Categories of Log Schemes with Archimedean Structures

By

Shinichi Mochizuki*

Contents

Abstract

In this paper, we generalize the main result of [Mzk2] (to the effect that very general noetherian log schemes may be reconstructed from naturally associated categories) to the case of log schemes locally of finite type over Zariski localizations of the ring of rational integers which are, moreover, equipped with certain "archimedean structures".

1. Introduction

As is discussed in the Introduction to [Mzk2], it is natural to ask to what extent various objects — such as *log schemes* — that occur in arithmetic geometry may be *represented by categories*, i.e., to what extent one may *reconstruct* the original object solely from the category-theoretic structure of a category naturally associated to the object. As is explained in *loc. cit.*, this point of view is partially motivated by the *anabelian philosophy* of Grothendieck.

In the present paper, we extend the theory of [Mzk2], which only concerns log schemes, to obtain a theory that proves a *similar categorical representability result* [cf. Theorem 5.1 below] for what we call *"arithmetic log schemes"* [cf. Definitions 4.1, 4.2 below], i.e., log schemes that are locally of finite type over a Zariski localization of the ring of rational integers and, moreover, are equipped with certain *"archimedean structures"* at archimedean primes.

In §3, we review the theory of [Mzk2], and revise the formulation of the main theorem of [Mzk2] slightly [cf. Theorem 3.1]. In §4, we define the notion of an *archimedean structure* on a fine, saturated log scheme which is of finite type over a Zariski localization of $\mathbb{Z}$. Finally, in §5, we generalize Theorem 3.1 [cf. Theorem 5.1] so as to take into account these archimedean structures.

Acknowledgements:

I would like to thank *Akio Tamagawa* and *Makoto Matsumoto* for many helpful comments concerning the material presented in this paper.

2. Notations and Conventions

Numbers:

We will denote by $\mathbb{N}$ the set (or, occasionally, the commutative monoid) of *natural numbers*, by which we take to consist set of the integers $n \geq 0$. A *number field* is defined to be a finite extension of the field of rational numbers $\mathbb{Q}$. The field of *real numbers* (respectively, *complex numbers*) will be denoted by $\mathbb{R}$ (respectively, $\mathbb{C}$). The topological group of *complex numbers of unit norm* will be denoted by $\mathbb{S}^1 \subseteq \mathbb{C}$.

We shall say that a scheme S is a *Zariski localization of* $\mathbb{Z}$ if $S = \mathrm{Spec}(R)$, where $R = M^{-1} \cdot \mathbb{Z}$, for some *multiplicative subset* $M \subseteq \mathbb{Z}$.

Topological Spaces:

In this paper, the term *"compact"* is to be understood to *include* the assumption that the topological space in question is *Hausdorff*. (The author wishes to thank *A. Tamagawa* for his comments concerning the importance of making this assumption *explicit*.)

Also, when a topological space H is equipped with an *involution* σ (typically an action of "complex conjugation"), we shall denote by

$$H^{\mathbb{R}}$$

(i.e., a superscript "$\mathbb{R}$") the *quotient topological space* of "σ-orbits".

Categories:

Let $\mathcal{C}$ be a *category*. We shall denote the collection of *objects* of $\mathcal{C}$ by:

$$\mathrm{Ob}(\mathcal{C})$$

If $A \in \mathrm{Ob}(\mathcal{C})$ is an *object* of $\mathcal{C}$, then we shall denote by

$$\mathcal{C}_A$$

the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose morphisms (from an object $B_1 \to A$ to an object $B_2 \to A$) are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$. Thus, we have a *natural functor*

$$(j_A)_! : \mathcal{C}_A \to \mathcal{C}$$

(given by forgetting the structure morphism to A). Similarly, if $f : A \to B$ is a *morphism* in $\mathcal{C}$, then f defines a *natural functor*

$$f_! : \mathcal{C}_A \to \mathcal{C}_B$$

by mapping an arrow (i.e., an object of $\mathcal{C}_A$) $C \to A$ to the object of $\mathcal{C}_B$ given by the composite $C \to A \to B$ with f.

If the category $\mathcal{C}$ *admits finite products*, then $(j_A)_!$ is *left adjoint* to the *natural functor*

$$j_A^* : \mathcal{C} \to \mathcal{C}_A$$

given by taking the *product with* A, and $f_!$ is *left adjoint* to the *natural functor*

$$f^* : \mathcal{C}_B \to \mathcal{C}_A$$

given by taking the *fibered product over* B *with* A.

We shall call an object $A \in \mathrm{Ob}(\mathcal{C})$ *terminal* if for every object $B \in \mathrm{Ob}(\mathcal{C})$, there exists a unique arrow $B \to A$ in $\mathcal{C}$. We shall call an object $A \in \mathrm{Ob}(\mathcal{C})$ *quasi-terminal* if for every object $B \in \mathrm{Ob}(\mathcal{C})$, there exists an arrow $\phi : B \to A$ in $\mathcal{C}$, and, moreover, for every other arrow $\psi : B \to A$, there exists an automorphism α of A such that $\psi = \alpha \circ \phi$.

We shall refer to a *natural transformation* between functors all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. A functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ between categories $\mathcal{C}_1$, $\mathcal{C}_2$ will be called *rigid* if ϕ has no nontrivial automorphisms. A category $\mathcal{C}$ will be called *slim* if the natural functor $\mathcal{C}_A \to \mathcal{C}$ is *rigid*, for every $A \in \mathrm{Ob}(\mathcal{C})$.

If $\mathcal{C}$ if a *category* and $\mathcal{S}$ is a *collection of arrows in* $\mathcal{C}$, then we shall say that an arrow $A \to B$ is *minimal-adjoint to* $\mathcal{S}$ if every factorization $A \to C \to B$ of this arrow $A \to B$ in $\mathcal{C}$ such that $A \to C$ lies in $\mathcal{S}$ satisfies the property that $A \to C$ is, in fact, an *isomorphism*. Often, the collection $\mathcal{S}$ will be taken to be the collection of arrows satisfying a *particular property* $\mathcal{P}$; in this case, we shall refer to the property of being "minimal-adjoint to $\mathcal{S}$" as the *minimal-adjoint notion to* $\mathcal{P}$.

3. Review of the Theory for Log Schemes

We begin our discussion by reviewing the *theory for log schemes* developed in [Mzk2]. Also, we give a slight extension of this theory (to the case of locally noetherian log schemes and morphisms which are locally of finite type). In the context of this extension, it is natural to modify the notation used in [Mzk2] slightly as follows:

Let us denote by

$$\mathrm{Sch}^{\log}$$

the category of all *locally noetherian fine saturated log schemes* and *locally finite type morphisms*, and by

$$\mathrm{NSch}^{\log}$$

the category of all *noetherian fine saturated log schemes* and *finite type morphisms.* Note that

$$\mathrm{NSch}^{\log} \subseteq \mathrm{Sch}^{\log}$$

may be characterized as the *full subcategory* consisting of the $X^{\log}$ for which X is *noetherian.*

If $X^{\log}$ is a *fine saturated log scheme* whose underlying scheme X is *locally noetherian*, then we shall write

$$\mathrm{Sch}^{\log}(X^{\log}) \stackrel{\mathrm{def}}{=} (\mathrm{Sch}^{\log})_{X^{\log}}$$

and

$$\mathrm{NSch}^{\log}(X^{\log}) \subseteq \mathrm{Sch}^{\log}(X^{\log})$$

for the *full subcategory* consisting of the $Y^{\log} \to X^{\log}$ for which Y is *noetherian.* Thus, when X is *noetherian*, we have $\mathrm{NSch}^{\log}(X^{\log}) = (\mathrm{NSch}^{\log})_{X^{\log}}$.

To simplify terminology, we shall often refer to the *domain* $Y^{\log}$ of an arrow $Y^{\log} \to X^{\log}$ which is an object of $\mathrm{Sch}^{\log}(X^{\log})$ or $\mathrm{NSch}^{\log}(X^{\log})$ as an "object of $\mathrm{Sch}^{\log}(X^{\log})$ or $\mathrm{NSch}^{\log}(X^{\log})$".

If $X^{\log}$, $Y^{\log}$ are *locally noetherian fine saturated log schemes*, then denote the set of isomorphisms of log schemes $X^{\log} \xrightarrow{\sim} Y^{\log}$ by:

$$\mathrm{Isom}(X^{\log}, Y^{\log})$$

Then the *main result* of [Mzk2] [cf. [Mzk2], Theorem 2.19] states that the natural map

$$\mathrm{Isom}(X^{\log}, Y^{\log}) \to \mathrm{Isom}(\mathrm{NSch}^{\log}(Y^{\log}), \mathrm{NSch}(X^{\log}))$$

given by $f^{\log} \mapsto \mathrm{NSch}^{\log}(f^{\log})$ [i.e., mapping an isomorphism to the induced equivalence between "$\mathrm{NSch}^{\log}(-)$'s"] is *bijective.* (Here, the "Isom" on the right is to be understood to denote *isomorphism classes of equivalences* between the two categories in parentheses.) This result generalizes immediately to the case of "$\mathrm{Sch}^{\log}(-)$":

Theorem 3.1. (Categorical Reconstruction of Locally Noetherian Fine Saturated Log Schemes) *Let* $X^{\log}$*,* $Y^{\log}$ *be* **locally noetherian fine saturated log schemes***. Then the natural map*

$$\mathrm{Isom}(X^{\log}, Y^{\log}) \to \mathrm{Isom}(\mathrm{Sch}^{\log}(Y^{\log}), \mathrm{Sch}^{\log}(X^{\log}))$$

is **bijective***.*

Proof. Indeed, by *functoriality* and [Mzk2], Theorem 2.19, it suffices to show that the subcategory

$$\mathrm{NSch}^{\log}(X^{\log}) \subseteq \mathrm{Sch}^{\log}(X^{\log})$$

may be recovered *"category-theoretically"*.

To see this, let us first observe that the proof given in [Mzk2] [cf. [Mzk2], Corollary 2.14] of the category-theoreticity of the property that a morphism in $\mathrm{NSch}^{\log}(X^{\log})$ be *"scheme-like"* (i.e., that the log structure on the domain is the pull-back of the log structure on the codomain) is entirely valid in $\mathrm{Sch}^{\log}(X^{\log})$. (Indeed, the proof essentially only involves morphisms among "one-pointed objects", which are the same in $\mathrm{NSch}^{\log}(X^{\log})$, $\mathrm{Sch}^{\log}(X^{\log})$.) Moreover, once one knows which morphisms are scheme-like, the *open immersions* may be characterized category-theoretically as in [Mzk2], Corollary 1.3.

Next, let us first observe that the property that a collection of open immersions

$$Y^{\log}_{\alpha} \to Y^{\log}$$

(where α ranges over the elements of some index set A) in $\mathrm{Sch}^{\log}(X^{\log})$ be *surjective* is *category-theoretic*. Indeed, this follows from the fact that this collection is surjective if and only if, for any morphism $Z^{\log} \to Y^{\log}$, where $Z^{\log}$ is *nonempty*, the *fiber product* $Y^{\log}_{\alpha} \times_{Y^{\log}} Z^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ [cf. [Mzk2], Lemma 2.6] is *nonempty* for some α [cf. also [Mzk2], Proposition 1.1, (i), applied to the complement of the union of the images of the $Y^{\log}_{\alpha}$].

Thus, it suffices to observe that an object $Y^{\log}$ is *noetherian* if and only if, for any *surjective collection* of open immersions (in $\mathrm{Sch}^{\log}(X^{\log})$) $Y^{\log}_{\alpha} \to Y^{\log}$ (where α ranges over the elements of some index set A), there exists a *finite subset* $B \subseteq A$ such that the collection $\{Y^{\log}_{\beta} \to Y^{\log}\}_{\beta \in B}$ is *surjective.* ☐

Remark 1. Similar [but easier] results hold for

Sch (respectively, NSch)

— i.e., the category of all *locally noetherian schemes* and *locally finite type morphisms* (respectively, all *noetherian log schemes* and *finite type morphisms*).

4. Archimedean Structures

In this §, we generalize the categories defined in [Mzk2] so as to include *archimedean primes*. In particular, we prepare for the proof in §5 below of a *global arithmetic analogue* [cf. Theorem 5.1] of Theorem 3.1.

Let $X^{\log}$ be a *fine, saturated locally noetherian log scheme* (with underlying scheme X).

Definition 4.1. We shall say that X is *arithmetically (locally) of finite type* if X is (locally) of finite type over a Zariski localization of $\mathbb{Z}$. Similarly, we shall say that $X^{\log}$ is *arithmetically (locally) of finite type* if X is.

Suppose that $X^{\log}$ is *arithmetically locally of finite type*. Then $X^{\log}_{\mathbb{Q}} \stackrel{\text{def}}{=} X^{\log} \otimes_{\mathbb{Z}} \mathbb{Q}$ is locally of finite type over $\mathbb{Q}$. In particular, the set of $\mathbb{C}$-valued points

$$X(\mathbb{C})$$

is equipped with a natural *topology* (induced by the topology of $\mathbb{C}$), together with an *involution* $\sigma_X : X(\mathbb{C}) \to X(\mathbb{C})$ induced by the complex conjugation automorphism on $\mathbb{C}$. Similarly, in the *logarithmic context*, it is natural to consider the topological space

$$X^{\log}(\mathbb{C}) \stackrel{\text{def}}{=} \{(x,\theta) \mid x \in X(\mathbb{C}),\ \theta \in \mathrm{Hom}(M^{\mathrm{gp}}_{X,x}, \mathbb{S}^1) \tag{4.1}$$

$$\text{s.t. } \theta(f) = f(x)/|f(x)|,\ \forall f \in \mathcal{O}^{\times}_{X,x}\} \tag{4.2}$$

[cf. [KN], §1.2]. Here, we use the notation M_X to denote the *monoid that defines the log structure* of $X^{\log}$ [cf. [Mzk2], §2]. Thus, we have a natural *surjection*

$$X^{\log}(\mathbb{C}) \to X(\mathbb{C})$$

whose *fibers* are (noncanonically) isomorphic to products of finitely many copies of $\mathbb{S}^1$. Also, we observe that it follows immediately from the definition that σ_X extends to an involution $\sigma_{X^{\log}}$ on $X^{\log}(\mathbb{C})$.

Definition 4.2.

(i) Let $H \subseteq X(\mathbb{C})$ be a compact subset stabilized by σ_X. Then we shall refer to a pair $\overline{X} = (X, H)$ as an *arithmetic scheme*, and H as the *archimedean structure* on $\overline{X}$. We shall say that an archimedean structure $H \subseteq X(\mathbb{C})$ is *trivial* (respectively, *total*) if $H = \emptyset$ (respectively, $H = X(\mathbb{C})$).

(ii) Let $H \subseteq X^{\log}(\mathbb{C})$ be a compact subset stabilized by $\sigma_{X^{\log}}$. Then we shall refer to a pair $\overline{X}^{\log} = (X^{\log}, H)$ as an *arithmetic log scheme*, and H as the *archimedean structure* on $\overline{X}^{\log}$. We shall say that an archimedean structure $H \subseteq X^{\log}(\mathbb{C})$ is *trivial* (respectively, *total*) if $H = \emptyset$ (respectively, $H = X^{\log}(\mathbb{C})$).

Remark 2. The idea that *"integral structures at archimedean primes"* should be given by *compact/bounded subsets* of the set of complex valued points may be seen in the discussion of [Mzk1], p. 9; cf. also Remark 8 below.

Remark 3. Relative to Definition 4.2, one may think of the case where "H" is *open* as the case of an *ind-arithmetic (log) scheme* [or, alternatively, an "ind-archimedean structure"], i.e., the inductive system of arithmetic (log) schemes [or, alternatively, archimedean structures] determined by considering *all compact subsets* that lie inside the given open.

Let us denote the *category of all arithmetic log schemes* by:

$$\overline{\mathrm{Sch}}^{\log}$$

Thus, a morphism $\overline{X}_1^{\log} = (X_1^{\log}, H_1) \to \overline{X}_2^{\log} = (X_2^{\log}, H_2)$ in this category is a locally finite type morphism $X_1^{\log} \to X_2^{\log}$ such that the induced map $X_1^{\log}(\mathbb{C}) \to X_2^{\log}(\mathbb{C})$ maps H_1 into H_2. The full subcategory of *noetherian objects* of $\overline{\mathrm{Sch}}^{\log}$ [i.e., objects whose underlying scheme is noetherian] will be denoted by:

$$\overline{\mathrm{NSch}}^{\log} \subseteq \overline{\mathrm{Sch}}^{\log}$$

Similarly, if we forget about log structures, we obtain categories $\overline{\mathrm{NSch}}$, $\overline{\mathrm{Sch}}$.

Definition 4.3.

(i) An arithmetic (log) scheme will be called *purely nonarchimedean* if its archimedean structure is trivial.

(ii) A morphism between arithmetic (log) schemes will be called *purely archimedean* if the underlying morphism between (log) schemes is an isomorphism.

Denote by

$$\underline{\mathrm{Sch}}^{\log} \subseteq \mathrm{Sch}^{\log}$$

the *full subcategory* determined by those objects which are *arithmetically locally of finite type*. Then note that by considering *purely nonarchimedean* objects, we obtain a *natural embedding*

$$\underline{\mathrm{Sch}}^{\log} \hookrightarrow \overline{\mathrm{Sch}}^{\log}$$

of $\underline{\mathrm{Sch}}^{\log}$ as a *full subcategory* of $\overline{\mathrm{Sch}}^{\log}$.

If $\overline{X}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$, then we shall write

$$\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}) \stackrel{\mathrm{def}}{=} (\overline{\mathrm{Sch}}^{\log})_{\overline{X}^{\log}}$$

[cf. §3] and

$$\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})^{\mathrm{arch}} \subseteq \overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$$

for the *subcategory* whose objects $\overline{Y}^{\log} \to \overline{X}^{\log}$ are *purely archimedean* arrows of $\overline{\mathrm{Sch}}^{\log}$. (Thus, the morphisms $\overline{Y}_1^{\log} \to \overline{Y}_2^{\log}$ of this subcategory are also necessarily purely archimedean.)

On the other hand, if T is a *topological space*, then let us write

$$\mathrm{Open}(T) \text{ (respectively, } \mathrm{Closed}(T))$$

for the category whose objects are *open subsets* $U \subseteq T$ (respectively, *closed subsets* $F \subseteq T$) and whose morphisms are inclusions of subsets of T. Thus, one verifies easily (by taking *complements*!) that $\mathrm{Closed}(T)$ is the *opposite category* $\mathrm{Open}(T)^{\mathrm{opp}}$ associated to $\mathrm{Open}(T)$. Also, let us write

$$\mathrm{Shv}(T)$$

for the category of *sheaves on* T (valued in sets).

Now we have the following:

Proposition 4.1. (Conditional Reconstruction of the Archimedean Topological Space)

(i) If H is the **archimedean structure** *on $\overline{X}^{\log}$, then the functor*

$$\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})^{\mathrm{arch}} \to \mathrm{Closed}(H^{\mathbb{R}}) \; (\stackrel{\sim}{\to} \mathrm{Open}(H^{\mathbb{R}})^{\mathrm{opp}})$$

[cf. §2 for more on the superscript "$\mathbb{R}$"] given by assigning to an arrow $\overline{Y}^{\log} \to \overline{X}^{\log}$ the image of the archimedean structure of $\overline{Y}^{\log}$ in $H^{\mathbb{R}} \subseteq X^{\log}(\mathbb{C})^{\mathbb{R}}$ is an equivalence.

(ii) Let $\overline{X}_1^{\log}, \overline{X}_2^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$. Suppose that

$$\Phi : \overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}) \stackrel{\sim}{\to} \overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log})$$

is an **equivalence of categories** *that preserves purely archimedean arrows (i.e., an arrow f in $\overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log})$ is purely archimedean if and only if $\Phi(f)$ is purely archimedean). Then one can construct, for every object $\overline{Y}_1^{\log} = (Y_1^{\log}, K_1) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}))$ that maps via Φ to an object $\overline{Y}_2^{\log} = (Y_2^{\log}, K_2) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log}))$, a* **homeomorphism**

$$K_1^{\mathbb{R}} \xrightarrow{\sim} K_2^{\mathbb{R}}$$

which is **functorial** *in $Y_1^{\log}$.*

Proof. Assertion (i) is a formal consequence of the definitions. To prove assertion (ii), let us first observe that (for an arbitrary topological space T) $\mathrm{Shv}(T)$ may be reconstructed functorially from $\mathrm{Open}(T)$, since *coverings* of objects of $\mathrm{Open}(T)$ may be characterized as collections of objects whose *inductive limit* (a purely categorical notion!) is isomorphic to the object to be covered. Thus, our assumption on Φ, together with assertion (i), implies that (for $i = 1, 2$) $\mathrm{Shv}(K_i^{\mathbb{R}})$ may be *reconstructed category-theoretically* from $Y_i^{\log}$ in a fashion which is *functorial* in $Y_i^{\log}$. Moreover, since $K_i^{\mathbb{R}}$ is clearly a *sober* topological space, we thus conclude [by a well-known result from "topos theory" — cf., e.g., [Mzk2], Theorem 1.4] that the *topological space* $K_i^{\mathbb{R}}$ itself may be *reconstructed category-theoretically* from $Y_i^{\log}$ in a fashion which is *functorial* in $Y_i^{\log}$, as desired. $\square$

Before proceeding, we observe the following:

Lemma 4.1. (Finite Products of Arithmetic Log Schemes) *The category $\overline{\mathrm{Sch}}^{\log}$ admits finite products.*

Proof. Indeed, if, for $i = 1, 2, 3$, we are given objects $\overline{X}_i^{\log} = (X_i^{\log}, H_i) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$ and morphisms $X_1^{\log} \to X_2^{\log}$, $X_3^{\log} \to X_2^{\log}$ in $\overline{\mathrm{Sch}}^{\log}$, then we may form the product of $X_1^{\log}$, $X_3^{\log}$ over $X_2^{\log}$ by equipping the log scheme

$$X_1^{\log} \times_{X_2^{\log}} X_3^{\log}$$

(which is easily seen to be arithmetically locally of finite type) with the *archimedean structure* given by the *inverse image* of

$$H_1 \times_{H_2} H_3 \subseteq X_1^{\log}(\mathbb{C}) \times_{X_2^{\log}(\mathbb{C})} X_3^{\log}(\mathbb{C})$$

(where we note that $H_1 \times_{H_2} H_3$ is *compact*, since H_2 is *Hausdorff*) via the natural map:

$$(X_1^{\log} \times_{X_2^{\log}} X_3^{\log})(\mathbb{C}) \to X_1^{\log}(\mathbb{C}) \times_{X_2^{\log}(\mathbb{C})} X_3^{\log}(\mathbb{C})$$

Note that this last map is *proper* [i.e., inverse images of compact sets are compact], since, for *any* $Y^{\log}$ which is arithmetically locally of finite type, the map $Y^{\log}(\mathbb{C}) \to Y(\mathbb{C})$ is *proper*, and, moreover, the map induced on $\mathbb{C}$-valued points of underlying schemes by

$$X_1^{\log} \times_{X_2^{\log}} X_3^{\log} \to X_1 \times_{X_2} X_3$$

[i.e., where the domain is equipped with the trivial log structure] is *finite* [cf. [Mzk2], Lemma 2.6], hence *proper*. $\square$

Thus, if $\overline{X}^{\log}, \overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$, then any morphism $\overline{X}^{\log} \to \overline{Y}^{\log}$ in $\overline{\mathrm{Sch}}^{\log}$ induces a *natural functor*

$$\overline{\mathrm{Sch}}^{\log}(\overline{Y}^{\log}) \to \overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$$

(by sending an object $\overline{Z}^{\log} \to \overline{Y}^{\log}$ to the fibered product $\overline{Z}^{\log} \times_{\overline{Y}^{\log}} \overline{X}^{\log} \to \overline{X}^{\log}$ — cf. the discussion of §2).

Next, we would like to show, in the following discussion [cf. Corollary 4.1, (ii) below], that the *hypothesis* of Proposition 4.1, (ii), is *automatically satisfied*.

Let $\overline{X}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$.

Proposition 4.2. (Minimal Objects) *An object* $\overline{Y}^{\log}$ *of* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *will be called* **minimal** *if it is nonempty and satisfies the property that any monomorphism* $\overline{Z}^{\log} \rightarrowtail \overline{Y}^{\log}$ *(where* $\overline{Z}^{\log}$ *is nonempty) in* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *is necessarily an isomorphism. An object* $\overline{Y}^{\log}$ *of* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *is minimal if and only if it is* **purely nonarchimedean** *and* **log scheme-theoretically minimal** *[i.e., the underlying object* $Y^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *is minimal as an object of* $\mathrm{Sch}(X^{\log})$ *— cf. [Mzk2], Proposition 2.4].*

Proof. The *sufficiency* of this condition is clear, since the domain of any morphism in $\overline{\mathrm{Sch}}^{\log}$ to a purely nonarchimedean object is necessarily itself purely nonarchimedean [i.e., no nonempty set maps to an empty set]. That this condition is *necessary* is evident from the definitions (e.g., if a nonempty object fails to be purely nonarchimedean, then it can always be "made smaller" [but still nonempty!] by setting the archimedean structure equal to the empty set, thus precluding "minimality"). $\square$

Proposition 4.3. (Characterization of One-Pointed Objects) *We shall call an object of* $\overline{\mathrm{Sch}}^{\log}$ **one-pointed** *if the underlying topological space of its underlying scheme consists of precisely one point. The one-pointed objects* $\overline{Y}^{\log}$ *of* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *may be characterized category-theoretically as the nonempty objects which satisfy the following property: For any two morphisms* $\overline{S}_i^{\log} \to \overline{Y}^{\log}$ *(for* $i = 1, 2$*), where* $\overline{S}_i^{\log}$ *is a* **minimal** *object, the product* $\overline{S}_1^{\log} \times_{\overline{Y}^{\log}} \overline{S}_2^{\log}$ *(in* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$*) is* **nonempty**.

Proof. This is a formal consequence of the definitions; Proposition 4.2; and [Mzk2], Corollary 2.9. □

Proposition 4.4. (Minimal Hulls) *Let* $\overline{Y}^{\log}$ *be a* **one-pointed object** *of the category* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$*. Then a monomorphism* $\overline{Z}^{\log} \rightarrowtail \overline{Y}^{\log}$ *will be called a* **hull** *for* $\overline{Y}^{\log}$ *if every morphism* $\overline{S}^{\log} \to \overline{Y}^{\log}$ *from a* **minimal** *object* $\overline{S}^{\log}$ *to* $\overline{Y}^{\log}$ *factors (necessarily uniquely!) though* $\overline{Z}^{\log}$*. A hull* $\overline{Z}^{\log} \rightarrowtail \overline{Y}^{\log}$ *will be called a* **minimal hull** *if every monomorphism* $\overline{Z}_1^{\log} \rightarrowtail \overline{Z}^{\log}$ *for which the composite* $\overline{Z}_1^{\log} \rightarrowtail \overline{Y}^{\log}$ *is a hull is necessarily an isomorphism. A one-pointed object* $\overline{Z}^{\log}$ *will be called a* **minimal hull** *if the identity morphism* $\overline{Z}^{\log} \to \overline{Z}^{\log}$ *is a minimal hull for* $\overline{Z}^{\log}$*.*

(i) An object $\overline{Y}^{\log}$ *of* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *is a minimal hull if and only if it is* **purely nonarchimedean** *and* **log scheme-theoretically a minimal hull** *[i.e., the underlying object* $Y^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *is a minimal hull in the sense of [Mzk2], Proposition 2.7; cf. also [Mzk2], Corollary 2.10].*

(ii) Any two minimal hulls of an object $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *are* **isomorphic** *(via a unique isomorphism over* $\overline{Y}^{\log}$*).*

(iii) If $\overline{Y}_1^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}))$*,* $\overline{Y}_2^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log}))$*, and*

$$\Phi : \overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}) \xrightarrow{\sim} \overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log})$$

is an **equivalence of categories** *such that* $\Phi(Y_1^{\log}) = \overline{Y}_2^{\log}$*, then* $\overline{Y}_1^{\log}$ *is a minimal hull if and only if* $Y_2^{\log}$ *is. That is to say, the condition that an object* $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *be a minimal hull is* **"category-theoretic"**.

Proof. Assertion (i) (respectively, (ii); (iii)) is a formal consequence of Proposition 4.2 (respectively, assertion (i); Proposition 4.3) [and the definitions of the terms involved]. □

Proposition 4.5. **(Purely Archimedean Morphisms of Reduced One-Pointed Objects)** *Let* $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *be one-pointed; let* $\overline{Z}^{\log} \rightarrowtail \overline{Y}^{\log}$ *be a* **minimal hull** *which factors as a composite of monomorphisms* $\overline{Z}^{\log} \rightarrowtail \overline{Z}_1^{\log} \rightarrowtail \overline{Y}^{\log}$. *Then the following are equivalent:*

(i) $\overline{Z}_1^{\log}$ *is* **reduced**.

(ii) $\overline{Z}^{\log} \to \overline{Z}_1^{\log}$ *is* **purely archimedean**.

(iii) $\overline{Z}^{\log} \to \overline{Z}_1^{\log}$ *is an* **epimorphism** *in* $\overline{\mathrm{Sch}}^{\log}(\overline{Z}_1^{\log})$ *[i.e., two sections* $\overline{Z}_1^{\log} \to \overline{S}^{\log}$ *of a morphism* $\overline{S}^{\log} \to \overline{Z}_1^{\log}$ *coincide if and only if they coincide after restriction to* $\overline{Z}^{\log}$*].*

Proof. The equivalence of (i), (ii) is a formal consequence of [Mzk2], Proposition 2.3; [Mzk2], Proposition 2.7, (ii), (iii); [Mzk2], Corollary 2.10. That (ii) implies (iii) is a formal consequence of the definitions. Finally, that (iii) implies (i) follows, for instance, by taking $\overline{S}^{\log} \to \overline{Z}_1^{\log}$ to be the *projective line* over $\overline{Z}_1^{\log}$ (so sections that lies in the open sub-log scheme of $S^{\log}$ determined by the affine line correspond to elements of $\Gamma(Z_1, \mathcal{O}_{Z_1})$). (Here, we equip the projective line with the archimedean structure which is the inverse image of the archimedean structure of $\overline{Z}_1^{\log}$.) ☐

Note that condition (iii) of Proposition 4.5 is *"category-theoretic"*. This implies the following:

Corollary 4.1. **(Characterization of Purely Nonarchimedean One-Pointed Objects and Purely Archimedean Morphisms)**

(i) A one-pointed object $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *is* **purely nonarchimedean** *if and only if it satisfies the following "category-theoretic" condition: Every minimal hull* $\overline{Z}^{\log} \rightarrowtail \overline{Y}^{\log}$ *is* **minimal-adjoint** *[cf. §2] to the collection of arrows* $\overline{Z}^{\log} \to \overline{Z}_1^{\log}$ *which satisfy the equivalent conditions of Proposition* 4.5.

(ii) A morphism $\zeta : \overline{Y}^{\log} \to \overline{Z}^{\log}$ *in* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *is* **purely archimedean** *if and only if it satisfies the following "category-theoretic" condition: The morphism* ζ *is a* **monomorphism** *in* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$*, and, moreover, for every morphism* $\phi : \overline{S}^{\log} \to \overline{Z}^{\log}$ *in* $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$*, where* $\overline{S}^{\log}$ *is* **one-pointed** *and* **purely nonarchimedean***, there exists a* **unique** *morphism* $\psi : \overline{S}^{\log} \to \overline{Y}^{\log}$ *such that* $\phi = \zeta \circ \psi$.

Proof. Assertion (i) is a formal consequence of Proposition 4.5 [and the definitions of the terms involved]. As for assertion (ii), the *necessity* of the condition is a formal consequence of the definitions of the terms involved. To prove *sufficiency*, let us first observe that by [Mzk2], Lemma 2.2; [Mzk2], Proposition 2.3, it follows from this condition that the underlying morphism of log schemes $Y^{\log} \to Z^{\log}$ is *scheme-like* [i.e., the log structure on $Y^{\log}$ is the pull-back of the log structure on $Z^{\log}$]. Thus, this condition implies that the underlying morphism of schemes $Y \to Z$ is *smooth* [cf. [Mzk2], Corollary 1.2] and *surjective*. But this implies [cf. [Mzk2], Corollary 1.3] that $Y \to Z$ is a *surjective open immersion*, hence that it is an *isomorphism of schemes*. Since $Y^{\log} \to Z^{\log}$ is *scheme-like*, we thus conclude that $Y^{\log} \to Z^{\log}$ is an *isomorphism of log schemes*, as desired. $\square$

Thus, Corollary 4.1, (ii), implies that the hypothesis of Proposition 4.1 is *automatically satisfied*. This allows us to conclude the following:

Corollary 4.2. (Unconditional Reconstruction of the Archimedean Topological Space) *The* $\mathbb{R}$*-superscripted topological space determined by the ar-chimedean structure on an object* $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *may be reconstructed* **category-theoretically** *in a fashion which is* **functorial** *in* $\overline{Y}^{\log}$ *[cf. Proposition* 4.1*, (ii)]. In particular, the condition that* $\overline{Y}^{\log}$ *be* **purely nonarchimedean** *is category-theoretic in nature.*

Corollary 4.3. (Reconstruction of the Underlying Log Scheme) *The full subcategory*

$$\mathrm{Sch}^{\log}(Y^{\log}) \subseteq \overline{\mathrm{Sch}}^{\log}(\overline{Y}^{\log}) = \overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})_{\overline{Y}^{\log}}$$

[i.e., consisting of arrows $\overline{Z}^{\log} \to \overline{Y}^{\log}$ *for which* $\overline{Z}^{\log}$ *is purely nonarchimedean] associated to an object* $\overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *is a* **category-theoretic** *invariant of the data* $(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}), \overline{Y}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})))$*. In particular, [cf. Theorem* 3.1*] the underlying log scheme* $Y^{\log}$ *associated to* $\overline{Y}^{\log}$ *may be reconstructed* **category-theoretically** *from this data in a fashion which is* **functorial** *in* $\overline{Y}^{\log}$*.*

Remark 4. Thus, by Corollary 4.3, one may functorially reconstruct the *underlying log scheme* $Y^{\log}$ of an object $\overline{Y}^{\log} = (Y^{\log}, K) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$, hence the *topological space* $Y^{\log}(\mathbb{C})$ from category-theoretic data. On the other hand, by Corollary 4.2, one may also reconstruct the topological space $K^{\mathbb{R}}$ ($\subseteq Y^{\log}(\mathbb{C})^{\mathbb{R}}$). Thus, the question arises:

> Is the reconstruction of $K^{\mathbb{R}}$ via Corollary 4.2 *compatible* with the reconstruction of $Y^{\log}(\mathbb{C})^{\mathbb{R}}$ via Corollary 4.3?

More precisely, given objects $\overline{X}_1^{\log}, \overline{X}_2^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log})$; objects

$\overline{Y}_1^{\log} = (Y_1^{\log}, K_1) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}))$; $\quad \overline{Y}_2^{\log} = (Y_2^{\log}, K_2) \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log}))$

and an *equivalence of categories*

$$\Phi : \overline{\mathrm{Sch}}^{\log}(\overline{X}_1^{\log}) \xrightarrow{\sim} \overline{\mathrm{Sch}}^{\log}(\overline{X}_2^{\log})$$

such that $\Phi(\overline{Y}_1^{\log}) = \overline{Y}_2^{\log}$, we wish to know whether or not the *diagram*

$$\begin{array}{ccc} K_1^{\mathbb{R}} & \xrightarrow{\sim} & K_2^{\mathbb{R}} \\ \downarrow & & \downarrow \\ Y_1^{\log}(\mathbb{C})^{\mathbb{R}} & \xrightarrow{\sim} & Y_2^{\log}(\mathbb{C})^{\mathbb{R}} \end{array}$$

— where the *vertical* morphisms are the *natural inclusions*; the *upper horizontal* morphism is the homeomorphism arising from Corollary 4.2; and the *lower horizontal* morphism is the homeomorphism arising by taking "$\mathbb{C}$-valued points" of the isomorphism of log schemes obtained in Corollary 4.3 — *commutes*. This question will be answered in the affirmative in Lemmas 5.1, 5.2 below.

Definition 4.4. In the notation of Remark 4, let us suppose that $\overline{X}_1^{\log}$, $\overline{Y}_1^{\log}$ are *fixed*. Then:

(i) If the diagram of Remark 4 commutes for all $\overline{X}_2^{\log}$, $\overline{Y}_2^{\log}$, Φ as in Remark 4, then we shall say that $\overline{Y}_1^{\log}$ is *(logarithmically) globally compatible*.

(ii) If the composite of the diagram of Remark 4 with the commutative diagram

$$\begin{array}{ccc} Y_1^{\log}(\mathbb{C})^{\mathbb{R}} & \xrightarrow{\sim} & Y_2^{\log}(\mathbb{C})^{\mathbb{R}} \\ \downarrow & & \downarrow \\ Y_1(\mathbb{C})^{\mathbb{R}} & \xrightarrow{\sim} & Y_2(\mathbb{C})^{\mathbb{R}} \end{array}$$

commutes for all $\overline{X}_2^{\log}$, $\overline{Y}_2^{\log}$, Φ as in Remark 4, then we shall say that $\overline{Y}_1^{\log}$ is *nonlogarithmically globally compatible*.

5. The Main Theorem

In the following discussion, we complete the proof of the *main theorem* of the present paper by showing that the archimedean and scheme-theoretic data reconstructed in Corollaries 4.2, 4.3 are *compatible* with one another.

Definition 5.1. We shall say that an object $\overline{S}^{\log}$ of $\overline{\mathrm{Sch}}^{\log}$ is a *test object* if its underlying scheme is *affine*, *connected*, and *normal*, and, moreover, the $\mathbb{R}$-superscripted topological space determined by its archimedean structure consists of *precisely one point*.

Note that by Corollaries 4.2, 4.3, the notion of a "test object" is *"category-theoretic"*.

Lemma 5.1. **(Nonlogarithmic Global Compatibility)** *Let $\overline{X}^{\log}$ be an object in $\overline{\mathrm{Sch}}^{\log}$. Then every object $\overline{S}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ is* **nonlogarithmically globally compatible**.

Proof. By the *functoriality* of the diagram discussed in Remark 4, it follows immediately that it suffices to prove the nonlogarithmic global compatibility of *test objects* $\overline{S}^{\log} = (S^{\log}, H_S)$. Since S is assumed to be *affine*, write $S = \mathrm{Spec}(R)$. Then we may think of the single point of $H_S^{\mathbb{R}}$ as *defining an "archimedean valuation"* v_R on the ring R.

Write

$$\overline{Y}^{\log} = (Y^{\log}, H_Y) \to \overline{S}^{\log} = (S^{\log}, H_S)$$

for the *projective line* over $\overline{S}^{\log}$, equipped with the *log structure* obtained by pulling back the log structure of $S^{\log}$ and the *archimedean structure* which is the inverse image of the archimedean structure of $\overline{S}^{\log}$. Note that this archimedean structure may be characterized *"category-theoretically"* [cf. Corollaries 4.2, 4.3] as the archimedean structure which yields a *quasi-terminal object* [cf. §2] in the subcategory of $\overline{\mathrm{Sch}}^{\log}(\overline{S}^{\log})$ consisting of purely archimedean morphisms among objects with underlying log scheme isomorphic (over $S^{\log}$) to $Y^{\log}$.

Next, let us observe that to reconstruct the log scheme $S^{\log}$ via Corollary 4.3 amounts, in effect, to *applying the theory of [Mzk2]*. Moreover, in the theory of [Mzk2], the set underlying the ring $R = \Gamma(S, \mathcal{O}_S)$ is *reconstructed as the set of sections* $\overline{S}^{\log} \to \overline{Y}^{\log}$ that *avoid the ∞-section* (of the projective line Y). Moreover, the *topology* determined on R by the "archimedean valuation" v_R is precisely the topology on this set of sections determined by considering the *induced sections* $H_S^{\mathbb{R}} \to H_Y^{\mathbb{R}}$ [i.e., two sections $\overline{S}^{\log} \to \overline{Y}^{\log}$ are "close" if and only if their induced sections $H_S^{\mathbb{R}} \to H_Y^{\mathbb{R}}$ are "close"]. Thus, we conclude (via Corollary 4.2) that this *topology on R* is a *"category-theoretic invariant"*.

On the other hand, it is immediate that the point $R \to \mathbb{C}$ (considered up to complex conjugation) determined by $H_S^{\mathbb{R}}$ may be recovered from this *topology* — i.e., by *"completing"* with respect to this topology. This completes the proof of the asserted nonlogarithmic global compatibility. $\square$

Lemma 5.2. **(Logarithmic Global Compatibility)** *Let* $\overline{X}^{\log}$ *be an object in* $\overline{\mathrm{Sch}}^{\log}$*. Then every object* $\overline{S}^{\log} \in \mathrm{Ob}(\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$ *is* **globally compatible**.

Proof. The proof is entirely similar to the proof of Lemma 5.1. In particular, we reduce immediately to the case where $\overline{S}^{\log}$ is a *test object*. Since, by Corollary 4.3, the structure of the underlying log scheme $S^{\log}$ is already known to be category-theoretic, we may even assume, without loss of generality, that the monoid M_S is *generated by its global sections*. This time, instead of considering $\overline{Y}^{\log}$, we consider the object

$$\overline{Z}^{\log} = (Z^{\log}, H_Z) \to \overline{S}^{\log} = (S^{\log}, H_S)$$

obtained by *"appending"* to the log structure of $Y^{\log}$ the log structure determined by the divisor given by the *zero section* (of the projective line Y). As in the case of $\overline{Y}^{\log}$, we take the *archimedean structure* on $\overline{Z}^{\log}$ to be the inverse image of the archimedean structure of $\overline{S}^{\log}$. Also, just as in the case of $\overline{Y}^{\log}$, this archimedean structure may be characterized category-theoretically.

Now if we think of the unique point in $H_S^{\mathbb{R}}$ as a *pair* (up to complex conjugation) (s, θ) [cf. the discussion preceding Definition 4.2], then it remains to show that θ may be *"recovered category-theoretically"*. To this end, let us first recall that $s \in S(\mathbb{C})$ determines a morphism $\mathrm{Spec}(\mathbb{C}) \to S$ with respect to which one may pull-back the log structure on S to obtain a log structure on $\mathrm{Spec}(\mathbb{C})$. By Lemma 5.1, we may also assume, without loss of generality, that S is *"sufficiently [Zariski] local with respect to s"* in the sense that the image of $\Gamma(S, \mathcal{O}_S^\times)$ in $\mathbb{C}$ is *dense*. Moreover, this log structure on $\mathrm{Spec}(\mathbb{C})$ amounts to the datum of a *monoid*

$$M_{S,s}$$

containing the *unit circle* $\mathbb{S}^1 \subseteq \mathbb{C}$. Thus, relative to this notation, θ [cf. the discussion preceding Definition 4.2] may be thought of as the datum of a *surjective homomorphism*

$$\theta : M_{S,s}^{\mathrm{gp}} \twoheadrightarrow \mathbb{S}^1$$

[where surjectivity follows from the fact that this homomorphism restricts to the identity on $\mathbb{S}^1 \subseteq M_{S,s}^{\mathrm{gp}}$]. In fact, since θ is required to restrict to the *identity* on $\mathbb{S}^1 \subseteq M_{S,s} \subseteq M_{S,s}^{\mathrm{gp}}$, it follows that the surjection θ is *completely determined* by its *kernel*. Thus, in summary, θ may be thought of as being the datum of a certain *quotient* of the group $M_{S,s}^{\mathrm{gp}}$, or, indeed, as a certain *quotient* of the monoid $M_{S,s}$.

Next, let us recall [cf. the proof of Lemma 5.1] that in the theory of [Mzk2][cf. the discussion preceding [Mzk2], Lemma 2.16], the set

$$\Gamma(S, M_S)$$

is *reconstructed* as the set of sections $\overline{S}^{\log} \to \overline{Z}^{\log}$ that *avoid the* ∞*-section* (of the projective line Z). Observe that [just as in the proof of Lemma 5.1] this set of sections is equipped with a natural *topology* determined by the *induced sections* $H_S^{\mathbb{R}} \to H_Z^{\mathbb{R}}$ — i.e., two sections $\overline{S}^{\log} \to \overline{Z}^{\log}$ are "close" if and only if their induced sections $H_S^{\mathbb{R}} \to H_Z^{\mathbb{R}}$ are "close". Thus, from the point of view of elements of $\Gamma(S, M_S)$, two elements of $\Gamma(S, M_S)$ are *"close"* if and only if their *images* under the composite of the natural morphism $\Gamma(S, M_S) \to M_{S,s}^{\mathrm{gp}}$ with the surjection θ are "close". In particular, if we denote by

$$\Gamma(S, M_S)^{\theta}$$

the *completion* of the set $\Gamma(S, M_S)$ with respect to this [not necessarily separated] topology, then it follows immediately [from our assumption that S is *"sufficiently [Zariski] local with respect to s"*] that the image of $\Gamma(S, \mathcal{O}_S^{\times}) \subseteq \Gamma(S, M_S)$ in this completion may be identified with $\mathbb{S}^1$. Since, moreover, sequences of elements of $\Gamma(S, M_S)$ that converge to elements of $M_{S,s}$ that lie in the kernel of θ clearly map to 0 in the completion $\Gamma(S, M_S)^{\theta}$, we conclude that the closure of the image of $\Gamma(S, \mathcal{O}_S^{\times})$ in $\Gamma(S, M_S)^{\theta}$ [which may be identified with a copy of $\mathbb{S}^1$] is, in fact, equal to $\Gamma(S, M_S)^{\theta}$, and, moreover, that relative to this *identification* of $\Gamma(S, M_S)^{\theta}$ with $\mathbb{S}^1$, the natural completion morphism

$$\Gamma(S, M_S) \to \Gamma(S, M_S)^{\theta} = \mathbb{S}^1$$

may be identified with the composite of the natural morphism $\Gamma(S, M_S) \to M_{S,s}^{\mathrm{gp}}$ with θ. That is to say, [in light of our assumption that the monoid M_S is *generated by its global sections*] the kernel of θ, hence θ *itself*, may be recovered from the following data: the log scheme $S^{\log}$ [as reconstructed in [Mzk2]], together with the *topology* considered above on $\Gamma(S, M_S)$. Since this topology is *"category-theoretic"* by Corollary 4.2, this completes the proof of Lemma 5.2. □

We are now ready to state the *main result* of the present §, i.e., the following *global arithmetic analogue* of Theorem 3.1:

Theorem 5.1. (Categorical Reconstruction of Arithmetic Log Schemes) *Let* $\overline{X}^{\log}$, $\overline{Y}^{\log}$ *be* **arithmetic log schemes**. *Then the categories* $\overline{\mathrm{Sch}}^{\log}(\overline{Y}^{\log})$, $\overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log})$ *are* **slim** *[cf. §2], and the natural map*

$$\mathrm{Isom}(\overline{X}^{\log}, \overline{Y}^{\log}) \to \mathrm{Isom}(\overline{\mathrm{Sch}}^{\log}(\overline{Y}^{\log}), \overline{\mathrm{Sch}}^{\log}(\overline{X}^{\log}))$$

is **bijective**.

Proof. Indeed, this is a formal consequence of Corollaries 4.2, 4.3; Lemma 5.2; [Mzk2], Theorem 2.20. □

Remark 5. The natural map of Theorem 5.1 is obtained by considering the *natural functors* mentioned in the discussion following Lemma 4.1.

Remark 6. Of course, similar [but easier!] arguments yield the expected versions of Theorem 5.1 for $\overline{\mathrm{NSch}}^{\log}$, $\overline{\mathrm{Sch}}$, $\overline{\mathrm{NSch}}$:

(i) If $\overline{X}^{\log}$, $\overline{Y}^{\log}$ are *noetherian arithmetic log schemes*, then the categories $\overline{\mathrm{NSch}}^{\log}(\overline{Y}^{\log})$, $\overline{\mathrm{NSch}}^{\log}(\overline{X}^{\log})$ are *slim*, and the natural map

$$\mathrm{Isom}(\overline{X}^{\log}, \overline{Y}^{\log}) \to \mathrm{Isom}(\overline{\mathrm{NSch}}^{\log}(\overline{Y}^{\log}), \overline{\mathrm{NSch}}^{\log}(\overline{X}^{\log}))$$

is *bijective.*

(ii) If $\overline{X}$, $\overline{Y}$ are *arithmetic schemes*, then the categories $\overline{\mathrm{Sch}}(\overline{Y})$, $\overline{\mathrm{Sch}}(\overline{X})$ are *slim*, and the natural map

$$\mathrm{Isom}(\overline{X}, \overline{Y}) \to \mathrm{Isom}(\overline{\mathrm{Sch}}(\overline{Y}), \overline{\mathrm{Sch}}(\overline{X}))$$

is *bijective.*

(iii) If $\overline{X}, \overline{Y}$ are *noetherian arithmetic schemes*, then the categories $\overline{\mathrm{NSch}}(\overline{Y})$, $\overline{\mathrm{NSch}}(\overline{X})$ are *slim*, and the natural map

$$\mathrm{Isom}(\overline{X}, \overline{Y}) \to \mathrm{Isom}(\overline{\mathrm{NSch}}(\overline{Y}), \overline{\mathrm{NSch}}(\overline{X}))$$

is *bijective.*

Example 5.1. (Arithmetic Vector Bundles)

(i) Let F be a *number field*; denote the associated *ring of integers* by $\mathcal{O}_F$; write $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_F)$. Equip S with the *archimedean structure* given by the whole of $S(\mathbb{C})$; denote the resulting *arithmetic scheme* by $\overline{S}$. Let $\mathcal{E}$ be a *vector bundle* on S. Write $V \to S$ for the result of *blowing up* the associated *geometric vector bundle* along its *zero section*; denote the resulting *exceptional divisor* [i.e., the inverse image of the zero section via the blow-up morphism] by $D \subseteq V$. If $\mathcal{E}$ is equipped with a *Hermitian metric* at each archimedean prime (up to complex conjugation) of F, then, by taking the *"archimedean structure"* on V to be the complex-valued points of V that correspond to sections of $\mathcal{E}$ with *norm* (relative to this Hermitian metric) ≤ 1 [hence include the complex-valued points of D], we obtain an *arithmetic scheme* $\overline{V}$ over $\overline{S}$. Now suppose that S is equipped with a *log structure* defined by some finite set Σ of closed points of S; denote the resulting *arithmetic log scheme* by $\overline{S}^{\log}$. Equip V with the log structure obtained by "appending" to the log structure pulled back from $S^{\log}$ the log structure determined by the divisor $D \subseteq V$. Thus, we obtain a *morphism of arithmetic log schemes*:

$$\overline{V}^{\log} \to \overline{S}^{\log}$$

The sections $\overline{S}^{\log} \to \overline{V}^{\log}$ of this morphism correspond naturally to the elements of $\Gamma(S, \mathcal{E})$ which are *nonzero* away from Σ and have *norm* ≤ 1 at all the archimedean primes.

(ii) For $i = 1, 2$, let $\overline{V}_i^{\log} \to \overline{S}_i^{\log}$ be constructed as in (i) above. Then (by Theorem 5.1) the *isomorphism classes of equivalences of categories*

$$\overline{\mathrm{Sch}}^{\log}(\overline{V}_1^{\log}) \xrightarrow{\sim} \overline{\mathrm{Sch}}^{\log}(\overline{V}_2^{\log})$$

correspond naturally to the following data: an *isometric isomorphism of vector bundles* $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ lying over an *isomorphism of log schemes* $S_1^{\log} \xrightarrow{\sim} S_2^{\log}$.

(iii) We shall refer to a subset

$$A \subseteq \mathbb{C}$$

as an *angular region* if there exists a $\rho \in \mathbb{R}_{>0}$ [where $\mathbb{R}_{>0} \subseteq \mathbb{R}$ is the subset of real numbers > 0] and a subset $A_{\mathbb{S}^1} \subseteq \mathbb{S}^1 \subseteq \mathbb{C}$ such that $A = \{\lambda \cdot u \mid \lambda \in [0, \rho], u \in A_{\mathbb{S}^1}\}$. We shall say that the angular region A is *open* (respectively, *closed*; *isotropic*) [i.e., as an angular region] if the subset $A_{\mathbb{S}^1} \subseteq \mathbb{S}^1$ is open (respectively, closed; equal to $\mathbb{S}^1$); we shall refer to ρ as the *radius* of the angular region A. Thus, if we write

$$\mathrm{Ang}(\mathbb{C}) \stackrel{\mathrm{def}}{=} \mathbb{C}^\times/\mathbb{R}_{>0}$$

[so the natural composite $\mathbb{S}^1 \hookrightarrow \mathbb{C} \twoheadrightarrow \mathrm{Ang}(\mathbb{C})$ is a homeomorphism], then the projection

$$\mathrm{Ang}(A) \subseteq \mathrm{Ang}(\mathbb{C})$$

of A [i.e., $A\backslash\{0\}$] to $\mathrm{Ang}(\mathbb{C}) \cong \mathbb{S}^1$ is simply $A^1_{\mathbb{S}}$. Note that the notion of an angular region (respectively, open angular region; closed angular region; $\mathrm{Ang}(-)$; radius of an angular region) extends immediately to the case where "$\mathbb{C}$" is replaced by an an arbitrary 1-dimensional complex vector space (respectively, vector space; vector space; vector space; vector space equipped with a Hermitian metric).

In particular, in the notation of (i), when $\mathcal{E}$ is a *line bundle*, the choice of a(n) *closed (respectively, open) angular region of radius* 1 at each of the complex archimedean primes of F determines a(n) *(ind-)archimedean structure* [cf. Remark 3] on $V^{\log}$. Thus, the (ind-)arithmetic log schemes discussed in (i) correspond to the case where all of the angular regions chosen are *isotropic*.

Remark 7. When the vector bundle $\mathcal{E}$ of Example 5.1 is a *line bundle* [i.e., of rank one], the blow-up used to construct V is an *isomorphism*. That is to say, in this case, V is simply the *geometric line bundle* associated to $\mathcal{E}$, and $D \subseteq V$ is its *zero section*.

Remark 8. Some readers may wonder *why*, in Definition 4.2, we took H to be a *compact* set, as opposed to, say, an *open* set (or, perhaps, an open set which is, in some sense, "bounded"). One reason for this is the following: If H were required to be *open*, then we would be obliged, in Example 5.1, to take the "archimedean structure" on V to be the open set defined by sections of *norm* < 1. In particular, if $\mathcal{E}$ is taken to be the *trivial line bundle*, then it would follow that the section of V defined by the section "1" of the trivial bundle would *fail* to define a morphism in the "category of arithmetic log schemes" — a situation which the author found to be unacceptable.

Another motivating reason for Definition 4.2 comes from *rigid geometry*: That is to say, in the context of rigid geometry, perhaps the most basic example of an *integral structure* on the affine line $\mathrm{Spec}(\mathbb{Q}_p[T])$ is that given by the *ring*

$$\mathbb{Z}_p[T]^\wedge$$

(where the "$\wedge$" denotes p-adic completion). Then the continuous *homomorphisms* $\mathbb{Z}_p[T]^\wedge \to \mathbb{C}_p$ [i.e., the "$\mathbb{C}_p$-valued points of the integral structure"] correspond precisely to the elements of $\mathbb{C}_p$ with absolute value ≤ 1.

Remark 9. If $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_F)$ [where $\mathcal{O}_F$ is the ring of integers of a number field F], and we equip S with the *log structure* associated to the chart $\mathbb{N} \ni 1 \mapsto 0 \in \mathcal{O}_S$, then an *archimedean structure* on $S^{\log}$ is *not* the same as a *choice of Hermitian metrics on the trivial line bundle over* $\mathcal{O}_S$ at various archimedean primes of S. This is somewhat *counter-intuitive*, from the point of view of the usual theory of log schemes. More generally:

> The definition of an *archimedean structure* [cf. Definition 4.2] adopted in this paper is perhaps *not so satisfactory* when one wishes to consider the archimedean aspects of *log structures* or other *infinitesimal deformations* (e.g., nilpotent thickenings) in detail.

For instance, the possible choices of an archimedean structure are *invariant* with respect to nilpotent thickenings. Thus, depending on the situation in which one wishes to apply the theory of the present paper, it may be *desirable to modify* Definition 4.2 so as to deal with archimedean structures on log structures or nilpotent thickenings in a more satisfactory matter — *perhaps by making use of the constructions of Example* 5.1 *[including "angular regions"!], applied to the* *various line bundles or vector bundles that form the log structures or nilpotent thickenings under consideration.*

At the time of writing, however, it is *not clear to the author* how to construct such a theory. Indeed, many of the complications that appear to arise if one is to construct such a theory seem to be related to the fact that *archimedean (integral) structures*, unlike their nonarchimedean counterparts, typically *fail* to be closed under *addition*. Since, however, such a theory is beyond the scope of the present paper, we shall not discuss this issue further in the present paper.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
motizuki@kurims.kyoto-u.ac.jp

References

[KN] K. Kato and C. Nakayama, Log Betti Cohomology, Log Étale Cohomology, and Log de Rham Cohomology of Log Schemes over $\mathbb{C}$, *Kodai Math. J.* **22** (1999), pp. 161-186.

[Mzk1] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[Mzk2] S. Mochizuki, *Categorical Representation of Locally Noetherian Log Schemes*, to appear in *Adv. Math.*