Categories of Log Schemes with Archimedean Structures

By

Shinichi Mochizuki

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Abstract

In this paper, we generalize the main result of [Mzk2] (to the effect that very general noetherian log schemes may be reconstructed from naturally associated categories) to the case of log schemes locally of finite type over Zariski localizations of the ring of rational integers which are, moreover, equipped with certain “archimedean structures”.

1. Introduction

As is discussed in the Introduction to [Mzk2], it is natural to ask to what extent various objects — such as log schemes — that occur in arithmetic geometry may be represented by categories, i.e., to what extent one may reconstruct the original object solely from the category-theoretic structure of a category naturally associated to the object. As is explained in loc. cit., this point of view is partially motivated by the anabelian philosophy of Grothendieck.
In the present paper, we extend the theory of [Mzk2], which only concerns log schemes, to obtain a theory that proves a similar categorical representability result [cf. Theorem 5.1 below] for what we call “arithmetic log schemes” [cf. Definitions 4.1, 4.2 below], i.e., log schemes that are locally of finite type over a Zariski localization of the ring of rational integers and, moreover, are equipped with certain “archimedean structures” at archimedean primes.

In §3, we review the theory of [Mzk2], and revise the formulation of the main theorem of [Mzk2] slightly [cf. Theorem 3.1]. In §4, we define the notion of an archimedean structure on a fine, saturated log scheme which is of finite type over a Zariski localization of \( \mathbb{Z} \). Finally, in §5, we generalize Theorem 3.1 [cf. Theorem 5.1] so as to take into account these archimedean structures.

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2. Notations and Conventions

Numbers:

We will denote by \( \mathbb{N} \) the set (or, occasionally, the commutative monoid) of natural numbers, by which we take to consist set of the integers \( n \geq 0 \). A number field is defined to be a finite extension of the field of rational numbers \( \mathbb{Q} \). The field of real numbers (respectively, complex numbers) will be denoted by \( \mathbb{R} \) (respectively, \( \mathbb{C} \)). The topological group of complex numbers of unit norm will be denoted by \( S^1 \subseteq \mathbb{C} \).

We shall say that a scheme \( S \) is a Zariski localization of \( \mathbb{Z} \) if \( S = \text{Spec}(R) \), where \( R = M^{-1} \cdot \mathbb{Z} \), for some multiplicative subset \( M \subseteq \mathbb{Z} \).

Topological Spaces:

In this paper, the term “compact” is to be understood to include the assumption that the topological space in question is Hausdorff. (The author wishes to thank A. Tamagawa for his comments concerning the importance of making this assumption explicit.)

Also, when a topological space \( H \) is equipped with an involution \( \sigma \) (typically an action of “complex conjugation”), we shall denote by

\[
H^\mathbb{R}
\]

(i.e., a superscript “\( \mathbb{R} \)”) the quotient topological space of “\( \sigma \)-orbits”.

Categories:

Let \( \mathcal{C} \) be a category. We shall denote the collection of objects of \( \mathcal{C} \) by:

\[
\text{Ob}(\mathcal{C})
\]
If $A \in \text{Ob}(C)$ is an object of $C$, then we shall denote by

$$C_A$$

the category whose objects are morphisms $B \to A$ of $C$ and whose morphisms (from an object $B_1 \to A$ to an object $B_2 \to A$) are $A$-morphisms $B_1 \to B_2$ in $C$. Thus, we have a natural functor

$$(j_A): C_A \to C$$

(given by forgetting the structure morphism to $A$). Similarly, if $f: A \to B$ is a morphism in $C$, then $f$ defines a natural functor

$$f_1: C_A \to C_B$$

by mapping an arrow (i.e., an object of $C_A$) $C \to A$ to the object of $C_B$ given by the composite $C \to A \to B$ with $f$.

If the category $C$ admits finite products, then $(j_A)_!$ is left adjoint to the natural functor

$$j^*_A: C \to C_A$$

given by taking the product with $A$, and $f_1$ is left adjoint to the natural functor

$$f^*: C_B \to C_A$$

given by taking the fibered product over $B$ with $A$.

We shall call an object $A \in \text{Ob}(C)$ terminal if for every object $B \in \text{Ob}(C)$, there exists a unique arrow $B \to A$ in $C$. We shall call an object $A \in \text{Ob}(C)$ quasi-terminal if for every object $B \in \text{Ob}(C)$, there exists an arrow $\phi: B \to A$ in $C$, and, moreover, for every other arrow $\psi : B \to A$, there exists an automorphism $\alpha$ of $A$ such that $\psi = \alpha \circ \phi$.

We shall refer to a natural transformation between functors all of whose component morphisms are isomorphisms as an isomorphism between the functors in question. A functor $\phi: C_1 \to C_2$ between categories $C_1, C_2$ will be called rigid if $\phi$ has no nontrivial automorphisms. A category $C$ will be called slim if the natural functor $C_A \to C$ is rigid, for every $A \in \text{Ob}(C)$.

If $C$ is a category and $S$ is a collection of arrows in $C$, then we shall say that an arrow $A \to B$ is minimal-adjoint to $S$ if every factorization $A \to C \to B$ of this arrow $A \to B$ in $C$ such that $A \to C$ lies in $S$ satisfies the property that $A \to C$ is, in fact, an isomorphism. Often, the collection $S$ will be taken to be the collection of arrows satisfying a particular property $P$; in this case, we shall refer to the property of being “minimal-adjoint to $S$” as the minimal-adjoint notion to $P$.

### 3. Review of the Theory for Log Schemes

We begin our discussion by reviewing the theory for log schemes developed in [Mzk2]. Also, we give a slight extension of this theory (to the case of locally...
noetherian log schemes and morphisms which are locally of finite type). In the
case of this extension, it is natural to modify the notation used in [Mzk2]
slightly as follows:

Let us denote by

\[ \text{Sch}^{\log} \]

the category of all \textit{locally noetherian fine saturated log schemes} and \textit{locally finite type morphisms}, and by

\[ \text{NSch}^{\log} \]

the category of all \textit{noetherian fine saturated log schemes} and \textit{finite type morphisms}. Note that

\[ \text{NSch}^{\log} \subseteq \text{Sch}^{\log} \]

may be characterized as the \textit{full subcategory} consisting of the \( X^{\log} \) for which \( X \) is \textit{noetherian}.

If \( X^{\log} \) is a \textit{fine saturated log scheme} whose underlying scheme \( X \) is \textit{locally noetherian}, then we shall write

\[ \text{Sch}^{\log}(X^{\log}) \overset{\text{def}}{=} (\text{Sch}^{\log})_{X^{\log}} \]

and

\[ \text{NSch}^{\log}(X^{\log}) \subseteq \text{Sch}^{\log}(X^{\log}) \]

for the \textit{full subcategory} consisting of the \( Y^{\log} \to X^{\log} \) for which \( Y \) is \textit{noetherian}. Thus, when \( X \) is \textit{noetherian}, we have \( \text{NSch}^{\log}(X^{\log}) = (\text{NSch}^{\log})_{X^{\log}} \).

To simplify terminology, we shall often refer to the \textit{domain} \( Y^{\log} \) of an arrow \( Y^{\log} \to X^{\log} \) which is an object of \( \text{Sch}^{\log}(X^{\log}) \) or \( \text{NSch}^{\log}(X^{\log}) \) as an “\textit{object of} \( \text{Sch}^{\log}(X^{\log}) \) or \( \text{NSch}^{\log}(X^{\log}) \)”.

If \( X^{\log}, Y^{\log} \) are \textit{locally noetherian fine saturated log schemes}, then denote the set of isomorphisms of log schemes \( X^{\log} \cong Y^{\log} \) by:

\[ \text{Isom}(X^{\log}, Y^{\log}) \]

Then the \textit{main result} of [Mzk2] [cf. [Mzk2], Theorem 2.19] states that the natural map

\[ \text{Isom}(X^{\log}, Y^{\log}) \to \text{Isom}(\text{NSch}^{\log}(Y^{\log}), \text{NSch}(X^{\log})) \]

given by \( f^{\log} \mapsto \text{NSch}^{\log}(f^{\log}) \) [i.e., mapping an isomorphism to the induced equivalence between “\( \text{NSch}^{\log}(\cdot) \)”s”] is \textit{bijective}. (Here, the “\text{Isom}” on the right is to be understood to denote \textit{isomorphism classes of equivalences} between the two categories in parentheses.) This result generalizes immediately to the case of “\( \text{Sch}^{\log}(\cdot) \)”:
Theorem 3.1. (Categorical Reconstruction of Locally Noetherian Fine Saturated Log Schemes) Let $X^{\log}, Y^{\log}$ be locally noetherian fine saturated log schemes. Then the natural map

$$\text{Isom}(X^{\log}, Y^{\log}) \to \text{Isom}(\text{Sch}^{\log}(Y^{\log}), \text{Sch}^{\log}(X^{\log}))$$

is bijective.

Proof. Indeed, by functoriality and [Mzk2], Theorem 2.19, it suffices to show that the subcategory $\text{NSch}^{\log}(X^{\log}) \subseteq \text{Sch}^{\log}(X^{\log})$ may be recovered “category-theoretically”.

To see this, let us first observe that the proof given in [Mzk2] [cf. [Mzk2], Corollary 2.14] of the category-theoreticity of the property that a morphism in $\text{NSch}^{\log}(X^{\log})$ be “scheme-like” (i.e., that the log structure on the domain is the pull-back of the log structure on the codomain) is entirely valid in $\text{Sch}^{\log}(X^{\log})$. (Indeed, the proof essentially only involves morphisms among “one-pointed objects”, which are the same in $\text{NSch}^{\log}(X^{\log}), \text{Sch}^{\log}(X^{\log})$.) Moreover, once one knows which morphisms are scheme-like, the open immersions may be characterized category-theoretically as in [Mzk2], Corollary 1.3.

Next, let us first observe that the property that a collection of open immersions

$$Y^{\log}_\alpha \to Y^{\log}$$

(where $\alpha$ ranges over the elements of some index set $A$) in $\text{Sch}^{\log}(X^{\log})$ be surjective is category-theoretic. Indeed, this follows from the fact that this collection is surjective if and only if, for any morphism $Z^{\log} \to Y^{\log}$, where $Z^{\log}$ is nonempty, the fiber product $Y^{\log}_\alpha \times_{Y^{\log}} Z^{\log}$ in $\text{Sch}^{\log}(X^{\log})$ [cf. [Mzk2], Lemma 2.6] is nonempty for some $\alpha$ [cf. also [Mzk2], Proposition 1.1, (i), applied to the complement of the union of the images of the $Y^{\log}_\alpha$].

Thus, it suffices to observe that an object $Y^{\log}$ is noetherian if and only if, for any surjective collection of open immersions (in $\text{Sch}^{\log}(X^{\log})$) $Y^{\log}_\alpha \to Y^{\log}$ (where $\alpha$ ranges over the elements of some index set $A$), there exists a finite subset $B \subseteq A$ such that the collection $\{Y^{\log}_\beta \to Y^{\log}\}_{\beta \in B}$ is surjective. 

Remark 1. Similar [but easier] results hold for

$\text{Sch}$ (respectively, $\text{NSch}$)

— i.e., the category of all locally noetherian schemes and locally finite type morphisms (respectively, all noetherian log schemes and finite type morphisms).
4. Archimedean Structures

In this §, we generalize the categories defined in [Mzk2] so as to include archimedean primes. In particular, we prepare for the proof in §5 below of a global arithmetic analogue [cf. Theorem 5.1] of Theorem 3.1.

Let $X^{\text{log}}$ be a fine, saturated locally noetherian log scheme (with underlying scheme $X$).

**Definition 4.1.** We shall say that $X$ is arithmetically (locally) of finite type if $X$ is (locally) of finite type over a Zariski localization of $\mathbb{Z}$. Similarly, we shall say that $X^{\text{log}}$ is arithmetically (locally) of finite type if $X$ is.

Suppose that $X^{\text{log}}$ is arithmetically locally of finite type. Then $X^{\text{log}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is locally of finite type over $\mathbb{Q}$. In particular, the set of $\mathbb{C}$-valued points

$$X(\mathbb{C})$$

is equipped with a natural topology (induced by the topology of $\mathbb{C}$), together with an involution $\sigma_X : X(\mathbb{C}) \to X(\mathbb{C})$ induced by the complex conjugation automorphism on $\mathbb{C}$. Similarly, in the logarithmic context, it is natural to consider the topological space

$$X^{\text{log}}(\mathbb{C}) \overset{\text{def}}{=} \{(x, \theta) \mid x \in X(\mathbb{C}), \theta \in \text{Hom}(M^{\text{gp}}_{X,x}, \mathbb{S}^1) \text{ s.t. } \theta(f) = f(x)/|f(x)|, \forall f \in \mathcal{O}_{X,x}^\times \} \quad (4.1)$$

and

$$X^{\text{log}}(\mathbb{C}) \overset{\text{def}}{=} \{(x, \theta) \mid x \in X(\mathbb{C}), \theta \in \text{Hom}(M^{\text{gp}}_{X,x}, \mathbb{S}^1) \text{ s.t. } \theta(f) = f(x)/|f(x)|, \forall f \in \mathcal{O}_{X,x}^\times \} \quad (4.2)$$

[cf. [KN], §1.2]. Here, we use the notation $M_X$ to denote the monoid that defines the log structure of $X^{\text{log}}$ [cf. [Mzk2], §2]. Thus, we have a natural surjection

$$X^{\text{log}}(\mathbb{C}) \to X(\mathbb{C})$$

whose fibers are (noncanonically) isomorphic to products of finitely many copies of $\mathbb{S}^1$. Also, we observe that it follows immediately from the definition that $\sigma_X$ extends to an involution $\sigma_{X^{\text{log}}}$ on $X^{\text{log}}(\mathbb{C})$.

**Definition 4.2.**

(i) Let $H \subseteq X(\mathbb{C})$ be a compact subset stabilized by $\sigma_X$. Then we shall refer to a pair $\overline{X} = (X, H)$ as an arithmetic scheme, and $H$ as the archimedean structure on $\overline{X}$. We shall say that an archimedean structure $H \subseteq X(\mathbb{C})$ is trivial (respectively, total) if $H = \emptyset$ (respectively, $H = X(\mathbb{C})$).

(ii) Let $H \subseteq X^{\text{log}}(\mathbb{C})$ be a compact subset stabilized by $\sigma_{X^{\text{log}}}$. Then we
shall refer to a pair $\mathfrak{X}^\log = (X^\log, H)$ as an arithmetic log scheme, and $H$ as the archimedean structure on $\mathfrak{X}^\log$. We shall say that an archimedean structure $H \subseteq X^\log(\mathbb{C})$ is trivial (respectively, total) if $H = \emptyset$ (respectively, $H = X^\log(\mathbb{C})$).

**Remark 2.** The idea that “integral structures at archimedean primes” should be given by compact/bounded subsets of the set of complex valued points may be seen in the discussion of [Mzk1], p. 9; cf. also Remark 8 below.

**Remark 3.** Relative to Definition 4.2, one may think of the case where “$H$” is open as the case of an ind-arithmetic (log) scheme [or, alternatively, an “ind-archimedean structure”], i.e., the inductive system of arithmetic (log) schemes [or, alternatively, archimedean structures] determined by considering all compact subsets that lie inside the given open.

Let us denote the category of all arithmetic log schemes by:

$$\text{Sch}^\log$$

Thus, a morphism $\mathfrak{X}_1^\log = (X_1^\log, H_1) \rightarrow \mathfrak{X}_2^\log = (X_2^\log, H_2)$ in this category is a locally finite type morphism $X_1^\log \rightarrow X_2^\log$ such that the induced map $X_1^\log(\mathbb{C}) \rightarrow X_2^\log(\mathbb{C})$ maps $H_1$ into $H_2$. The full subcategory of noetherian objects of $\text{Sch}^\log$ [i.e., objects whose underlying scheme is noetherian] will be denoted by:

$$\text{NSch}^\log \subseteq \text{Sch}^\log$$

Similarly, if we forget about log structures, we obtain categories $\text{NSch}$, $\text{Sch}$.

**Definition 4.3.**

(i) An arithmetic (log) scheme will be called purely nonarchimedean if its archimedean structure is trivial.

(ii) A morphism between arithmetic (log) schemes will be called purely archimedean if the underlying morphism between (log) schemes is an isomorphism.

Denote by

$$\text{Sch}^\log \subseteq \text{Sch}^\log$$
the full subcategory determined by those objects which are arithmetically locally of finite type. Then note that by considering purely nonarchimedean objects, we obtain a natural embedding

\[
\text{Sch}^\log \hookrightarrow \text{Sch}^\log
\]

of \(\text{Sch}^\log\) as a full subcategory of \(\text{Sch}^\log\). If \(\bar{X}^\log \in \text{Ob}(\text{Sch}^\log)\), then we shall write

\[
\text{Sch}^\log(\bar{X}^\log) \overset{\text{def}}{=} (\text{Sch}^\log)_{\bar{X}^\log}
\]

[cf. §3] and

\[
\text{Sch}^\log(\bar{X}^\log)_{\text{arch}} \subseteq \text{Sch}^\log(\bar{X}^\log)
\]

for the subcategory whose objects \(\bar{Y}^\log \to \bar{X}^\log\) are purely archimedean arrows of \(\text{Sch}^\log\). (Thus, the morphisms \(\bar{Y}_1^\log \to \bar{Y}_2^\log\) of this subcategory are also necessarily purely archimedean.)

On the other hand, if \(T\) is a topological space, then let us write

\[
\text{Open}(T) \quad \text{(respectively, Closed}(T))
\]

for the category whose objects are open subsets \(U \subseteq T\) (respectively, closed subsets \(F \subseteq T\)) and whose morphisms are inclusions of subsets of \(T\). Thus, one verifies easily (by taking complements!) that \(\text{Closed}(T)\) is the opposite category \(\text{Open}(T)^{\text{opp}}\) associated to \(\text{Open}(T)\). Also, let us write

\[
\text{Shv}(T)
\]

for the category of sheaves on \(T\) (valued in sets).

Now we have the following:

**Proposition 4.1.** (Conditional Reconstruction of the Archimedean Topological Space)

(i) If \(H\) is the archimedean structure on \(\bar{X}^\log\), then the functor

\[
\text{Sch}^\log(\bar{X}^\log)_{\text{arch}} \to \text{Closed}(H^R) \quad \overset{\sim}{\to} \text{Open}(H^R)^{\text{opp}}
\]

[cf. §2 for more on the superscript \(\mathbb{R}\)] given by assigning to an arrow \(\bar{Y}^\log \to \bar{X}^\log\) the image of the archimedean structure of \(\bar{Y}^\log\) in \(H^R \subseteq X^\log(\mathbb{C})^\mathbb{R}\) is an equivalence.

(ii) Let \(\bar{X}_1^\log, \bar{X}_2^\log \in \text{Ob}(\text{Sch}^\log)\). Suppose that

\[
\Phi : \text{Sch}^\log(\bar{X}_1^\log) \overset{\sim}{\to} \text{Sch}^\log(\bar{X}_2^\log)
\]
is an equivalence of categories that preserves purely archimedean arrows (i.e., an arrow $f$ in $\text{Sch}^{\log}(X^{\log}_1)$ is purely archimedean if and only if $\Phi(f)$ is purely archimedean). Then one can construct, for every object $Y^{\log}_1 = (Y^{\log}_1, K_1) \in \text{Ob}(\text{Sch}^{\log}(X^{\log}_1))$ that maps via $\Phi$ to an object $Y^{\log}_2 = (Y^{\log}_2, K_2) \in \text{Ob}(\text{Sch}^{\log}(X^{\log}_2))$, a homeomorphism

$$K^{R}_1 \sim K^{R}_2$$

which is functorial in $Y^{\log}_1$.

Proof. Assertion (i) is a formal consequence of the definitions. To prove assertion (ii), let us first observe that (for an arbitrary topological space $T$) $\text{Shv}(T)$ may be reconstructed functorially from $\text{Open}(T)$, since coverings of objects of $\text{Open}(T)$ may be characterized as collections of objects whose inductive limit (a purely categorical notion!) is isomorphic to the object to be covered. Thus, our assumption on $\Phi$, together with assertion (i), implies that (for $i = 1, 2$) $\text{Shv}(K^{R}_i)$ may be reconstructed category-theoretically from $Y^{\log}_i$ in a fashion which is functorial in $Y^{\log}_i$. Moreover, since $K^{R}_i$ is clearly a sober topological space, we thus conclude [by a well-known result from “topos theory” — cf., e.g., [Mzk2], Theorem 1.4] that the topological space $K^{R}_i$ itself may be reconstructed category-theoretically from $Y^{\log}_i$ in a fashion which is functorial in $Y^{\log}_i$, as desired.

Before proceeding, we observe the following:

Lemma 4.1. (Finite Products of Arithmetic Log Schemes) The category $\text{Sch}^{\log}$ admits finite products.

Proof. Indeed, if, for $i = 1, 2, 3$, we are given objects $X^{\log}_i = (X^{\log}_i, H_i) \in \text{Ob}(\text{Sch}^{\log})$ and morphisms $X^{\log}_1 \to X^{\log}_2, X^{\log}_3 \to X^{\log}_2$ in $\text{Sch}^{\log}$, then we may form the product of $X^{\log}_1, X^{\log}_3$ over $X^{\log}_2$ by equipping the log scheme

$$X^{\log}_1 \times X^{\log}_3$$

(which is easily seen to be arithmetically locally of finite type) with the archimedean structure given by the inverse image of

$$H_1 \times H_2 \times H_3 \subseteq X^{\log}_1(C) \times X^{\log}_2(C) \times X^{\log}_3(C)$$
(where we note that $H_1 \times_{H_2} H_3$ is compact, since $H_2$ is Hausdorff) via the natural map:

$$(X_1^{\log} \times_{X_2^{\log}} X_3^{\log})(\mathbb{C}) \to X_1^{\log}(\mathbb{C}) \times_{X_2^{\log}(\mathbb{C})} X_3^{\log}(\mathbb{C})$$

Note that this last map is proper [i.e., inverse images of compact sets are compact], since, for any $Y^{\log}$ which is arithmetically locally of finite type, the map $Y^{\log}(\mathbb{C}) \to Y(\mathbb{C})$ is proper, and, moreover, the map induced on $\mathbb{C}$-valued points of underlying schemes by

$$X_1^{\log} \times_{X_2^{\log}} X_3^{\log} \to X_1 \times X_2 X_3$$

[i.e., where the domain is equipped with the trivial log structure] is finite [cf. [Mzk2], Lemma 2.6], hence proper.

Thus, if $X^{\log}, Y^{\log} \in \text{Ob}(\text{Sch}^{\log})$, then any morphism $X^{\log} \to Y^{\log}$ in $\text{Sch}^{\log}$ induces a natural functor

$$\text{Sch}^{\log}(Y^{\log}) \to \text{Sch}^{\log}(X^{\log})$$

(by sending an object $Z^{\log} \to Y^{\log}$ to the fibered product $Z^{\log} \times_{Y^{\log}} X^{\log} \to X^{\log}$ — cf. the discussion of §2).

Next, we would like to show, in the following discussion [cf. Corollary 4.1, (ii) below], that the hypothesis of Proposition 4.1, (ii), is automatically satisfied.

Let $X^{\log} \in \text{Ob}(\text{Sch}^{\log})$.

**Proposition 4.2. (Minimal Objects)** An object $Y^{\log}$ of $\text{Sch}^{\log}(X^{\log})$ will be called **minimal** if it is nonempty and satisfies the property that any monomorphism $Z^{\log} \to Y^{\log}$ (where $Z^{\log}$ is nonempty) in $\text{Sch}^{\log}(X^{\log})$ is necessarily an isomorphism. An object $Y^{\log}$ of $\text{Sch}^{\log}(X^{\log})$ is minimal if and only if it is purely nonarchimedean and log scheme-theoretically minimal [i.e., the underlying object $Y^{\log}$ of $\text{Sch}^{\log}(X^{\log})$ is minimal as an object of $\text{Sch}(X^{\log})$ — cf. [Mzk2], Proposition 2.4].

**Proof.** The sufficiency of this condition is clear, since the domain of any morphism in $\text{Sch}^{\log}$ to a purely nonarchimedean object is necessarily itself purely nonarchimedean [i.e., no nonempty set maps to an empty set]. That this condition is necessary is evident from the definitions (e.g., if a nonempty object fails to be purely nonarchimedean, then it can always be “made smaller” [but still nonempty!] by setting the archimedean structure equal to the empty set, thus precluding “minimality”).
Proposition 4.3. (Characterization of One-Pointed Objects) We shall call an object of $\text{Sch}^{\text{log}}$ one-pointed if the underlying topological space of its underlying scheme consists of precisely one point. The one-pointed objects $\overline{Y}^{\text{log}}$ of $\text{Sch}^{\text{log}}(X^{\text{log}})$ may be characterized category-theoretically as the nonempty objects which satisfy the following property: For any two morphisms $\overline{S}_1^{\text{log}} \to \overline{Y}^{\text{log}}$ (for $i = 1, 2$), where $\overline{S}_i^{\text{log}}$ is a minimal object, the product $\overline{S}_1^{\text{log}} \times_{\overline{Y}^{\text{log}}} \overline{S}_2^{\text{log}}$ (in $\text{Sch}^{\text{log}}(X^{\text{log}})$) is nonempty.

Proof. This is a formal consequence of the definitions; Proposition 4.2; and [Mzk2], Corollary 2.9.

Proposition 4.4. (Minimal Hulls) Let $\overline{Y}^{\text{log}}$ be a one-pointed object of the category $\text{Sch}^{\text{log}}(X^{\text{log}})$. Then a monomorphism $\overline{Z}^{\text{log}} \to \overline{Y}^{\text{log}}$ will be called a hull for $\overline{Y}^{\text{log}}$ if every morphism $\overline{S}^{\text{log}} \to \overline{Y}^{\text{log}}$ from a minimal object $\overline{S}^{\text{log}}$ to $\overline{Y}^{\text{log}}$ factors (necessarily uniquely!) through $\overline{Z}^{\text{log}}$. A hull $\overline{Z}^{\text{log}} \to \overline{Y}^{\text{log}}$ will be called a minimal hull if every monomorphism $\overline{Z}_1^{\text{log}} \to \overline{Z}^{\text{log}}$ for which the composite $\overline{Z}_1^{\text{log}} \to \overline{Y}^{\text{log}}$ is a hull is necessarily an isomorphism. A one-pointed object $\overline{Z}^{\text{log}}$ will be called a minimal hull if the identity morphism $\overline{Z}^{\text{log}} \to \overline{Z}^{\text{log}}$ is a minimal hull for $\overline{Z}^{\text{log}}$.

(i) An object $\overline{Y}^{\text{log}}$ of $\text{Sch}^{\text{log}}(X^{\text{log}})$ is a minimal hull if and only if it is purely nonarchimedean and log scheme-theoretically a minimal hull [i.e., the underlying object $Y^{\text{log}}$ of $\text{Sch}^{\text{log}}(X^{\text{log}})$ is a minimal hull in the sense of [Mzk2], Proposition 2.7; cf. also [Mzk2], Corollary 2.10].

(ii) Any two minimal hulls of an object $\overline{Y}^{\text{log}} \in \text{Ob}(\text{Sch}^{\text{log}}(X^{\text{log}}))$ are isomorphic (via a unique isomorphism over $\overline{Y}^{\text{log}}$).

(iii) If $\overline{Y}_1^{\text{log}} \in \text{Ob}(\text{Sch}^{\text{log}}(X_1^{\text{log}}))$, $\overline{Y}_2^{\text{log}} \in \text{Ob}(\text{Sch}^{\text{log}}(X_2^{\text{log}}))$, and

$$\Phi : \text{Sch}^{\text{log}}(X_1^{\text{log}}) \xrightarrow{\sim} \text{Sch}^{\text{log}}(X_2^{\text{log}})$$

is an equivalence of categories such that $\Phi(\overline{Y}_1^{\text{log}}) = \overline{Y}_2^{\text{log}}$, then $\overline{Y}_1^{\text{log}}$ is a minimal hull if and only if $\overline{Y}_2^{\text{log}}$ is. That is to say, the condition that an object $\overline{Y}^{\text{log}} \in \text{Ob}(\text{Sch}^{\text{log}}(X^{\text{log}}))$ be a minimal hull is “category-theoretic”.

Proof. Assertion (i) (respectively, (ii); (iii)) is a formal consequence of Proposition 4.2 (respectively, assertion (i); Proposition 4.3) [and the definitions of the terms involved].
Proposition 4.5. (Purely Archimedean Morphisms of Reduced One-Pointed Objects) Let $Y^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log}))$ be one-pointed; let $Z^{\log} \rightarrow Y^{\log}$ be a minimal hull which factors as a composite of monomorphisms $Z \rightarrow Z_1 \rightarrow Y^{\log}$. Then the following are equivalent:

(i) $Z_1^{\log}$ is reduced.

(ii) $Z^{\log} \rightarrow Z_1^{\log}$ is purely archimedean.

(iii) $Z^{\log} \rightarrow Z_1^{\log}$ is an epimorphism in $\text{Sch}^{\log}(X^{\log})$ [i.e., two sections $Z_1^{\log} \rightarrow S^{\log}$ of a morphism $S^{\log} \rightarrow Z_1^{\log}$ coincide if and only if they coincide after restriction to $Z_1^{\log}$].

Proof. The equivalence of (i), (ii) is a formal consequence of [Mzk2], Proposition 2.3; [Mzk2], Proposition 2.7, (ii), (iii); [Mzk2], Corollary 2.10. That (ii) implies (iii) is a formal consequence of the definitions. Finally, that (iii) implies (i) follows, for instance, by taking $S^{\log} \rightarrow Z_1^{\log}$ to be the projective line over $Z_1^{\log}$ (so sections that lies in the open sub-log scheme of $S^{\log}$ determined by the affine line correspond to elements of $\Gamma(Z_1, O_{Z_1})$). (Here, we equip the projective line with the archimedean structure which is the inverse image of the archimedean structure of $Z_1^{\log}$.)

Note that condition (iii) of Proposition 4.5 is “category-theoretic”. This implies the following:

Corollary 4.1. (Characterization of Purely Nonarchimedean One-Pointed Objects and Purely Archimedean Morphisms)

(i) A one-pointed object $Y^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log}))$ is purely nonarchimedean if and only if it satisfies the following “category-theoretic” condition: Every minimal hull $Z^{\log} \rightarrow Y^{\log}$ is minimal-adjoint [cf. §2] to the collection of arrows $Z^{\log} \rightarrow Z_1^{\log}$ which satisfy the equivalent conditions of Proposition 4.5.

(ii) A morphism $\zeta : Y^{\log} \rightarrow Z^{\log}$ in $\text{Sch}^{\log}(X^{\log})$ is purely archimedean if and only if it satisfies the following “category-theoretic” condition: The morphism $\zeta$ is a monomorphism in $\text{Sch}^{\log}(X^{\log})$, and, moreover, for every morphism $\phi : S^{\log} \rightarrow Z^{\log}$ in $\text{Sch}^{\log}(X^{\log})$, where $S^{\log}$ is one-pointed and purely nonarchimedean, there exists a unique morphism $\psi : S^{\log} \rightarrow Y^{\log}$ such that $\phi = \zeta \circ \psi$. 
Proof. Assertion (i) is a formal consequence of Proposition 4.5 [and the definitions of the terms involved]. As for assertion (ii), the necessity of the condition is a formal consequence of the definitions of the terms involved. To prove sufficiency, let us first observe that by [Mzk2], Lemma 2.2; [Mzk2], Proposition 2.3, it follows from this condition that the underlying morphism of log schemes \( Y^{\log} \to Z^{\log} \) is scheme-like [i.e., the log structure on \( Y^{\log} \) is the pull-back of the log structure on \( Z^{\log} \)]. Thus, this condition implies that the underlying morphism of schemes \( Y \to Z \) is smooth [cf. [Mzk2], Corollary 1.2] and surjective. But this implies [cf. [Mzk2], Corollary 1.3] that \( Y \to Z \) is a surjective open immersion, hence that it is an isomorphism of schemes. Since \( Y^{\log} \to Z^{\log} \) is scheme-like, we thus conclude that \( Y^{\log} \to Z^{\log} \) is an isomorphism of log schemes, as desired.

Thus, Corollary 4.1, (ii), implies that the hypothesis of Proposition 4.1 is automatically satisfied. This allows us to conclude the following:

**Corollary 4.2.** (Unconditional Reconstruction of the Archimedean Topological Space) The \( \mathbb{R} \)-superscripted topological space determined by the archimedean structure on an object \( Y^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log})) \) may be reconstructed category-theoretically in a fashion which is functorial in \( Y^{\log} \) [cf. Proposition 4.1, (ii)]. In particular, the condition that \( Y^{\log} \) be purely nonarchimedean is category-theoretic in nature.

**Corollary 4.3.** (Reconstruction of the Underlying Log Scheme) The full subcategory
\[
\text{Sch}^{\log}(Y^{\log}) \subseteq \text{Sch}^{\log}(Y^{\log}) = \text{Sch}^{\log}(X^{\log})_{\text{spec}}
\]
[i.e., consisting of arrows \( Z^{\log} \to Y^{\log} \) for which \( Z^{\log} \) is purely nonarchimedean] associated to an object \( Y^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log})) \) is a category-theoretic invariant of the data \( (\text{Sch}^{\log}(X^{\log}), Y^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log}))) \). In particular, [cf. Theorem 3.1] the underlying log scheme \( Y^{\log} \) associated to \( Y^{\log} \) may be reconstructed category-theoretically from this data in a fashion which is functorial in \( Y^{\log} \).

**Remark 4.** Thus, by Corollary 4.3, one may functorially reconstruct the underlying log scheme \( Y^{\log} \) of an object \( Y^{\log} = (Y^{\log}, K) \in \text{Ob}(\text{Sch}^{\log}(X^{\log})) \), hence the topological space \( Y^{\log}(\mathbb{C}) \) from category-theoretic data. On the other hand, by Corollary 4.2, one may also reconstruct the topological space \( K^{\mathbb{R}} \) (\( \subseteq Y^{\log}(\mathbb{C})^{\mathbb{R}} \)). Thus, the question arises:
Is the reconstruction of $K^R$ via Corollary 4.2 compatible with the reconstruction of $Y^{\log}(C)^R$ via Corollary 4.3?

More precisely, given objects $X_1^{\log}, X_2^{\log} \in \text{Ob}(\text{Sch}^{\log})$; objects
\[ Y_1^{\log} = (Y_1^{\log}, K_1) \in \text{Ob}(\text{Sch}^{\log}(X_1^{\log})); \quad Y_2^{\log} = (Y_2^{\log}, K_2) \in \text{Ob}(\text{Sch}^{\log}(X_2^{\log})) \]
and an equivalence of categories
\[ \Phi : \text{Sch}^{\log}(X_1^{\log}) \sim \rightarrow \text{Sch}^{\log}(X_2^{\log}) \]
such that $\Phi(Y_1^{\log}) = Y_2^{\log}$, we wish to know whether or not the diagram
\[
\begin{array}{ccc}
K_1^R & \sim & K_2^R \\
\downarrow & & \downarrow \\
Y_1^{\log}(C)^R & \sim & Y_2^{\log}(C)^R
\end{array}
\]
— where the vertical morphisms are the natural inclusions; the upper horizontal morphism is the homeomorphism arising from Corollary 4.2; and the lower horizontal morphism is the homeomorphism arising by taking “$C$-valued points” of the isomorphism of log schemes obtained in Corollary 4.3 — commutes. This question will be answered in the affirmative in Lemmas 5.1, 5.2 below.

**Definition 4.4.** In the notation of Remark 4, let us suppose that $X_1^{\log}, Y_1^{\log}$ are fixed. Then:

(i) If the diagram of Remark 4 commutes for all $X_2^{\log}, Y_2^{\log}, \Phi$ as in Remark 4, then we shall say that $Y_1^{\log}$ is (logarithmically) globally compatible.

(ii) If the composite of the diagram of Remark 4 with the commutative diagram
\[
\begin{array}{ccc}
Y_1^{\log}(C)^R & \sim & Y_2^{\log}(C)^R \\
\downarrow & & \downarrow \\
Y_1(C)^R & \sim & Y_2(C)^R
\end{array}
\]
commutes for all $X_2^{\log}, Y_2^{\log}, \Phi$ as in Remark 4, then we shall say that $Y_1^{\log}$ is nonlogarithmically globally compatible.

5. **The Main Theorem**

In the following discussion, we complete the proof of the main theorem of the present paper by showing that the archimedean and scheme-theoretic data reconstructed in Corollaries 4.2, 4.3 are compatible with one another.
Definition 5.1. We shall say that an object $S^{\log}$ of $\text{Sch}^{\log}$ is a test object if its underlying scheme is affine, connected, and normal, and, moreover, the $\mathbb{R}$-superscripted topological space determined by its archimedean structure consists of precisely one point.

Note that by Corollaries 4.2, 4.3, the notion of a “test object” is “category-theoretic”.

Lemma 5.1. (Nonlogarithmic Global Compatibility) Let $X^{\log}$ be an object in $\text{Sch}^{\log}$. Then every object $S^{\log} \in \text{Ob}(\text{Sch}^{\log}(X^{\log}))$ is nonlogarithmically globally compatible.

Proof. By the functoriality of the diagram discussed in Remark 4, it follows immediately that it suffices to prove the nonlogarithmic global compatibility of test objects $S^{\log} = (S^{\log}, H_S)$. Since $S$ is assumed to be affine, write $S = \text{Spec}(R)$. Then we may think of the single point of $H_S^R$ as defining an “archimedean valuation” $v_R$ on the ring $R$.

Write $Y^{\log} = (Y^{\log}, H_Y) \rightarrow S^{\log} = (S^{\log}, H_S)$ for the projective line over $S^{\log}$, equipped with the log structure obtained by pulling back the log structure of $S^{\log}$ and the archimedean structure which is the inverse image of the archimedean structure of $S^{\log}$. Note that this archimedean structure may be characterized “category-theoretically” [cf. Corollaries 4.2, 4.3] as the archimedean structure which yields a quasi-terminal object [cf. §2] in the subcategory of $\text{Sch}^{\log}(S^{\log})$ consisting of purely archimedean morphisms among objects with underlying log scheme isomorphic (over $S^{\log}$) to $Y^{\log}$.

Next, let us observe that to reconstruct the log scheme $S^{\log}$ via Corollary 4.3 amounts, in effect, to applying the theory of [Mzk2]. Moreover, in the theory of [Mzk2], the set underlying the ring $R = \Gamma(S, O_S)$ is reconstructed as the set of sections $S^{\log} \rightarrow Y^{\log}$ that avoid the $\infty$-section (of the projective line $Y$). Moreover, the topology determined on $R$ by the “archimedean valuation” $v_R$ is precisely the topology on this set of sections determined by considering the induced sections $H_S^R \rightarrow H_Y^R$ [i.e., two sections $S^{\log} \rightarrow Y^{\log}$ are “close” if and only if their induced sections $H_S^R \rightarrow H_Y^R$ are “close”]. Thus, we conclude (via Corollary 4.2) that this topology on $R$ is a “category-theoretic invariant”.

On the other hand, it is immediate that the point $R \rightarrow \mathbb{C}$ (considered up to complex conjugation) determined by $H_S^R$ may be recovered from this topology — i.e., by “completing” with respect to this topology. This completes the proof of the asserted nonlogarithmic global compatibility. \qed
Lemma 5.2. (Logarithmic Global Compatibility) Let $\mathcal{X}^{\log}$ be an object in $\text{Sch}^{\log}$. Then every object $\mathcal{S}^{\log} \in \text{Ob}(\text{Sch}^{\log}(\mathcal{X}^{\log}))$ is globally compatible.

Proof. The proof is entirely similar to the proof of Lemma 5.1. In particular, we reduce immediately to the case where $\mathcal{S}^{\log}$ is a test object. Since, by Corollary 4.3, the structure of the underlying log scheme $\mathcal{S}^{\log}$ is already known to be category-theoretic, we may even assume, without loss of generality, that the monoid $M_S$ is generated by its global sections. This time, instead of considering $Y^{\log}$, we consider the object $Z^{\log} = (Z^{log}, H_Z) \rightarrow (\mathcal{S}^{\log}, H_S)$ obtained by “appending” to the log structure of $Y^{\log}$ the log structure determined by the divisor given by the zero section (of the projective line $Y$). As in the case of $Y^{\log}$, we take the archimedean structure on $Z^{\log}$ to be the inverse image of the archimedean structure of $\mathcal{S}^{\log}$. Also, just as in the case of $Y^{\log}$, this archimedean structure may be characterized category-theoretically.

Now if we think of the unique point in $H^S_\mathbb{R}$ as a pair (up to complex conjugation) $(s, \theta)$ [cf. the discussion preceding Definition 4.2], then it remains to show that $\theta$ may be “recovered category-theoretically”. To this end, let us first recall that $s \in S(\mathbb{C})$ determines a morphism $\text{Spec}(\mathbb{C}) \rightarrow S$ with respect to which one may pull-back the log structure on $S$ to obtain a log structure on $\text{Spec}(\mathbb{C})$. By Lemma 5.1, we may also assume, without loss of generality, that $S$ is “sufficiently [Zariski] local with respect to $s$” in the sense that the image of $\Gamma(S, \mathcal{O}_S^S)$ in $\mathbb{C}$ is dense. Moreover, this log structure on $\text{Spec}(\mathbb{C})$ amounts to the datum of a monoid $M_{S,s}$ containing the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$. Thus, relative to this notation, $\theta$ [cf. the discussion preceding Definition 4.2] may be thought of as the datum of a surjective homomorphism

$$\theta : M^\text{gp}_{S,s} \twoheadrightarrow \mathbb{S}^1$$

[where surjectivity follows from the fact that this homomorphism restricts to the identity on $\mathbb{S}^1 \subseteq M^\text{gp}_{S,s}$]. In fact, since $\theta$ is required to restrict to the identity on $\mathbb{S}^1 \subseteq M_{S,s} \subseteq M^\text{gp}_{S,s}$, it follows that the surjection $\theta$ is completely determined by its kernel. Thus, in summary, $\theta$ may be thought of as being the datum of a certain quotient of the group $M^\text{gp}_{S,s}$, or, indeed, as a certain quotient of the monoid $M_{S,s}$.

Next, let us recall [cf. the proof of Lemma 5.1] that in the theory of [Mzk2][cf. the discussion preceding [Mzk2], Lemma 2.16], the set

$$\Gamma(S, M_S)$$
is reconstructed as the set of sections $\mathcal{S}^\log \to \mathbb{Z}^\log$ that avoid the $\infty$-section (of the projective line $\mathbb{Z}$). Observe that [just as in the proof of Lemma 5.1] this set of sections is equipped with a natural topology determined by the induced sections $H^\log_S \to H^\log_\mathcal{S}$ — i.e., two sections $\mathcal{S}^\log \to \mathbb{Z}^\log$ are “close” if and only if their induced sections $H^\log_S \to H^\log_\mathcal{S}$ are “close”. Thus, from the point of view of elements of $\Gamma(S, M_S)$, two elements of $\Gamma(S, M_S)$ are “close” if and only if their images under the composite of the natural morphism $\Gamma(S, M_S) \to M_{S,s}^\gp$ with the surjection $\theta$ are “close”. In particular, if we denote by $\Gamma(S, M_S)^\theta$ the completion of the set $\Gamma(S, M_S)$ with respect to this [not necessarily separated] topology, then it follows immediately [from our assumption that $S$ is “sufficiently Zariski local with respect to $s$”] that the image of $\Gamma(S, \mathcal{O}_S^\log) \subseteq \Gamma(S, M_S)$ in this completion may be identified with $\mathbb{S}^1$. Since, moreover, sequences of elements of $\Gamma(S, M_S)$ that converge to elements of $M_{S,s}$ that lie in the kernel of $\theta$ clearly map to 0 in the completion $\Gamma(S, M_S)^\theta$, we conclude that the closure of the image of $\Gamma(S, \mathcal{O}_S^\log)$ in $\Gamma(S, M_S)^\theta$ [which may be identified with a copy of $\mathbb{S}^1$] is, in fact, equal to $\Gamma(S, M_S)^\theta$, and, moreover, that relative to this identification of $\Gamma(S, M_S)^\theta$ with $\mathbb{S}^1$, the natural completion morphism

$$\Gamma(S, M_S) \to \Gamma(S, M_S)^\theta = \mathbb{S}^1$$

may be identified with the composite of the natural morphism $\Gamma(S, M_S) \to M_{S,s}^\gp$ with $\theta$. That is to say, [in light of our assumption that the monoid $M_S$ is generated by its global sections] the kernel of $\theta$, hence $\theta$ itself, may be recovered from the following data: the log scheme $\mathcal{S}^\log$ [as reconstructed in [Mzk2]], together with the topology considered above on $\Gamma(S, M_S)$. Since this topology is “category-theoretic” by Corollary 4.2, this completes the proof of Lemma 5.2.

We are now ready to state the main result of the present §, i.e., the following global arithmetic analogue of Theorem 3.1:

**Theorem 5.1.** (Categorical Reconstruction of Arithmetic Log Schemes) Let $X^\log$, $Y^\log$ be arithmetic log schemes. Then the categories $\text{Sch}^\log(Y^\log)$, $\text{Sch}^\log(X^\log)$ are slim [cf. §2], and the natural map

$$\text{Isom}(X^\log, Y^\log) \to \text{Isom}(\text{Sch}^\log(Y^\log), \text{Sch}^\log(X^\log))$$

is bijective.

**Proof.** Indeed, this is a formal consequence of Corollaries 4.2, 4.3; Lemma 5.2; [Mzk2], Theorem 2.20.
**Remark 5.** The natural map of Theorem 5.1 is obtained by considering the natural functors mentioned in the discussion following Lemma 4.1.

**Remark 6.** Of course, similar [but easier!] arguments yield the expected versions of Theorem 5.1 for $\text{NSch}^{\log}$, $\text{Sch}$, $\text{NSch}$:

(i) If $\overline{X}^{\log}$, $\overline{Y}^{\log}$ are noetherian arithmetic log schemes, then the categories $\text{NSch}^{\log}(\overline{Y}^{\log})$, $\text{NSch}^{\log}(\overline{X}^{\log})$ are slim, and the natural map

$$\text{Isom}(\overline{X}^{\log}, \overline{Y}^{\log}) \to \text{Isom}(\text{NSch}^{\log}(\overline{Y}^{\log}), \text{NSch}^{\log}(\overline{X}^{\log}))$$

is bijective.

(ii) If $\overline{X}$, $\overline{Y}$ are arithmetic schemes, then the categories $\text{Sch}(\overline{Y})$, $\text{Sch}(\overline{X})$ are slim, and the natural map

$$\text{Isom}(\overline{X}, \overline{Y}) \to \text{Isom}(\text{Sch}(\overline{Y}), \text{Sch}(\overline{X}))$$

is bijective.

(iii) If $\overline{X}$, $\overline{Y}$ are noetherian arithmetic schemes, then the categories $\overline{\text{NSch}}(\overline{Y})$, $\overline{\text{NSch}}(\overline{X})$ are slim, and the natural map

$$\text{Isom}(\overline{X}, \overline{Y}) \to \text{Isom}(\overline{\text{NSch}}(\overline{Y}), \overline{\text{NSch}}(\overline{X}))$$

is bijective.

**Example 5.1.** (Arithmetic Vector Bundles)

(i) Let $F$ be a number field; denote the associated ring of integers by $\mathcal{O}_F$; write $S \equiv \text{Spec}(\mathcal{O}_F)$. Equip $S$ with the archimedean structure given by the whole of $S(\mathbb{C})$; denote the resulting arithmetic scheme by $\overline{S}$. Let $E$ be a vector bundle on $S$. Write $V \to S$ for the result of blowing up the associated geometric vector bundle along its zero section; denote the resulting exceptional divisor [i.e., the inverse image of the zero section via the blow-up morphism] by $D \subseteq V$. If $E$ is equipped with a Hermitian metric at each archimedean prime (up to complex conjugation) of $F$, then, by taking the "archimedean structure" on $V$ to be the complex-valued points of $V$ that correspond to sections of $E$ with norm (relative to this Hermitian metric) $\leq 1$ [hence include the complex-valued points of $D$], we obtain an arithmetic scheme $\overline{V}$ over $\overline{S}$. Now suppose that $S$ is equipped with a log structure defined by some finite set $\Sigma$ of closed points
of $S$; denote the resulting arithmetic log scheme by $S^\log$. Equip $V$ with the log structure obtained by “appending” to the log structure pulled back from $S^\log$ the log structure determined by the divisor $D \subseteq V$. Thus, we obtain a morphism of arithmetic log schemes:

$$V^\log \to S^\log$$

The sections $S^\log \to V^\log$ of this morphism correspond naturally to the elements of $\Gamma(S, E)$ which are nonzero away from $\Sigma$ and have norm $\leq 1$ at all the archimedean primes.

(ii) For $i = 1, 2$, let $V_i^\log \to S_i^\log$ be constructed as in (i) above. Then (by Theorem 5.1) the isomorphism classes of equivalences of categories

$$\text{Sch}^\log(V_1^\log) \simeq \text{Sch}^\log(V_2^\log)$$

correspond naturally to the following data: an isometric isomorphism of vector bundles $E_1 \simeq E_2$ lying over an isomorphism of log schemes $S_1^\log \simeq S_2^\log$.

(iii) We shall refer to a subset $A \subseteq C$ as an angular region if there exists a $\rho \in \mathbb{R}_{>0}$ [where $\mathbb{R}_{>0} \subseteq \mathbb{R}$ is the subset of real numbers $> 0$] and a subset $A_{S^1} \subseteq S^1 \subseteq C$ such that $A = \{ \lambda \cdot u \mid \lambda \in [0, \rho], u \in A_{S^1} \}$. We shall say that the angular region $A$ is open (respectively, closed; isotropic) [i.e., as an angular region] if the subset $A_{S^1} \subseteq S^1$ is open (respectively, closed; equal to $S^1$); we shall refer to $\rho$ as the radius of the angular region $A$. Thus, if we write

$$\text{Ang}(C) \overset{\text{def}}{=} C^\times / \mathbb{R}_{>0}$$

[so the natural composite $S^1 \hookrightarrow C \twoheadrightarrow \text{Ang}(C)$ is a homeomorphism], then the projection

$$\text{Ang}(A) \subseteq \text{Ang}(C)$$

of $A$ [i.e., $A \setminus \{0\}$] to $\text{Ang}(C) \cong S^1$ is simply $A_{S^1}$. Note that the notion of an angular region (respectively, open angular region; closed angular region; $\text{Ang}(-)$; radius of an angular region) extends immediately to the case where “$C$” is replaced by an an arbitrary 1-dimensional complex vector space (respectively, vector space; vector space; vector space; vector space equipped with a Hermitian metric).

In particular, in the notation of (i), when $E$ is a line bundle, the choice of a(n) closed (respectively, open) angular region of radius 1 at each of the complex archimedean primes of $F$ determines a(n) (ind-)archimedean structure [cf. Remark 3] on $V^\log$. Thus, the (ind-)arithmetic log schemes discussed in (i) correspond to the case where all of the angular regions chosen are isotropic.
Remark 7. When the vector bundle $E$ of Example 5.1 is a line bundle [i.e., of rank one], the blow-up used to construct $V$ is an isomorphism. That is to say, in this case, $V$ is simply the geometric line bundle associated to $E$, and $D \subseteq V$ is its zero section.

Remark 8. Some readers may wonder why, in Definition 4.2, we took $H$ to be a compact set, as opposed to, say, an open set (or, perhaps, an open set which is, in some sense, “bounded”). One reason for this is the following: If $H$ were required to be open, then we would be obliged, in Example 5.1, to take the “archimedean structure” on $V$ to be the open set defined by sections of norm $< 1$. In particular, if $E$ is taken to be the trivial line bundle, then it would follow that the section of $V$ defined by the section “1” of the trivial bundle would fail to define a morphism in the “category of arithmetic log schemes” — a situation which the author found to be unacceptable.

Another motivating reason for Definition 4.2 comes from rigid geometry. That is to say, in the context of rigid geometry, perhaps the most basic example of an integral structure on the affine line $\text{Spec}(\mathbb{Q}_p[T])$ is that given by the ring $\mathbb{Z}_p[T]^{\wedge}$ (where the “$\wedge$” denotes $p$-adic completion). Then the continuous homomorphisms $\mathbb{Z}_p[T]^{\wedge} \to \mathbb{C}_p$ [i.e., the “$\mathbb{C}_p$-valued points of the integral structure”] correspond precisely to the elements of $\mathbb{C}_p$ with absolute value $\leq 1$.

Remark 9. If $S \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_F)$ [where $\mathcal{O}_F$ is the ring of integers of a number field $F$], and we equip $S$ with the log structure associated to the chart $\mathbb{N} \ni 1 \mapsto 0 \in \mathcal{O}_S$, then an archimedean structure on $S^{\log}$ is not the same as a choice of Hermitian metrics on the trivial line bundle over $\mathcal{O}_S$ at various archimedean primes of $S$. This is somewhat counter-intuitive, from the point of view of the usual theory of log schemes. More generally:

The definition of an archimedean structure [cf. Definition 4.2] adopted in this paper is perhaps not so satisfactory when one wishes to consider the archimedean aspects of log structures or other infinitesimal deformations (e.g., nilpotent thickenings) in detail.

For instance, the possible choices of an archimedean structure are invariant with respect to nilpotent thickenings. Thus, depending on the situation in which one wishes to apply the theory of the present paper, it may be desirable to modify Definition 4.2 so as to deal with archimedean structures on log structures or nilpotent thickenings in a more satisfactory matter — perhaps by making use of the constructions of Example 5.1 [including “angular regions”!], applied to the
various line bundles or vector bundles that form the log structures or nilpotent thickenings under consideration.

At the time of writing, however, it is not clear to the author how to construct such a theory. Indeed, many of the complications that appear to arise if one is to construct such a theory seem to be related to the fact that archimedean (integral) structures, unlike their nonarchimedean counterparts, typically fail to be closed under addition. Since, however, such a theory is beyond the scope of the present paper, we shall not discuss this issue further in the present paper.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
motizuki@kurims.kyoto-u.ac.jp

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