

COMMENTS ON “A COMBINATORIAL VERSION OF THE GROTHENDIECK CONJECTURE”

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(1.) In the final sentence of Definition 1.1, (ii), the phrase “rank ≥ 2” should read “rank > 2”.

(2.) In Example 2.5, the set of primes Σ should be assumed to be nonempty.

(3.) In the second sentence of Proposition 1.2, the phrase “a vertex v_i (respectively, an edge e_i) of Π_G” should read “a vertex v_i (respectively, an edge e_i) of G”.

(4.) The arguments applied in Definition 1.4, (v), (vi), and Remarks 1.4.2, 1.4.3, and 1.4.4 to prove Theorem 1.6 are formulated in a somewhat confusing way and should be modified as follows:

(i) First of all, we remark that throughout the paper, as well as in the following discussion, a “Galois” finite étale covering is to be understood as being connected.

(ii) In the second sentence of Definition 1.4, (v), the cuspidal and nodal cases of the notion of a purely totally ramified covering are in fact unnecessary and may be deleted. Also, the terminology introduced in Definition 1.4, (vi), concerning finite étale coverings that descend is unnecessary and may be deleted.

(iii) The text of Remark 1.4.2 should be replaced by the following text:

Let \( \mathcal{G}' \rightarrow \mathcal{G} \) be a Galois finite étale covering of degree a positive power of \( l \), where \( \mathcal{G} \) is of pro-Σ PSC-type, \( Σ = \{ l \} \). Then one verifies immediately that, if we assume further that the covering \( \mathcal{G}' \rightarrow \mathcal{G} \) is cyclic, then \( \mathcal{G}' \rightarrow \mathcal{G} \) is cuspidally totally ramified if and only if the inequality

\[
 r(\mathcal{G}'') < l \cdot r(\mathcal{G})
\]

— where we write \( \mathcal{G}' \rightarrow \mathcal{G}'' \rightarrow \mathcal{G} \) for the unique [up to isomorphism] factorization of the finite étale covering \( \mathcal{G}' \rightarrow \mathcal{G} \) as a composite of finite étale coverings such that \( \mathcal{G}'' \rightarrow \mathcal{G} \) is of degree \( l \) — is satisfied. Suppose further that \( \mathcal{G}' \rightarrow \mathcal{G} \) is a [not necessarily cyclic!] \( \Pi_{\mathcal{G}}^{unr} \)-covering [so \( n(\mathcal{G}') = \)]
Then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is \textit{vertically purely totally ramified} if and only if the equality

$$i(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (i(\mathcal{G}) - 1) + 1$$

is satisfied. Also, we observe that this last inequality is equivalent to the following equality involving the expression “$i(\ldots) - n(\ldots)$” [cf. Remark 1.1.3]:

$$i(\mathcal{G}') - n(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (i(\mathcal{G}) - n(\mathcal{G}) - 1) + 1$$

(iv) The text of Remark 1.4.3 should be replaced by the following text:

Suppose that $\mathcal{G}$ is of pro-\Sigma PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that the cuspidal edge-like subgroups of $\Pi_{\mathcal{G}}$ may be characterized as the maximal [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following condition:

for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''}$, then the cyclic finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is cuspidally totally ramified.

[Indeed, the necessity of this characterization is immediate from the definitions; the sufficiency of this characterization follows by observing that since the set of cusps of a finite étale covering of $\mathcal{G}$ is always finite, the above condition implies that there exists a compatible system of cusps of the various $\mathcal{G}'$ that arise, each of which is stabilized by the action of $A$.] On the other hand, in order to characterize the unramified vertical subgroups of $\Pi_{\mathcal{G}}^{\unr}$, it suffices — by considering stabilizers of vertices of underlying semi-graphs of finite étale $\Pi_{\mathcal{G}}^{\unr}$-coverings of $\mathcal{G}$ — to give a functorial characterization of the set of vertices of $\mathcal{G}$ [i.e., which may also be applied to finite étale $\Pi_{\mathcal{G}}^{\unr}$-coverings of $\mathcal{G}$]. This may be done, for sturdy $\mathcal{G}$, as follows. Write $M_{\mathcal{G}}^{\unr}$ for the abelianization of $\Pi_{\mathcal{G}}^{\unr}$. For each vertex $v$ of the underlying semi-graph $\mathcal{G}$ of $\mathcal{G}$, write $M_{\mathcal{G}}^{\unr}[v] \subseteq M_{\mathcal{G}}^{\unr}$ for the image of the $\Pi_{\mathcal{G}}^{\unr}$-conjugacy class of unramified vertical subgroups of $\Pi_{\mathcal{G}}^{\unr}$ associated to $v$. Then one verifies immediately, by constructing suitable abelian $\Pi_{\mathcal{G}}^{\unr}$-coverings of $\mathcal{G}$ via suitable gluing operations [i.e., as in the proof of Proposition 1.2], that the inclusions $M_{\mathcal{G}}^{\unr}[v] \subseteq M_{\mathcal{G}}^{\unr}$ determine a split injection

$$\bigoplus_v M_{\mathcal{G}}^{\unr}[v] \hookrightarrow M_{\mathcal{G}}^{\unr}$$

[where $v$ ranges over the vertices of $\mathcal{G}$], whose image we denote by $M_{\mathcal{G}}^{\unr,\text{vert}} \subseteq M_{\mathcal{G}}^{\unr}$. Now we consider elementary abelian quotients

$$\phi: M_{\mathcal{G}}^{\unr} \twoheadrightarrow Q$$

— i.e., where $Q$ is an elementary abelian group. We identify such quotients whenever their kernels coincide and order such quotients by means of the
relation of "domination" [i.e., inclusion of kernels]. Then one verifies immediately that such a quotient \( \phi : \text{M}^{\text{unr}}_G \rightarrow Q \) corresponds to a \textit{vertically purely totally ramified} covering of \( G \) if and only if there exists a vertex \( v \) of \( G \) such that \( \phi(\text{M}^{\text{unr}}_G[v]) = Q, \phi(\text{M}^{\text{unr}}_G[v']) = 0 \) for all vertices \( v' \neq v \) of \( G \). In particular, one concludes immediately that

the elementary abelian quotients \( \phi : \text{M}^{\text{unr}}_G \rightarrow Q \) whose restriction to \( \text{M}^{\text{unr-vert}}_G \) surjects onto \( Q \) and has the same kernel as the quotient

\[
\text{M}^{\text{unr-vert}}_G \rightarrow \text{M}^{\text{unr}}_G[v] \rightarrow \text{M}^{\text{unr}}_G[v] \otimes F_l
\]

— where the first \( \rightarrow \) is the natural projection; the second \( \rightarrow \) is given by reduction modulo \( l \) — may be characterized as the \textit{maximal quotients} [i.e., relative to the relation of domination] among those elementary abelian quotients of \( \text{M}^{\text{unr}}_G \) that correspond to \textit{vertically purely totally ramified} coverings of \( G \).

Thus, since \( G \) is \textit{sturdy}, the set of vertices of \( G \) may be characterized as the set of \textit{nontrivial} quotients \( \text{M}^{\text{unr-vert}}_G \rightarrow \text{M}^{\text{unr}}_G[v] \otimes F_l \).

(v) The text of Remark 1.4.4 should be replaced by the following text:

Suppose that \( G \) is of pro-\( \Sigma \) PSC-type, where \( \Sigma = \{l\} \), and that \( G \) is \textit{noncuspidal}. Then, in the spirit of the \textit{cuspidal} portion of Remark 1.4.3, we observe the following: One verifies immediately that the \textit{nodal edge-like subgroups} of \( \Pi_G \) may be \textit{characterized} as the \textit{maximal} [cf. Proposition 1.2, (i)] closed subgroups \( A \subseteq \Pi_G \) isomorphic to \( \mathbb{Z}_l \) which satisfy the following condition:

for every characteristic open subgroup \( \Pi_{G'} \subseteq \Pi_G \), if we write \( G' \rightarrow G'' \rightarrow G \) for the finite \( \acute{e}tale \) coverings corresponding to \( \Pi_{G'} \subseteq \Pi_{G''} \), then the \textit{cyclic} finite \( \acute{e}tale \) covering \( G' \rightarrow G'' \) is \textit{nodally totally ramified}.

Here, we note further that [one verifies immediately that] the finite \( \acute{e}tale \) covering \( G' \rightarrow G'' \) is \textit{nodally totally ramified} if and only if it is \textit{module-wise nodal}.

(vi) The text of the \textit{second paragraph} of the proof of Theorem 1.6 should be replaced by the following text [which may be thought as being appended to the end of the \textit{first paragraph} of the proof of Theorem 1.6]:

Then the fact that \( \alpha \) is \textit{group-theoretically cuspidal} follows formally from the characterization of \textit{cuspidal edge-like subgroups} given in Remark 1.4.3 and the characterization of \textit{cuspidally totally ramified cyclic} finite \( \acute{e}tale \) coverings given in Remark 1.4.2.

(vii) The text of the \textit{final paragraph} of the proof of Theorem 1.6 should be replaced by the following text [which may be thought of as a sort of "easy version"]:
of the argument given in the proof of the implication “(iii) $\implies$ (i)” of [CbTpII], Proposition 1.5:

Finally, we consider assertion (iii). Sufficiency is immediate. On the other hand, necessity follows formally from the characterization of unramified vertical subgroups given in Remark 1.4.3 and the characterization of vertically purely totally ramified cyclic finite étale coverings given in Remark 1.4.2.

(5.) In Remarks 2.8.1, 2.8.2, one works “in the situation of Corollary 2.8”, despite the fact that the assumption “$p \not\in \Sigma$” in the statement of Corollary 2.8 is not satisfied in the situations considered in Remarks 2.8.1, 2.8.2. At first glance, this state of affairs may strike the reader as self-contradictory. The point, however, is that one thinks of Corollary 2.8 as being applied to the various maximal pro-$l$ quotients of open subgroups of the geometric fundamental groups that appear in Remark 2.8.1, 2.8.2, i.e., that one takes the “$\Sigma$” of Corollary 2.8 to be $\{l\}$ for a suitable prime number $l$ such that the assumptions in the statement of Corollary 2.8 are indeed satisfied.

Bibliography