INTER-UNIVERSAL TEICHMÜLLER
THEORY: A PROGRESS REPORT

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“Travel and Lectures”

§1. Comparison with Earlier “Teichmüller Theories”

§2. The Two Underlying Dimensions of Arithmetic Fields

§3. The Log-Theta-Lattice

§4. Inter-universality and Anabelian Geometry

§5. Expected Main Results
§1. **Comparison w/Earlier “Teich. Theories”**

**Classical Complex Teichmuller Theory:**

Relative to canonical coord. \( z = x + iy \) (assoc’d to a square diff.) on the Riemann surface, **Teichmüller deformations** given by

\[
\begin{align*}
z & \mapsto \zeta = \xi + i\eta = Kx + iy
\end{align*}
\]

— where \( 1 < K < \infty \) is the **dilation** factor.

**Key point:** **One** holomorphic dimension, but **two** underlying real dimensions, of which **one** is **dilated**, while the **other** is held **fixed**!
$p$-adic Teich. Theory:

• $p$-adic canon. liftings of a hyp. curve in pos. char. equipped with a nilp. ind. bun.
• Frobenius liftings over ord. locus of moduli stack, tautological curve — cf. Poincaré upper half-plane, Weil-Petersson metric/$\mathbb{C}$.

Analogy between IUTeich and $p$Teich:

scheme theory $\leftrightarrow$ scheme theory/$\mathbb{F}_p$
“log” no. field $\leftrightarrow$ pos. char. hyp. curve
once-punct’d ell. curve/NF $\leftrightarrow$ nilp. IB
log-$\Theta$-lattice $\leftrightarrow$ $p$-adic can. + Frob. lift.
§2. Two Underlying Dims. of Arith. Fields
Addition and Multiplication, Cohom. Dim.:

Regard ring structure of rings such as \( \mathbb{Z} \) as

one-dim. “arith. hol. str.”!

— which has

two underlying comb. dims.!

\[(\mathbb{Z}, +) \quad \sqcup \quad (\mathbb{Z}, \times)\]

1-comb. dim. 1-comb. dim.

— cf. two coh. dims. of abs. Gal. gp. of

• (totally imag.) no. field \( F/\mathbb{Q} < \infty \)
• \( p \)-adic local field \( k/\mathbb{Q}_p < \infty \)

as well as two underlying real dims. of

• \( \mathbb{C}^\times \)
Units and Value Group:

In case of $p$-adic local field $k/\mathbb{Q}_p < \infty$, one may also think of these two underlying comb. dims. as follows:

\[ \mathcal{O}_k^\times \subseteq k^\times \rightarrow k^\times / \mathcal{O}_k^\times \ (\cong \mathbb{Z}) \]

1-comb. dim. 1-comb. dim.

— cf. complex case: $\mathbb{C}^\times = S^1 \times \mathbb{R}_{>0}$

In IUTeich, we shall\n
**deform the hol. str. of the NF**

by

**dilating the val. gps. via the theta fn.**

while

keeping the **units undilated**
§3. The Log-Theta-Lattice

Noncomm. (!) Diagram of Hodge Theaters:

\[ \begin{array}{c}
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\end{array} \]

\[ \begin{array}{c}
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\end{array} \]

Analogy between IUTeich and pTeich:
each “HT” \( \bullet \) \( \longleftrightarrow \) scheme theory/\( \mathbb{F}_p \)
\[ \uparrow = \text{log-link} \longleftrightarrow \text{Frob. in pos. char.} \]
\[ \longrightarrow = \Theta\text{-link} \longleftrightarrow \left( \frac{p^n}{p^{n+1}} \rightsquigarrow \frac{p^{n+1}}{p^{n+2}} \right) \]
Thus, 2-dims. of diagram
\[ \leftrightarrow \] 2-cb. dims. of $p$-adic loc. fld.

**log-Link:**

At nonarch. $v$ of NF $F$, ring strs. on either side of log-link related by non-ring. hom.

\[ \log_v : \bar{k}^\times \rightarrow \bar{k} \]

— where $\bar{k}$ is an alg. cl. of $k \overset{\text{def}}{=} F_v$.

**Key point:** log-link is compatible with isom.

\[ \Pi_v \overset{\sim}{\rightarrow} \Pi_v \]

of arith. fund. gps. $\Pi_v$ on either side, with natural actions via $\Pi_v \rightarrow G_v \overset{\text{def}}{=} \text{Gal}(\bar{k}/k)$; also, compatible with global Galois gps.

At arch. $v$ of $F$, $\exists$ an analogous theory
**Θ-Link:**

At **bad nonarch.** \( v \) of NF \( F \), ring strs. on either side of Θ-link related by **non-ring. hom.**

\[
\mathcal{O}^\times_k \xrightarrow{\sim} \mathcal{O}^\times_k
\]

\[
\Theta|_{l\text{-tors}} = \left\{ q^{j^2} \right\}_{j=1,\ldots,(l-1)/2} \mapsto q
\]

— where \( \overline{k} \) is an alg. cl. of \( k \overset{\text{def}}{=} F_v \).

**Key point:** Θ-link is **compatible** with isom.

\[
G_v \xrightarrow{\sim} G_v
\]

— where \( G_v \overset{\text{def}}{=} \text{Gal}(\overline{k}/k) \) — and **natural actions** on \( \mathcal{O}^\times_k \).

At **good nonarch./arch.** \( v \) of \( F \), define analogously, using **product formula**.
Note: ring str. rigid wrt/log-link [cf. \( \Pi_v \! \)], but not wrt/\( \Theta \)-link [cf. \( G_v \! \) \( \hat{\mathbb{Z}}^\times \sim \mathcal{O}_{k}^\times \! \)]

Note: “Galois portion” of log-\( \Theta \)-lattice \( \rightsquigarrow \) étale-picture — cf. cartes. vs. polar coords. for Gaussian int. \( \int_{0}^{\infty} e^{-x^2} \, dx \):

\[
\begin{array}{ccc}
\text{arith. hol.} & \text{str. } \Pi_v \\
\ldots & \mid & \ldots \\
\end{array}
\]

\[
\begin{array}{c}
\text{arith. hol.} \\
\text{str. } \Pi_v \\
\ldots
\end{array} - \begin{array}{c}
\text{mono-analytic core } G_v \\
\ldots
\end{array} - \begin{array}{c}
\text{arith. hol.} \\
\text{str. } \Pi_v \\
\ldots
\end{array}
\]

Note that $\log$-link, $\Theta$-link [i.e., $\Theta$-dilation!] incompatible with ring strs.:

$$\log_v : \bar{k}^\times \to \bar{k}$$

$$\Theta|_{l\text{-tors}} = \left\{ q^{j^2} \right\}_{j=1,\ldots,(l-1)/2} \mapsto q$$

— hence with basepoints arising from

· scheme-theoretic pts., i.e., ring homs.!
· Gal. gps. regarded as field str. automs.!

Consequence: As one crosses $\log$-, $\Theta$-links, one only knows “$\Pi_v$”, “$G_v$” as abstract top. gps.! Thus, can only relate the bps., “universes”, ring/scheme theory in domain, codomain of $\log$-, $\Theta$-links by applying

anabelian geometry!
§5. Expected Main Results
(work in progress!!)

Apply theory/ideas of tempered anab. geo., Étale Theta Fn., Frobenioids, and Topics in Abs. Anab. Geo. III to conclude:

Expected Main Theorem: One can give an explicit, algorithmic description, up to mild indeterminacies, of the left-hand side of the Θ-link — i.e., of “Θ|_{l-tors}” — relative to the [a priori, “alien”!] ring str. on the right-hand side of the Θ-link.

Key point: coric nature of \( G_v \cong \mathcal{O}_k^\times \)
— cf. analogy with Gaussian integral: i.e., dfn. of Θ-link, log-Θ-latt. \( \longleftrightarrow \) cart. crds. algo. desc. via anab. geo. \( \longleftrightarrow \) pol. crds.
By performing a volume computation concerning the output of the algorithms of the Expected Main Theorem, one obtains:

**Expected Corollary:** Inequality of Szpiro $(\iff \text{ABC})$ Conjecture.

... cf.

- “Hasse invariant $= d(\text{Frob. lift.})$” in $p\text{Teich}$
- Gauss-Bonnet on a Riemann surface $S$

\[ - \int_S (\text{Poincaré metric}) = 4\pi(1 - g) \]