FIBERED PRODUCTS OF HOPF ALGEBRAS AND SEIFERT-VAN KAMPEN THEOREM FOR SEMI-GRAPHS OF TANNAKIAN CATEGORIES

YUKI KAWAGUCHI

ABSTRACT. It is known that Seifert-van Kampen theorem (for "good" topological spaces) can be showed by arguing the category of covering spaces. Similar arguments should be valid for abstract Galois categories (which Mochizuki calls "connected anabelioids") and neutral Tannakian categories. But when we try to state the theorem, the problem is the existence of amalgams (in other words, fibered coproducts) of profinite groups and that of affine group schemes (which is translated to the existence of fibered products of commutative Hopf algebras). A construction of amalgams of profinite groups can be found in Zalesskii [6]. We will construct fibered products of commutative Hopf algebras by using the explicit construction of cofree coalgebras which Hazewinkel gave in [3]. Another interest is the existence of so-called HNN extensions of affine group schemes, which we will also prove. By combining these two kinds of constructions, when we are given data of finitely many affine group schemes and a manner of composing them, we can describe the composite affine group scheme. The main theorem in this article is that, when we are given data of finitely many neutral Tannakian categories and a manner of glueing them, the fundamental group of the glued neutral Tannakian categories is isomorphic to the composition of the respective fundamental groups under the assumption that the data can be translated to the data of affine group schemes, which is not true in general unlike the case of Galois categories and profinite groups.

CONTENTS

1. FIBERED PRODUCTS OF HOPF ALGEBRAS

Throughout this article, k denotes a field. In this section, we construct the fibered product of A_1 and A_2 over A_0 for given Hopf algebras A_0, A_1, A_2 over k. We write $\mathbf{Vect}_k$ (resp. $\mathbf{Alg}_k, \mathbf{Coalg}_k, \mathbf{Bialg}_k, \mathbf{Hopf}_k, \mathbf{Aff}_k$ and $\mathbf{AGS}_k$) for the category of k-vector spaces (resp. commutative k-algebras, k-coalgebras, commutative k-bialgebras, commutative k-Hopf algebras, k-affine schemes and k-affine groups schemes). Note that the same argument as in this section will be valid for non-commutative k-algebras, k-bialgebras and k-Hopf algebras. Let $\mathbb{N}$ denotes $\{0, 1, 2, \ldots\}$.

Definition 1.1. *Let (C, Δ, ε) be a k-coalgebra, where C is a k-vector space, $\Delta : C \longrightarrow C \otimes C$ is a comultiplication and $\varepsilon : C \longrightarrow k$ is a counit. For $n \in \mathbb{N}$, $\Delta_n : C \longrightarrow C^{\otimes n}$ is defined by:*

$$\begin{aligned}
&\Delta_0 = \varepsilon : C \longrightarrow k = C^{\otimes 0},\\
&\Delta_1 = \mathrm{id} : C \longrightarrow C = C^{\otimes 1},\\
&\Delta_n = (\Delta \otimes \mathrm{id}^{\otimes n-2}) \circ \Delta_{n-1} : C \longrightarrow C^{\otimes n} \ (n \geq 2).
\end{aligned}$$

Remark 1.2. *By the coassociativity of Δ, it follows that for $n, t, s \in \mathbb{N}$ such that $t + s = n - 2 \geq 0$, $\left(\mathrm{id}^{\otimes t} \otimes \Delta \otimes \mathrm{id}^{\otimes s}\right) \circ \Delta_{n-1} = \Delta_n$.*

Furthermore, one can also show that for $t, s \in \mathbb{N}$, we have $(\Delta_t \otimes \Delta_s) \circ \Delta = \Delta_{t+s}$.

The following proposition is due to Hazewinkel [3].

Proposition 1.3. *The forgetful functor $\mathcal{F} : \mathbf{Coalg}_k \longrightarrow \mathbf{Vect}_k$ has a right adjoint functor $\mathcal{C}$ and we can construct it as below.*

Proof. Let $V \in \mathbf{Vect}_k$. We set

$$\hat{T}V = \prod_{n \in \mathbb{N}} V^{\otimes n},$$

$$\hat{T}V \hat{\otimes} \hat{T}V = \prod_{t,s \in \mathbb{N}} V^{\otimes t+s}$$

and

$$\Delta : \hat{T}V \longrightarrow \hat{T}V \hat{\otimes} \hat{T}V; (z_n)_n \mapsto (z_{t+s})_{t,s}.$$

An element $z \in \hat{T}V$ is called representative if $\Delta(z)$ lies in the image of the natural map

$$\hat{T}V \otimes \hat{T}V \longrightarrow \hat{T}V \hat{\otimes} \hat{T}V; (x_t)_t \otimes (y_s)_s \mapsto (x_t \otimes y_s)_{t,s}.$$

We write TV_{repr} for the set of representative elements of $\hat{T}V$. Then Δ restricts to $\Delta : TV_{\mathrm{repr}} \longrightarrow TV_{\mathrm{repr}} \otimes TV_{\mathrm{repr}}$ (see Hazewinkel [3] (3.12)). We define ε as the composition of

$$TV_{\mathrm{repr}} \longrightarrow \hat{T}V = \prod_{n \in \mathbb{N}} V^{\otimes n} \xrightarrow{\mathrm{pr}_0} k.$$

Let us show that $\mathrm{id} \otimes \Delta \circ \Delta = \Delta \otimes \mathrm{id} \circ \Delta$. For $(z_n)_n \in TV_{\mathrm{repr}}$, we write

$$\Delta((z_n)_n) = \sum_l \left(x_t^{(l)}\right)_t \otimes \left(y_s^{(l)}\right)_s$$

$$\Delta\left(\left(x_n^{(l)}\right)_n\right) = \sum_{l'} \left(p_t^{(l,l')}\right)_t \otimes \left(q_s^{(l,l')}\right)_s$$

and

$$\Delta\left(\left(y_n^{(l)}\right)_n\right) = \sum_{l'} \left(u_t^{(l,l')}\right)_t \otimes \left(v_s^{(l,l')}\right)_s.$$

Then we have

$$\mathrm{id} \otimes \Delta \circ \Delta((z_n)_n) = \sum_{l,l'} \left(x_t^{(l)}\right)_t \otimes \left(u_r^{(l,l')}\right)_r \otimes \left(v_s^{(l,l')}\right)_s$$

and the image of the right hand side in $\displaystyle\prod_{t,r,s \in \mathbb{N}} V^{\otimes t+r+s}$ is

$$\left(\sum_l x_t^{(l)} \otimes \left(\sum_{l'} u_r^{(l,l')} \otimes v_s^{(l,l')}\right)\right)_{t,r,s}.$$

Similarly we have

$$\Delta \otimes \mathrm{id} \circ \Delta((z_n)_n) = \sum_{l,l'} \left(p_t^{(l,l')}\right)_t \otimes \left(q_r^{(l,l')}\right)_r \otimes \left(y_s^{(l)}\right)_s$$

and the image of the right hand side in $\prod_{t,r,s \in \mathbb{N}} V^{\otimes t+r+s}$ is

$$\left(\sum_l \left(\sum_{l'} p_t^{(l,l')} \otimes q_r^{(l,l')}\right) \otimes y_s^{(l)}\right)_{t,r,s}.$$

Since

$$x_{t+s}^{(l)} = \sum_{l'} p_t^{(l,l')} \otimes q_s^{(l,l')}$$

and

$$y_{t+s}^{(l)} = \sum_{l'} u_t^{(l,l')} \otimes v_s^{(l,l')}$$

we see that

$$\begin{aligned} &\left(\sum_l x_t^{(l)} \otimes \left(\sum_{l'} u_r^{(l,l')} \otimes v_s^{(l,l')}\right)\right)_{t,r,s} \\ =&\left(\sum_l x_t^{(l)} \otimes y_{r+s}^{(l)}\right)_{t,r,s} \\ =&(z_{t+r+s})_{t,r,s} \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_l \left(\sum_{l'} p_t^{(l,l')} \otimes q_r^{(l,l')}\right) \otimes y_s^{(l)}\right)_{t,r,s} \\ =&\left(\sum_l x_{t+r}^{(l)} \otimes y_s^{(l)}\right)_{t,r,s} \\ =&(z_{t+r+s})_{t,r,s}. \end{aligned}$$

Let us show that $\mathrm{id} \otimes \varepsilon \circ \Delta = \varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}$. Under the same notation as above, we have for $(z_n)_n \in TV_{\mathrm{repr}}$

$$\mathrm{id} \otimes \varepsilon \circ \Delta((z_n)_n) = \sum_l \left(y_0^{(l)} x_t^{(l)}\right)_t = (z_t)_t$$

and thus $\mathrm{id} \otimes \varepsilon \circ \Delta = \mathrm{id}$. Similarly we have $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}$.

After all TV_{repr} becomes a k-coalgebra, which we denote $\mathcal{C}(V)$. Moreover, for a k-linear map $V \longrightarrow W$, we define a k-coalgebra homomorphism $\mathcal{C}(f) : \mathcal{C}(V) \longrightarrow \mathcal{C}(W); (z_n)_n \mapsto (f^{\otimes n}(z_n))_n$. Then we obtain a functor $\mathcal{C} : \mathbf{Vect}_k \longrightarrow \mathbf{Coalg}_k$.

Now let us show that $\mathcal{C}$ is a right adjoint functor of $\mathcal{F}$. We claim that, for $C \in \mathbf{Coalg}_k, V \in \mathbf{Vect}_k$, there is a functorial bijective map

$$\varphi_{C,V} : \mathrm{Hom}_{\mathbf{Vect}_k}(\mathcal{F}(C), V) \longrightarrow \mathrm{Hom}_{\mathbf{Coalg}_k}(C, \mathcal{C}(V)); g \mapsto \left(z \mapsto \left(g^{\otimes n} \circ \Delta_n(z)\right)_n\right).$$

First, we must check that $\big(g^{\otimes n}\circ\Delta_n(z)\big)_n$ is a representative element of $\hat{T}V$. When we write $\Delta(z)=\sum_l x^{(l)}\otimes y^{(l)}$ $\big(x^{(l)},y^{(l)}\in C\big)$, we have

$$\begin{aligned}\Delta\Big(\big(g^{\otimes n}\circ\Delta_n(z)\big)_n\Big) &= \big(g^{\otimes t+s}(\Delta_{t+s}(z))\big)_{t,s}\\ &= \Big(g^{\otimes t+s}\Big(\sum_l \Delta_t\big(x^{(l)}\big)\otimes\Delta_s\big(y^{(l)}\big)\Big)\Big)_{t,s}\\ &= \sum_l \big(g^{\otimes t}\circ\Delta_t\big(x^{(l)}\big)\otimes g^{\otimes s}\circ\Delta_s\big(y^{(l)}\big)\big)_{t,s}\end{aligned}$$

using Remark 1.2. Here the right hand side is the image of $\sum_l \big(g^{\otimes n}\circ\Delta_n(x^{(l)})\big)_n\otimes \big(g^{\otimes n}\circ\Delta_n(y^{(l)})\big)_n\in\hat{T}V\otimes\hat{T}V$, which implies that $(g^{\otimes n}\circ\Delta_n(z))_n$ is representative.

The above calculation shows also that the map $z\mapsto\big(g^{\otimes n}\circ\Delta_n(z)\big)_n$ is compatible with comultiplications. Moreover, it is easy to see that the map $z\mapsto\big(g^{\otimes n}\circ\Delta_n(z)\big)_n$ is counit preserving and hence a k-coalgebra homomorphism.

For bijectivity of $\varphi_{C,V}$, we will show that the map $\psi_{C,V}: f\mapsto \mathrm{pr}_1\circ f$ is the inverse. Obviously we have $\psi_{C,V}\circ\varphi_{C,V}=\mathrm{id}$. To see that $\varphi_{C,V}\circ\psi_{C,V}=\mathrm{id}$, we have to show that, for $f\in\mathrm{Hom}_{\mathbf{Coalg}_k}(C,\mathcal{C}(V))$, $\mathrm{pr}_n\circ f$ is determined by $\mathrm{pr}_1\circ f$ for each $n\in\mathbb{N}$. Since $\mathrm{pr}_0\circ f=\varepsilon$, $\mathrm{pr}_0\circ f$ is determined. It is enough to show that for $n\geq 1$, if $\mathrm{pr}_n\circ f$ is determined by $\mathrm{pr}_1\circ f$, then $\mathrm{pr}_{n+1}\circ f$ is also determined. For $z\in C$, we fix a presentation $\Delta(z)=\sum_l x^{(l)}\otimes y^{(l)}$ and write $f\big(x^{(l)}\big)=\big(\alpha_n^{(l)}\big)_n$, $f\big(y^{(l)}\big)=\big(\beta_n^{(l)}\big)_n$ and $f(z)=(\gamma_n)_n$. Then we have

$$\begin{aligned}f\otimes f\circ\Delta(z) &= \sum_l f\big(x^{(l)}\big)\otimes f\big(y^{(l)}\big)\\ &= \sum_l \big(\alpha_n^{(l)}\big)_n\otimes\big(\beta_n^{(l)}\big)_n\end{aligned}$$

and the image of the right hand side in $\prod_{t,s\in\mathbb{N}} V^{\otimes t+s}$ is $\sum_l\big(\alpha_t^{(l)}\otimes\beta_s^{(l)}\big)_{t,s}$. On the other hand the image of $\Delta\circ f(z)$ is $(\gamma_{t+s})_{t,s}$. Therefore we have for $t,s\in\mathbb{N}$

$$\gamma_{t+s}=\sum_l \alpha_t^{(l)}\otimes\beta_s^{(l)}$$

and in particular, $\gamma_{n+1}=\sum_l\alpha_n^{(l)}\otimes\beta_1^{(l)}$. Then since $\alpha_n^{(l)}$ and $\beta_1^{(l)}$ are determined by $\mathrm{pr}_1\circ f$ and z by assumption, γ_{n+1} is also determined and hence we are done. □

Remark 1.4. *For $C\in\mathbf{Coalg}_k$, the natural k-coalgebra homomorphism $C\longrightarrow\mathcal{C}(\mathcal{F}(C))$ obtained from the adjointness is given by $z\mapsto(\Delta_n(z))_n$ and injective. Thus we can always regard C as a k-subcoalgebra of $\mathcal{C}(\mathcal{F}(C))$.*

Lemma 1.5. *Let $C\in\mathbf{Coalg}_k$ and $V\subset C$ be a k-linear subspace. Then there is the largest k-subcoalgebra of C contained in V.*

Proof. For k-subcoalgebras $C_1'',C_2''\subset C$, we see that $C_1''+C_2''\subset C$ is a k-subcoalgebra. Thus $C'=\bigcup\{C''\subset C: k\text{-subcoalgebra}\mid C''\subset V\}$ is the largest k-subcoalgebra of C contained in V. □

Proposition 1.6. *For $C_1, C_2 \in \mathbf{Coalg}_k$, a direct product $C_1 \star C_2$ of C_1 and C_2 in $\mathbf{Coalg}_k$ exists and we can construct it as below.*

Proof. We regard C_i as a k-subcoalgebra of $\mathcal{C}(\mathcal{F}(C_i))$. We define $C_1 \star C_2$ as the largest k-subcoalgebra of $\mathcal{C}(\mathcal{F}(C_1) \oplus \mathcal{F}(C_2))$ contained in $\mathcal{C}(\mathrm{pr}_1)^{-1}(C_1) \cap \mathcal{C}(\mathrm{pr}_2)^{-1}(C_2)$ where pr_i denotes the i-th projection $\mathcal{F}(C_1) \oplus \mathcal{F}(C_2) \longrightarrow \mathcal{F}(C_i)$. Then for an arbitrary k-coalgebra D, we see that

$$\begin{aligned}
&\mathrm{Hom}(D, C_1 \star C_2) \\
\cong& \{h \in \mathrm{Hom}(D, \mathcal{C}(\mathcal{F}(C_1) \oplus \mathcal{F}(C_2))) \mid \mathrm{Im}\, \mathcal{C}(\mathrm{pr}_1) \circ h \subset C_1, \mathrm{Im}\, \mathcal{C}(\mathrm{pr}_2) \circ h \subset C_2\} \\
\cong& \{f \in \mathrm{Hom}(D, \mathcal{C}(\mathcal{F}(C_1))) \mid \mathrm{Im}\, f \subset C_1\} \times \{g \in \mathrm{Hom}(D, \mathcal{C}(\mathcal{F}(C_2))) \mid \mathrm{Im}\, g \subset C_2\} \\
\cong& \mathrm{Hom}(D, C_1) \times \mathrm{Hom}(D, C_2)
\end{aligned}$$

using adjointness of $\mathcal{F}$ and $\mathcal{C}$ and the fact that the image of a k-coalgebra homomorphism is k-subcoalgebra. This implies that $C_1 \star C_2$ satisfies the universality of a direct product of C_1 and C_2. $\square$

Proposition 1.7. *The functor $\mathcal{C}$ induces a right adjoint functor of the forgetful functor $\mathcal{F} : \mathbf{Bialg}_k \longrightarrow \mathbf{Alg}_k$. We also write $\mathcal{C}$ for that functor.*

Proof. This follows from almost the same argument with the proof of Proposition 1.3. Note that for $A \in \mathbf{Alg}_k$, $\hat{T}A$ has a natural structure of k-algebra (the direct product of k-algebras $A^{\otimes n}$, not the tensor algebra of A) and TA_{repr} is its k-subalgebra. $\square$

Lemma 1.8. *Let $A \in \mathbf{Bialg}_k$ and $B \subset A$ be a k-subalgebra. Then the largest k-subcoalgebra of A contained in B is the largest k-subbialgebra of A contained in B.*

Proof. Let A' be the largest k-subcoalgebra of A contained in B. We claim that A' is also a k-subalgebra. Let A'' be the k-subalgebra of A generated by A', i.e., the k-linear subspace of A spanned by elements $x_1 x_2 \cdots x_n$ with $x_i \in A'$. For $x_1, x_2, \ldots, x_n \in A'$, since $\Delta_A(x_i) \in A' \otimes A'$ $(i = 1, 2, \ldots n)$, we have $\Delta_A(x_1 x_2 \cdots x_n) \in A'' \otimes A''$. Therefore A'' is a k-subbialgebra contained in B and containing A'. We conclude that $A' = A''$ and hence A' is the largest k-subbialgebra of A. $\square$

Proposition 1.9. *For $A_1, A_2 \in \mathbf{Bialg}_k$, we can equip $A_1 \star A_2$ constructed in Lemma 1.6 with a k-bialgebra structure which makes it a direct product of A_1 and A_2 in $\mathbf{Bialg}_k$.*

Proof. Note that $\mathcal{C}(\mathrm{pr}_1)^{-1}(A_1) \cap \mathcal{C}(\mathrm{pr}_2)^{-1}(A_2)$ is a k-subalgebra of $\mathcal{C}(\mathcal{F}(A_1) \oplus \mathcal{F}(A_2))$. Since $A_1 \star A_2$ is the largest k-subcoalgebra of $\mathcal{C}(\mathcal{F}(A_1) \oplus \mathcal{F}(A_2))$ contained in $\mathcal{C}(\mathrm{pr}_1)^{-1}(A_1) \cap \mathcal{C}(\mathrm{pr}_2)^{-1}(A_2)$, it is the largest k-subbialgebra of $\mathcal{C}(\mathcal{F}(A_1) \oplus \mathcal{F}(A_2))$ contained in $\mathcal{C}(\mathrm{pr}_1)^{-1}(A_1) \cap \mathcal{C}(\mathrm{pr}_2)^{-1}(A_2)$ by Proposition 1.8. By the similar argument as (1.6), we see that $A_1 \star A_2$ satisfies the universality of a direct product of A_1 and A_2 (note that the image of a k-bialgebra homomorphism is k-subbialgebra). $\square$

Proposition 1.10. *For $A_1, A_2 \in \mathbf{Hopf}_k$, we can eqiuip $A_1 \star A_2$ constructed in Proposition 1.9 with a k-Hopf algebra structure which makes it a direct product of A_1 and A_2 in $\mathbf{Hopf}_k$.*

Proof. We write

$$S = S_{A_1} \times S_{A_2} : \mathcal{F}(A_1) \times \mathcal{F}(A_2) \longrightarrow \mathcal{F}(A_1) \times \mathcal{F}(A_2)$$

where S_{A_i} denotes the antipode of A_i $(i = 1, 2)$ and

$$S : \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \longrightarrow \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)); (z_n)_n \mapsto (\tau_n \circ S^{\otimes n}(z_n))_n$$

where τ_n denotes the map

$$(\mathcal{F}(A_1) \times \mathcal{F}(A_2))^{\otimes n} \longrightarrow (\mathcal{F}(A_1) \times \mathcal{F}(A_2))^{\otimes n}; z_{n,1} \otimes \cdots \otimes z_{n,n} \mapsto z_{n,n} \otimes \cdots \otimes z_{n,1}.$$

We claim that $S(\mathcal{C}(\mathcal{F}(A_1)\times\mathcal{F}(A_2))) \subset \mathcal{C}(\mathcal{F}(A_1)\times\mathcal{F}(A_2))$. For $(z_n)_n \in \mathcal{C}(\mathcal{F}(A_1)\times \mathcal{F}(A_2))$, since it is representative, we can write

$$(z_{t+s})_{t,s} = \sum_l \left(x_t^{(l)} \otimes y_s^{(l)}\right) \quad \left(\left(x_n^{(l)}\right)_n, \left(y_n^{(l)}\right)_n \in \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))\right)$$

Then we have $\Delta(S((z_n)_n)) = \Delta\left(\left(\tau_n \circ S^{\otimes n}(z_n)\right)_n\right) = \left(\tau_{t+s} \circ S^{\otimes t+s}(z_{t+s})\right)_{t,s} = \sum_l \left(\tau_s \circ S^{\otimes s} y_s^{(l)} \otimes \tau_t \circ S^{\otimes t} x_t^{(l)}\right)_{t,s}$, which implies $S((z_n)_n)$ is representative.

Next we claim that $S(A_1 \star A_2) \subset A_1 \star A_2$. It is enough to show that $S(A_1 \star A_2)$ is a k-subbialgebra of $\mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))$ and $\mathcal{C}(\mathrm{pr}_i)(S(A_1 \star A_2)) \subset A_i$ $(i = 1, 2)$. It is clear that $S(A_1 \star A_2)$ is a k-subalgebra. Since the diagram

$$\begin{tikzcd}
\mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \arrow[r, "\Delta"] \arrow[d, "S"'] & \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \otimes \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \arrow[d, "\tau\circ S\otimes S"] \\
\mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \arrow[r, "\Delta"'] & \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \otimes \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))
\end{tikzcd}$$

is commutative (where $\tau = \tau_2$), we have $\Delta(S(A_1 \star A_2)) = \tau(S \otimes S(\Delta(A_1 \star A_2))) \subset \tau(S\otimes S(A_1\star A_2 \otimes A_1\star A_2)) \subset \tau(S(A_1\star A_2)\otimes S(A_1\star A_2)) \subset S(A_1\star A_2)\otimes S(A_1\star A_2)$, which implies that $S(A_1\star A_2)$ is a k-subbialgebra. Moreover, we see that for $i = 1, 2$, $\mathcal{C}(\mathrm{pr}_i)(S(A_1 \star A_2)) \subset A_i$ since the diagram

$$\begin{tikzcd}
A_1 \star A_2 \arrow[r] \arrow[d, "S"'] & \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \arrow[r, "\mathcal{C}(\mathrm{pr}_i)"] \arrow[d, "S"'] & \mathcal{C}(\mathcal{F}(A_i)) \arrow[d, "S"'] & A_i \arrow[l] \arrow[d, "S"] \\
S(A_1 \star A_2) \arrow[r] & \mathcal{C}(\mathcal{F}(A_1) \times \mathcal{F}(A_2)) \arrow[r, "\mathcal{C}(\mathrm{pr}_i)"'] & \mathcal{C}(\mathcal{F}(A_i)) & A_i \arrow[l]
\end{tikzcd}$$

is commutative.

Now we will show that $(A_1 \star A_2, m, e, \Delta, \varepsilon, S)$ is a k-Hopf algebra where m and e denote the multiplication and the unit respectively. We have to show that the diagram

$$\begin{tikzcd}
A_1 \star A_2 \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & (A_1 \star A_2) \otimes (A_1 \star A_2) \arrow[d, "m\circ\mathrm{id}\otimes S"] \\
k \arrow[r, "e"'] & A_1 \star A_2
\end{tikzcd}$$

is commutative. Let us consider $(z_n)_n \in A_1\star A_2$. Note that there is a pair of $a \in A_1$ and $b \in A_2$ such that $\mathrm{pr}_1^{\otimes n}(z_n) = \Delta_n(a)$ and $\mathrm{pr}_2^{\otimes n}(z_n) = \Delta_n(b)$ for $n \in \mathbb{N}$. We can write

$$(z_{t+s})_{t,s} = \sum_l \left(x_t^{(l)} \otimes y_s^{(l)}\right)_{t,s} \quad \left(\left(x_n^{(l)}\right)_n, \left(y_n^{(l)}\right)_n \in \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))\right)$$

and thus

$$\Delta((z_n)_n) = \sum_l \big(x_n^{(l)}\big)_n \otimes \big(y_n^{(l)}\big)_n.$$

Then we have

$$\begin{aligned} m \circ (\mathrm{id} \otimes S) \circ \Delta((z_n)_n) &= m \circ \mathrm{id} \otimes S\bigg(\sum_l \big(x_n^{(l)}\big)_n \otimes \big(y_n^{(l)}\big)_n \bigg) \\ &= m\bigg(\sum_l (x_n^{(l)})_n \otimes (\tau_n \circ S^{\otimes n}(y_n^{(l)}))_n \bigg) \\ &= \bigg(\sum_l x_n^{(l)} \cdot \tau_n \circ S^{\otimes n}(y_n^{(l)}) \bigg)_n \\ &= \bigg(\sum_l m \circ \big(\mathrm{id}^{\otimes n} \otimes \tau_n \circ S^{\otimes n}\big)\big(x_n^{(l)} \otimes y_n^{(l)}\big) \bigg)_n \\ &= \big(m \circ \big(\mathrm{id}^{\otimes n} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big)(z_{2n})\big)_n. \end{aligned}$$

Thus it is enough to show that for $n \in \mathbb{N}$

$$m \circ \big(\mathrm{id}^{\otimes n} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big)(z_{2n}) = z_0(1,1)^{\otimes n}.$$

Clearly this holds for $n = 0$. We consider the case $n = 1$. When we write

$$z_2 = \sum_l \big(\alpha_1^{(l)}, \beta_1^{(l)}\big) \otimes \big(\alpha_2^{(l)}, \beta_2^{(l)}\big)$$

we have

$$\sum_l \alpha_1^{(l)} \otimes \alpha_2^{(l)} = \mathrm{pr}_1 \otimes \mathrm{pr}_1(z_2) = \Delta(a),$$

$$\sum_l \beta_1^{(l)} \otimes \beta_2^{(l)} = \mathrm{pr}_2 \otimes \mathrm{pr}_2(z_2) = \Delta(b)$$

and thus

$$\sum_l \alpha_1^{(l)} S\big(\alpha_2^{(l)}\big) = m \circ \mathrm{id} \otimes S \circ \Delta(a) = \varepsilon(a) = z_0,$$

$$\sum_l \beta_1^{(l)} S\big(\beta_2^{(l)}\big) = m \circ \mathrm{id} \otimes S \circ \Delta(b) = \varepsilon(b) = z_0.$$

Hence $m \circ \mathrm{id} \otimes S(z_2) = \sum_l \big(\alpha_1^{(l)} S\big(\alpha_2^{(l)}\big), \beta_1^{(l)} S\big(\beta_2^{(l)}\big)\big) = (z_0, z_0) = z_0(1,1)$. We will show that if $n \geq 1$ and the claim holds for n then it also holds for $n+1$. We can write

$$(y_{t+s}^{(l)})_{t,s} = \sum_{l,l'} \big(p_t^{(l,l')} \otimes q_s^{(l,l')}\big) \quad \big(\big(p_n^{(l,l')}\big)_n, \big(q_n^{(l,l')}\big)_n \in \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))\big)$$

and

$$(p_{t+s}^{(l,l')})_{t,s} = \sum_{l,l',l''} \big(u_t^{(l,l',l'')} \otimes v_s^{(l,l',l'')}\big) \quad \big(\big(u_n^{(l,l',l'')}\big)_n, \big(v_n^{(l,l'l'')}\big)_n \in \hat{T}(\mathcal{F}(A_1) \times \mathcal{F}(A_2))\big).$$

Then we have

$$\begin{aligned} z_{2(n+1)} &= \sum_{l} x_n^{(l)} \otimes y_{n+2}^{(l)} \\ &= \sum_{l} x_n^{(l)} \otimes \Big(\sum_{l'} p_2^{(l,l')} \otimes q_n^{(l,l')} \Big) \\ &= \sum_{l} x_n^{(l)} \otimes \Big(\sum_{l'} \Big(\sum_{l''} u_1^{(l,l',l'')} \otimes v_1^{(l,l',l'')} \Big) \otimes q_n^{(l,l')} \Big) \\ &= \sum_{l,l',l''} x_n^{(l)} \otimes u_1^{(l,l',l'')} \otimes v_1^{(l,l',l'')} \otimes q_n^{(l,l')} \end{aligned}$$

and thus

$$\begin{aligned} & m \circ \big(\mathrm{id}^{\otimes n+1} \otimes \big(\tau_{n+1} \circ S^{\otimes n+1}\big)\big)(z_{2(n+1)}) \\ =& \sum_{l,l',l''} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(q_n^{(l,l')}\big)\big) \otimes m\big(u_1^{(l,l',l'')} \otimes S\big(v_1^{(l,l',l'')}\big)\big) \\ =& \sum_{l,l'} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(q_n^{(l,l')}\big)\big) \otimes m\Big(\sum_{l''} u_1^{(l,l',l'')} \otimes S\big(v_1^{(l,l',l'')}\big) \Big) \\ =& \sum_{l,l'} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(q_n^{(l,l')}\big)\big) \otimes m \circ (\mathrm{id} \otimes S)\big(p_2^{(l,l')}\big) \\ =& \sum_{l,l'} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(q_n^{(l,l')}\big)\big) \otimes p_0^{(l,l')}(1,1) \\ =& \sum_{l,l'} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(p_0^{(l,l')} q_n^{(l,l')}\big)\big) \otimes (1,1) \\ =& \sum_{l} m\big(x_n^{(l)} \otimes \big(\tau_n \circ S^{\otimes n}\big)\big(y_n^{(l)}\big)\big) \otimes (1,1) \\ =& \, m \circ \mathrm{id}^{\otimes n} \otimes \big(\tau_n \circ S^{\otimes n}\big)(z_{2n}) \otimes (1,1) \\ =& \, z_0 (1,1)^{\otimes n+1}. \end{aligned}$$

Hence we have $m \circ (\mathrm{id} \otimes S) \circ \Delta = e \circ \varepsilon$, as well as $m \circ (S \otimes \mathrm{id}) \circ \Delta = e \circ \varepsilon$.

To see that $A_1 \star A_2$ satisfies the universality, let $B \in \mathbf{Hopf}_k$ and $f_i : B \longrightarrow A_i$ be a k-Hopf algebra homomorphism ($i = 1, 2$). Then the map

$$g : B \longrightarrow A_1 \star A_2 ; w \mapsto \big((\mathcal{F}(f_1) \times \mathcal{F}(f_2))^{\otimes n} \circ \Delta_n(w) \big)_n$$

is the unique k-bialgebra homomorphism such that $\mathcal{C}(\mathrm{pr}_i) \circ g = f_i$ ($i = 1, 2$). Thus it is enough to show that h is a k-Hopf algebra homomorphism, i.e., $g \circ S = S \circ g$. This is true because

$$\begin{aligned} g \circ S(w) =& \big((\mathcal{F}(f_1) \times \mathcal{F}(f_2))^{\otimes n} \circ \Delta_n(S(w)) \big)_n \\ =& \big((\mathcal{F}(f_1) \times \mathcal{F}(f_2))^{\otimes n} \circ \tau_n \circ S^{\otimes n} \circ \Delta_n(w) \big)_n \\ =& \big(\tau_n \circ S^{\otimes n} \circ (\mathcal{F}(f_1) \times \mathcal{F}(f_2))^{\otimes n} \circ \Delta_n(w) \big)_n \\ =& S \circ g(w). \end{aligned}$$

Note that, in general, for a Hopf algebra A, $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ and thus $\Delta_n \circ S = \tau_n \circ S^{\otimes n} \circ \Delta_n$ (see Abe [1] Theorem 2.1.4). $\square$

Lemma 1.11. *Let $A \in \mathbf{Hopf}_k$ and $B \subset A$ be a k-subalgebra. Then there is the largest k-subHopf algebra of A contained in B.*

Proof. We set $B' = \{x \in B \mid S(x) \in B\} \subset B$, which is a k-subalgebra. The largest k-subbialgebra A' contained in B' is a k-subHopf algebra. Indeed, since $S(A')$ is a k-subbialgebra (see Abe [1] Theorem 2.1.4) and contained in $S(B') = B'$, we see that $S(A') \subset A'$. A' satisfies the property. □

Proposition 1.12. *For $A_0, A_1 \in \mathbf{Hopf}_k$ and k-algebra homomorphism $f : A_1 \longrightarrow A_0$, there exists a unique (upto isomorphism) pair of $B \in \mathbf{Hopf}_k$ and a k-Hopf algebra homomorphism $g : B \longrightarrow A_1$ that satisfies the following universal properties:*

- *The composition of $B \xrightarrow{g} A_1 \xrightarrow{f} A_0$ coincides with the composition of $B \xrightarrow{counit} k \longrightarrow A_0$.*
- *For any pair of $B' \in \mathbf{Hopf}_k$ and a k-Hopf algebra morphism $g' : B' \longrightarrow A_1$, if the composition of $B' \xrightarrow{g'} A_1 \xrightarrow{f} A_0$ coincides with the composition of $B' \xrightarrow{counit} k \longrightarrow A_0$, then there is a unique k-Hopf algebra homomorphism $h : B' \longrightarrow B$ such that $g \circ h = g'$.*

Proof. We define B as the largest k-subHopf algebra of A_1 contained in the equalizer of f and the composition of $A_1 \xrightarrow{\text{counit}} k \longrightarrow A_0$, and g as the inclusion $B \longrightarrow A_1$. Then B and g clearly satisfiy the first property. For the second property, let $g' : B' \longrightarrow A_1$ be such a homomorphism. The image of g' is a k-subHopf algebra of A_1 contained in the equalizer of f and the composition of $A_1 \xrightarrow{\text{counit}} k \longrightarrow A_0$ and hence contained in B. Therefore g' induces a k-Hopf algebra homomorphism $h : B' \longrightarrow B$ such that $g \circ h = g'$. Uniqueness of such h is clear. □

Remark 1.13. *Note that the morphism $g : B \longrightarrow A_1$ in the statement of the proposition is injective. Thus for $V, W \in \mathbf{Comodf}_B$, a morphism $V \longrightarrow W$ of $\mathbf{Comodf}_{A_1}$ is also a morphism of $\mathbf{Comodf}_B$.*

Proposition 1.14. *For $A_0, A_1, A_2 \in \mathbf{Hopf}_k$ and k-Hopf algebra homomorphisms $f_1 : A_1 \longrightarrow A_0, f_2 : A_2 \longrightarrow A_0$, a fibered product $A_1 \star_{A_0} A_2$ of A_1 and A_2 over A_0 in $\mathbf{Hopf}_k$ exists and is constructed as below.*

Proof. Let $A_1 \star A_2$ be a direct product of A_1 and A_2 in $\mathbf{Hopf}_k$ and $\mathrm{pr}_i : A_1 \star A_2 \longrightarrow A_i$ be the i-th projection. We apply Proposition 1.12 to the k-algebra homomorphism

$$A_1 \star A_2 \xrightarrow{\quad\Delta\quad} (A_1 \star A_2) \otimes_k (A_1 \star A_2) \xrightarrow{m\circ((f_1\circ \mathrm{pr}_1)\otimes(f_2\circ \mathrm{pr}_2\circ S))} A_0$$

and obtain $g : B \longrightarrow A_1 \star A_2$. Then B is a fibered product of A_1 and A_2 over A_0. Indeed, since

$$m \circ ((f_1 \circ p_1) \otimes (f_2 \circ p_2 \circ S)) \circ \Delta \circ g = \varepsilon$$

we have

$$
\begin{aligned}
& f_1 \circ \mathrm{pr}_1 \circ g \\
= & ((f_1 \circ \mathrm{pr}_1) \otimes \varepsilon) \circ \Delta \circ g \\
= & ((f_1 \circ \mathrm{pr}_1) \otimes (\varepsilon \circ f_2 \circ \mathrm{pr}_2)) \circ \Delta \circ g \\
= & m \circ ((f_1 \circ \mathrm{pr}_1) \otimes (m \circ (S \otimes \mathrm{id}) \circ \Delta \circ f_2 \circ \mathrm{pr}_2)) \circ \Delta \circ g \\
= & m \circ ((f_1 \circ \mathrm{pr}_1) \otimes (m \circ (f_2 \circ \mathrm{pr}_2 \circ S \otimes f_2 \circ \mathrm{pr}_2) \circ \Delta)) \circ \Delta \circ g \\
= & m \circ (\mathrm{id} \otimes m) \circ ((f_1 \circ \mathrm{pr}_1) \otimes (f_2 \circ \mathrm{pr}_2 \circ S) \otimes (f_2 \circ \mathrm{pr}_2)) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \circ g \\
= & m \circ (m \otimes \mathrm{id}) \circ ((f_1 \circ \mathrm{pr}_1) \otimes (f_2 \circ \mathrm{pr}_2 \circ S) \otimes (f_2 \circ \mathrm{pr}_2)) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \circ g \\
= & m \circ ((m \circ (f_1 \circ \mathrm{pr}_1 \otimes f_2 \circ \mathrm{pr}_2 \circ S) \circ \Delta) \otimes (f_2 \circ \mathrm{pr}_2)) \circ \Delta \circ g \\
= & m \circ (m \circ (f_1 \circ \mathrm{pr}_1 \otimes f_2 \circ \mathrm{pr}_2 \circ S) \circ \Delta \circ g) \otimes (f_2 \circ \mathrm{pr}_2 \circ g)) \circ \Delta \\
= & m \circ (\varepsilon \otimes (f_2 \circ \mathrm{pr}_2 \circ g)) \circ \Delta \\
= & f_2 \circ \mathrm{pr}_2 \circ g.
\end{aligned}
$$

Moreover, for a k-Hopf algebra homomorphism $g' : B' \longrightarrow A_1 \star A_2$ such that $f_1 \circ p_1 \circ g' = f_2 \circ p_2 \circ g'$, since

$$
\begin{aligned}
& m \circ ((f_1 \circ \mathrm{pr}_1) \otimes (f_2 \circ \mathrm{pr}_2 \circ S)) \circ \Delta \circ g' \\
= & m \circ ((f_1 \circ \mathrm{pr}_1 \circ g') \otimes (f_2 \circ \mathrm{pr}_2 \circ g' \circ S)) \circ \Delta \\
= & m \circ ((f_1 \circ \mathrm{pr}_1 \circ g') \otimes (f_1 \circ \mathrm{pr}_1 \circ g' \circ S)) \circ \Delta \\
= & m \circ (\mathrm{id} \otimes S) \circ \Delta \circ f_1 \circ \mathrm{pr}_1 \circ g' \\
= & \varepsilon \circ f_1 \circ \mathrm{pr}_1 \circ g' \\
= & \varepsilon
\end{aligned}
$$

there is a unique k-Hopf algebra homomorphism $h : B' \longrightarrow B$ such that $g \circ h = g'$. □

Remark 1.15. *We did not use commutativity of algebras in the above arguments. Thus it is almost clear that the same claim as Proposition 1.7-Proposition 1.14 holds even if we replace* $\mathbf{Alg}_k$ *(resp.* $\mathbf{Bialg}_k, \mathbf{Hopf}_k$*) by the category of non-commutative k-algebras (resp.k-bialgebras, k-Hopf algebras).*

By the contravariant equivalences between $\mathbf{Alg}_k$ and $\mathbf{Aff}_k$, and between $\mathbf{Hopf}_k$ and $\mathbf{AGS}_k$, we immediately obtain the following corollaries.

Corollary 1.16. *For $G_1, G_2 \in \mathbf{AGS}_k$, a free product (direct coproduct) $G_1 * G_2$ of G_1 and G_2 in $\mathbf{AGS}_k$ exists.*

Corollary 1.17. *For $G_0, G_1 \in \mathbf{AGS}_k$ and k-scheme morphism $\rho : G_0 \longrightarrow G_1$, there exists a unique (upto isomorphism) pair of $H \in \mathbf{AGS}_k$ and a k-affine group scheme morphism $\varpi : G_1 \longrightarrow H$ that satisfies the following universal properties:*

- *The composition of $G_0 \xrightarrow{\rho} G_1 \xrightarrow{\varpi} H$ coincides with the composition of $G_0 \longrightarrow \operatorname{Spec} k \xrightarrow{unit} H$.*
- *For any pair of $H' \in \mathbf{AGS}_k$ and a k-affine group scheme morphism $\varpi' : G_1 \longrightarrow H'$, if the composition of $G_0 \xrightarrow{\rho} G_1 \xrightarrow{\varpi'} H'$ coincides with the composition of $G_0 \longrightarrow \operatorname{Spec} k \xrightarrow{unit} H'$, then there is a unique k-affine group scheme morphism $\xi : H \longrightarrow H'$ such that $\xi \circ \varpi = \varpi'$.*

Corollary 1.18. *For $G_0, G_1, G_2 \in \mathbf{AGS}_k$ and k-affine group scheme morphisms $\rho_1 : G_0 \longrightarrow G_1, \rho_2 : G_0 \longrightarrow G_2$, an amalgam (fibered coproduct) $G_1 *_{G_0} G_2$ of G_1 and G_2 over G_0 in $\mathbf{AGS}_k$ exists.*

2. The category of Tannakian categories

In this section, we will define the category $\mathbf{Tann}_k$ of neutral Tannakian categories over k as a quotient category of the category of pairs of neutral Tannakian categories over k, and its neutral fiber functors. Then we will get category equivalences among $\mathbf{AGS}_k, \mathbf{Hopf}_k$ and $\mathbf{Tann}_k$.

We write $\mathbf{Vectf}_k$ for the category of the finite dimensional k-vector spaces. For $A \in \mathbf{Hopf}_k$, let $\mathbf{Comodf}_A$ denotes the category of finite dimensional (right) A-comodules over k and for $G \in \mathbf{AGS}_k$, let $\mathbf{Repf}_G$ denotes the category of finite dimensional representation of G over k.

Definition 2.1. *We write $\mathbf{Tann}'_k$ for the category whose object is a pair $(\mathcal{C}, \omega)$ where $\mathcal{C}$ is a neutral Tannakian category over k and ω is a neutral fiber functor on $\mathcal{C}$. For $(\mathcal{C}, \omega), (\mathcal{C}', \omega') \in \mathbf{Tann}'_k$, a morphism between $(\mathcal{C}, \omega)$ and $(\mathcal{C}', \omega')$ is defined to be a pair (F, φ) where F is an exact faithful k-linear tensor functor $\mathcal{C} \longrightarrow \mathcal{C}'$ and φ is a tensor functor isomorphism $F^*\omega' \longrightarrow \omega$. For $(F, \varphi) : (\mathcal{C}, \omega) \longrightarrow (\mathcal{C}', \omega')$ and $(F', \varphi') : (\mathcal{C}', \omega') \longrightarrow (\mathcal{C}'', \omega'')$, we set $(F', \varphi') \circ (F, \varphi) = (F' \circ F, \varphi \circ F^*\varphi')$.*

We define an equivalence relation $\sim$ on $\mathrm{Hom}_{\mathbf{Tann}'}((\mathcal{C}, \omega), (\mathcal{C}', \omega'))$ so that $(F_1, \varphi_1) \sim (F_2, \varphi_2)$ if and only if there exists a tensor functor isomorphism $\mu : F_1 \longrightarrow F_2$ such that

$$\begin{tikzcd} \omega' \circ F_1 \arrow[rr, "\omega'_*\mu"] \arrow[dr, "\varphi_1"'] & & \omega' \circ F_2 \arrow[dl, "\varphi_2"] \\ & \omega & \end{tikzcd}$$

commutes. Then we can define the quotient category $\mathbf{Tann}_k = \mathbf{Tann}'_k / \sim$.

Consider the functor $\mathbf{AGS}_k \longrightarrow \mathbf{Tann}'_k$ defined by sending G to $(\mathbf{Repf}_G, \omega_G)$ where $\omega_G : \mathbf{Repf}_G \longrightarrow \mathbf{Vectf}_k$ is the forgetful functor, and $\rho : G \longrightarrow G'$ to (ρ^*, id). C denotes the composition of that functor followed by the natural functor $\mathbf{Tann}'_k \longrightarrow \mathbf{Tann}_k$.

Similarly, consider the functor $\mathbf{Hopf}_k \longrightarrow \mathbf{Tann}'_k$ defined by sending A to $(\mathbf{Comodf}_A, \omega_A)$ where $\omega_A : \mathbf{Comodf}_A \longrightarrow \mathbf{Vectf}_k$ is the forgetful functor, and $f : A \longrightarrow A'$ to (f_*, id). D denotes the composition of that functor followed by the natural functor $\mathbf{Tann}'_k \longrightarrow \mathbf{Tann}_k$.

Remark 2.2. *The category equivalence $\mathbf{Hopf}_k \longrightarrow \mathbf{AGS}_k; A \mapsto \mathrm{Spec}\, A$ makes the following diagram commute:*

$$\begin{tikzcd} \mathbf{AGS}_k \arrow[rr, "\cong"] \arrow[dr, "C"'] & & \mathbf{Hopf}_k \arrow[dl, "D"] \\ & \mathbf{Tann}_k & \end{tikzcd}.$$

Proposition 2.3. *We consider a functor from $\mathbf{Tann}'_k$ to the category of group valued functors on $\mathbf{Alg}_k$ defined by sending $(\mathcal{C}, \omega)$ to $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ and $(F, \varphi) : (\mathcal{C}, \omega) \longrightarrow (\mathcal{C}', \omega')$ to $\rho_{F,\varphi}$ where, for $R \in \mathbf{Alg}_k$, $(\rho_{F,\varphi})_R$ maps $\sigma \in \underline{\mathrm{Aut}}^{\otimes}(\omega')(R)$ to $\varphi \otimes R \circ$*

$F^*\sigma \circ \varphi^{-1} \otimes R \in \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$. *Then the functor factors as*

$$\mathbf{Tann}'_k \longrightarrow \mathbf{Tann}_k \xrightarrow{\pi_1} \mathbf{AGS}_k \longrightarrow (\textit{group valued functors on } \mathbf{Alg}_k)$$

Proof. It is a well-known fact that the functor factors through $\mathbf{AGS}_k$. Hence it is enough to show that the functor factors through $\mathbf{Tann}_k$. Consider $(F_1, \varphi_1), (F_2, \varphi_2) : (\mathcal{C}, \omega) \longrightarrow (\mathcal{C}', \omega')$ and a tensor functor isomorphism $\mu : F_1 \longrightarrow F_2$ such that $\varphi_2 \circ \omega'_* \mu = \varphi_1$. Then for $R \in \mathbf{Alg}_k$ and $\sigma \in \underline{\mathrm{Aut}}^{\otimes}(\omega')(R)$, since

$$\begin{tikzcd}
\omega' \circ F_1 \otimes R \arrow[r, "F_1^*\sigma"] \arrow[d, "\omega'_*\mu \otimes R"'] & \omega' \circ F_1 \otimes R \arrow[d, "\omega'_*\mu \otimes R"] \\
\omega' \circ F_2 \otimes R \arrow[r, "F_2^*\sigma"'] & \omega' \circ F_2 \otimes R
\end{tikzcd}$$

commutes, we see that $\varphi_1 \otimes R \circ F_1^*\sigma \circ \varphi_1^{-1} \otimes R = \varphi_2 \otimes R \circ \omega'_*\mu \otimes R \circ F_1^*\sigma \circ \omega'_*\mu^{-1} \otimes R \circ \varphi_2^{-1} \otimes R = \varphi_2 \otimes R \circ F_2^*\sigma \circ \varphi_2^{-1} \otimes R$. □

Theorem 2.4. *π_1 is a quasi-inverse of C and consequently $\mathbf{AGS}_k, \mathbf{Hopf}_k$ and $\mathbf{Tann}_k$ are equivalent.*

Proof. For $G \in \mathbf{AGS}$ and $R \in \mathbf{Alg}_k$, we take the morphism $G(R) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}(\omega_G)(R)$ which maps σ to $\widetilde{\sigma}$ where $\widetilde{\sigma}$ is the association $\mathbf{Repf}_G \ni V \mapsto \sigma_V : \omega_G(V) \otimes R \longrightarrow \omega_G(V) \otimes R$. This gives a functor morphism $\mathrm{id}_{\mathbf{AGS}_k} \longrightarrow \pi_1 \circ C$, which is in fact a functor isomorphism by [4] Proposition 2.8.

Next we will show that $\mathrm{id}_{\mathbf{Tann}_k}$ is isomorphic to $C \circ \pi_1$. Let $(\mathcal{C}, \omega) \in \mathbf{Tann}_k$ and $G = \pi_1(\mathcal{C}, \omega)$. Then for each $X \in \mathcal{C}$, we see $\omega(X)$ has natural structure of finite dimensional representation of G. Therefore ω factors as

$$\mathcal{C} \xrightarrow{F_{\mathcal{C},\omega}} \mathbf{Repf}_G \xrightarrow{\omega_G} \mathbf{Vectf}_k\,.$$

Here $F_{\mathcal{C},\omega}$ is a equivalence of k-linear tensor categories by [4] Theorem 2.11. We want to show that the association $(\mathcal{C}, \omega) \mapsto [F_{\mathcal{C},\omega}, \mathrm{id}] \in \mathrm{Hom}_{\mathbf{Tann}_k}\big((\mathcal{C}, \omega), C\big(\pi_1(\mathcal{C}, \omega)\big)\big)$ gives a functor morphism $\mathrm{id}_{\mathbf{Tann}_k} \longrightarrow C \circ \pi_1$, i.e., for each morphism $[F, \varphi] : (\mathcal{C}, \omega) \longrightarrow (\mathcal{C}', \omega')$ in $\mathbf{Tann}_k$, the diagram

$$\begin{tikzcd}
(\mathcal{C}, \omega) \arrow[r, "{[F_{\mathcal{C},\omega}, \mathrm{id}]}"] \arrow[d, "{[F,\varphi]}"'] & (\mathbf{Repf}_G, \omega_G) \arrow[d, "{[\rho^*, \mathrm{id}]}"] \\
(\mathcal{C}', \omega') \arrow[r, "{[F_{\mathcal{C}',\omega'}, \mathrm{id}]}"'] & (\mathbf{Repf}_{G'}, \omega_{G'})
\end{tikzcd}$$

where $G = \pi_1(\mathcal{C}, \omega), G' = \pi_1(\mathcal{C}', \omega')$ and $\rho = \pi_1(F, \varphi)$, is commutative. Since $(\rho^*, \mathrm{id}) \circ (F_{\mathcal{C},\omega}, \mathrm{id}) = (\rho^* \circ F_{\mathcal{C},\omega}, \mathrm{id})$ and $(F_{\mathcal{C}',\omega'}, \mathrm{id}) \circ (F, \varphi) = (F_{\mathcal{C}',\omega'} \circ F, \varphi)$, what we have to show is that there exists a tensor functor isomorphism $\mu : F_{\mathcal{C}',\omega'} \circ F \longrightarrow \rho^* \circ F_{\mathcal{C},\omega}$ that makes the diagram

$$\begin{tikzcd}
\omega_{G'} \circ F_{\mathcal{C}',\omega'} \circ F \arrow[rr, "\omega_{G'*}\mu"] \arrow[d, "="'] & & \omega_{G'} \circ \rho^* \circ F_{\mathcal{C},\omega} \arrow[d, "="] \\
\omega' \circ F \arrow[r, "\varphi"'] & \omega & \omega_G \circ F_{\mathcal{C},\omega} \arrow[l, "="]
\end{tikzcd}$$

commute (as a diagram of functors). By the definition of π_1, for $X \in \mathcal{C}, R \in \mathbf{Alg}_k$ and $\sigma \in G'(R)$,we see the diagram

$$\begin{array}{ccc} \omega' \circ F(X) \otimes R & \xrightarrow{\varphi_X \otimes R} & \omega(X) \otimes R \\ {\scriptstyle \sigma_{F(X)}} \Big\downarrow & & \Big\downarrow {\scriptstyle (\pi_1(F,\varphi)_R(\sigma))_X} \\ \omega' \circ F(X) \otimes R & \xrightarrow[\varphi_X \otimes R]{} & \omega(X) \otimes R \end{array}$$

is commutative. This means that $\varphi_X : \omega' \circ F(X) \longrightarrow \omega(X)$ is a morphism in $\mathbf{Repf}_{G'}$ under the natural action of G', namely, the existence of μ that we are looking for.

It remains to show that, in general, for $(\mathcal{C}, \omega), (\mathcal{C}', \omega') \in \mathbf{Tann}_k$ and a k-linear tensor category equivalence $F : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $F^*\omega' = \omega$, $[F, \mathrm{id}] : (\mathcal{C}, \omega) \longrightarrow (\mathcal{C}', \omega')$ is an isomorphism in $\mathbf{Tann}_k$. We take a quasi-inverse F^{-1} of F and a tensor functor isomorphism $\delta : F \circ F^{-1} \longrightarrow \mathrm{id}_{\mathcal{C}'}$. Then $(F, \mathrm{id}) \circ (F^{-1}, \omega'_*\delta) = (F \circ F^{-1}, \omega'_*\delta) \sim (\mathrm{id}, \mathrm{id})$. When we write ε for the element of $\mathrm{Hom}(F^{-1} \circ F, \mathrm{id})$ that corresponds to $F^*\delta \in \mathrm{Hom}(F \circ F^{-1} \circ F, F)$ by $F_* : \mathrm{Hom}(F^{-1} \circ F, \mathrm{id}) \xrightarrow{\cong} \mathrm{Hom}(F \circ F^{-1} \circ F, F)$, ε is an isomorphism and $\omega_*\varepsilon = \omega'_*F_*\varepsilon = \omega'_*F^*\delta = F^*\omega'_*\delta : \omega \circ F^{-1} \circ F \longrightarrow \omega$. This implies $(F^{-1}, \omega'_*\delta) \circ (F, \mathrm{id}) = (F^{-1} \circ F, F^*\omega'_*\delta) \sim (\mathrm{id}, \mathrm{id})$ and thus, we are done. $\square$

3. Semi-graphs of Tannakian categories

By the result of the preceding section, we may expect that the Tannakian fundamental group of the "fibered product" of Tannakian categories should be isomorphic to the amalgam (fibered coproduct) of their Tannakian fundamental groups. This is an analogy of Seifert-van Kampen theorem. In addition, we may consider the "HNN extension" of affine group schemes. In this section we will generalize these two constructions by introducing "semi-graphs of Tannakian categories". The idea of semi-graph of Tannakian categories is an analogy of "semi-graphs of anabelioids" introduced by Mochizuki in [5].

Definition 3.1. *(1) A semi-graph $\mathbb{G}$ consists of the following data:*

- *a set $\mathcal{V}$, whose elements we refer to as "vertices",*
- *a set $\mathcal{E}$, whose elements we refer to as "edges", each of whose elements e is a set of cardinality 2 satisfying the property "$e \neq e' \implies e \cap e' = \emptyset$",*
- *a collection ζ of maps, one for each edge e, such that $\zeta_e : e \longrightarrow \mathcal{V} \cup \{\mathcal{V}\}$ is a map from the set e to the set $\mathcal{V} \cup \{\mathcal{V}\}$.*

We refer to an edge e of a semi-graph $(\mathcal{V}, \mathcal{E}, \zeta)$ such that the inverse image of the subset $\mathcal{V} \subset \mathcal{V} \cup \{\mathcal{V}\}$ has cardinality 2 as closed. We say a semi-graph $\mathbb{G}$ is connected if it is connected with respect to its natural topology. See [5] *for more details.*

(2) A semi-graph $\mathfrak{C}$ of Tannakian categories over k consists of the following data:

- *a semi-graph $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta)$,*
- *for each vertex v of $\mathbb{G}$, a Tannakian category $\mathcal{C}_v$ over k,*

– *for each edge e of $\mathbb{G}$, a Tannakian category $\mathcal{C}_e$ over k, together with, for each pair $b \in e$ and $v \in \mathcal{V}$ such that $\zeta_e(b) = v$, a exact faithful k-linear tensor functor $F_b : \mathcal{C}_v \longrightarrow \mathcal{C}_e$.*

Let $\mathfrak{C} = (\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ be a semi-graph of Tannakian categories over k. We say $\mathfrak{C}$ is connected if its underlying semi-graph $\mathbb{G}$ is connected. If moreover there exists an edge e such that $\mathcal{C}_e$ is a neutral Tannakian category, i.e., it admits some neutral fiber functor, then we say $\mathfrak{C}$ is neutral.

(3) *Let $\mathfrak{C} = (\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ be a connected semi-graph of Tannakian categories over k. If $\mathbb{G}$ has at least one vertex, $\mathcal{B}(\mathfrak{C})$ denotes the category whose object is $((X_v)_{v\in\mathcal{V}}, (m_e)_{e:\textit{closed edge}})$ where for each $v \in \mathcal{V}$, $X_v \in \mathcal{C}_v$ and for each closed edge $e = \{b_1, b_2\}$, $m_e : F_{b_2}(X_{\zeta_e(b_2)}) \longrightarrow F_{b_1}(X_{\zeta_e(b_1)})$ is an isomorphism in $\mathcal{C}_e$ and whose morphism is $(F_v)_{v\in\mathcal{V}}$ that is compatible with m_e's. If $\mathbb{G}$ has no vertices and hence precisely one edge, say e, then we set $\mathcal{B}(\mathfrak{C}) = \mathcal{C}_e$.*

Note that for each vertex v there is a natural forgetful functor $P_v : \mathcal{B}(\mathfrak{C}) \longrightarrow \mathcal{C}_v$.

Proposition 3.2. *If a semi-graph $\mathfrak{C} = (\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ of Tannakian categories is neutral, then $\mathcal{B}(\mathfrak{C})$ has a natural structure of a rigid k-linear abelian tensor category whose unit object 1 satisfies $\mathrm{End}(1) \cong k$ and admits a neutral fiber functor, and hence becomes a neutral Tannakian category over k. Moreover, for each vertex v, the forgetful functor P_v is an exact faithful k-linear tensor functor.*

Proof. Of course we may assume $\mathfrak{C}$ has at least one vertex. It is clear that $\mathcal{B}(\mathfrak{C})$ has natural structure of k-linear abelian category.

To see that $\mathcal{B}(\mathfrak{C})$ is a tensor category, for $X = ((X_v), (m_e)), X' = ((X'_v), (m'_e)) \in \mathcal{B}(\mathfrak{C})$, we set $X \otimes X' = ((X_v \otimes X'_v), (m_e \otimes m'_e))$ where, for $e = \{b_1, b_2\}$, $m_e \otimes m'_e$ denotes the composition of $F_{b_2}(X_{\zeta(b_2)} \otimes X'_{\zeta(b_2)}) \cong F_{b_2}(X_{\zeta(b_2)}) \otimes F_{b_2}(X'_{\zeta(b_2)}) \overset{m_e\otimes m'_e}{\cong} F_{b_1}(X_{\zeta(b_1)}) \otimes F_{b_1}(X'_{\zeta(b_1)}) \cong F_{b_1}(X_{\zeta(b_1)} \otimes X'_{\zeta(b_1)})$. Obviously this makes $\mathcal{B}(\mathfrak{C})$ into tensor category.

To see that $\mathcal{B}(\mathfrak{C})$ has a unit object, we set $1 = ((1_v), (\iota_e))$ where 1_v denotes a unit object in $\mathcal{C}_v$ for each vertex v, and ι_e denotes the canonical isomorphism $F_{b_2}(1_{v_2}) \longrightarrow F_{b_1}(1_{v_1})$ for each $e = \{b_1, b_2\}$ (note that exact faithful k-linear tensor functor maps a unit object to a unit object). Moreover, there is a canonical isomorphism $1 = ((1_v), (\iota_e)) \longrightarrow ((1_v \otimes 1_v), (\iota_e \otimes \iota_e)) = 1 \otimes 1$. We can easily check that $\mathcal{B}(\mathfrak{C}) \longrightarrow \mathcal{B}(\mathfrak{C}); X \mapsto 1 \otimes X$ is an equivalence of categories.

For rigidity, we must construct internal homs. For $X = ((X_v), (m_e)), Y = ((Y_v), (n_e)) \in \mathcal{B}(\mathfrak{C})$, we set $\underline{\mathrm{Hom}}(X, Y) = ((\underline{\mathrm{Hom}}(X_v, Y_v)), (m_e, n_e)))$ where for $e = \{b_1, b_2\}$, (m_e, n_e) denotes the composition of

$$\begin{aligned} F_{b_2}(\underline{\mathrm{Hom}}(X_{\zeta(b_2)}, Y_{\zeta(b_2)})) &\cong \underline{\mathrm{Hom}}(F_{b_2}(X_{\zeta(b_2)}), F_{b_2}(Y_{\zeta(b_2)})) \\ &\cong F_{b_2}(X_{\zeta(b_2)})^\vee \otimes F_{b_2}(Y_{\zeta(b_2)}) \\ &\overset{(m_e^\vee)^{-1}\otimes n_e}{\cong} F_{b_1}(X_{\zeta(b_1)})^\vee \otimes F_{b_1}(Y_{\zeta(b_1)}) \\ &\cong \underline{\mathrm{Hom}}(F_{b_1}(X_{\zeta(b_1)}), F_{b_1}(Y_{\zeta(b_1)})) \\ &\cong F_{b_1}(\underline{\mathrm{Hom}}(X_{\zeta(b_1)}, Y_{\zeta(b_1)})). \end{aligned}$$

Then it is clear that for $X_1, X_2, Y_1, Y_2 \in \mathcal{B}(\mathfrak{C})$ the natural morphism $\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \longrightarrow \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ is an isomorphism and that for $X \in \mathcal{B}(\mathfrak{C})$ the natural morphism $X \longrightarrow X^{\vee\vee}$ is an isomorphism.

We claim that $\mathrm{End}(1) \cong k$. Note that

$$\mathrm{End}(1) = \left\{ (f_v) \in \prod_v \mathrm{End}(1_v) \mid \iota_e \circ F_{b_2}(f_{\zeta(b_2)}) = F_{b_1}(f_{\zeta(b_1)}) \circ \iota_e \right\}.$$

Since for a vertex v_0 the composition of $k \longrightarrow \mathrm{End}(1) \xrightarrow{\mathrm{pr}_{v_0}} \mathrm{End}(1_{v_0}) \cong k$ equals to identity, it is enough to show that pr_{v_0} is injective. Let $(f_v), (f'_v) \in \mathrm{End}(1)$ such that $f_{v_0} = f'_{v_0}$. We set $\mathcal{W} = \{v \in \mathcal{V} \mid f_v = f'_v\} \subset \mathcal{V}$. Since $\mathcal{W}$ is non-empty and $\mathbb{G}$ is connected, if $\mathcal{W} \neq \mathcal{V}$, there exists an edge $e = \{b_1, b_2\}$ such that $\zeta(b_1) \in \mathcal{W}$ and $\zeta(b_2) \notin \mathcal{W}$. But then $F_{b_1}(f_{\zeta(b_1)}) = F_{b_1}(f'_{\zeta(b_1)})$ and hence $F_{b_2}(f_{\zeta(b_2)}) = F_{b_2}(f'_{\zeta(b_2)})$. Since the composition of $k \cong \mathrm{End}(1_{\zeta(b_2)}) \xrightarrow{F_{b_2}} \mathrm{End}(1_e) \cong k$ is identity, $f_{\zeta(b_2)} = f'_{\zeta(b_2)}$, which is absurd.

For existence of a neutral fiber functor, we may assume, for an edge $e_0 = \{b_1, b_2\}$, $\mathcal{C}_{e_0}$ have a neutral fiber functor ω_0 and $\zeta(b_1)$ is a vertex. We claim that $\omega = \omega_0 \circ P_{\zeta(b_1)}$ is a neutral fiber functor on $\mathcal{B}(\mathfrak{C})$. It is enough to show that, for each vertex v, $P_v : \mathcal{B}(\mathfrak{C}) \longrightarrow \mathcal{C}_v$ is an exact faithful k-linear tensor functor. The faithfulness can be checked similarly to the argument in the preceding paragraph and the other conditions are clear. □

Proposition 3.3. *Let $\mathfrak{C} = (\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ be a neutral semi-graph of Tannakian categories. Let us consider the situation that $\mathcal{V} = \{v_1, v_2\}$, $\mathcal{E} = \{e = \{b_1, b_2\}\}$, $\zeta_e(b_i) = v_i$ $(i = 1, 2)$ and ω_e is a neutral fiber functor on $\mathcal{C}_e$. We write $\omega_{v_i} = F^*_{b_i}\omega_e$, $\omega = P^*_{v_1}\omega_{v_1}$ $G_0 = \pi_1(\mathcal{C}_e, \omega_e)$ and $G_i = \pi_1(\mathcal{C}_{v_i}, \omega_{v_i})$ $(i = 1, 2)$. Then $\pi_1(\mathcal{B}(\mathfrak{C}), \omega)$ is isomorphic to $G_1 *_{G_0} G_2$.*

Proof. We rewrite $\mathcal{C}_e$ (resp. $\mathcal{C}_{v_i}$, $\mathcal{B}(\mathfrak{C})$, F_{b_i}, P_{v_i}, ω_e and ω_{v_i}) as $\mathcal{C}_0$ (resp. $\mathcal{C}_i$, $\mathcal{C}$, F_i, P_i, ω_0 and ω_i). Note that $\mathcal{C}$ equals to the fibered product category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$. There is a natural functor isomorphism $\mu : F_2 \circ P_2 \longrightarrow F_1 \circ P_1$ that associate to $(X_1, X_2, m) \in \mathcal{C}$ the isomorphism $F_2(P_2(X_1, X_2, m)) = F_2(X_2) \xrightarrow{m} F_1(X_1) = F_1(P_1(X_1, X_2, m))$. This is indeed a tensor functor isomorphism since for $(X_1, X_2, m), (Y_1, Y_2, n) \in C$, the diagram

$$\begin{tikzcd}
F_2 \circ P_2(X_1 \otimes Y_1, X_2 \otimes Y_2, m \otimes n) \arrow[r, "\cong"] \arrow[d, equal] & F_2 \circ P_2(X_1, X_2, m) \otimes F_2 \circ P_2(Y_1, Y_2, n) \arrow[d, equal] \\
F_2(X_2 \otimes Y_2) \arrow[r, "\cong"] \arrow[d] & F_2(X_2) \otimes F_2(Y_2) \arrow[d, "m\otimes n"] \\
F_1(X_1 \otimes Y_1) \arrow[r, "\cong"] \arrow[d, equal] & F_1(X_1) \otimes F_1(Y_1) \arrow[d, equal] \\
F_1 \circ P_1(X_1 \otimes Y_1, X_2 \otimes Y_2, m \otimes n) \arrow[r, "\cong"'] & F_1 \circ P_1(X_1, X_2, m) \otimes F_1 \circ P_1(Y_1, Y_2, n)
\end{tikzcd}$$

is commutative. Then we can easily see

$$\begin{tikzcd}
(\mathcal{C},\omega) \arrow[r, "{[P_2,\omega_{0*}\mu]}"] \arrow[d, "{[P_1,\mathrm{id}]}"'] & (\mathcal{C}_2,\omega_2) \arrow[d, "{[F_2,\mathrm{id}]}"] \\
(\mathcal{C}_1,\omega_1) \arrow[r, "{[F_1,\mathrm{id}]}"'] & (\mathcal{C}_0,\omega_0)
\end{tikzcd}$$

is a commutative diagram in $\mathbf{Tann}_k$. We set $G_i = \pi_1(\mathcal{C}_i,\omega_i)$, $\rho_i = \pi_1\big([F_i,\mathrm{id}]\big)$, $\nu_1 = \pi_1\big([P_1,\mathrm{id}]\big)$ and $\nu_2 = \pi_1\big([P_2,\omega_{0*}\mu]\big)$. Let H be a amalgam of G_1 and G_2 over G_0. Then we get a commutative diagram

$$\begin{tikzcd}
G & & \\
& H \arrow[ul] & G_2 \arrow[l, "\iota_2"'] \arrow[ull, bend right, "\nu_2"'] \\
& G_1 \arrow[u, "\iota_1"] \arrow[uul, bend left, "\nu_1"] & G_0 \arrow[l, "\rho_1"] \arrow[u, "\rho_2"']
\end{tikzcd}$$

in AGS_k and furthermore a commutative diagram

$$\begin{tikzcd}
\mathbf{Repf}_G \arrow[dr] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\
& \mathbf{Repf}_H \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \mathbf{Repf}_{G_1} \times_{\mathbf{Repf}_{G_0}} \mathbf{Repf}_{G_2} \arrow[r] \arrow[d] & \mathbf{Repf}_{G_2} \arrow[d] \\
& & \mathbf{Repf}_{G_1} \arrow[r] & \mathbf{Repf}_{G_0}
\end{tikzcd}$$

of Tannakian categories. Since $C_i \cong \mathbf{Repf}_{G_i}$ $(i = 0, 1, 2)$, we have $\mathcal{C} \cong \mathbf{Repf}_{G_1} \times_{\mathbf{Repf}_{G_0}} \mathbf{Repf}_{G_2}$. We are going to show that $\mathbf{Repf}_G \cong \mathbf{Repf}_H$. Considering the commutative diagram

$$\begin{tikzcd}
& & \mathcal{C} \arrow[dll, "\cong"'] \arrow[d, "\cong"] \\
\mathbf{Repf}_G \arrow[r] & \mathbf{Repf}_H \arrow[r] & \mathbf{Repf}_{G_1} \times_{\mathbf{Repf}_{G_0}} \mathbf{Repf}_{G_2}
\end{tikzcd}$$

it is enough to show that the natural functor $\mathbf{Repf}_H \longrightarrow \mathbf{Repf}_{G_1} \times_{\mathbf{Repf}_{G_0}} \mathbf{Repf}_{G_2}$ is fully faithful. We deduce it from the following lemma. □

Lemma 3.4. *Let $A_i \in \mathbf{Hopf}_k$ $(i = 0, 1, 2)$, $f_i : A_i \longrightarrow A_0$ be Hopf algebra homomorphisms $(i = 1, 2)$ and A be a fibered product of A_1 and A_2 over A_0. Then for $V, W \in \mathbf{Comodf}_A$ and a k-linear map $g : V \longrightarrow W$, g is an A-comodule homomorphism if and only if it is an A_1-comodule homomorphism and A_2-comodule homomorphism.*

Proof. Let $\rho_V : V \longrightarrow V \otimes_k A$ and $\rho_W : W \longrightarrow W \otimes_k A$ be structure morphisms of V and W as A-comodules. What we have to show is that $(g \otimes \mathrm{id}_A) \circ \rho_V = \rho_W \circ g$

if $(g \otimes \mathrm{id}_{A_i}) \circ (\mathrm{id}_V \otimes \mathrm{pr}_i) \circ \rho_V = (\mathrm{id}_W \otimes \mathrm{pr}_i) \circ \rho_W \circ g$ $(i = 1, 2)$ where pr_i denotes the i-th projection $A \longrightarrow A_i$ $(i = 1, 2)$.

Take bases $\{e_i\} \subset V, \{e'_j\} \subset W$. We write

$$\begin{aligned} g(e_i) &= \sum_j t_{ij} e'_j \ (t_{ij} \in k) \\ \rho_V(e_i) &= \sum_l e_l \otimes a_{il} \ (a_{il} \in A) \\ \rho_W(e'_j) &= \sum_m e'_m \otimes b_{jm} \ (b_{jm} \in A). \end{aligned}$$

Let I be the ideal of A generated by

$$\left\{ \sum_l t_{lm} a_{il} - \sum_j t_{ij} b_{jm}, S_A\left(\sum_l t_{lm} a_{il} - \sum_j t_{ij} b_{jm} \right) \right\}_{i,m}$$

where S_A denotes the antipode of A. We claim that I is a Hopf ideal, i.e., (1) $\Delta_A(I) \subset I \otimes_k A + A \otimes_k I$, (2) $S_A(I) \subset I$, (3) $\varepsilon_A(I) = 0$.

Let us check that $\Delta_A(I) \subset I \otimes_k A + A \otimes_k I$. Since Δ_A is a k-algebra homomorphism it is enough to show that Δ_A maps the generators of I into $I \otimes_k A + A \otimes_k I$. Using the condition that $(\mathrm{id} \otimes \Delta_A) \circ \rho_V = (\rho_V \otimes \mathrm{id}) \circ \rho_V$, we know that $\Delta_A(a_{il}) = \sum_s a_{sl} \otimes a_{is}$ and similarly $\Delta_A(b_{jm}) = \sum_r b_{rm} \otimes b_{jr}$. Then we can achieve (1) by the caluculation

$$\begin{aligned} &\Delta_A\left(\sum_l t_{lm} a_{il} - \sum_j t_{ij} b_{jm} \right) \\ =&\sum_l \left(t_{lm} \sum_s a_{sl} \otimes a_{is} \right) - \sum_j \left(t_{ij} \sum_r b_{rm} \otimes b_{jr} \right) \\ =&\sum_s \left(\sum_l t_{lm} a_{sl} - \sum_j t_{sj} b_{jm} \right) \otimes a_{is} + \sum_{s,j} t_{sj} b_{jm} \otimes a_{is} \\ &+ \sum_r b_{rm} \otimes \left(\sum_l t_{lr} a_{il} - \sum_j t_{ij} b_{jr} \right) - \sum_{r,l} t_{lr} b_{rm} \otimes a_{il} \\ =&\sum_s \left(\sum_l t_{lm} a_{sl} - \sum_j t_{sj} b_{jm} \right) \otimes a_{is} + \sum_r b_{rm} \otimes \left(\sum_l t_{lr} a_{il} - \sum_j t_{ij} b_{jr} \right) \\ &\in I \otimes_k A + I \otimes_k A \end{aligned}$$

and

$$\begin{aligned} &\Delta_A\left(S_A\left(\sum_l t_{lm} a_{il} - \sum_j t_{ij} b_{jm} \right) \right) \\ =&\tau \circ (S_A \otimes S_A) \circ \Delta_A\left(\sum_l t_{lm} a_{il} - \sum_j t_{ij} b_{jm} \right) \\ &\in \tau \circ (S_A \otimes S_A)(I \otimes_k A + A \otimes_k I) \subset \tau(I \otimes_k A + A \otimes_k I) \subset I \otimes_k A + A \otimes_k I. \end{aligned}$$

(2) is obvious since S_A is a k-algebra homomorphism. To check (3), similarly to (1), it is enough to show that ε_A maps the generators of I to 0. Note that

$\varepsilon_A \circ S_A = \varepsilon_A$ ([1] Theorem 2.1.4). For each i and m, since

$$\begin{aligned}&\sum_m e'_m \otimes \varepsilon\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big)\\ =&(\mathrm{id}\otimes\varepsilon)\circ((g\otimes\mathrm{id})\circ\rho_V - \rho_W\circ g)(e_i)\\ =&((g\otimes\varepsilon)\circ\rho_V - (\mathrm{id}\otimes\varepsilon)\circ\rho_W\circ g)(e_i)\\ =&(g-g)(e_i)=0\end{aligned}$$

and $\{e'_m\}$ is a basis, we see that $\varepsilon\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big) = 0$.

Next we claim that (4) $I \subset \operatorname{Ker}\mathrm{pr}_1 \cap \operatorname{Ker}\mathrm{pr}_2$. For i and m, since

$$\begin{aligned}&\sum_m e'_m \otimes \mathrm{pr}_i\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big)\\ =&(\mathrm{id}\otimes\mathrm{pr}_i)\circ((g\otimes\mathrm{id})\circ\rho_V - \rho_W\circ g)(e_i)=0\end{aligned}$$

and $\{e'_m\}$ is a basis, we see that $\mathrm{pr}_i\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big) = 0$. Moreover, since pr_i is a k-Hopf algebra homomorphism, $\mathrm{pr}_i\Big(S\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big)\Big) = 0$.

By (1),(2),(3) and (4) we obtain a commutative diagram

$$\begin{tikzcd} A/I \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & A_0 \end{tikzcd}$$

in $\mathbf{Hopf}_k$ and hence a k-Hopf algebra homomorphism $A/I \longrightarrow A$ such that the composition of $A \longrightarrow A/I \longrightarrow A$ is id_A using the universality of A. Thus $I = 0$, which implies that for each i and m, we have $\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big) = 0$ and so

$$\begin{aligned}&((g\otimes\mathrm{id})\circ\rho_V - \rho_W\circ g)(e_i)\\ =&\Big(\sum_{l,m} t_{lm}e'_m\otimes a_{il}\Big) - \Big(\sum_{j,m} t_{ij}e'_m\otimes b_{jm}\Big)\\ =&\sum_m e'_m\otimes\Big(\sum_l t_{lm}a_{il} - \sum_j t_{ij}b_{jm}\Big)=0.\end{aligned}$$

Since $\{e_i\}$ is a basis, we conclude that $(g\otimes\mathrm{id})\circ\rho_V - \rho_W\circ g = 0$. □

Proposition 3.5. *Let* $\mathfrak{C} = (\mathbb{G} = (\mathcal{V},\mathcal{E},\zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ *be a neutral semi-graph of Tannakian categories. Let us consider the situation that* $\mathcal{V} = \{v\}$, $\mathcal{E} = \{e = \{b_1, b_2\}\}$, $\zeta_e(b_i) = v$ $(i = 1,2)$ *and* ω_e *is a neutral fiber functor on* $\mathcal{C}_e$. *We set* $\omega_i = F^*_{b_i}\omega_e$ *and* $\omega = P^*_v\omega_1$. *We assume there is a tensor functor isomorphism* $\varphi : \omega_2 \longrightarrow \omega_1$. *We write* $G_0 = \pi_1(\mathcal{C}_e, \omega_e)$, $G_1 = \pi_1(\mathcal{C}_v, \omega_1)$, $\rho_1 = \pi_1(F_{b_1}, \mathrm{id}), \rho_2 = \pi_1(F_{b_2}, \varphi) : G_0 \longrightarrow G_1$ *and* $\mathcal{C} = \mathcal{B}(\mathfrak{C})$. *Then* $\pi_1(\mathcal{C}, \omega)$ *is isomorphic to the* k*-affine* *group scheme G obtained by applying Corollary 1.17 to the composition of*

$$\begin{aligned} G_0 &\xrightarrow{\rho_2\times u\times(i\circ\rho_1)\times(i\circ u)} G_1\times\mathbb{Z}^{\mathrm{alg}}\times G_1\times\mathbb{Z}^{\mathrm{alg}} \\ &\xrightarrow{\qquad\qquad\qquad} (G_1*\mathbb{Z}^{\mathrm{alg}})\times(G_1*\mathbb{Z}^{alg})\times(G_1*\mathbb{Z}^{\mathrm{alg}})\times(G_1*\mathbb{Z}^{\mathrm{alg}}) \\ &\xrightarrow{\qquad m\qquad} G_1*\mathbb{Z}^{\mathrm{alg}} \end{aligned}$$

where i and m denotes the inverse and multiplication repectively, $\mathbb{Z}^{\mathrm{alg}}$ is the algebraic hull of $\mathbb{Z}$ and $u : \operatorname{Spec} k \longrightarrow \mathbb{Z}^{\mathrm{alg}}$ is the image of 1 by the natural group homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}^{\mathrm{alg}}(k)$.

Proof. We may assume $\mathcal{C}_e = \mathbf{Repf}_{G_0}, \mathcal{C}_v = \mathbf{Repf}_{G_1}$ and ω_e is the forgetful functor ω_{G_0}. It is enough to show that there is a equivalence $\mathcal{C} \longrightarrow \mathbf{Repf}_G$ of tensor k-linear abelian categories that makes the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\cong"] \arrow[dr, "\omega"'] & & \mathbf{Repf}_G \arrow[dl, "\omega_G"] \\ & \mathbf{Vectf}_k & \end{tikzcd}$$

commutes. Let $(X, m) \in \mathcal{C}$, i.e., $X \in \mathbf{Repf}_{G_1}$ and m is an isomorphism $\rho_2^* X \longrightarrow \rho_1^* X$ in $\mathbf{Repf}_{G_0}$. Let $\mathbb{Z}^{\mathrm{alg}} \longrightarrow \mathrm{GL}(\omega_{G_1}(X))$ be the representation of $\mathbb{Z}^{\mathrm{alg}}$ corresponding to the representation $\mathbb{Z} \longrightarrow \mathrm{GL}(\omega_{G_1}(X))(k); 1 \mapsto m$ of $\mathbb{Z}$. Then we obtain $G_1 * \mathbb{Z}^{\mathrm{alg}} \longrightarrow \mathrm{GL}(\omega_1(X))$. We claim that the diagram

$$\begin{tikzcd} G_0 \arrow[r, "m\circ(\rho_1\times u\times(i\circ\rho_2)\times(i\circ u))"] \arrow[d] & G_1 * \mathbb{Z}^{\mathrm{alg}} \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \mathrm{GL}(\omega_1(X)) \end{tikzcd}$$

is commutative. For $R \in \mathbf{Alg}_k$ and $\sigma \in G_0(R)$, since the diagram

$$\begin{tikzcd} \omega_{G_1}(X)\otimes R \arrow[r, "(\rho_2)_R(\sigma)"] \arrow[d, "m\otimes R"'] & \omega_{G_1}(X)\otimes R \arrow[d, "m\otimes R"] \\ \omega_{G_1}(X)\otimes R \arrow[r, "(\rho_1)_R(\sigma)"'] & \omega_{G_1}(X)\otimes R \end{tikzcd}$$

is commutative, we have

$$(\rho_1)_R(\sigma)\cdot u_R(e)\cdot(i\circ(\rho_2)_R(\sigma))\cdot(i\circ u_R(e)) = \mathrm{id}_{\omega_{G_1}(X)\otimes R}\,.$$

Thus the diagram

$$\begin{tikzcd}
G_0(R) \arrow[dr] \arrow[drr, bend left] \arrow[dddd] & & \\
& G_1(R)\times\mathbb{Z}^{alg}(R)\times G_1(R)\times\mathbb{Z}^{alg}(R) \arrow[r, phantom, "="] \arrow[d] & (G_1\times\mathbb{Z}^{alg}\times G_1\times\mathbb{Z}^{alg})(R) \arrow[d] \\
& (G_1(R)*\mathbb{Z}^{alg}(R))^4 \arrow[r] \arrow[d] & (G_1*\mathbb{Z}^{alg})^4(R) \arrow[d] \\
& G_1(R)*\mathbb{Z}^{alg}(R) \arrow[r] \arrow[dr] & (G_1*\mathbb{Z}^{alg})(R) \arrow[d] \\
\operatorname{Spec} k(R) \arrow[rr] & & \mathrm{GL}(\omega_{G_1}(X))(R)
\end{tikzcd}$$

is commutative, which proves the claim. Then by Corollary 1.17 we obtain an object $\Phi(X,m) = (G \longrightarrow \mathrm{GL}(\omega_1(X))$ of $\mathbf{Repf}_G$. Let $f : (X,m) \longrightarrow (Y,n)$ be a morphism of $\mathcal{C}$, i.e., f is a morphism of $\mathbf{Repf}_{G_1}$ such that the diagram

$$\begin{tikzcd}
\rho_2^*X \arrow[r, "f"] \arrow[d, "m"'] & \rho_2^*Y \arrow[d, "n"] \\
\rho_1^*X \arrow[r, "f"'] & \rho_1^*Y
\end{tikzcd}$$

is commutative, which implies that the k-linear map f is a morphism of $\mathbf{Repf}_{\mathbb{Z}^{\mathrm{alg}}}$. By Lemma 3.4, we see that f is a morphism of $\mathbf{Repf}_{G_1*\mathbb{Z}^{\mathrm{alg}}}$ and furthermore a morphism of $\mathbf{Repf}_G$ by Remark 1.13. Hence we can define a functor $\Phi : \mathcal{C} \longrightarrow \mathbf{Repf}_G$. Clearly, Φ is a k-linear tensor funtor and $\omega_G \circ \Phi = \omega$.

Conversely, Let $Z = (G \longrightarrow \mathrm{GL}(V)) \in \mathbf{Repf}_G$. We set $X = (G_1 \longrightarrow G \longrightarrow \mathrm{GL}(V))$. Let m be the image of 1 by $\mathbb{Z} \longrightarrow \mathbb{Z}^{\mathrm{alg}}(k) \longrightarrow \mathrm{GL}(V)(k)$. We set $\Psi(Z) = (X,m)$. Clearly Ψ becomes a functor $\mathbf{Repf}_G \longrightarrow \mathcal{C}$ and gives a quasi-inverse of Φ. □

Theorem 3.6. *Let $\mathfrak{C} = (\mathbb{G} = (\mathcal{V},\mathcal{E},\zeta),(\mathcal{C}_v)_{v\in\mathcal{V}},(\mathcal{C}_e)_{e\in\mathcal{E}},(F_b)_b)$ be a neutral semi-graph of Tannakian categories such that $\mathcal{V}$ and $\mathcal{E}$ are finite. We consider the case that $\mathcal{C}_v = \mathbf{Repf}_{G_v}, \mathcal{C}_e = \mathbf{Repf}_{G_e}$ and $F_b = \rho_b^*$ where $G_v, G_e \in \mathbf{AGS}_k$ $(v \in \mathcal{V}, e \in \mathcal{E})$ and ρ_b is a k-affine group scheme morphism $G_e \longrightarrow G_{\zeta(b)}$ $(b \in e, \zeta(b) \in \mathcal{V})$. We can calculate $\pi_1(\mathcal{B}(\mathfrak{C}))$ as follows.*

1. *(1) If there is a non-closed edge e_0, we consider a new semi-graph of Tanakian categories $\mathfrak{C}' = (\mathbb{G}' = (\mathcal{V},\mathcal{E}\backslash\{e_0\},\zeta),(\mathcal{C}_v)_v,(\mathcal{C}_e)_{e\in\mathcal{E}\backslash\{e_0\}},(F_b)_b)$. Then $\mathcal{B}(\mathfrak{C}')$ is naturally isomorphic to $\mathcal{B}(\mathfrak{C})$. We replace $\mathfrak{C}$ by $\mathfrak{C}'$. We repeat this argument and assume there is no non-closed edge.*
2. *(2) We consider the case that there is a closed edge $e_0 = \{b_1,b_2\}$ such that $\zeta(b_1) \neq \zeta(b_2)$. We set*

$$\mathcal{V}' = \mathcal{V} \setminus \{\zeta(b_1),\zeta(b_2)\} \amalg \{v_0\}$$

$$\mathcal{E}' = \mathcal{E} \setminus \{e_0\}$$

$$\zeta'(b) = \left(\begin{array}{ll} \zeta(b) & (\textit{if } \zeta(b) \notin \{\zeta(b_1),\zeta(b_2)\}) \\ v_0 & (\textit{if } \zeta(b) \in \{\zeta(b_1),\zeta(b_2)\}) \end{array} \right. \quad (b \in e \in \mathcal{E}')$$

$$\mathcal{C}_{v_0} = \mathbf{Repf}_{G_{\zeta(b_1)} *_{G_e} G_{\zeta(b_2)}} \cong \mathcal{B}\left(\mathbf{Repf}_{G_{\zeta(b_1)}} \bullet \underset{\mathbf{Repf}_{G_e}}{\overline{\qquad\qquad\qquad}} \bullet \mathbf{Repf}_{G_{\zeta(b_2)}} \right)$$

$$\rho'_b = \left(\begin{array}{ll} \rho_b & (\textit{if } \zeta(b) \notin \{\zeta(b_1), \zeta(b_2)\}) \\ (G_e \xrightarrow{\rho_b} G_{\zeta(b)} \longrightarrow G_{\zeta(b_1)} *_{G_e} G_{\zeta(b_2)}) & (\textit{if } \zeta(b) \in \{\zeta(b_1), \zeta(b_2)\}) \end{array} \right. \quad (b \in e \in \mathcal{E}')$$

$$\mathfrak{C}' = ((\mathcal{V}', \mathcal{E}', \zeta'), (\mathcal{C}_v), (\mathcal{C}_e), (\rho'^*_b))$$

Then $\mathcal{B}(\mathfrak{C}')$ is naturally isomorphic to $\mathcal{B}(\mathfrak{C})$. We substitute $\mathfrak{C}'$ for $\mathfrak{C}$. We repeat this argument and assume there is no edge $e = \{b_1, b_2\}$ such that $\zeta(b_1) \neq \zeta(b_2)$.

(3) *We may assume there is a unique vertex v and all edges are closed. We consider the case that there is a edge e_0. We set*

$$\mathcal{E}' = \mathcal{E} \setminus \{e_0\}$$

$$\mathcal{C}'_v = \mathbf{Repf}_G \cong \mathcal{B}\left(\mathbf{Repf}_{G_e} \bigcirc\!\!\bullet\, \mathbf{Repf}_{G_v} \right)$$

$$\rho'_b = (G_e \xrightarrow{\rho_b} G_v \longrightarrow G) \ (b \in e \in \mathcal{E}')$$

$$\mathfrak{C}' = ((\mathcal{V}, \mathcal{E}', \zeta), (\mathcal{C}'_v), (\mathcal{C}_e), (\rho'^*_b))$$

then $\mathcal{B}(\mathfrak{C}')$ is naturally isomorphic to $\mathcal{B}(\mathfrak{C})$. We substitute $\mathfrak{C}'$ for $\mathfrak{C}$. We repeat this argument and assume there is a unique vertex and no edge.

After these three steps, $\mathfrak{C}$ consists of one Tannakian category $\mathbf{Repf}_G$ where G is a k-affine group scheme calculated from the first data $\mathfrak{C}$. We conclude that G is what we want.

Corollary 3.7. *Let $\mathfrak{C} = (\mathbb{G} = (\mathcal{V}, \mathcal{E}, \zeta), (\mathcal{C}_v)_{v\in\mathcal{V}}, (\mathcal{C}_e)_{e\in\mathcal{E}}, (F_b)_b)$ be a neutral semi-graph of Tannakian categories such that $\mathcal{V}$ and $\mathcal{E}$ are finite. Suppose we are given, for each closed edge e, neutral fiber functor ω_e on $\mathcal{C}_e$ and, for each triple $(v, e = (b_1, b_2), e' = (b'_1, b'_2))$ such that $\zeta(b_1) = \zeta(b'_1) = v$, isomorphism between $F^*_{b_1}\omega_e$ and $F^*_{b'_1}\omega_{e'}$. In this case $\mathfrak{C}$ is isomorphic to a neutral semi-graph of Tannakian categories of the form like one of the theorem. Thus we can calculate the fundamental group of $\mathfrak{C}$ in the manner of the theorem.*

Remark 3.8. *Let $\mathcal{C}$ be a neutral Tannakian category over an algebraically closed field k. Suppose there is a set J of objects which generate $\mathcal{C}$ as a tensor category satisfying one of the following two conditions:*

- *J is countable.*
- *The cardinality of J is less than the one of k.*

Then any two neutral fiber functors on $\mathcal{C}$ are isomorphic ([2])*. Thus for a neutral semi-graph $\mathfrak{C}$ of Tannakian categories over an algebraically closed field k, if the Tannakian category associated to each edge admits a neutral fiber functor and the Tannakian category associated to each vertex satisfies one of the conditions above, we may apply the corollary.*

References

[1] Eiichi Abe, Hopf algebras, Cambridge Uniersity Press, 1982.
[2] Christopher Deninger, A remark on the structure of torsors under an affine group scheme, http://arxiv.org/abs/1304.6216, 2013.
[3] Michiel Hazewinkel, Cofree coalgebras and multivariable recursiveness, Journal of Pure and Applied Algebra 183, pp 61 - 103, 2003.
[4] P.Deligne, J.Milne, Tannakian Categories, http://www.jmilne.org/math/xnotes/tc.pdf, 2012.
[5] Shinnichi Mochizuki, Semi-graphs of Anabelioids, Publ. Res. Inst. Math. Sci. 42 , 2006.
[6] Luis Ribes, Pavel Zalesskii, Profinite Groups, Springer-Verlag Berlin Heidelberg, 2010.