the case with the *Kummer theory* of *κ-coric rational functions* and *theta functions/mono-theta environments* discussed in §3.4, the discussion of *Galois evaluation* given above does not mention any **multiradiality** properties.

- Ultimately, one is interested in relating [the *Kummer theory* surrounding] *Frobenius-like* structures — such as the *theta values* and *copies of NF's* that arise from *Galois evaluation* — in the *domain* of the Θ-link to [the *Kummer theory* surrounding] *Frobenius-like* structures in the *codomain* of the Θ-link, i.e., in accordance with the discussion of the technique of **mono-anabelian transport** in §2.7, §2.9. On the other hand, since the *theta values* and *copies of NF's* that arise from *Galois evaluation* are not [necessarily] *local units* at the various $\underline{v} \in \underline{\mathbb{V}}$, it is by no means clear how to relate this *Galois evaluation output data* to the *codomain* of the Θ-link using the **horizontally coric** portion — i.e., the **[local] unit group portion** (a^{Θ}) and (a^{q}) — of the Θ-link.

In fact, these two shortcomings are closed related: That is to say, the existence of the *obstruction* discussed in §3.4, (i), to the approach to *Kummer theory* taken in Example 2.12.1, (ii), that arises from the *natural action* "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$" implies — in light of the *nontrivial extension structure* that exists between the *value groups* and *units* of finite subextensions of "k" in "$\overline{k}$" [cf. the discussion in the final portion of Example 2.12.1, (iii)] — that, at least in any sort of *a priori* or *natural* sense,

> the **output data** — i.e., **theta values** and **copies of NF's** — of the **Galois evaluation** operations discussed in (ii), (iii) above [i.e., which lies in various copies of "k"!] is **by no means multiradial** [cf. [IUTchII], Remark 2.9.1, (iii); [IUTchII], Remark 3.4.1, (ii); [IUTchII], Remark 3.7.1; [IUTchIII], Remark 2.2.1, (iv)].

One of the *fundamental ideas* of inter-universal Teichmüller theory is that

> one may apply the theory of the $\mathfrak{log}$**-link** and **log-shells** to obtain a **solution** to these closely related shortcomings.

More precisely, from the point of view of the *log-theta-lattice*, the $\mathfrak{log}$*-link* from the lattice point $(n, m-1)$ to the lattice point (n, m) [where $n, m \in \mathbb{Z}$] allows one to construct [cf. the notation of Example 2.12.3, (iv)] a **holomorphic Frobenius-like log-shell** "$\mathcal{I}$" at (n, m) from the "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{k}}$" at $(n, m-1)$. Thus,

> the **output data** — i.e., **theta values** and **copies of NF's** — of the **Galois evaluation** operations discussed in (ii), (iii) above at (n, m) **acts naturally** on the "$\mathcal{I} \otimes \mathbb{Q}$" [i.e., the copy of "$k$"] at (n, m) that arises from this $\mathfrak{log}$**-link** from $(n, m-1)$ to (n, m) [cf. [IUTchIII], Proposition 3.3, (i); [IUTchIII], Proposition 3.4, (ii); [IUTchIII], Definition 3.8, (ii)].

On the other hand, this gives rise to a *fundamental dilemma*:

> Since the construction of the $\mathfrak{log}$-link — i.e., at a more concrete level, the *formal power series* of the *natural logarithm* — can only be defined

> if *both* the additive *and* the multiplicative structures of the [topological] rings involved are available, the $\mathfrak{log}$-link and hence, in particular, the construction of *log-shells* just discussed, at, say, the lattice point (n, m), are **meaningless**, at least in any *a priori* sense, from the point of view of the lattice point $(n+1, m)$, i.e., the *codomain* of the **Θ-link** from (n, m) to $(n+1, m)$ [cf. [IUTchIII], Remark 3.11.3; Steps (iii) and (iv) of the proof of [IUTchIII], Corollary 3.12].

Another of the *fundamental ideas* of inter-universal Teichmüller theory is the following [cf. the discussion in §3.3, (ii), of the **symmetry** of the *[infinite!] vertical lines* of the log-theta-lattice with respect to arbitrary **vertical translations**!]:

> By considering structures that are **invariant** with respect to **vertical shifts** of the log-theta-lattice — i.e., *vertically coric* structures such as **holomorphic/mono-analytic étale-like** *log-shells* that serve as *containers* for Frobenius-like objects such as **holomorphic Frobenius-like** *log-shells* [cf. Fig. 3.15 below] or *notions of vertical invariance* such as **upper semi-commutativity** or **log-volume compatibility** [cf. Example 2.12.3, (iv)] — and then *transporting* these invariant structures to the opposite side of the Θ-link by means of **mono-analytic Frobenius-like** *log-shells* [i.e., which may be constructed directly from the data "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\underline{k}}$" that appears in the *horizontally coric* data (a^{Θ}), (a^{q}) of the Θ-link], one may construct **multiradial containers** for the output data — i.e., **theta values** and **copies of NF's** — of the Galois evaluation operations discussed above.

$$\begin{array}{ccccccccc} \dots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

Fig. 3.15: A *vertical line* of the log-theta-lattice [shown *horizontally*]: *holomorphic Frobenius-like structures* "$\bullet$" at each lattice point related, via various *Kummer isomorphisms* [i.e., vertical or diagonal arrows], to *vertically coric holomorphic étale-like structures* "$\circ$"

Here, we observe that each of the *four* types of log-shells discussed in Example 2.12.3, (iv), plays an *indispensable role* in the theory [cf. [IUTchIII], Remark 3.12.2, (iv), (v)]:

- the **holomorphic Frobenius-like log-shells** satisfy the property [unlike their *mono-analytic* counterparts!] that the $\mathfrak{log}$-*link* — whose construction requires the use of the *topological ring structure* on these log-shells! — may be applied to them, as well as the property [unlike their *étale-like* counterparts!] that they belong to a *fixed vertical position* of a vertical line of the log-theta-lattice, hence are *meaningful* even in the *absence* of the $\mathfrak{log}$-link and, in particular, may be *related directly* to the Θ-link;
- the **holomorphic étale-like log-shells** allow one [unlike their *Frobenius-like* counterparts!] to relate holomorphic Frobenius-like log-shells at *different vertical positions* of a vertical line of the log-theta-lattice to one

another in a fashion [unlike their *mono-analytic* counterparts!] that takes into account the $\mathfrak{log}$*-link* [whose construction requires the use of the *topological ring structure* on these log-shells!];

· the **mono-analytic Frobenius-like log-shells** satisfy the property [unlike their *holomorphic* counterparts!] that they may be *constructed directly* from the data "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\underline{k}}$" that appears in the *horizontally coric* data (a^{Θ}), (a^{q}) of the Θ-link, as well as the property [unlike their *étale-like* counterparts!] that they belong to a *fixed vertical position* of a vertical line of the log-theta-lattice, hence are *meaningful* even in the *absence* of the $\mathfrak{log}$-link and, in particular, may be *related directly* to the Θ-link [cf. the discussion in the final portion of §3.3, (ii)];

· the **mono-analytic étale-like log-shells** satisfy the property [unlike their *holomorphic* counterparts!] that they may be *constructed directly* from the data "G_k" that appears in the *horizontally coric* data (a^{Θ}), (a^{q}) of the Θ-link, as well as the property [unlike their *Frobenius-like* counterparts, when taken *alone*!] that they may be used, in *conjunction* with their *Frobenius-like* counterparts, to relate *mono-analytic Kummer theory* [i.e., Kummer isomorphisms between mono-analytic Frobenius-like/étale-like log-shells] to *holomorphic Kummer theory* [i.e., Kummer isomorphisms between holomorphic Frobenius-like/étale-like log-shells].

In particular, the significance of working with *mono-analytic Frobenius-like log-shells* may be understood as follows. If, instead of working with mono-analytic Frobenius-like log-shells, one simply passes from *holomorphic Frobenius-like log-shells* at *arbitrary vertical positions* [in a single vertical line of the log-theta-lattice] to *holomorphic étale-like log-shells* [via *Kummer isomorphisms*] and then to *mono-analytic étale-like log-shells* [by *forgetting* various structures] — i.e.,

$$\begin{array}{ccccc} & & (\text{Ind3}) \curvearrowright & & (\text{Ind1}) \curvearrowright \\ \boxed{\begin{array}{c}\text{holomorphic}\\ \text{Frobenius-like}\\ \text{log-shells}\end{array}} & \rightsquigarrow & \boxed{\begin{array}{c}\text{holomorphic}\\ \text{étale-like}\\ \text{log-shells}\end{array}} & \rightsquigarrow & \boxed{\begin{array}{c}\text{mono-analytic}\\ \text{étale-like}\\ \text{log-shells}\end{array}} \end{array}$$

— then the relationship between the *mono-analytic étale-like log-shells* [whose *vertical position* is *indeterminate*!] and the various *holomorphic Frobenius-like log-shells* [in the vertical line under consideration] is **subject simultaneously** to **indeterminacies** arising from *both* the $\mathfrak{log}$**-link** [i.e., "(Ind3)" — cf. the discussion of §3.7, (i), below] *and* the **Θ-link** [i.e., "(Ind1)" — cf. the discussion of §3.7, (i), below]. Relative to the analogy discussed in the final portion of §3.3, (ii), between the log-theta-lattice and "$\mathbb{C}^{\times} \backslash GL^{+}(V)/\mathbb{C}^{\times}$", such *simultaneous indeterminacies* correspond to indeterminacies with respect to the action of the *subgroup of* $GL^{+}(V)$ *generated by* $\mathbb{C}^{\times}$ *and* "$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$". By contrast, by stipulating that the passage from *holomorphic étale-like log-shells* to *mono-analytic étale-like log-shells* be executed in *conjunction* with the *Kummer isomorphisms* [implicit in the data that is glued together in definition of the Θ-link] with *corresponding Frobenius-like log-shells at the vertical position* "0" — i.e.,

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{holomorphic}\\ \text{Frobenius-like}\\ \text{log-shells}\end{array}} \arrow[r, squiggly] & \boxed{\begin{array}{c}\text{holomorphic}\\ \text{étale-like}\\ \text{log-shells}\end{array}} & \boxed{\begin{array}{c}\text{mono-analytic}\\ \text{étale-like}\\ \text{log-shells}\end{array}} \\
& \boxed{\begin{array}{c}\text{holomorphic}\\ \text{Frobenius-like}\\ \text{log-shells at 0}\end{array}} \arrow[u, "(\mathrm{Ind3})\ \curvearrowright"] \arrow[r, phantom, "\rightsquigarrow"] & \boxed{\begin{array}{c}\text{mono-analytic}\\ \text{Frobenius-like}\\ \text{log-shells at 0}\end{array}} \arrow[u, "\curvearrowleft\ (\mathrm{Ind1}, 2)"']
\end{tikzcd}
$$

[where the vertical arrows denote the respective Kummer isomorphisms] — one obtains a **"decoupling"** of the $\mathfrak{log}$-link/Θ-link indeterminacies, i.e.,

- a **partially rigid relationship** between the **holomorphic/mono-analytic étale-like log-shells** and the **holomorphic/mono-analytic Frobenius-like log-shells** at the **vertical position 0** [in the vertical line under consideration] that is *subject only* to *indeterminacies* arising from the **Θ-link** [i.e., "(Ind1), (Ind2)" — cf. the discussion of §3.7, (i), below], together with

- a **partially rigid relationship** between the **holomorphic Frobenius-like log-shells** at the **vertical position 0** and the various **holomorphic Frobenius-like log-shells** at **arbitrary vertical positions** [in the vertical line under consideration], i.e., via **holomorphic étale-like log-shells**, that is *subject only* to *indeterminacies* arising from the **$\mathfrak{log}$-link** [i.e., "(Ind3)" — cf. the discussion of §3.7, (i), below].

Relative to the analogy discussed in the final portion of §3.3, (ii), between the log-theta-lattice and "$\mathbb{C}^\times \backslash GL^+(V)/\mathbb{C}^\times$", this *"decoupling"* of indeterminacies corresponds to an *indeterminacy* with respect to an action of $\mathbb{C}^\times$ *from the left*, together with a *distinct action* of "$\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$" *from the right*.

(v) <u>**Processions:**</u> In order to achieve the **multiradiality** discussed in (iv) for *Galois evaluation output data*, one further technique must be introduced, namely, the use of **processions** [cf. [IUTchI], Definition 4.10], which serve as a sort of **mono-analytic** substitute for the various **labels** in $\mathbb{F}_l^{\circledast}$, $|\mathbb{F}_l|$ $(= \mathbb{F}_l^{\circledast} \cup \{0\})$, or $\mathbb{F}_l$ discussed in (i). That is to say, since these labels are closely related to the various **cuspidal inertia subgroups** of the *geometric fundamental groups* "Δ" of the hyperbolic orbicurves involved [cf. the discussion of (i), (ii); §2.13; §3.3, (v)], it follows that these labels are **not horizontally coric** [i.e., *not directly visible* to the opposite side of the Θ-link — cf. the discussion of [IUTchIV], Remark 3.6.3, (ii)] and indeed do **not** even admit, at least in any *a priori* sense, any natural **multiradial** formulation. The approach taken in inter-universal Teichmüller theory to dealing with this state of affairs is to consider the *diagram of inclusions* of finite sets

$$\mathbb{S}^{\pm}_{1} \hookrightarrow \mathbb{S}^{\pm}_{1+1=2} \hookrightarrow \dots \hookrightarrow \mathbb{S}^{\pm}_{j+1} \hookrightarrow \dots \hookrightarrow \mathbb{S}^{\pm}_{1+l^{\circledast}=l^{\pm}}$$

— where we write $\mathbb{S}^{\pm}_{j+1} \stackrel{\mathrm{def}}{=} \{0, 1, \dots, j\}$, for $j = 0, \dots, l^{\circledast}$, and we think of each of these finite sets as being subject to *arbitrary permutation automorphisms*. That is to say, we think of

> the set $\mathbb{S}^{\pm}_{j+1}$ as a **container** for the labels $0, 1, \ldots, j$, and of the **label** "j" as **"some"** element of this container set, i.e., for each j, there is an *indeterminacy of* $j+1$ *possibilities* for the element of this container set that corresponds to j.

Here, we note in passing that this sort of indeterminacy is *substantially milder* than the indeterminacies that occur if one considers each j only as "some" element of $\mathbb{S}_{l^{\pm}}$, in which case *every* j is subject to an indeterminacy of $l^{\pm}$ possibilities — cf. [IUTchI], Proposition 6.9, (i), (ii). One then regards

> each such *container set* as an **index set** for a collection — which is referred to as a **"capsule"** [cf. [IUTchI], §0] — of copies of some sort of **étale-like mono-analytic prime-strip**.

An *étale-like mono-analytic prime-strip* is, roughly speaking, a collection of copies of data "G_k" indexed by $\underline{v} \in \underline{\mathbb{V}}$ [cf. [IUTchI], Fig. I1.2, and the surrounding discussion; [IUTchI], Definition 4.1, (iii)]. Now each *étale-like mono-analytic prime-strip*, in a *capsule*, as described above, gives rise, at each $\underline{v} \in \underline{\mathbb{V}}$, to

> *mono-analytic étale-like log-shells*, on which the **Galois evaluation output data acts**, in the fashion described in (iv), up to various **indeterminacies** that arise from the passage from *holomorphic Frobenius-like log-shells* to *mono-analytic étale-like log-shells* [cf. Figs. 3.16, 3.17 below, where each "$/^{\pm}$" denotes an *étale-like mono-analytic prime-strip*].

These *indeterminacies* will be discussed in more detail in §3.7, (i), below. In fact, ultimately, from the point of various **log-volume** computations, it is more natural to consider the *Galois evaluation output data* as acting, up to various indeterminacies, on certain **tensor products** of the various log-shells indexed by a particular container set $\mathbb{S}^{\pm}_{j+1}$. Such tensor products are referred to as **tensor packets** [cf. [IUTchIII], Propositions 3.1, 3.2].

$$
\begin{array}{ccccccccccc}
 & & \underline{\underline{q}}^{1} \curvearrowright & & & & \underline{\underline{q}}^{j^2} \curvearrowright & & & & \underline{\underline{q}}^{(l^*)^2} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots /^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots \ldots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}
$$

Fig. 3.16: *Theta values* acting on *tensor packets*

$$
\begin{array}{ccccccccccc}
 & & (F^{\times}_{\mathrm{mod}})_{1} \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_{j} \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_{l^*} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots /^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots \ldots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}
$$

Fig. 3.17: *Copies of* $F^{\times}_{\mathrm{mod}}$ acting on *tensor packets*

§**3.7.** **Log-volume estimates via the multiradial representation:** In the following, we outline the **statements** of and **relationships** between the **main results** [cf. [IUTchIII], Theorem 3.11; [IUTchIII], Corollary 3.12; [IUTchIV], Theorem 1.10; [IUTchIV], Corollaries 2.2, 2.3] of inter-universal Teichmüller theory.

(i) <u>**Multiradial representation of the Θ-pilot object up to mild indeterminacies:**</u> The content of the discussion of §3.4, §3.5, and §3.6 may be summarized as follows [cf. [IUTchIII], Theorem A; [IUTchIII], Theorem 3.11]:

> the data in the *domain* (a^{Θ}), (b^{Θ}), and (c^{Θ}) [cf. §3.3, (vii)] of the **Θ-link** may be expressed in a fashion that is **multiradial**, when considered up to certain **indeterminacies** (Ind1), (Ind2), (Ind3) [cf. the discussion below], with respect to the *radial algorithm*
>
> $$(a^{\Theta}),\ (b^{\Theta}),\ (c^{\Theta}) \quad \mapsto \quad (a^{\Theta})$$
>
> [cf. the discussion at the beginning of §3.4] by regarding this [**Frobenius-like**!] data (a^{Θ}), (b^{Θ}), and (c^{Θ}) [up to the indeterminacies (Ind1), (Ind2), (Ind3)] as data [cf. Fig. 3.18 below] that is **constructed** by
>
> · *first* applying the **Kummer theory** and **multiradial decouplings/cyclotomic rigidity** of **κ-coric rational functions** in the case of (c^{Θ}) [cf. §3.4, (ii), (v)] and of **theta functions/mono-theta environments** in the case of (b^{Θ}) [cf. §3.4, (iii), (iv), (v)]; §3.5; and
>
> · *then* applying the theory of **Galois evaluation** and **symmetrizing isomorphisms** at **l-torsion points**, together with the **$\mathbb{F}_l^{\rtimes\pm}$-symmetry** in the case of (b^{Θ}) [cf. §3.6, (i), (ii), (iv), (v)] and at **decomposition groups** corresponding to F_{mod}-/F_{sol}**-rational points**, together with the **$\mathbb{F}_l^{\circledast}$-symmetry** in the case of (c^{Θ}) [cf. §3.6, (i), (iii), (iv), (v)].

$$\boxed{\begin{array}{c} \underline{\underline{q}}^{1} \\[1em] \underline{\underline{q}}^{j^2} \\[1em] \underline{\underline{q}}^{(l^{\circledast})^2} \end{array}} \quad \curvearrowright \quad \boxed{\begin{array}{l} /^{\pm} \ \hookrightarrow \\ \quad /^{\pm}/^{\pm} \ \hookrightarrow \\ \qquad \dots \ \hookrightarrow \\ \qquad\quad /^{\pm}/^{\pm} \ \dots \ /^{\pm} \ \hookrightarrow \\ \qquad\qquad \dots \ \hookrightarrow \\ \qquad\qquad\quad /^{\pm}/^{\pm} \ \dots \ \dots \ /^{\pm} \end{array}} \quad \curvearrowleft \quad \boxed{\begin{array}{c} (F^{\times}_{\mathrm{mod}})_1 \\[1em] (F^{\times}_{\mathrm{mod}})_j \\[1em] (F^{\times}_{\mathrm{mod}})_{l^{\circledast}} \end{array}}$$

Fig. 3.18: The full *multiradial representation*

Here, we recall from §3.6, (iv), (v), that the data (b^{Θ}) and (c^{Θ}) act on **processions** of **tensor packets** that arise from the **mono-analytic étale-like log-shells** constructed from the data (a^{Θ}). The *indeterminacies* (Ind1),(Ind2), (Ind3) referred to above act on these *log-shells* and may be described as follows:

(Ind1) These indeterminacies are the **étale-transport** indeterminacies [cf. §2.7, (vi); Example 2.12.3, (i)] that occur as a result of the *automorphisms* [which, as was discussed in Example 2.12.3, (i), do *not*, in general, preserve the *ring structure*!] of the various "G_k's" that appear in the data (a^{Θ}).

(Ind2) These indeterminacies are the **Kummer-detachment** indeterminacies [cf. §2.7, (vi)] that occur as a result of the *identification* of, or *confusion* between, *mono-analytic Frobenius-like* and *mono-analytic étale-like log-shells* [cf. the discussion of §3.6, (iv)]. At a more concrete level, these indeterminacies arise from the action of the group of **"isometries"** — which is often denoted

$$\text{"Ism}(-)\text{"}$$

[cf. [IUTchII], Example 1.8, (iv)] — of the data "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$" [which we regard as equipped with a system of *integral structures*, or *lattices*] of (a^{Θ}), i.e., the **compact** *topological group* of G_k-equivariant automorphisms of the ind-topological module $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$ that, for each open subgroup $H \subseteq G_k$, *preserve* the *lattice* $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}^H} \subseteq (\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}})^H$.

(Ind3) These indeterminacies are the **Kummer-detachment** indeterminacies [cf. §2.7, (vi)] that occur as a result of the **upper semi-commutativity** [cf. Example 2.12.3, (iv); §3.6, (iv)] of the **$\mathfrak{log}$-Kummer correspondence** [cf. [IUTchIII], Remark 3.12.2, (iv), (v)], i.e., the system of **$\mathfrak{log}$-links** and **Kummer isomorphisms** of a particular *vertical line* of the log-theta-lattice [cf. Fig. 3.15].

In addition to these *"explicitly visible"* indeterminacies, there are also **"invisible indeterminacies"** [cf. [IUTchIII], Remark 3.11.4] that in fact arise, but may be *ignored* in the above description, essentially as a *formal consequence* of the way in which the various objects that appear are *defined*:

- The various **theta values** and **copies of** $F^{\times}_{\text{mod}}$ that occur as *Frobenius-like Galois evaluation output data* at various *vertical positions* of the $\mathfrak{log}$-*Kummer correspondence* [cf. the discussion of §3.6, (iv), (v)] **satisfy** an important **"non-interference"** property [cf. [IUTchIII], Proposition 3.5, (ii), (c); [IUTchIII], Proposition 3.10, (ii)]: namely, the *intersection* of such output data with the product of the *local units* [i.e., "$\mathcal{O}^{\times}$"] at those elements of $\underline{\mathbb{V}}$ at which the output data in question occurs consists only of **roots of unity**. As a result, the only *"possible confusion"*, or *"indeterminacy"*, that occurs as a consequence of possibly applying iterates of the $\mathfrak{log}$-link to the various *local units* consists of a *possible multiplication by a root of unity*. On the other hand, since the *theta values* and *copies of* $F^{\times}_{\text{mod}}$ that occur as *Frobenius-like Galois evaluation output data* are **defined** in such a way as to be **stable** under the action by multiplication by such roots of unity, this *indeterminacy* may, in fact, be *ignored* [cf. the discussion of [IUTchIII], Remark 3.11.4, (i)].
- The *indeterminacy* of possible **multiplication by** ± 1 in the **cyclotomic rigidity isomorphism** that is applied in the *Kummer theory* of *κ-coric rational functions* [cf. the final portion of the discussion of §3.4, (ii)] may be ignored since the *global Frobenioids* related to the data (c^{Θ}), i.e., that arise from copies of F_{mod}, only require the use of the **totality** of [copies of] the **multiplicative monoid** $F^{\times}_{\text{mod}}$, which is *stabilized* by the operation of *inversion* [cf. the discussion of [IUTchIII], Remark 3.11.4, (i)].

At this point, it is useful to recall [cf. the discussion at the beginning of §3.4, (i)] that it was possible to define, in §3.3, (vii), the **gluing isomorphisms** that constitute the **Θ-link** between the *domain* data (a^{Θ}), (b^{Θ}), (c^{Θ}) and the *codomain* data (a^q), (b^q), (c^q) *precisely* because we worked with various *abstract monoids* or *global realified Frobenioids*, i.e., as opposed to the *"conventional scheme-like representations"* of this data (a^{Θ}), (b^{Θ}), (c^{Θ}) in terms of *theta values* and *copies of*

NF's. In particular, one way to *interpret* the *multiradial representation* discussed above [cf. the discussion of **"simultaneous execution"** at the beginning of §2.9 and §3.4, (i)] is as follows:

> This **multiradial representation** may be understood as the [*somewhat surprising*!] assertion that not only the *domain* data (a^{Θ}), (b^{Θ}), (c^{Θ}), but also the *codomain* data (a^{q}), (b^{q}), (c^{q}) — or, indeed,
>
> **any** collection of data
> [i.e., not just the *codomain* data (a^{q}), (b^{q}), (c^{q})!]
> that is isomorphic to the *domain* data (a^{Θ}), (b^{Θ}), (c^{Θ})
>
> [cf., e.g., the discussion concerning "$\underline{\underline{q}}^{\lambda}$" in the second to last display of §3.8, below] — can, when regarded up to suitable **indeterminacies**, be represented via the **"conventional scheme-like representation"** of (a^{Θ}), (b^{Θ}), (c^{Θ}) in a fashion that is **compatible** with the **original** *"conventional scheme-like representation"* of the given collection of data [i.e., such as the "conventional scheme-like representation" of the data (a^{q}), (b^{q}), (c^{q})].

If we specialize this *interpretation* to the case of the data (a^{q}), (b^{q}), (c^{q}), then we obtain the following [again *somewhat surprising*!] *interpretation* of the *multiradial representation* discussed above [cf. the discussion of §2.4]:

> If one takes a **symmetrized average** over $N = 1^2, 2^2, \ldots, j^2, \ldots, (l^{*})^2$, and one works up to suitable **indeterminacies**, then
>
> the **arithmetic line bundle** determined by "$\underline{\underline{q}}$"
>
> [i.e., the $2l$-th roots of the q-parameter at the valuations $\in \underline{\mathbb{V}}^{\text{bad}}$, or, alternatively, the *"q-pilot object"*] may be **identified** — i.e., from the point of view of performing any sort of *computation* that takes into account the "suitable *indeterminacies*" — with
>
> the **arithmetic line bundle** determined by "$\underline{\underline{q}}^{N}$"
>
> [i.e., the *"Θ-pilot object"*] obtained by raising the *arithmetic line bundle* determined by "$\underline{\underline{q}}$" to the ***N*-th tensor power**

[cf. Fig. 3.19 below, where "LHS" and "RHS" denote, respectively, the *left-hand* and *right-hand sides*, i.e., the *domain* and *codomain*, of the Θ-link].

$$\begin{array}{c} \boxed{\{\underline{\underline{q}}^{j^2}_{\text{LHS}}\}_j} \\ \vdots \\ \boxed{\underline{\underline{q}}_{\text{LHS}}} \end{array} \ \underset{\text{“}=\text{”}}{\overset{\Theta}{\longrightarrow}} \ \begin{array}{c} \boxed{\{\underline{\underline{q}}^{j^2}_{\text{RHS}}\}_j} \\ \vdots \\ \boxed{\underline{\underline{q}}_{\text{RHS}}} \end{array} \quad \rightsquigarrow$$

$$\begin{array}{ccc} \boxed{\{\underline{\underline{q}}^{j^2}_{\text{LHS}}\}_j} & \overset{!!}{\leftrightarrows} & \boxed{\{\underline{\underline{q}}^{j^2}_{\text{RHS}}\}_j} \\ \searrow & & \swarrow \\ & \boxed{\text{mono-analytic log-shells}} & \end{array}$$

Fig. 3.19: The *gluing*, or *tautological identification* "=", of the Θ-*link* from the point of view of the *multiradial representation*

(ii) **Log-volume estimates:** The *interpretation* discussed in the final portion of (i) leads naturally to an **estimate** of the **arithmetic degree** of the **q-pilot object** [cf. [IUTchIII], Theorem B; [IUTchIII], Corollary 3.12], as follows [cf. Steps (x), (xi) of the proof of [IUTchIII], Corollary 3.12; [IUTchIII], Fig. 3.8]:

(1^{est}) One starts with the *Frobenius-like* version of the **q-pilot object** — i.e., "$\underline{\underline{q}}_{\mathrm{RHS}}$" — on the RHS of the **$\Theta$-link**. *All subsequent computations* are to be understood as computations that are performed relative to the **fixed arithmetic holomorphic structure** of this RHS of the Θ-link.

(2^{est}) The isomorphism class determined by this *q-pilot object* in the **global realified Frobenioid** of (c^q) [cf. §3.3, (vii)] is sent, via the Θ-*link*, to the isomorphism class determined by the **Θ-pilot object** — i.e., "$\{\underline{\underline{q}}_{\mathrm{LHS}}^{j^2}\}_j$" — in the **global realified Frobenioid** of (c^{Θ}) [cf. §3.3, (vii)].

(3^{est}) One then applies the **multiradial representation** discussed in (i) [cf. Fig. 3.19].

(4^{est}) One observes that the **log-volume**, suitably *normalized*, on the log-shells that occur in this *multiradial representation* is **invariant** with respect to the indeterminacies (Ind1) and (Ind2), as well as with respect to the *invisible indeterminacies*, discussed in (i).

(5^{est}) On the other hand, the **upper semi-commutativity** indeterminacy (Ind3) — i.e., "commutativity" at the level of *inclusions of regions in log-shells* [cf. the discussion of Example 2.12.3, (iv)] — may be understood as asserting that the log-volume of the multiradial representation of the Θ-pilot object must be interpreted as an **upper bound**.

(6^{est}) The **multiradial representation** of the Θ-*pilot object* "$\{\underline{\underline{q}}_{\mathrm{LHS}}^{j^2}\}_j$" can only be **compared** to the **isomorphism class** determined by the *original q-pilot object "$\underline{\underline{q}}_{\mathrm{RHS}}$"* in the global realified Frobenioid of (c^q), i.e., not to the *specific arithmetic line bundle* given by a copy of the trivial arithmetic line bundle with *fixed local trivializations* "1" multiplied by $2l$-th roots of q-parameters [cf. [IUTchIII], Remark 3.12.2, (v)].

(7^{est}) In particular, in order to perform such a **comparison** between the **multiradial representation** of the Θ-*pilot object* "$\{\underline{\underline{q}}_{\mathrm{LHS}}^{j^2}\}_j$" and the **isomorphism class** determined by the *original q-pilot object "$\underline{\underline{q}}_{\mathrm{RHS}}$"* in the global realified Frobenioid of (c^q), it is necessary to make the *output data* of the *multiradial representation* into a collection of *"$\mathcal{O}_k$-modules"* [where we use the notation "$\mathcal{O}_k$" to denote the various completions of the ring of integers of K at, for simplicity, the valuations $\in \mathbb{V}^{\mathrm{non}}$], i.e., relative to the **arithmetic holomorphic structure** of the **RHS** of the Θ-link!

(8^{est}) Such "$\mathcal{O}_k$-modules" are obtained by, essentially [cf. [IUTchIII], Remark 3.9.5, for more details], forming the **"$\mathcal{O}_k$-modules generated by"** the various *tensor packets of log-shells* [cf. the discussion of §3.6, (iv), (v)] that appear in the multiradial representation, i.e., which, *a priori*, are [up to a factor given by a suitable power of "p"] just *topological modules*

$$\text{“}\log(\mathcal{O}_k^{\times}) \otimes \log(\mathcal{O}_k^{\times}) \otimes \ldots \otimes \log(\mathcal{O}_k^{\times})\text{”}$$

— that is to say, tensor products of $j+1$ copies of "$\log(\mathcal{O}_k^{\times})$" at the portion of the multiradial representation *labeled by j*. This operation yields a "slightly enlarged multiradial representation", which is referred to as the

holomorphic hull [cf. [IUTchIII], Corollary 3.12; [IUTchIII], Remark 3.9.5] of the multiradial representation.

(9^{est}) The **log-volume**, when applied to the *original q-pilot object* "$\underline{\underline{q}}_{\mathrm{RHS}}$" or to the Θ*-pilot object* "$\{\underline{\underline{q}}^{j^2}_{\mathrm{LHS}}\}_j$", may be interpreted as the **arithmetic degree** of these objects [cf. §2.2; [IUTchIII], Remark 1.5.2, (i), (iii); [IUTchIII], Proposition 3.9, (iii); [IUTchIII], Remark 3.10.1, (iv)].

(10^{est}) In particular, any **upper bound** on the *log-volume* of the *holomorphic hull* of the *multiradial representation* of the **Θ-pilot object** "$\{\underline{\underline{q}}^{j^2}_{\mathrm{LHS}}\}_j$" may be **interpreted** [cf. [IUTchIII], Remark 3.12.2, (i), (ii)] as an upper bound on the log-volume of the *original [Frobenius-like] q-pilot object* "$\underline{\underline{q}}_{\mathrm{RHS}}$".

(11^{est}) This comparison of log-volumes was obtained by considering the *images* of various *Frobenius-like* objects in the *étale-like* tensor packets of log-shells of the multiradial representation. In particular, one must apply the **$\mathfrak{log}$-Kummer correspondence** on *both* the LHS [in the case of the Θ*-pilot object* "$\{\underline{\underline{q}}^{j^2}_{\mathrm{LHS}}\}_j$"] *and* the RHS [in the case of the *original [Frobenius-like] q-pilot object* "$\underline{\underline{q}}_{\mathrm{RHS}}$"] of the Θ-link. On the other hand, this does not affect the *resulting inequality*, in light of the **compatibility** of **log-volumes** with the arrows of the $\mathfrak{log}$-Kummer correspondence [cf. Example 2.12.3, (iv); §3.6, (iv); [IUTchIII], Remark 3.12.2, (iv), (v)].

(12^{est}) Thus, in summary, one obtains an **inequality** of **log-volumes** [cf. [IUTchIII], Theorem B; [IUTchIII], Corollary 3.12]

$$\text{log-vol.}\left(\begin{array}{c}\textbf{Θ-pilot object}\text{ up to}\\ \textbf{mild indeterminacies},\\ \text{i.e., (Ind1), (Ind2), (Ind3),}\\ \text{plus formation of}\\ \textbf{holomorphic hull}\end{array}\right) \;\geq\; \text{log-vol.}\left(\textbf{q-pilot object}\right) \quad (\approx 0)$$

— where the *log-volume of the q-pilot object* on the *right-hand side* of the inequality is *negative* and of *negligible* absolute value by comparison to the terms of interest [to be discussed in more detail in (iv) below] on the *left-hand side* of the inequality.

(iii) **<u>Comparison with a result of Stewart-Yu</u>:** Recall that in [StYu], an *inequality* is obtained which may be thought of as a sort of *"weak version of the ABC Conjecture"*, i.e., which is, roughly speaking, *weaker* than the inequality of the usual ABC Conjecture in that it contains an **undesired exponential** operation "exp(−)" in its upper bound. This sort of *deviation* from the inequality of the usual ABC Conjecture is of interest from the point of view of the **"vertical shift"** discussed in §3.6, (iv), which, on the one hand, gives rise to the **indeterminacy** (Ind3) [cf. the discussion of (i); (ii), (5^{est})] and, on the other hand, arises from the fact that the **horizontally coric** portion of the data related by the Θ-link *differs* from the sort of data in which one is ultimately interested *precisely* by a **single iterate** of the **$\mathfrak{log}$-link**, i.e., a **single vertical shift** in the log-theta-lattice.

(iv) **<u>Computation of log-volumes</u>:** Let us return to the discussion of (ii). It remains to compute the *left-hand side* of the *inequality* of (ii), (12^{est}), in more elementary terms. This is done in [IUTchIV], Theorem 1.10. This computation yields a **rather strong version** of the **Szpiro Conjecture inequality**, in the case of *elliptic curves over NF's* that admit **initial Θ-data** [cf. §3.3, (i)] that satisfies *certain technical conditions*. The **existence** of such *initial Θ-data* that satisfies certain technical conditions is then verified in [IUTchIV], Corollary 2.2, (ii), for *elliptic curves over NF's* that satisfy *certain technical conditions* by applying the techniques of [GenEll], §3, §4. Here, we remark in passing that the **prime number** l that appears in this initial Θ-data constructed in [IUTchIV], Corollary 2.2, (ii), is roughly of the **order** of the **square root** of the **height** of the elliptic curve under consideration [cf. [IUTchIV], Corollary 2.2, (ii), (C1)]. This initial Θ-data yields a version of the **Szpiro Conjecture inequality** [cf. [IUTchIV], Corollary 2.2, (ii), (iii)], which, although somewhat weaker and less effective than the inequality of [IUTchIV], Theorem 1.10, is still *rather strong* in the sense that it implies that, if we restrict, for simplicity, to the case of *elliptic curves over* $\mathbb{Q}$, then

> the **"ϵ terms"** that appear in the Szpiro Conjecture inequality concerning the **height** h may be **bounded above** by terms of the order of
>
> $$h^{1/2} \cdot \log(h)$$
>
> — i.e., at least in the case of elliptic curves over $\mathbb{Q}$ whose **moduli** are **"compactly bounded"**, in the sense that the moduli lie inside given **fixed compact subsets** of the sets of *rational points* of the moduli of elliptic curves over $\mathbb{R}$ and $\mathbb{Q}_2$

[cf. [IUTchIV], Remark 1.10.5, (ii), (iii); [IUTchIV], Remark 2.2.1, (i), (ii)]. Here, we recall from these Remarks in [IUTchIV] [cf. also [Mss]; [vFr], §2] that

> This "$\frac{1}{2}$" in the exponent of h is of interest in light of the existence of sequences of *"abc sums"* for which this "$\frac{1}{2}$" is **asympototically attained**, i.e., as a **bound from below**, but *only if* one works with abc sums that correspond to elliptic curves whose moduli are **not necessarily compactly bounded**.

This prompts the following *question*:

> *Can one construct sequences of* **abc sums** *with* **similar asymptotic behavior***, but which correspond to elliptic curves whose moduli are indeed* **compactly bounded***?*

At the time of writing, it appears that *no definitive answer to this question is known*, although there does exist some *preliminary work* in this direction [cf. [Wada]]. In this context, it is also of interest to recall [cf. the discussion of [IUTchIV], Remark 2.2.1; [vFr], §2] that this "$\frac{1}{2}$" is **highly reminiscent** of the "$\frac{1}{2}$" that appears in the **Riemann hypothesis**. So far, in the above discussion, we have restricted ourselves to versions of the *Szpiro Conjecture inequality* for elliptic curves over NF's that satisfy various *technical conditions*. On the other hand,

> by applying the theory of **noncritical Belyi maps** [cf. the discussion in the final portion of §2.1; [GenEll], Theorem 2.1; [IUTchIV], Corollary 2.3; [IUTchIV], Theorem A] — which may be thought of as a sort of **arithmetic version** of **analytic continuation** [cf. the discussion of §3.3, (vi); [IUTchI], Remark 5.1.4; [IUTchIV], Remark 2.2.4, (iii)] — one

may derive the inequalities of the **Vojta/Szpiro/ABC Conjectures** in their usual form.

We refer to [Fsk], §1.3, §2.12, for another — and, in certain respects, more detailed — discussion of these aspects of inter-universal Teichmüller theory. Although a detailed discussion of the *somewhat technical*, but *entirely elementary* computation of the *left-hand side* of the inequality of (ii), (12^{est}), lies beyond the scope of the present paper, we conclude the present (iv) with a summary of the *very simple computation* of the **leading term** of the *left-hand side* of the inequality of (ii), (12^{est}):

· First, one notes [cf. [IUTchIV], Proposition 1.2, (i); [IUTchIV], Proposition 1.3, (i); the second to last display of Step (v) of the proof of [IUTchIV], Theorem 1.10] that, if one *ignores* [since we are only interested in computing the *leading term*!] the *archimedean* valuations of K, as well as the *nonarchimedean* valuations of K whose *ramification index* over $\mathbb{Q}$ is *"large"* [i.e., is $\geq$ the cardinality of the set of nonzero elements of the *residue field* of the corresponding prime of $\mathbb{Q}$], then the resulting **log-volume** [suitably *normalized*!] of the **tensor packet** of **log-shells** corresponding to the **label** $j \in \{1, 2, \ldots, l^*\}$ is

$$(j+1) \cdot \log(\mathfrak{d}^K)$$

— where we write $\log(\mathfrak{d}^K)$ for the *arithmetic degree* [suitably *normalized*, so as to be *invariant* with respect to finite extensions — cf. [IUTchIV], Definition 1.9, (i)] of the *arithmetic divisor* determined by the **different ideal** of the number field K over $\mathbb{Q}$.

· On the other hand, the effect — i.e., on the *tensor packet* of *log-shells* corresponding to the *label* j — of *multiplying* by the **theta value** "$\underline{\underline{q}}^{j^2}$" [cf. Figs. 3.16, 3.18] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the second to last display of Step (v) of the proof of [IUTchIV], Theorem 1.10] is given by

$$-\tfrac{j^2}{2l} \cdot \log(\mathfrak{q})$$

— where we write $\log(\mathfrak{q})$ for the *arithmetic degree* [again suitably *normalized*] of the *arithmetic divisor* determined by the ***q*-parameters** of the elliptic curve E over F at the valuations that lie over valuations $\in \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$.

· Thus, the **leading term** of the **log-volume** of the *left-hand side* of the inequality of (ii), (12^{est}), is given by

$$\begin{aligned}\sum_{j=1}^{l^*} (j+1) \cdot \log(\mathfrak{d}^K) \;-\; \tfrac{j^2}{2l} \cdot \log(\mathfrak{q}) \approx \;& \tfrac{1}{2} \cdot (\tfrac{l}{2})^2 \cdot \log(\mathfrak{d}^K) \;-\; \tfrac{1}{3 \cdot 2l} \cdot (\tfrac{l}{2})^3 \cdot \log(\mathfrak{q}) \\ = \;& \tfrac{l^2}{8} \cdot \log(\mathfrak{d}^K) \;-\; \tfrac{l^2}{48} \cdot \log(\mathfrak{q}) \\ = \;& \tfrac{l^2}{48}\{6 \cdot \log(\mathfrak{d}^K) \;-\; \log(\mathfrak{q})\}\end{aligned}$$

— where the notation "$\approx$" denotes a possible *omission* of terms that do not affect the *leading term*. That is to say, one obtains a *"large positive constant"* $\frac{l^2}{48}$ times *precisely the quantity* — i.e.,

$$6 \cdot \log(\mathfrak{d}^K) \;-\; \log(\mathfrak{q})$$

— that one wishes to **bound from below** in order to conclude [a suitable version of] the **Szpiro Conjecture inequality**.

As discussed in [IUTchIV], Remark 1.10.1 [cf. also the discussion of §3.9, (i), (ii), below], this *"computation of the leading term"*, which was originally motivated by the **scheme-theoretic Hodge-Arakelov theory** of [HASurI], [HASurII], was in fact known to the author around the year 2000. Put another way, one of the *primary motivations* for the development of inter-universal Teichmüller theory was precisely the problem of establishing a **suitable framework**, or **geometry**, in which this computation could be performed.

§**3.8.** **<u>Comparison with the Gaussian integral</u>:** At this point, it is of interest to *compare* and *contrast* the theory of **"arithmetic changes of coordinates"** [i.e., §2] and **multiradial representations** [i.e., the present §3] discussed so far in the present paper with the classical computation of the **Gaussian integral**, as discussed in §1. In the following, the various "Steps" refer to the "Steps" in the computation of the Gaussian integral, as reviewed in §1.

(1^{gau}) The **naive change of coordinates** "$e^{-x^2} \rightsquigarrow u$" of Step 1 [cf. also Step 2] is *formally reminiscent* [cf. [IUTchII], Remark 1.12.5, (ii)] of the assignment

$$\{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*} \mapsto \underline{\underline{q}}$$

that appears in the definition of the **Θ-link** [cf. §2.4; §3.3, (vii), (b^{Θ}), (b^{q})].

(2^{gau}) The introduction of **two "mutually alien" copies** of the Gaussian integral in Step 3 may be thought of as corresponding to the appearance of the **two $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters "•"** in the *domain* and *codomain* of the Θ-link [cf. §2.7, (i); Fig. 3.6], which may be thought of as representing *two "mutually alien" copies* of the **conventional scheme theory** surrounding the elliptic curve under consideration, i.e., the elliptic curve that appears in the *initial Θ-data* of §3.3, (i).

(3^{gau}) The **two dimensions** of the Euclidean space $\mathbb{R}^2$ that appears in Step 4 may be thought of as corresponding to the *two dimensions* of the **log-theta-lattice** [cf. §3.3, (ii)], which are closely related to the **two underlying combinatorial dimensions** of a **ring**. Here, we recall from §2.11 that these two underlying combinatorial dimensions of a ring may be understood quite *explicitly* in the case of *NF's*, *MLF's*, or the *field of complex numbers*.

(4^{gau}) The point of view discussed in Step 5 that integrals may be thought of as computations of **net masses** as *limits of sums of infinitesimals of* **zero mass** may be understood as consisting of *two aspects*: First of all, the *summation of local contributions* that occurs in an integral may be regarded as corresponding to the use of **prime-strips** [i.e., *local data* indexed by elements of $\underline{\mathbb{V}}$ — cf. the discussion of §3.3, (iv)] and the computation of **heights** in terms of **log-volumes**, as discussed in §2.2. On the other hand, the *limit* aspect of an *integral*, which involves the consideration of some sort of notion of **infinitesimals** [i.e., *"zero mass"* objects], may be thought of as corresponding to the **fundamental dichotomy** between

Frobenius-like and **étale-like** objects [cf. §2.2; §2.7, (ii), (iii); §2.9; §3.3, (iii); §3.4, (i); §3.5].

(5^{gau}) The generalities concerning the effect on **integrals** of **changes of coordinates** on *local patches* of the *Euclidean plane* in Step 6 may be thought of as corresponding to the generalities on the *computational technique* of **mono-anabelian transport** as a mechanism for computing the effect of **"arithmetic changes of coordinates"** [cf. the discussion of §2.7, §2.8, §2.9, §2.10, §2.11]. This technique is motivated by the *concrete examples* given in §2.5, §2.6 of *changes of coordinates* related to **positive characteristic Frobenius morphisms**, as well as by the discussion of examples of **finite discrete approximations** of harmonic analysis given in §2.14.

(6^{gau}) The fundamental examples given in Step 7 of **linear changes of coordinates** in the Euclidean plane — i.e., **unipotent** linear transformations, **toral dilations**, and **rotations** — may be thought of as corresponding to the fundamental examples of **arithmetic changes of coordinates** in the case of **MLF's** that were discussed in §2.12 [cf. also the discussion of §2.3, §2.4, and §2.13; §3.3, (vi)].

<u>*Classical computation of the Gaussian integral*</u>	<u>*Inter-universal Teichmüller theory*</u>
naive change of coordinates $e^{-x^2} \rightsquigarrow u$	**Θ-link** $\{\underline{\underline{q}}^{j^2}\}_{j=1,\ldots,l^*} \mapsto \underline{\underline{q}}$
two "mutually alien" copies of the Gaussian integral	**domain** and **codomain** "•" of the Θ-link
two dimensions of the Euclidean plane $\mathbb{R}^2$	**two dimensions** of the **log-theta-lattice/ rings/NF's/MLF's/$\mathbb{C}$**
net mass = lim $\sum$ (infinitesimals of *zero mass*)	**prime-strips**; **heights** as **log-volumes**; **Frobenius-like/étale-like** objects
the effect on **integrals** of local Euclidean **changes of coordinates**	computation via **mono-anabelian transport** of the effect of **arithmetic changes of coordinates**
unipotent linear transformations, **toral dilations**, and **rotations**	examples for **MLF's** of **arithmetic changes of coordinates**

Fig. 3.20.1: Comparison between *inter-universal Teichmüller theory* and the classical computation of the *Gaussian integral*

(7^{gau}) The **passage** from **planar cartesian** to **polar coordinates** discussed in Step 8 may be understood as a sort of **rotation**, or **continuous deformation**, between the **two mutually alien copies** of the Gaussian integral introduced in Step 3, i.e., which correspond to the x- and y-axes. Thus, this passage to polar coordinates may be regarded as corresponding [cf. the discussion at the beginning of §3.1; §3.1, (iv); §3.2; the discussion surrounding Figs. 3.1, 3.3, 3.5, 3.6, 3.12, 3.13, 3.19; the discussion of [IUTchII], Remark 1.12.5] to the "deformation", or **"parallel transport"**, between distinct collections of **radial data** that appears in the definition of the notion of **multiradiality** and, in particular, to the passage from the **log-theta-lattice** to the **étale-picture** [cf. §3.6, (i)] and, ultimately, to the **multiradial representation** of §3.7, (i).

Classical computation of the Gaussian integral	*Inter-universal Teichmüller theory*
passage from **planar cartesian** to **polar coordinates**	passage from the **log-theta-lattice** to the **étale-picture**; **multiradiality**
decoupling into **radial** and **angular** coordinates	**Kummer theory** for **functions** via **multiradial decouplings/ cyclotomic rigidity**
evaluation of the **radial integral** via the **quadraticity** of the **exponent** in the Gaussian distribution	**multiradiality/ mono-theta rigidity** via the **quadraticity** of **theta group commutators**; **value group portion** of the **Θ-link**
evaluation of the **angular integral** via the **natural logarithm** on the complex **units**	**log-shells** as **containers** for **Galois evaluation** output; iterated **rotations/juggling** of the $\mathfrak{log}$**-link** acting on [local] **units**
naive change of coordinates *"justifiable"* up to a suitable **"error factor"** $\sqrt{\pi}$ arising from the square root of the **angular integral** over the complex **units**	**naive approach** to **bounding heights** via **"Gaussian Frobenius morphisms"** on NF's *"justifiable"* up to a suitable **"log-different error factor"** arising from the **indeterminacies** (Ind1), (Ind2), (Ind3) acting on **log-shells**

Fig. 3.20.2: Comparison between *inter-universal Teichmüller theory* and the classical computation of the *Gaussian integral*

(8^{gau}) The fundamental **decoupling** into **radial** and **angular** coordinates discussed in Step 9 may be understood as corresponding to the discussion in

§3.4 of **Kummer theory** for **special types** of **functions** via **multiradial decouplings/cyclotomic rigidity** [cf. also the discussion of *unit group* and *value group* portions in §2.11; §3.3, (vii)].

(9^{gau}) The efficacy of the **change of coordinates** that renders possible the evaluation of the **radial integral** in Step 10 may be understood as an *essentially formal* consequence of the **quadratic** nature of the **exponent** that appears in the Gaussian distribution. This *fundamental aspect* of the computation of the Gaussian integral may be regarded as corresponding to the fact that the **rigidity** properties of **mono-theta environments** that underlie the **multiradial decouplings/cyclotomic rigidity** discussed in §3.4, (iii), (iv) are, in essence, *formal consequences* [cf. the discussion in the final portion of §3.4, (iv)] of the **quadratic** structure of the **commutators** of the **theta groups** associated to the ample line bundles that appear in the theory [cf. the discussion of [IUTchIII], Remark 2.1.1; [IUTchIV], Remark 2.2.2; the discussion of the *functional equation of the theta function* in [Pano], §3]. In particular, the evaluation of the *radial integral* in Step 10 corresponds to the portion of inter-universal Teichmüller theory that relates to the [local] **value group portion** (b^{Θ}) of the **Θ-link** [cf. (1^{gau})].

(10^{gau}) The **angular integral** of Step 11 is an integral over the **unit group** of the field of complex numbers that is evaluated by executing the **change of coordinates** determined by the *imaginary part* of the **natural logarithm**. This *important aspect* of the computation of the Gaussian integral may be regarded as corresponding to the theory of **Galois evaluation** and **log-shells** exposed in §3.6 — cf., especially, the theory involving *iterates* of the **$\mathfrak{log}$-link** discussed in §3.6, (iv). In this context, it is of interest to recall [cf. Example 2.12.3, (v); §3.3, (ii), (vi)] that the $\mathfrak{log}$-*link* may be understood as a sort of **arithmetic rotation**, or **juggling**, of the *two underlying combinatorial dimensions of a ring* that essentially concerns the [local] **unit group portion**, i.e., (a^{Θ}), (a^{q}), of the **Θ-link** [cf. §3.3, (vii); §3.6, (iv)].

(11^{gau}) The **final computation** of the Gaussian integral in Step 12 may be summarized [cf. §1.7] as asserting that the **naive change of coordinates** of (1^{gau}) may in fact be *"justified"*, provided that one allows for a suitable **"error factor"** given by the *square root* of the *angular integral* of (10^{gau}). This conclusion may be understood as corresponding to the **computation**, discussed in §3.7, (ii), (iii), (iv), of the *left-hand side* of the *inequality* of §3.7, (ii), (12^{est}). This computation may be summarized as asserting that one obtains a **bound** on the **height** of the elliptic curve under consideration whose *leading term* is the **"log-different** $\log(\mathfrak{d}^K)$**"** [cf. the final portion of §3.7, (iv)] that one expects from [a suitable version of] the **Szpiro Conjecture**. Put another way, this computation may be summarized as asserting that

> the **naive approach** outlined in §2.3, §2.4 to **bounding heights** via **"Gaussian Frobenius morphisms"** on NF's may in fact be *"justified"*, provided that one allows for a suitable **"error factor"** that arises from the **indeterminacies** (Ind1), (Ind2), (Ind3) acting on the **log-shells** — i.e., the [local] **unit group**

portion (a^{Θ}), (a^{q}) of the **Θ-link** [cf. the discussion of (10^{gau})] — that appear in the **multiradial representation** discussed in §3.7, (i) [cf. also the discussion of [IUTchIV], Remark 2.2.2].

The various *observations* discussed in the present §3.8 are summarized in Figs. 3.20.1, 3.20.2, above. Finally, with regard to (1^{gau}), we note that

> the **left-hand side** "$\{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*}$" of the assignment discussed in (1^{gau}) **cannot** be **replaced** by
>
> $$\text{“}\underline{\underline{q}}^{\lambda}\text{” for } 1 \neq \lambda \in \mathbb{Q}_{>0}$$
>
> or by
>
> $$\text{“}\{\underline{\underline{q}}^{N\cdot j^2}\}_{j=1,\dots,l^*}\text{” for } 2 \leq N \in \mathbb{N}.$$

Indeed, *this property* of the *left-hand side* of the assignment discussed in (1^{gau}) is, in the case of "$\underline{\underline{q}}^{\lambda}$", a consequence of the

· the **lack** [i.e., in the case of "$\underline{\underline{q}}^{\lambda}$"] of a theory of **multiradial decouplings/cyclotomic rigidity** of the sort [cf. the discussion of §3.4, (iii), (iv)] that exists in the case of **theta functions** and **mono-theta environments**

[cf. the discussion of [IUTchIII], Remark 2.2.2, (i), (ii), (iii)] and, in the case of "$\{\underline{\underline{q}}^{N\cdot j^2}\}_{j=1,\dots,l^*}$", a consequence of the

· *special role* [cf. the discussion of §3.4, (iii)] played by the **first power** of [reciprocals of l-th roots of the] **theta function**;
· the condition [cf. the discussion at the beginning of §3.6] that the *assignment* "abstract functions $\mapsto$ values" that occurs in the passage from theta functions to theta values be obtained by applying the technique of **Galois evaluation**

[cf. [IUTchII], Remark 3.6.4, (iii), (iv); [IUTchIII], Remark 2.1.1, (iv); the discussion of the final portion of Step (xi) of the proof of [IUTchIII], Corollary 3.12]. Moreover, the *negation of this property* of the *left-hand side* of the assignment discussed in (1^{gau}) would imply a *stronger version of the Szpiro Conjecture inequality* that is in fact **false** [cf. [IUTchIV], Remark 2.3.2, (ii)]. By contrast,

> the **right-hand side** "$\underline{\underline{q}}$" of the assignment discussed in (1^{gau}) **can** be **replaced** by "$\underline{\underline{q}}^{\lambda}$" for $1 \neq \lambda \in \mathbb{Q}_{>0}$, *without any substantive effect* on the theory; moreover, doing so does **not** result in **any substantive improvement** in the estimates discussed in §3.7, (ii), (iii), (iv)

[cf. [IUTchIII], Remark 3.12.1, (ii)]. In this context, it is of interest to observe that:

> This sort of *qualitative difference* between the *left-* and *right-* hand sides of the assignment $\{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*} \mapsto \underline{\underline{q}}$ is *reminiscent* of the **qualitative difference** — e.g., the presence or absence of the **exponential**! — between the *left-* and *right-* hand sides of the *naive change of coordinates* $e^{-x^2} \rightsquigarrow u$.

§**3.9.** **Relation to scheme-theoretic Hodge-Arakelov theory:** Finally, we pause to reconsider the theory of **multiradial representations** developed in the present §3 from the point of view of the **scheme-theoretic Hodge-Arakelov theory** discussed in Example 2.14.3 — a theory which, as discussed in the final portion of §3.7, (iv), played a *central role* in motivating the development of inter-universal Teichmüller theory.

(i) **Hodge filtrations and theta trivializations:** We begin by examining the *natural isomorphism of F-vector spaces of dimension l^2*

$$\Gamma(E^\dagger, \mathcal{L}|_{E^\dagger})^{<l} \quad \stackrel{\sim}{\to} \quad \mathcal{L}|_{E[l]}$$

that constitutes the **fundamental theorem of Hodge-Arakelov theory** discussed in Example 2.14.3 in a bit more detail [cf., e.g., the discussion surrounding [Pano], Theorem 1.1] in the case where F is an **NF**. First of all, we observe that, although both the domain and codomain of this isomorphism are F-vector spaces of *dimension l^2*, by considering the natural action of suitable *theta groups* on the domain and codomain and applying the well-known theory of *irreducible representations of theta groups*, one may conclude that, up to *"uninteresting redudancies"*, this isomorphism may in fact be ["essentially"] regarded as an isomorphism between F-vector spaces of *dimension l*. The *left-hand side* of this isomorphism of l-dimensional F-vector spaces admits a natural **Hodge filtration** that arises by considering the subspaces of *relative degree* $< t$, for $t = 0, \ldots, l-1$. Moreover, one verifies easily that, if we write ω_E for the *cotangent space* of E at the origin of E, and τ_E for the *dual* of ω_E, then the *adjacent subquotients* of this Hodge filtration are the 1-dimensional F-vector spaces

$$\tau_E^{\otimes t}$$

for $t = 0, \ldots, l-1$, tensored with some *fixed* 1-dimensional F-vector space [i.e., which is *independent* of t], which we shall *ignore* since its *arithmetic degree* [i.e., when regarded as being equipped with *natural integral structures* at the nonarchimedean valuations of F and *natural Hermitian structures* at the archimedean valuations of F] is *sufficiently small* that its omission does not affect the computation of the *leading terms* of interest. On the other hand, the *right-hand side* of the isomorphism under consideration admits a natural **theta trivialization** [i.e., a natural isomorphism with the F-vector space $F^{\oplus l}$ given by the direct sum of l copies of the F-vector space F, which we think of as being *labeled* by $t = 0, \ldots, l-1$], which is compatible — up to contributions that are *sufficiently small* as to have no effect on the computation of the *leading terms* of interest — with the various *natural integral structures* and *natural Hermitian metrics* involved, except at the valuations where E has *potentially multiplicative reduction*, where one must *adjust* the natural integral structure [i.e., the integral structure determined by the *ring of integers* $\mathcal{O}_F$ of F] on the copy of F in $F^{\oplus l}$ *labeled* $t \in \{0, \ldots, l-1\}$ by a factor of

$$\underline{\underline{q}}^{t^2/4}$$

— where the notation "$\underline{\underline{q}}$" denotes a $2l$-th root of the *q-parameter* of E at the valuation under consideration. Next, let us observe that [again up to contributions that are *sufficiently small* as to have no effect on the computation of the *leading terms* of interest] one may *replace* the label $t \in \{0, \ldots, l-1\}$ by a *label* $j \in \{1, \ldots, l^*\}$, where we think of t as $\approx 2j$. Write $\Omega_E^{\log} \stackrel{\text{def}}{=} \omega_E^{\otimes 2}$. Then the 1-*dimensional F-vector* *spaces* — i.e., which we think of as *arithmetic line bundles* by equipping these 1-dimensional F-vector spaces with *natural integral structures* and *natural Hermitian metrics* — corresponding to the *label* $j \in \{1, \ldots, l^*\}$ on the *left-* and *right-hand* sides of the *natural isomorphism* of l-dimensional F-vector spaces determined by the *fundamental theorem of Hodge-Arakelov theory* assume the form

$$\{(\Omega_E^{\log})^{\otimes j}\}^\vee, \quad \underline{\underline{q}}^{j^2} \cdot F$$

— where the notation "$\vee$" denotes the *dual*. Put another way, if we tensor the *dual* of the *left-hand side* contribution at $j \in \{1, \ldots, l^*\}$ with the *right-hand side* contribution at $j \in \{1, \ldots, l^*\}$, then we obtain the conclusion that the *natural isomorphism* under consideration may be thought of *"at a very rough level"* — i.e., by replacing the *Hodge filtration* with its *semi-simplification*, etc. — as a sort of

> **global section** of some sort of **weighted average** over j of the [arithmetic line bundles corresponding to the] 1-dimensional F-vector spaces
>
> $$\underline{\underline{q}}^{j^2} \cdot (\Omega_E^{\log})^{\otimes j}$$
>
> — where j ranges over the elements of $\{1, \ldots, l^*\}$.

In fact, the above discussion may be translated into *purely geometric terms* by working with the **tautological one-dimensional semi-abelian scheme** over the **natural compactification of the moduli stack of elliptic curves** [over, say, a field of characteristic zero]. Then "$\Omega_E^{\log}$" may be thought of as the line bundle of **logarithmic differentials** on this compactified moduli stack. Moreover, one can *compute global degrees* "deg$(-)$"

$$\begin{aligned}
\sum_{j=1}^{l^*} \deg((\Omega_E^{\log})^{\otimes j}) \ - \ \deg(\tfrac{j^2}{2l} \cdot [\infty]) &= \sum_{j=1}^{l^*} j \cdot \deg(\Omega_E^{\log}) \ - \ \tfrac{j^2}{2l} \cdot \deg([\infty]) \\
&\approx \tfrac{1}{2} \cdot (\tfrac{l}{2})^2 \cdot \deg(\Omega_E^{\log}) \ - \ \tfrac{1}{3 \cdot 2l} \cdot (\tfrac{l}{2})^3 \cdot \deg([\infty]) \\
&= \tfrac{l^2}{8} \cdot \deg(\Omega_E^{\log}) \ - \ \tfrac{l^2}{48} \cdot \deg([\infty]) \\
&= \tfrac{l^2}{48} \{\deg((\Omega_E^{\log})^{\otimes 6}) \ - \ \deg([\infty])\}
\end{aligned}$$

— where the notation "$\approx$" denotes a possible *omission* of terms that do not affect the *leading term*; "$[\infty]$" denotes the effective divisor on the compactified moduli stack under consideration determined by the *point at infinity*, i.e., the scheme-theoretic zero locus of the q-parameter; by abuse of notation, we use the same notation for *"compactified moduli stack versions"* of the corresponding objects introduced in the discussion of elliptic curves over NF's. That is to say, in summary,

> the **determinant** of the **natural isomorphism** that appears in the **fundamental theorem of Hodge-Arakelov theory** is simply [an invertible constant multiple of] some positive tensor power of the well-known **discriminant modular form** of weight 12, i.e., a global section of $\omega_E^{\otimes 12} = (\Omega_E^{\log})^{\otimes 6}$ whose unique zero is a zero of order 1 at the *point at infinity* of the compactified moduli stack of elliptic curves

[cf. [Pano], §1; the discussion of the final portion of [HASurI], §1.2, for more details].

(ii) **Comparison with inter-universal Teichmüller theory:** First, we begin with the observation that, relative to the *classical analogy* between *NF's* and *one-dimensional functions fields* [over some field of constants], it is natural to think of

· **log-shells** as *localized absolute arithmetic analogues* of the notion of the sheaf of **logarithmic differentials**.

Indeed, this point of view is supported by the fact that the *log-shell* associated to a finite extension of $\mathbb{Q}_p$ [for some prime number p] whose *absolute ramification index* is $\leq p-2$ *coincides* with the *dual fractional ideal* to the *different ideal* of the given finite extension of $\mathbb{Q}_p$ [cf. [IUTchIV], Proposition 1.2, (i); [IUTchIV], Proposition 1.3, (i)]. Thus, it is natural to regard

· the [arithmetic line bundle corresponding to the] 1-dimensional F-vector space

$$\underline{\underline{q}}^{j^2} \cdot (\Omega_E^{\log})^{\otimes j}$$

— where j ranges over the elements of $\{1, \ldots, l^*\}$ — of the discussion of (i) as a sort of **scheme-theoretic analogue**, or **precursor**, of the portion *labeled* by j of the **multiradial representation** discussed in §3.7, (i) [cf. also the explicit *display* of §3.7, (ii), (8^{est})];

· the resulting *computation of global degrees* "deg(−)" given in (i) as a sort of **scheme-theoretic analogue**, or **precursor**, of the computation of the *leading term* of the **log-volume** of the *left-hand side* of the inequality of §3.7, (ii), (12^{est}) [cf. the final portion of §3.7, (iv)].

Indeed, this was *precisely* the point of view of the author around the year 2000 that *motivated* the author to develop inter-universal Teichmüller theory.

(iii) **Analytic torsion interpretation:** In conventional *Arakelov theory* for varieties over NF's, *analytic torsion* refers to a *metric invariant*, at the archimedean valuations of an NF, that measures the way in which the *space of global [holomorphic/algebraic] sections of a line bundle* — which is regarded, by means of various considerations in *harmonic analysis*, as a subspace of the *Hilbert space of L^2-class sections* of the line bundle — is *embedded* inside this ambient Hilbert space of L^2-class sections.

Since this ambient Hilbert space of L^2-class sections may be regarded as a **topological invariant**, i.e., which is *unaffected* by *deformations of the holomorphic moduli* of the variety under consideration, the notion of **analytic torsion** may be understood as a measure of the way in which the subspace constituted by the space of global algebraic sections — which depends, in a quite essential fashion, on the **holomorphic moduli** of the variety under consideration — is **embedded** inside this topological invariant.

When formulated in this way,

the notion of **analytic torsion** becomes *highly reminiscent* of the *computational technique* of **mono-analytic transport** [cf. the discussion of §2.7, §2.9] and, in particular, of the use of **log-shells** to construct the **"multiradial containers"** [cf. the discussion of §3.6, (iv)] for the various

arithmetic holomorphic structures that appear in the **multiradial representation** discussed in §3.7, (i).

Indeed, from this point of view, *scheme-theoretic Hodge-Arakelov theory* may be understood as a sort of *intermediate step* — i.e., a **finite discrete approximation**, in the spirit of the discussion of §2.14, which is, moreover, [unlike the classical notion of analytic torsion!] **defined over NF's** — between the *classical notion of analytic torsion* and *inter-universal Teichmüller theory.* Put another way,

- the *natural isomorphism* that appears in the *fundamental theorem* of **scheme-theoretic Hodge-Arakelov theory** may be understood as a sort of **polynomial-theoretic discretization** of the theory surrounding the classical notion of **analytic torsion**, while
- **inter-universal Teichmüller theory** may be understood as a sort of **global Galois-theoretic version over NF's** of the theory surrounding the classical notion of **analytic torsion**

[cf. the discussion of [IUTchIV], Remark 1.10.4].

Section 4: Historical comparisons and analogies

§**4.1.** **<u>Numerous connections to classical theories</u>:** Many discussions of inter-universal Teichmüller theory exhibit a tendency to emphasize the *novelty* of many of the *ideas* and *notions* that constitute the theory. On the other hand, another important aspect of many of these ideas and notions of inter-universal Teichmüller theory is their quite substantial relationship to *numerous classical theories.* One notable consequence of this latter aspect of inter-universal Teichmüller theory is the following:

> one obstacle that often hampers the progress of mathematicians in their study of inter-universal Teichmüller theory is a **lack of familiarity** with such **classical theories**, many of which date back to the 1960's or 1970's [or even earlier]!

(i) **<u>Contrast with classical numerical computations</u>:** We begin our discussion by recalling

> the *famous computation* in the late nineteenth century by *William Shanks* of π *to 707 places*, which was later found, with the advent of digital computing devices in the twentieth century, to be *correct only up to 527 places*!

The work that went into this sort of computation may strike some mathematicians as being reminiscent, in a certain sense, of the *sheer number of pages* of the various papers — i.e., such as [Semi], [FrdI], [FrdII], [EtTh], [GenEll], [AbsTopIII], [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] — that one must study in order to achieve a thorough understanding of inter-universal Teichmüller theory. In fact, however, inter-universal Teichmüller theory differs quite *fundamentally* from the computation of Shanks in that, as was discussed throughout the present paper, and especially in §3.8, the *central ideas* of inter-universal Teichmüller theory are rather **compact** and **conceptual** in nature and revolve around the issue of

comparison, by means of the notions of **mono-anabelian transport** and **multiradiality**, of **mutually alien copies** of miniature models of **conventional scheme theory** in a fashion that exhibits remarkable similarities to the *compact* and *conceptual* nature of the **classical computation** of the **Gaussian integral** by means of the introduction of two *"mutually alien copies"* of this integral.

One way of briefly summarizing these *remarkable similarities* [i.e., which are discussed in more detail in §3.8, as well as in the Introduction to the present paper] is as follows:

· the **"theta portion"** — i.e., the **Θ-link** — of the *log-theta-lattice* of inter-universal Teichmüller theory may be thought of as a sort of **statement** of the **main computational problem** of inter-universal Teichmüller theory and may be understood as corresponding to the various **Gaussians** that appear in the classical computation of the Gaussian integral, while
· the **"log portion"** — i.e., the **$\mathfrak{log}$-link** — of the *log-theta-lattice* of inter-universal Teichmüller theory may be thought of as a sort of **solution** of the **main computational problem** of inter-universal Teichmüller theory and may be understood as corresponding to the **angular** portion of the representation via *polar coordinates* of the [square of the] Gaussian integral.

In this context, it is of interest to recall the *remarkable similarities* of certain aspects of inter-universal Teichmüller theory to the theory surrounding the **functional equation** of the **theta function** — i.e., **"Jacobi's identity"** [cf. the discussion of the final portion of [Pano], §3; the discussion preceding [Pano], Theorem 4.1] — which may be thought of as a sort of **function-theoretic** version of the computation of the **Gaussian integral** that may be obtained, roughly speaking, by *interpreting* this computation of the Gaussian integral in the context of the **hyperbolic geometry** of the **upper half-plane**. As discussed in §2.4; Example 3.2.1, (iii); §3.3, (v), the analogy between certain aspects of inter-universal Teichmüller theory and the *hyperbolic geometry* of the *upper half-plane* has not been exposed in the present paper in much detail since it has already been exposed in substantial detail in [BogIUT]. On the other hand, it is of interest to recall that *classically*,

Jacobi's identity was often appreciated as a relatively *compact* and *conceptual* way to achieve a **startling improvement** in **computational accuracy** in the context of **explicit numerical calculations** of **values** of the **theta function**

[cf. the discussion preceding [Pano], Theorem 4.1].

(ii) **Explicit examples of connections to classical theories:** Next, we review various explicit examples of *connections* between inter-universal Teichmüller theory, as exposed thus far in the present paper, and various *classical theories*:

(1^{cls}) Recall from the discussion of §2.10 that the notion of a **"universe"**, as well as the use of **multiple universes** within the discussion of a single set-up in arithmetic geometry, already occurs in the mathematics of the 1960's, i.e., in the mathematics of *Galois categories* and *étale topoi* associated to schemes [cf. [SGA1], [SGA4]].

($2^{\rm cls}$) One important aspect of the appearance of *universes* in the theory of **Galois categories** is the **inner automorphism indeterminacies** that occur when one relates Galois categories associated to distinct schemes via a morphism between such schemes [cf. [SGA1], Exposé V, §5, §6, §7]. These indeterminacies may be regarded as **distant ancestors**, or **prototypes**, of the more *drastic* indeterminacies — cf., e.g., the indeterminacies (Ind1), (Ind2), (Ind3) discussed in §3.7, (i) — that occur in inter-universal Teichmüller theory.

($3^{\rm cls}$) The theory of *Tate* developed in the 1960's [cf. [Serre], Chapter III, Appendix] concerning **Hodge-Tate representations** plays a *fundamental role* in the theory of [$\mathbb{Q}_p$GC] [cf. also [AbsTopI], §3]. This theory of [$\mathbb{Q}_p$GC] and [AbsTopI], §3, may be regarded as a *precursor* of the theory of **log-shells** developed in [AbsTopIII], §3, §4, §5.

($4^{\rm cls}$) The approach of *Faltings* to ***p*-adic Hodge theory** via the technique of *almost étale extensions* [cf. [Falt2]] plays a *central role* in the ***p*-adic anabelian geometry** developed in [pGC]. Here, we recall that this theory of [pGC] constitutes the *crucial technical tool* that underlies the **Belyi** and **elliptic cuspidalizations** of [AbsTopII], §3, which, in turn, play a quite essential role in the theory of [AbsTopIII], hence, in particular, in inter-universal Teichmüller theory.

($5^{\rm cls}$) The work of *Tate* in the 1960's concerning **theta functions** on uniformizations of **Tate curves** [cf. [Mumf2], §5] plays a *fundamental role* in [EtTh], hence also in inter-universal Teichmüller theory.

($6^{\rm cls}$) The **scheme-theoretic Hodge-Arakelov theory** discussed in Example 2.14.3 and §3.9 may be regarded as a *natural extension* of [the portion concerning *elliptic curves* of] *Mumford's* theory of **algebraic theta functions** [cf. [Mumf1]].

($7^{\rm cls}$) The **invariance of the étale site**, up to isomorphism, with respect to the **Frobenius morphism in positive characteristic** [cf. the discussion of Example 2.6.1, (i)] was well-known [cf. [SGA1], Exposé IX, Théorème 4.10] to the Grothendieck school in the 1960's. As discussed in §2.6, §2.7, this phenomenon, taken together with the *fundamental work* of *Uchida* [cf. [Uchi]] in the 1970's concerning the **anabelian geometry** of one-dimensional function fields over a finite field, may be regarded as the *fundamental prototype* for the apparatus of **mono-anabelian transport** — and, in particular, for the terms **"Frobenius-like"** and **"étale-like"** — which plays a *central role* in inter-universal Teichmüller theory.

($8^{\rm cls}$) The use of **abstract [commutative] monoids**, e.g., in the theory of **Frobenioids** [cf. §3.3, (iii); §3.5], which plays a *fundamental role* in inter-universal Teichmüller theory, was motivated by the use of such monoids in the theory of **log schemes** [cf. [Kato1], [Kato2]], which, in turn, was motivated by the use of such monoids in the classical theory of **toric varieties** developed in the 1970's [cf. [KKMS]].

($9^{\rm cls}$) Recall from the discussion of §3.1, (iv), that the notion of **multiradiality**, which plays a *fundamental role* in inter-universal Teichmüller theory, may be regarded as a sort of *abstract combinatorial analogue* of the **Grothendieck definition** of a **connection**, i.e., which plays a central role in the classical theory of the **crystalline site** and dates back to the 1960's [cf. [GrCrs]].

(iii) **Monoids and Galois theory:** With regard to (ii), (8^{cls}), we observe that the important role played by *abstract [commutative] monoids* in inter-universal Teichmüller theory is reminiscent of the way in which such monoids are used by many mathematicians in research related to **"geometry over $\mathbb{F}_1$"** [i.e., the fictitious *"field with one element"*]. On the other hand, the way in which such monoids are used in inter-universal Teichmüller theory *differs fundamentally* from the way in which such monoids are used in conventional research on *geometry over* $\mathbb{F}_1$ in the following respect:

· in *inter-universal Teichmüller theory*, various **anabelian** and **Kummer-theoretic** aspects of **Galois** or **arithmetic fundamental groups** that *act* on such monoids play a *fundamental role* [cf. the discussion of **mono-anabelian transport** in §2.7, §2.9!];
· by contrast, at least to the author's knowledge at the time of writing, research on *geometry over* $\mathbb{F}_1$ does not involve, in any sort of essential way, such *anabelian* or *Kummer-theoretic* aspects of *Galois* or *arithmetic fundamental groups* acting on monoids.

Indeed, this fundamental difference between inter-universal Teichmüller theory and conventional research on geometry over $\mathbb{F}_1$ might give rise to various interesting questions and hence stimulate further research.

(iv) **Techniques to avoid stacks and 2-categories:** Some mathematicians have a *strong aversion* to the use of such notions as *"categories of categories"* or *algebraic stacks* — i.e., notions that obligate one to work with **2-categories** — in arithmetic geometry. Here, we observe that the substantive mathematical phenomenon that obligates one, in such situations, to work with 2-categories is essentially the same phenomenon as the phenomenon constituted by the **inner automorphism indeterminacies** discussed in (ii), (2^{cls}). On the other hand, in inter-universal Teichmüller theory, *various "general nonsense" techniques* are applied that allow one, in such situations, to work with **categories** [i.e., as opposed to 2-*categories*] and thus *avoid the cumbersome complications* that arise from working with 2-*categories*:

· In inter-universal Teichmüller theory, one typically works with **slim categories** such as Galois categories that arise from *slim profinite groups* [i.e., profinite groups for which the centralizer of every open subgroup is trivial] or *temp-slim tempered groups* [cf. [Semi], Remark 3.4.1]. The use of slim categories allows one, in effect, to think of "categories of categories" as *categories* [i.e., rather than 2-*categories*]. Indeed, in the case of slim profinite groups, this point of view is precisely the point of view that underlies the theory of [GeoAnbd]. Generalities concerning *slim categories* may be found in [FrdI], Appendix.
· Another important *"general nonsense"* technique that is used in inter-universal Teichmüller theory to *keep track explicitly* of the various types of **indeterminacies** that occur is the notion of a **poly-morphism** [cf. [IUTchI], §0], i.e., a [possibly empty] subset of the set of arrows between two objects in a category. Thus, there is a natural way to *compose* two poly-morphisms [i.e., that consist of composable arrows] to obtain a new

poly-morphism. Consideration of such composites of poly-morphisms allows one to *trace how various indeterminacies interact with one another*.

In this context, it is perhaps useful to observe that, from a more *classical point of view*,

> the **inner automorphism indeterminacies** discussed in (ii), (2^{cls}), correspond to the indeterminacy in the choice of a **basepoint** of a [say, connected, locally contractible] **topological space**.

That is to say, in **anabelian geometry**, working with

slim anabelioids as opposed to **slim profinite groups**

corresponds, in essence, to working, in **classical topology**, with

topological spaces as opposed to **pointed topological spaces**.

Since many natural maps between topological spaces — i.e., such as **localization** maps! — are *not* compatible with choices of *distinguished points*, it is often more natural, in many discussions of classical topology, to make use [not only of the notion of a *"pointed topological space"*, but also] of the notion of a *"topological space"* [i.e., that is *not* equipped with the choice of a *distinguished point*!]. It is precisely for this reason that in many discussions — i.e., such as those that occur in inter-universal Teichmüller theory, for instance, in the case of **localizations** at various **primes of an NF**! [cf. the discussion of §3.3, (iv), (v), (vi)] — involving the *geometry of categories*, it is much more natural and less cumbersome to work with *slim categories* such as *slim anabelioids* [i.e., as opposed to *profinite groups*].

(v) <u>**Notational complexity and mutually alien copies**</u>**:** Some readers of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] have expressed bafflement at the degree of **complexity of the notation** — i.e., by comparison to the degree of complexity of notation that is typical in conventional papers on arithmetic geometry — that appears in these papers. This complexity of notation may be understood as a *natural consequence* of

· the need to distinguish between objects that belong to **distinct copies**, i.e., *distinct "miniature models"*, of *conventional scheme theory* [cf., e.g., the labels "n, m" for the various *lattice points* "$\bullet$" in the *log-theta-lattice*, as discussed in §3.3, (ii); §3.6, (iv); [IUTchIII], Definition 3.8, (iii)], together with

· the need to distinguish between distinct objects — such as **distinct cyclotomes** related by nontrivial **cyclotomic rigidity isomorphisms** [cf., e.g., the discussion of §2.6, §2.12, §2.13, §3.4, as well as the discussion of §4.2, (i), below] — within a *single* miniature model of conventional scheme theory that are related to one another via structures that are *"taken for granted"* in conventional discussions of arithmetic geometry, but whose *precise specification* is in fact *highly nontrivial* in the context of situations where one considers *multiple* miniature models of conventional scheme theory.

Put another way, this complexity of notation may be regarded as an *inevitable consequence* of the *central role* played in inter-universal Teichmüller theory by **"mutually alien copies/multiple miniature models"** of conventional scheme theory and the resulting **inter-universality** issues that arise [cf. the discussion of §2.7, (i), (ii); §2.10; §3.8]. In particular, this complexity of notation is *by no means superfluous.*

§**4.2. <u>Contrasting aspects of class field theory and Kummer theory</u>:** We begin our discussion by observing that the *role* played by **local class field theory** [cf. the discussion of §2.11; §2.12, especially Example 2.12.1, (ii), (iii); §3.4, (v)] in inter-universal Teichmüller theory is, in many respects, *not particularly prominent*, while **global class field theory** [for NF's] is **entirely absent** from inter-universal Teichmüller theory. This situation for class field theory *contrasts sharply* with the *very central role* played by **Kummer theory** in inter-universal Teichmüller theory [cf. the discussion of **mono-anabelian transport** in §2.7, §2.9!]. In fact, this state of affairs is both **natural** and indeed somewhat **inevitable** for a number of reasons, which we pause to survey in the discussion to follow [cf. also Fig. 4.1 below].

(i) **<u>Strong functoriality properties and the central role of cyclotomic rigidity in Kummer theory</u>:** Perhaps the *most conspicuous difference* between *class field theory* and *Kummer theory* is the fact that, whereas

- **class field theory** may only be formulated for a certain **special class of arithmetic fields** [a class which in fact includes the function fields of all the integral schemes that appear in inter-universal Teichmüller theory], e.g., for *global fields* [i.e., fields that are finitely generated over an *NF* or a *finite field*] or certain types of *completions or localizations* of such global fields,
- **Kummer theory**, by contrast, may be formulated, by using the *Kummer exact sequence* in étale cohomology [cf., e.g., the discussion at the beginning of [Cusp], §2], for [essentially] **arbitrary** types of schemes and even for **abstract monoids** [cf. [FrdII], Definition 2.1, (ii)] that satisfy relatively weak conditions and do not necessarily arise from the multiplicative structure of a commutative ring.

A closely related difference between class field theory and Kummer theory is the fact that, whereas

- **class field theory** only satisfies very **limited functoriality** properties, i.e., for finite separable field extensions and certain types of localization operations associated to a valuation,
- **Kummer theory** satisfies **very strong functoriality** properties, for [essentially] *arbitrary morphisms* between [essentially] *arbitrary schemes* or between *abstract monoids* that satisfy suitable, relatively weak conditions.

These properties of Kummer theory make

> **Kummer theory** much more suitable for use in **anabelian geometry**, where it is natural to consider morphisms between arithmetic fundamental groups that correspond to quite *general morphisms* between quite *general schemes*, i.e., where by "quite general", we mean *by comparison* to the restrictions that arise if one attempts to apply class field theory.

Perhaps the *most fundamental example* of this sort of situation [i.e., that is of interest in anabelian geometry, but to which class field theory cannot, at least in any immediate way, be applied] is the situation that arises if one considers the operation of

> **evaluation** of various types of **functions** on, say, a curve, at a closed **point** of the curve

[cf. the discussion of (ii) below; Example 2.13.1, (iv); §2.14; §3.6; [IUTchIV], Remark 2.3.3, (vi)]. On the other hand, one *highly nontrivial* and *quite delicate* aspect of *Kummer theory* that does *not* appear in *class field theory* is the issue of

> establishing **cyclotomic rigidity isomorphisms** between *cyclotomes* constructed from the various *rings, monoids, Galois groups, or arithmetic fundamental groups* that appear in a particular situation.

Various *examples* of such isomorphisms between cyclotomes may be seen in the theory discussed in [PrfGC], [the discussion preceding] Lemma 9.1; [AbsAnab], Lemma 2.5; [Cusp], Proposition 1.2, (ii); [FrdII], Theorem 2.4, (ii); [EtTh], Corollary 2.19, (i); [AbsTopIII], Corollary 1.10, (ii); [AbsTopIII], Remarks 3.2.1, 3.2.2 [cf. also Example 2.12.1, (ii); Example 2.13.1, (ii); §3.4, (ii), (iii), (iv), (v), of the present paper]. All of these *examples* concern **"Kummer-faithful"** situations [cf. [AbsTopIII], Definition 1.5], i.e., situations in which

> the **Kummer map** on the multiplicative monoid [e.g., which arises from the multiplicative structure of a ring] of interest is **injective**.

Then the cyclotomic rigidity issues that arise typically involve the *cyclotomes* obtained by considering the *torsion subgroups* of such multiplicative monoids. This sort of situation **contrasts sharply** with the sort of *highly "non-Kummer-faithful"* situation considered in [PopBog], i.e., where one works with function fields over *algebraic closures of finite fields* [cf. [PopBog], Theorem I]. That is to say, in the sort of situation considered in [PopBog], the Kummer map *vanishes* on the *roots of unity* of the base field, and the *Kummer theory* that is applied [cf. [PopBog], §5.2] does *not* revolve around the issue of *establishing cyclotomic rigidity isomorphisms*. In particular, in the context of this sort of application of Kummer theory, it is natural to think of the image of the Kummer map as a sort of *projective space*, i.e., a quotient by the action of multiplication by nonzero elements of the base field. Thus, in summary, the relationship just discussed between "Kummer-faithful Kummer theory" and "non-Kummer-faithful Kummer theory" may be thought of as the difference between

"**injective** *Kummer theory*" and "**projective** *Kummer theory*".

(ii) **The functoriality of Kummer theory with respect to evaluation of special functions at torsion points:** As mentioned in (i), the operation of **evaluation** of various types of [**special**] **functions** on, say, a curve, at various types of [**special**] **points** of the curve plays a *fundamental role* in the **Kummer theory** that is applied in inter-universal Teichmüller theory [cf. the discussion of Example 2.13.1, (iv); §2.14; §3.6; [IUTchIV], Remark 2.3.3, (vi)]. This *contrasts sharply* with the fact that

> **class field theory** may only be related to the operation of *evaluation of special functions at special points* in very *restricted classical cases*, namely,

the theory of **exponential functions** in the case of $\mathbb{Q}$ or **modular and elliptic functions** in the case of imaginary quadatic fields.

Indeed, the goal of generalizing the theory that exists in these very restricted cases to the case of **arbitrary NF's** is precisely the content of *Kronecker's Jugendtraum*, or *Hilbert's twelfth problem* [cf. the discussion of [IUTchIV], Remark 2.3.3, (vii)]. Indeed, in light of this state of affairs, one is tempted to regard **inter-universal Teichmüller theory** as a sort of

> **"realization/solution"** of the "version" of **Kronecker's Jugendtraum**, or **Hilbert's twelfth problem**, that one obtains if one replaces *class field theory* by **Kummer theory**!

<u>*Class field theory*</u>	<u>*Kummer theory*</u>
may be formulated only for **special arithmetic fields**	may be formulated for very general abstract **multiplicative monoids**
satisfies only very **limited functoriality** properties	satisfies very **strong functoriality** properties
limited range of applicability to **anabelian geometry**	**wide** range of applicability to **anabelian geometry**
cyclotomic rigidity isomorphisms are **irrelevant**	**cyclotomic rigidity isomorphisms** play a *central role*
no known compatibility with *evaluation* of *functions* at *points*	*compatible* with **evaluation** of **functions** at **points**
closely related to the **arithmetic holomorphic structure** of *very restricted types* of **arithmetic fields**	applicable to **mono-analytic structures** such as abstract **multiplicative monoids**
incompatible with local *unit group/value group decouplings*	*compatible* with local **unit group/ value group decouplings**
related to **global Dirichlet density of primes**	naturally applied in conjunction with **Prime Number Theorem**
verification, involving **cyclotomic extensions**, of the **global reciprocity law**	**global cyclotomic rigidity** algorithms via $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$

Fig. 4.1: Comparison between *class field theory* and *Kummer theory*

(iii) <u>**The arithmetic holomorphicity of global class field theory versus the mono-analyticity of Kummer theory:**</u> Another important aspect of the *fundamental differences* between *class field theory* and *Kummer theory* that were highlighted in the discussion of (i) is the following:

· whereas the essential content of **class field theory** reflects various *delicate arithmetic properties* that are closed related to the **arithmetic holomorphic structure** of the very restricted types of *arithmetic fields* to which it may be applied,
· the *very general* and *strongly functorial* nature of **Kummer theory** makes Kummer theory more suited to treating the sorts of **mono-analytic structures** that arise in inter-universal Teichmüller theory

[cf. the discussion of §2.7, (vii)]. Indeed, at a very naive level, this phenomenon may be seen in the *difference* between the **"input data"** for class field theory and Kummer theory, i.e., very restricted arithmetic *fields* in the case of *class field theory* versus very general types of *abstract multiplicative monoids* in the case of *Kummer theory* [cf. the discussion of (i)]. Another important instance of this phenomenon may be seen in the fact that whereas

· the **global reciprocity law**, which plays a *central role* in **class field theory** for NF's, involves a *nontrivial "intertwining" relationship*, for any *prime number* l, between the *local unit* determined by l at nonarchimedean valuations of residue characteristic $\neq l$ and the nonzero element of the *value group* determined by l at nonarchimedean valuations of residue characteristic l,
· the *compatibility* of the **Kummer theory** applied in inter-universal Teichmüller theory with various **"splittings"/"decouplings"** between the local **unit group** and **value group** portions of this Kummer theory plays a *central role* in inter-universal Teichmüller theory

— cf. the discussion of §3.4; §3.8, (8^{gau}); [IUTchIV], Remark 2.3.3, (v). A closely related fact is the fact that such local *unit group/value group splittings* are **incompatible** with [the multiplicative version of] **Hilbert's Theorem 90**, which plays a *central role* in **class field theory**: that is to say, in the notation of Examples 2.12.1, 2.12.2, one verifies immediately that whereas $H^1(G_k, \overline{k}^\times) = 0$,

$$H^1(G_k, \boldsymbol{\mu}_{\overline{k}}) \neq 0, \quad H^1(G_k, \mathcal{O}^\times_{\overline{k}}) \neq 0, \quad H^1(G_k, \boldsymbol{\mu}_{\overline{k}} \cdot \pi_k^{\mathbb{Q}}) \neq 0$$

— where we write $\boldsymbol{\mu}_{\overline{k}} \cdot \pi_k^{\mathbb{Q}} \subseteq \overline{k}^\times$ for the subgroup of elements for which *some positive power* $\in \pi_k^{\mathbb{Z}}$. Finally, we recall from the discussion of [IUTchIV], Remark 2.3.3, (i), (ii), (iv), that a sort of *analytic number theory* version of this phenomenon may be seen in the fact that whereas

· **class field theory** is closely related — especially if one takes the point of view of *early approaches* to class field theory such as the approach attributed to *Weber* — to the **"coherent aggregrations"** of primes that appear in discussions of the **Dirichlet density** of primes, e.g., in the context of the **Tchebotarev density theorem**,
· the **Kummer-theoretic** approach of inter-universal Teichmüller theory gives rise to the **multiradial representation** discussed in §3.7, (i), which leads to *log-volume estimates* [cf. the discussion of §3.7, (ii), (iv); the application of [IUTchIV], Proposition 1.6, and [IUTchIV], Proposition 2.1, (ii), in the explicit calculations of [IUTchIV], §1, §2] that involve, in an essential way, the **Prime Number Theorem**, i.e., which, so to speak, *counts primes "one by one"*, in effect *"deactivating the coherent aggre-*

grations of primes" that appear in discussions of the Dirichlet density of primes.

(iv) **Global reciprocity law versus global cyclotomic rigidity:** Finally, we recall from the discussion of ["(b-4)" in] [IUTchIV], Remark 2.3.3, (i), (ii), that

· the use of **cyclotomic extensions** in classical approaches to verifying the **global reciprocity law** in **class field theory** for NF's, i.e., to verifying that, in effect, the *reciprocity map vanishes* on idèles that arise from elements of the NF under consideration,

may be thought of as corresponding to

· the approach taken in inter-universal Teichmüller theory to constructing **cyclotomic rigidity isomorphisms** for the **Kummer theory** related to NF's [cf. §3.4, (ii), (v)], i.e., in effect, by applying the elementary fact that $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$ [cf. the discussion of the latter portion of [IUTchIII], Remark 3.12.1, (iii)].

Indeed, both of these phenomena concern the fact that some version of the **product formula** — that is to say, which, *a priori* [or from a more naive, elementary point of view], is only known to hold for the *Frobenius-like multiplicative monoids* that arise from NF's — in fact *holds* [i.e., in the form of the *global reciprocity law* or the elementary fact that $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$] at the level of [*étale-like*!] **profinite Galois groups**.

§4.3. Arithmetic and geometric versions of the Mordell Conjecture:

(i) **Rough qualitative connections with Faltings' proof of the Mordell Conjecture:** First, we begin by observing [cf. [IUTchIV], Remark 2.3.3, (i), (ii), for more details] that there are numerous *rough, qualitative correspondences* — some of which are closely related to the topics discussed in §4.1 and §4.2 — between various components of the proof of the Mordell Conjecture given in [Falt1] and inter-universal Teichmüller theory:

(1^{flt}) Various well-known aspects of classical algebraic number theory related to the **"geometry of numbers"**, such as the theory of *heights* and the *Hermite-Minkowski theorem*, are applied in [Falt1]. Similar aspects of classical algebraic number theory may be seen in the *"non-interference"* property [i.e., the fact that the only nonzero elements of an NF that are *integral* at all nonarchimedean and archimedean valuations of the NF are the *roots of unity*] for copies of the number field F_{mod} discussed in §3.7, (i), as well as in the use of *global realified Frobenioids* associated to NF's [i.e., which are essentially an abstract category-theoretic version of the classical notions of *arithmetic degrees* and *heights*].

(2^{flt}) **Global class field theory** for NF's, as well as the closely related notion of **Dirichlet density of primes**, plays an important role in [Falt1]. These aspects of [Falt1] are compared and contrasted in substantial detail in the discussion of §4.2 with the *Kummer theory* that plays a central role in inter-universal Teichmüller theory.

(3^{flt}) The theory of **Hodge-Tate decompositions** of p-adic Tate modules of abelian varieties over MLF's plays an important role both in [Falt1] and, as discussed in §4.1, (ii), (4^{cls}), in inter-universal Teichmüller theory.

(4^{flt}) The computations, applied in [Falt1], of the ramification that occurs in the theory surrounding **finite flat group schemes** bear a rough resemblance to the ramification computations involving *log-shells* in [AbsTopIII]; [IUTchIV], Propositions 1.1, 1.2, 1.3, 1.4.

(5^{flt}) The *hidden endomorphisms* [cf. the discussion of [AbsTopII], Introduction] that underlie the theory of *Belyi* and *elliptic cuspidalizations* [cf. the discussion of §3.3, (vi); §3.4, (iii)], which play an important role in inter-universal Teichmüller theory, as well as the theory of *noncritical Belyi maps* that is applied [cf. the discussion of §3.7, (iv)], via [GenEll], §2, in [IUTchIV], §2, may be thought of as a sort of *analogue* for *hyperbolic curves* of the theory of **isogenies** and **Tate modules** of **abelian varieties** that plays a central role in [Falt1].

(6^{flt}) The important role played by **polarizations** of **abelian varieties** in [Falt1] may be compared to the quite *central role* played by *commutators* of *theta groups* in the theory of *rigidity properties* of *mono-theta environments*, and hence in inter-universal Teichmüller theory as a whole [cf. the discussion of §3.4, (iv); §3.8, (9^{gau}); §4.1, (ii), (5^{cls})].

(7^{flt}) The **logarithmic geometry** of **toroidal compactifications**, which plays an important role in [Falt1], may be compared to the *logarithmic geometry* of special fibers of stable curves. The latter instance of logarithmic geometry is the starting point for the *combinatorial anabelian geometry* of *tempered fundamental groups* developed in [Semi], which plays an important role throughout inter-universal Teichmüller theory.

(ii) **Arithmetic holomorphicity versus mono-analyticity/multiradiality:** One way to summarize the discussion of (i), as well as a substantial portion of the discussion of §4.2 [cf. [IUTchIV], Remark 2.3.3, (iii)], is as follows:

> **inter-universal Teichmüller theory** may be understood, to a substantial extent, as a sort of **hyperbolic, mono-analytic/multiradial** analogue of the the **abelian**, **arithmetic holomorphic** theory of [Falt1].

Indeed, this is precisely the point of view of the discussion of §4.2, (iii), concerning the relationship between the essentially *arithmetic holomorphic* nature of **global class field theory** and the essentially *mono-analytic* nature of **Kummer theory**. If one takes the point of view [cf. the discussion of §2.3, §2.4, §2.5, §2.6; Examples 2.14.2, 2.14.3] that *Galois* or *arithmetic fundamental groups* should be thought of as *"arithmetic tangent bundles"*, then the point of view of the present discussion may be formulated in the following way [cf. the discussion of the final portion of [IUTchI], §I2]: Many results in the **conventional** framework of arithmetic geometry that concern *Galois* or *arithmetic fundamental groups* may be understood as results to the effect that some sort of

$$\text{“}H^0\text{(arithmetic tangent bundle)”}$$

does indeed coincide with some sort of *very small collection* of **scheme-theoretic** — i.e., **arithmetic holomorphic** — auto-/endo-morphisms. Indeed, examples of this sort of phenomenon include

(1^{hol}) the version of the **Tate Conjecture** proven in [Falt1];

(2^{hol}) various **bi-anabelian** results [cf. the discussion of §2.7, (v)] — i.e., *fully* *faithfulness* results in the style of various versions of the *"Grothendieck Conjecture"* — in anabelian geometry;

(3^{hol}) the *"tiny" special case* of the theory of [Falt1] discussed in §2.3 to the effect that **"Frobenius endomorphisms of NF's"** of the desired type [i.e., that yield *bounds on heights* — cf. the discussion of §2.4!] **cannot exist**, i.e., so long as one *restricts* oneself to working within the framework of **conventional scheme theory**;

(4^{hol}) the results of [Wiles] concerning *Galois representations* [cf. the discussion of [IUTchI], §I5], which may be summarized as asserting, in essence, that, roughly speaking, **nontrivial deformations** of **Galois representations** that satisfy suitable natural conditions **do not exist**.

All of the results just stated assert some sort of

"arithmetic holomorphic nonexistence"

[up to a very small number of exceptions], hence lie in a *fundamentally different* direction from the content of **inter-universal Teichmüller theory**, which, in effect, concerns the *construction* — or

"non-arithmetic holomorphic existence"

— of a **"Frobenius endomorphism of an NF"**, by working *outside* the framework of *conventional scheme theory*, i.e., by considering suitable **mono-analytic/ multiradial deformations** of the **arithmetic holomorphic structure**, that is to say, at the level of suggestive notation, by considering

"H^1(arithmetic tangent bundle)".

(iii) **Comparison with the metric proofs of Parshin and Bogomolov in the complex case:** **Parshin** [cf. [Par]] and **Bogomolov** [cf. [ABKP], [Zh]] have given proofs of *geometric versions* over the **complex numbers** of the **Mordell** and **Szpiro Conjectures**, respectively [cf. the discussion of [IUTchIV], Remarks 2.3.4, 2.3.5]. Parshin's proof of the geometric version of the Mordell Conjecture is discussed in detail in [IUTchIV], Remark 2.3.5, while Bogomolov's proof of the geometric version of the Szpiro Conjecture is discussed in detail in [BogIUT]. The relationships of these two proofs to *one another*, as well as to the *arithmetic theory*, may be summarized as follows:

(1^{PB}) Both proofs revolve around the consideration of **metric estimates** of the **displacements** that arise from various *natural actions* of elements of the [usual topological] **fundamental groups** that appear [cf. the discussion at the beginning of [IUTchIV], Remark 2.3.5].

(2^{PB}) Both *Parshin's* and *Bogomolov's proofs* concern the **metric geometry** of the complex spaces that appear. On the other hand, these two proofs *differ fundamentally* in that whereas the metric geometry that appears in Parshin's proof concerns the **holomorphic geometry** that arises from the **Kobayashi distance** — i.e., in effect, the **Schwarz lemma** of elementary complex analysis — the metric geometry that appears in Bogomolov's proof concerns the **real analytic hyperbolic geometry** of the upper half-plane [cf. [IUTchIV], Remark 2.3.5, (PB1)].

(3^{PB}) The difference observed in (2^{PB}) is interesting in that in corresponds *precisely* to the difference discussed in (ii) above between the proof of the **arithmetic Mordell Conjecture** [for NF's!] in [Falt1] and **inter-universal Teichmüller theory** [cf. the discussion of [IUTchIV], Remark 2.3.5, (i)].

(4^{PB}) *Parshin's proof* concerns, as one might expect from the *statement* of the Mordell Conjecture, **rough, qualitative estimates**. This state of affairs *contrasts sharply*, again as one might expect from the *statement* of the Szpiro Conjecture, with *Bogomolov's proof*, which concerns **effective, quantitative estimates** [cf. the discussion of [IUTchIV], Remark 2.3.5, (ii)].

(5^{PB}) The appearance of the *Kobayashi distance* — i.e., in essence, the *Schwarz lemma* of elementary complex analysis — in (2^{PB}) is of interest in light of the point of view discussed in §3.3, (vi); §3.7, (iv), concerning the correspondence between the use of **Belyi maps** in inter-universal Teichmüller theory, i.e., in the context of **Belyi cuspidalizations** or **height estimates**, as a sort of means of **arithmetic analytic continuation**, and the classical complex theory surrounding the **Schwarz lemma** [cf. the discussion of [IUTchIV], Remark 2.3.5, (iii)].

§4.4. Atavistic resemblance in the development of mathematics:

(i) **Questioning strictly linear models of evolution:** Progress in mathematics is often portrayed as a **strictly linear** affair — a process in which old theories or ideas are rendered *essentially obsolete*, and hence *forgotten*, as soon as the essential content of those theories or ideas is *"suitably extracted/absorbed"* and formulated in a more *"modern form"*, which then becomes known as the *"state of the art"*. The historical development of mathematics is then envisioned as a sort of *towering edifice* that is subject to a perpetual appending of higher and higher floors, as *new "states of the art"* are discovered. On the other hand, it is often overlooked that there is in fact *no intrinsic justification* for this sort of *strictly linear model of evolution*. Put another way, there is

> **no rigorous justification** for excluding the possibility that a *particular approach to mathematical research* that happens to be embraced without doubt by a particular community of mathematicians as the **path forward** in this sort of **strictly linear evolutionary model** may in fact be nothing more than a **dramatic "wrong turn"**, i.e., a sort of *unproductive march into a meaningless cul de sac*.

Indeed, Grothendieck's original idea that **anabelian geometry** could shed light on **diophantine geometry** [cf. the discussion at the beginning of [IUTchI], §I5] suggests precisely this sort of skepticism concerning the *"linear evolutionary model"* that arose in the 1960's to the effect that progress in arithmetic geometry was best understood as a sort of

> **strictly linear march** toward the goal of realizing the theory of **motives**, i.e., a sort of idealized version of the notion of a **Weil cohomology**.

In more recent years, another major *"linear evolutionary model"* that has arisen, partly as a result of the influence of the work of Wiles [cf. [Wiles]] concerning **Galois representations**, asserts that progress in arithmetic geometry is best understood as a sort of

strictly linear march toward the goal of realizing the **representation-theoretic** approach to arithmetic geometry constituted by the **Langlands program**.

As discussed in §4.3, (ii); [IUTchI], §I5,

(a^{app}) the **"mono-anabelian"** approach to arithmetic geometry constituted by *inter-universal Teichmüller theory*

differs fundamentally not only from

(b^{app}) the *motive-/cohomology-theoretic* and *representation-theoretic* approaches to arithmetic geometry just discussed — both of which may be characterized as **"abelian"**! —

but also from

(c^{app}) the **"bi-anabelian"** approach involving the **section conjecture** that was apparently originally envisioned by Grothendieck.

Here, we recall, moreover, the point of view of the *dichotomy* discussed in §4.3, (ii), concerning **arithmetic holomorphicity** and **mono-analyticity/multiradiality**, i.e., to the effect that the *difference* between (a^{app}), on the one hand, and *both* (b^{app}) and (c^{app}), on the other, may be understood [if, for the sake of brevity, one applies the term "holomorphic" as an abbreviation of the term "arithmetically holomorphic"] as the difference between

non-holomorphic existence and **holomorphic nonexistence**.

Another way to understand, at a *very rough level*, the *difference* between (a^{app}) and (b^{app}) is as a reflection of the *deep structural differences* between

[discrete or profinite] **free groups** and **matrix groups** [with discrete or profinite coefficients]

— cf. the discussion of [IUTchI], §I5. Finally, at a *much more elementary level*, we note that the theory of **Galois groups** — which may be thought of as a mechanism that allows one to pass from *field theory* to *group theory* — plays a *fundamental role* in (a^{app}), (b^{app}), and (c^{app}). From this point of view, the difference between (a^{app}), on the one hand, and (b^{app}) [and, to a slightly lesser extent, (c^{app})], on the other, may be understood as the difference between the *"inequalities"*

$$\textbf{group theory} \ggg \textbf{field theory} \text{ and}$$
$$\textbf{field theory} \ggg \textbf{group theory}$$

— i.e., the issue of whether one regards [abstract] *group theory* as the *central object of interest*, while *field theory* [which we understand as including *vector spaces* over fields, hence also *representation theory*] is relegated to playing only a *subordinate role*, or *vice versa*.

(ii) **<u>Examples of atavistic development</u>:** An alternative point of view to the sort of *strictly linear evolutionary model* discussed in (i) is the point of view that progress in mathematics is best understood as a much more complicated **family tree**, i.e., not as a tree that consists solely of a *single trunk without branches* that continues to grow upward in a strictly linear manner, but rather as

> a *much more complicated* **organism**, whose growth is sustained by an *intricate mechanism of interaction* among a vast *multitude of branches*, some of which sprout *not from branches of relatively recent vintage*, but rather from **much older, more ancestral branches** of the organism that were entirely irrelevant to the recent growth of the organism.

In the context of the present paper, it is of interest to note that this point of view, i.e., of

> **substantially different multiple evolutionary branches** that sprout from a *single common ancestral branch*, is reminiscent of the notion of **"mutually alien copies"**, which forms a *central theme* of the present paper [cf. the discussion of §2.7, (i), (ii); §3.8].

Phenomena that support this point of view of an **"atavistic model of mathematical development"** may be seen in many of the examples discussed in §4.1, §4.2, and §4.3 such as the following:

(1^{atv}) The very elementary construction of **Belyi maps** in the early 1980's, or indeed **noncritical Belyi maps** in [NCBelyi], could easily have been discovered in the late nineteenth century [cf. §4.3, (ii), (5^{flt}); §4.3, (iii), (5^{PB})].

(2^{atv}) The application of Belyi maps to **Belyi cuspidalization** [cf. [AbsTopII], §3] could easily have been discovered in the mid-1990's [cf. also (1^{atv})].

(3^{atv}) The application of noncritical Belyi maps to **height estimates** in [GenEll], §2, could easily have been discovered in the mid-1980's [cf. also (1^{atv})].

(4^{atv}) The **Galois-theoretic** interpretation of the **Gaussian integral** or **Jacobi's identity** furnished by inter-universal Teichmüller theory [cf. the discussion of §3.8; the discussion at the end of §3.9, (iii); the discussion of the final portion of §4.1, (i)] could easily have been discovered much earlier than in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV].

(5^{atv}) The interpretation of **changes of universe** in the context of **non-ring-theoretic** *"arithmetic changes of coordinates"* as in the discussion of §2.10 is entirely elementary and could easily have been discovered in the 1960's [cf. §4.1, (ii), (1^{cls}), (2^{cls})].

(6^{atv}) The use of **Hodge-Tate representations** as in [$\mathbb{Q}_p$GC] or [AbsTopI], §3, could easily have been discovered in the 1960's [cf. §4.1, (ii), (3^{cls})].

(7^{atv}) The use of **Hodge-Tate decompositions** as in [pGC] could easily have been discovered in the 1980's [cf. §4.1, (ii), (4^{cls})].

(8^{atv}) The **anabelian** approach to **theta functions** on **Tate curves** taken in [EtTh] [cf. §3.4, (iii), (iv)] could easily have been discovered in the mid-1990's [cf. §4.1, (ii), (5^{cls})].

(9^{atv}) The **non-representation-theoretic** use of the structure of **theta groups** in the theory of [EtTh] [cf. the discussion at the end of §3.4, (iv)] could easily have been discovered in the 1980's [cf. §4.1, (ii), (5^{cls}), (6^{cls})].

(10^{atv}) **Scheme-theoretic Hodge-Arakelov theory**, which may be regarded as a *natural extension* of the [the portion concerning *elliptic curves* of] *Mumford's* theory of **algebraic theta functions**, could easily have been discovered in the late 1960's [cf. §4.1, (ii), (6^{cls})].

(11^{atv}) The technique of **mono-anabelian transport** in the context of *positive characteristic anabelian geometry*, i.e., in the style of Example 2.6.1, could easily have been discovered in the 1980's [cf. §4.1, (ii), (7^{cls})].

(12^{atv}) The use of **monoids** as in the theory of **Frobenioids** could easily have been discovered in the mid-1990's [cf. §4.1, (ii), (8^{cls})].

(13^{atv}) The notion of **multiradiality** is *entirely elementary* and could easily have been discovered in the late 1960's [cf. §4.1, (ii), (9^{cls})].

(14^{atv}) The point of view of taking a **Kummer-theoretic** approach to **Kronecker's Jugendtraum**, i.e., as discussed in §4.2, (ii), could easily have been discovered much earlier than in the series of papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV].

In this context, we note that the *atavistic model of mathematical development* just discussed also suggests the possibility that the theory of **Frobenioids** — which, as was discussed in §3.5, was originally developed for reasons that were [related to, but, strictly speaking] *independent* of inter-universal Teichmüller theory, and is, in fact, only used in inter-universal Teichmüller theory in a *relatively weak* sense — may give rise, at some *distant future date*, to further developments of interest that are not directly related to inter-universal Teichmüller theory.

(iii) <u>**Escaping from the cage of deterministic models of mathematical development:**</u> The adoption of **strictly linear evolutionary models** of progress in mathematics of the sort discussed in (i) tends to be *highly attractive* to many mathematicians in light of the **intoxicating simplicity** of such strictly linear evolutionary models, by comparison to the *more complicated point of view* discussed in (ii). This *intoxicating simplicity* also makes such strictly linear evolutionary models — together with *strictly linear numerical evaluation devices* such as the "number of papers published", the "number of citations of published papers", or other like-minded **narrowly defined data formats** that have been concocted for *measuring progress in mathematics* — *highly enticing* to administrators who are charged with the tasks of **evaluating**, **hiring**, or **promoting** mathematicians. Moreover, this state of affairs that regulates the collection of individuals who are granted the license and resources necessary to actively engage in mathematical research tends to have the effect, over the long term, of **stifling efforts** by young researchers to conduct **long-term mathematical research** in directions that **substantially diverge** from the **strictly linear evolutionary models** that have been adopted, thus making it exceedingly difficult for *new* **"unanticipated"** *evolutionary branches* in the development of mathematics to sprout. Put another way,

> *inappropriately narrowly defined* **strictly linear evolutionary models** of progress in mathematics exhibit a *strong* and *unfortunate* tendency in the profession of mathematics as it is currently practiced to become something of a **self-fulfilling prophecy** — a "prophecy" that is often zealously rationalized by dubious bouts of **circular reasoning**.

In particular, the issue of

> **escaping** from the **cage** of such *narrowly defined* **deterministic models** of mathematical development stands out as an issue of *crucial strategic importance* from the point of view of charting a **sound, sustainable course** in the future development of the field of mathematics, i.e., a course that *cherishes the priviledge to foster genuinely* **novel** *and* **unforeseen evolutionary branches** *in its development.*

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