# COMMENTS ON "A COMBINATORIAL VERSION OF THE GROTHENDIECK CONJECTURE"

Shinichi Mochizuki

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(1.) In the final sentence of Definition 1.1, (ii), the phrase "rank $\geq 2$" should read "rank $> 2$".

(2.) In Example 2.5, the set of primes $\Sigma$ should be assumed to be *nonempty*.

(3.) In the *second sentence* of Proposition 1.2, the phrase "*a vertex* $v_i$ *(respectively, an edge* $e_i$*) of* $\Pi_{\mathcal{G}}$" should read "*a vertex* $v_i$ *(respectively, an edge* $e_i$*) of* $\mathbb{G}$".

(4.) The arguments applied in Definition 1.4, (v), (vi), and Remarks 1.4.2, 1.4.3, and 1.4.4 to prove Theorem 1.6 are formulated in a somewhat confusing way and should be modified as follows:

(i) First of all, we remark that throughout the paper, as well as in the following discussion, a *"Galois"* finite étale covering is to be understood as being *connected*.

(ii) In the *second sentence* of Definition 1.4, (v), the *cuspidal* and *nodal* cases of the notion of a *purely totally ramified* covering are in fact *unnecessary* and *may be deleted*. Also, the terminology introduced in Definition 1.4, (vi), concerning finite étale coverings that *descend* is *unnecessary* and *may be deleted*.

(iii) The text of Remark 1.4.2 should be replaced by the following text:

> Let $\mathcal{G}' \to \mathcal{G}$ be a Galois finite étale covering of degree a *positive power of* $l$, where $\mathcal{G}$ is of pro-$\Sigma$ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that, if we assume further that the covering $\mathcal{G}' \to \mathcal{G}$ is *cyclic*, then $\mathcal{G}' \to \mathcal{G}$ is *cuspidally totally ramified* if and only if the inequality
>
> $$\underline{r}(\mathcal{G}'') < l \cdot \underline{r}(\mathcal{G})$$
>
> — where we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the *unique* [up to isomorphism] *factorization* of the finite étale covering $\mathcal{G}' \to \mathcal{G}$ as a composite of finite étale coverings such that $\mathcal{G}'' \to \mathcal{G}$ is of degree $l$ — is satisfied. Suppose further that $\mathcal{G}' \to \mathcal{G}$ is a [*not necessarily cyclic*!] $\Pi_{\mathcal{G}}^{\mathrm{unr}}$*-covering* [so $\underline{n}(\mathcal{G}') =$

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$\deg(\mathcal{G}'/\mathcal{G}) \cdot \underline{n}(\mathcal{G})]$. Then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is *verticially purely totally ramified* if and only if the equality

$$\underline{i}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - 1) + 1$$

is satisfied. Also, we observe that this last inequality is equivalent to the following equality involving the expression "$\underline{i}(\dots) - \underline{n}(\dots)$" [cf. Remark 1.1.3]:

$$\underline{i}(\mathcal{G}') - \underline{n}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - \underline{n}(\mathcal{G}) - 1) + 1$$

(iv) The text of Remark 1.4.3 should be replaced by the following text:

Suppose that $\mathcal{G}$ is of pro-$\Sigma$ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that the *cuspidal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:

> for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \stackrel{\mathrm{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *cuspidally totally ramified.*

[Indeed, the *necessity* of this *characterization* is immediate from the definitions; the *sufficiency* of this *characterization* follows by observing that since the set of cusps of a finite étale covering of $\mathcal{G}$ is always *finite*, the above *condition* implies that there exists a *compatible system of cusps* of the various $\mathcal{G}'$ that arise, each of which is *stabilized* by the action of $A$.] On the other hand, in order to characterize the *unramified verticial subgroups* of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$, it suffices — by considering *stabilizers* of *vertices* of underlying semi-graphs of finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-coverings of $\mathcal{G}$ — to give a *functorial characterization* of the *set of vertices* of $\mathcal{G}$ [i.e., which may also be applied to finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-coverings of $\mathcal{G}$]. This may be done, for *sturdy* $\mathcal{G}$, as follows. Write $M_{\mathcal{G}}^{\mathrm{unr}}$ for the *abelianization* of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$. For each vertex $v$ of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$, write $M_{\mathcal{G}}^{\mathrm{unr}}[v] \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$ for the image of the $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-conjugacy class of unramified verticial subgroups of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$ associated to $v$. Then one verifies immediately, by *constructing suitable abelian* $\Pi_{\mathcal{G}}^{\mathrm{unr}}$*-coverings* of $\mathcal{G}$ via suitable *gluing* operations [i.e., as in the proof of Proposition 1.2], that the inclusions $M_{\mathcal{G}}^{\mathrm{unr}}[v] \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$ determine a *split injection*

$$\bigoplus_v M_{\mathcal{G}}^{\mathrm{unr}}[v] \hookrightarrow M_{\mathcal{G}}^{\mathrm{unr}}$$

[where $v$ ranges over the vertices of $\mathbb{G}$], whose image we denote by $M_{\mathcal{G}}^{\text{unr-vert}} \subseteq M_{\mathcal{G}}^{\mathrm{unr}}$. Now we consider *elementary abelian quotients*

$$\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$$

— i.e., where $Q$ is an *elementary abelian group.* We *identify* such quotients whenever their *kernels coincide* and *order* such quotients by means of the relation of *"domination"* [i.e., inclusion of kernels]. Then one verifies immediately that such a quotient $\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$ corresponds to a *verticially purely totally ramified* covering of $\mathcal{G}$ if and only if there exists a vertex $v$ of $\mathbb{G}$ such that $\phi(M_{\mathcal{G}}^{\mathrm{unr}}[v]) = Q$, $\phi(M_{\mathcal{G}}^{\mathrm{unr}}[v']) = 0$ for all vertices $v' \neq v$ of $\mathbb{G}$. In particular, one concludes immediately that

> the elementary abelian quotients $\phi : M_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow Q$ whose restriction to $M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}}$ surjects onto $Q$ and has the same kernel as the quotient
>
> $$M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}} \twoheadrightarrow M_{\mathcal{G}}^{\mathrm{unr}}[v] \twoheadrightarrow M_{\mathcal{G}}^{\mathrm{unr}}[v] \otimes \mathbb{F}_l$$
>
> — where the first "$\twoheadrightarrow$" is the natural projection; the second "$\twoheadrightarrow$" is given by reduction modulo $l$ — may be characterized as the *maximal quotients* [i.e., relative to the relation of domination] among those elementary abelian quotients of $M_{\mathcal{G}}^{\mathrm{unr}}$ that correspond to *verticially purely totally ramified* coverings of $\mathcal{G}$.

Thus, since $\mathcal{G}$ is *sturdy*, the *set of vertices* of $\mathcal{G}$ may be characterized as the *set of [nontrivial!] quotients* $M_{\mathcal{G}}^{\mathrm{unr\text{-}vert}} \twoheadrightarrow M_{\mathcal{G}}^{\mathrm{unr}}[v] \otimes \mathbb{F}_l$.

(v) The text of Remark 1.4.4 should be replaced by the following text:

Suppose that $\mathcal{G}$ is of pro-$\Sigma$ PSC-type, where $\Sigma = \{l\}$, and that $\mathcal{G}$ is *noncuspidal*. Then, in the spirit of the *cuspidal* portion of Remark 1.4.3, we observe the following: One verifies immediately that the *nodal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:

> for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \stackrel{\mathrm{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified*.

Here, we note further that [one verifies immediately that] the finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified* if and only if it is *module-wise nodal*.

(vi) The text of the *second paragraph* of the proof of Theorem 1.6 should be replaced by the following text [which may be thought as being appended to the end of the *first paragraph* of the proof of Theorem 1.6]:

Then the fact that $\alpha$ is *group-theoretically cuspidal* follows formally from the characterization of *cuspidal edge-like subgroups* given in Remark 1.4.3 and the characterization of *cuspidally totally ramified* cyclic finite étale coverings given in Remark 1.4.2.

(vii) The text of the *final paragraph* of the proof of Theorem 1.6 should be replaced by the following text [which may be thought of as a sort of *"easy version"* of the argument given in the proof of the implication "(iii) $\Longrightarrow$ (i)" of [CbTpII], Proposition 1.5]:

> Finally, we consider assertion (iii). *Sufficiency* is immediate. On the other hand, *necessity* follows formally from the characterization of *unramified verticial subgroups* given in Remark 1.4.3 and the characterization of *verticially purely totally ramified* cyclic finite étale coverings given in Remark 1.4.2.

## Bibliography

[CbTpII] Y. Hoshi, S. Mochizuki, *Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization*, RIMS Preprint **1762** (November 2012).