COMMENTS ON A COMBINATORIAL VERSION OF THE SECTION CONJECTURE AND THE MAIN THEOREM OF POP-STIX

Shinichi Mochizuki

Updated March 3, 2011

In the "Comments on the Main Theorem of Pop-Stix" dated November 15, 2010 (i.e., [PSCom]), we discussed a remark of Y. André to the effect that this theorem of Pop-Stix allows one to reduce the *Profinite p-adic Section Conjecture* to its *tempered* counterpart. In the present note, we observe that this *reduction* may also be obtained as a consequence of an entirely *elementary* observation concerning *actions of finite groups on a finite graph*, hence, in particular, without resorting to the use of *highly nontrivial arithmetic* results such as Tamagawa's *"resolution of nonsingularities"* [i.e., the main result of [Tama]].

(1) Let Σ be a nonempty *set of prime numbers*, G a *finite cyclic group* of order a *power of a prime* $\in \Sigma$, Γ a *finite graph* equipped with an *action by* G. Suppose, for simplicity, that the action of G on Γ *does not switch the branches* of any edge of Γ. Observe that one may form a *quotient graph*

$$\Gamma/G$$

whose *vertices* are precisely the G-orbits of vertices of Γ, and whose *edges* are precisely the G-orbits of edges of Γ; moreover, one has a natural morphism $\Gamma \to \Gamma/G$. On the other hand, one may also form a *"quotient orbigraph"*

$$\Gamma//G$$

— i.e., a quotient of Γ by G "in the sense of stacks". In the present discussion, we shall only be interested in the *pro-*Σ *fundamental group* [relative to a suitable basepoint]

$$\pi_1^{\Sigma}(\Gamma//G)$$

of $\Gamma//G$. To define this profinite group, it suffices to describe the connected finite étale Galois coverings of $\Gamma//G$. [That is to say, then arbitrary finite étale coverings of $\Gamma//G$ may be described as coproducts of subcoverings of such connected finite étale Galois coverings.] A connected finite étale Galois covering of $\Gamma//G$ consists of a connected finite étale Galois covering $\Gamma^* \to \Gamma$ equipped with the action of a finite group G^* and an augmentation [i.e., a surjective homomorphism] $\epsilon^* : G^* \twoheadrightarrow G$ such

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX that the action of $N^* \stackrel{\text{def}}{=} \text{Ker}(\epsilon^*)$ on Γ^* induces an isomorphism $N^* \xrightarrow{\sim} \text{Gal}(\Gamma^*/\Gamma)$, and the induced action of $G^*/N^* \xrightarrow{\sim} G$ on Γ is compatible with the original action of G on Γ. Thus, we have a *natural exact sequence*

$$1 \rightarrow \pi_1^{\Sigma}(\Gamma) \rightarrow \pi_1^{\Sigma}(\Gamma//G) \rightarrow G \rightarrow 1$$

of profinite groups. Now let us observe the following elementary graph-theoretic assertion — which may be thought of as a sort of *"Combinatorial Section Conjecture"*:

($*_{\text{CSC}}$) Suppose that the natural surjection $\Pi_G \stackrel{\text{def}}{=} \pi_1^{\Sigma}(\Gamma//G) \twoheadrightarrow G$ admits a *section* $\sigma : G \rightarrow \Pi_G$. Then there exists a vertex v of Γ that is *fixed* by G.

Indeed, we may assume without loss of generality that the action of G on Γ is *faithful*. Let us first consider the case where the order of G is *prime*. In this case, if ($*_{\text{CSC}}$) is *false*, then it follows that the action of G on Γ is *free*, hence that the natural morphism $\Gamma \rightarrow \Gamma/G$ is a *finite étale covering*, so $\Gamma//G$ may be *identified* with Γ/G; but since Γ/G is itself a graph, it follows that $\pi_1^{\Sigma}(\Gamma/G) \xrightarrow{\sim} \pi_1^{\Sigma}(\Gamma//G)$ is a *free pro-Σ group*, hence *torsion-free*, in contradiction to the existence of the section σ. Next, we consider the case of G of *arbitrary non-prime order*. Let $H \subseteq G$ be the unique subgroup such that $Q \stackrel{\text{def}}{=} G/H$ is of prime order. Write $\Gamma_Q \stackrel{\text{def}}{=} \Gamma/H$. Then we have a diagram of graphs and group actions

$$\begin{array}{ccc} G \curvearrowright & & Q \curvearrowright \\ \Gamma & \longrightarrow & \Gamma_Q \end{array}$$

which induces an outer homomorphism $\Pi_G \rightarrow \Pi_Q \stackrel{\text{def}}{=} \pi_1^{\Sigma}(\Gamma_Q//Q)$, whose kernel we denote by N. The restriction of this outer homomorphism to $\sigma(H) \subseteq \Pi_G$ determines an outer homomorphism $\sigma(H) \rightarrow \pi_1^{\Sigma}(\Gamma_Q)$. Since $\pi_1^{\Sigma}(\Gamma_Q)$ is a free pro-Σ group, hence torsion-free, we thus conclude that this homomorphism $\sigma(H) \rightarrow \pi_1^{\Sigma}(\Gamma_Q)$ is *trivial*, hence that σ determines a section $\sigma_Q : Q \rightarrow \Pi_Q$ of the natural surjection $\Pi_Q \twoheadrightarrow Q$. In particular, by applying ($*_{\text{CSC}}$) in the case of Q [which has already been verified], we thus conclude that there exists a vertex v_Q of Γ_Q that is *fixed* by the action of Q. Let v be a vertex of Γ that lifts v_Q, $g \in G$ a generator of G. Then since Q fixes v_Q, it follows that $v^g = v^h$, for some $h \in H$, hence that v is *fixed* by $g \cdot h^{-1} \in G$. On the other hand, since $g \cdot h^{-1}$ generates G, we thus conclude that v is fixed by G. This completes the proof of ($*_{\text{CSC}}$).

(2) Let Σ be a nonempty *set of primes*, $l \in \Sigma$, k a perfect field of characteristic $\neq l$, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$. Then we shall say that k is *l-cyclotomically full* if the image of the l-adic cyclotomic character $G_k \rightarrow \mathbb{Z}_l^{\times}$ is *open* in $\mathbb{Z}_l^{\times}$. Write $T^{\log}$ for the log scheme obtained by equipping $T \stackrel{\text{def}}{=} \text{Spec}(k)$ with the log structure determined by the chart $\mathbb{N} \ni 1 \mapsto 0$. Let $Z^{\log} \rightarrow T^{\log}$ be a *stable log curve* [cf., e.g., [NodNon], §0], $X^{\log}$ a *connected covering* of $Z^{\log}$ that arises from the *logarithmic fundamental group* of $Z^{\log}$, and $S^{\log} \rightarrow T^{\log}$ the resulting covering of $T^{\log}$. Here, we assume further, to simplify notation, that the underlying morphism of schemes $S \rightarrow T$ is an *isomorphism*. Write $\Pi_{X/k}$, $\Pi_{S/k}$ for the respective *maximal*

pro-l quotients of the *logarithmic fundamental groups* $\pi_1(X^{\log} \times_k \overline{k})$, $\pi_1(S^{\log} \times_k \overline{k})$ [relative to suitable basepoints]. Write Π_X, Π_S for the respective quotients of the *logarithmic fundamental groups* $\pi_1(X^{\log})$, $\pi_1(S^{\log})$ [relative to suitable basepoints] of the natural surjections $\pi_1(X^{\log} \times_k \overline{k}) \twoheadrightarrow \Pi_{X/k}$, $\pi_1(S^{\log} \times_k \overline{k}) \twoheadrightarrow \Pi_{S/k}$. Thus, we have *natural exact sequences of profinite groups*

$$1 \to \Pi_{S/k} \to \Pi_S \to G_k \to 1; \quad 1 \to \Pi_{X/k} \to \Pi_X \to G_k \to 1$$

$$1 \to \Delta_{X/S} \to \Pi_{X/k} \to \Pi_{S/k} \to 1; \quad 1 \to \Delta_{X/S} \to \Pi_X \to \Pi_S \to 1$$

— where $\Delta_{X/S}$ is defined so as to render the final two sequences exact. Write Γ_X for the *dual graph* of $X^{\log} \times_k \overline{k}$. Suppose, for simplicity, that the natural action of $\pi(S^{\log})$ on Γ_X *factors* through the quotient $\pi(S^{\log}) \twoheadrightarrow \Pi_S$ and, moreover, *does not switch the branches* of any edge of Γ_X. Write $\widetilde{\Gamma}_X$ for the *pro-graph* determined by the profinite universal covering $\widetilde{X} \to X$ of X corresponding to Π_X. Then let us observe the following consequence — which may be thought of as a sort of *"Log-scheme-theoretic Section Conjecture"* — of the *purely combinatorial result* of (1):

($*_{\mathrm{LSC}}$) Suppose that k is *l-cyclotomically full*, and that the natural surjection $\Pi_X \twoheadrightarrow \Pi_S$ admits a *section* $\sigma : \Pi_S \to \Pi_X$. Then there exists a vertex $\tilde{v}$ of $\widetilde{\Gamma}_X$ such that, if we write $D_{\tilde{v}} \subseteq \Pi_X$ for the *decomposition group* associated to $\tilde{v}$, then $\sigma(\Pi_S) \subseteq D_{\tilde{v}}$. Finally, the collection of possibilities for $\tilde{v}$ determines *"star"* in $\widetilde{\Gamma}_X$, i.e., a [connected] tree in $\widetilde{\Gamma}_X$ that admits a vertex $\tilde{v}_*$ such that every vertex that of the tree is connected to $\tilde{v}_*$ by a path of length ≤ 1.

Indeed, write Δ_Γ for the *maximal pro-l quotient* of $\pi_1^{\Sigma}(\Gamma_X)$, where we take $\Sigma \stackrel{\mathrm{def}}{=} \{l\}$. By replacing $X^{\log}$ by an appropriate covering arising from Π_X, one verifies immediately that one may assume without loss of generality that Δ_Γ is *nonabelian.* The natural action of Π_S on Γ_X factors through a *finite quotient* $\Pi_S \twoheadrightarrow Q_S$. Thus, one obtains a *natural outer action* of Q_S on Δ_Γ, hence a profinite group Π_Γ that fits into a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \Pi_X & \longrightarrow & \Pi_S & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_\Gamma & \longrightarrow & \Pi_\Gamma & \longrightarrow & Q_S & \longrightarrow & 1 \end{array}$$

in which the vertical arrows are surjections. Write $\Pi_{S/k} \twoheadrightarrow Q_{S/k}$, $G_k \twoheadrightarrow Q_k$ for the *natural surjections* determined by the natural surjection $\Pi_S \twoheadrightarrow Q_S$. Thus, these natural surjections determine an *exact sequence of quotients* $1 \to Q_{S/k} \to Q_S \to Q_k \to 1$, as well as a corresponding *exact sequence of kernels* $1 \to N_{S/k} \to N_S \to N_k \to 1$, hence, in particular, a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_{S/k} & \longrightarrow & N_S & \longrightarrow & N_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{S/k} & \longrightarrow & \Pi_S & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Q_{S/k} & \longrightarrow & Q_S & \longrightarrow & Q_k & \longrightarrow & 1 \end{array}$$

— where we observe that $N_{S/k}$ is an open subgroup of $\Pi_{S/k}$, hence noncanonically isomorphic to $\mathbb{Z}_l$. Thus, if we set $\Pi_{\Gamma/k} \stackrel{\mathrm{def}}{=} \Pi_\Gamma \times_{Q_S} Q_{S/k}$, then we obtain an exact sequence

$$1 \to \Delta_\Gamma \to \Pi_{\Gamma/k} \to Q_{S/k} \to 1$$

as well as compatible *natural surjections* $\Pi_{X/k} \twoheadrightarrow \Pi_{\Gamma/k}$, $\Pi_X \twoheadrightarrow \Pi_\Gamma$. Next, let us consider the homomorphisms

$$N_{S/k} \to \Delta_\Gamma \ (\subseteq \Pi_{\Gamma/k}); \quad N_S \to \Pi_\Gamma$$

obtained by composing these natural surjections with the restriction of σ to $N_{S/k} \subseteq N_S \subseteq \Pi_{S/k} \subseteq \Pi_S$. Thus, any section $\tau : N_k \to N_S$ of the natural surjection $N_S \twoheadrightarrow N_k$ determines *compatible actions* of N_k on $N_{S/k}$ and Δ_Γ. Now let us observe that $N_k \subseteq G_k$ acts on $N_{S/k}$ via the *l-adic cyclotomic character*, while the action of N_k on Δ_Γ is *trivial*. Thus, it follows immediately from our assumption that k is *l-cyclotomically full* that the above homomorphism $N_{S/k} \to \Delta_\Gamma$ is *trivial*. In particular, the homomorphism $\Pi_{S/k} \to \Pi_{\Gamma/k}$ determined by composing the restriction of σ to $\Pi_{S/k} \subseteq \Pi_S$ with the natural surjection $\Pi_{X/k} \twoheadrightarrow \Pi_{\Gamma/k}$ *factors* through the quotient $\Pi_{S/k} \twoheadrightarrow Q_{S/k}$, hence determines a *section* $Q_{S/k} \to \Pi_{\Gamma/k}$ of the natural surjection $\Pi_{\Gamma/k} \twoheadrightarrow Q_{S/k}$. Thus, by applying the observation $(*_{\mathrm{CSC}})$ of (1) above, we *conclude* that the natural action of $\Pi_{S/k}$ [i.e., of $Q_{S/k}$] on Γ_X *fixes* some vertex of Γ_X. By applying this *conclusion* [obtained in the case of Π_X] to the various *open subgroups* of Π_X, we thus obtain that there exists a *vertex* $\tilde{v}$ of $\widetilde{\Gamma}_X$ such that, if we write $D_{\tilde{v}} \subseteq \Pi_X$ for the *decomposition group* associated to $\tilde{v}$, then $\sigma(\Pi_{S/k}) \subseteq D_{\tilde{v}}$. Moreover, by [NodNon], Proposition 3.9, (i), it follows immediately that the collection of possibilities for $\tilde{v}$ determines a [connected] *tree* in $\widetilde{\Gamma}_X$ such that any two vertices of the tree are connected by a path of length ≤ 2. One verifies immediately that such a tree is necessarily a *"star"*. Since, by assumption, the action of $\sigma(\Pi_S)$ on $\widetilde{\Gamma}_X$ *does not switch the branches* of any edge of $\widetilde{\Gamma}_X$, it follows immediately that this star admits at least one vertex *fixed* by the action of $\sigma(\Pi_S)$. In particular, one may choose $\tilde{v}$ such that $\sigma(\Pi_S) \subseteq D_{\tilde{v}}$. This completes the proof of $(*_{\mathrm{LSC}})$.

(3) It is not difficult to verify that the observation $(*_{\mathrm{LSC}})$ of (2) generalizes immediately, for Σ an arbitrary nonempty set of primes [i.e., not necessarily of cardinality one], to the case of the *geometrically pro-Σ fundamental groups* associated to *arbitrary nodally nondegenerate outer representations*, i.e., that do not necessarily arise from a stable log curve over a log point as in (2) [cf. the theory of [NodNon]]. We leave the routine details to the reader.

(4) Now let us consider the situation discussed in [PSCom], (7). That is to say, let k be an *arbitrary complete discrete valuation field of mixed characteristic* whose residue characteristic we denote by p, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, Σ a *set of primes* that contains a prime $l \neq p$, X a *proper hyperbolic curve* over k. Write

$$\Pi_X \twoheadrightarrow \Pi_X^{(\Sigma)}$$

for the *geometrically pro-Σ quotient* of Π_X and

$$\Pi_X^{\mathrm{tp},(\Sigma)} \subseteq \Pi_X^{(\Sigma)}$$

for the *"Σ-tempered fundamental group"*, i.e., the image of the tempered fundamental group Π_X^{tp} of X in $\Pi_X^{(\Sigma)}$. Thus, we have natural surjections $\Pi_X^{(\Sigma)} \twoheadrightarrow G_k$, $\Pi_X^{\mathrm{tp},(\Sigma)} \twoheadrightarrow G_k$. Then the discussion of (2) has the following immediate consequence:

($*_{\mathrm{PTS}}$) The natural map

$$\mathrm{Sect}(\Pi_X^{\mathrm{tp},(\Sigma)}/G_k) \ \rightarrow \ \mathrm{Sect}(\Pi_X^{(\Sigma)}/G_k)$$

— i.e., from $\Pi_X^{\mathrm{tp},(\Sigma)}$-conjugacy classes of sections of $\Pi_X^{\mathrm{tp},(\Sigma)} \rightarrow G_k$ to $\Pi_X^{(\Sigma)}$-conjugacy classes of sections of $\Pi_X^{(\Sigma)} \rightarrow G_k$ — is *injective*. If, moreover, k is *l-cyclotomically full* for some $l \in \Sigma$ that is $\neq p$, then this natural map is *bijective*.

Indeed, the proof of the asserted *injectivity* is discussed in [PSCom], (7). On the other hand, the asserted *surjectivity* is an immediate consequence of ($*_{\mathrm{LSC}}$) [i.e., applied to the special fibers of stable models of finite étale coverings of X].

(5) Thus, in the situation of (4) [cf. also the discussion of [PSCom], (2)], if one is given a cofinal system of finite étale connected Galois coverings of X with stable reduction

$$\ldots \ \rightarrow \ X_{i+1} \ \rightarrow \ X_i \ \rightarrow \ \ldots$$

[where i ranges over the positive integers] and a *section* $s : G_k \rightarrow \Pi_X^{(\Sigma)}$ of $\Pi_X^{(\Sigma)} \twoheadrightarrow G_k$, then, by applying ($*_{\mathrm{LSC}}$) to the special fibers of stable models of the X_i, one concludes that [after possibly passing to a cofinal subsystem] there exists *either* a [not necessarily unique] *system of vertices*

$$\ldots \ \rightsquigarrow \ v_{i+1} \ \rightsquigarrow \ v_i \ \rightsquigarrow \ \ldots$$

or a [not necessarily unique] *system of edges*

$$\ldots \ \rightsquigarrow \ e_{i+1} \ \rightsquigarrow \ e_i \ \rightsquigarrow \ \ldots$$

of X — i.e., each v_i (respectively, e_i) is an irreducible component (respectively, node) of the special fiber of the stable model of X_i that is *fixed* by the natural action of the image $\mathrm{Im}(s)$ of the section s; the image of the irreducible component v_{i+1} (respectively, node e_{i+1}) in X_i is contained in the irreducible component v_i (respectively, node e_i). The central issue discussed in [PSCom] is precisely the issue of

(Q1) whether or not the main theorem of Pop-Stix yields any *essentially new* information concerning the above situation — i.e., information that cannot already be derived from the above *systems of vertices or edges*.

Since

(i) it appears that aside from various *valuation-theoretic techniques*, the main technically nontrivial input into the proof of the main theorem of Pop-Stix is Tamagawa's *resolution of nonsingularities*, and, moreover,

(ii) it is difficult to see how Tamagawa's *resolution of nonsingularities* can lead to any *essentially stronger information* than the *existence of a system of vertices or edges* as discussed above,

it seems reasonable to suspect that

(Q2) it should be possible to *derive* the main theorem of Pop-Stix directly from the *existence of a system of vertices or edges* as discussed above, together with various purely *valuation-theoretic techniques*.

On the other hand, since I am not familiar with these valuation-theoretic techniques, it is not clear to me how to obtain a proof of the main theorem of Pop-Stix as in (Q2).

Bibliography

[Semi] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[CmbGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455-479.

[NodNon] Y. Hoshi, S. Mochizuki, On the Combinatorial Anabelian Geometry of Nodally Nondegenerate Outer Representations, to appear in *Hiroshima Math. J.*

[AbsTopII] S. Mochizuki, *Topics in Absolute Anabelian Geometry II: Decomposition Groups*, RIMS Preprint **1625** (March 2008).

[PSCom] S. Mochizuki, *Comments on the Main Theorem of Pop-Stix*, manuscript dated November 15, 2010.

[PS] F. Pop and J. Stix, *Arithmetic in the fundamental group of a p-adic curve — On the p-adic section conjecture for curves*, preprint, Philadelphia-Heidelberg-Cambridge, August 2010.

[Tama] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 1291-1336.