Abstract. The present paper constitutes the third paper in a series of four papers and may be regarded as the culmination of the abstract conceptual portion of the theory developed in the series. In the present paper, we study the theory surrounding the log-theta-lattice, a highly non-commutative two-dimensional diagram of “miniature models of conventional scheme theory”, called $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theaters}}$. Here, we recall that $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theaters}}$ were associated, in the first paper of the series, to certain data, called initial $\Theta$-data, that includes an elliptic curve $E_F$ over a number field $F$, together with a prime number $l \geq 5$. Each arrow of the log-theta-lattice corresponds to a certain gluing operation between the $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theaters}}$ in the domain and codomain of the arrow. The horizontal arrows of the log-theta-lattice are defined as certain versions of the “$\Theta$-link” that was constructed, in the second paper of the series, by applying the theory of Hodge-Arakelov-theoretic evaluation — i.e., evaluation in the style of the scheme-theoretic Hodge-Arakelov theory established by the author in previous papers — of the reciprocal of the $l$-th root of the theta function at $l$-torsion points. In the present paper, we focus on the theory surrounding the log-link between $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theaters}}$. The log-link is obtained, roughly speaking, by applying, at each valuation of the number field under consideration, the local $p$-adic logarithm. The significance of the log-link lies in the fact that it allows one to construct log-shells, i.e., roughly speaking, slightly adjusted forms of the image of the local units at the valuation under consideration via the local $p$-adic logarithm. The theory of log-shells was studied extensively in a previous paper by the author. The vertical arrows of the log-theta-lattice are given by the log-link. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of multiradial algorithms for constructing “splitting monoids of logarithmic Gaussian procession monoids”. Here, we recall that “multiradial algorithms” are algorithms that make sense from the point of view of an “alien arithmetic holomorphic structure”, i.e., the ring/scheme structure of a $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theater}}$ related to a given $\Theta^{\pm \text{ell}}_{\text{NF-Hodge theater}}$ by means of a non-ring/scheme-theoretic horizontal arrow of the log-theta-lattice. These logarithmic Gaussian procession monoids, or LGP-monoids, for short, may be thought of as the log-shell-theoretic versions of the Gaussian monoids that were studied in the second paper of the series. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain estimates for the log-volume of these LGP-monoids. Explicit computations of these estimates will be applied, in the fourth paper of the series, to derive various diophantine results.
Introduction

In the following discussion, we shall continue to use the notation of the Introduction to the first paper of the present series of papers [cf. [IUTchI], §II]. In particular, we assume that are given an elliptic curve $E_F$ over a number field $F$, together with a prime number $l \geq 5$. In the first paper of the series, we introduced and studied the basic properties of $\Theta^{\pm \text{ell}}_F$-Hodge theaters, which may be thought of as miniature models of the conventional scheme theory surrounding the given elliptic curve $E_F$ over the number field $F$. In the present paper, which forms the third paper of the series, we study the theory surrounding the log-link between $\Theta^{\pm \text{ell}}_F$-Hodge theaters. The log-link induces an isomorphism between the underlying $D$-$\Theta^{\pm \text{ell}}_F$-Hodge theaters and, roughly speaking, is obtained by applying, at each [say, for simplicity, nonarchimedean] valuation $v \in V$, the local $p_v$-adic logarithm to the local units [cf. Proposition 1.3, (i)]. The significance of the log-link lies in the fact that it allows one to construct log-shells, i.e., roughly speaking, slightly adjusted forms of the image of the local units at $v \in V$ via the local $p_v$-adic logarithm. The theory of log-shells was studied extensively in [AbsTopIII]. The introduction of log-shells leads naturally to the construction of new versions — namely, the $\Theta_{\text{LGP}}^{\times \mu}$-/$\Theta_{\text{LGP}}^{\times \mu}$-links [cf. Definition 3.8, (ii)] — of the $\Theta$-/$\Theta_{\text{gau}}^{\times \mu}$-links studied in [IUTchI], [IUTchII]. The resulting [highly non-commutative!] diagram of iterates of the log- [i.e., the vertical arrows] and $\Theta_{\text{LGP}}^{\times \mu}$-/$\Theta_{\text{gau}}^{\times \mu}$-links [i.e., the horizontal arrows] — which we refer to as the log-theta-lattice [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1 below, in the case of the $\Theta_{\text{LGP}}^{\times \mu}$-link] — plays a central role in the theory of the present series of papers.

Fig. I.1: The [LGP-Gaussian] log-theta-lattice
Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing “splitting monoids of logarithmic Gaussian procession monoids” [cf. Theorem A below]. Here, we recall that “multiradial algorithms” [cf. the discussion of [IUTchII], Introduction] are algorithms that make sense from the point of view of an “alien arithmetic holomorphic structure”, i.e., the ring/scheme structure of a $\Theta^{\pm\text{ell}}\text{NF-Hodge theater}$ related to a given $\Theta^{\pm\text{ell}}\text{NF-Hodge theater}$ by means of a non-ring/scheme-theoretic $\Theta^-/\Theta^\times/\Theta^\times_{\text{gau}}/\Theta^\times_{\text{LGP}}/\Theta^\times_{\text{lgp}}$-link. These logarithmic Gaussian procession monoids, or **LGP-monoids**, for short, may be thought of as the log-shell-theoretic versions of the Gaussian monoids that were studied in [IUTchII]. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain estimates for the log-volume of these LGP-monoids [cf. Theorem B below]. These estimates will be applied to verify various diophantine results in [IUTchIV].

Recall [cf. [IUTchI], §I1] the notion of an $F$-prime-strip. An $F$-prime-strip consists of data indexed by the valuations $v \in \mathcal{V}$; roughly speaking, the data at each $v$ consists of a Frobenioid, i.e., in essence, a system of monoids over a base category. For instance, at $v \in \mathcal{V}_{\text{bad}}$, this data may be thought of as an isomorphic copy of the monoid with Galois action

$$\Pi_v \act \mathcal{O}^\times_{\overline{F}_v}$$

where we recall that $\mathcal{O}^\times_{\overline{F}_v}$ denotes the multiplicative monoid of nonzero integral elements of the completion of an algebraic closure $\overline{F}$ of $F$ at a valuation lying over $v$ [cf. [IUTchI], §I1, for more details]. The $p_v$-adic logarithm $\log_v : \mathcal{O}^\times_{\overline{F}_v} \to \overline{F}_v$ at $v$ then defines a natural $\Pi_v$-equivariant isomorphism of ind-topological modules

$$(\mathcal{O}^\times_{\overline{F}_v} \otimes \mathbb{Q} \cong ) \mathcal{O}^\times_{\overline{F}_v} \otimes \mathbb{Q} \cong \overline{F}_v$$

where we recall the notation “$\mathcal{O}^\times_{\overline{F}_v} = \mathcal{O}^\times_{\overline{F}_v}/\mathcal{O}^\mu_{\overline{F}_v}$” from the discussion of [IUTchI], §1 — which allows one to equip $\mathcal{O}^\times_{\overline{F}_v} \otimes \mathbb{Q}$ with the field structure arising from the field structure of $\overline{F}_v$. The portion at $v$ of the log-link associated to an $F$-prime-strip [cf. Definition 1.1, (iii); Proposition 1.2] may be thought of as the correspondence

$$\begin{cases}
\Pi_v \act \mathcal{O}^\times_{\overline{F}_v} \\
\Pi_v \act \mathcal{O}^\times_{\overline{F}_v}
\end{cases} \xrightarrow{\log} \begin{cases}
\Pi_v \act \mathcal{O}^\times_{\overline{F}_v} \\
\Pi_v \act \mathcal{O}^\times_{\overline{F}_v}
\end{cases}$$

in which one thinks of the copy of “$\mathcal{O}^\times_{\overline{F}_v}$” on the right as obtained from the field structure induced by the $p_v$-adic logarithm on the tensor product with $\mathbb{Q}$ of the copy of the units “$\mathcal{O}^\times_{\overline{F}_v} \subseteq \mathcal{O}^\times_{\overline{F}_v}$” on the left. Since this correspondence induces an isomorphism of topological groups between the copies of $\Pi_v$ on either side, one may think of $\Pi_v$ as “immune to”/“neutral with respect to” — or, in the terminology of the present series of papers, “coric” with respect to — the transformation constituted by the log-link. This situation is studied in detail in [AbsTopIII], §3, and reviewed in Proposition 1.2 of the present paper.

By applying various results from **absolute anabelian geometry**, one may algorithmically reconstruct a copy of the data “$\Pi_v \act \mathcal{O}^\times_{\overline{F}_v}$” from $\Pi_v$. Moreover,
by applying Kummer theory, one obtains natural isomorphisms between this "coric version" of the data "\( \Pi_v \acts O_{F_v}^\times \)" and the copies of this data that appear on either side of the log-link. On the other hand, one verifies immediately that these Kummer isomorphisms are not compatible with the coricity of the copy of the data "\( \Pi_v \acts O_{F_v}^\times \)" algorithmically constructed from \( \Pi_v \). This phenomenon is, in some sense, the central theme of the theory of [AbsTopIII], §3, and is reviewed in Proposition 1.2, (iv), of the present paper.

The introduction of the log-link leads naturally to the construction of log-shells at each \( v \in \mathbb{V} \). If, for simplicity, \( v \in \mathbb{V}_{bad} \), then the log-shell at \( v \) is given, roughly speaking, by the compact additive module

\[
I_v \overset{\text{def}}{=} p_v^{-1} \cdot \log_v (O_{K_v}^\times) \subseteq K_v \subseteq F_v
\]

[cf. Definition 1.1, (i), (ii); Remark 1.2.2, (i), (ii)]. One has natural functorial algorithms for constructing various versions of the notion of a log-shell — i.e., mono-analytic/holomorphic and étale-like/Frobenius-like — from \( \mathcal{D}^+/\mathcal{D}-/\mathcal{F}^+/\mathcal{F}-\)-prime-strips [cf. Proposition 1.2, (v), (vi), (vii), (viii), (ix)]. Although, as discussed above, the relevant Kummer isomorphisms are not compatible with the log-link "at the level of elements", the log-shell \( I_v \) at \( v \) satisfies the important property

\[
O_{K_v}^\times \subseteq I_v; \quad \log_v (O_{K_v}^\times) \subseteq I_v
\]

— i.e., it contains the images of the Kummer isomorphisms associated to both the domain and the codomain of the log-shell [cf. Proposition 1.2, (v); Remark 1.2.2, (i), (ii)]. In light of the compatibility of the log-link with log-volumes [cf. Propositions 1.2, (iii); 3.9, (iv)], this property will ultimately lead to upper bounds — i.e., as opposed to "precise equalities" — in the computation of log-volumes in Corollary 3.12 [cf. Theorem B below]. Put another way, although iterates [cf. Remark 1.1.1] of the log-link fail to be compatible with the various Kummer isomorphisms that arise, one may nevertheless consider the entire diagram that results from considering such iterates of the log-link and related Kummer isomorphisms [cf. Proposition 1.2, (x)]. We shall refer to such diagrams

\[
\begin{array}{cccc}
\cdots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \cdots \\
\cdots & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
\circ & & & & & & & & &
\end{array}
\]

— i.e., where the horizontal arrows correspond to the log-links [that is to say, to the vertical arrows of the log-theta-lattice]; the "\( \bullet \)'s" correspond to the Frobenioid-theoretic data within a \( \Theta^{\pm\text{ell}} \)-NF-Hodge theater; the "\( \circ \)" corresponds to the coric version of this data [that is to say, in the terminology discussed below, vertically coric data of the log-theta-lattice]; the vertical/diagonal arrows correspond to the various Kummer isomorphisms — as log-Kummer correspondences [cf. Theorem 3.11, (ii); Theorem A, (ii), below]. Then the inclusions of the above display may be interpreted as a sort of "upper semi-commutativity" of such diagrams [cf. Remark 1.2.2, (iii)], which we shall also refer to as the "upper semi-compatibility" of the log-link with the relevant Kummer isomorphisms — cf. the discussion of the "indeterminacy" (Ind3) in Theorem 3.11, (ii).
By considering the \( \log \)-links associated to the various \( \mathcal{F} \)-prime-strips that occur in a \( \Theta_{\text{mod}}^{\pm} \text{NF-Hodge theater} \), one obtains the notion of a \textit{log-link} between \( \Theta_{\text{mod}}^{\pm} \text{NF-Hodge theaters} \)

\[
\dagger \mathcal{H} \Theta_{\text{mod}}^{\pm} \text{NF} \xrightarrow{\log} \dagger \mathcal{H} \Theta_{\text{mod}}^{\pm} \text{NF}
\]

[cf. Proposition 1.3, (i)]. As discussed above, by considering the iterates of the \( \log \)-[i.e., the vertical arrows] and \( \Theta_{\text{gau}}^{\pm \mu}/\Theta_{\text{LGp}}^{\pm \mu}/\Theta_{\text{lgp}}^{\pm \mu} \)-links [i.e., the horizontal arrows], one obtains a diagram which we refer to as the \textit{log-theta-lattice} [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1, in the case of the \( \Theta_{\text{LGp}}^{\mu} \)-link]. As discussed above, this diagram is \textit{highly noncommutative}, since the definition of the \( \log \)-link depends, in an essential way, on both the \textit{additive} and the \textit{multiplicative} structures — i.e., on the \textit{ring structure} — of the various local rings at \( \mathfrak{v} \in \mathbb{V} \), structures which are \textit{not preserved} by the \( \Theta_{\text{gau}}^{\pm \mu}/\Theta_{\text{LGp}}^{\pm \mu}/\Theta_{\text{lgp}}^{\pm \mu} \)-links [cf. Remark 1.4.1, (i)]. So far, in the Introductions to [IUTchI], [IUTchII], as well as in the present Introduction, we have discussed various “\textit{coricity}” properties — i.e., properties of \textit{invariance} with respect to various types of “transformations” — in the context of \( \Theta_{\text{gau}}^{\pm \mu}/\Theta_{\text{LGp}}^{\pm \mu}/\Theta_{\text{lgp}}^{\pm \mu} \)-links, as well as in the context of \( \log \)-links. In the context of the log-theta-lattice, it becomes necessary to distinguish between various types of coricity. That is to say, coricity with respect to \( \log \)-links [i.e., the vertical arrows of the log-theta-lattice] will be referred to as \textit{vertical coricity}, while coricity with respect to \( \Theta_{\text{gau}}^{\pm \mu}/\Theta_{\text{LGp}}^{\pm \mu}/\Theta_{\text{lgp}}^{\pm \mu} \)-links [i.e., the horizontal arrows of the log-theta-lattice] will be referred to as \textit{horizontal coricity}. On the other hand, coricity properties that hold with respect to \textit{all} of the arrows of the log-theta-lattice will be referred to as \textit{bi-coricity} properties.

Relative to the analogy between the theory of the present series of papers and \( p \)-adic Teichmüller theory [cf. [IUTchI], §I4], we recall that a \( \Theta_{\text{mod}}^{\pm} \text{NF-Hodge theater} \), which may be thought of as a miniature model of the \textit{conventional scheme theory} surrounding the given elliptic curve \( E_F \) over the number field \( F \), corresponds to the \textit{positive characteristic scheme theory} surrounding a hyperbolic curve over a positive characteristic perfect field that is equipped with a nilpotent ordinary indigenous bundle [cf. Fig. I.2 below]. Then the \textit{rotation}, or “\textit{juggling}”, effected by the \( \log \)-link of the \textit{additive} and \textit{multiplicative} structures of the conventional scheme theory represented by a \( \Theta_{\text{mod}}^{\pm} \text{NF-Hodge theater} \) may be thought of as corresponding to the \textit{Frobenius morphism} in \textit{positive characteristic} [cf. the discussion of [AbsTopIII], §I1, §I3, §I5]. Thus, just as the Frobenius morphism is completely well-defined in \textit{positive characteristic}, the \( \log \)-link may be thought of as a phenomenon that occurs within a \textit{single arithmetic holomorphic structure}, i.e., a \textit{vertical} line of the log-theta-lattice. By contrast, the essentially \textit{non-ring/scheme-theoretic} relationship between \( \Theta_{\text{mod}}^{\pm} \text{NF-Hodge theaters} \) constituted by the \( \Theta_{\text{gau}}^{\pm \mu}/\Theta_{\text{LGp}}^{\pm \mu}/\Theta_{\text{lgp}}^{\pm \mu} \)-links corresponds to the relationship between the “\textit{mod} \( p^n \)” and “\textit{mod} \( p^{n+1} \)” portions of the ring of Witt vectors, in the context of a \textit{canonical lifting} of the original positive characteristic data [cf. the discussion of Remark 1.4.1, (iii); Fig. I.2 below]. Thus, the \textit{log-theta-lattice}, taken as a whole, may be thought of as corresponding to the \textit{canonical lifting} of the original positive characteristic data, equipped with a corresponding \textit{canonical Frobenius action/lifting} [cf. Fig. I.2 below]. Finally, the \textit{non-commutativity} of the log-theta-lattice may be thought of as corresponding to the complicated “\textit{intertwining}” that occurs in the theory of Witt vectors and canonical liftings between the Frobenius morphism in positive
characteristic and the mixed characteristic nature of the ring of Witt vectors [cf. the discussion of Remark 1.4.1, (ii), (iii)].

One important consequence of this “noncommutative intertwining” of the two dimensions of the log-theta-lattice is the following. Since each horizontal arrow of the log-theta-lattice [i.e., the $\Theta/\Theta^\times/\Theta_{\text{gau}}/\Theta^\times_{\text{LGP}}/\Theta^\times_{\text{lgp}}$] may only be used to relate — i.e., via various Frobenoids — the multiplicative portions of the ring structures in the domain and codomain of the arrow, one natural approach to relating the additive portions of these ring structures is to apply the theory of log-shells. That is to say, since each horizontal arrow is compatible with the canonical splittings [up to roots of unity] discussed in [IUTchII], Introduction, of the theta/Gaussian monoids in the domain of the horizontal arrow into unit group and value group portions, it is natural to attempt to relate the ring structures on either side of the horizontal arrow by applying the canonical splittings to

- relate the multiplicative structures on either side of the horizontal arrow by means of the value group portions of the theta/Gaussian monoids;

- relate the additive structures on either side of the horizontal arrow by means of the unit group portions of the theta/Gaussian monoids, shifted once via a vertical arrow, i.e., the log-link, so as to “render additive” the [a priori] multiplicative structure of these unit group portions.

Indeed, this is the approach that will ultimately be taken in Theorem 3.11 [cf. Theorem A below] to relating the ring structures on either side of a horizontal arrow. On the other hand, in order to actually implement this approach, it will be necessary to overcome numerous technical obstacles. Perhaps the most immediately obvious such obstacle lies in the observation [cf. the discussion of Remark 1.4.1, (ii)] that, precisely because of the “noncommutative intertwining” nature of the log-theta-lattice,

any sort of algorithmic construction concerning objects lying in the domain of a horizontal arrow that involves vertical shifts [e.g., such as the approach to relating additive structures in the fashion described above] cannot be “translated” in any immediate sense into an algorithm that makes sense from the point of view of the codomain of the horizontal arrow.

In a word, our approach to overcoming this technical obstacle consists of working with objects in the vertical line of the log-theta-lattice that contains the domain of the horizontal arrow under consideration that satisfy the crucial property of being invariant with respect to vertical shifts

— i.e., shifts via iterates of the log-link [cf. the discussion of Remarks 1.2.2, (iii); 1.4.1, (ii)]. For instance, étale-like objects that are vertically coric satisfy this invariance property. On the other hand, as discussed in the beginning of [IUTchII], Introduction, in the theory of the present series of papers, it is of crucial importance to be able to relate corresponding Frobenius-like and étale-like structures to one another via Kummer theory. In particular, in order to obtain structures
that are *invariant* with respect to *vertical shifts*, it is necessary to consider *log-Kummer correspondences*, as discussed above. Moreover, in the context of such log-Kummer correspondences, typically, one may only obtain structures that are invariant with respect to vertical shifts if one is willing to admit some sort of *indeterminacy*, e.g., such as the *"upper semi-compatibility"* [cf. the discussion of the *"indeterminacy"* (Ind3) in Theorem 3.11, (ii)] discussed above.

<table>
<thead>
<tr>
<th><strong>Inter-universal Teichmüller theory</strong></th>
<th><strong>$p$-adic Teichmüller theory</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>number field</td>
<td>hyperbolic curve $C$ over a positive characteristic perfect field</td>
</tr>
<tr>
<td>$F$</td>
<td>nilpotent ordinary indigenous bundle $P$ over $C$</td>
</tr>
<tr>
<td>[once-punctured] elliptic curve $X$ over $F$</td>
<td></td>
</tr>
<tr>
<td><strong>$\Theta$</strong>-link arrows of the <em>log-theta-lattice</em></td>
<td>mixed characteristic extension structure of a ring of Witt vectors</td>
</tr>
<tr>
<td><strong>$\log$</strong>-link arrows of the <em>log-theta-lattice</em></td>
<td>the <strong>Frobenius morphism</strong> in positive characteristic</td>
</tr>
<tr>
<td>the entire <strong>log-theta-lattice</strong></td>
<td>the resulting <strong>canonical lifting</strong> + <strong>canonical Frobenius action</strong>; <strong>canonical Frobenius lifting</strong> over the ordinary locus</td>
</tr>
<tr>
<td>relatively straightforward original construction of $\Theta_{LGP}$-link</td>
<td>relatively straightforward original construction of canonical liftings</td>
</tr>
<tr>
<td>highly nontrivial <em>description of alien arithmetic holomorphic structure</em> via absolute anabelian geometry</td>
<td>highly nontrivial <em>absolute anabelian reconstruction of canonical liftings</em></td>
</tr>
</tbody>
</table>

Fig. I.2: Correspondence between inter-universal Teichmüller theory and $p$-adic Teichmüller theory
One important property of the log-link, and hence, in particular, of the construction of log-shells, is its compatibility with the $\mathbb{F}_l^{\times \pm}$-symmetry discussed in the Introductions to [IUTchI], [IUTchII] — cf. Remark 1.3.2. Here, we recall from the discussion of [IUTchII], Introduction, that the $\mathbb{F}_l^{\times \pm}$-symmetry allows one to relate the various $\mathcal{F}$-prime-strips — i.e., more concretely, the various copies of the data $\Pi_{\mathfrak{v}} \vee \mathcal{O}_{\mathfrak{P}_{\mathfrak{v}}}^\triangleright$ at $\mathfrak{v} \in \mathbb{V}^{\text{bad}}$ [and their analogues for $\mathfrak{v} \in \mathbb{V}^{\text{good}}$] — associated to the various labels $\in \mathbb{F}_l$ that appear in the Hodge-Arakelov-theoretic evaluation of [IUTchII] in a fashion that is compatible with

- the distinct nature of distinct labels $\in \mathbb{F}_l$;
- the Kummer isomorphisms used to relate Frobenius-like and étale-like versions of the $\mathcal{F}$-prime-strips that appear, i.e., more concretely, the various copies of the data $\Pi_{\mathfrak{v}} \vee \mathcal{O}_{\mathfrak{P}_{\mathfrak{v}}}^\triangleright$ at $\mathfrak{v} \in \mathbb{V}^{\text{bad}}$ [and their analogues for $\mathfrak{v} \in \mathbb{V}^{\text{good}}$];
- the structure of the underlying $\mathcal{D}$-prime-strips that appear, i.e., more concretely, the various copies of the [arithmetic] tempered fundamental group $\Pi_{\mathfrak{v}}$ at $\mathfrak{v} \in \mathbb{V}^{\text{bad}}$ [and their analogues for $\mathfrak{v} \in \mathbb{V}^{\text{good}}$];

— cf. the discussion of [IUTchII], Introduction; Remark 1.5.1; Step (vii) of the proof of Corollary 3.12 of the present paper. This compatibility with the $\mathbb{F}_l^{\times \pm}$-symmetry gives rise to the construction of

- vertically coric $\mathcal{F}^{\times \mu}$-prime-strips, log-shells by means of the arithmetic holomorphic structures under consideration;
- mono-analytic $\mathcal{F}^{\times \mu}$-prime-strips, log-shells which are bi-coric

— cf. Theorem 1.5. These bi-coric mono-analytic log-shells play a central role in the theory of the present paper.

One notable aspect of the compatibility of the log-link with the $\mathbb{F}_l^{\times \pm}$-symmetry in the context of the theory of Hodge-Arakelov-theoretic evaluation developed in [IUTchII] is the following. One important property of mono-theta environments is the property of “isomorphism class compatibility”, i.e., in the terminology of [EtTh], “compatibility with the topology of the tempered fundamental group” [cf. the discussion of Remark 2.1.1]. This “isomorphism class compatibility” allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the ring-theoretic basepoints that occur on either side of the log-link [cf. Remark 2.1.1, (ii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the log-Kummer correspondences discussed above. Here, we recall that the significance of working with such “ring-theoretic basepoints” lies in the fact that the full ring structure of the local rings involved [i.e., as opposed to, say, just the multiplicative portion of this ring structure] is necessary in order to construct the log-link. That is to say, it is precisely by establishing the conjugate synchronization arising from the $\mathbb{F}_l^{\times \pm}$-symmetry relative to these basepoints that occur on either side of the log-link that one is able to conclude the crucial compatibility of this conjugate synchronization with the log-link discussed in Remark 1.3.2. Thus, in
summary, one important consequence of the “isomorphism class compatibility” of mono-theta environments is the simultaneous compatibility of

- the Kummer theory of mono-theta environments;
- the conjugate synchronization arising from the $\mathbb{F}_p^{\times \pm}$-symmetry;
- the construction of the log-link.

This simultaneous compatibility is necessary in order to perform the construction of the [crucial!] splitting monoids of LGP-monoids referred to above — cf. the discussion of Step (vi) of the proof of Corollary 3.12.

In §2 of the present paper, we continue our preparation for the multiradial construction of splitting monoids of LGP-monoids given in §3 [of the present paper] by presenting a global formulation of the essentially local theory at $v \in \mathcal{V}_{\text{bad}}$ [cf. [IUTchII], §1, §2, §3] concerning the interpretation, via the notion of multiradiality, of various rigidity properties of mono-theta environments. That is to say, although much of the [essentially routine!] task of formulating the local theory of [IUTchII], §1, §2, §3, in global terms was accomplished in [IUTchII], §4, the [again essentially routine!] task of formulating the portion of this local theory that concerns multiradiality was not explicitly addressed in [IUTchII], §4. One reason for this lies in the fact that, from the point of view of the theory to be developed in §3 of the present paper, this global formulation of multiradiality properties of the mono-theta environment may be presented most naturally in the framework developed in §1 of the present paper, involving the log-theta-lattice [cf. Theorem 2.2; Corollary 2.3]. Indeed, the étale-like versions of the mono-theta environment, as well as the various objects constructed from the mono-theta environment, may be interpreted, from the point of view of the log-theta-lattice, as vertically coric structures, and are Kummer-theoretically related to their Frobenius-like [i.e., Frobenioid-theoretic] counterparts, which arise from the [Frobenioid-theoretic portions of the] various $\Theta^{\pm\ell}$NF-Hodge theaters in a vertical line of the log-theta-lattice [cf. Theorem 2.2, (ii); Corollary 2.3, (ii), (iii), (iv)]. Moreover, it is precisely the horizontal arrows of the log-theta-lattice that give rise to the $\mathbb{Z}^{\times}$-indeterminacies acting on copies of $\mathcal{O}^{\times \mu}$ that play a prominent role in the local multiradiality theory developed in [IUTchII] [cf. the discussion of [IUTchII], Introduction]. In this context, it is useful to recall from the discussion of [IUTchII], Introduction [cf. also Remark 2.2.1 of the present paper], that the essential content of this local multiradiality theory consists of the observation [cf. Fig. 1.3 below] that, since mono-theta-theoretic cyclotomic and constant multiple rigidity only require the use of the portion of $\mathcal{O}^{\times} \mathcal{F}_{\mathbb{F}_v}$, for $v \in \mathcal{V}_{\text{bad}}$, given by the torsion subgroup $\mathcal{O}^{\mu}_{\mathcal{F}_{\mathbb{F}_v}} \subseteq \mathcal{O}^{\times} \mathcal{F}_{\mathbb{F}_v}$ [i.e., the roots of unity], the triviality of the composite of natural morphisms

$$\mathcal{O}^{\mu}_{\mathcal{F}_{\mathbb{F}_v}} \hookrightarrow \mathcal{O}^{\times} \mathcal{F}_{\mathbb{F}_v} \twoheadrightarrow \mathcal{O}^{\mu \times \mu}_{\mathcal{F}_{\mathbb{F}_v}}$$

has the effect of insulating the Kummer theory of the étale theta function — i.e., via the theory of the mono-theta environments developed in [EtTh] — from the $\mathbb{Z}^{\times}$-indeterminacies that act on the copies of $\mathcal{O}^{\times \mu}$ that arise in the $\mathcal{F}^{\times \mu}$-prime-strips that appear in the $\Theta-/\mathcal{O}^{\times \mu} \mathcal{F}_{\mathbb{F}_{\text{gau}}} / \mathcal{O}^{\mu \times \mu}_{\mathcal{F}_{\text{lGP}}} / \Theta^{\times \mu}_{\text{trp}}$-link.
In §3 of the present paper, which, in some sense, constitutes the conclusion of the theory developed thus far in the present series of papers, we present the construction of the [splitting monoids of] **LGP-monoids**, which may be thought of as a **multiradial** version of the [splitting monoids of] **Gaussian monoids** that were constructed via the theory of **Hodge-Arakelov-theoretic evaluation** developed in [IUTchII]. In order to achieve this multiradiality, it is necessary to “multiradialize” the various components of the construction of the Gaussian monoids given in [IUTchII]. The first step in this process of “multiradialization” concerns the **labels** $j \in F^*$ that occur in the Hodge-Arakelov-theoretic evaluation performed in [IUTchII]. That is to say, the construction of these labels, together with the closely related theory of $F^*$-**symmetry**, depend, in an essential way, on the **full arithmetic tempered fundamental groups** “$\Pi_v$” at $v \in V^{\text{bad}}$, i.e., on the portion of the arithmetic holomorphic structure within a $\Theta^{\text{ell}}$-Hodge theater which is **not shared** by an **alien arithmetic holomorphic structure** [i.e., an arithmetic holomorphic structure related to the original arithmetic holomorphic structure via a horizontal arrow of the log-theta-lattice]. One naive approach to remedying this state of affairs is to simply consider the underlying set, of cardinality $l^*$, associated to $F^*$, which we regard as being equipped with the full set of **symmetries** given by arbitrary permutation automorphisms of this underlying set. The problem with this approach is that it yields a situation in which, for each label $j \in F^*$, one must contend with an **indeterminacy of $l^*$ possibilities** for the element of this underlying set that corresponds to $j$ [cf. [IUTchI], Propositions 4.11, (i); 6.9, (i)]. From the point of view of the **log-volume computations** to be performed in [IUTchIV], this degree of indeterminacy gives rise to log-volumes which are **“too large”**, i.e., to estimates that are not sufficient for deriving the various **diophantine results** obtained in [IUTchIV]. Thus, we consider the following alternative approach, via **processions** [cf. [IUTchI], Propositions, 4.11, 6.9]. Instead of working just with the underlying set associated to $F^*$, we consider the **diagram of inclusions** of finite sets

$$S^\pm_1 \hookrightarrow S^\pm_{1+1=2} \hookrightarrow \ldots \hookrightarrow S^\pm_{j+1} \hookrightarrow \ldots \hookrightarrow S^\pm_{1+l^*=l^\pm}$$

— where we write $S^\pm_{j+1} \overset{\text{def}}{=} \{0, 1, \ldots, j\}$, for $j = 0, \ldots, l^*$, and we think of each of these finite sets as being subject to arbitrary permutation automorphisms. That is to say, we think of the set $S^\pm_{j+1}$ as a **container** for the labels $0, 1, \ldots, j$. Thus, for each $j$, one need only contend with an **indeterminacy of $j+1$ possibilities** for the element of this container that corresponds to $j$. In particular, if one allows $j = 0, \ldots, l^*$ to vary, then this approach allows one to **reduce** the resulting label indeterminacy from a total of $(l^\pm)^{l^\pm}$ possibilities [where we write $l^\pm = 1 + l^* =$...
\((l+1)/2\) to a total of \(l^\pm!\) possibilities. It turns out that this reduction will yield just the right estimates in the log-volume computations to be performed in [IUTchIV]. Moreover, this approach satisfies the important property of insulating the "core label 0" from the various label indeterminacies that occur.

Each element of each of the containers \(S_{j+1}^\pm\) may be thought of as parametrizing an \(F\)- or \(D\)-prime-strip that occurs in the Hodge-Arakelov-theoretic evaluation of [IUTchII]. In order to render the construction multiradial, it is necessary to replace such holomorphic \(F/-D\)-prime-strips by mono-analytic \(\mathcal{F}^+/\mathcal{D}^+\)-prime-strips. In particular, as discussed above, one may construct, for each such \(\mathcal{F}^+/\mathcal{D}^+\)-prime-strip, a collection of log-shells associated to the various \(v \in \mathcal{V}\). Then, in order to obtain objects that are immune to the various label indeterminacies discussed above, we consider, for each element \(* \in S_{j+1}^\pm\), and for each \([\text{say, for simplicity, nonarchimedean}]\) \(v_Q \in \mathcal{V}_Q\),

- the direct sum of the log-shells associated to the prime-strip labeled by the given element \(* \in S_{j+1}^\pm\) at the \(v \in \mathcal{V}\) that lie over \(v_Q\);

we then form

- the tensor product, over the elements \(* \in S_{j+1}^\pm\), of these direct sums.

This collection of tensor products associated to \(v_Q \in \mathcal{V}_Q\) will be referred to as the tensor packet associated to the collection of prime-strips indexed by elements of \(S_{j+1}^\pm\). One may carry out this construction of the tensor packet either for holomorphic \(F/-D\)-prime-strips [cf. Proposition 3.1] or for mono-analytic \(\mathcal{F}^+/\mathcal{D}^+\)-prime-strips [cf. Proposition 3.2].

The tensor packets associated to \(D\)-prime-strips will play a crucial role in the theory of §3, as "multiradial mono-analytic containers" for the principal objects of interest [cf. the discussion of Remark 3.12.2, (ii)], namely,

- the action of the splitting monoids of the LGP-monoids — i.e., the monoids generated by the theta values \(\{q_j^{\pm}\}_{j=1}^{l^*}\) — on the portion of the tensor packets just defined at \(v \in \mathcal{V}_{\text{bad}}\) [cf. Fig. I.4 below; Propositions 3.4, 3.5; the discussion of [IUTchII], Introduction];

- the action of copies \("(F_{\text{mod}}^\times)^j\)" of [the multiplicative monoid of nonzero elements of] the number field \(F_{\text{mod}}\) labeled by \(j = 1, \ldots, l^*\) on the product, over \(v_Q \in \mathcal{V}_Q\), of the portion of the tensor packets just defined at \(v_Q\) [cf. Fig. I.5 below; Propositions 3.3, 3.7, 3.10].

\[
\begin{array}{cccccccccc}
/ & /^\pm & \leadsto & /^\pm & /^\pm & \leadsto & /^\pm & /^\pm & \leadsto & /^\pm \\
S_1^\pm & S_{1+1=2}^\pm & \cdots & S_{j+1}^\pm & \cdots & S_{j+1}^\pm & \cdots & S_{j+1}^\pm & \cdots & S_{l^*+l^*=t^\pm}^\pm
\end{array}
\]

Fig. I.4: Splitting monoids of LGP-monoids acting on tensor packets
Indeed, these [splitting monoids of] LGP-monoids and copies “\((F^\times_{\text{mod}})^j\)” of [the multiplicative monoid of nonzero elements of] the number field \(F^\times_{\text{mod}}\) admit natural embeddings into/actions on the various tensor packets associated to labeled \(F\)-prime-strips in each \(\Theta^\pm\text{ell}\)NF-Hodge theater \(n,mH\Theta^\pm\text{ell}\)NF of the log-theta-lattice. One then obtains vertically coric versions of these splitting monoids of LGP-monoids and labeled copies “\((F^\times_{\text{mod}})^j\)” of [the multiplicative monoid of nonzero elements of] the number field \(F^\times_{\text{mod}}\) by applying suitable Kummer isomorphisms between

- log-shells/tensor packets associated to [labeled] \(F\)-prime-strips and
- log-shells/tensor packets associated to [labeled] \(D\)-prime-strips.

Finally, by passing to the

- log-shells/tensor packets associated to [labeled] \(D^\tau\)-prime-strips

— i.e., by forgetting the arithmetic holomorphic structure associated to a specific vertical line of the log-theta-lattice — one obtains the desired multiradial representation, i.e., description in terms that make sense from the point of view of an alien arithmetic holomorphic structure, of the splitting monoids of LGP-monoids and labeled copies of the number field \(F^\times_{\text{mod}}\) discussed above. This passage to the multiradial representation is obtained by admitting the following three types of indeterminacy:

(Ind1): This is the indeterminacy that arises from the automorphisms of processions of \(D^\tau\)-prime-strips that appear in the multiradial representation — i.e., more concretely, from permutation automorphisms of the label sets \(\mathcal{S}^\pm_{j+1}\) that appear in the processions discussed above, as well as from the automorphisms of the \(D^\tau\)-prime-strips that appear in these processions.

(Ind2): This is the [“non-(Ind1) portion” of the] indeterminacy that arises from the automorphisms of the \(F^\times\times^\mu\)-prime-strips that appear in the \(\Theta-/\Theta^\times\mu\)-/\(\Theta^\times\mu\)-/\(\Theta^\times\mu\)-link — i.e., in particular, at [for simplicity] \(g \in \mathcal{V}^\text{non}\), the \(\hat{\mathbb{Z}}^\times\)-indeterminacies acting on local copies of “\(O^\times\mu\)” [cf. the above discussion].

(Ind3): This is the indeterminacy that arises from the upper semi-compatibility of the log-Kummer correspondences associated to the specific vertical line of the log-theta-lattice under consideration [cf. the above discussion].
A detailed description of this multiradial representation, together with the indeterminacies (Ind1), (Ind2) is given in Theorem 3.11, (i) [and summarized in Theorem A, (i), below; cf. also Fig. I.6 below].

Fig. I.6: The full multiradial representation

One important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, log-shells, splitting monoids of LGP-monoids, and number fields — of this multiradial representation and the log-Kummer correspondence of the specific vertical line of the log-theta-lattice under consideration. This property — which may be thought of as a sort of “non-interference”, or “mutual compatibility”, property — asserts that the multiplicative monoids constituted by the splitting monoids of LGP-monoids and copies of \( F_\text{mod} \times \) “do not interfere”, relative to the various arrows that occur in the log-Kummer correspondence, with the local units at \( v \in V \) that give rise to the log-shells. In the case of splitting monoids of LGP-monoids, this non-interference/mutual compatibility property is, in essence, a formal consequence of the existence of the canonical splittings [up to roots of unity] of the theta/Gaussian monoids that appear into unit group and value group portions [cf. the discussion of [IUTchII], Introduction]. Here, we recall that, in the case of the theta monoids, these canonical splittings are, in essence, a formal consequence of the constant multiple rigidity property of mono-theta environments reviewed above. In the case of copies of \( F_\text{mod} \), this non-interference/mutual compatibility property is, in essence, a formal consequence of the well-known fact in elementary algebraic number theory that any nonzero element of a number field that is integral at every valuation of the number field is necessarily a root of unity. These mutual compatibility properties are described in detail in Theorem 3.11, (ii), and summarized in Theorem A, (ii), below.

Another important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, log-shells, splitting monoids of LGP-monoids, and number fields — of this multiradial representation and the \( \Theta_{\text{LGP}} \)-links, i.e., the horizontal arrows of the log-theta-lattice under consideration. This property — which may be thought of as a property of compatibility with the \( \Theta_{\text{LGP}} \)-link — asserts that the cyclotomic rigidity isomorphisms that appear in the Kummer theory surrounding the splitting monoids of LGP-monoids and copies of \( F_\text{mod} \times \) are immune to the \( \hat{\mathbb{Z}} \)-indeterminacies that act on the copies of “\( \mathcal{O}_\mu \)” that arise in the \( \mathcal{F}_\mu \)-prime-strips that appear in the \( \Theta_{\text{LGP}} \)-link. In the case of splitting monoids of LGP-monoids, this property amounts precisely to the multiradiality theory developed in \S 2 [cf. the above
discussion], i.e., in essence, to the **mono-theta-theoretic cyclotomic rigidity** property reviewed in the above discussion. In the case of copies of \( F_{\text{mod}}^\times \), this property follows from the theory surrounding the construction of the cyclotomic rigidity isomorphisms discussed in [IUTchI], Example 5.1, (v). These compatibility properties are described in detail in Theorem 3.11, (iii), and summarized in Theorem A, (iii), below.

At this point, we pause to observe that although considerable attention has been devoted so far in the present series of papers, especially in [IUTchII], to the theory of **Gaussian monoids**, not so much attention has been devoted [i.e., outside of [IUTchI], §5; [IUTchII], Corollaries 4.7, 4.8] to [the multiplicative monoids constituted by] copies of \( F_{\text{mod}}^\times \). These copies of \( F_{\text{mod}}^\times \) enter into the theory of the **multiradial representation** discussed above in the form of various types of **global Frobenioids** in the following way. If one starts from the *number field* \( F_{\text{mod}} \), one natural Frobenioid that can be associated to \( F_{\text{mod}} \) is the Frobenioid \( F_{\text{mod}}^\times \) of [stack-theoretic] *arithmetic line bundles* on [the spectrum of the ring of integers of] \( F_{\text{mod}} \) discussed in [IUTchI], Example 5.1, (iii) [cf. also Example 3.6 of the present paper]. From the point of view of the theory surrounding the **multiradial representation** discussed above, there are **two natural ways** to approach the construction of “\( F_{\text{mod}}^\times \)”:

\[
(\otimes_{\text{MOD}}) \quad \text{(Rational Function Torsor Version): This approach consists of considering the category } F_{\text{MOD}}^\otimes \text{ of } F_{\text{mod}}^\times \text{-torsors equipped with trivializations at each } \mathfrak{v} \in \mathcal{V} \text{ [cf. Example 3.6, (i), for more details].}
\]

\[
(\otimes_{\text{mod}}) \quad \text{(Local Fractional Ideal Version): This approach consists of considering the category } F_{\text{mod}}^\otimes \text{ of collections of integral structures on the various completions } K_{\mathfrak{v}} \text{ at } \mathfrak{v} \in \mathcal{V} \text{ and morphisms between such collections of integral structures that arise from multiplication by elements of } F_{\text{mod}}^\times \text{ [cf. Example 3.6, (ii), for more details].}
\]

Then one has **natural isomorphisms of Frobenioids**

\[
F_{\text{mod}}^\otimes \xrightarrow{\sim} F_{\text{MOD}}^\otimes \xrightarrow{\sim} F_{\text{mod}}^\otimes
\]

that induce the respective *identity morphisms* \( F_{\text{mod}}^\times \to F_{\text{mod}}^\times \to F_{\text{mod}}^\times \) on the associated *rational function monoids* [cf. [FrdII], Corollary 4.10]. In particular, at first glance, \( F_{\text{MOD}}^\otimes \) and \( F_{\text{mod}}^\otimes \) appear to be “essentially equivalent” objects.

On the other hand, when regarded from the point of view of the **multiradial representations** discussed above, these two constructions exhibit a number of significant differences — cf. Fig. I.7 below; the discussion of Remarks 3.6.2, 3.10.1. For instance, whereas the construction of \((\otimes_{\text{MOD}})\) depends only on the **multiplicative** structure of \( F_{\text{mod}}^\times \), the construction of \((\otimes_{\text{mod}})\) involves the *module*, i.e., the **additive**, structure of the localizations \( K_{\mathfrak{v}} \). The global portion of the \( \Theta_{\text{LGP}}^{\times \mu} \)-link (respectively, the \( \Theta_{\text{LGP}}^{\times \mu} \)-link) is, by definition [cf. Definition 3.8, (ii)], constructed by means of the *realification* of the Frobenioid that appears in the construction of \((\otimes_{\text{MOD}})\) (respectively, \((\otimes_{\text{mod}})\)). This means that the construction of the global portion of the \( \Theta_{\text{LGP}}^{\times \mu} \)-link — which is the version of the \( \Theta \)-link that is in fact ultimately used in the theory of the multiradial representation — depends only on the **multiplicative** monoid structure of a copy of \( F_{\text{mod}}^\times \), together with the various valuation
homomorphisms $F^\times_{\text{mod}} \to \mathbb{R}$ associated to $\nu \in \mathcal{V}$. Thus, the *mutual compatibility* [discussed above] of copies of $F^\times_{\text{mod}}$ with the log-Kummer correspondence implies that one may perform this construction of the global portion of the $\Theta^\times_{\text{LGP}}$-link in a fashion that is *immune* to the “upper semi-compatibility” indeterminacy (Ind3) [discussed above]. By contrast, the construction of $(\otimes_{\text{mod}})$ involves integral structures on the underlying local additive modules “$K^\nu_{\text{L}}$”, i.e., from the point of view of the multiradial representation, integral structures on log-shells and tensor packets of log-shells, which are subject to the “upper semi-compatibility” indeterminacy (Ind3) [discussed above]. In particular, the log-Kummer correspondence subjects the construction of $(\otimes_{\text{mod}})$ to “substantial distortion”. On the other hand, the essential role played by local integral structures in the construction of $(\otimes_{\text{mod}})$ enables one to compute the *global arithmetic degree* of the arithmetic line bundles constituted by objects of the category “$\mathcal{F}_{\text{mod}}^\otimes$” in terms of log-volumes on log-shells and tensor packets of log-shells [cf. Proposition 3.9, (iii)]. This property of the construction of $(\otimes_{\text{mod}})$ will play a *crucial role* in deriving the *explicit estimates* for such log-volumes that are obtained in Corollary 3.12 [cf. Theorem B below].

<table>
<thead>
<tr>
<th>$\mathcal{F}_{\text{MOD}}^\otimes$</th>
<th>$\mathcal{F}_{\text{mod}}^\otimes$</th>
</tr>
</thead>
<tbody>
<tr>
<td>biased toward <strong>multiplicative</strong> structures</td>
<td>biased toward <strong>additive</strong> structures</td>
</tr>
<tr>
<td>easily related to value group/non-coric portion “$(-)^{##}$” of $\Theta^\times_{\text{LGP}}$-link</td>
<td>easily related to unit group/coric portion “$(-)^{##}$” of $\Theta^\times_{\text{LGP}}/\Theta^{\times}_{\text{LGP}}$-link, i.e., mono-analytic log-shells</td>
</tr>
<tr>
<td>admits precise log-Kummer correspondence</td>
<td>only admits “upper semi-compatible” log-Kummer correspondence</td>
</tr>
<tr>
<td>rigid, but not suited to explicit computation</td>
<td>subject to substantial distortion, but suited to explicit estimates</td>
</tr>
</tbody>
</table>

Fig. I.7: $\mathcal{F}_{\text{MOD}}^\otimes$ versus $\mathcal{F}_{\text{mod}}^\otimes$

Thus, in summary, the natural isomorphism $\mathcal{F}_{\text{MOD}}^\otimes \cong \mathcal{F}_{\text{mod}}^\otimes$ discussed above plays the important role, in the context of the *multiradial representation* discussed above, of relating

- the **multiplicative** structure of the global number field $F_{\text{mod}}$ to the **additive** structure of $F_{\text{mod}}$,
the unit group/coric portion \((-)^{×\mu}\) of the \(\Theta^{×\mu}_{\text{LGP}}\)-link to the value group/non-coric portion \((-)^{\bullet}\)
portion \((-)^{×\mu}\) of the \(\Theta^{×\mu}_{\text{LGP}}\)-link.

Finally, in Corollary 3.12 [cf. also Theorem B below], we apply the multiradial representation discussed above to estimate certain log-volumes as follows. We begin by introducing some terminology [cf. Definition 3.8, (i)]. We shall refer to the object that arises in any of the versions [including realifications] of the global Frobenioid \(\tilde{\mathcal{F}}_{\text{mod}}^\circ\) discussed above — such as, for instance, the global realified Frobenioid that occurs in the codomain of the \(\Theta^{×\mu}_{\text{gau}}/\Theta^{×\mu}_{\text{LGP}}/\Theta^{×\mu}_{\text{lgp}}\)-link — by considering the arithmetic divisor determined by the zero locus of the elements \(\tilde{q}\) at \(v \in \mathbb{V}^{\text{bad}}\) as a \(q\)-pilot object. The log-volume of the \(q\)-pilot object will be denoted by

\[-|\log(\tilde{q})| \in \mathbb{R}\]

— so \(|\log(q)| > 0\) [cf. Corollary 3.12; Theorem B]. In a similar vein, we shall refer to the object that arises in the global realified Frobenioid that occurs in the domain of the \(\Theta^{×\mu}_{\text{gau}}/\Theta^{×\mu}_{\text{LGP}}/\Theta^{×\mu}_{\text{lgp}}\)-link by considering the arithmetic divisor determined by the zero locus of the collection of theta values \(\{q_j^2\}_{j=1,...,l}\) at \(v \in \mathbb{V}^{\text{bad}}\) as a \(\Theta\)-pilot object. The log-volume of the holomorphic hull — cf. Remark 3.9.5, (i); Step (xi) of the proof of Corollary 3.12 — of the union of the collection of possible images of the \(\Theta\)-pilot object in the multiradial representation — i.e., where we recall that these “possible images” are subject to the indeterminacies (Ind1), (Ind2), (Ind3) — will be denoted by

\[-|\log(\Theta)| \in \mathbb{R} \cup \{+\infty\}\]

[cf. Corollary 3.12; Theorem B]. Here, the reader might find the use of the notation “\(-\)” and “\(\ldots\)” confusing [i.e., since this notation suggests that \(-|\log(\Theta)|\) is a non-positive real number, which would appear to imply that the possibility that \(-|\log(\Theta)| = +\infty\) may be excluded from the outset]. The reason for the use of this notation, however, is to express the point of view that \(-|\log(\Theta)|\) should be regarded as a positive real multiple of \(-|\log(q)|\) [i.e., which is indeed a negative real number!] plus a possible error term, which [a priori!] might be equal to +\(\infty\). Then the content of Corollary 3.12, Theorem B may be summarized, roughly speaking [cf. Remark 3.12.1, (ii)], as a result concerning the

**negativity of the \(\Theta\)-pilot log-volume \(|\log(\Theta)|\)**

— i.e., where we write \(|\log(\Theta)| \overset{\text{def}}{=} -(−|\log(\Theta)|) \in \mathbb{R} \cup \{-\infty\}\). Relative to the analogy between the theory of the present series of papers and complex/p-adic Teichmüller theory [cf. [IUTchI], §14], this result may be thought of as a statement to the effect that

**“the pair consisting of a number field equipped with an elliptic curve is metrically hyperbolic, i.e., has negative curvature”**.

That is to say, it may be thought of as a sort of analogue of the inequality

\[\chi_S = -\int_S d\mu_S < 0\]
arising from the classical Gauss-Bonnet formula on a hyperbolic Riemann surface of finite type $S$ [where we write $\chi_S$ for the Euler characteristic of $S$ and $d\mu_S$ for the Kähler metric on $S$ determined by the Poincaré metric on the upper half-plane — cf. the discussion of Remark 3.12.3], or, alternatively, of the inequality

$$(1 - p)(2g_X - 2) \leq 0$$

that arises by computing global degrees of line bundles in the context of the Hasse invariant that arises in $p$-adic Teichmüller theory [where $X$ is a smooth, proper hyperbolic curve of genus $g_X$ over the ring of Witt vectors of a perfect field of characteristic $p$ which is canonical in the sense of $p$-adic Teichmüller theory — cf. the discussion of Remark 3.12.4, (v)].

The proof of Corollary 3.12 [i.e., Theorem B] is based on the following fundamental observation: the multiradial representation discussed above yields

**two tautologically equivalent ways to compute**

the $q$-pilot log-volume $- |\log(q)|$ — cf. Fig. I.8 below; Step (xi) of the proof of Corollary 3.12. That is to say, suppose that one starts with the $q$-pilot object in the $\Theta^{\pm,\text{ell}}\text{NF}$-Hodge theater $\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF}$ at $(1, 0)$, which we think of as being represented, via the approach of $(@\text{mod})$, by means of the action of the various $\mathbb{Q}$, for $\mathbb{Q} \in \mathbb{Q}\text{bad}$, on the log-shells that arise, via the log-link $\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF} \xrightarrow{\log} 1.0\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF}$, from the various local “$\mathcal{O}\times \mu$” in the $\Theta^{\pm,\text{ell}}\text{NF}$-Hodge theater $\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF}$ at $(1, -1)$. Thus, if one considers the value group “$(-)^{\bullet}$” and unit group “$(-)^{\times}$” portions of the codomain of the $\Theta^{\times,\mu,\text{LGP}}\text{-link}$ $\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF} \xrightarrow{\Theta^{\times,\mu,\text{LGP}}} 1.0\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF}$ in the context of the arithmetic holomorphic structure of the vertical line $(1, \circ)$, this action on log-shells may be thought of as a somewhat intricate “intertwining” between these value group and unit group portions [cf. Remark 3.12.2, (ii)]. On the other hand, the $\Theta^{\times,\mu,\text{LGP}}\text{-link}$ $\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF} \xrightarrow{\Theta^{\times,\mu,\text{LGP}}} 1.0\mathcal{H}\Theta^{\pm,\text{ell}}\text{NF}$ in the context of the arithmetic holomorphic structures associated to the vertical lines $(0, \circ)$ and $(1, \circ)$ that is based on

**forgetting this intricate intertwining**, i.e., by working solely with abstract isomorphisms of $\mathbb{F}\times\mathbb{F}^{-}\mu^{-}\text{-prime-strips}$.  

Thus, in order to relate the arithmetic holomorphic structures, say, at $(0, 0)$ and $(1, 0)$, one must apply the multiradial representation discussed above. That is to say, one starts by applying the theory of bi-coric mono-analytic log-shells given in Theorem 1.5. One then applies the Kummer theory surrounding the splitting monoids of theta/Gaussian monoids and copies of the number field $F\text{mod}$, which allows one to pass from the Frobenius-like versions of various objects that appear in — i.e., that are necessary in order to consider — the $\Theta^{\times,\mu,\text{LGP}}\text{-link}$ to the corresponding étale-like versions of these objects that appear in the multiradial representation. This passage from Frobenius-like versions to étale-like versions is referred to as the operation of Kummer-detachment [cf. Fig. I.8; Remark 1.5.4, (i)]. As discussed above, this operation of Kummer-detachment is possible precisely
as a consequence of the **compatibility** of the multiradial representation with the **indeterminacies** (Ind1), (Ind2), (Ind3), hence, in particular, with the $\Theta_{\text{LGP}}^{\times \mu}$-**link**. Here, we recall that since the log-theta-lattice is, as discussed above, **far from commutative**, in order to represent the various “log-link-conjugates” at $(0,m)$ [for $m \in \mathbb{Z}$] in terms that may be understood from the point of view of the arithmetic holomorphic structure at $(1,0)$, one must work [not only with the Kummer isomorphisms at a single $(0,m)$, but rather] with the **entire log-Kummer correspondence**. In particular, one must take into account the **indeterminacy** (Ind3). Once one completes the operation of Kummer-detachment so as to obtain **vertically coric** versions of objects on the vertical line $(0,\circ)$, one then passes to **multiradial objects**, i.e., to the “final form” of the **multiradial representation**, by taking into account [once again] the **indeterminacy** (Ind1), i.e., that arises from working with [mono-analytic!$D^\prec$ - [as opposed to $D^\succ$!]] prime-strips. Finally, one computes the **log-volume** of the holomorphic hull of this “final form” multiradial representation of the $\Theta$-pilot object — i.e., subject to the **indeterminacies** (Ind1), (Ind2), (Ind3)! — and concludes the desired estimates from the **tautological observation** that

*the log-theta-lattice — and, in particular, the “gluing isomorphism” constituted by the $\Theta_{\text{LGP}}^{\times \mu}$-**link** — were constructed precisely in such a way as to ensure that the computation of the log-volume of the holomorphic hull of the union of the collection of possible images of the $\Theta$-pilot object [cf. the definition of $|\log(\Theta)|$] necessarily amounts to a computation of [an upper bound for] $|\log(q)|$*
— cf. Fig. I.8; Step (xi) of the proof of Corollary 3.12. That is to say, the “gluing isomorphism” constituted by the $\Theta_{LGP}^{\times \mu}$-link relates two distinct “arithmetic holomorphic structures”, i.e., two distinct copies of conventional ring/scheme theory, that are glued together precisely by means of a relation that identifies the $\Theta$-pilot object in the domain of the $\Theta_{LGP}^{\times \mu}$-link with the $q$-pilot object in the codomain of the $\Theta_{LGP}^{\times \mu}$-link. Thus, once one sets up such an apparatus, the computation of the log-volume of the holomorphic hull of the union of possible images of the $\Theta$-pilot object in the domain of the $\Theta_{LGP}^{\times \mu}$-link amounts — tautologically! — to the computation of the log-volume of the $q$-pilot object in terms of itself, i.e., to a computation that reflects certain intrinsic properties of this $q$-pilot object. This is the content of Corollary 3.12 [i.e., Theorem B]. As discussed above, this sort of “computation of intrinsic properties” in the present context of a number field equipped with an elliptic curve may be regarded as analogous to the “computations of intrinsic properties” reviewed above in the classical complex and $p$-adic cases.

We conclude the present Introduction with the following summaries of the main results of the present paper.

**Theorem A.** (Multiradial Algorithms for Logarithmic Gaussian Procession Monoids) Fix a collection of initial $\Theta$-data $(\mathcal{F}/F, X_F, l, C_{K}, \mathcal{V}, \mathcal{V}_{\text{mod}}^{\text{bad}}, \varepsilon)$ as in [IUTchI], Definition 3.1. Let

$$\{n,m\mathcal{H}T^{\Theta_{LGP}^{\pm \text{ell}NF}}\}_{n,m \in \mathbb{Z}}$$

be a collection of distinct $\Theta_{LGP}^{\pm \text{ell}NF}$-Hodge theaters [relative to the given initial $\Theta$-data] — which we think of as arising from a LGP-Gaussian log-theta-lattice [cf. Definition 3.8, (iii)]. For each $n \in \mathbb{Z}$, write

$$n,\circ\mathcal{H}T^{D,\Theta_{LGP}^{\pm \text{ell}NF}}$$

for the $D,\Theta_{LGP}^{\pm \text{ell}NF}$-Hodge theater determined, up to isomorphism, by the various $n,m\mathcal{H}T^{\Theta_{LGP}^{\pm \text{ell}NF}}$, where $m \in \mathbb{Z}$, via the vertical coricity of Theorem 1.5, (i) [cf. Remark 3.8.2].

(i) (Multiradial Representation) Write

$$n,\circ\mathcal{R}^{LGP}$$

for the collection of data consisting of

(a) tensor packets of log-shells;

(b) splitting monoids of LGP-monoids acting on the tensor packets of (a);

(c) copies, labeled by $j \in \mathbb{F}_{l}^{\times}$, of [the multiplicative monoid of nonzero elements of] the number field $F_{\text{mod}}$ acting on the tensor packets of (a)
the indeterminacies induced by the automorphisms of the procession of $\mathcal{D}^\tau$-prime-strips $\text{Prc}(n^\circ \mathcal{D}^\tau_T)$ that gives rise to the tensor packets of (a);

(Ind2) the [“non-(Ind1) portion” of the] indeterminacies that arise from the automorphisms of the $\mathcal{F}^\tau$×μ-prime-strips that appear in the $\Theta^\times_{\text{LGP}}$-link, i.e., in particular, at [for simplicity] $v \in \mathcal{V}_{\text{non}}$, the $\mathcal{O}^\times_{\mu}$-indeterminacies acting on local copies of $\mathcal{O}^\times_{\mu}$

— cf. Theorem 3.11, (i), for more details. Then $n^\circ R_{\text{LGP}}$ may be constructed via an algorithm in the procession of $\mathcal{D}^\tau$-prime-strips $\text{Prc}(n^\circ \mathcal{D}^\tau_T)$, which is functorial with respect to isomorphisms of processions of $\mathcal{D}^\tau$-prime-strips. For $n, n' \in \mathbb{Z}$, the permutation symmetries of the étale-picture discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce compatible poly-isomorphisms

$$\text{Prc}(n^\circ \mathcal{D}^\tau_T) \xrightarrow{\sim} \text{Prc}(n'^\circ \mathcal{D}^\tau_T); \quad n^\circ R_{\text{LGP}} \xrightarrow{\sim} n'^\circ R_{\text{LGP}}$$

which are, moreover, compatible with the bi-coricity poly-isomorphisms

$$n^\circ \mathcal{D}^\tau_0 \xrightarrow{\sim} n'^\circ \mathcal{D}^\tau_0$$

of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].

(ii) (log-Kummer Correspondence) For $n, m \in \mathbb{Z}$, the inverses of the Kummer isomorphisms associated to the various $\mathcal{F}$-prime-strips and NF-bridges that appear in the $\Theta_{\text{cell}}\mathcal{H}^\tau_{\text{cell}}$-Hodge theater $n,m \mathcal{H}^\tau_{\Theta_{\text{cell}}\mathcal{H}^\tau_{\text{cell}}}$ induce “inverse Kummer” isomorphisms between the vertically coric data (a), (b), (c) of (i) and the corresponding Frobenioid-theoretic data arising from each $\Theta_{\text{cell}}\mathcal{H}^\tau_{\text{cell}}$-Hodge theater $n,m \mathcal{H}^\tau_{\Theta_{\text{cell}}\mathcal{H}^\tau_{\text{cell}}}$ [cf. Theorem 3.11, (ii), (a), (b), (c), for more details]. Moreover, as one varies $m \in \mathbb{Z}$, the corresponding Kummer isomorphisms [i.e., inverses of “inverse Kummer” isomorphisms] of splitting monoids of LGP-monoids [cf. (i), (b)] and labeled copies of the number field $F_{\text{mod}}$ [cf. (i), (c)] are mutually compatible, relative to the log-links of the $n$-th column of the LGP-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the [Frobenioid-theoretic] domains of these Kummer isomorphisms that are possibly related to one another via the log-links consist of roots of unity in the domains of the log-links [multiplication by which corresponds, via the log-link, to an “addition by zero” indeterminacy, i.e., to no indeterminacy!] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii); Theorem 3.11, (ii), for more details. On the other hand, the Kummer isomorphisms of tensor packets of log-shells [cf. (i), (a)] are subject to a certain “indeterminacy” as follows:

(Ind3) as one varies $m \in \mathbb{Z}$, these Kummer isomorphisms of tensor packets of log-shells are “upper semi-compatible”, relative to the log-links of the
n-th column of the LGP-Gaussian log-theta-lattice under consideration, in a sense that involves certain natural inclusions \( \subseteq \) at \( v_Q \in \mathbb{V}^m_Q \) and certain natural surjections \( 
rightarrow \) at \( v_Q \in \mathbb{V}^G_Q \) — cf. Proposition 3.5, (ii), (a), (b); Theorem 3.11, (ii), for more details.

Finally, as one varies \( m \in \mathbb{Z} \), these Kummer isomorphisms of tensor packets of log-shells are [precisely!] compatible, relative to the log-links of the n-th column of the LGP-Gaussian log-theta-lattice under consideration, with the respective log-volumes [cf. Proposition 3.9, (iv)].

(iii) \((\Theta^\mu_{\text{LGP}}\text{-Link Compatibility})\) The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various horizontal arrows — i.e., \( \Theta^\mu_{\text{LGP}} \)-links — of the LGP-Gaussian log-theta-lattice under consideration as follows: The tensor packets of log-shells [cf. (i), (a)] are compatible, relative to the relevant Kummer isomorphisms, with [the unit group portion \((-)^{-\times} \mu \text{ of} \) the \( \Theta^\mu_{\text{LGP}} \)-link [cf. the indeterminacy \( \text{(Ind2)} \) of (i)]; we refer to Theorem 3.11, (iii), (a), (b), for more details. The identity automorphism on the objects that appear in the construction of the splitting monoids of LGP-monoids via mono-theta environments [cf. (i), (b)] is compatible, relative to the relevant Kummer isomorphisms and isomorphisms of mono-theta environments, with the \( \Theta^\mu_{\text{LGP}} \)-link [cf. the indeterminacy \( \text{(Ind2)} \) of (i)]; we refer to Theorem 3.11, (iii), (c), for more details. The identity automorphism on the objects that appear in the construction of the labeled copies of the number field \( F_{\text{mod}} \) [cf. (i), (c)] is compatible, relative to the relevant Kummer isomorphisms and cyclotomic rigidity isomorphisms [cf. the discussion of Remark 2.3.2; the constructions of [IUTchI], Example 5.1, (v)], with the \( \Theta^\mu_{\text{LGP}} \)-link [cf. the indeterminacy \( \text{(Ind2)} \) of (i)]; we refer to Theorem 3.11, (iii), (d), for more details.

**Theorem B.** (Log-volume Estimates for Multiradially Represented Splitting Monoids of Logarithmic Gaussian Procession Monoids) Suppose that we are in the situation of Theorem A. Write

\[
\neg \log(\Theta) \in \mathbb{R} \cup \{+\infty\}
\]

for the procession-normalized mono-analytic log-volume [where the average is taken over \( j \in \mathbb{F}^*_l \) — cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a), for more details] of the holomorphic hull [cf. Remark 3.9.5, (i)] of the union of the possible images of a \( \Theta \)-pilot object [cf. Definition 3.8, (i)], relative to the relevant Kummer isomorphisms [cf. Theorems A, (ii); 3.11, (ii)], in the multiradial representation of Theorems A, (i); 3.11, (i), which we regard as subject to the indeterminacies \( \text{(Ind1), (Ind2), (Ind3)} \) described in Theorems A, (i), (ii); 3.11, (i), (ii). Write

\[
\neg \log(q) \in \mathbb{R}
\]

for the procession-normalized mono-analytic log-volume of the image of a \( q \)-pilot object [cf. Definition 3.8, (i)], relative to the relevant Kummer isomorphisms [cf. Theorems A, (ii); 3.11, (ii)], in the multiradial representation of
Theorems A, (i); 3.11, (i), which we do not regard as subject to the indeterminacies (Ind1), (Ind2), (Ind3) described in Theorems A, (i), (ii); 3.11, (i), (ii). Here, we recall the definition of the symbol “△” as the result of identifying the labels “0” and “(F̂ ∗)” [cf. [IUTchII], Corollary 4.10, (i)]. In particular, |log(q)| > 0 is easily computed in terms of the various q-parameters of the elliptic curve EF [cf. [IUTchI], Definition 3.1, (b)] at v ∈ V̂ bad (≠ ∅). Then it holds that − |log(Θ)| ∈ ℜ, and

− |log(Θ)| ≥ − |log(q)|

− i.e., CΘ ≥ −1 for any real number CΘ ∈ ℜ such that − |log(Θ)| ≤ CΘ · |log(q)|.

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Notations and Conventions:

We shall continue to use the “Notations and Conventions” of [IUTchI], §0.
Section 1: The Log-theta-lattice

In the present §1, we discuss various enhancements to the theory of log-shells, as developed in [AbsTopIII]. In particular, we develop the theory of the log-link [cf. Definition 1.1; Propositions 1.2, 1.3], which, together with the $\Theta\times\mu$ and $\Theta_{gau}$-links of [IUTchII], Corollary 4.10, (iii), leads naturally to the construction of the log-theta-lattice, an apparatus that is central to the theory of the present series of papers. We conclude the present §1 with a discussion of various coric structures associated to the log-theta-lattice [cf. Theorem 1.5].

In the following discussion, we assume that we have been given initial $\Theta$-data as in [IUTchI], Definition 3.1. We begin by reviewing various aspects of the theory of log-shells developed in [AbsTopIII].

Definition 1.1. Let $\overset{\dagger}{\mathfrak{F}} = \{\overset{\dagger}{\mathfrak{F}}_v\}_{v \in V}$ be an $\mathcal{F}$-prime-strip [relative to the given initial $\Theta$-data — cf. [IUTchI], Definition 5.2, (i)]. Write

$\overset{\dagger}{\mathfrak{F}}^+ = \{\overset{\dagger}{\mathfrak{F}}^+_v\}_{v \in V}; \quad \overset{\dagger}{\mathfrak{F}}^{+\times\mu} = \{\overset{\dagger}{\mathfrak{F}}^{+\times\mu}_v\}_{v \in V}; \quad \overset{\dagger}{\mathfrak{D}} = \{\overset{\dagger}{\mathfrak{D}}_v\}_{v \in V}$

for the associated $\mathcal{F}^+, \mathcal{F}^{+\times\mu}, \mathcal{D}$-prime-strips [cf. [IUTchI], Remark 5.2.1, (ii); [IUTchII], Definition 4.9, (vi), (vii); [IUTchI], Remark 5.2.1, (i)]. Recall the functorial algorithm of [IUTchII], Corollary 4.6, (i), in the $\mathcal{F}$-prime-strip $\overset{\dagger}{\mathfrak{F}}$ for constructing the assignment $\Psi_{\text{cns}}(\overset{\dagger}{\mathfrak{F}})$ given by

\[ \forall_{\text{non}} \ni v \mapsto \Psi_{\text{cns}}(\overset{\dagger}{\mathfrak{F}})_v \overset{\text{def}}{=} \left\{ G_v(\overset{\dagger}{\Pi}_v) \hookrightarrow \Psi_{\dagger\mathfrak{F}}_v \right\} \]

\[ \forall_{\text{arc}} \ni v \mapsto \Psi_{\text{cns}}(\overset{\dagger}{\mathfrak{F}})_v \overset{\text{def}}{=} \Psi_{\dagger\mathfrak{F}}_v \]

— where the data in brackets “{−}” is to be regarded as being well-defined only up to a $\overset{\dagger}{\Pi}_v$-conjugacy indeterminacy [cf. [IUTchII], Corollary 4.6, (i), for more details]. In the following, we shall write

\[ (\neg)^{\text{gp}} \overset{\text{def}}{=} (\neg)^{\text{gp}} \bigcup \{0\} \]

for the formal union with $\{0\}$ of the groupification $(\neg)^{\text{gp}}$ of a [multiplicatly written] monoid “(−).” Thus, by setting the product of all elements of $(\neg)^{\text{gp}}$ with 0 to be equal to 0, one obtains a natural monoid structure on $(\neg)^{\text{gp}}$.

(i) Let $\underline{v} \in \forall_{\text{non}}$. Write

\[ (\Psi_{\dagger\mathfrak{F}}_v \ni \Psi_{\dagger\mathfrak{F}}^\times_\underline{v} \rightarrow) \quad \Psi_{\dagger\mathfrak{F}}^\times_\underline{v} \overset{\text{def}}{=} (\Psi_{\dagger\mathfrak{F}}^\times_\underline{v})^{\text{pf}} \]

for the perfection $(\Psi_{\dagger\mathfrak{F}}^\times_\underline{v})^{\text{pf}}$ of the submonoid of units $\Psi_{\dagger\mathfrak{F}}^\times_\underline{v}$ of $\Psi_{\dagger\mathfrak{F}}_v$. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 3.1, (iv); [AbsTopIII], Proposition 3.2, (iii), (v)] that the natural, algorithmically constructible
ind-topological field structure on $\Psi^{gp}_{\log(\dagger F_v)}$ allows one to define a $p_v$-adic logarithm on $\Psi_{\dagger F_v}$, which, in turn, yields a functorial algorithm in the Frobenioid $\dagger F_v$ for constructing an ind-topological field structure on $\Psi_{\dagger F_v}$. Write

$$\Psi_{\log(\dagger F_v)} \subseteq \Psi_{\dagger F_v}$$

for the resulting multiplicative monoid of nonzero integers. Here, we observe that the resulting diagram

$$\Psi_{\dagger F_v} \supseteq \Psi_{\dagger F_v} \otimes \Psi_{\log(\dagger F_v)}$$

is compatible with the various natural actions of $\dagger \Pi_v \to G_v(\dagger \Pi_v)$ on each of the four “$\Psi$’s” appearing in this diagram. The pair $\{\dagger \Pi_v \otimes \Psi_{\log(\dagger F_v)}\}$ now determines a Frobenioid

$$\log(\dagger F_v)$$

[cf. [AbsTopIII], Remark 3.1.1; [IUTchI], Remark 3.3.2] — which is, in fact, naturally isomorphic to the Frobenioid $\dagger F_v$, but which we wish to think of as being related to $\dagger F_v$ via the above diagram. We shall denote this diagram by means of the notation

$$\dagger F_v \xrightarrow{\log} \log(\dagger F_v)$$

and refer to this relationship between $\dagger F_v$ and $\log(\dagger F_v)$ as the tautological log-link associated to $\dagger F_v$ [or, when $\dagger \mathfrak{F}$ is fixed, at $\mathfrak{F}$]. If $\log(\dagger F_v) \cong \dagger F_v$ is any [poly-]isomorphism of Frobenioids, then we shall write

$$\dagger F_v \xrightarrow{\log} \dagger F_v$$

for the diagram obtained by post-composing the tautological log-link associated to $\dagger F_v$ with the given [poly-]isomorphism $\log(\dagger F_v) \cong \dagger F_v$ and refer to this relationship between $\dagger F_v$ and $\dagger F_v$ as a log-link from $\dagger F_v$ to $\dagger F_v$; when the given [poly-]isomorphism $\log(\dagger F_v) \cong \dagger F_v$ is the full poly-isomorphism, then we shall refer to the resulting log-link as the full log-link from $\dagger F_v$ to $\dagger F_v$. Finally, we recall from [AbsTopIII], Definition 3.1, (iv), that the image in $\Psi_{\dagger F_v}$ of the submonoid of $G_v(\dagger \Pi_v)$-invariants of $\Psi_{\dagger F_v}$ constitutes a compact topological module, which we shall refer to as the pre-log-shell. Write $p_v^* = p_v$ when $p_v$ is odd and $p_v^* = p_v^2$ when $p_v$ is even. Then we shall refer to the result of multiplying the pre-log-shell by the factor $(p_v^*)^{-1}$ as the log-shell

$$\mathcal{I}_{\dagger F_v} \subseteq \Psi_{\dagger F_v} = \Psi^{gp}_{\log(\dagger F_v)}$$

[cf. [AbsTopIII], Definition 5.4, (iii)]. In particular, by applying the natural, algorithmically constructible ind-topological field structure on $\Psi^{gp}_{\log(\dagger F_v)}$ [cf. [AbsTopIII], Proposition 3.2, (iii)], it thus follows that one may think of this log-shell as an object associated to the codomain of any [that is to say, not necessarily tautological!] log-link

$$\dagger F_v \xrightarrow{\log} \dagger F_v$$
— i.e., an object that is determined by the image of a certain portion [namely, the $G_{\mathbb{U}}(1)$-invariants of $\Psi_{+\mathbb{F}_{+}}^\sim$] of the domain of this log-link.

(ii) Let $v \in V_{\text{arc}}$. For $N \in \mathbb{N}_{\geq 1}$, write $\Psi_{+\mathbb{F}_{+}}^\mu_{N} \subseteq \Psi_{+\mathbb{F}_{+}}^X \subseteq \Psi_{+\mathbb{F}_{+}}^{\text{gp}}$ for the subgroup of $N$-th roots of unity and $\Psi_{+\mathbb{F}_{+}}^\sim \to \Psi_{+\mathbb{F}_{+}}^{\text{gp}}$ for the [pointed] universal covering of the topological group determined by the groupification $\Psi_{+\mathbb{F}_{+}}^{\text{gp}}$ of the topological monoid $\Psi_{+\mathbb{F}_{+}}$. Then one verifies immediately that one may think of the composite covering of topological groups

$$
\Psi_{+\mathbb{F}_{+}}^\sim \to \Psi_{+\mathbb{F}_{+}}^{\text{gp}} \to \Psi_{+\mathbb{F}_{+}}^{\text{gp}}/\Psi_{+\mathbb{F}_{+}}^\mu_{N}
$$

— where the second “$\to$” is the natural surjection — as a [pointed] universal covering of $\Psi_{+\mathbb{F}_{+}}^{\text{gp}}/\Psi_{+\mathbb{F}_{+}}^\mu_{N}$. That is to say, one may think of $\Psi_{+\mathbb{F}_{+}}^\sim$ as an object constructed from $\Psi_{+\mathbb{F}_{+}}^{\text{gp}}/\Psi_{+\mathbb{F}_{+}}^\mu_{N}$ [cf. also Remark 1.2.1, (i), below]. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 4.1, (iv); [AbsTopIII], Proposition 4.2, (i), (ii)] that the natural, algorithmically constructible [i.e., starting from the collection of data $+\mathbb{F}_{+}$ — cf. [IUTchI], Definition 5.2, (i), (b)] topological field structure on $\Psi_{+\mathbb{F}_{+}}^{\text{gp}}$ allows one to define a [complex archimedean] logarithm on $\Psi_{+\mathbb{F}_{+}}^\sim$, which, in turn, yields a functorial algorithm in the collection of data $+\mathbb{F}_{+}$ [cf. [IUTchI], Definition 5.2, (i), (b)] for constructing a topological field structure on $\Psi_{+\mathbb{F}_{+}}^\sim$, together with a $\Psi_{+\mathbb{F}_{+}}^\sim$-Kummer structure on $+\mathbb{U}_{+}$ [cf. [AbsTopIII], Definition 4.1, (iv); [IUTchII], Proposition 4.4, (i)]. Write

$$
\Psi_{\log (+\mathbb{F}_{+})} \subseteq \Psi_{+\mathbb{F}_{+}}^\sim
$$

for the resulting multiplicative monoid of nonzero integral elements [i.e., elements of norm $\leq 1$]. Here, we observe that the resulting diagram

$$
\Psi_{+\mathbb{F}_{+}} \subseteq \Psi_{+\mathbb{F}_{+}}^{\text{gp}} \to \Psi_{+\mathbb{F}_{+}}^\sim = \Psi_{\log (+\mathbb{F}_{+})}
$$

is compatible [cf. the discussion of [AbsTopIII], Definition 4.1, (iv)] with the co-holomorphicizations determined by the natural $\Psi_{+\mathbb{F}_{+}}^{\text{gp}}$-Kummer [cf. [IUTchII], Proposition 4.4, (i)] and $\Psi_{+\mathbb{F}_{+}}^\sim$-Kummer [cf. the above discussion] structures on $+\mathbb{U}_{+}$. The triple of data consisting of the topological monoid $\Psi_{\log (+\mathbb{F}_{+})}$, the Automorphic space $+\mathbb{U}_{+}$, and the $\Psi_{+\mathbb{F}_{+}}^\sim$-Kummer structure on $+\mathbb{U}_{+}$ discussed above determines a collection of data [i.e., as in [IUTchI], Definition 5.2, (i), (b)]

$$
\log (+\mathbb{F}_{+})
$$

which is, in fact, naturally isomorphic to the collection of data $+\mathbb{F}_{+}$, but which we wish to think of as being related to $+\mathbb{F}_{+}$ via the above diagram. We shall denote this diagram by means of the notation

$$
+\mathbb{F}_{+} \xrightarrow{\log} \log (+\mathbb{F}_{+})
$$

and refer to this relationship between $+\mathbb{F}_{+}$ and $\log (+\mathbb{F}_{+})$ as the tautological log-link associated to $+\mathbb{F}_{+}$ [or, when $+\mathbb{F}_{+}$ is fixed, at $v$]. If $\log (+\mathbb{F}_{+}) \cong +\mathbb{F}_{+}$ is any
[poly]-isomorphism of collections of data [i.e., as in [IUTchI], Definition 5.2, (i), (b)], then we shall write

\[ \log^\# \mathcal{F}_v \xrightarrow{\log} \mathcal{F}_v \]

for the diagram obtained by post-composing the tautological log-link associated to \( \mathcal{F}_v \) with the given [poly]-isomorphism \( \log(\mathcal{F}_v) \sim \mathcal{F}_v \), and refer to this relationship between \( \mathcal{F}_v \) and \( \mathcal{F}_v \) as a log-link from \( \mathcal{F}_v \) to \( \mathcal{F}_v \); when the given [poly]-isomorphism \( \log(\mathcal{F}_v) \sim \mathcal{F}_v \) is the full poly-isomorphism, then we shall refer to the resulting log-link as the full log-link from \( \mathcal{F}_v \) to \( \mathcal{F}_v \). Finally, we recall from [AbsTopIII], Definition 4.1, (iv), that the submonoid of units \( \Psi \times \mathcal{F}_v \subseteq \Psi \mathcal{F}_v \) determines a compact topological subquotient of \( \Psi \mathcal{F}_v \), which we shall refer to as the pre-log-shell. We shall refer to the \( \Psi \mathcal{F}_v \)-orbit of the [uniquely determined] closed line segment of \( \Psi \mathcal{F}_v \) which is preserved by multiplication by \( \pm 1 \) and whose endpoints differ by a generator of the kernel of the natural surjection \( \Psi \mathcal{F}_v \twoheadrightarrow \Psi \mathcal{F}_v / \Psi \mu_N \mathcal{F}_v \) — or, equivalently, the \( \Psi \mathcal{F}_v \)-orbit of the result of multiplying by \( N \) the [uniquely determined] closed line segment of \( \Psi \mathcal{F}_v \) which is preserved by multiplication by \( \pm 1 \) and whose endpoints differ by a generator of the kernel of the natural surjection \( \Psi \mathcal{F}_v \twoheadrightarrow \Psi \mathcal{F}_v / \Psi \mu_N \mathcal{F}_v \) — as the log-shell

\[ \mathcal{F}_v \subseteq \Psi \mathcal{F}_v = \Psi \log(\mathcal{F}_v) \]

[cf. [AbsTopIII], Definition 5.4, (v)]. Thus, one may think of the log-shell as an object constructed from \( \Psi \mathcal{F}_v / \Psi \mu_N \mathcal{F}_v \). Moreover, by applying the natural, algorithmically constructible topological field structure on \( \Psi \log(\mathcal{F}_v) (= \Psi \mathcal{F}_v) \), it thus follows that one may think of this log-shell as an object associated to the codomain of any [that is to say, not necessarily tautological!] log-link

\[ \mathcal{F}_v \xrightarrow{\log} \mathcal{F}_v \]

— i.e., an object that is determined by the image of a certain portion [namely, the subquotient \( \Psi \mathcal{F}_v \) of \( \Psi \mathcal{F}_v \)] of the domain of this log-link.

(iii) Write

\[ \log(\mathcal{F}_v) \overset{\text{def}}{=} \{ \log(\mathcal{F}_v) \overset{\text{def}}{=} \Psi \mathcal{F}_v \}_{v \in V} \]

for the collection of ind-topological modules constructed in (i), (ii) above indexed by \( v \in V \) — where the group structure arises from the additive portion of the field structures on \( \Psi \mathcal{F}_v \) discussed in (i), (ii); for \( v \in V^{\text{non}} \), we regard \( \Psi \mathcal{F}_v \) as equipped with its natural \( G_v(\bar{\Pi}_v) \)-action. Write

\[ \log(\mathcal{F}_v) \overset{\text{def}}{=} \{ \log(\mathcal{F}_v) \}_{v \in V} \]

for the \( \mathcal{F} \)-prime-strip determined by the data \( \log(\mathcal{F}_v) \) constructed in (i), (ii) for \( v \in V \). We shall denote by

\[ \mathcal{F}_v \xrightarrow{\log} \log(\mathcal{F}_v) \]
the collection of diagrams \( \{ \dag F_v \stackrel{\log}{\rightarrow} \log(\dag F_v) \}_{v \in V} \) constructed in (i), (ii) for \( v \in V \) and refer to this relationship between \( \dag \mathfrak{F} \) and \( \log(\dag \mathfrak{F}) \) as the **tautological log-link associated to** \( \dag \mathfrak{F} \). If \( \log(\dag \mathfrak{F}) \sim \dag \mathfrak{G} \) is any [poly-]isomorphism of \( \mathcal{F} \)-prime-strips, then we shall write \( \dag \mathfrak{F} \stackrel{\log}{\rightarrow} \dag \mathfrak{G} \) for the diagram obtained by post-composing the tautological log-link associated to \( \dag \mathfrak{F} \) with the given [poly-]isomorphism \( \log(\dag \mathfrak{F}) \sim \dag \mathfrak{G} \) and refer to this relationship between \( \dag \mathfrak{F} \) and \( \dag \mathfrak{G} \) as a **log-link** from \( \dag \mathfrak{F} \) to \( \dag \mathfrak{G} \); when the given [poly-]isomorphism \( \log(\dag \mathfrak{F}) \sim \dag \mathfrak{G} \) is the full poly-isomorphism, then we shall refer to the resulting log-link as the **full log-link** from \( \dag \mathfrak{F} \) to \( \dag \mathfrak{G} \). Finally, we shall write

\[
I_{\dag \mathfrak{F}} \overset{\text{def}}{=} \{ I_{\dag \mathfrak{F}_v} \}_{v \in V}
\]

for the collection of log-shells constructed in (i), (ii) for \( v \in V \) and refer to this collection as the **log-shell associated to** \( \dag \mathfrak{F} \) and [by a slight abuse of notation]

\[
I_{\dag \mathfrak{F}} \subseteq \log(\dag \mathfrak{F})
\]

for the collection of natural inclusions indexed by \( v \in V \). In particular, [cf. the discussion of (i), (ii)], it thus follows that one may think of this log-shell as an object associated to the **codomain of any** [that is to say, not necessarily tautological!] log-link

\[
\dag \mathfrak{F} \stackrel{\log}{\rightarrow} \dag \mathfrak{G}
\]

— i.e., an object that is determined by the image of a certain portion [cf. the discussion of (i), (ii)] of the domain of this log-link.

(iv) Let \( v \in V^{\text{non}} \). Then observe that it follows immediately from the constructions of (i) that the **ind-topological modules with** \( G_v(\dag \Pi_v) \)-action \( I_{\dag \mathfrak{F}_v} \subseteq \log(\dag \mathfrak{F}_v) \) may be constructed solely from the collection of data \( \dag \mathfrak{F}_v \times \mu \) [i.e., the portion of the \( \mathcal{F} \times \mu \)-prime-strip \( \dag \mathfrak{F}_v \times \mu \) labeled by \( v \)]. That is to say, in light of the definition of a \( \times \mu \)-Kummer structure [cf. [IUTchII], Definition 4.9, (i), (ii), (iv), (vi), (vii)], these constructions only require the perfection “\( \mu^\text{perf} \)” of the units and are manifestly unaffected by the operation of forming the quotient by a torsion subgroup of the units. Write

\[
I_{\dag \mathfrak{F}_v \times \mu} \subseteq \log(\dag \mathfrak{F}_v \times \mu)
\]

for the resulting ind-topological modules with \( G_v(\dag \Pi_v) \)-action, regarded as objects constructed from \( \dag \mathfrak{F}_v \times \mu \).

(v) Let \( v \in V^{\text{arc}} \). Then by applying the algorithms for constructing “\( k^\sim(G) \), “\( \mathcal{I}(G) \)” given in [AbsTopIII], Proposition 5.8, (v), to the [object of the category “\( \mathbb{T}\mathcal{M}^\text{sym} \)” of split topological monoids discussed in [IUTchII], Example 3.4, (ii), determined by the] split Frobenioid portion of the collection of data \( \dag \mathfrak{F}_v \), one obtains a **functorial algorithm** in the collection of data \( \dag \mathfrak{F}_v \) for constructing a **topological module** \( \log(\dag \mathfrak{F}_v) \) [i.e., corresponding to “\( k^\sim(G) \)”] and a **topological subspace** \( I_{\mathfrak{F}_v} \) [i.e., corresponding to “\( \mathcal{I}(G) \)”]. In fact, this functorial algorithm only makes use of the **unit portion** of this split Frobenioid, together with a **pointed universal covering**
of this unit portion. Moreover, by arguing as in (ii), one may in fact regard this
functorial algorithm as an algorithm that only makes use of the quotient of this unit
portion by its $N$-torsion subgroup, for $N \in \mathbb{N}_{\geq 1}$, together with a pointed universal
covering of this quotient. That is to say, this functorial algorithm may, in fact,
be regarded as a functorial algorithm in the collection of data $\mathcal{I} \mathfrak{F}^{v,\mu}_{\mathbb{L}}$ [cf. Remark
1.2.1, (i), below; [IUTchII], Definition 4.9, (v), (vi), (vii)]. Write

$$\mathcal{I}_v \mathfrak{F}^{v,\mu}_{\mathbb{L}} \subseteq \log(\mathfrak{F}^{v,\mu}_{\mathbb{L}})$$

for the resulting topological module equipped with a closed subspace, regarded as
objects constructed from $\mathfrak{F}^{v,\mu}_{\mathbb{L}}$.

(vi) Finally, just as in (iii), we shall write

$$\mathcal{I}_v \mathfrak{F}^{v,\mu}_{\mathbb{L}} \triangleq \{\mathcal{I}_v \mathfrak{F}^{v,\mu}_{\mathbb{L}}\}_{v \in \mathbb{Y}} \subseteq \log(\mathfrak{F}^{v,\mu}_{\mathbb{L}}) \triangleq \{\log(\mathfrak{F}^{v,\mu}_{\mathbb{L}})\}_{v \in \mathbb{Y}}$$

for the resulting collections of data constructed solely from the $\mathfrak{F}^{v,\mu}_{\mathbb{L}}$-prime-strip
$\mathfrak{F}^{v,\mu}_{\mathbb{L}}$ [i.e., which we do not regard as objects constructed from $\mathfrak{F}^{v,\mu}_{\mathbb{L}}$].

**Remark 1.1.1.**

(i) Thus, $\log$-links may be thought of as correspondences between certain
portions of the ind-topological monoids in the domain of the $\log$-link and certain
portions of the ind-topological monoids in the codomain of the $\log$-link. Frequently,
in the theory of the present paper, we shall have occasion to consider “iterates” of
$\log$-links. The $\log$-links — i.e., correspondences between certain portions of the ind-
topological monoids in the domains and codomains of the $\log$-links — that appear
in such iterates are to be understood as being defined only on the [local] units [cf.
also (ii) below, in the case of $v \in \mathbb{Y}_{\text{arc}}$] that appear in the domains of these $\log$-links.
Thus, for instance, when considering [the nonzero elements of] a global number field
embedded in an “idèlic” product [indexed by the set of all valuations of the number
field] of localizations, we shall regard the $\log$-links that appear as being defined
only on the product [indexed by the set of all valuations of the number field] of the
groups of local units that appear in the domains of these $\log$-links. Indeed, in the
theory of the present paper, the only reason for the introduction of $\log$-links is to
render possible

the construction of the log-shells from the various [local] units.

That is to say, the construction of log-shells does not require the use of the “non-
unit” — i.e., the local and global “value group” — portions of the various monoids
in the domain. Thus, when considering the effect of applying various iterates of $\log$
links, it suffices, from the point of view of computing the effect of the construction
of the log-shells from the local units, to consider the effect of such iterates on the
various groups of local units that appear.

(ii) Suppose that we are in the situation of the discussion of [local] units in
(i), in the case of $v \in \mathbb{Y}_{\text{arc}}$. Then, when thinking of Kummer structures in
terms of Aut-holomorphic structures and co-holomorphicizations, as in the
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discussion of [IUTchI], Remark 3.4.2, it is natural to regard the [local] units that appear as being, in fact, “Aut-holomorphic semi-germs”, that is to say,

- projective systems of arbitrarily small neighborhoods of the [local] units [i.e., of “$\mathbb{S}^1$” in “$\mathbb{C}^\times$”, or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of “$\mathcal{O}^\times(C_\omega)$” in “$\mathcal{O}^\triangledown(C_\omega)^{gp}$”], equipped with

  - the Aut-holomorphic structures induced by restricting the ambient Aut-holomorphic structure [i.e., of “$\mathbb{C}^\times$”, or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of “$\mathcal{O}^\triangledown(C_\omega)^{gp}$”],

  - the group structure [germ] induced by restricting the ambient group structure [i.e., of “$\mathbb{C}^\times$”, or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of “$\mathcal{O}^\triangledown(C_\omega)^{gp}$”], and

  - a choice of one of the two connected components of the complement of the units in a sufficiently small neighborhood [i.e., determined by “$\mathcal{O}^\triangledown(C_\omega) \setminus \mathcal{O}^\times(C_\omega) \subseteq \mathcal{O}^\triangledown(C_\omega)^{gp} \setminus \mathcal{O}^\times(C_\omega)$”].

Indeed, one verifies immediately that such “Aut-holomorphic semi-germs” are rigid in the sense that they do not admit any nontrivial holomorphic automorphisms. In particular, by thinking of the [local] units as “Aut-holomorphic semi-germs” in this way, the approach to Kummer structures and co-holomorphicizations discussed in [IUTchI], Remark 3.4.2, carries over without change [cf. [AbsTopIII], Corollary 2.3, (i)]. Moreover, in light of the well-known discreteness of the image of the units of a number field via the logarithms of the various archimedean valuations of the number field [cf., e.g., [Lang], p. 144, Theorem 5], it follows that the “id`elic” aspects discussed in (i) are also unaffected by thinking in terms of Aut-holomorphic semi-germs.

**Remark 1.1.2.**

(i) In the notation of Definition 1.1, (i), we observe that the tautological log-link

$$\dagger \mathcal{F}_\omega \xrightarrow{\log} \log(\dagger \mathcal{F}_\omega)$$

satisfies the property that there is a tautological identification between the $\dagger \Pi_\omega$’s that appear in the data that gives rise to the domain [i.e., $\{\dagger \Pi_\omega \leftrightsquigarrow \Psi_{\dagger \mathcal{F}_\omega}\}$] and the data that gives rise to the codomain [i.e., $\{\dagger \Pi_\omega \leftrightsquigarrow \Psi_{\log(\dagger \mathcal{F}_\omega)}\}$] of the tautological log-link. By contrast, the $\dagger \Pi_\omega$ that appears in the data that gives rise to the domain of the full log-link

$$\dagger \mathcal{F}_\omega \xrightarrow{\log} \dagger \mathcal{F}_\omega$$

is related to the $\dagger \Pi_\omega$ [where we use analogous notational conventions for objects associated to $\dagger \mathcal{F}$ to the notational conventions already in force for objects associated to $\dagger \mathcal{F}$] that appears in the data that gives rise to the codomain of the full log-link by means of a full poly-isomorphism $\dagger \Pi_\omega \leftrightsquigarrow \dagger \Pi_\omega$. In this situation,

the specific isomorphism between the $\dagger \Pi_\omega$’s in the domain and codomain of the tautological log-link may be thought of as a sort of specific “rigidifying path” between the $\dagger \Pi_\omega$’s in the domain and codomain of the tautological log-link that is constructed precisely by using [in an essential
way!] Frobenius-like monoids that are related via the $p_v$-adic logarithm [cf. the construction of Definition 1.1, (i)], i.e., by applying the Galois-equivariance of the power series defining the $p_v$-adic logarithm to relate automorphisms of the monoid $\Psi_{\dagger F_v}$ to [induced!] automorphisms of the monoid $\Psi_{\sim \dagger F_v} = \Psi_{\log(\dagger F_v)}$. Here, the use of the term “path” is intended to be in the spirit of the notion of a path in the étale groupoid [i.e., in the context of the classical theory of the étale fundamental group], except that, in the present context, we allow arbitrary automorphism indeterminacies, instead of just inner automorphism indeterminacies. In the present paper, we shall work mainly with the full log-link [i.e., not with the tautological log-link!] since, in the context of the multiradial algorithms to be developed in §3 below, it will be of crucial importance to be able to express the relationship between the étale-like $(-)\Pi_v$’s in the domain and codomain of the log-links that appear in purely étale-like terms, i.e., in a fashion that is [unlike the specific “rigidifying path” discussed above!] free of any dependence on the Frobenius-like monoids involved.

(ii) An analogous discussion to that of (i) may be given in the situation of Definition 1.1, (ii), i.e., in the case of $v \in V_{\text{arc}}$. We leave the routine details to the reader.

From the point of view of the present series of papers, the theory of [AbsTopIII] may be summarized as follows.

**Proposition 1.2. (log-links Between $F$-prime-strips)** Let

$$\dagger \mathcal{F} = \{\dagger F_v\}_{v \in V}; \quad \dagger \mathcal{G} = \{\dagger G_v\}_{v \in V}$$

be $F$-prime-strips [relative to the given initial $\Theta$-data — cf. [IUTchI], Definition 5.2, (i)] and

$$\dagger \mathcal{F} \xrightarrow{\log} \dagger \mathcal{G}$$

a log-link from $\dagger \mathcal{F}$ to $\dagger \mathcal{G}$. Write $\dagger \mathcal{F}^{+ \times \mu}, \dagger \mathcal{G}^{+ \times \mu}$ for the associated $\mathcal{F}^{+ \times \mu}$-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii)]; $\dagger \mathcal{D}, \dagger \mathcal{G}$ for the associated $\mathcal{D}$-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)]; $\dagger \mathcal{D}^+, \dagger \mathcal{G}^+$ for the associated $\mathcal{D}^+$-prime-strips [cf. [IUTchII], Definition 4.1, (iv)]. Also, let us recall the diagrams

$$\Psi_{\dagger F_v} \supset \Psi_{\dagger F_v}^{X} \xrightarrow{\log(\dagger F_v)} \Psi_{\log(\dagger F_v)}^{\log(\dagger F_v)} \xrightarrow{\sim} \Psi_{\dagger F_v}^{\log(\dagger F_v)} \quad (\ast_{\text{non}})$$

$$\Psi_{\dagger F_v} \subset \Psi_{\dagger F_v}^{X} \xleftarrow{\log(\dagger F_v)} \Psi_{\log(\dagger F_v)}^{\log(\dagger F_v)} \xrightarrow{\sim} \Psi_{\dagger F_v}^{\log(\dagger F_v)} \quad (\ast_{\text{arc}})$$

— where the $v$ of $(\ast_{\text{non}})$ (respectively, $(\ast_{\text{arc}})$) belongs to $V_{\text{non}}$ (respectively, $V_{\text{arc}}$), and the [poly-]isomorphisms on the right are induced by the “$\xrightarrow{\log}$” — of Definition 1.1, (i), (ii).
(i) (Coricity of Associated $\mathcal{D}$-Prime-Strips) The log-link $\dagger \mathfrak{F} \xrightarrow{\log} \dagger \mathfrak{F}$ induces [poly-]isomorphisms
\[
\dagger \mathcal{D} \sim \dagger \mathcal{D}; \quad \dagger \mathcal{D}^\dagger \sim \dagger \mathcal{D}^\dagger
\]
between the associated $\mathcal{D}$- and $\mathcal{D}^\dagger$-prime-strips. In particular, the [poly-]isomorphism $\dagger \mathcal{D} \sim \dagger \mathcal{D}$ induced by $\dagger \mathfrak{F} \xrightarrow{\log} \dagger \mathfrak{F}$ induces a [poly-]isomorphism
\[
\Psi_{\text{cns}}(\dagger \mathcal{D}) \sim \Psi_{\text{cns}}(\dagger \mathcal{D})
\]
between the collections of monoids equipped with auxiliary data of [IUTchII], Corollary 4.5, (i).

(ii) (Simultaneous Compatibility with Ring Structures) At $v \in \mathbb{V}^{\text{non}}$, the natural $\dagger \mathbb{P}_v$-actions on the “$\Psi$’s” appearing in the diagram $(\ast_{\text{non}})$ are compatible with the ind-topological ring structures on $\Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}}$ and $\Psi^{\text{arc}}_{\log(\dagger \mathbb{F}^v_\mathbb{L})}$. At $v \in \mathbb{V}^{\text{arc}}$, the co-holomorphicizations determined by the natural $\Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}}$ and $\Psi^{\text{arc}}_{\log(\dagger \mathbb{F}^v_\mathbb{L})}$ (in $\Psi_{\dagger \mathbb{F}^v_\mathbb{L}}$)-Kummer structures on $\dagger \mathbb{U}_\mathbb{L}$ — which [cf. the discussion of Definition 1.1, (ii)] are compatible with the diagram $(\ast_{\text{arc}})$ — are compatible with the topological ring structures on $\Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}}$ and $\Psi^{\text{arc}}_{\log(\dagger \mathbb{F}^v_\mathbb{L})}$.

(iii) (Simultaneous Compatibility with Log-volumes) At $v \in \mathbb{V}^{\text{non}}$, the diagram $(\ast_{\text{non}})$ is compatible with the natural $p_v$-adic log-volumes [cf. [AbsTopIII], Proposition 5.7, (i), (c); [AbsTopIII], Corollary 5.10, (ii)] on the subsets of $\dagger \mathbb{P}_v$-invariants of $\Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}}$ and $\Psi^{\text{arc}}_{\log(\dagger \mathbb{F}^v_\mathbb{L})}$. At $v \in \mathbb{V}^{\text{arc}}$, the diagram $(\ast_{\text{arc}})$ is compatible with the natural angular log-volume [cf. Remark 1.2.1, (i), below; [AbsTopIII], Proposition 5.7, (ii); [AbsTopIII], Corollary 5.10, (ii)] on $\Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}}$ and the natural radial log-volume [cf. [AbsTopIII], Proposition 5.7, (ii), (c); [AbsTopIII], Corollary 5.10, (ii)] on $\Psi^{\text{arc}}_{\log(\dagger \mathbb{F}^v_\mathbb{L})}$ — cf. also Remark 1.2.1, (ii), below.

(iv) (Kummer theory) The Kummer isomorphisms
\[
\Psi_{\text{cns}}(\dagger \mathfrak{F}) \sim \Psi_{\text{cns}}(\dagger \mathcal{D}); \quad \Psi_{\text{cns}}(\dagger \mathfrak{F}) \sim \Psi_{\text{cns}}(\dagger \mathcal{D})
\]
of [IUTchII], Corollary 4.6, (i), fail to be compatible with the [poly-]isomorphism $\Psi_{\text{cns}}(\dagger \mathcal{D}) \sim \Psi_{\text{cns}}(\dagger \mathcal{D})$ of (i), relative to the diagrams $(\ast_{\text{non}})$, $(\ast_{\text{arc}})$ [and the notational conventions of Definition 1.1] — cf. [AbsTopIII], Corollary 5.5, (iv). [Here, we regard the diagrams $(\ast_{\text{non}})$, $(\ast_{\text{arc}})$ as diagrams that relate $\Psi_{\dagger \mathbb{F}^v_\mathbb{L}}$ and $\Psi_{\dagger \mathbb{F}^v_\mathbb{L}}$, via the [poly-]isomorphism $\log(\dagger \mathfrak{F}) \sim \dagger \mathfrak{F}$ that determines the log-link $\dagger \mathfrak{F} \xrightarrow{\log} \dagger \mathfrak{F}$.]

(v) (Holomorphic Log-shells) At $v \in \mathbb{V}^{\text{non}}$, the log-shell
\[
\mathcal{I}_{\dagger \mathbb{F}^v_\mathbb{L}} \subseteq \log(\dagger \mathbb{F}^v_\mathbb{L}) \quad (\sim \Psi^{\text{arc}}_{\dagger \mathbb{F}^v_\mathbb{L}})
\]
satisfies the following properties: $(a_{\text{non}})$ $\mathcal{I}_{\dagger \mathbb{F}^v_\mathbb{L}}$ is compact, hence of finite log-volume [cf. [AbsTopIII], Corollary 5.10, (i)]; $(b_{\text{non}})$ $\mathcal{I}_{\dagger \mathbb{F}^v_\mathbb{L}}$ contains the submonoid
of $\dagger \Pi_{\underline{v}}$-invariants of $\Psi_{\log(\dagger F_{\underline{v}})}$ [cf. [AbsTopIII], Definition 5.4, (iii)]; $(c_{\text{non}})$ $\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}}$ contains the image of the submonoid of $\dagger \Pi_{\underline{v}}$-invariants of $\Psi_{\dagger \mathcal{F}_{\underline{v}}}$. At $v \in \underline{V}_{\text{arc}}$, the log-shell

$$
\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}} \subseteq \log(\dagger F_{\underline{v}}) \quad (\sim \Psi^{\text{gp}}_{\dagger \mathcal{F}_{\underline{v}}})
$$

satisfies the following properties: $(a_{\text{arc}})$ $\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}}$ is compact, hence of finite radial log-volume [cf. [AbsTopIII], Corollary 5.10, (i)]; $(b_{\text{arc}})$ $\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}}$ contains $\Psi_{\log(\dagger F_{\underline{v}})}$ [cf. [AbsTopIII], Definition 5.4, (v)]; $(c_{\text{arc}})$ the image of $\mathcal{I}_{\dagger \mathcal{F}_{\underline{v}}}$ in $\Psi_{\dagger \mathcal{F}_{\underline{v}}}$ contains $\Psi_{\times \dagger F_{\underline{v}}}$ [i.e., in essence, the pre-log-shell].

$(vi)$ (Nonarchimedean Mono-analytic Log-shells) At $v \in \underline{V}_{\text{non}}$, if we write $\dagger D_{\underline{v}} = B(\dagger G_{\underline{v}})^{0}$ for the portion of $\dagger \mathcal{D}_{\underline{v}}$ indexed by $\underline{v}$ [cf. the notation of [IUTchII], Corollary 4.5], then the algorithms for constructing "$k^\sim(G)"$, "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (ii), yield a functorial algorithm in the category $\dagger \mathcal{D}_{\underline{v}}$ for constructing an ind-topological module equipped with a continuous $\dagger G_{\underline{v}}$-action

$$
\log(\dagger D_{\underline{v}}) \overset{\text{def}}{=} \left\{ \dagger G_{\underline{v}} \curvearrowright k^\sim(\dagger G_{\underline{v}}) \right\}
$$

and a topological submodule — i.e., a "mono-analytic log-shell" —

$$
\mathcal{I}_{\dagger D_{\underline{v}}} \overset{\text{def}}{=} \mathcal{I}(\dagger G_{\underline{v}}) \subseteq k^\sim(\dagger G_{\underline{v}})
$$

equipped with a $p_{\underline{v}}$-adic log-volume [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural functorial algorithm [cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data $\dagger F_{\underline{v}} \times \mu$ [i.e., the portion of $\dagger \mathcal{S}_{\underline{v}} \times \mu$ labeled by $\underline{v}$] for constructing an Ism-orbit of isomorphisms [cf. [IUTchII], Example 1.8, (iv); [IUTchII], Definition 4.9, (i), (vii)]

$$
\log(\dagger D_{\underline{v}}) \overset{\sim}{\rightarrow} \log(\dagger F_{\underline{v}} \times \mu)
$$

of ind-topological modules [cf. Definition 1.1, (iv)], as well as a functorial algorithm [cf. [AbsTopIII], Corollary 5.10, (iv), (c), (d); the fourth display of [IUTchII], Corollary 4.5, (ii); the final display of [IUTchII], Corollary 4.6, (i)] in the collection of data $\dagger \mathcal{F}_{\underline{v}}$ for constructing isomorphisms

$$
\log(\dagger D_{\underline{v}}) \overset{\sim}{\rightarrow} \log(\dagger F_{\underline{v}} \times \mu) \overset{\sim}{\rightarrow} \log(\dagger F_{\underline{v}}) \quad (\sim \Psi^{\text{gp}}_{\dagger \mathcal{F}_{\underline{v}}})
$$

of ind-topological modules. The various isomorphisms of the last two displays are compatible with one another, as well as with the respective $\dagger G_{\underline{v}}$- and $G_{\underline{v}}(\dagger \Pi_{\underline{v}})$-actions [relative to the natural identification $\dagger G_{\underline{v}} = G_{\underline{v}}(\dagger \Pi_{\underline{v}})$] that arises from regarding $\dagger D_{\underline{v}}$ as an object constructed from $\dagger \mathcal{D}_{\underline{v}}$, the respective log-shells, and the respective log-volumes on these log-shells.

$(vii)$ (Archimedean Mono-analytic Log-shells) At $v \in \underline{V}_{\text{arc}}$, the algorithms for constructing "$k^\sim(G)"$, "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (v), yield a functorial algorithm in $\dagger \mathcal{D}_{\underline{v}}$ [regarded as an object of the category
“\(\mathcal{TM}^-\)” of split topological monoids discussed in [IUTchI], Example 3.4, (ii) for constructing a topological module

\[
\log(\uparrow D_u^-) \overset{\text{def}}{=} k^\sim(\uparrow G_u^-)
\]

and a topological subspace — i.e., a “mono-analytic log-shell” —

\[
I_{1-D_u^-} \overset{\text{def}}{=} I(\uparrow G_u^-) \subseteq k^\sim(\uparrow G_u^-)
\]

equipped with angular and radial log-volumes [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural functorial algorithm [cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data \(\uparrow F_u^+ \times \mu\) for constructing a poly-isomorphism [i.e., an orbit of isomorphisms with respect to the independent actions of \(\{\pm 1\}\) on each of the direct factors that occur in the construction of [AbsTopIII], Proposition 5.8, (v)]

\[
\log(\uparrow D_u^-) \sim \log(\uparrow F_u^+ \times \mu)
\]

of topological modules. The various isomorphisms of the last two displays are compatible with one another, as well as with the respective log-shells and the respective angular and radial log-volumes on these log-shells.

(viii) (Mono-analytic Log-shells) The various poly-isomorphisms of (vi), (vii) [cf. also Definition 1.1, (iii), (vi)] yield collections of poly-isomorphisms indexed by \(v \in \mathcal{V}\)

\[
\log(\uparrow D_u^-) \overset{\text{def}}{=} \{\log(\uparrow D_u^-)_v\}_{v \in \mathcal{V}} \sim \log(\uparrow \mathcal{S}^+ \times \mu) \overset{\text{def}}{=} \{\log(\uparrow F_u^+ \times \mu)\}_{v \in \mathcal{V}}
\]

\[
I_{1-D_u^-} \overset{\text{def}}{=} \{I_{1-D_u^-}_v\}_{v \in \mathcal{V}} \sim I_{1-\mathcal{S}^+ \times \mu} \overset{\text{def}}{=} \{I_{1-F_u^+ \times \mu}\}_{v \in \mathcal{V}}
\]

\[
\log(\uparrow D_u^-) \sim \log(\uparrow \mathcal{S}^+ \times \mu) \sim \log(\uparrow \mathcal{S}) \overset{\text{def}}{=} \{\log(\uparrow F_u)_v\}_{v \in \mathcal{V}}
\]

\[
I_{1-D_u^-} \sim I_{1-\mathcal{S}^+ \times \mu} \sim I_{1-\mathcal{S}} \overset{\text{def}}{=} \{I_{1-F_u}_v\}_{v \in \mathcal{V}}
\]

— where, in the definition of “\(\Psi_{\text{cns}}^{\text{gp}}(\uparrow \mathcal{S})\)”, we regard each \(\Psi_{\text{cns}}^{\text{gp}}(\uparrow \mathcal{S})_v\), for \(v \in \mathcal{V}_{\text{non}}\), as being equipped with its natural \(G_u(\uparrow \Pi_u^-)\)-action [cf. the discussion at the beginning of Definition 1.1].
(ix) (Coric Holomorphic Log-shells) Let $\mathcal{D}$ be a $\mathcal{D}$-prime-strip; write
$$\text{coric holomorphic log-shells}$$
for the $\mathcal{F}$-prime-strip naturally determined by $\Psi_{\text{coric}}(\mathcal{D})$ [cf. [IUTchII], Remark 4.5.1, (i)]. Suppose that $\mathcal{F} / \mathcal{D} = \mathcal{D}$ and that the given log-link $\mathcal{F} \log \longrightarrow \mathcal{D}$ is the full log-link. Then there exists a functorial algorithm in the $\mathcal{D}$-prime-strip $\mathcal{F}$ for constructing a collection of topological subspaces — i.e., a collection of “coric holomorphic log-shells” —

$$I_{\mathcal{D}} \overset{\text{def}}{=} I_{\mathcal{F}}$$

of the collection $\Psi_{\text{coric}}(\mathcal{D})$, which may be naturally identified with $\Psi_{\text{coric}}(\mathcal{F})$, together with a collection of natural isomorphisms [cf. (viii); the fourth display of [IUTchII], Corollary 4.5, (ii)]

$$I_{\mathcal{D}} \overset{\sim}{\rightarrow} I_{\mathcal{F}}$$

— where we write $\mathcal{F}$ for the $\mathcal{D}$-prime-strip determined by $\mathcal{F}$.

(x) (Frobenius-picture) Let $\{\mathcal{F}_{n}\}_{n \in \mathbb{Z}}$ be a collection of distinct $\mathcal{F}$-prime-strips [relative to the given initial $\Theta$-data — cf. [IUTchI], Definition 5.2, (i)] indexed by the integers. Write $\{\mathcal{D}_{n}\}_{n \in \mathbb{Z}}$ for the associated $\mathcal{D}$-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)] and $\{\mathcal{D}_{n}\}_{n \in \mathbb{Z}}$ for the associated $\mathcal{D}$-prime-strips [cf. [IUTchI], Definition 4.1, (iv)]. Then the full log-links $\mathcal{F} \log \longrightarrow (n+1)\mathcal{F}$, for $n \in \mathbb{Z}$, give rise to an infinite chain

$$\ldots \mathcal{F} \log \longrightarrow (n+1)\mathcal{F} \log \longrightarrow n\mathcal{F} \log \longrightarrow (n+1)\mathcal{F} \log \longrightarrow \ldots$$

of log-linked $\mathcal{F}$-prime-strips which induces chains of full poly-isomorphisms

$$\ldots \overset{\sim}{\rightarrow} n\mathcal{D} \overset{\sim}{\rightarrow} (n+1)\mathcal{D} \overset{\sim}{\rightarrow} \ldots$$

and

$$\ldots \overset{\sim}{\rightarrow} n\mathcal{D} \overset{\sim}{\rightarrow} (n+1)\mathcal{D} \overset{\sim}{\rightarrow} \ldots$$

on the associated $\mathcal{D}$- and $\mathcal{D}$-prime-strips [cf. (i)]. These chains may be represented symbolically as an oriented graph $\Gamma$ [cf. [AbsTopIII], §0]

$$\ldots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \ldots$$

— i.e., where the horizontal arrows correspond to the “$\log \longrightarrow$’s”; the “$\bullet$’s” correspond to the “$\mathcal{F}$”; the “$\circ$” corresponds to the “$\mathcal{D}$”, identified up to isomorphism; the vertical/diagonal arrows correspond to the Kummer isomorphisms of (iv). This oriented graph $\Gamma$ admits a natural action by $\mathbb{Z}$ [cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a translation symmetry — that fixes the “core” $\circ$, but it does not admit arbitrary permutation symmetries. For instance, $\Gamma$ does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.
Proof. The various assertions of Proposition 1.2 follow immediately from the definitions and the references quoted in the statements of these assertions. □

Remark 1.2.1.

(i) Suppose that we are in the situation of Definition 1.1, (ii). Then at the level of metrics — i.e., which give rise to angular log-volumes as in Proposition 1.2, (iii) — we suppose that $\Psi_{\mathcal{F}_1}^{\mathcal{P}}/\Psi_{\mathcal{F}_2}^{\mu N}$ is equipped with the metric obtained by descending the metric of $\Psi_{\mathcal{F}_2}^{\mathcal{P}}$, but we regard the object $\Psi_{\mathcal{F}_1}^{\mathcal{P}}/\Psi_{\mathcal{F}_2}^{\mu N}$ [or $\Psi_{\mathcal{F}_2}^{\mathcal{P}}/\Psi_{\mathcal{F}_2}^{\mu N}$] as being equipped with a “weight $N$” — i.e., which has the effect of ensuring that the log-volume of $\Psi_{\mathcal{F}_1}^{\mathcal{P}}/\Psi_{\mathcal{F}_2}^{\mu N}$ is equal to that of $\Psi_{\mathcal{F}_2}^{\mathcal{P}}$. That is to say, this convention concerning “weights” ensures that working with $\Psi_{\mathcal{F}_1}^{\mathcal{P}}/\Psi_{\mathcal{F}_2}^{\mu N}$ does not have any effect on various computations of log-volume.

(ii) Although, at first glance, the compatibility with archimedean log-volumes discussed in Proposition 1.2, (iii), appears to relate “different objects” — i.e., angular versus radial log-volumes — in the domain and codomain of the log-link under consideration, in fact, this compatibility property may be regarded as an invariance property — i.e., that relates “similar objects” in the domain and codomain of the log-link under consideration — by reasoning as follows. Let $k$ be a complex archimedean field. Write $O_{k}^{\times} \subseteq k$ for the group of elements of absolute value $= 1$ and $k^{\times} \subseteq k$ for the group of nonzero elements. In the following, we shall use the term “metric on $k$” to refer to a Riemannian metric on the real analytic manifold determined by $k$ that is compatible with the two natural almost complex structures on this real analytic manifold and, moreover, is invariant with respect to arbitrary additive translation automorphisms of $k$. In passing, we note that any metric on $k$ is also invariant with respect to multiplication by elements $\in O_{k}^{\times}$. Next, let us observe that the metrics on $k$ naturally form a torsor over $\mathbb{R} > 0$. In particular, if we write $k^{\times} \cong O_{k}^{\times} \times \mathbb{R} > 0$ for the natural direct product decomposition, then one verifies immediately that

any metric on $k$ is uniquely determined either by its restriction to $O_{k}^{\times} \subseteq k$ or by its restriction to $\mathbb{R} > 0 \subseteq k$.

Thus, if one regards the compatibility property concerning angular and radial log-volumes discussed in Proposition 1.2, (iii), as a property concerning the respective restrictions of the corresponding uniquely determined metrics [i.e., the metrics corresponding to the respective standard norms on the complex archimedean fields under consideration — cf. [AbsTopIII], Proposition 5.7, (ii), (a)], then this compatibility property discussed in Proposition 1.2, (iii), may be regarded as a property that asserts the invariance of the respective natural metrics with respect to the “transformation” constituted by the log-link.

Remark 1.2.2. Before proceeding, we pause to consider the significance of the various properties discussed in Proposition 1.2, (v). For simplicity, we suppose
that “\(\mathfrak{F}\)” is the \(\mathcal{F}\)-prime-strip that arises from the data constructed in [IUTchI], Examples 3.2, (iii); 3.3, (i); 3.4, (i) [cf. [IUTchI], Definition 5.2, (i)].

(i) Suppose that \(v \in \mathcal{V}^{\text{non}}\). Thus, \(K_v\) [cf. the notation of [IUTchI], Definition 3.1, (e)] is a mixed-characteristic nonarchimedean local field. Write \(k \overset{\text{def}}{=} K_v\), \(O_k \subseteq k\) for the ring of integers of \(k\), \(O_k^\times \subseteq O_k\) for the group of units, and \(\log_k : O_k^\times \to k\) for the \(p_v\)-adic logarithm. Then, at a more concrete level — i.e., relative to the notation of the present discussion — the log-shell “\(\mathcal{I}_{\mathcal{F}_v}\)” corresponds to the submodule

\[\mathcal{I}_k \overset{\text{def}}{=} (p_v^*)^{-1} \cdot \log_k(O_k^\times) \subseteq k\]

— where \(p_v^* = p_v\) if \(p_v\) is odd, \(p_v^* = p_v^2\) if \(p_v\) is even — while the properties \((b_{\text{non}}), (c_{\text{non}})\) of Proposition 1.2, (v), correspond, respectively, to the evident inclusions

\[O_k^{\gg} \overset{\text{def}}{=} O_k \setminus \{0\} \subseteq O_k \subseteq \mathcal{I}_k; \quad \log_k(O_k^\times) \subseteq \mathcal{I}_k\]

of subsets of \(k\).

(ii) Suppose that \(v \in \mathcal{V}^{\text{arc}}\). Thus, \(K_v\) [cf. the notation of [IUTchI], Definition 3.1, (e)] is a complex archimedean field. Write \(k \overset{\text{def}}{=} K_v\), \(O_k \subseteq k\) for the subset of elements of absolute value \(\leq 1\), \(O_k^\times \subseteq O_k\) for the group of elements of absolute value \(= 1\), and \(\exp_k : k \to k^\times\) for the exponential map. Then, at a more concrete level — i.e., relative to the notation of the present discussion — the log-shell “\(\mathcal{I}_{\mathcal{F}_v}\)” corresponds to the subset

\[\mathcal{I}_k \overset{\text{def}}{=} \{a \in k \mid |a| \leq \pi\} \subseteq k\]

of elements of absolute value \(\leq \pi\), while the properties \((b_{\text{arc}}), (c_{\text{arc}})\) of Proposition 1.2, (v), correspond, respectively, to the evident inclusions

\[O_k^{\gg} \overset{\text{def}}{=} O_k \setminus \{0\} \subseteq O_k \subseteq \mathcal{I}_k; \quad O_k^\times \subseteq \exp_k(\mathcal{I}_k)\]

— where we note the slightly different roles played, in the archimedean [cf. the present (ii)] and nonarchimedean [cf. (i)] cases, by the exponential and logarithmic functions, respectively [cf. [AbsTopIII], Remark 4.5.2].

(iii) The diagram represented by the oriented graph \(\Gamma\) of Proposition 1.2, (x), is, of course, far from commutative [cf. Proposition 1.2, (iv)]! Ultimately, however, [cf. the discussion of Remark 1.4.1, (ii), below] we shall be interested in

(a) constructing invariants with respect to the \(\mathbb{Z}\)-action on \(\Gamma\) — i.e., in effect, constructing objects via functorial algorithms in the coric \(D\)-prime-strips “\(n\mathfrak{D}\)” —

while, at the same time,

(b) relating the corically constructed objects of (a) to the non-coric “\(n\mathfrak{F}\)” via the various Kummer isomorphisms of Proposition 1.2, (iv).
That is to say, from the point of view of (a), (b), the content of the inclusions discussed in (i) and (ii) above may be interpreted, at \( v \in \mathcal{V}^{\text{non}} \), as follows:

the coric holomorphic log-shells of Proposition 1.2, (ix), contain not only the images, via the Kummer isomorphisms [i.e., the vertical/diagonal arrows of \( \vec{\Gamma} \)], of the various \( \mathcal{O}^{\geq} \) at \( v \in \mathcal{V}^{\text{non}} \), but also the images, via the composite of the Kummer isomorphisms with the various iterates [cf. Remark 1.1.1] of the log-link [i.e., the horizontal arrows of \( \vec{\Gamma} \)], of the portions of the various \( \mathcal{O}^{\geq} \) at \( v \in \mathcal{V}^{\text{non}} \) on which these iterates are defined.

An analogous statement in the case of \( v \in \mathcal{V}^{\text{arc}} \) may be formulated by adjusting the wording appropriately so as to accommodate the latter portion of this statement, which corresponds to a certain surjectivity — we leave the routine details to the reader. Thus, although the diagram [corresponding to] \( \vec{\Gamma} \) fails to be commutative,

the coric holomorphic log-shells involved exhibit a sort of “upper semi-commutativity” with respect to containing/surjecting onto the various images arising from composites of arrows in \( \vec{\Gamma} \).

(iv) Note that although the diagram \( \vec{\Gamma} \) admits a natural “upper semi-commutativity” interpretation as discussed in (iii) above, it fails to admit a corresponding “lower semi-commutativity” interpretation. Indeed, such a “lower semi-commutativity” interpretation would amount to the existence of some sort of collection of portions of the various \( \mathcal{O}^{\geq} 's \) involved [cf. the discussion of (i), (ii) above] — i.e., a sort of “core” — that are mapped to one another isomorphically by the various maps “log_k”/“exp_k” [cf. the discussion of (i), (ii) above] in a fashion that is compatible with the various Kummer isomorphisms that appear in the diagram \( \vec{\Gamma} \). On the other hand, it is difficult to see how to construct such a collection of portions of the various \( \mathcal{O}^{\geq} 's \) involved.

(v) Proposition 1.2, (iii), may be interpreted in the spirit of the discussion of (iii) above. That is to say, although the diagram corresponding to \( \vec{\Gamma} \) fails to be commutative, it is nevertheless “commutative with respect to log-volumes”, in the sense discussed in Proposition 1.2, (iii). This “commutativity with respect to log-volumes” allows one to work with log-volumes in a fashion that is consistent with all composites of the various arrows of \( \vec{\Gamma} \). Log-volumes will play an important role in the theory of §3, below, as a sort of mono-analytic version of the notion of the degree of a global arithmetic line bundle [cf. the theory of [AbsTopIII], §5].

(vi) As discussed in [AbsTopIII], §I3, the log-links of \( \vec{\Gamma} \) may be thought of as a sort of “juggling of \( \oplus \), \( \otimes \)” [i.e., of the two combinatorial dimensions of the ring structure constituted by addition and multiplication]. The “arithmetic holomorphic structure” constituted by the coric \( \mathcal{D} \)-prime-strips is immune to this juggling, and hence may be thought as representing a sort of quotient of the horizontal arrow portion of \( \vec{\Gamma} \) by the action of \( \mathbb{Z} \) [cf. (iii), (a)] — i.e., at the level of abstract oriented graphs, as a sort of “oriented copy of \( \mathbb{S}^1 \)”. That is to say, the horizontal arrow portion of \( \vec{\Gamma} \) may be thought of as a sort of “unraveling” of
this “oriented copy of $S^1$”, which is subject to the “juggling of $\Box, \otimes$” constituted by the $\mathbb{Z}$-action. Here, it is useful to recall that

(a) the Frobenius-like structures constituted by the monoids that appear in the horizontal arrow portion of $\vec{\Gamma}$ play the crucial role in the theory of the present series of papers of allowing one to construct such “non-ring/scheme-theoretic filters” as the $\Theta$-link [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

By contrast,

(b) the étale-like structures constituted by the coric $D$-prime-strips play the crucial role in the theory of the present series of papers of allowing one to construct objects that are capable of “functorially permeating” such non-ring/scheme-theoretic filters as the $\Theta$-link [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

Finally, in order to relate the theory of (a) to the theory of (b), one must avail oneself of Kummer theory [cf. (iii), (b), above].

<table>
<thead>
<tr>
<th>mono-anabelian coric étale-like structures</th>
<th>invariant differential $d\theta$ on $S^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>post-anabelian Frobenius-like structures</td>
<td>coordinate functions $\int_{\cdot} d\theta$ on $\bar{\Gamma}$</td>
</tr>
</tbody>
</table>

Fig. 1.1: Analogy with the differential geometry of $S^1$

(vii) From the point of view of the discussion in (vi) above of the “oriented copy of $S^1$” obtained by forming the quotient of the horizontal arrow portion of $\bar{\Gamma}$ by $\mathbb{Z}$, one may think of the coric étale-like structures of Proposition 1.2, (i) — as well as the various objects constructed from these coric étale-like structures via the various mono-anabelian algorithms discussed in [AbsTopIII] — as corresponding to the “canonical invariant differential $d\theta$” on $S^1$ [which is, in particular, invariant with respect to the action of $\mathbb{Z}!$]. On the other hand, the various post-anabelian Frobenius-like structures obtained by forgetting the mono-anabelian algorithms applied to construct these objects — cf., e.g., the $\Psi_{\text{cns}}(\hat{\mathfrak{I}})$ that appear in the Kummer isomorphisms of Proposition 1.2, (iv) — may be thought of as coordinate functions on the horizontal arrow portion of $\bar{\Gamma}$ [which are not invariant with respect to the action of $\mathbb{Z}!$] of the form “$\int_{\cdot} d\theta$” obtained by integrating the invariant differential $d\theta$ along various paths of $\bar{\Gamma}$ that emanate from some fixed vertex “$\cdot$” of $\bar{\Gamma}$.

This point of view is summarized in Fig. 1.1 above. Finally, we observe that this point of view is reminiscent of the discussion of [AbsTopIII], §I5, concerning the analogy between the theory of [AbsTopIII] and the construction of canonical
Remark 1.2.3.

(i) Observe that, relative to the notation of Remark 1.2.2, (i), any multiplicative indeterminacy with respect to the action on \( O_k^\times \) of some subgroup \( H \subset O_k^\times \) at some “•” of the diagram \( \Gamma \) gives rise to an additive indeterminacy with respect to the action of \( \log_k(H) \) on the copy of “\( O_k^\times \)” that corresponds to the subsequent “•” of the diagram \( \Gamma \). In particular, if \( H \) consists of roots of unity, then \( \log_k(H) = \{0\} \), so the resulting additive indeterminacy ceases to exist. This observation will play a crucial role in the theory of §3, below, when it is applied in the context of the constant multiple rigidity properties constituted by the canonical splittings of theta and Gaussian monoids discussed in [IUTchII], Proposition 3.3, (i); [IUTchII], Corollary 3.6, (iii) [cf. also [IUTchII], Corollary 1.12, (ii); the discussion of [IUTchII], Remark 1.12.2, (iv)].

(ii) In the theory of §3, below, we shall consider global arithmetic line bundles. This amounts, in effect, to considering multiplicative translates by \( f \in F_\mathfrak{mod}^\times \) of the product of the various “\( O_k^\times \)” of Remark 1.2.2, (i), (ii), as \( \mathfrak{y} \) ranges over the elements of \( \mathbb{V}_\mathfrak{y} \). Such translates are disjoint from one another, except in the case where \( f \) is a unit at all \( \mathfrak{y} \in \mathbb{V} \). By elementary algebraic number theory [cf., e.g., [Lang], p. 144, the proof of Theorem 5], this corresponds precisely to the case where \( f \) is a root of unity. In particular, to consider quotients by this multiplicative action by \( F_\mathfrak{mod}^\times \) at one “•” of the diagram \( \Gamma \) [where we allow \( \mathfrak{y} \) to range over the elements of \( \mathbb{V} \)] gives rise to an additive indeterminacy by “logarithms of roots of unity” at the subsequent “•” of the diagram \( \Gamma \). In particular, at \( \mathfrak{y} \in \mathbb{V}_\mathfrak{mod}^{\text{non}} \), the resulting additive indeterminacy ceases to exist [cf. the discussion of (i); Definition 1.1, (iv)]; at \( \mathfrak{y} \in \mathbb{V}_\mathfrak{mod}^{\text{arc}} \), the resulting indeterminacy corresponds to considering certain quotients of the copies of “\( O_k^\times \)” — i.e., of “\( S^1 \)” — that appear by some finite subgroup [cf. the discussion of Definition 1.1, (ii)]. These observations will be of use in the development of the theory of §3, below.

Remark 1.2.4.

(i) At this point, we pause to recall the important observation that the \( \log \)-link is incompatible with the ring structures of \( \Psi^\mathfrak{gp}_{/F_\mathfrak{y}}^{\text{dp}} \) and \( \Psi^\mathfrak{gp}_{\log(1/F_\mathfrak{y})}^{\text{dp}} \) [cf. the notation of Proposition 1.2, (ii)], in the sense that it does not arise from a ring homomorphism between these two rings. The barrier constituted by this incompatibility between the ring structures on either side of the \( \log \)-link is precisely what is referred to as the “\( \log \)-wall” in the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I4]. This incompatibility with the respective ring structures implies that it is not possible, a priori, to transport objects whose structure depends on these ring structures via the \( \log \)-link by invoking the principle of “transport of structure”. From the point of view of the theory of the present series of papers, this means, in particular, that the \( \log \)-wall is incompatible with conventional scheme-theoretic base-points, which are defined by means of geometric points [i.e., ring homomorphisms of a certain type] coordinates via integration of Frobenius-invariant differentials in the classical p-adic theory.
In this context, it is useful to recall that étale fundamental groups — i.e., Galois groups — are defined as certain automorphism groups of fields/rings; in particular, the definition of such a Galois group “as a certain automorphism group of some ring structure” is incompatible, in a quite essential way, with the log-wall. In a similar vein, Kummer theory, which depends on the multiplicative structure of the ring under consideration, is also incompatible, in a quite essential way, with the log-wall [cf. Proposition 1.2, (iv)]. That is to say, in the context of the log-link,

the only structure of interest that is manifestly compatible with the log-link [cf. Proposition 1.2, (i), (ii)] is the associated $D$-prime-strip

— i.e., the abstract topological groups [isomorphic to “$\Pi$” — cf. the notation of [IUTchI], Definition 3.1, (e), (f)] at $v \in \mathcal{V}^{\text{non}}$ and abstract Aut-holomorphic spaces [isomorphic to “$\mathcal{U}$” — cf. the notation of [IUTchII], Proposition 4.3] at $v \in \mathcal{V}^{\text{arc}}$. Indeed, this observation is precisely the starting point of the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I1, §I4].

(ii) Other important examples of structures which are incompatible with the log-wall include:

(a) the additive structure on the image of the Kummer map [cf. the discussion of [AbsTopIII], Remark 3.7.5];

(b) in the “birational” situation — i.e., where one replaces “$\Pi$” by the absolute Galois group $\Gamma^{\text{birat}}$ of the function field of the affine curve that gave rise to $\Pi$ — the datum of the collection of closed points that determines the affine curve [cf. [AbsTopIII], Remark 3.7.7, (ii)].

Note, for instance in the case of (b), when, say, for simplicity, $v \in \mathcal{V}^{\text{good}} \setminus \mathcal{V}^{\text{non}}$, that one may think of the additional datum under consideration as consisting of the natural outer surjection $\Gamma^{\text{birat}} \twoheadrightarrow \Pi$, that arises from the scheme-theoretic morphism from the spectrum of the function field to the given affine curve. On the other hand, just as in the case of the discussion of scheme-theoretic basepoints in (i), the construction of such an object $\Gamma^{\text{birat}} \twoheadrightarrow \Pi$ whose structure depends, in an essential way, on the scheme [i.e., ring!] structures involved necessarily fails to be compatible with the log-link [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)].

(iii) One way to understand the incompatibility discussed in (ii), (b), is as follows. Write $\Delta^{\text{birat}}, \Delta$ for the respective kernels of the natural surjections $\Gamma^{\text{birat}} \twoheadrightarrow G, \Pi \twoheadrightarrow G$. Then if one forgets about the scheme-theoretic basepoints discussed in (i), $G, \Delta^{\text{birat}},$ and $\Delta$ may be understood on both sides of the log-wall as “some topological group”, and each of the topological groups $\Delta^{\text{birat}},$ $\Delta$ may be understood on both sides of the log-wall as being equipped with “some outer $G$-action” — cf. the two diagonal arrows of Fig. 1.2 below. On the other hand, the datum of a particular outer surjection $\Delta^{\text{birat}} \twoheadrightarrow \Delta$ [cf. the dotted line in Fig. 1.2] relating these two diagonal arrows — which depends, in an essential way, on the scheme [i.e., ring] structures involved! — necessarily fails to be compatible with the log-link [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)]. This issue
of “triangular compatibility between independent indeterminacies” is formally reminiscent of the issue of compatibility of outer homomorphisms discussed in [IUTchI], Remark 4.5.1, (i) [cf. also [IUTchII], Remark 2.5.2, (ii)].

\[
\begin{align*}
\text{indep.} \\
\text{bp. indet.} \\
\sim
\end{align*}
\]

\[
\begin{array}{c}
\text{nonarch. local} \\
\text{abs. Galois group } G_v
\end{array}
\]

\[
\begin{align*}
\text{indep.} \\
\text{bp. indet.} \\
\sim
\end{align*}
\]

\[
\begin{array}{c}
\text{birational geom.} \\
\text{fund. gp. } \Delta_{\text{birat}}
\end{array}
\]

\[
\begin{align*}
\text{indep.} \\
\text{bp. indet.} \\
\sim
\end{align*}
\]

\[
\begin{array}{c}
\text{affine geom.} \\
\text{fund. gp. } \Delta_v
\end{array}
\]

\[
\begin{array}{c}
\text{birational geom.} \\
\text{fund. gp. } \Delta_{\text{birat}}
\end{array}
\]

\[
\begin{array}{c}
? \\
\sim
\end{array}
\]

Fig. 1.2: Independent basepoint indeterminacies obstruct relationship between birational and affine geometric fundamental groups

**Remark 1.2.5.** The discussion in Remark 1.2.4 of the incompatibility of the log-wall with various structures that arise from ring/scheme-theory is closely related to the issue of avoiding the use of fixed ring/scheme-theoretic reference models in mono-anabelian construction algorithms [cf. the discussion of [IUTchI], Remark 3.2.1, (i); [AbsTopIII], §I4]. Put another way, at least in the context of the log-link [i.e., situations of the sort considered in [AbsTopIII], as well as in the present paper], mono-anabelian construction algorithms may be understood as algorithms whose dependence on data arising from such fixed ring/scheme-theoretic reference models is “invariant”, or “coric”, with respect to the action of log on such models.

A substantial portion of [AbsTopIII], §3, is devoted precisely to the task of giving a precise formulation of this concept of “invariance” by means of such notions as observables, families of homotopies, and telecores. For instance, one approach to formulating the failure of the ring structure of a fixed reference model to be “coric” with respect to log may be seen in [AbsTopIII], Corollary 3.6, (iv); [AbsTopIII], Corollary 3.7, (iv).

**Proposition 1.3.** (log-links Between $\Theta^{\pm\text{ell}}$NF-Hodge Theaters) Let

\[
\begin{align*}
\uparrow H^\Theta^{\pm\text{ell}}NF, \\
\downarrow H^\Theta^{\pm\text{ell}}NF
\end{align*}
\]
be $\Theta^{\pm\text{ell}}\text{NF}$-Hodge theaters [relative to the given initial $\Theta$-data] — cf. [IUTchI], Definition 6.13, (i). Write $\hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF}$, $\hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF}$ for the associated $\mathcal{D}$-$\Theta^{\pm\text{ell}}\text{NF}$-Hodge theaters — cf. [IUTchI], Definition 6.13, (ii). Then:

(i) (Construction of the log-Link) Fix an isomorphism

$$\Xi : \hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF} \sim \to \hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF}$$

of $\mathcal{D}$-$\Theta^{\pm\text{ell}}\text{NF}$-Hodge theaters. Let $\hat{\mathcal{F}}$ be one of the $\mathcal{F}$-prime-strips that appear in the $\Theta$- and $\Theta^{\pm\text{ell}}$-bridges that constitute $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ — i.e., either one of the $\mathcal{F}$-prime-strips $\hat{\mathcal{F}}, \hat{\mathcal{F}} >$ or one of the constituent $\mathcal{F}$-prime-strips of the capsules $\hat{\mathcal{F}}_J, \hat{\mathcal{F}}_T$

[cf. [IUTchI], Definition 5.5, (ii); [IUTchI], Definition 6.11, (i)]. Write $\hat{\mathcal{F}}$ for the corresponding $\mathcal{F}$-prime-strip of $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$. Then the poly-isomorphism determined by $\Xi$ between the $\mathcal{D}$-prime-strips associated to $\hat{\mathcal{F}}, \hat{\mathcal{F}} >$ uniquely determines a poly-isomorphism $\log(\hat{\mathcal{F}}) \sim \to \hat{\mathcal{F}} [\text{cf. Definition 1.1, (iii); [IUTchI], Corollary 5.3, (ii)}]$, hence a log-link $\hat{\mathcal{F}} \to \hat{\mathcal{F}} [\text{cf. Definition 1.1, (iii)}]$. We shall denote by

$$\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}} \to \log \hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$$

and refer to as a log-link from $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ to $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ the collection of data consisting of $\Xi$, together with the collection of log-links $\hat{\mathcal{F}} \to \log \hat{\mathcal{F}}$, as “$\square$” ranges over all possibilities for the $\mathcal{F}$-prime-strips in question. When $\Xi$ is replaced by a poly-isomorphism $\hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF} \sim \to \hat{\mathcal{H}}^D, \Theta^{\pm\text{ell}}\text{NF}$, we shall also refer to the resulting collection of log-links [i.e., corresponding to each constituent isomorphism of the poly-isomorphism $\Xi$] as a log-link from $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ to $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$. When $\Xi$ is the full poly-isomorphism, we shall refer to the resulting log-link as the full log-link.

(ii) (Coricity) Any log-link $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}} \to \log \hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ satisfies, for each $\mathcal{F}$-prime-strip $\hat{\mathcal{F}}$, properties corresponding to the properties of Proposition 1.2, (ii), (iii), (iv), (v), (vi), (vii), (vii), (ix), i.e., concerning simultaneous compatibility with ring structures and log-volumes, Kummer theory, and log-shells.

(iii) (Further Properties of the log-Link) In the notation of (i), any log-link $\hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}} \to \log \hat{\mathcal{H}}^{\Theta^{\pm\text{ell}}\text{NF}}$ satisfies, for each $\mathcal{F}$-prime-strip $\hat{\mathcal{F}}$, properties corresponding to the properties of Proposition 1.2, (ii), (iii), (iv), (v), (vi), (vii), (vii), (ix), i.e., concerning simultaneous compatibility with ring structures and log-volumes, Kummer theory, and log-shells.
(iv) (Frobenius-picture) Let \( \{ n \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \}_{n \in \mathbb{Z}} \) be a collection of distinct \( \Theta^\pm \ell \text{NF} \)-Hodge theaters [relative to the given initial \( \Theta \)-data] indexed by the integers. Write \( \{ n \mathcal{H}^p_{D,\Theta^\pm \ell \text{NF}} \}_{n \in \mathbb{Z}} \) for the associated \( D,\Theta^\pm \ell \text{NF} \)-Hodge theaters. Then the full log-links \( n \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \xrightarrow{\log} (n+1) \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \), for \( n \in \mathbb{Z} \), give rise to an infinite chain\(^1\)

\[
... \xrightarrow{\log} (n-1) \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \xrightarrow{\log} n \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \xrightarrow{\log} (n+1) \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \xrightarrow{\log} ...
\]

of log-linked \( \Theta^\pm \ell \text{NF} \)-Hodge theaters which induces a chain of full poly-isomorphisms

\[
... \xrightarrow{\sim} n \mathcal{H}^p_{D,\Theta^\pm \ell \text{NF}} \xrightarrow{\sim} (n+1) \mathcal{H}^p_{D,\Theta^\pm \ell \text{NF}} \xrightarrow{\sim} ...
\]

on the associated \( D,\Theta^\pm \ell \text{NF} \)-Hodge theaters. These chains may be represented symbolically as an oriented graph \( \vec{\Gamma} \) [cf. [AbsTopIII], §0]

\[
... \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow ... \\
... \downarrow \downarrow \check{\downarrow} ...
\]

— i.e., where the horizontal arrows correspond to the “\( \log \)”s; the “\( \bullet \)”s correspond to the “\( \mathcal{H}^p_{\Theta^\pm \ell \text{NF}} \)”s; the “\( \circ \)” corresponds to the “\( \mathcal{H}^p_{D,\Theta^\pm \ell \text{NF}} \)”s, identified up to isomorphism; the vertical/diagonal arrows correspond to the Kummer isomorphisms implicit in the statement of (iii). This oriented graph \( \vec{\Gamma} \) admits a natural action by \( \mathbb{Z} \) [cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a translation symmetry — that fixes the “core” \( \circ \), but it does not admit arbitrary permutation symmetries. For instance, \( \vec{\Gamma} \) does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.

**Proof.** The various assertions of Proposition 1.3 follow immediately from the definitions and the references quoted in the statements of these assertions. \( \Box \)

**Remark 1.3.1.** Note that in Proposition 1.3, (i), it was necessary to carry out the given construction of the log-link first for a single \( \Xi \) [i.e., as opposed to a poly-isomorphism \( \Xi \)], in order to maintain compatibility with the crucial “\( \pm \)-synchronization” [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] inherent in the structure of a \( \Theta^\pm \ell \)-Hodge theater.

**Remark 1.3.2.** In the construction of Proposition 1.3, (i), the constituent \( \mathcal{F} \)-prime-strips \( \dagger \mathfrak{F}_t \), for \( t \in T \), of the capsule \( \dagger \mathfrak{F}_T \) are considered without regard to the \( \mathbb{F}_t^\pm \)-symmetries discussed in [IUTchII], Corollary 4.6, (iii). On the other hand, one verifies immediately that the log-links associated, in the construction of Proposition 1.3, (i), to these \( \mathcal{F} \)-prime-strips \( \dagger \mathfrak{F}_t \), for \( t \in T \) — i.e., more precisely, associated to the labeled collections of monoids \( \Psi_{\text{cus}}(\dagger \mathfrak{F}_t) \) of [IUTchII], Corollary 4.6, (iii) — are in fact compatible with the \( \mathbb{F}_t^\pm \)-symmetrizing isomorphisms discussed in [IUTchII], Corollary 4.6, (iii), hence also with the conjugate synchronization determined by these \( \mathbb{F}_t^\pm \)-symmetrizing isomorphisms — cf. the discussion of Step...
(vi) of the proof of Corollary 3.12 of §3 below. We leave the routine details to the reader.

**Remark 1.3.3.**

(i) In the context of Proposition 1.3 [cf. also the discussion of Remarks 1.2.4, 1.3.1, 1.3.2], it is of interest to observe that the relationship between the various **Frobenioid-theoretic** [i.e., Frobenius-like] portions of the $\Theta^\pm_{\text{ell}}$NF-Hodge theaters in the domain and codomain of the log-link of Proposition 1.3, (i),

does not include any data — i.e., of the sort discussed in Remark 1.2.4, (ii), (a), (b); Remark 1.2.4, (iii) — that is incompatible, relative to the relevant Kummer isomorphisms, with the coricity property for étale-like structures given in Proposition 1.3, (ii).

Indeed, this observation may be understood as a consequence of the fact [cf. Remarks 1.3.1, 1.3.2; [IUTchI], Corollary 5.3, (i), (ii), (iv); [IUTchII], Corollary 5.6, (i), (ii), (iii)] that these Frobenioid-theoretic portions of the $\Theta^\pm_{\text{ell}}$NF-Hodge theaters under consideration are completely [i.e., fully faithfully] controlled [cf. the discussion of (ii) below for more details], via **functorial algorithms**, by the corresponding étale-like structures, i.e., structures that appear in the associated $\mathcal{D}$-$\Theta^\pm_{\text{ell}}$NF-Hodge theaters, which satisfy the crucial coricity property of Proposition 1.3, (ii).

(ii) In the context of (i), it is of interest to recall that the global portion of the underlying $\Theta^\text{ell}$-bridges is defined [cf. [IUTchI], Definition 6.11, (ii)] in such a way that it does not contain any global Frobenioid-theoretic data! In particular, the issue discussed in (i) concerns only the Frobenioid-theoretic portions of the following:

(a) the various $\mathcal{F}$-prime-strips that appear;
(b) the underlying $\Theta$-Hodge theaters of the $\Theta^\pm_{\text{ell}}$NF-Hodge theaters under consideration;
(c) the global portion of the underlying NF-bridges of the $\Theta^\pm_{\text{ell}}$NF-Hodge theaters under consideration.

Here, the Frobenioid-theoretic data of (c) gives rise to **independent** [i.e., for corresponding portions of the $\Theta^\pm_{\text{ell}}$NF-Hodge theaters in the domain and codomain of the log-link] basepoints with respect to the $\mathbb{F}^*_p$-symmetry [cf. [IUTchI], Corollary 5.6, (iii); [IUTchII], Remark 6.12.6, (iii); [IUTchII], Remark 4.7.6]. On the other hand, the independent basepoints that arise from the Frobenioid-theoretic data of (b), as well as of the portion of (a) that lies in the underlying $\Theta$NF-Hodge theater, do not cause any problems [i.e., from the point of view of the sort of incompatibility discussed in (i)] since this data is only subject to relationships defined by means of full poly-isomorphisms [cf. [IUTchI], Examples 4.3, 4.4]. That is to say, the $\mathcal{F}$-prime-strips that lie in the underlying $\Theta^\pm_{\text{ell}}$-Hodge theater constitute the most delicate [i.e., relative to the issue of independent basepoints!] portion of the Frobenioid-theoretic data of a $\Theta^\pm_{\text{ell}}$NF-Hodge theater. This delicacy revolves
around the global synchronization of \( \pm \)-indeterminacies in the underlying \( \Theta^{\pm \text{ell}} \)-Hodge theater [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)]. On the other hand, this delicacy does not in fact cause any problems [i.e., from the point of view of the sort of incompatibility discussed in (i)] since [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] the synchronizations of \( \pm \)-indeterminacies in the underlying \( \Theta^{\pm \text{ell}} \)-Hodge theater are defined [not by means of scheme-theoretic relationships, but rather] by applying the intrinsic structure of the underlying \( D\Theta^{\pm \text{ell}} \)-Hodge theater, which satisfies the crucial coricity property of Proposition 1.3, (ii) [cf. the discussion of (i); Remarks 1.3.1, 1.3.2].

The diagrams discussed in the following Definition 1.4 will play a central role in the theory of the present series of papers.

**Definition 1.4.** We maintain the notation of Proposition 1.3 [cf. also [IUTchII], Corollary 4.10, (iii)]. Let \( \{^{n,m} \mathcal{HT}^{\pm \text{ell}}_{\text{NF}}\}_{n,m \in \mathbb{Z}} \) be a collection of distinct \( \Theta^{\pm \text{ell}} \text{NF} \)-Hodge theaters [relative to the given initial \( \Theta \)-data] indexed by pairs of integers. Then we shall refer to either of the diagrams

\[
\begin{array}{cccccc}
\vdots & & \vdots & \downarrow \log & \downarrow \log & \\
\Theta^{\times \mu} & n, m+1 & \Theta^{\times \mu} & n+1, m+1 & \Theta^{\times \mu} & \\
\downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \\
\vdots & & \vdots & & \vdots & \\
\Theta^{\times \mu} & n, m & \Theta^{\times \mu} & n, m & \Theta^{\times \mu} & \\
\downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \\
\vdots & & \vdots & & \vdots & \\
\Theta^{\times \mu}_{\text{gau}} & n, m+1 & \Theta^{\times \mu}_{\text{gau}} & n+1, m+1 & \Theta^{\times \mu}_{\text{gau}} & \\
\downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \\
\vdots & & \vdots & & \vdots & \\
\Theta^{\times \mu}_{\text{gau}} & n, m & \Theta^{\times \mu}_{\text{gau}} & n, m & \Theta^{\times \mu}_{\text{gau}} & \\
\downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \downarrow \log & \\
\vdots & & \vdots & & \vdots & \\
\end{array}
\]

— where the vertical arrows are the full log-links, and the horizontal arrows are the \( \Theta^{\times \mu} \)- and \( \Theta^{\times \mu}_{\text{gau}} \)-links of [IUTchII], Corollary 4.10, (iii) — as the log-theta-lattice. We shall refer to the log-theta-lattice that involves the \( \Theta^{\times \mu} \) (respectively, \( \Theta^{\times \mu}_{\text{gau}} \))
links as non-Gaussian (respectively, Gaussian). Thus, either of these diagrams may be represented symbolically by an oriented graph

\[
\begin{array}{c}
\vdots \\
\uparrow \\
\vdots \\
\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\uparrow \\
\vdots \\
\end{array}
\]

- where the "●'s" correspond to the \(n,m\mathcal{H}^{\Theta_{\pm\ell}^{\text{NF}}}\).

**Remark 1.4.1.**

(i) One fundamental property of the log-theta-lattices discussed in Definition 1.4 is the following:

the various squares that appear in each of the log-theta-lattices discussed in Definition 1.4 are far from being [1-]commutative!

Indeed, whereas the vertical arrows in each log-theta-lattice are constructed by applying the various logarithms at \(v \in \mathcal{V}\) — i.e., which are defined by means of power series that depend, in an essential way, on the local ring structures at \(v \in \mathcal{V}\) — the horizontal arrows in each log-theta-lattice [i.e., the \(\Theta^{\times\mu}-\), \(\Theta^{\times\mu}_{\text{gau}}\)-links] are incompatible with these local ring structures at \(v \in \mathcal{V}\) in an essential way [cf. [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) Whereas the horizontal arrows in each log-theta-lattice [i.e., the \(\Theta^{\times\mu}-\), \(\Theta^{\times\mu}_{\text{gau}}\)-links] allow one, roughly speaking, to identify the respective "\(\mathcal{O}^{\times\mu}\)'s" at [for simplicity] \(v \in \mathcal{V}^{\text{non}}\) on either side of the horizontal arrow [cf. [IUTchII], Corollary 4.10, (iv)], in order to avail oneself of the theory of log-shells — which will play an essential role in the multiradial representation of the Gaussian monoids to be developed in §3 below — it is necessary for the "●" [i.e., \(\Theta^{\pm\ell}\text{NF-Hodge theater}\)] in which one operates to appear as the codomain of a log-link, i.e., of a vertical arrow of the log-theta-lattice [cf. the discussion of [AbsTopIII], Remark 5.10.2, (iii)]. That is to say, from the point of view of the goal of constructing the multiradial representation of the Gaussian monoids that is to be developed in §3 below,

each execution of a horizontal arrow of the log-theta-lattice necessarily obligates a subsequent execution of a vertical arrow of the log-theta-lattice.

On the other hand, in light of the noncommutativity observed in (i), this "intertwining" of the horizontal and vertical arrows of the log-theta-lattice means
that the desired **multiradiality** — i.e., **simultaneous compatibility** with the arithmetic holomorphic structures on **both sides of a horizontal arrow** of the log-theta-lattice — can only be realized [cf. the discussion of Remark 1.2.2, (iii)] if one works with objects that are **invariant with respect to the vertical arrows** [i.e., with respect to the action of $\mathbb{Z}$ discussed in Proposition 1.3, (iv)], that is to say, with **“vertical cores”**, of the log-theta-lattice.

(iii) From the point of view of the analogy between the theory of the present series of papers and **$p$-adic Teichmüller theory** [cf. [AbsTopIII], §I5], the **vertical arrows** of the log-theta-lattice correspond to the **Frobenius morphism in positive characteristic**, whereas the **horizontal arrows** of the log-theta-lattice correspond to the “transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$”, i.e., the **mixed characteristic extension structure of a ring of Witt vectors** [cf. [IUTchI], Remark 3.9.3, (i)]. These correspondences are summarized in Fig. 1.3 below. In particular, the “**intertwining of horizontal and vertical arrows** of the log-theta-lattice” discussed in (ii) above may be thought of as the analogue, in the context of the theory of the present series of papers, of the well-known “intertwining between the mixed characteristic extension structure of a ring of Witt vectors and the Frobenius morphism in positive characteristic” that appears in the classical $p$-adic theory.

<table>
<thead>
<tr>
<th><strong>horizontal arrows of the log-theta-lattice</strong></th>
<th><strong>mixed characteristic extension structure of a ring of Witt vectors</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>vertical arrows of the log-theta-lattice</strong></td>
<td><strong>the Frobenius morphism in positive characteristic</strong></td>
</tr>
</tbody>
</table>

Fig. 1.3: Analogy between the log-theta-lattice and $p$-adic Teichmüller theory

**Remark 1.4.2.**

(i) The horizontal and vertical arrows of the log-theta-lattices discussed in Definition 1.4 share the common property of being **incompatible with the local ring structures**, hence, in particular, with the **conventional scheme-theoretic basepoints** on either side of the arrow in question [cf. the discussion of [IUTchII], Remark 3.6.3, (i)]. On the other hand, whereas the linking data of the **vertical arrows** [i.e., the log-link] is **rigid** and corresponds to a **single fixed, rigid arithmetic holomorphic structure** in which **addition and multiplication** are subject to “**rotations**” [cf. the discussion of [AbsTopIII], §I3], the linking data of the **horizontal arrows** [i.e., the $\Theta^{x\mu}$, $\Theta^{x\mu\text{gau}}$-links] — i.e., more concretely, the “$\mathcal{O}^{x\mu}$’s” at [for simplicity] $v \in \mathcal{V}^{\text{non}}$ — is subject to a **$\mathbb{Z}^\times$-indeterminacy**, which has the effect of **obliterating the arithmetic holomorphic structure associated to a vertical line** of the log-theta-lattice [cf. the discussion of [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) If, in the spirit of the discussion of [IUTchII], Remark 1.11.2, (ii), one attempts to “**force**” the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow by
declaring — in the style of the log-link! — that these horizontal arrows induce an isomorphism of the respective “Π _v’s” at [for simplicity] v ∈ V _{non}, then one must contend with a situation in which the “common arithmetic holomorphic structure rigidified by the isomorphic copies of Π _v” is obliterated each time one takes into account the action of a nontrivial element of $\hat{\mathbb{Z}}^\times$ [i.e., that arises from the $\hat{\mathbb{Z}}^\times$-indeterminacy involved] on the corresponding “$O^\times$”. In particular, in order to keep track of the arithmetic holomorphic structure currently under consideration, one must, in effect, consider paths that record the sequence of “Π _v-rigidifying” and “$\hat{\mathbb{Z}}^\times$-indeterminacy” operations that one invokes. On the other hand, the horizontal lines of the log-theta-lattices given in Definition 1.4 amount, in effect, to universal covering spaces of the loops — i.e., “unraveling paths of the loops” [cf. the discussion of Remark 1.2.2, (vi)] — that occur as one invokes various series of “Π _v-rigidifying” and “$\hat{\mathbb{Z}}^\times$-indeterminacy” operations. Thus, in summary, any attempt as described above to “force” the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow does not result in any substantive simplification of the theory of the present series of papers. We refer the reader to [IUTchIV], Remark 3.6.3, for a discussion of a related topic.

We are now ready to state the **main result** of the present §1.

**Theorem 1.5. (Bi-cores of the Log-theta-lattice)** Fix a collection of initial Θ-data $(\mathcal{F}/F, X_F, l, C_K, Y, Y_{\text{bad}}^\mod, \xi)$ as in [IUTchI], Definition 3.1. Then any Gaussian log-theta-lattice corresponding to this collection of initial Θ-data [cf. Definition 1.4] satisfies the following properties:

(i) **(Vertical Coricity)** The vertical arrows of the Gaussian log-theta-lattice induce full poly-isomorphisms between the respective associated $D^{\Theta^\pm\text{ell}}_{\mathcal{F}}$-Hodge theaters

\[ \ldots \rightsquigarrow n,m \mathcal{H}^{D^{\Theta^\pm\text{ell}}}_{\mathcal{F}} \rightsquigarrow n,m+1 \mathcal{H}^{D^{\Theta^\pm\text{ell}}}_{\mathcal{F}} \rightsquigarrow \ldots \]

[cf. Proposition 1.3, (ii)]. Here, n ∈ $\mathbb{Z}$ is held fixed, while m ∈ $\mathbb{Z}$ is allowed to vary.

(ii) **(Horizontal Coricity)** The horizontal arrows of the Gaussian log-theta-lattice induce full poly-isomorphisms between the respective associated $F^{\times\mu}$-prime-strips

\[ \ldots \rightsquigarrow n,m \mathcal{F}^{\times\mu}_{\Delta} \rightsquigarrow n+1,m \mathcal{F}^{\times\mu}_{\Delta} \rightsquigarrow \ldots \]

[cf. [IUTchII], Corollary 4.10, (iv)]. Here, m ∈ $\mathbb{Z}$ is held fixed, while n ∈ $\mathbb{Z}$ is allowed to vary.

(iii) **(Bi-coric $F^{\times\mu}$-Prime-Strips)** For n, m ∈ $\mathbb{Z}$, write $n,m \mathcal{D}^\Delta_{\mathcal{F}}$ for the $D^\Delta$-prime-strip associated to the $F^{\times\mu}$-prime-strip $n,m \mathcal{F}^{\times\mu}_{\Delta}$ labeled “Δ” of the $\Theta^\pm\text{ell}$ NF-Hodge theater $n,m \mathcal{H}^{\Theta^\pm\text{ell}}_{\mathcal{F}}$ [cf. [IUTchII], Corollary 4.10, (i)]; $n,m \mathcal{D}_{\succ}$ for the $D$-prime-strip labeled “$\succ$” of the $\Theta^\pm\text{ell}$ NF-Hodge theater $n,m \mathcal{H}^{\Theta^\pm\text{ell}}_{\mathcal{F}}$ [cf. [IUTchI],
Definitions 6.11, (i), (iii); 6.13, (i). Let us identify [cf. [IUTchII], Corollary 4.10, (ii)] the collections of data
\[
\Psi_{\text{cns}}^{(n,m \mathcal{D}_\prec)}_0 \text{ and } \Psi_{\text{cns}}^{(n,m \mathcal{D}_\succ)}(\mathcal{F}^i_\prec)
\]
via the isomorphism of the final display of [IUTchII], Corollary 4.5, (iii), and denote by
\[
\Xi \triangleleft (n,m \mathcal{D}_\succ)
\]
the resulting \(\mathcal{F}^\times\) -prime-strip. [Thus, it follows immediately from the constructions involved — cf. the discussion of [IUTchII], Corollary 4.10, (i) — that there is a natural identification isomorphism \(\Xi \triangleleft (n,m \mathcal{D}_\succ) \cong \Xi \triangleleft (n,m \mathcal{D}_\succ)\), where we write \(\Xi \triangleleft (n,m \mathcal{D}_\succ)\) for the \(\mathcal{F}^\times\) -prime-strip determined by \(\Psi_{\text{cns}}^{(n,m \mathcal{D}_\succ)}\).] Write
\[
\Xi \times \triangleright (n,m \mathcal{D}_\succ), \quad \Xi \times \mu (n,m \mathcal{D}_\succ)
\]
for the \(\mathcal{F}^\times\), \(\mathcal{F}^{\times \mu}\) -prime-strips determined by \(\Xi \triangleleft (n,m \mathcal{D}_\succ)\) [cf. [IUTchII], Definition 4.9, (vi), (vii)]. Thus, by applying the isomorphisms “\(\Psi_{\text{cns}}(\triangleright)^x_{\mathcal{U}} \cong \Psi_{\text{cns}}^x(\triangleright)^x_{\mathcal{U}}\)” for \(\mathcal{U} \in \mathcal{U}\) of [IUTchII], Corollary 4.5, (ii), [it follows immediately from the definitions that] there exists a functorial algorithm in the \(\mathcal{D}^\times\) -prime-strip \(n,m \mathcal{D}_\triangleleft\) for constructing an \(\mathcal{F}^\times\) -prime-strip \(\Xi \times \triangleright (n,m \mathcal{D}_\triangleleft)\), together with a functorial algorithm in the \(\mathcal{D}\) -prime-strip \(n,m \mathcal{D}_\succ\) for constructing a natural isomorphism
\[
\Xi \times \triangleright (n,m \mathcal{D}_\succ) \cong \Xi \times \triangleright (n,m \mathcal{D}_\succ)
\]
— i.e., in more intuitive terms, \(\Xi \times \triangleright (n,m \mathcal{D}_\succ)\)”, hence also the associated \(\mathcal{F}^{\times \mu}\) -prime-stripe \(\Xi \times \mu (n,m \mathcal{D}_\succ)\”, may be naturally regarded, up to isomorphism, as objects constructed from \(n,m \mathcal{D}_\triangleleft\). Then the poly-isomorphisms of (i) [cf. Remark 1.3.2], (ii) induce, respectively, poly-isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips
\[
\ldots \cong \Xi \times \mu (n,m \mathcal{D}_\succ) \cong \Xi \times \mu (n,m+1 \mathcal{D}_\succ) \cong \ldots
\]
\[
\ldots \cong \Xi \times \mu (n,m \mathcal{D}_\succ) \cong \Xi \times \mu (n+1,m \mathcal{D}_\succ) \cong \ldots
\]
— where we note that, relative to the natural isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips \(\Xi \times (n,m \mathcal{D}_\succ) \cong \Xi \times (n,m \mathcal{D}_\succ)\) discussed above, the collection of isomorphisms that constitute the poly-isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips of the first line of the display is, in general, strictly smaller than the collection of isomorphisms that constitute the poly-isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips of the second line of the display [cf. the existence of non-scheme-theoretic automorphisms of absolute Galois groups of MLF’s, as discussed in [AbsTopIII], §31]; the poly-isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips of the second line of the display are not full [cf. [IUTchII], Remark 1.8.1]. In particular, by composing these isomorphisms, one obtains poly-isomorphisms of \(\mathcal{F}^{\times \mu}\) -prime-strips
\[
\Xi \times \mu (n,m \mathcal{D}_\succ) \cong \Xi \times \mu (n',m' \mathcal{D}_\succ)
\]
for arbitrary \(n', m' \in \mathbb{Z}\). That is to say, in more intuitive terms, the \(\mathcal{F}^{\times \mu}\) -prime-stripe \(\Xi \times (n,m \mathcal{D}_\succ)\)”, regarded up to a certain class of isomorphisms, is an
invariant — which we shall refer to as “bi-coric” — of both the horizontal and the vertical arrows of the Gaussian log-theta-lattice. Finally, the Kummer isomorphisms “\(\Psi_{\text{cns}}(\frac{1}{2}\mathfrak{g}) \rightarrow \Psi_{\text{cns}}(\frac{1}{2}\mathcal{D})\)” of [IUTchII], Corollary 4.6, (i), determine Kummer isomorphisms

\[
n,m\mathfrak{g}_{\Delta}^{\times \mu} \sim \mathfrak{g}_{\Delta}^{\times \mu}(n,m\mathcal{D}_{\Delta})
\]

which are compatible with the poly-isomorphisms of (ii), as well as with the \(\times \mu\)-Kummer structures at the \(v \in \mathcal{V}_{\text{non}}\) of the various \(\mathcal{F}^{\times \mu}\)-prime-strips involved [cf. [IUTchII], Definition 4.9, (vi), (vii)]; a similar compatibility holds for \(v \in \mathcal{V}_{\text{arc}}\) [cf. the discussion of the final portion of [IUTchII], Definition 4.9, (v)].

(iv) (Bi-coric Mono-analytic Log-shells) The poly-isomorphisms that constitute the bi-coricity property discussed in (iii) induce poly-isomorphisms

\[
\left\{ \mathcal{I}_{n,m}^{\mathcal{D}_{\Delta}} \subseteq \log(n,m\mathcal{D}_{\Delta}) \right\} \sim \left\{ \mathcal{I}_{n',m'}^{\mathcal{D}_{\Delta}} \subseteq \log(n',m'\mathcal{D}_{\Delta}) \right\}
\]

\[
\left\{ \mathcal{I}_{\mathfrak{g}_{\Delta}^{\times \mu}(n,m\mathcal{D}_{\Delta})} \subseteq \log(\mathfrak{g}_{\Delta}^{\times \mu}(n,m\mathcal{D}_{\Delta})) \right\} \sim \left\{ \mathcal{I}_{\mathfrak{g}_{\Delta}^{\times \mu}(n',m'\mathcal{D}_{\Delta})} \subseteq \log(\mathfrak{g}_{\Delta}^{\times \mu}(n',m'\mathcal{D}_{\Delta})) \right\}
\]

for arbitrary \(n, m, n', m' \in \mathbb{Z}\) that are compatible with the natural poly-isomorphisms

\[
\left\{ \mathcal{I}_{n,m}^{\mathcal{D}_{\Delta}} \subseteq \log(n,m\mathcal{D}_{\Delta}) \right\} \sim \left\{ \mathcal{I}_{\mathfrak{g}_{\Delta}^{\times \mu}(n,m\mathcal{D}_{\Delta})} \subseteq \log(\mathfrak{g}_{\Delta}^{\times \mu}(n,m\mathcal{D}_{\Delta})) \right\}
\]

of Proposition 1.2, (viii). On the other hand, by applying the constructions of Definition 1.1, (i), (ii), to the collections of data “\(\Psi_{\text{cns}}(\frac{1}{2}\mathfrak{g}_{\Delta})_0\)” and “\(\Psi_{\text{cns}}(\frac{1}{2}\mathfrak{g}_{\Delta})(\varphi^*)\)” used in [IUTchII], Corollary 4.10, (i), to construct \(n,m\mathfrak{g}_{\Delta}^{\times \mu}\) [cf. Remark 1.3.2], one obtains a [“holomorphic”] log-shell, together with an enveloping \(\log(-)\) [cf. the pair “\(\mathcal{I}_{\mathfrak{g}} \subseteq \log(\mathfrak{g})\)” of Definition 1.1, (iii)], which we denote by

\[
\mathcal{I}_{n,m\mathfrak{g}_{\Delta}} \subseteq \log(n,m\mathfrak{g}_{\Delta})
\]

[by means of a slight abuse of notation, since no \(\mathcal{F}\)-prime-strip “\(n,m\mathfrak{g}_{\Delta}\)” has been defined!]. Then one has natural poly-isomorphisms

\[
\left\{ \mathcal{I}_{n,m}^{\mathcal{D}_{\Delta}} \subseteq \log(n,m\mathcal{D}_{\Delta}) \right\} \sim \left\{ \mathcal{I}_{n,m\mathfrak{g}_{\Delta}^{\times \mu}} \subseteq \log(n,m\mathfrak{g}_{\Delta}^{\times \mu}) \right\}
\]

\[
\sim \left\{ \mathcal{I}_{n,m\mathfrak{g}_{\Delta}} \subseteq \log(n,m\mathfrak{g}_{\Delta}) \right\}
\]

[cf. the poly-isomorphisms obtained in Proposition 1.2, (viii)]; here, the first “\(\sim\)” may be regarded as being induced by the Kummer isomorphisms of (iii) and is compatible with the poly-isomorphisms induced by the poly-isomorphisms of (ii).

(v) (Bi-coric Mono-analytic Global Realified Frobenioids) Let \(n, m, n', m' \in \mathbb{Z}\). Then the poly-isomorphisms of \(\mathcal{D}^{\times \mu}\)-prime-strips \(n,m\mathcal{D}_{\Delta}^{\times \mu} \sim n',m'\mathcal{D}_{\Delta}^{\times \mu}\) induced by the full poly-isomorphisms of (i), (ii) induce [cf. [IUTchII], Corollaries 4.5, (ii); 4.10, (v)] an isomorphism of collections of data

\[
(\mathcal{D}^{\times \mu}(n,m\mathcal{D}_{\Delta}^{\times \mu}), \text{Prime}(\mathcal{D}^{\times \mu}(n,m\mathcal{D}_{\Delta}^{\times \mu})) \rightarrow \mathcal{V}, \{n,m\rho_{\mathcal{D}^{\times \mu}}\}_{\mathcal{V} \in \mathcal{V}}) \sim (\mathcal{D}^{\times \mu}(n',m'\mathcal{D}_{\Delta}^{\times \mu}), \text{Prime}(\mathcal{D}^{\times \mu}(n',m'\mathcal{D}_{\Delta}^{\times \mu})) \rightarrow \mathcal{V}, \{n',m'\rho_{\mathcal{D}^{\times \mu}}\}_{\mathcal{V} \in \mathcal{V}})
\]
— i.e., consisting of a Frobenioid, a bijection, and a collection of isomorphisms of topological monoids indexed by $V$. Moreover, this isomorphism of collections of data is compatible, relative to the horizontal arrows of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of (ii)], with the $\mathbb{R}_{>0}$-orbits of the isomorphisms of collections of data

\[
(n,m)^{\text{Frobenius}} \mapsto \mathcal{V}, \{n,m \rho_{\Delta, \underline{u}}\}_{\underline{u} \in \mathcal{V}} \\
\sim \\
(n,m)^{\text{D}} \mapsto \mathcal{V}, \{n,m \rho_{\mathcal{D}, \underline{u}}\}_{\underline{u} \in \mathcal{V}}
\]

obtained by applying the functorial algorithm discussed in the final portion of [IUTchII], Corollary 4.6, (ii) [cf. also the latter portions of [IUTchII], Corollary 4.10, (i), (v)].

**Proof.** The various assertions of Theorem 1.5 follow immediately from the definitions and the references quoted in the statements of these assertions. 

**Remark 1.5.1.**

(i) Note that the theory of conjugate synchronization developed in [IUTchII] [cf., especially, [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)] plays an essential role in establishing the bi-coricity properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., at a more technical level, in constructing the objects equipped with a subscript “$\Delta$” that appear in Theorem 1.5, (iii); [IUTchII], Corollary 4.10, (i). That is to say, the conjugate synchronization determined by the various symmetrizing isomorphisms of [IUTchII], Corollaries 4.5, (iii); 4.6, (iii), may be thought of as a sort of descent mechanism that allows one to descend data that, a priori, is label-dependent [i.e., depends on the labels “$t \in \text{LabCusp}^{\pm}(\mathbb{R})$”] to data that is label-independent. Here, it is important to recall that these labels depend, in an essential way, on the “arithmetic holomorphic structures” involved — i.e., at a more technical level, on the geometric fundamental groups involved — hence only make sense within a vertical line of the log-theta-lattice. That is to say, the significance of this transition from label-dependence to label-independence lies in the fact that this transition is precisely what allows one to construct objects that make sense in horizontally adjacent “$\bullet$’s” of the log-theta-lattice, i.e., to construct horizontally coric objects [cf. Theorem 1.5, (ii); the second line of the fifth display of Theorem 1.5, (iii)]. On the other hand, in order to construct the horizontal arrows of the log-theta-lattice, it is necessary to work with Frobenius-like structures [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)]. In particular, in order to construct vertically coric objects [cf. the first line of the fifth display of Theorem 1.5, (iii)], it is necessary to pass to étale-like structures [cf. the discussion of Remark 1.2.4, (i)] by means of Kummer isomorphisms [cf. the final display of Theorem 1.5, (iii)]. Thus, in summary,

the bi-coricity properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., roughly speaking, the bi-coricity of the various “$\mathcal{O}^{\times \mu}$” at $\underline{u} \in \mathcal{V}^{\text{non}}$ — may be thought of as a consequence of the intricate interplay of various aspects of the theory of Kummer-compatible conjugate synchronization established in [IUTchII], Corollaries 4.5, (iii); 4.6, (iii).
(ii) In light of the central role played by the theory of conjugate synchronization in the constructions that underlie Theorem 1.5 [cf. the discussion of (i)], it is of interest to examine in more detail to what extent the highly technically nontrivial theory of conjugate synchronization may be replaced by a simpler apparatus. One naive approach to this problem is the following. Let $G$ be a topological group [such as one of the absolute Galois groups $G_v$ associated to $v \in V^{\text{non}}$]. Then one way to attempt to avoid the application of the theory of conjugate synchronization — which amounts, in essence, to the construction of a diagonal embedding

$$G \hookrightarrow G \times \ldots \times G$$

[cf. the notation “$(|F_l|)$”, “($F_l^\#)$” that appears in [IUTchII], Corollaries 3.5, 3.6, 4.5, 4.6] in a product of copies of $G$ that, $a$ priori, may only be identified with one another up to conjugacy [i.e., up to composition with an inner automorphism] — is to try to work, instead, with the $(G \times \ldots \times G)$-conjugacy class of such a diagonal. Here, to simplify the notation, let us assume that the above products of copies of $G$ are, in fact, products of two copies of $G$. Then to identify the diagonal embedding $G \hookrightarrow G \times G$ with its $(G \times G)$-conjugates implies that one must consider identifications

$$(g,g) \sim (g,hgh^{-1}) = (g,[h,g] \cdot g)$$

[where $g,h \in G$] — i.e., one must identify $(g,g)$ with the product of $(g,g)$ with $(1,[h,g])$. On the other hand, the original purpose of working with distinct copies of $G$ lies in considering distinct Galois-theoretic Kummer classes — corresponding to distinct theta values [cf. [IUTchII], Corollaries 3.5, 3.6] — at distinct components. That is to say, to identify elements of $G \times G$ that differ by a factor of $(1,[h,g])$ is incompatible, in an essential way, with the convention that such a factor $(1,[h,g])$ should correspond to distinct elements [i.e., “1” and “[h,g]”] at distinct components [cf. the discussion of Remark 1.5.3, (ii), below]. Here, we note that this incompatibility may be thought of as an essential consequence of the highly nonabelian nature of $G$, e.g., when $G$ is taken to be a copy of $G_v$, for $v \in V^{\text{non}}$. Thus, in summary, this naive approach to replacing the theory of conjugate synchronization by a simpler apparatus is inadequate from the point of view of the theory of the present series of papers.

(iii) At a purely combinatorial level, the notion of conjugate synchronization is reminiscent of the label synchronization discussed in [IUTchI], Remark 4.9.2, (i), (ii). Indeed, both conjugate and label synchronization may be thought of as a sort of combinatorial representation of the arithmetic holomorphic structure associated to a single vertical line of the log-theta-lattice [cf. the discussion of [IUTchI], Remark 4.9.2, (iv)].

Remark 1.5.2.

(i) Recall that unlike the case with the action of the $\mathbb{F}_l^{x^\pm}$-symmetry on the various labeled copies of the absolute Galois group $G_v$, for $v \in V^{\text{non}}$ [cf. [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)], it is not possible to establish an analogous theory of conjugate synchronization in the case of the $\mathbb{F}_l^\#$-symmetry for labeled copies of $\mathbb{F}$ [cf. [IUTchII], Remark 4.7.2]. This is to say, the closest analogue of the conjugate synchronization obtained in the local case relative to the $\mathbb{F}_l^{x^\pm}$-symmetry is the
action of the $F^*_l$-symmetry on labeled copies of the subfields $F_{\text{mod}} \subseteq F_{\text{sol}} \subseteq \overline{F}$ and
the pseudo-monoid of $\kappa$-coric rational functions, i.e., as discussed in [IUTchII], Corollaries 4.7, (ii); 4.8, (ii). One consequence of this incompatibility of the $F^*_l$-symmetry with the full algebraic closure $\overline{F}$ of $F_{\text{mod}}$ is that, as discussed in [IUTchI], Remark 5.1.5, the reconstruction of the ring structure on labeled copies of the subfield $F_{\text{sol}} \subseteq F$ subject to the $F^*_l$-symmetry [cf. [IUTchII], Corollaries 4.7, (ii); 4.8, (ii)], fails to be compatible with the various localization operations that occur in the structure of a $\mathcal{D}$-$\Theta$NF-Hodge theater. This is one quite essential reason why it is not possible to establish bi-coricity properties for, say, $F^\times_{\text{sol}}$ [which we regard as being equipped with the ring structure on the union of $F^\times_{\text{sol}}$ with $\{0\}$ — without which the abstract pair $\text{Gal}(F_{\text{sol}}/F_{\text{mod}}) \rtimes F^\times_{\text{sol}}$ consisting of an abstract module equipped with the action of an abstract topological group is not very interesting] that are analogous to the bi-coricity properties established in Theorem 1.5, (iii), for $\mathcal{O}^\times \mu$ [cf. the discussion of Remark 1.5.1, (i)]. From this point of view,

the bi-coric mono-analytic global realified Frobenioids of Theorem 1.5, (v) — i.e., in essence, the notion of “log-volume” [cf. the point of view of Remark 1.2.2, (v)] — may be thought of as a sort of “closest possible approximation” to such a “bi-coric $F^\times_{\text{sol}}$” [i.e., which does not exist].

Alternatively, from the point of view of the theory to be developed in §3 below,

we shall apply the bi-coric “$\mathcal{O}^\times \mu$’s” of Theorem 1.5, (iii) — i.e., in the form of the bi-coric mono-analytic log-shells of Theorem 1.5, (iv) — to construct “multiradial containers” for the labeled copies of $F_{\text{mod}}$ discussed above by applying the localization functors discussed in [IUTchII], Corollaries 4.7, (iii); 4.8, (iii).

That is to say, such “multiradial containers” will play the role of a transportation mechanism for “$F^\times_{\text{mod}}$” — up to certain indeterminacies! — between distinct arithmetic holomorphic structures [i.e., distinct vertical lines of the log-theta-lattice].

(ii) In the context of the discussion of “multiradial containers” in (i) above, we recall [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)] that, in general, Kummer theory plays a crucial role precisely in situations in which one performs constructions — such as, for instance, the construction of the $\Theta_\pm$, $\Theta^\times \mu_\pm$, or $\Theta^\times \mu_{\text{gal}}$-links — that are “not bound to conventional scheme theory”. That is to say, in the case of the labeled copies of “$F^\times_{\text{mod}}$” discussed in (i), the incompatibility of “solvable reconstructions” of the ring structure with the localization operations that occur in a $\mathcal{D}$-$\Theta$NF-Hodge theater [cf. [IUTchI], Remark 5.1.5] may be thought of as a reflection of the dismantling of the global prime-tree structure of a number field [cf. the discussion of [IUTchII], Remark 4.11.2, (iv)] that underlies the construction of the $\Theta^\pm$NF-Hodge theater performed in [IUTchI], [IUTchII], hence, in particular, as a reflection of the requirement of establishing a Kummer-compatible theory of conjugate synchronization relative to the $F^*_l$-$\pm$-symmetry [cf. the discussion of Remark 1.5.1, (i)].

(iii) Despite the failure of labeled copies of “$F^\times_{\text{mod}}$” to admit a natural bi-coric structure — a state of affairs that forces one to resort to the use of “multiradial
containers” in order to transport such labeled copies of “$F_{\text{mod}}^\times$” to alien arithmetic holomorphic structures [cf. the discussion of (i) above] — the global Frobenioids associated to copies of “$F_{\text{mod}}^\times$” nevertheless possess important properties that are not satisfied, for instance, by the bi-coric global realified Frobenioids discussed in Theorem 1.5, (v) [cf. also [IUTchI], Definition 5.2, (iv); [IUTchII], Corollary 4.5, (ii); [IUTchIII], Corollary 4.6, (ii)]. Indeed, unlike the objects contained in the realified global Frobenioids that appear in Theorem 1.5, (v), the objects contained in the global Frobenioids associated to copies of “$F_{\text{mod}}^\times$” correspond to genuine “conventional arithmetic line bundles”. In particular, by applying the ring structure of the copies of “$F_{\text{mod}}^\times$” under consideration, one can push forward such arithmetic line bundles so as to obtain arithmetic vector bundles over [the ring of rational integers] $\mathbb{Z}$ and then form tensor products of such arithmetic vector bundles. Such operations will play a key role in the theory of §3 below, as well as in the theory to be developed in [IUTchIV].

Remark 1.5.3.

(i) In [QuCnf] [cf. also [AbsTopIII], Proposition 2.6; [AbsTopIII], Corollary 2.7], a theory was developed concerning deformations of holomorphic structures on Riemann surfaces in which holomorphic structures are represented by means of squares or rectangles on the surface, while quasiconformal Teichmüller deformations of holomorphic structures are represented by parallelograms on the surface. That is to say, relative to suitable choices of local coordinates, quasiconformal Teichmüller deformations may be thought of as affine linear deformations in which one of the two underlying real dimensions of the Riemann surface is dilated by some factor $\in \mathbb{R}_{>0}$, while the other underlying real dimensions is left undeformed. From this point of view, the theory of conjugate synchronization — which may be regarded as a sort of rigidity that represents the arithmetic holomorphic structure associated to a vertical line of the log-theta-lattice [cf. the discussion given in [IUTchII], Remarks 4.7.3, 4.7.4, of the uniradiality of the $F_{\mathbb{P}}^\times_{\pm}$-symmetry that underlies the phenomenon of conjugate synchronization] — may be thought of as a sort of nonarchimedean arithmetic analogue of the representation of holomorphic structures by means of squares/rectangles referred to above. That is to say, the right angles which are characteristic of squares/rectangles may be thought of as a sort of synchronization between the metrics of the two underlying real dimensions of a Riemann surface [i.e., metrics which, a priori, may differ by some dilating factor] — cf. Fig. 1.4 below. Here, we mention in passing that this point of view is reminiscent of the discussion of [IUTchII], Remark 3.6.5, (ii), in which the point of view is taken that the phenomenon of conjugate synchronization may be thought of as a reflection of the coherence of the arithmetic holomorphic structures involved.

(ii) Relative to the point of view discussed in (i), the approach described in Remark 1.5.1, (ii), to “avoiding conjugate synchronization by identifying the various conjugates of the diagonal embedding” corresponds — in light of the highly non-abelian nature of the groups involved! [cf. the discussion of Remark 1.5.1, (ii)] — to thinking of a holomorphic structure on a Riemann surface as an “equivalence class of holomorphic structures in the usual sense relative to the equivalence relation of differing by a Teichmüller deformation”! That is to say, such an [unconventional!]
approach to the definition of a holomorphic structure allows one to circumvent the issue of *rigidifying* the relationship between the metrics of the two underlying real dimensions of the Riemann surface — but only at the cost of rendering unfeasible any meaningful theory of “deformations of a holomorphic structure”!

(iii) The analogy discussed in (i) between conjugate synchronization [which arises from the $F_l^\pm \times \text{-symmetry}!$] and the representation of a complex holomorphic structure by means of squares/rectangles may also be applied to the “$\kappa$-sol-conjugate synchronization” [cf. the discussion of [IUTchI], Remark 5.1.5] given in [IUTchII], Corollary 4.7, (ii); [IUTchII], Corollary 4.8, (ii), between, for instance, the various labeled non-realified and realified global Frobenioids by means of the $F_l^\times \times \text{-symmetry}$. Indeed, this analogy is all the more apparent in the case of the realified global Frobenioids — which admit a natural $\mathbb{R}_{>0}$-action. Here, we observe in passing that, just as the theory of conjugate synchronization [via the $F_l^\times \times \text{-symmetry}!$] plays an essential role in the construction of the local portions of the $\Theta^\times \times \mu$-, $\Theta^\times \times \mu$-links given in [IUTchII], Corollary 4.10, (i), (ii), (iii), the *synchronization* of global realified Frobenioids by means of the $F_l^\times \times \text{-symmetry}$ may be related — via the isomorphisms of Frobenioids of the second displays of [IUTchII], Corollary 4.7, (iii); [IUTchII], Corollary 4.8, (iii) [cf. also the discussion of [IUTchII], Remark 4.8.1] — to the construction of the global realified Frobenioid portion of the $\Theta^\times \times \mu$-link given in [IUTchII], Corollary 4.10, (ii).

On the other hand, the synchronization involving the non-realified global Frobenioids may be thought of as a sort of further rigidification of the global realified Frobenioids. As discussed in Remark 1.5.2, (iii), this “further rigidification” will play an important role in the theory of §3 below.

\[ G_v \ldots G_v \quad G_v \quad G_v \quad \ldots \quad G_v \]
\[ \sim \quad \ldots \quad \sim \quad \sim \quad \ldots \quad \sim \]
\[ -l^* \quad \ldots \quad -1 \quad 0 \quad 1 \quad \ldots \quad l^* \]
\[ \vdots \]
\[ \mathbb{R}_{>0} \quad \sim \quad \vdots \quad \nearrow \quad \searrow \]
\[ \vdots \quad \nearrow \quad \searrow \]
\[ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \sim \quad \text{id} \]
\[ \vdots \quad \searrow \quad \nearrow \]
\[ \vdots \quad \searrow \quad \nearrow \]

Fig. 1.4: Analogy between conjugate synchronization and the representation of complex holomorphic structures via squares/rectangles
Remark 1.5.4.

(i) As discussed in [IUTchII], Remark 3.8.3, (iii), one of the main themes of the present series of papers is the goal of giving an explicit description of what one arithmetic holomorphic structure — i.e., one vertical line of the log-theta-lattice — looks like from the point of view of a distinct arithmetic holomorphic structure — i.e., another vertical line of the log-theta-lattice — that is only related to the original arithmetic holomorphic structure via some mono-analytic core, e.g., the various bi-coric structures discussed in Theorem 1.5, (iii), (iv), (v). Typically, the objects of interest that are constructed within the original arithmetic holomorphic structure are Frobenius-like structures [cf. the discussion of [IUTchII], Remark 3.6.2], which, as we recall from the discussion of Remark 1.5.2, (ii) [cf. also the discussion of [IUTchII], Remark 3.6.2, (ii)], are necessary in order to perform constructions — such as, for instance, the construction of the $\Theta_\cdot$, $\Theta^{\times \mu}$-, or $\Theta^{\times \mu}_{\text{gau}}$-links — that are “not bound to conventional scheme theory”. Indeed, the main example of such an object of interest consists precisely of the Gaussian monoids discussed in [IUTchII], §3, §4. Thus, the operation of describing such an object of interest from the point of view of a distinct arithmetic holomorphic structure may be broken down into two steps:

(a) passing from Frobenius-like structures to étale-like structures via various Kummer isomorphisms;

(b) transporting the resulting étale-like structures from one arithmetic holomorphic structure to another by means of various multiradiality properties.

In particular, the computation of what the object of interest looks like from the point of view of a distinct arithmetic holomorphic structure may be broken down into the computation of the indeterminacies or “departures from rigidity” that arise — i.e., the computation of “what sort of damage is incurred to the object of interest” — during the execution of each of these two steps (a), (b). We shall refer to the indeterminacies that arise from (a) as Kummer-detachment indeterminacies and to the indeterminacies that arise from (b) as étale-transport indeterminacies.

(ii) Étale-transport indeterminacies typically amount to the indeterminacies that occur as a result of the execution of various “anabelian” or “group-theoretic” algorithms. One fundamental example of such indeterminacies is constituted by the indeterminacies that occur in the context of Theorem 1.5, (iii), (iv), as a result of the existence of automorphisms of the various [copies of] local absolute Galois groups $G_v$, for $v \in \mathcal{V}^\text{non}$, which are not of scheme-theoretic origin [cf. the discussion of [AbsTopIII], §I3].

(iii) On the other hand, one important example, from the point of view of the theory of the present series of papers, of a Kummer-detachment indeterminacy is constituted by the Frobenius-picture diagrams given in Propositions 1.2, (x); 1.3, (iv) — i.e., the issue of which path one is to take from a particular “•” to the coric “◦”. That is to say, despite the fact that these diagrams fail to be commutative, the “upper semi-commutativity” property satisfied by the coric holomorphic
log-shells involved [cf. the discussion of Remark 1.2.2, (iii)] may be regarded as a sort of computation, in the form of an upper estimate, of the Kummer-detachment indeterminacy in question. Another important example, from the point of view of the theory of the present series of papers, of a Kummer-detachment indeterminacy is given by the $\hat{\mathbb{Z}}\times$-indeterminacies discussed in Remark 1.4.2 [cf. also the Kummer isomorphisms of the final display of Theorem 1.5, (iii)].
Section 2: Multiradial Theta Monoids

In the present §2, we globalize the multiradial portion of the local theory of theta monoids developed in [IUTchII], §1, §3, at $v \in V^{\text{bad}}$ [cf., especially, [IUTchII], Corollary 1.12; [IUTchII], Proposition 3.4] so as to cover the theta monoids/Frobenioids of [IUTchII], Corollaries 4.5, (iv), (v); 4.6, (iv), (v), and explain how the resulting theory may be fit into the framework of the log-theta-lattice developed in §1.

In the following discussion, we assume that we have been given initial $\Theta$-data as in [IUTchI], Definition 3.1. Let $\hat{H}^\Theta_{\Theta^{\pm\text{ell}}\text{NF}}$ be a $\Theta^{\pm\text{ell}}$-NF-Hodge theater [relative to the given initial $\Theta$-data — cf. [IUTchI], Definition 6.13, (i)] and

$$\{n,m\hat{H}^\Theta_{\Theta^{\pm\text{ell}}\text{NF}}\}_{n,m \in \mathbb{Z}}$$

a collection of distinct $\Theta^{\pm\text{ell}}$-NF-Hodge theaters [relative to the given initial $\Theta$-data] indexed by pairs of integers, which we think of as arising from a Gaussian log-theta-lattice, as in Definition 1.4. We begin by reviewing the theory of theta monoids developed in [IUTchII].

Proposition 2.1. (Vertical Coricity and Kummer Theory of Theta Monoids) We maintain the notation introduced above. Also, we shall use the notation $\text{Aut}_{\mathcal{F}^\nu}(-)$ to denote the group of automorphisms of the $\mathcal{F}^\nu$-prime-strip in parentheses. Then:

(i) (Vertically Coric Theta Monoids) In the notation of [IUTchII], Corollary 4.5, (iv), (v) [cf. also the assignment \( \nu, \nu' \mapsto \nu'' \) of [IUTchI], Proposition 6.7], there are functorial algorithms in the $\mathcal{D}^\nu$ and $\mathcal{D}^\nu^\nu$-prime-strips $\hat{\mathcal{D}}^\nu$, $\hat{\mathcal{D}}^\nu^\nu$ associated to the $\Theta^{\pm\text{ell}}$-NF-Hodge theater $\hat{H}^\Theta_{\Theta^{\pm\text{ell}}\text{NF}}$ for constructing collections of data indexed by $\mathbb{V}$

$$\mathbb{V} \ni \nu \mapsto \Psi_{\text{env}}(\hat{\mathcal{D}}^\nu)\nu; \quad \mathbb{V} \ni \nu \mapsto \infty \Psi_{\text{env}}(\hat{\mathcal{D}}^\nu)\nu$$

as well as a global realified Frobenioid

$$D_{\text{env}}^\nu(\hat{\mathcal{D}}^\nu)$$

equipped with a bijection $\text{Prime}(D_{\text{env}}^\nu(\hat{\mathcal{D}}^\nu)) \sim \mathbb{V}$ and corresponding local isomorphisms, for each $\nu \in \mathbb{V}$, as described in detail in [IUTchII], Corollary 4.5, (v). In particular, each isomorphism of the full poly-isomorphism induced [cf. Theorem 1.5, (i)] by a vertical arrow of the Gaussian log-theta-lattice under consideration induces a compatible collection of isomorphisms

$$\Psi_{\text{env}}(n,m\mathcal{D}^\nu) \sim \Psi_{\text{env}}(n,m+1\mathcal{D}^\nu); \quad \infty \Psi_{\text{env}}(n,m\mathcal{D}^\nu) \sim \infty \Psi_{\text{env}}(n,m+1\mathcal{D}^\nu)$$

$$D_{\text{env}}^\nu(n,m\mathcal{D}^\nu) \sim D_{\text{env}}^\nu(n,m+1\mathcal{D}^\nu)$$
— where the final isomorphism of Frobenioids is compatible with the respective bijections involving “Prime(−)”, as well as with the respective local isomorphisms for each \( v \in V \).

**(ii) (Kummer Isomorphisms)** In the notation of [IUTchII], Corollary 4.6, (iv), (v), there are functorial algorithms in the \( \Theta^{\pm\text{ell}}\text{NF-Hodge theater} \) for constructing collections of data indexed by \( \mathbb{V} \)

\[
\forall \ni u \mapsto \Psi_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger)_{u}; \quad \forall \ni v \mapsto \infty \Psi_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger)_{v}
\]

as well as a global realified Frobenioid

\[
C_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger)
\]

equipped with a bijection \( \text{Prime}(\mathcal{C}_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger))) \xrightarrow{\sim} \mathbb{V} \) and corresponding local isomorphisms, for each \( v \in \mathbb{V} \), as described in detail in [IUTchII], Corollary 4.6, (v). Moreover, there are functorial algorithms in \( \Theta^{\pm\text{ell}}\text{NF} \) for constructing Kummer isomorphisms

\[
\Psi_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger) \xrightarrow{\sim} \Psi_{\text{env}}^{\dagger}(\mathcal{D}_{\Theta}^{\dagger}); \quad \infty \Psi_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger) \xrightarrow{\sim} \infty \Psi_{\text{env}}^{\dagger}(\mathcal{D}_{\Theta}^{\dagger})
\]

— where the final isomorphism of Frobenioids is compatible with the respective bijections involving “Prime(−)”, as well as with the respective data discussed in (i) [cf. [IUTchII], Corollary 4.6, (iv), (v)]. Finally, the collection of data \( \Psi_{\text{env}}^{\dagger}(\mathcal{D}_{\Theta}^{\dagger}) \) gives rise, in a natural fashion, to an \( F^{\dagger}-\text{prime-stripe} \)

\[
C_{\text{env}}^{\dagger}(\mathcal{T}_{\Theta}^\dagger) \xrightarrow{\sim} \mathcal{D}_{\text{env}}^{\dagger}(\mathcal{D}_{\Theta}^{\dagger})
\]

of \( F^{\dagger}, \mathcal{F}^{\dagger}-\text{prime-strips} \).

**(iii) (Kummer Theory at Bad Primes)** The portion at \( v \in \mathbb{V}^{\text{bad}} \) of the Kummer isomorphisms of (ii) is obtained by composing the Kummer isomorphisms of [IUTchII], Proposition 3.3, (i) — which, we recall, were defined by forming Kummer classes in the context of mono-theta environments that arise from tempered Frobenioids — with the isomorphisms on cohomology classes induced [cf. the upper left-hand portion of the first display of [IUTchII], Proposition 3.4, (i)] by the full poly-isomorphism of projective systems of mono-theta environments \( \mathcal{M}_{\Theta}^{\dagger}(\mathcal{D}_{\Theta}^{\dagger}) \xrightarrow{\sim} \mathcal{M}_{\Theta}^{\dagger}(\mathcal{F}_{\Theta}^{\dagger}) \) [cf. [IUTchII], Proposition 3.4; [IUTchII], Remark 4.2.1, (iv)] between projective systems of mono-theta environments that arise from tempered Frobenioids [i.e., “\( \mathcal{F}_{\Theta}^{\dagger} \) ”] and projective systems of mono-theta environments.
environments that arise from the tempered fundamental group \([i.e., \overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{D_{\Delta}}}}}]\) — cf. the left-hand portion of the third display of [IUTchII], Corollary 3.6, (ii), in the context of the discussion of [IUTchII], Remark 3.6.2, (i). Here, each “isomorphism on cohomology classes” is induced by the isomorphism on exterior cyclotomes

\[
\Pi_\mu(M_\star(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{D_{\Delta}}}}))) \cong \Pi_\mu(M_\star(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{F_{\Delta}}}})))
\]
determined by each of the isomorphisms that constitutes the full poly-isomorphism of projective systems of mono-theta environments discussed above. In particular, the composite map

\[
\Pi_\mu(M_\star(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{F_{\Delta}}}}))) \otimes \mathbb{Q}/\mathbb{Z} \to (\Psi_1 F_{\Delta})^\times \mu
\]
obtained by composing the result of applying \(\otimes \mathbb{Q}/\mathbb{Z}\) to this isomorphism on exterior cyclotomes with the natural inclusion

\[
\Pi_\mu(M_\star(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{F_{\Delta}}}}))) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow (\Psi_1 F_{\Delta})^\times
\]
\[\text{[cf. the notation of [IUTchII], Proposition 3.4, (i); the description given in [IUTchII], Proposition 1.3, (i), of the exterior cyclotome of a mono-theta environment that arises from a tempered Frobenioid] and the natural projection} \]
\[\text{(Ψ}_{1\mathbb{F}_{\Delta}})^{\times} \to (Ψ_{1\mathbb{F}_{\Delta}})^{\times \mu}
\]
is equal to the zero map.

\((iv)\) (Kummer Theory at Good Nonarchimedean Primes) The unit portion at \(v \in \mathbf{V}_{\text{good}} \cap \mathbf{V}_{\text{non}}\) of the Kummer isomorphisms of (ii) is obtained \([cf.\ [IUTchII],\ Proposition\ 4.2,\ (iv)]\) as the unit portion of a “labeled version” of the isomorphism of ind-topological monoids equipped with a topological group action — i.e., in the language of [AbsTopIII], Definition 3.1, (ii), the isomorphism of “MLF-Galois \(\mathbb{T}\)-pairs” — discussed in [IUTchII], Proposition 4.2, (i) \([cf.\ also\ [IUTchII],\ Remark\ 1.11.1,\ (i),\ (a);\ [AbsTopIII],\ Proposition\ 3.2,\ (iv)])\]. In particular, the portion at \(v \in \mathbf{V}_{\text{good}} \cap \mathbf{V}_{\text{non}}\) of the \(\text{Aut}_{\mathbb{F}_{\Delta}}(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{\Delta}}}})\)-orbit of the second isomorphism of the final display of (ii) may be obtained as a “labeled version” of the “Kummer poly-isomorphism of semi-simplifications” given in the final display of [IUTchII], Proposition 4.2, (ii).

\((v)\) (Kummer Theory at Archimedean Primes) The unit portion at \(v \in \mathbf{V}_{\text{arc}}\) of the Kummer isomorphisms of (ii) is obtained \([cf.\ [IUTchII],\ Proposition\ 4.4,\ (iv)]\) as the unit portion of a “labeled version” of the isomorphism of topological monoids discussed in [IUTchII], Proposition 4.4, (i). In particular, the portion at \(v \in \mathbf{V}_{\text{arc}}\) of the \(\text{Aut}_{\mathbb{F}_{\Delta}}(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{\Delta}}}})\)-orbit of the second isomorphism of the final display of (ii) may be obtained as a “labeled version” of the “Kummer poly-isomorphism of semi-simplifications” given in the final display of [IUTchII], Proposition 4.4, (ii) \([cf.\ also\ [IUTchII],\ Remark\ 4.6.1])\).

\((vi)\) (Compatibility with Constant Monoids) The definition of the unit portion of the theta monoids involved \([cf.\ [IUTchII],\ Corollary\ 4.10,\ (iv)]\) gives rise to natural isomorphisms

\[
\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{\Theta_{\Delta}}}}} \cong \overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\Theta_{\text{env}}}}};\quad \overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{\Delta}}}}(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{D_{\Delta}}}}}) \cong \overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\Theta_{\text{env}}}}}(\overset{\wedge}{\overset{\oplus}{\overset{\wedge}{\overset{\wedge}{D_{\Delta}}}}})
\]
where the morphism induced on $F^\times\mu$-prime-strips by the first displayed
isomorphism is precisely the isomorphism of the first display of [IUTchII], Corollary
4.10, (iv) — of the respective associated $F^\times$-prime-strips [cf. the notation of
Theorem 1.5, (iii), where the label “$n,m$” is replaced by the label “$\dagger$”]. Moreover,
these natural isomorphisms are compatible with the Kummer isomorphisms of
(ii) above and Theorem 1.5, (iii).

Proof. The various assertions of Proposition 2.1 follow immediately from the
definitions and the references quoted in the statements of these assertions. 

Remark 2.1.1. The theory of mono-theta environments [cf. Proposition 2.1,
(iii)] will play a crucial role in the theory of the present §2 [cf. Theorem 2.2, (ii);
Corollary 2.3, (iv), below] in the passage from Frobenius-like to étale-like structures [cf. Remark 1.5.4, (i), (a)] at bad primes. In particular, the various rigidity
properties of mono-theta environments established in [EtTh] play a fundamental
role in ensuring that the resulting “Kummer-detachment indeterminacies” [cf. the
discussion of Remark 1.5.4, (i)] are sufficiently mild so as to allow the establishment
of the various reconstruction algorithms of interest. For this reason, we pause to
review the main properties of mono-theta environments established in [EtTh] [cf.
[EtTh], Introduction] — namely,

(a) cyclotomic rigidity
(b) discrete rigidity
(c) constant multiple rigidity
(d) isomorphism class compatibility
(e) Frobenioid structure compatibility

— and the roles played by these main properties in the theory of the present series of
papers. Here, we remark that “isomorphism class compatibility” [i.e., (d)] refers to
compatibility with the convention that various objects of the tempered Frobenioids
[and their associated base categories] under consideration are known only up to
isomorphism [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. In the
Introduction to [EtTh], instead of referring to (d) in this form, we referred to the
property of compatibility with the topology of the tempered fundamental group. In
fact, however, this compatibility with the topology of the tempered fundamental
group is a consequence of (d) [cf. [EtTh], Remarks 5.12.1, 5.12.2]. On the other
hand, from the point of view of the present series of papers, the essential property
of interest in this context is best understood as being the property (d).

(i) First, we recall that the significance, in the context of the theory of the
present series of papers, of the compatibility with the Frobenioid structure of the
tempered Frobenioids under consideration [i.e., (e)] — i.e., in particular, with the
monoidal portion, equipped with its natural Galois action, of these Frobenioids
— lies in the role played by this “Frobenius-like” monoidal portion in performing
constructions — such as, for instance, the construction of the $\log_\ast$, $\Theta_\ast$, $\Theta^{\times\mu}_\ast$, or
$\Theta^{\times\mu}_\gau$-links — that are “not bound to conventional scheme theory”, but may
be related, via Kummer theory, to various étale-like structures [cf. the discussions
of Remark 1.5.4, (i); [IUTchII], Remark 3.6.2, (ii); [IUTchII], Remark 3.6.4, (ii),
(v)].
(ii) Next, we consider isomorphism class compatibility [i.e., (d)]. As discussed above, this compatibility corresponds to regarding each of the various objects of the tempered Frobenioids [and their associated base categories] under consideration as being known only up to isomorphism [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. As discussed in [IUTchII], Remark 3.6.4, (i), the significance of this property (d) in the context of the present series of papers lies in the fact that — unlike the case with the projective systems constituted by Kummer towers constructed from N-th power morphisms, which are compatible with only the multiplicative, but not the additive structures of the \( p \)-adic local fields involved — each individual object in such a Kummer tower corresponds to a single field [i.e., as opposed to a projective system of multiplicative groups of fields]. This field/ring structure is necessary in order to apply the theory of the \( \log \)-link developed in §1 — cf. the vertical coricity discussed in Proposition 2.1, (i). Note, moreover, that, unlike the \( \log \), \( \Theta \cdot \), \( \Theta^\times \), or \( \Theta^\times \mu \)-links, the N-th power morphisms that appear in a Kummer tower are “algebraic”, hence compatible with the conventional scheme theory surrounding the \( \acute{e} \)tale [or tempered] fundamental group. In particular, since the tempered Frobenioids under consideration may be constructed from such scheme-theoretic categories, the fundamental groups on either side of such an N-th power morphism may be related up to an indeterminacy arising from an inner automorphism of the tempered fundamental group [i.e., the “fundamental group” of the base category] under consideration — cf. the discussion of [IUTchII], Remark 3.6.3, (ii). On the other hand, the objects that appear in these Kummer towers necessarily arise from nontrivial line bundles [indeed, line bundles all of whose positive tensor powers are nontrivial!] on tempered coverings of a Tate curve — cf. the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9]; the crucial role played by the commutator “\([-,-]\)” in the theory of cyclotomic rigidity [i.e., (a)] reviewed in (iv) below. In particular, the extraction of various N-th roots in a Kummer tower necessarily leads to mutually non-isomorphic line bundles, i.e., mutually non-isomorphic objects in the Kummer tower. From the point of view of reconstruction algorithms, such non-isomorphic objects may be naturally — i.e., algorithmically — related to another only via indeterminate isomorphisms [cf. (d)!]. This point of view is precisely the starting point of the discussion of — for instance, “constant multiple indeterminacy” in — [EtTh], Remarks 5.12.2, 5.12.3.

(iii) Next, we recall that the significance of constant multiple rigidity [i.e., (c)] in the context of the present series of papers lies in the construction of the canonical splittings of theta monoids via restriction to the zero section discussed, for instance, in [IUTchII], Corollary 1.12, (ii); [IUTchII], Proposition 3.3, (i); [IUTchII], Remark 1.12.2, (iv) [cf. also Remark 1.2.3, (i), of the present paper].

(iv) Next, we review the significance of cyclotomic rigidity [i.e., (a)] in the context of the present series of papers. First, we recall that this cyclotomic rigidity is essentially a consequence of the nondegenerate nature of the commutator “\([-,-]\)” of the theta groups involved [cf. the discussion of [EtTh], Introduction; [EtTh], Remark 2.19.2]. Put another way, since this commutator is quadratic in nature, one may think of this nondegenerate nature of the commutator as a statement to the effect that “the degree of the commutator is precisely 2”. At a more concrete level, the cyclotomic rigidity arising from a mono-theta environment consists of
a certain specific isomorphism between the interior and exterior cyclotomes [cf. the discussion of [IUTchII], Definition 1.1, (ii); [IUTchII], Remark 1.1.1]. Put another way, one may think of this cyclotomic rigidity isomorphism as a sort of rigidification of a certain “projective line of cyclotomes”, i.e., the projectivization of the direct sum of the interior and exterior cyclotomes [cf. the computations that underlie [EtTh], Proposition 2.12]. In particular, this rigidification is fundamentally nonlinear in nature. Indeed, if one attempts to compose it with an $N$-th power morphism, then one is obliged to sacrifice constant multiple rigidity [i.e., (c)] — cf. the discussion of [EtTh], Remark 5.12.3. That is to say, the distinguished nature of the “first power” of the cyclotomic rigidity isomorphism is an important theme in the theory of [EtTh] [cf. the discussion of [EtTh], Remark 5.12.5; [IUTchII], Remark 3.6.4, (iii), (iv)]. The multiradiality of mono-theta-theoretic cyclotomic rigidity [cf. [IUTchII], Corollary 1.10] — which lies in stark contrast with the indeterminacies that arise when one attempts to give a multiradial formulation [cf. [IUTchII], Corollary 1.11; the discussion of [IUTchII], Remark 1.11.3] of the more classical “MLF-Galois pair cyclotomic rigidity” arising from local class field theory — will play a central role in the theory of the present §2 [cf. Theorem 2.2, Corollary 2.3 below].

(v) Finally, we review the significance of discrete rigidity [i.e., (b)] in the context of the present series of papers. First, we recall that, at a technical level, whereas cyclotomic rigidity may be regarded [cf. the discussion of (iv)] as a consequence of the fact that “the degree of the commutator is precisely 2”, discrete rigidity may be regarded as a consequence of the fact that “the degree of the commutator is $\leq 2$” [cf. the statements and proofs of [EtTh], Proposition 2.14, (ii), (iii)]. At a more concrete level, discrete rigidity assures one that one may restrict one’s attentions to $\mathbb{Z}$-multiples/powers — as opposed to $\hat{\mathbb{Z}}$-multiples/powers — of divisors, line bundles, and rational functions [such as, for instance, the $q$-parameter!] on the tempered coverings of a Tate curve that occur in the theory of [EtTh] [cf. [EtTh], Remark 2.19.4]. This prompts the following question:

Can one develop a theory of $\hat{\mathbb{Z}}$-divisors/line bundles/rational functions in, for instance, a parallel fashion to the way in which one considers perfections and realifications of Frobenioids in the theory of [FrdII]?

As far as the author can see at the time of writing, the answer to this question is “no”. Indeed, unlike the case with $\mathbb{Q}$ or $\mathbb{R}$, there is no notion of positivity [or negativity] in $\hat{\mathbb{Z}}$. For instance, $-1 \in \hat{\mathbb{Z}}$ may be obtained as a limit of positive integers. In particular, if one had a theory of $\hat{\mathbb{Z}}$-divisors/line bundles/rational functions, then such a theory would necessarily require one to “confuse” positive [i.e., effective] and negative divisors, hence to work birationally. But to work birationally means, in particular, that one must sacrifice the conventional structure of isomorphisms [e.g., automorphisms] between line bundles — which plays an indispensable role, for instance, in the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9; the crucial role played by the commutator $[\cdot, \cdot]$ in the theory of cyclotomic rigidity [i.e., (a)] reviewed in (iv) above].
Remark 2.1.2.

(i) In the context of the discussion of Remark 2.1.1, (v), it is of interest to recall [cf. [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii)] that the essential role played, in the context of the context of the $F^s_{l\pm}$-symmetry, by the “global bookkeeping operations” involving the labels of the evaluation points gives rise, in light of the profinite nature of the global étale fundamental groups involved, to a situation in which one must apply the “complements on tempered coverings” developed in [IUTchI], §2. That is to say, in the notation of the discussion given in [IUTchII], Remark 2.1.1, (i), of the various tempered coverings that occur at $v \in V_{\text{bad}}$, these “complements on tempered coverings” are applied precisely so as to allow one to restrict one’s attention to the $\mathbb{Z}$-conjugates — i.e., as opposed to $\hat{\mathbb{Z}}$-conjugates where we write $\hat{\mathbb{Z}}$ for the profinite completion of $\mathbb{Z}$ — of the theta functions involved. In particular, although such “evaluation-related issues”, which will become relevant in the context of the theory of §3 below, do not play a role in the theory of the present §2, the role played by the theory of [IUTchI], §2, in the theory of the present series of papers may also be thought of as a sort of “discrete rigidity” — which we shall refer to as “evaluation discrete rigidity” — i.e., a sort of rigidity that is concerned with similar issues to the issues discussed in the case of “mono-theta-theoretic discrete rigidity” in Remark 2.1.1, (v), above.

(ii) Next, let us suppose that we are in the situation discussed in [IUTchII], Proposition 2.1. Fix $v \in V_{\text{bad}}$. Write $\Pi \overset{\text{def}}{=} \Pi_v$; $\hat{\Pi}$ for the profinite completion of $\Pi$. Thus, we have natural surjections $\Pi \twoheadrightarrow l \cdot \mathbb{Z} (\subseteq \mathbb{Z})$, $\hat{\Pi} \twoheadrightarrow l \cdot \hat{\mathbb{Z}} (\subseteq \hat{\mathbb{Z}})$. Write $\Pi^\dagger \overset{\text{def}}{=} \hat{\Pi} \times_\hat{\mathbb{Z}} \mathbb{Z} \subseteq \hat{\Pi}$. Next, we observe that from the point of view of the evaluation points, the evaluation discrete rigidity discussed in (i) corresponds to the issue of whether, relative to some arbitrarily chosen basepoint, the “coordinates” [i.e., element of the “torsor over $\mathbb{Z}$” discussed in [IUTchII], Remark 2.1.1, (i)] of the evaluation point lie in $\mathbb{Z}$ or in $\hat{\mathbb{Z}}$. Thus, if one is only concerned with the issue of arranging for these coordinates to lie in $\mathbb{Z}$, then one is led to pose the following question:

Is it possible to simply use the “partially tempered fundamental group” $\Pi^\dagger$ instead of the “full” tempered fundamental group $\Pi$ in the theory of the present series of papers?

The answer to this question is “no”. One way to see this is to consider the [easily verified] natural isomorphism

$$N_{\hat{\Pi}}(\Pi^\dagger)/\Pi^\dagger \cong \hat{\mathbb{Z}}/\mathbb{Z}$$

involving the normalizer $N_{\hat{\Pi}}(\Pi^\dagger)$ of $\Pi^\dagger$ in $\hat{\Pi}$. One consequence of this isomorphism is that — unlike the tempered fundamental group $\Pi$ [cf., e.g., [SemiAnbd], Theorems 6.6, 6.8] — the topological group $\Pi^\dagger$ fails to satisfy various fundamental absolute anabelian properties which play a crucial role in the theory of [EtTh], as well as in the present series of papers [cf., e.g., the theory of [IUTchII], §2]. At a more concrete level, unlike the case with the tempered fundamental group $\Pi$, the profinite conjugacy indeterminacies that act on $\Pi^\dagger$ give rise to $\hat{\mathbb{Z}}$-translation
indeterminacies acting on the coordinates of the evaluation points involved. That is to say, in the case of \( \Pi \), such \( \hat{\mathbb{Z}} \)-translation indeterminacies are avoided precisely by applying the “complements on tempered coverings” developed in [IUTchI], §2 — i.e., in a word, as a consequence of the “highly anabelian nature” of the [full] tempered fundamental group \( \Pi \).

**Theorem 2.2. (Kummer-compatible Multiradiality of Theta Monoids)**

Fix a collection of initial \( \Theta \)-data

\[
(\mathcal{F}/F, X_F, l, \mathbb{C}_K, \Psi, \Psi_{\text{mod}}^{\text{bad}}, \xi)
\]

as in [IUTchI], Definition 3.1. Let \( ^{\dagger}H^{\Theta}_{\text{ell}} \mathbb{NF} \) be a \( \Theta \text{-ell} \mathbb{NF} \)-Hodge theater [relative to the given initial \( \Theta \)-data — cf. [IUTchI], Definition 6.13, (i)]. For \( \square \in \{ \llbracket, \llbracket ] \times \mu, \llbracket ]' \times \mu \} \), write \( \text{Aut}_{\mathcal{F}(\square)}(-) \) for the group of automorphisms of the \( \mathcal{F}(\square) \)-prime-strip in parentheses [cf. [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (vi), (vii), (viii)].

(i) **(Automorphisms of Prime-strips)** The natural functors determined by assigning to an \( \mathcal{F}_{\square}^{\dagger} \)-prime-strip the associated \( \mathcal{F}_{\square}^{\dagger} \times \mu \)- and \( \mathcal{F}_{\square}^{\dagger} \times \mu \)-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] and then composing with the natural isomorphisms of Proposition 2.1, (vi), determine natural homomorphisms

\[
\text{Aut}_{\mathcal{F}_{\square}^{\dagger}}(\mathfrak{g}_{\text{env}}^{\dagger}(\mathbb{D}_>)_\mathfrak{g}) \to \text{Aut}_{\mathcal{F}_{\square}^{\dagger} \times \mu}(\mathfrak{g}_{\text{env}}^{\dagger}(\mathbb{D}_>)_\mathfrak{g} \times \mu(\mathbb{D}_{\Delta})) \to \text{Aut}_{\mathcal{F}_{\square}^{\dagger} \times \mu}(\mathfrak{g}_{\text{env}}^{\dagger}(\mathbb{D}_{\Delta}))
\]

— where the second arrows in each line are surjections — that are compatible with the Kummer isomorphisms of Proposition 2.1, (ii), and Theorem 1.5, (iii) [cf. the final portions of Proposition 2.1, (iv), (v), (vi)].

(ii) **(Kummer Aspects of Multiradiality at Bad Primes)** Let \( \underline{\Psi} \in \Psi_{\text{mod}}^{\text{bad}} \).

Write

\[
\infty \Psi_{\text{env}}^{\dagger}(\mathbb{D}_>)_{\underline{\Psi}} \subseteq \infty \Psi_{\text{env}}(\mathbb{D}_>)_{\underline{\Psi}}; \quad \infty \Psi_{\mathcal{F}_{\square}^{\dagger}}(\mathbb{H}^{\Theta})_{\underline{\Psi}} \subseteq \infty \Psi_{\mathcal{F}_{\square}^{\dagger}}(\mathbb{H}^{\Theta})_{\underline{\Psi}}
\]

for the submonoids corresponding to the respective splittings [cf. [IUTchII], Corollaries 3.5, (iii); 3.6, (iii)], i.e., the submonoids generated by \( \infty \theta_{\text{env}}^{\dagger}(\mathbb{M}^{\Theta}) \) [cf. the notation of [IUTchII], Proposition 3.1, (i)] and the respective torsion subgroups.

Now consider the commutative diagram

\[
\begin{array}{ccc}
\infty \Psi_{\text{env}}^{\dagger}(\mathbb{D}_>)_{\underline{\Psi}} & \supseteq & \infty \Psi_{\text{env}}(\mathbb{D}_>)_{\underline{\Psi}} \\
\downarrow & & \downarrow \\
\infty \Psi_{\mathcal{F}_{\square}^{\dagger}}(\mathbb{H}^{\Theta})_{\underline{\Psi}} & \supseteq & \infty \Psi_{\mathcal{F}_{\square}^{\dagger}}(\mathbb{H}^{\Theta})_{\underline{\Psi}}
\end{array}
\]

\[
\begin{array}{ccc}
\rightarrow & \infty \Psi_{\text{env}}(\mathbb{D}_>)_{\underline{\Psi}} \times \mu & \xrightarrow{\sim} \Psi_{\text{cns}}^{\text{ss}}(\mathbb{D}_>)_{\underline{\Psi}} \\
\downarrow & & \downarrow \\
\rightarrow & \infty \Psi_{\mathcal{F}_{\square}^{\dagger}}(\mathbb{H}^{\Theta})_{\underline{\Psi}} \times \mu & \xrightarrow{\sim} \Psi_{\text{cns}}^{\text{ss}}(\mathbb{D}_{\Delta})_{\underline{\Psi}}
\end{array}
\]
— where the inclusions “⊇”, “⊆” are the natural inclusions; the surjections “↠” are the natural surjections; the superscript “µ” denotes the torsion subgroup; the superscript “×” denotes the group of units; the superscript “×µ” denotes the quotient “(−)×/(−)µ”; the first four vertical arrows are the isomorphisms determined by the inverse of the second Kummer isomorphism of the third display of Proposition 2.1, (ii); \( 1\Sigma^\mu_\Delta \) is as discussed in Theorem 1.5, (iii); \( 1\Sigma^\mu_\Delta \) is as discussed in [IUTchII], Corollary 4.10, (i); the final vertical arrow is the inverse of the “Kummer poly-isomorphism” determined by the second displayed isomorphism of [IUTchII], Corollary 4.6, (ii); the final upper horizontal arrow is the poly-isomorphism determined by composing the isomorphism determined by the inverse of the second displayed natural isomorphism of Proposition 2.1, (vi), with the collection of automorphisms of \( \text{Aut}_G(\Sigma^\mu_\Delta) \) — where “\( \sim \)” denotes the natural surjection; “\( \sim \)” denotes the full poly-automorphism of \( G_\Sigma(\Sigma^\mu_\Delta) \). Finally, each of the various composite maps

\[
\Pi X(M^\Delta_\Delta(1D_{>,\underline{\omega}})) \to G_\Sigma(M^\Delta_\Delta(1D_{>,\underline{\omega}})) = G_\Sigma(M^\Delta_\Delta(1D_{>,\underline{\omega}})) \to G_\Sigma(M^\Delta_\Delta(1D_{>,\underline{\omega}}))
\]

is equal to the zero map [cf. (b)] below; the final portion of Proposition 2.1, (iii). In particular, the identity automorphism on the following objects is compatible, relative to the various natural morphisms involved [cf. the above commutative diagram], with the collection of automorphisms of \( \Psi_{\text{cns}}^{\subseteq}(1\Sigma^\Delta_\mu) \times \mu \) induced by arbitrary automorphisms \( \in \text{Aut}_{\Sigma^\mu}(1\Sigma^\Delta_\mu) \) [cf. [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i)]:

- (a) \( 1\Sigma^\mu_\Delta(M^\Delta_\Delta(1D_{>,\underline{\omega}})) \otimes \mathbb{Q}/\mathbb{Z} \) [cf. the discussion of Proposition 2.1, (iii)], relative to the natural isomorphism \( \Pi_\mu(M^\Delta_\Delta(1D_{>,\underline{\omega}})) \otimes \mathbb{Q}/\mathbb{Z} \sim \Pi_\mu(M^\Delta_\Delta(1D_{>,\underline{\omega}})) \otimes \mathbb{Q}/\mathbb{Z} \) of [IUTchII], Remark 1.5.2 [cf. (a)];

- (c) the projective system of mono-theta environments \( M^\Delta_\Delta(1D_{>,\omega}) \) [cf. (b)];

- (d) the splittings \( \Pi_\mu(M^\Delta_\Delta(1D_{>,\omega})) \otimes \mathbb{Q}/\mathbb{Z} \) [cf. (a)] by means of restriction to zero-labeled evaluation points [cf. [IUTchII], Proposition 3.1, (i)].

Proof. The various assertions of Theorem 2.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ☐
Remark 2.2.1. In light of the central importance of Theorem 2.2, (ii), in the theory of the present §2, we pause to examine the significance of Theorem 2.2, (ii), in more conceptual terms.

(i) In the situation of Theorem 2.2, (ii), let us write [for simplicity] \( \Pi_\nu \overset{\text{def}}{=} \Pi_X(\mathcal{M}^\Theta(\mathcal{D}_{\Delta,\nu})) \), \( G_\nu \overset{\text{def}}{=} G(\mathcal{M}^\Theta(\mathcal{D}_{\Delta,\nu})) \), \( \Pi_\mu \overset{\text{def}}{=} \Pi_\mu(\mathcal{M}^\Theta(\mathcal{D}_{\Delta,\nu})) \) [cf. (b_\nu)]. Also, for simplicity, we write \( (l \cdot \Delta_\Theta) \overset{\text{def}}{=} (l \cdot \Delta_\Theta)(\mathcal{M}^\Theta(\mathcal{D}_{\Delta,\nu})) \) [cf. IUTchII, Proposition 1.5, (iii)]. Here, we recall that in fact, \( (l \cdot \Delta_\Theta) \) may be thought of as an object constructed from \( \Pi_\nu \) [cf. IUTchII, Proposition 1.4]. Then the projective system of mono-theta environments \( \mathcal{M}^\Theta(\mathcal{D}_{\Delta,\nu}) \) [cf. (c_\nu)] may be thought of as a sort of “amalgamation of \( \Pi_\nu \) and \( \Pi_\mu \),” where the amalgamation is such that it allows the reconstruction of the mono-theta-theoretic cyclotomic rigidity isomorphism

\[
(l \cdot \Delta_\Theta) \overset{\sim}{\rightarrow} \Pi_\mu
\]

[cf. IUTchII, Proposition 1.5, (iii)] — i.e., not just the \( \mathbb{Z}^\times \)-orbit of this isomorphism!

(ii) Now, in the notation of (i), the Kummer classes \( \infty \Psi^\perp_{\text{ev}}(\mathcal{D}_{\Delta,\nu}) \) [cf. (a_\nu)] constituted by the various étale theta functions may be thought of, for a suitable characteristic open subgroup \( H \subseteq \Pi_\nu \) as twisted homomorphisms

\[
(\Pi_\nu \supseteq H) \rightarrow \Pi_\mu
\]

whose restriction to \( (l \cdot \Delta_\Theta) \) coincides with the cyclotomic rigidity isomorphism \( (l \cdot \Delta_\Theta) \overset{\sim}{\rightarrow} \Pi_\mu \) discussed in (i). Then the essential content of Theorem 2.2, (ii), lies in the observation that

since the Kummer-theoretic link between étale-like data and Frobenius-like data at \( \nu \in \mathcal{Y}_{\text{bad}} \) is established by means of projective systems of mono-theta environments [cf. the discussion of Proposition 2.1, (iii)] — i.e., which do not involve the various monoids “\( (\cdot)^\times \times \mu \)”! — the mono-theta-theoretic cyclotomic rigidity isomorphism [i.e., not just the \( \mathbb{Z}^\times \)-orbit of this isomorphism!] is immune to the various automorphisms of the monoids “\( (\cdot)^\times \times \mu \)” which, from the point of view of the multiradial formulation to be discussed in Corollary 2.3 below, arise from isomorphisms of coric data.

Put another way, this “immunity” may be thought of as a sort of decoupling of the “geometric” [i.e., in the sense of the geometric fundamental group \( \Delta_\nu \subseteq \Pi_\nu \) and “base-field-theoretic” [i.e., associated to the local absolute Galois group \( \Pi_\nu \rightarrow G_\nu \)] data which allows one to treat the exterior cyclotome \( \Pi_\mu \) — which, a priori, “looks base-field-theoretic” — as being part of the “geometric” data. From the point of view of the multiradial formulation to be discussed in Corollary 2.3 below [cf. also the discussion of IUTchII, Remark 1.12.2, (vi)], this decoupling may be thought of as a sort of splitting into purely radial and purely coric components — i.e., with respect to which \( \Pi_\mu \) is “purely radial”, while the various monoids “\( (\cdot)^\times \times \mu \)” are “purely coric”.
(iii) Note that the immunity to automorphisms of the monoids “$(-)^{\times \mu}$” discussed in (ii) lies in stark contrast to the $\hat{\mathbb{Z}}^{\times}$-indeterminacies that arise in the case of the cyclotomic rigidity isomorphisms constructed from MLF-Galois pairs in a fashion that makes essential use of the monoids “$(-)^{\times \mu}$”, as discussed in [IUTchII], Corollary 1.11; [IUTchII], Remark 1.11.3. In the following discussion, let us write “$\mathcal{O}^{\times \mu}$” for the various monoids “$(-)^{\times \mu}$” that occur in the situation of Theorem 2.2; also, we shall use similar notation “$\mathcal{O}^{\mu}$”, “$\mathcal{O}^{\times}$”, “$\mathcal{O}^{>}$”, “$\mathcal{O}^{\text{gp}}$”, “$\mathcal{O}^{\hat{\text{gp}}}$” [cf. the notational conventions of [IUTchII], Example 1.8, (ii), (iii), (iv), (vii)]. Thus, we have a diagram

$$
\begin{align*}
\mathcal{O}^{\mu} &\subseteq \mathcal{O}^{\times} &\subseteq &\mathcal{O}^{>} &\subseteq &\mathcal{O}^{\text{gp}} &\subseteq &\mathcal{O}^{\hat{\text{gp}}} \\
\downarrow & & & & & & & \\
\mathcal{O}^{\times \mu}
\end{align*}
$$

of natural morphisms between monoids equipped with $\Pi_\mu$-actions. Relative to this notation, the essential input data for the cyclotomic rigidity isomorphism constructed from an MLF-Galois pair is given by “$\mathcal{O}^{>}$” [cf. [IUTchII], Corollary 1.11, (a)]. On the other hand — unlike the case with $\mathcal{O}^{\mu}$ — a $\hat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times \mu}$ does not lie under an identity action on $\mathcal{O}^{\times}$. That is to say, a $\hat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times \mu}$ can only be lifted naturally to $\hat{\mathbb{Z}}^{\times}$-indeterminacies on $\mathcal{O}^{\times}$, $\mathcal{O}^{\hat{\text{gp}}}$ [cf. Fig. 2.1 below; [IUTchII], Corollary 1.11, (a), in the case where one takes “$\Gamma$” to be $\hat{\mathbb{Z}}^{\times}$; [IUTchII], Remark 1.11.3, (ii)]. In the presence of such $\hat{\mathbb{Z}}^{\times}$-indeterminacies, one can only recover the $\hat{\mathbb{Z}}^{\times}$-orbit of the MLF-Galois-pair-theoretic cyclotomic rigidity isomorphism.

$$
\begin{align*}
\hat{\mathbb{Z}}^{\times} &\twoheadrightarrow \hat{\mathbb{Z}}^{\times} &\twoheadrightarrow \hat{\mathbb{Z}}^{\times} &\twoheadrightarrow \hat{\mathbb{Z}}^{\times} \\
\mathcal{O}^{\times \mu} &\twoheadrightarrow &\mathcal{O}^{\times} &\subseteq &\mathcal{O}^{>} &\subseteq &\mathcal{O}^{\text{gp}} &\subseteq &\mathcal{O}^{\hat{\text{gp}}} \\
& & & & & & & & (\supseteq \mathcal{O}^{\mu})
\end{align*}
$$

Fig. 2.1: Induced $\hat{\mathbb{Z}}^{\times}$-indeterminacies in the case of MLF-Galois pair cyclotomic rigidity

$$
\begin{align*}
\Pi_\mu &\sim \mathcal{O}^{\mu} \\
\mathcal{O}^{\times \mu} &\rightarrow
\end{align*}
$$

Fig. 2.2: Insulation from $\hat{\mathbb{Z}}^{\times}$-indeterminacies in the case of mono-theta-theoretic cyclotomic rigidity

(iv) Thus, in summary, [cf. Fig. 2.2 above]
The techniques that underlie the resulting multiradiality of theta monoids [cf. Corollary 2.3 below], cannot, however, be applied immediately to the case of Gaussian monoids. That is to say, the corresponding multiradiality of Gaussian monoids, to be discussed in §3 below, requires one to apply the theory of log-shells developed in §1 [cf. [IUTchII], Remark 2.9.1, (iii); [IUTchII], Remark 3.4.1, (ii); [IUTchII], Remark 3.7.1]. On the other hand, as we shall see in §3 below, the multiradiality of Gaussian monoids depends in an essential way on the multiradiality of theta monoids discussed in the present §2 as a sort of “essential first step” constituted by the decoupling discussed in (ii) above. Indeed, if one tries to consider the Kummer theory of the theta values [i.e., the “q^2" — cf. [IUTchII], Remark 2.5.1, (i)] just as elements of the base field — i.e., without availing oneself of the theory of the étale theta function — then it is difficult to see how to rigidify the cyclotomes involved by any means other than the theory of MLF-Galois pairs discussed in (iii) above. But, as discussed in (iii) above, this approach to cyclotomic rigidity gives rise to \(\hat{\mathbb{Z}}^\times\)-indeterminacies — i.e., to confusion between the theta values “q^2" and their \(\hat{\mathbb{Z}}^\times\)-powers, which is unacceptable from the point of view of the theory of the present series of papers! For another approach to understanding the indispensability of the multiradiality of theta monoids, we refer to Remark 2.2.2 below.

Remark 2.2.2.

(i) One way to understand the very special role played by the theta values [i.e., the values of the theta function] in the theory of the present series of papers is to consider the following naive question:

Can one develop a similar theory to the theory of the present series of papers in which one replaces the \(\Theta_{\text{gau}}^{x\mu}\)-link

\[
q \leftrightarrow q \left( \begin{array}{c} 2 \\ \vdots \\ (l^\ast)^2 \end{array} \right)
\]

[cf. [IUTchII], Remark 4.11.1] by a correspondence of the form

\[
q \leftrightarrow q^\lambda
\]

— where \(\lambda\) is some arbitrary positive integer?

The answer to this question is “no”. Indeed, such a correspondence does not come equipped with the extensive multiradiality machinery — such as mono-theta-theoretic cyclotomic rigidity and the splittings determined by zero-labeled
evaluation points — that has been developed for the étale theta function [cf. the discussion of Step (vi) of the proof of Corollary 3.12 of §3 below]. For instance, the lack of mono-theta-theoretic cyclotomic rigidity means that one does not have an apparatus for insulating the Kummer classes of such a correspondence from the \(\hat{\mathbb{Z}}\times\)-indeterminacies that act on the various monoids \((-)^{\times \mu}\) [cf. the discussion of Remark 2.2.1, (iv)]. The splittings determined by zero-labeled evaluation points also play an essential role in decoupling these monoids \((-)^{\times \mu}\) — i.e., the coric log-shells — from the “purely radial” [or, put another way, “value group”] portion of such a correspondence \(q \mapsto q^\lambda\) [cf. the discussion of (iii) below; Remark 2.2.1, (ii); [IUTchII], Remark 1.12.2, (vi)]. Note, moreover, that if one tries to realize such a multiradial splitting via evaluation — i.e., in accordance with the principle of “Galois evaluation” [cf. the discussion of [IUTchII], Remark 1.12.4] — for a correspondence \(q \mapsto q^\lambda\) by, for instance, taking \(\lambda\) to be one of the \(j^2\) [where \(j\) is a positive integer] that appears as a value of the étale theta function, then one must contend with issues of symmetry between the zero-labeled evaluation point and the evaluation point corresponding to \(\lambda\) — i.e., symmetry issues that are resolved in the theory of the present series of papers by means of the theory surrounding the \(\mathbb{F}_t^\pm\)-symmetry [cf. the discussion of [IUTchII], Remarks 2.6.2, 3.5.2]. As discussed in [IUTchII], Remark 2.6.3, this sort of situation leads to numerous conditions on the collection of evaluation points under consideration. In particular, ultimately, it is difficult to see how to construct a theory as in the present series of papers for any collection of evaluation points other than the collection that is in fact adopted in the definition of the \(\Theta_{\mathbb{R}_{\text{gau}}}\)-link.

(ii) As discussed in Remark 2.2.1, (iv), we shall be concerned, in §3 below, with developing multiradial formulations for Gaussian monoids. These multiradial formulations will be subject to certain indeterminacies, which — although sufficiently mild to allow the execution of the volume computations that will be the subject of [IUTchIV] — are, nevertheless, substantially more severe than the indeterminacies that occur in the multiradial formulation given for theta monoids in the present §2 [cf. Corollary 2.3 below]. Indeed, the indeterminacies in the multiradial formulation given for theta monoids in the present §2 — which essentially consist of multiplication by roots of unity [cf. [IUTchII], Proposition 3.1, (i)] — are essentially negligible and may be regarded as a consequence of the highly nontrivial Kummer theory surrounding mono-theta environments [cf. Proposition 2.1, (iii); Theorem 2.2, (ii)], which, as discussed in Remark 2.2.1, (iv), cannot be mimicked for “theta values regarded just as elements of the base field”. That is to say, the quite exact nature of the multiradial formulation for theta monoids — i.e., which contrasts sharply with the somewhat approximate nature of the multiradial formulation for Gaussian monoids to be developed in §3 — constitutes another important ingredient of the theory of the present paper that one must sacrifice if one attempts to work with correspondences \(q \mapsto q^\lambda\) as discussed in (i), i.e., correspondences which do not come equipped with the extensive multiradiality machinery that arises as a consequence of the theory of the étale theta function developed in [EtTh].

(iii) One way to understand the significance, in the context of the discussions of (i) and (ii) above, of the multiradial coric/radial decouplings furnished by the splittings determined by the zero-labeled evaluation points is as follows. Ultimately, in order to establish, in §3 below, multiradial formulations for Gaussian
monoids, it will be of crucial importance to pass from the **Frobenius-like theta monoids** that appear in the **domain** of the $\Theta_{\text{gau}}^{x,\mu}$-link to **vertically coric étale-like** objects by means of **Kummer theory** [cf. the discussions of Remarks 1.2.4, (i); 1.5.4, (i), (iii)], in the context of the relevant **log-Kummer correspondences**, as discussed, for instance, in Remark 3.12.2, (iv), (v), below [cf. also IUTchII, Remark 1.12.2, (iv)]. On the other hand, in order to obtain formulations expressed in terms that are meaningful from the point of view of the **codomain** of the $\Theta_{\text{gau}}^{x,\mu}$-link, it is necessary [cf. the discussion of Remark 3.12.2, (iv), (v), below] to relate this **Kummer theory of theta monoids** in the **domain** of the $\Theta_{\text{gau}}^{x,\mu}$-link to the Kummer theory constituted by the $\times \mu$-Kummer structures that appear in the **horizontally coric** portion of the data that constitutes the $\Theta_{\text{gau}}^{x,\mu}$-link [cf. Theorem 1.5, (ii)]. This is precisely what is achieved by the **Kummer-compatibility** of the multiradial splitting via **evaluation** — i.e., in accordance with the principle of “**Galois evaluation**” [cf. the discussion of IUTchII, Remark 1.12.4].

This state of affairs [cf., especially, the two displays of IUTchII, Corollary 1.12, (ii); the final arrow of the diagram “$(\mathfrak{t}_{\mu,\times \mu})$” of IUTchII, Corollary 1.12, (iii)] is illustrated in Fig. 2.3 below.

Here, the **multiple arrows** [i.e., indicated by means of the “$\rightarrow$’s” separated by vertical dots] in the **lower portion** of the diagram correspond to the fact that the “$\mathcal{O}^\times$” on the left-hand side of this lower portion is related to the “$\mathcal{O}^{x,\mu}$” on the right-hand side via an Ism-orbit of morphisms; the analogous arrow in the **upper portion** of the diagram consists of a **single arrow** [i.e., “$\rightarrow$”] and corresponds to the fact that the restriction of the multiple arrows in the lower portion of the diagram to “$\Theta_{\mu}$” amounts to a single arrow, i.e., precisely as a consequence of the fact that $\Theta_{\mu} \mapsto 1 \in \mathcal{O}^{x,\mu}$ [cf. the situation illustrated in Fig. 2.2]. On the other hand, the “$\Pi/\Delta$’s” on the left-hand side of both the upper and the lower portions of the
The multiradial structure [cf. the lower portion of Fig. 2.3] on the “theta monoid $O^\times \cdot \infty^\theta$” furnished by the splittings via Galois evaluation into coric/radial components is compatible with the relationship between the respective Kummer theories of the “$O^\times$” portion of “$O^\times \cdot \infty^\theta$” [on the left] and the coric “$O^\times \mu$” [on the right].

This state of affairs lies in stark contrast to the situation that arises in the case of a naive correspondence of the form “$q \mapsto q^\lambda$” as discussed in (i): That is to say, in the case of such a naive correspondence, the corresponding arrows “$\mapsto$” of the analogue of Fig. 2.3 map $q^\lambda \mapsto 1 \in O^\times \mu$

and hence are fundamentally incompatible with passage to Kummer classes, i.e., since the Kummer class of $q^\lambda$ in a suitable cohomology group of $\Pi/\Delta$ is by no means mapped, via the poly-isomorphism $\Pi/\Delta \sim G$, to the trivial element of the relevant cohomology group of $G$.

We conclude the present §2 with the following multiradial interpretation [cf. [IUTchII], Remark 4.1.1, (iii); [IUTchII], Remark 4.3.1] — in the spirit of the étale-picture of $D$-$Theta^\pm$NF-Hodge theaters of [IUTchII], Corollary 4.11, (ii) — of the theory surrounding Theorem 2.2.

**Corollary 2.3.** (Étale-picture of Multiradial Theta Monoids) In the notation of Theorem 2.2, let

$$\{n,m \mathcal{HT}^{\Theta^\pm_{\text{ell}}\text{NF}}\}_{n,m \in \mathbb{Z}}$$

be a collection of distinct $Theta^\pm_{\text{ell}}\text{NF}$-Hodge theaters [relative to the given initial $Theta$-data] — which we think of as arising from a Gaussian log-theta-lattice [cf. Definition 1.4]. Write $n,m \mathcal{HT}^{D,Theta^\pm_{\text{ell}}\text{NF}}$ for the $D$-$Theta^\pm_{\text{ell}}\text{NF}$-Hodge theater associated to $n,m \mathcal{HT}^{Theta^\pm_{\text{ell}}\text{NF}}$. Consider the radial environment [cf. [IUTchII], Example 1.7, (ii)] defined as follows. We define a collection of radial data

$$\uparrow \mathcal{R} = (\uparrow \mathcal{HT}^{D,Theta^\pm_{\text{ell}}\text{NF}}, \mathcal{F}^\mu_{\text{env}}(\uparrow D_\triangle), \mathcal{F}^\mu_{\text{bad}}(\uparrow D_\triangle), \mathcal{F}^{\uparrow \times \mu}_{\text{env}}(\uparrow D_\triangle) \sim \mathcal{F}^{\uparrow \times \mu}(\uparrow D_\triangle))$$

to consist of

$$(a_{\mathcal{R}})$$ a $D$-$Theta^\pm_{\text{ell}}\text{NF}$-Hodge theater $\uparrow \mathcal{HT}^{D,Theta^\pm_{\text{ell}}\text{NF}}$;

$$(b_{\mathcal{R}})$$ the $\mathcal{F}^\mu$-prime-strip $\mathcal{F}^\mu_{\text{env}}(\uparrow D_\triangle)$ associated to $\uparrow \mathcal{HT}^{D,Theta^\pm_{\text{ell}}\text{NF}}$ [cf. Proposition 2.1, (ii)].
(c) the data \((a_v), (b_v), (c_v), (d_v)\) of Theorem 2.2, (ii), for \(v \in V^{\text{bad}}\), which we denote by \(\text{R}^{\text{bad}}\);

(d) the \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\) associated to \(\mathcal{H}^D_\Theta\pm^{\text{ell}}\) [cf. Theorem 1.5, (iii)];

(e) the full poly-isomorphism of \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\)

We define a morphism between two collections of radial data \(\text{R} \to \text{R}\) [where we apply the evident notational conventions with respect to “†” and “‡”] to consist of data as follows:

(a) an isomorphism of \(D^\perp\)-prime-strips \(\mathfrak{F}^\perp\)

(b) the \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\)

(c) the isomorphism between collections of data \(\text{R}^{\text{bad}} \sim \text{R}^{\text{bad}}\)

(d) an isomorphism of \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\)

(e) we observe that the isomorphisms of (b) and (d) are necessarily compatible with the poly-isomorphisms of (e) for “†”, “‡”.

We define a collection of coric data
\[\text{C} = (\mathfrak{F}^+\times \Delta^\perp)\]
to consist of

(a) a \(D^\perp\)-prime-strip \(\mathfrak{F}^\perp\)

(b) the \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\)

We define a morphism between two collections of coric data \(\text{C} \to \text{C}\) [where we apply the evident notational conventions with respect to “†” and “‡”] to consist of data as follows:

(a) an isomorphism of \(D^\perp\)-prime-strips \(\mathfrak{F}^\perp\)

(b) an isomorphism of \(F^+\times \Delta\)-prime-strips \(\mathfrak{F}^+\times \Delta^\perp\)

We define the radial algorithm by the assignment
\[\text{R} = (\mathcal{H}^D_\Theta\pm^{\text{ell}}\mathfrak{F}^+\times \Delta^\perp)\]
\[\to \text{C} = (\mathfrak{F}^+\times \Delta^\perp)\]
— together with the assignment on morphisms determined by the data of \((d_{\text{Mor}_R})\).

Then:

(i) The functor associated to the radial algorithm defined above is full and essentially surjective. In particular, the radial environment defined above is multiradial.

(ii) Each \(D-\Theta^{\pm\text{ell}}\) NF-Hodge theater \(n,m \mathcal{H}_D^\pm \Theta^{\pm\text{ell}}\)NF, for \(n,m \in \mathbb{Z}\), defines, in an evident way, an associated collection of radial data \(n,m \mathcal{R}\). The poly-isomorphisms induced by the vertical arrows of the Gaussian log-theta-lattice under consideration [cf. Theorem 1.5, (i)] induce poly-isomorphisms of radial data \(\sim n,m \mathcal{R} \sim n,m+1 \mathcal{R} \sim \ldots\). Write \(n,^0 \mathcal{R}\) for the collection of radial data obtained by identifying the various \(n,m \mathcal{R}\), for \(m \in \mathbb{Z}\), via these poly-isomorphisms and \(n,^0 \mathcal{C}\) for the collection of coric data associated, via the radial algorithm defined above, to the radial data \(n,^0 \mathcal{R}\). In a similar vein, the horizontal arrows of the Gaussian log-theta-lattice under consideration induce full poly-isomorphisms \(\sim n,m \mathcal{D}_\Delta \sim n+1,m \mathcal{D}_\Delta \sim \ldots\) of \(D^\prime\)-prime-strips [cf. Theorem 1.5, (ii)]. Write \(\circ,^0 \mathcal{C}\) for the collection of coric data obtained by identifying the various \(n,^0 \mathcal{C}\), for \(n \in \mathbb{Z}\), via these poly-isomorphisms. Thus, by applying the radial algorithm defined above to each \(n,^0 \mathcal{R}\), for \(n \in \mathbb{Z}\), we obtain a diagram — i.e., an étale-picture of radial data — as in Fig. 2.4 below. This diagram satisfies the important property of admitting arbitrary permutation symmetries among the spokes [i.e., the labels \(n \in \mathbb{Z}\)] and is compatible, in the evident sense, with the étale-picture of \(D-\Theta^{\pm\text{ell}}\) NF-Hodge theaters of [IUTchII], Corollary 4.11, (ii).

\[
\begin{array}{ccc}
\mathcal{F}^\dag_{\text{env}}(n,^0 \mathcal{D}^>) + n,^0 \mathcal{R}_{\text{bad}} + \ldots \\
\vdots \\
\mathcal{F}^\dag_{\text{env}}(n',^0 \mathcal{D}^>) + n',^0 \mathcal{R}_{\text{bad}} + \ldots \\
\vdots \\
\mathcal{F}^\dag_{\text{env}}(n'',^0 \mathcal{D}^>) + n'',^0 \mathcal{R}_{\text{bad}} + \ldots
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}^\dag_{\Delta} \times \mu(n,^0 \mathcal{D}_\Delta^0) \\
\vdots \\
\mathcal{F}^\dag_{\Delta} \times \mu(n',^0 \mathcal{D}_\Delta^0) \\
\vdots \\
\mathcal{F}^\dag_{\Delta} \times \mu(n'',^0 \mathcal{D}_\Delta^0)
\end{array}
\]

Fig. 2.4: Étale-picture of radial data
(iii) The [poly-]isomorphisms of $\mathcal{F}^\times \times \mu$-prime-strips of/induced by $\langle e_{\mathcal{R}} \rangle$, $\langle b_{\text{Mor}_{\mathcal{R}}} \rangle$, $\langle d_{\text{Mor}_{\mathcal{R}}} \rangle$ [cf. also $\langle e_{\mathcal{R}_{\text{Mor}}} \rangle$] are compatible, relative to the Kummer isomorphisms of Proposition 2.1, (ii) [cf. also Proposition 2.1, (vi)], and Theorem 1.5, (iii), with the poly-isomorphisms — arising from the horizontal arrows of the Gaussian log-theta-lattice — of Theorem 1.5, (ii).

(iv) The algorithmic construction of the isomorphisms $\mathfrak{S}_{\text{env}}^{\dagger}(\uparrow \mathcal{D} \succ \dagger) \simeq \mathfrak{S}_{\text{env}}^{\dagger}(\uparrow \mathcal{D} \succ \dagger)$, $\uparrow \mathcal{R} \overset{\sim}{\to} \uparrow \mathcal{R}$ of $\langle b_{\text{Mor}_{\mathcal{R}}} \rangle$, $\langle c_{\text{Mor}_{\mathcal{R}}} \rangle$, as well as of the Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii)], and Theorem 1.5, (iii), (v), are compatible [cf. the final portions of Theorems 1.5, (v); 2.2, (ii)] with the horizontal arrows of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of Theorem 1.5, (ii)], in the sense that these constructions are stabilized/equivariant functorial with respect to arbitrary automorphisms of the domain and codomain of these horizontal arrows of the Gaussian log-theta-lattice.

Proof. The various assertions of Corollary 2.3 follow immediately from the definitions and the references quoted in the statements of these assertions. \(\Box\)

Remark 2.3.1.

(i) In the context of the étale-picture of Fig. 2.4, it is of interest to recall the point of view of the discussion of [IUTchII], 1.12.5, (i), (ii), concerning the analogy between étale-pictures in the theory of the present series of papers and the polar coordinate representation of the classical Gaussian integral.

(ii) The étale-picture discussed in Corollary 2.3, (ii), may be thought of as a sort of canonical splitting of the portion of the Gaussian log-theta-lattice under consideration that involves theta monoids [cf. the discussion of [IUTchI], §I1, preceding Theorem A].

(iii) The portion of the multiradiality discussed in Corollary 2.3, (iv), at $\mathfrak{v} \in \mathfrak{V}_{\text{bad}}$ corresponds, in essence, to the multiradiality discussed in [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i).

Remark 2.3.2. A similar result to Corollary 2.3 may be formulated concerning the multiradiality properties satisfied by the Kummer theory of $\kappa$-coric structures as discussed in [IUTchII], Corollary 4.8. That is to say, the Kummer theory of the localization poly-morphisms

$$\left\{ \{\pi_1^{\text{sol}}(\uparrow \mathcal{D} \overset{\dagger}{\succ} \dagger) \} j \rightarrow \uparrow \mathcal{M}_{\kappa^\times \mathfrak{v}_j} \subseteq \uparrow \mathcal{M}_{\kappa^\times \mathfrak{v}_j} \}ight\}_{\mathfrak{v} \in \mathfrak{V}}$$

discussed in [IUTchII], Corollary 4.8, (iii), is based on the cyclotomic rigidity isomorphisms for $\kappa$-coric structures discussed in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) [cf. also the discussion of [IUTchII], Corollary 4.8, (i)], which satisfy “insulation” properties analogous to the properties discussed in Remark 2.2.1 in the case of mono-theta-theoretic cyclotomic rigidity.
Moreover, the reconstruction of \( \infty \kappa \text{-coric structures} \) from \( \infty \kappa \times \text{-structures} \) via restriction of Kummer classes

\[
\hat{\mathbb{M}}_{\infty \kappa v_j} \subset \hat{\mathbb{M}}_{\infty \kappa \times v_j} \rightarrow \hat{\mathbb{M}}_{\infty \kappa v_j}^\times \sim \hat{\mathbb{M}}_{v_j}^\times
\]

as discussed in [IUTchI], Definition 5.2, (vi), (viii) — i.e., a reconstruction in accordance with the principle of Galois evaluation [cf. [IUTchII], Remark 1.12.4] — may be regarded as a decoupling into

- radial [i.e., \( \{ \pi^\text{sol} (\hat{\mathbb{D}}^\circ) \} \sim \hat{\mathbb{M}}_{\infty \kappa v_j}^\circ \), \( \hat{\mathbb{M}}_{\infty \kappa v_j} \), \( \hat{\mathbb{M}}_{\infty \kappa v_j}^\times \) and]
- coric [i.e., the quotient of \( \hat{\mathbb{M}}_{\infty \kappa \times v_j}^\times \sim \hat{\mathbb{M}}_{\infty \kappa v_j}^\circ \) by its torsion subgroup]

components, i.e., in an entirely analogous fashion to the mono-theta-theoretic case discussed in Remark 2.2.2, (iii). The Galois evaluation that gives rise to the \textit{theta values} \( \overline{q_j}^2 \) in the case of theta monoids corresponds to the construction via Galois evaluation of the monoids \( \hat{\mathbb{M}}_{\mod}^\circ \), i.e., via the operation of restricting Kummer classes associated to elements of \( \infty \kappa \text{-coric structures} \), as discussed in [IUTchI], Example 5.1, (v); [IUTchII], Corollary 4.8, (i) [cf. also [IUTchI], Definition 5.2, (vi), (viii)]. We leave the routine details of giving a formulation in the style of Corollary 2.3 to the reader.

**Remark 2.3.3.** In the context of Remark 2.3.2, it is of interest to compare and contrast the \textit{multiradiality} properties that hold in the \textit{theta} [cf. Remarks 2.2.1, 2.2.2; Corollary 2.3] and \textit{number field} [cf. Remark 2.3.2] cases, as follows.

(i) One important similarity between the \textit{theta} and \textit{number field} cases lies in the establishment of \textit{multiradiality} properties, i.e., such as the \textit{radial/coric decoupling} discussed in Remarks 2.2.2, (iii); 2.3.2, by using the \textit{geometric dimension} of the elliptic curve under consideration as a sort of

\textit{“multiradial geometric container”} for the \textit{radial arithmetic data} of interest, i.e., \textit{theta values} \( \overline{q_j}^2 \) or copies of the \textit{number field} \( F_{\mod} \).

That is to say, in the theta case, the theory of \textit{theta functions} on Tate curves as developed in [EtTh] furnishes such a geometric container for the theta values, while in the number field case, the absolute anabelian interpretation developed in [AbsTopIII] of the theory of \textit{Belyi maps} as \textit{Belyi cuspidalizations} [cf. [IUTchI], Remark 5.1.4] furnishes such a geometric container for copies of \( F_{\mod} \). In this context, another important similarity is the passage from such a geometric container to the radial arithmetic data of interest by means of Galois evaluation [cf. Remark 2.2.2, (i), (iii); Remark 2.3.2].

(ii) One important theme of the present series of papers is the point of view of \textit{dismantling the two underlying combinatorial dimensions} of [the ring of integers of] a number field — cf. the discussion of Remark 3.12.2 below. As discussed in [IUTchI], Remark 6.12.3 [cf. also [IUTchI], Remark 6.12.6], this dismantling may be compared to the dismantling of the \textit{single complex holomorphic dimension} of the \textit{upper half-plane} into \textit{two underlying real dimensions}. If one considers this dismantling from such a classical point of view, then one is tempted to attempt to understand the dismantling into two underlying real dimensions, by, in effect,
base-changing from \( \mathbb{R} \) to \( \mathbb{C} \), so as to obtain two-dimensional complex holomorphic objects, which we regard as being equipped with some sort of descent data arising from the base-change from \( \mathbb{R} \) to \( \mathbb{C} \).

Translating this approach back into the case of number fields, one obtains a situation in which one attempts to understand the dismantling of the two underlying combinatorial dimensions of [the ring of integers of] a number field by working with two-dimensional scheme-theoretic data — i.e., such as an elliptic curve over [a suitable localization of the ring of integers of] a number field — equipped with “suitable descent data”. From this point of view, one may think of

the “multiradial geometric containers” discussed in (i) as a sort of realization of such two-dimensional scheme-theoretic data,

and of

the accompanying Galois evaluation operations, i.e., the multiradial representations up to certain mild indeterminacies obtained in Theorem 3.11, below [cf. also the discussion of Remark 3.12.2, below], as a sort of realization of the corresponding “suitable descent data”.

This sort of interpretation is reminiscent of the interpretation of multiradiality in terms of parallel transport via a connection as discussed in [IUTchII], Remark 1.7.1, and the closely related interpretation given in the discussion of [IUTchII], Remark 1.9.2, (iii), of the tautological approach to multiradiality in terms of PD-envelopes in the style of the \( p \)-adic theory of the crystalline site.

(iii) Another fundamental similarity between the theta and number field cases may be seen in the fact that the associated Galois evaluation operations — i.e., that give rise to the theta values “\( q^{2v} \)” [cf. [IUTchII], Corollary 3.6] or copies of the number field “\( F_{\text{mod}} \)” [cf. [IUTchII], Corollary 4.8, (i), (ii)] — are performed in the context of the \( \log \)-link, which depends, in a quite essential way, on the arithmetic holomorphic [i.e., ring!] structures of the various local fields involved — cf., for instance, the discussion of the relevant \( \log \)-Kummer correspondences in Remark 3.12.2, (iv), (v), below. On the other hand, one fundamental difference between the theta and number field cases may be observed in the fact that whereas

- the output data in the theta case — i.e., the theta values “\( q^{2v} \)” — depends, in an essential way, on the labels \( j \in \mathbb{F}_l^* \),
- the output data in the number field case — i.e., the copies of the number field “\( F_{\text{mod}} \)” — is independent of these labels \( j \in \mathbb{F}_l^* \).

In this context, let us recall that these labels \( j \in \mathbb{F}_l^* \) correspond, in essence, to collections of cuspidal inertia groups [cf. [IUTchII], Definition 4.1, (ii)] of the local geometric fundamental groups that appear [i.e., in the notation of the discussion of Remark 2.2.2, (iii), the subgroup “\( \Delta (\subseteq \Pi) \)” of the local arithmetic fundamental group \( \Pi \)]. On the other hand, let us recall that, in the context of these local arithmetic fundamental groups \( \Pi \), the arithmetic holomorphic structure also depends, in an essential way, on the geometric fundamental group portion [i.e.,
“Δ ⊆ Π” of Π [cf., e.g., the discussion of [AbsTopIII], Theorem 1.9, in [IUTchI], Remark 3.1.2, (ii); the discussion of [AbsTopIII], §13]. In particular, it is a quite nontrivial fact that

the Galois evaluation and Kummer theory in the theta case may be performed [cf. [IUTchII], Corollary 3.6] in a consistent fashion that is compatible with both the labels \( j \in \mathbb{F}_l^* \) [cf. also the associated symmetries discussed in [IUTchII], Corollary 3.6, (i)] and the arithmetic holomorphic structures involved

— i.e., both of which depend on “Δ” in an essential way. By contrast,

the corresponding Galois evaluation and Kummer theory operations in the number field case are performed [cf. [IUTchII], Corollary 4.8, (i), (ii)] in a way that is compatible with the arithmetic holomorphic structures involved, but yields output data [i.e., copies of the number field “\( \mathbb{F}_{mod} \)”] that is free of any dependence on the labels \( j \in \mathbb{F}_l^* \).

Of course, the global realified Gaussian Frobenioids constructed in [IUTchII], Corollary 4.6, (v), which also play an important role in the theory of the present series of papers, involve global data that depends, in an essential way, on the labels \( j \in \mathbb{F}_l^* \), but this dependence occurs only in the context of global realified Frobenioids, i.e., which [cf. the notation “\( \sqcup \)” as it is used in [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (viii), as well as in Definition 2.4, (iii), below] are mono-analytic in nature [i.e., do not depend on the arithmetic holomorphic structure of copies of the number field “\( \mathbb{F}_{mod} \)”].

(iv) In the context of the observations of (iii), we make the further observation that it is a highly nontrivial fact that the construction algorithm for the mono-theta-theoretic cyclotomic rigidity isomorphism applied in the theta case admits \( \mathbb{F}_l^{\times \pm} \)-symmetries [cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchII], Corollary 3.6, (i)] in a fashion that is consistent with the dependence of the theta values on the labels \( j \in \mathbb{F}_l^* \). As discussed in [IUTchII], Remark 1.1.1, (v), this state of affairs differs quite substantially from the state of affairs that arises in the case of the approach to cyclotomic rigidity taken in [IUTchI], Example 5.1, (v), which is based on a rather “straightforward” or “naïve” utilization of the Kummer classes of rational functions. That is to say, the “highly nontrivial” fact just observed in the theta case would amount, from the point of view of this “naïve Kummer approach” to cyclotomic rigidity, to the existence of a rational function [or, alternatively, a collection of rational functions without “labels”] that is invariant [up to, say, multiples by roots of unity] with respect to the \( \mathbb{F}_l^{\times \pm} \)-symmetries that appear, but nevertheless attains values on some \( \mathbb{F}_l^{\times \pm} \)-orbit of points that have distinct valuations at distinct points — a situation that is clearly self-contradictory!

(v) One way to appreciate the nontriviality of the “highly nontrivial” fact observed in (iv) is as follows. One possible approach to realizing the apparently “self-contradictory” state of affairs constituted by a “symmetric rational function with non-symmetric values” consists of replacing the local arithmetic fundamental group “\( \Pi \)” [cf. the notation of the discussion of (iii)] by some suitable closed subgroup of infinite index of \( \Pi \). That is to say, if one works with such infinite index closed
subgroups of \( \Pi \), then the possibility arises that the Kummer classes of those rational functions that constitute the \textit{obstruction to symmetry} in the case of some given rational function of interest [i.e., at a more concrete level, the rational functions that arise as \textit{quotients} of the given rational function by its \( F^{\pm} \)-conjugates] vanish upon \textit{restriction} to such infinite index closed subgroups of \( \Pi \). On the other hand, this approach has the following \textit{"fundamental deficiencies"}, both of which relate to an apparently \textbf{fatal lack of compatibility} with the \textit{arithmetic holomorphic structures} involved:

- It is not clear that the \textbf{absolute anabelian} results of [AbsTopIII], §1 — i.e., which play a \textit{fundamental role} in the theory of the present series of papers — admit generalizations to the case of such infinite index closed subgroups of \( \Pi \).

- The vanishing of \textbf{Kummer classes} of certain rational functions that occurs when one \textit{ restricts} to such infinite index closed subgroups of \( \Pi \) will not, in general, be compatible with the \textbf{ring structures} involved [i.e., of the rings/fields of rational functions that appear].

In particular, this approach does not appear to be likely to give rise to a meaningful theory.

(vi) Another possible approach to realizing the apparently \textit{"self-contradictory"} state of affairs constituted by a \textit{symmetric} rational function with \textit{non-symmetric} values consists of working with \textbf{distinct rational functions}, i.e., one \textit{symmetric} rational function [or collection of rational functions] for constructing \textit{cyclotomic rigidity isomorphisms} via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), and one \textit{non-symmetric} rational function to which one applies \textit{Galois evaluation} operations to construct the analogue of \textit{“theta values”}. On the other hand, this approach has the following \textit{“fundamental deficiency"}, which again relates to a sort of \textbf{fatal lack of compatibility} with the \textit{arithmetic holomorphic structures} involved: The crucial \textbf{absolute anabelian} results of [AbsTopIII], §1 [cf. also the discussion of [IUTchI], Remark 3.1.2, (ii), (iii)], depend, in an essential way, on the use of \textbf{numerous cyclotomes} [i.e., copies of \( \hat{\mathbb{Z}}(1) \)] — which, for simplicity, we shall denote by

\[
\mu^\ast_{et}
\]

in the present discussion — that arise from the various \textbf{cuspidal inertia groups} at the \textbf{cusps} \( \# \) of [the various cuspidalizations associated to] the hyperbolic curve under consideration. These cyclotomes \( \mu^\ast_{et} \) [i.e., for various cusps \( \# \)] may be \textbf{naturally identified} with one another, i.e., via the \textit{natural isomorphisms} of [AbsTopIII], Proposition 1.4, (ii); write

\[
\mu^\forall_{et}
\]

for the cyclotome resulting from this natural identification. Moreover, since the various [pseudo-]monoids constructed by applying these anabelian results are constructed as sub[pseudo-]monoids of first [group] cohomology modules with coefficients in the \textit{cyclotome} \( \mu^\forall_{et} \), it follows [cf. the discussion of [IUTchII], Remark 1.5.2] that the \textit{cyclotome}

\[
\mu_{Fr}
\]
determined by [i.e., the cyclotome obtained by applying \( \text{Hom}(\mathbb{Q}/\mathbb{Z}, -) \) to the \textit{torsion subgroup} of] such a [pseudo-]monoid may be \textbf{tautologically identified} — i.e.,
whenever the [pseudo-]monoid under consideration is regarded [not just as an abstract “Frobenius-like” [pseudo-]monoid, but rather] as the “étale-like” output data of an anabelian construction of the sort just discussed — with the cyclotome $\mu_{\text{et}}$.

In the context of the relevant log-Kummer correspondences [i.e., as discussed in Remark 3.12.2, (iv), (v), below; Theorem 3.11, (ii), below], we shall work with various Kummer isomorphisms between such Frobenius-like and étale-like versions of various [pseudo-]monoids, i.e., in the notation of the final display of Proposition 1.3, (iv), between various objects associated to the Frobenius-like “$\cdot$’s” and corresponding objects associated to the étale-like “$\circ$”. Now so long as one regards these various Frobenius-like “$\cdot$’s” and the étale-like “$\circ$” as distinct labels for corresponding objects, the diagram constituted by the relevant log-Kummer correspondence does not result in any “vicious circles” or “loops”. On the other hand, ultimately in the theory of §3 [cf., especially, the final portion of Theorem 3.11, (iii), (c), (d), below; the proof of Corollary 3.12 below], we shall be interested in applying the theory to the task of constructing algorithms to describe objects of interest of one arithmetic holomorphic structure in terms of some alien arithmetic holomorphic structure [cf. Remark 3.11.1] by means of “multiradial containers” [cf. Remark 3.12.2, (ii)]. These multiradial containers arise from étale-like versions of objects, but are ultimately applied as containers for Frobenius-like versions of objects. That is to say,

in order for such multiradial containers to function as containers, it is necessary to contend with the consequences of identifying the Frobenius-like and étale-like versions of various objects under consideration, e.g., in the context of the above discussion, of identifying $\mu_{\text{Fr}}$ with $\mu_{\text{et}}$.

On the other hand, let us recall that the approach to constructing cyclotomic rigidity isomorphisms associated to rational functions via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), amounts in effect [i.e., in the context of the above discussion], to “identifying” various “$\mu_{\text{et}}$’s” with various “sub-cyclotomes” of “$\mu_{\text{Fr}}$” via morphisms that differ from the usual natural identification precisely by multiplication by the order $[\in \mathbb{Z}]$ at “$\ast$” of the zeroes/poles of the rational function under consideration. That is to say,

to execute such a cyclotomic rigidity isomorphism construction in a situation subject to the further identification of $\mu_{\text{Fr}}$ with $\mu_{\text{et}}$ [which, we recall, was obtained by identifying the various “$\mu_{\text{et}}$’s”!] does indeed result — at least in an a priori sense! — in “vicious circles”/“loops” [cf. the discussion of [IUTchIV], Remark 3.3.1, (i); the reference to this discussion in [IUTchI], Remark 4.3.1, (ii)]. That is to say, in order to avoid any possible contradictions that might arise from such “vicious circles”/“loops”, it is necessary to work with objects that are “invariant”, or “coric”, with respect to such “vicious circles”/“loops”, i.e., to regard

the cyclotome $\mu_{\text{et}}$ as being subject to indeterminacies with respect to multiplication by elements of the submonoid

$$\mathbb{I}^{\text{ord}} \subseteq \pm \mathbb{N}_{\geq 1} \overset{\text{def}}{=} \mathbb{N}_{\geq 1} \times \{\pm 1\}$$

generated by the orders $[\in \mathbb{Z}]$ of the zeroes/poles of the rational function(s) that appear in the cyclotomic rigidity isomorphism construction under consideration.
In the following discussion, we shall also write $\mathbb{I}_{\geq 1} \subseteq \mathbb{N}_{\geq 1}$, $\mathbb{I}_\pm \subseteq \{\pm 1\}$ for the respective images of $\mathbb{I}_{\text{ord}}$ via the natural projections to $\mathbb{N}_{\geq 1}$, $\{\pm 1\}$. This sort of indeterminacy is fundamentally incompatible, for numerous reasons, with any sort of construction that purports to be analogous to the construction of the “theta values” in the theory of the present series of papers, i.e., at least whenever the resulting indeterminacy submonoid $\mathbb{I}_{\text{ord}} \subseteq \pm \mathbb{N}_{\geq 1}$ is nontrivial. For instance, it follows immediately, by considering the effect of independent indeterminacies of this type on valuations at distinct $\varphi \in \mathcal{V}$, that such independent indeterminacies are incompatible with the “product formula” [i.e., with the structure of the global realified Frobenioids involved — cf. [IUTchI], Remark 3.5.1, (ii)]. Here, we observe that this sort of indeterminacy does not occur in the theta case [cf. Fig. 2.5 below] — i.e., the resulting indeterminacy submonoid

\[
(\pm \mathbb{N}_{\geq 1} \supseteq) \mathbb{I}_{\text{ord}} = \{1\}
\]

— precisely as a consequence of the fact [which is closely related to the symmetry properties discussed in [IUTchII], Remark 1.1.1, (v)] that

the order $[\varphi \in \mathbb{Z}]$ of the zeroes/poles of the theta function at every cusp is equal to 1

[cf. [EtTh], Proposition 1.4, (i); [IUTchI], Remark 3.1.2, (ii), (iii)] — a state of affairs that can never occur in the case of an algebraic rational function [i.e., since the sum of the orders $[\varphi \in \mathbb{Z}]$ of the zeroes/poles of an algebraic rational function is always equal to 0]! On the other hand, in the number field case [cf. Fig. 2.6 below], the portion of the indeterminacy under consideration that is constituted by $\mathbb{I}_{\geq 1}$ is avoided precisely [cf. the discussion of [IUTchI], Example 5.1, (v)] by applying the property

$\mathbb{Q}_{> 0} \cap \hat{\mathbb{Z}}^\times = \{1\}$

[cf. also the discussion of (vii) below!], which has the effect of isolating the $\hat{\mathbb{Z}}^\times$-torsor of interest [i.e., some specific isomorphism between cyclotomes] from the subgroup of $\mathbb{Q}_{> 0}$ generated by $\mathbb{I}_{\geq 1}$. This technique for avoiding the indeterminacy constituted by $\mathbb{I}_{\geq 1}$ remains valid even after the identification discussed above of $\mu_{\text{Fr}}$ with $\mu_{\text{et}}^\vee$. By contrast, the portion of the indeterminacy under consideration that is constituted by $\mathbb{I}_{\text{ord}}$ is avoided in the construction of [IUTchI], Example 5.1, (v), precisely by applying the fact that the inverse of a nonconstant $\kappa$-coric rational function is never $\kappa$-coric [cf. the discussion of [IUTchI], Remark 3.1.7, (i)] — a technique that depends, in an essential way, on distinguishing cusps “$\ast$” at which the orders $[\varphi \in \mathbb{Z}]$ of the zeroes/poles of the rational function(s) under consideration are distinct. In particular, this technique is fundamentally incompatible with the identification discussed above of $\mu_{\text{Fr}}$ with $\mu_{\text{et}}^\vee$. That is to say, in summary,

in the number field case, in order to regard étale-like versions of objects as containers for Frobenius-like versions of objects, it is necessary to regard the relevant cyclotomic rigidity isomorphisms — hence also the output data of interest in the number field case, i.e., copies of [the union with $\{0\}$ of] the group “$F_{\text{mod}}^\times$” — as being subject to an indeterminacy constituted by [possible] multiplication by $\{\pm 1\}$.

This does not result in any additional technical obstacles, however, since
the output data of interest in the number field case — i.e., copies of [the union with \{0\} of] the group \(F_{\text{mod}}^\times\) — is [unlike the case with the theta values \(\frac{\varphi^2}{\varphi}\) stabilized by the action of \{\pm 1\}]
— cf. the discussion of Remark 3.11.4 below. Moreover, we observe in passing, in the context of the Galois evaluation operations in the number field case, that the copies of \{the group\} \(F_{\text{mod}}^\times\) are constructed globally and in a fashion compatible with the \(F_{\text{\hat{\ell}}}^\times\)-symmetry [cf. [IUTchII], Corollary 4.8, (i), (ii)], hence, in particular, in a fashion that does not require the establishment of compatibility properties [e.g., relating to the “product formula”] between constructions at distinct \(v \in \mathcal{V}\).

\[
\begin{array}{cccccccc}
+1 & +1 & +1 & +1 & \ldots & +1 & +1 & +1 \\
* & * & * & * & \ldots & * & * & * \\
\end{array}
\]

Fig. 2.5: Orders \([\in \mathbb{Z}]\) of zeroes/poles of the theta function at the cusps “*”

\[
\begin{array}{cccccccc}
0 & 0 & +8 & -5 & \ldots & -6 & +3 & 0 \\
* & * & * & * & \ldots & * & * & * \\
\end{array}
\]

Fig. 2.6: Orders \([\in \mathbb{Z}]\) of zeroes/poles of an algebraic rational function at the cusps “*”

(vii) In the context of the discussion of (vi), we observe that the indeterminacy issues discussed in (vi) may be thought of as a sort of “multiple cusp version” of the “\(N\)-th power versus first power” and “linearity” issues discussed in [IUTchII], Remark 3.6.4, (iii). Also, in this context, we recall from the discussion at the beginning of Remark 2.1.1 that the theory of mono-theta-theoretic cyclotomic rigidity satisfies the important property of being compatible with the topology of the tempered fundamental group. Such a compatibility contrasts sharply with the cyclotomic rigidity algorithms discussed in [IUTchI], Example 5.1, (v), which depend [cf. the discussion of (vi) above!], in an essential way, on the property

\[
\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}
\]

— i.e., which is fundamentally incompatible with the topology of the profinite groups involved [as can be seen, for instance, by considering the fact that \(N \geq 1\) forms a dense subset of \(\hat{\mathbb{Z}}\). This close relationship between cyclotomic rigidity and [a sort of] discrete rigidity [i.e., the property of the above display] is reminiscent of the discussion given in [IUTchII], Remark 2.8.3, (ii), of such a relationship in the case of mono-theta environments.

(viii) In the context of the discussion of (vi), (vii), we observe that the indeterminacy issues discussed in (vi) also occur in the case of the cyclotomic rigidity algorithms discussed in [IUTchI], Definition 5.2, (vi), i.e., in the context of mixed-characteristic local fields. On the other hand, [cf. [IUTchII], Proposition 4.2, (i)] these algorithms in fact yield the same cyclotomic rigidity isomorphism as the cyclotomic rigidity isomorphisms that are applied in [AbsTopIII], Proposition 3.2, (iv) [i.e., the cyclotomic rigidity isomorphisms discussed in [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1]. Moreover, these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] are manifestly compatible with the topology of the profinite groups involved. From the point of view of the discussion of (vi),
this sort of “de facto” compatibility with the topology of the profinite groups involved may be thought of as a reflection of the fact that these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] amount, in essence, to applying the approach to cyclotomic rigidity by considering the Kummer theory of algebraic rational functions [i.e., the approach of (vi), or, alternatively, of [IUTchI], Example 5.1, (v)], in the case where the algebraic rational functions are taken to be the uniformizers — i.e., “rational functions” [any one of which is well-defined up to a unit] with precisely one zero of order 1 and no poles [cf. the discussion of the theta function in (vi)!] — of the mixed-characteristic local field under consideration. Put another way, this sort of “de facto” compatibility may be regarded as a reflection of the fact that, unlike number fields [i.e., “NF’s”] or one-dimensional function fields [i.e., “one-dim. FF’s”], mixed-characteristic local fields [i.e., “MLF’s”] are equipped with a uniquely determined “canonical valuation” — a situation that is reminiscent of the fact that the order $[\in \mathbb{Z}]$ of the zeroes/poles of the theta function at every cusp is equal to 1 [i.e., the fact that “the set of equivalences classes of cusps relative to the equivalence relationship on cusps determined by considering the order $[\in \mathbb{Z}]$ of the zeroes/poles of the theta function is of cardinality one”]. From the point of view of “geometric containers” discussed in (i) and (ii), this state of affairs may be summarized as follows:

the indeterminacy issues that occur in the context of the discussion of cyclotomic rigidity isomorphisms in (vi) exhibit similar qualitative behavior in the

\[ \text{MLF/mono-theta} \quad (\longleftrightarrow \quad \text{one valuation/cusp}) \]

[i.e., where the expression “one cusp” is to be understood as referring to “one equivalence class of cusps”, as discussed above] cases, as well as in the

\[ \text{NF/one-dim. FF} \quad (\longleftrightarrow \quad \text{global collection of valuations/cusps}) \]

cases.

Put another way, at least at the level of the theory of valuations,

the theory of theta functions (respectively, one-dimensional function fields) serves as an accurate “qualitative geometric model” of the theory of mixed-characteristic local fields (respectively, number fields).

Finally, we observe that in this context, the crucial property “$\mathbb{Q}_>0 \bigcap \hat{\mathbb{Z}} = \{1\}$” that occurs in the discussion of the number field/one-dimensional function field cases is highly reminiscent of the global nature of number fields [i.e., such as $\mathbb{Q}$] — cf. the discussion of Remark 3.12.1, (iii), below].

(ix) The comparison given in (viii) of the special properties satisfied by the theta function with the corresponding properties of the algebraic rational functions that appear in the number field case is reminiscent of the analogy discussed in [IUTchI], Remark 6.12.3, (iii), with the classical upper half-plane. That is to say, the eigenfunction for the additive symmetries of the upper half-plane [i.e., which corresponds to the theta case]

\[ q \overset{\text{def}}{=} e^{2\pi iz} \]
<table>
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<td>algebraic rational function of ( z ): ( w \overset{\text{def}}{=} \frac{z-i}{z+i} )</td>
</tr>
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Fig. 2.7: Comparison between the theta and number field cases
is highly transcendental in the coordinate $z$, whereas the eigenfunction for the multiplicative symmetries of the upper half-plane [i.e., which corresponds to the number field case]

$$w \overset{\text{def}}{=} \frac{z-i}{z+i}$$

is an algebraic rational function in the coordinate $z$.

(x) The various properties discussed above in the theta and number field cases are summarized in Fig. 2.7 above.

**Remark 2.3.4.** Before proceeding, it is perhaps of interest to review once more the essential content of [EtTh] in light of the various observations made in Remark 2.3.3.

(i) The starting point of the relationship between the theory of [EtTh] and the theory of the present series of papers lies [cf. the discussion of Remark 2.1.1, (i); [IUTchII], Remark 3.6.2, (ii)] in the various non-ring/scheme-theoretic filters [i.e., log-links and various types of Θ-links] between distinct ring/scheme theories that are constructed in the present series of papers. Such non-scheme-theoretic filters may only be constructed by making use of Frobenius-like structures. On the other hand, étale-like structures are important in light of their ability to relate structures on opposite sides of such non-scheme-theoretic filters. Then Kummer theory is applied to relate corresponding Frobenius-like and étale-like structures. Moreover, it is crucial that this Kummer theory be conducted in a multiradial fashion. This is achieved by means of certain radial/coric decouplings, by making use of multiradial geometric containers, as discussed in Remark 2.3.3, (i), (ii). That is to say, it is necessary to make use of such multiradial geometric containers and then to pass to theta values or number fields by means of Galois evaluation, since direct use of such theta values or number fields results in a Kummer theory that does not satisfy the desired multiradiality properties [cf. Remarks 2.2.1, 2.3.2].

(ii) The most naive approach to the Kummer theory of the “functions” that are to be used as “multiradial geometric containers” may be seen in the approach involving algebraic rational functions on the various algebraic curves under consideration, i.e., in the fashion of [IUTchI], Example 5.1, (v) [cf. also [IUTchI], Definition 5.2, (vi)]. On the other hand, in the context of the local theory at $v \in \mathbb{V}^{\text{bad}}$, this approach suffers from the fatal drawback of being incompatible with the profinite topology of the profinite fundamental groups involved [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii); Figs. 2.5, 2.6]. Thus, in order to maintain compatibility with the profinite/tempered topology of the profinite/tempered fundamental groups involved, one is obliged to work with the Kummer theory of theta functions, truncated modulo $N$. On the other hand, the naive approach to this sort of [truncated modulo $N$] Kummer theory of theta functions suffers from the fatal drawback of being incompatible with discrete rigidity [cf. Remark 2.1.1, (v)]. This incompatibility with discrete rigidity arises from a lack of “shifting automorphisms” as in [EtTh], Proposition 2.14, (ii) [cf. also [EtTh], Remark 2.14.3], and is closely related to the incompatibility of this naive approach with the $F_{l}^{\pm}$-symmetry [cf. the discussion of [IUTchII], Remark 1.1.1,
(iv), (v)]. In order to surmount such incompatibilities, one is obliged to consider not the Kummer theory of theta functions in the naive sense, but rather, so to speak, the Kummer theory of the first Chern classes of the line bundles associated to theta functions [cf. the discussion of [IUTchII], Remark 1.1.1, (v)]. Thus, in summary:

[truncated] Kummer theory of theta [not algebraic rational] functions

⇒ compatible with profinite/tempered topologies;

[truncated] Kummer theory of [first Chern classes of] line bundles [not rational functions!]

⇒ compatible with discrete rigidity, $F_l^{\times\pm}$-symmetry.

(iii) To consider the “[truncated] Kummer theory of line bundles [associated to the theta function]” amounts, in effect, to considering the [partially truncated] arithmetic fundamental group of the $\mathbb{G}_m$-torsor determined by such a line bundle in a fashion that is compatible with the various tempered Frobenioids and tempered fundamental groups under consideration. Such a “[partially truncated] arithmetic fundamental group” corresponds precisely to the “topological group” portion of the data that constitutes a mono-theta or bi-theta environment [cf. [EtTh], Definition 2.13, (ii), (a); [EtTh], Definition 2.13, (iii), (a)]. In the context of the theory of theta functions, such “[partially truncated] arithmetic fundamental groups” are equipped with two natural distinguished [classes of] sections, namely, theta sections and algebraic sections. If one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped neither with data corresponding to theta sections nor with data corresponding to algebraic sections, then the resulting mathematical object is necessarily subject to indeterminacies arising from multiplication by constant units [i.e., “$O^\times$” of the base local field], hence, in particular, suffers from the drawback of being incompatible with constant multiple rigidity [cf. Remark 2.1.1, (iii)]. On the other hand, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped both with data corresponding to theta sections and with data corresponding to algebraic sections, then the resulting mathematical object suffers from the same lack of symmetries as the [truncated] Kummer theory of theta functions [cf. the discussion of (ii)], hence, in particular, is incompatible with discrete rigidity [cf. Remark 2.1.1, (v)]. Finally, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped only with data corresponding to algebraic sections [i.e., but not with data corresponding to theta sections!], then the resulting mathematical object is not equipped with sufficient data to apply the crucial commutator property of [EtTh], Proposition 2.12 [cf. also the discussion of [EtTh], Remark 2.19.2], hence, in particular, is incompatible with cyclotomic rigidity [cf. Remark 2.1.1, (iv)]. That is to say, it is only by thinking of the [partially truncated] arithmetic fundamental groups under consideration as being equipped only with data corresponding to theta sections [i.e., but not with data corresponding to algebraic sections!] — i.e., in short, by working with mono-theta environments — that one may achieve a situation that is compatible with the tempered topology of the tempered fundamental groups involved, the $F_l^{\times\pm}$-symmetry, and all three types of rigidity [cf. the initial portion of Remark 2.1.1; [IUTchII], Remark 3.6.4, (ii)]. Thus, in summary:

working neither with theta sections nor with algebraic sections $\implies$
incompatible with constant multiple rigidity!

working with bi-theta environments, i.e.,
working simultaneously with both theta sections and algebraic sections \( \implies \) incompatible with discrete rigidity, \( \mathbb{F}_l^{\times^\pm} \)-symmetry!

working with algebraic sections but not theta sections \( \implies \) incompatible with cyclotomic rigidity!

working with mono-theta environments, i.e.,
working with theta sections but not algebraic sections \( \implies \) compatible with tempered topology, \( \mathbb{F}_l^{\times^\pm} \)-symmetry, all three rigidities!

(iv) Finally, we note that the approach of [EtTh] to the theory of theta functions differs substantially from more conventional approaches to the theory of theta functions such as

- the classical function-theoretic approach via explicit series representations, i.e., as given at the beginning of the Introduction to [IUTchII] [cf. also [EtTh], Proposition 1.4], and

- the representation-theoretic approach, i.e., by considering irreducible representations of theta groups.

Both of these more conventional approaches depend, in an essential way, on the ring structures — i.e., on both the additive and the multiplicative structures — of the various rings involved. [Here, we recall that explicit series are constructed precisely by adding and multiplying various functions on some space, whereas representations are, in effect, modules over suitable rings, hence, by definition, involve both additive and multiplicative structures.] In particular, although these more conventional approaches are well-suited to many situations in which one considers “the” theta function in some fixed model of scheme/ring theory, they are ill-suited to the situations treated in the present series of papers, i.e., where one must consider theta functions that appear in various distinct ring/scheme theories, which [cf. the discussion of (i)] may only be related to one another by means of suitable Frobenius-like and étale-like structures such as tempered Frobenioids and tempered fundamental groups. Here, we recall that these tempered Frobenioids correspond essentially to multiplicative monoid structures arising from the various rings of functions that appear, whereas tempered fundamental groups correspond to various Galois actions. That is to say, consideration of such multiplicative monoid structures and Galois actions is compatible with the dismantling of the additive and multiplicative structures of a ring, i.e., as considered in the present series of papers [cf. the discussion of Remark 3.12.2 below].

\[ \hat{\hat{\mathfrak{F}}}^+ = \{ \hat{\hat{\mathfrak{F}}}^+_\mathfrak{F} \}_{\mathfrak{F} \in \mathfrak{Y}} \]
be an $\mathcal{F}^+$-prime-strip. Then recall from the discussion of [IUTchII], Definition 4.9, (ii), that at each $w \in \mathcal{V}^{\text{triv}}$, the splittings of the split Frobenioid $\mathcal{O}_w^+$ determine submonoids $\mathcal{O}^+(-) \subseteq \mathcal{O}^\triangleright(-)$, as well as quotient monoids $\mathcal{O}^+(-) \rightarrow \mathcal{O}^\triangleright(-)$ [i.e., by forming the quotient of $\mathcal{O}^+(-)$ by its torsion subgroup]. In a similar vein, for each $w \in \mathcal{V}^{\text{good}}$, the splitting of the split Frobenioid determined by [indeed, “constituted by”, when $w \in \mathcal{V}^{\text{good}} \cap \mathcal{V}^{\text{non}}$ — cf. [IUTchI], Definition 5.2, (ii)] $\mathcal{O}_w^+$ determines a submonoid $\mathcal{O}^+(-) \subseteq \mathcal{O}^\triangleright(-)$ whose subgroup of units is trivial [cf. [IUTchII], Definition 4.9, (iv), when $w \in \mathcal{V}^{\text{good}} \cap \mathcal{V}^{\text{non}}$]; in this case, we set $\mathcal{O}^\triangleright(-) \equiv \mathcal{O}^+(-)$. Write

$$\mathcal{F}_w^{+\perp} = \left\{ \mathcal{F}_w^{+\perp} \right\}_{w \in \mathcal{V}}; \quad \mathcal{F}_w^{+\triangleright} = \left\{ \mathcal{F}_w^{+\triangleright} \right\}_{w \in \mathcal{V}}$$

for the collections of data obtained by replacing the split Frobenioid portion of each $\mathcal{F}_w^{+\perp}$ by the Frobenioids determined, respectively, by the subquotient monoids $\mathcal{O}^+(-) \subseteq \mathcal{O}^\triangleright(-)$, “$\mathcal{O}^\triangleright(-)$” just defined.

(ii) We define [in the spirit of [IUTchII], Definition 4.9, (vii)] an $\mathcal{F}^{+\perp}$-prime-strip to be a collection of data

$$*\mathcal{F}^{+\perp} = \left\{ *\mathcal{F}_{w}^{+\perp} \right\}_{w \in \mathcal{V}}$$

that satisfies the following conditions: (a) if $w \in \mathcal{V}^{\text{mon}}$, then $*\mathcal{F}_{w}^{+\perp}$ is a Frobenioid that is isomorphic to $\mathcal{F}_{w}^{+\perp}$ [cf. (i)]; (b) if $w \in \mathcal{V}^{\text{arc}}$, then $*\mathcal{F}_{w}^{+\perp}$ consists of a Frobenioid and an object of $\mathcal{V}C_{TM}^{\triangleright}$ [cf. [IUTchI], Definition 5.2, (ii)] such that $*\mathcal{F}_{w}^{+\perp}$ is isomorphic to $\mathcal{F}_{w}^{+\perp}$. In a similar vein, we define an $\mathcal{F}^{+\triangleright}$-prime-strip to be a collection of data

$$*\mathcal{F}^{+\triangleright} = \left\{ *\mathcal{F}_{w}^{+\triangleright} \right\}_{w \in \mathcal{V}}$$

that satisfies the following conditions: (a) if $w \in \mathcal{V}^{\text{mon}}$, then $*\mathcal{F}_{w}^{+\triangleright}$ is a Frobenioid that is isomorphic to $\mathcal{F}_{w}^{+\triangleright}$ [cf. (i)]; (b) if $w \in \mathcal{V}^{\text{arc}}$, then $*\mathcal{F}_{w}^{+\triangleright}$ consists of a Frobenioid and an object of $\mathcal{V}C_{TM}^{\triangleright}$ [cf. [IUTchI], Definition 5.2, (ii)] such that $*\mathcal{F}_{w}^{+\triangleright}$ is isomorphic to $\mathcal{F}_{w}^{+\triangleright}$. A morphism of $\mathcal{F}^{+\perp}$- (respectively, $\mathcal{F}^{+\triangleright}$-) prime-strips is defined to be a collection of isomorphisms, indexed by $\mathcal{V}$, between the various constituent objects of the prime-strips [cf. [IUTchI], Definition 5.2, (iii)].

(iii) We define [in the spirit of [IUTchII], Definition 4.9, (viii)] an $\mathcal{F}^{+\perp}$-prime-strip to be a collection of data

$$*\mathcal{F}^{+\perp} = \left( *\mathcal{C}^{+\perp}, \text{Prime}( *\mathcal{C}^{+\perp} ) \right) \simeq \mathcal{V}, *\mathcal{F}^{+\perp}, \left\{ *\rho_{w} \right\}_{w \in \mathcal{V}}$$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{+\perp}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^{+\triangleright}$-prime-strip is replaced by an $\mathcal{F}^{+\perp}$-prime-strip. [We leave the routine details to the reader.] In a similar vein, we define an $\mathcal{F}^{+\triangleright}$-prime-strip to be a collection of data

$$*\mathcal{F}^{+\triangleright} = \left( *\mathcal{C}^{+\triangleright}, \text{Prime}( *\mathcal{C}^{+\triangleright} ) \right) \simeq \mathcal{V}, *\mathcal{F}^{+\triangleright}, \left\{ *\rho_{w} \right\}_{w \in \mathcal{V}}$$
satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\dagger}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^\perp$-prime-strip is replaced by an $\mathcal{F}^\dagger\mathcal{F}^\perp$-prime-strip. [We leave the routine details to the reader.] A morphism of $\mathcal{F}^{\dagger, \perp}$- (respectively, $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$-) prime-strips is defined to be an isomorphism between collections of data as discussed above.

**Remark 2.4.1.**

(i) Thus, by applying the constructions of Definition 2.4, (i), to the underlying $\mathcal{F}^\perp$-prime-strips associated to the $\mathcal{F}^\dagger$-prime-strips that appear in Corollary 2.3, one may regard the multiradiality of Corollary 2.3, (i), as implying a corresponding multiradiality assertion concerning the associated $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$-prime-strips.

(ii) Suppose that we are in the situation discussed in (i). Then at $v \in \mathcal{V}^{\text{bad}}$, the submonoids $\mathcal{O}^{\dagger, \perp}(-) \subseteq \mathcal{O}^{\dagger, \mathcal{F}^\perp}(-)$ may be regarded, in a natural way [cf. Proposition 2.1, (ii); Theorem 2.2, (ii)], as submonoids of the monoids $\mathcal{F}^{\mathcal{F}^\dagger, \mathcal{F}^\perp}$ of Theorem 2.2, (ii), (a). Moreover, the resulting inclusion of monoids is compatible with the multiradiality discussed in (i) and the multiradiality of the data $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$ of Corollary 2.3, (ii), that is implied by the multiradiality of Corollary 2.3, (ii).

**Remark 2.4.2.**

(i) One verifies immediately that, just as one may associate to an $\mathcal{F}^{\dagger, \perp}$-prime-strip a pilot object in the global realified Frobenioid portion of the $\mathcal{F}^{\dagger, \perp}$-prime-strip [cf. [IUTchII], Definition 4.9, (vii)], one may associate to an $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$-prime-strip a pilot object in the global realified Frobenioid portion of the $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$-prime-strip [i.e., in the notation of the final display of Definition 2.4, (iii), the global realified Frobenioid *$\mathcal{C}$* of the $\mathcal{F}^{\dagger, \mathcal{F}^\perp}$-prime-strip *$\mathcal{C}$*].

(ii) For $u \in \mathcal{V}$ lying over $v \in \mathcal{V}_{\text{mod}}$ and $v_Q \in \mathcal{V}_Q \overset{\text{def}}{=} \mathcal{V}(\mathbb{Q})$, write

\[ r_u \overset{\text{def}}{=} [(F_{\text{mod}})_v : \mathcal{Q}_v] \cdot \log(p_u) \in \mathbb{R} \text{ if } u \in \mathcal{V}^{\text{good}}, \]

\[ r_v \overset{\text{def}}{=} [(F_{\text{mod}})_v : \mathcal{Q}_v] \cdot \text{ord}_u(q) \cdot \log(p_v) \in \mathbb{R} \text{ if } u \in \mathcal{V}^{\text{bad}} \]

— where, if $u \in \mathcal{V}^{\text{bad}}$, then $\text{ord}_u : K^\times_u \to \mathbb{Q}$ denotes the natural $p_u$-adic valuation normalized so that $\text{ord}_u(p_v) = 1$, and $q_u$ is as in [IUTchI], Example 3.2, (iv);

\[ r_u^{\dagger} \overset{\text{def}}{=} - \sum_{w \in \mathcal{V}^{\text{bad}}} r_{uw} \]

[cf. the constructions of [IUTchII], Example 3.5; [IUTchI], Remark 3.5.1; the discussion of weights in Remark 3.1.1, (ii), below].

(iii) In the notation of (ii), let $M$ be any ordered monoid isomorphic [as an ordered monoid] to $\mathbb{R}$ [endowed with the usual additive and order structures]. Then $M$ naturally determines a collection of data

\[ (M, \{M_v\}_{v \in \mathcal{V}}, \{\rho_{M_v} : M_v \rightsquigarrow M\}_{v \in \mathcal{V}}) \]
as follows: for each \(v \in V\), we take \(M_v\) to be a copy of \(M\) and \(\rho_{M_v} : M_v \to M\) to be the isomorphism of monoids [that reverses the ordering!] given by \textit{multiplying}
by \(\tau_{\frac{1}{v}} \in \mathbb{R}\).

(iv) In the notation of (ii), (iii), suppose, further, that we have been a \textbf{negative element}
\[\eta_M \in M\]
[i.e., an element \(<0\)], which we shall refer to as a \textbf{pilot element}. Then, since, for \(v \in V\), \(M_v\) is defined to be a copy of \(M\), \(\eta_M\) determines an element \(\eta_{M_v} \in M_v\).
Thus, the \textit{pair} \((M, \eta_M)\) naturally determines a \textit{collection of data}
\[(M, \{M^\tau_v\}_{v \in V}, \{\rho_{M^\tau_v} : M^\tau_v \hookrightarrow M\}_{v \in V})\]
as follows: for each \(v \in V_{\text{non}}\), we take \(M^\tau_v \subseteq M_v\) to be the submonoid [isomorphic to \(\mathbb{N}\)] generated by \(\eta_{M_v}\) and \(\rho_{M^\tau_v} : M^\tau_v \to M\) to be the restriction of \(\rho_{M_v}\) to \(M^\tau_v\);
for each \(v \in V_{\text{arc}}\), we take \(M^\tau_v \subseteq M_v\) to be the submonoid [isomorphic to \(\mathbb{R}_{\geq 0}\)]
given by the elements \(\leq 0\) and \(\rho_{M^\tau_v} : M^\tau_v \hookrightarrow M\) to be the restriction of \(\rho_{M_v}\) to \(M^\tau_v\). In particular, it follows immediately from the construction of this data that
\[
\rho_{M^\tau_v}(\eta_{M_v}) = \tau_{\frac{1}{v}} \cdot \eta_M
\]
for each \(v \in V\).

(v) Now we observe that the constructions of (iii) and (iv) allow one to give a sort of \textit{converse} to the construction of the \textit{pilot object} in (i). Indeed, consider the \(\mathcal{F}_{\text{arc}}\)-prime-strip \(*\mathcal{F}_\tau\) in the final display of Definition 2.4, (iii). Next, observe that the \textbf{Picard group} constructions \(\text{Pic}_\mathbb{R}(-)\) and \(\text{Pic}_\mathbb{C}(-)\) of [FrdI], Theorem 5.1, (i), applied to any object of the global realified Frobenioid \(*\mathcal{C}_\mathbb{R}\) yield canonically isomorphic groups for \textit{any object} of \(*\mathcal{C}_\mathbb{R}\). In particular, it makes sense to speak of \(\text{Pic}(\ast\mathcal{C}_\mathbb{R})\). Moreover, it follows from [FrdI], Theorem 6.4, (i), (ii), that \(\text{Pic}(\ast\mathcal{C}_\mathbb{R})\) is equipped with a canonical structure of \textit{ordered monoid}, with respect to which it is isomorphic to \(\mathbb{R}\) [endowed with the usual additive and order structures].
Relative to this structure of ordered monoid, the \textit{pilot object} discussed in (i) [cf. also the discussion of [IUTchII], Definition 4.9, (viii)] determines a \textbf{negative element}
\(\eta_{\ast\mathcal{C}_\mathbb{R}} \in \text{Pic}(\ast\mathcal{C}_\mathbb{R})\). Thus, one verifies immediately, by recalling the various definitions involved, that the \textit{collection of data} \(\{(M, \{M_v\}_{v \in V}, \{\rho_{M_v} : M_v \to M\}_{v \in V})\}\) constructed in (iii) from \(\textit{M}\) is already sufficient to reconstruct, i.e., by taking \(M \overset{\text{def}}{=} \text{Pic}(\ast\mathcal{C}_\mathbb{R})\), the \textit{collection of data}
\[(\ast\mathcal{C}_\mathbb{R}, \text{Prime}(\ast\mathcal{C}_\mathbb{R}) \to V)\]
[cf. the notation of the final display of Definition 2.4, (iii)], while the \textit{collection of data} \(\{(M, \{M^\tau_v\}_{v \in V}, \{\rho_{M^\tau_v} : M^\tau_v \hookrightarrow M\}_{v \in V})\}\) constructed in (iv) from the pair \(\textit{\{(M, \eta_M)\}}\) is sufficient to reconstruct, i.e., by taking \(M \overset{\text{def}}{=} \text{Pic}(\ast\mathcal{C}_\mathbb{R})\) and \(\eta_M \overset{\text{def}}{=} \eta_{\ast\mathcal{C}_\mathbb{R}}\), the \textit{collection of data}
\[(\ast\mathcal{C}_\mathbb{R}, \text{Prime}(\ast\mathcal{C}_\mathbb{R}) \to V, \{\Phi_{\ast\mathcal{F}_\tau}, \{\rho_{\ast\mathcal{F}_\tau}\}_{v \in V}, \{\ast\mathcal{C}_\mathbb{R}\}_{v \in V})\].
where, for $v \in \mathbb{V}$, we write $\Phi_* F_\downarrow v$ for the [constant!] divisor monoid [i.e., in effect, a single monoid isomorphic to $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$] determined by the Frobenioid structure [cf. [FrdI], Corollary 4.11, (iii); [FrdII], Theorem 1.2, (i)] on $F_\downarrow v$ [cf. the notation of the final display of Definition 2.4, (i)].

(vi) One immediate consequence of the discussion of (v) is the following: If one starts from $M = \text{Pic}(^* C^\text{fr})$, then the resulting collection of data

$$(^* C^\text{fr}, \text{Prime}(^* C^\text{fr}) \xrightarrow{\sim} \mathbb{V})$$

yields a common container, namely, the Frobenioid $^* C^\text{fr}$ [regarded as an object reconstructed from $M = \text{Pic}(^* C^\text{fr})$!], in which distinct choices of the [negative!] pilot element $\in M = \text{Pic}(^* C^\text{fr})$ — hence also the data

$$(^* C^\text{fr}, \text{Prime}(^* C^\text{fr}) \xrightarrow{\sim} \mathbb{V}, \{\Phi_* F_\downarrow v\}_{v \in \mathbb{V}}, \{^* \rho_v\}_{v \in \mathbb{V}})$$

[which may be thought of as a sort of “further rigidification” on $^* C^\text{fr}$] reconstructed from such distinct choices of pilot element — may be compared with one another.

By contrast, if one attempts to compare the constructions of (v) applied to positive and negative “$\eta_M \in M$” [i.e., which amounts to reversing the order structure on $M$!], then already the corresponding Frobenioids “$^* C^\text{fr}$” [i.e., attached to the same group “Pic($^* C^\text{fr}$)”], but with reversed order structures!] involve pre-steps [i.e., in effect, the category-theoretic version of the notion of an inclusion of line bundles — cf. [FrdI], Definition 1.2, (iii)] going in opposite directions. That is to say, such Frobenioids may only be compared with one another if they are embedded in some sort of larger ambient category in which the pre-steps are rendered invertible; but this already implies that all objects arising from such Frobenioids become isomorphic in the ambient category. That is to say, working in such a larger ambient category already renders any sort of argument that requires one to distinguish distinct elements of $\text{Pic}(^* C^\text{fr})$ — i.e., distinct arithmetic degrees/heights of arithmetic line bundles — meaningless [cf. the discussion of positivity in Remark 2.1.1, (v)].
Section 3: Multiradial Logarithmic Gaussian Procession Monoids

In the present §3, we apply the theory developed thus far in the present series of papers to give [cf. Theorem 3.11 below] multiradial algorithms for a slightly modified version of the Gaussian monoids discussed in [IUTchII], §4. This modification revolves around the combinatorics of processions, as developed in [IUTchI], §4, §5, §6, and is necessary in order to establish the desired multiradiality. At a more concrete level, these combinatorics require one to apply the theory of tensor packets [cf. Propositions 3.1, 3.2, 3.3, 3.4, 3.7, 3.9, below]. Finally, we observe in Corollary 3.12 that these multiradial algorithms give rise to certain estimates concerning the log-volumes of the logarithmic Gaussian procession monoids that occur. This observation forms the starting point of the theory to be developed in [IUTchIV].

In the following discussion, we assume that we have been given initial $\Theta$-data as in [IUTchI], Definition 3.1. Also, we shall write

$$V_Q \overset{\text{def}}{=} V(Q)$$

[cf. [IUTchI], §0] and apply the notation of Definition 1.1 of the present paper. We begin by discussing the theory of tensor packets, which may be thought of as a sort of amalgamation of the theory of log-shells developed in §1 with the theory of processions developed in [IUTchI], §4, §5, §6.

**Proposition 3.1.** (Local Holomorphic Tensor Packets) Let

$$\{^\alpha \mathcal{F}_{v_Q}\}_{v_Q} = \left\{ \{^\alpha \mathcal{F}_{v_Q}\}_{v_Q} \right\}_{\alpha \in A}$$

be an $n$-capsule, with index set $A$, of $F$-prime-strips [relative to the given initial $\Theta$-data — cf. [IUTchI], §0; [IUTchI], Definition 5.2, (i)]. Then [cf. the notation of Definition 1.1, (iii)] for $V_Q \ni v_Q$, by considering invariants with respect to the natural action of various open subgroups of the topological group $^\alpha \Pi_{V_Q}$, one may regard $\log(\mathcal{F}_{v_Q})$ as an inductive limit of topological modules, each of which is of finite dimension over $Q_{v_Q}$; we shall refer to the correspondence

$$V_Q \ni v_Q \mapsto \log(\mathcal{F}_{v_Q}) \overset{\text{def}}{=} \bigoplus_{V \ni v} \log(\mathcal{F}_{v})$$

as the [1]-tensor packet associated to the $F$-prime-strip $^\alpha \mathcal{F}$ and to the correspondence

$$V_Q \ni v_Q \mapsto \log(\mathcal{F}_{v_Q}) \overset{\text{def}}{=} \bigotimes_{\alpha \in A} \log(\mathcal{F}_{v_Q}) = \bigoplus_{\{v_{\alpha}\}_{\alpha \in A}} \left\{ \bigotimes_{\alpha} \log(\mathcal{F}_{v_{\alpha}}) \right\}$$

— where the tensor products are to be understood as tensor products of ind-topological modules [i.e., as discussed above], and the direct sum is over all collections $\{v_{\alpha}\}_{\alpha \in A}$ of [not necessarily distinct!] elements $v_{\alpha} \in V$ lying over $v_Q$ and indexed by $\alpha \in A$. —
as the \([n\text{-}t]\)ensor packet associated to the collection of \(\mathcal{F}\)-prime-strips \(\{\mathcal{F}_\alpha\}_{\alpha \in A}\). Then:

(i) (Ring Structures) The ind-topological field structures on the various \(\log(\alpha \mathcal{F}_v)\) [cf. Definition 1.1, (i), (ii), (iii)], for \(\alpha \in A\), determine an ind-topological ring structure on \(\log(A \mathcal{F}_{v_Q})\) with respect to which \(\log(A \mathcal{F}_{v_Q})\) may be regarded as an inductive limit of direct sums of ind-topological fields. Such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on \(\log(A \mathcal{F}_{v_Q})\) and, moreover, are compatible, for \(\alpha \in A\), with the natural action of the topological group \(\alpha \Pi_v\) [where \(V \ni v \mid v_Q\)] on the direct summand with subscript \(v\) of the factor labeled \(\alpha\).

(ii) (Integral Structures) Fix elements \(\alpha \in A\), \(v \in V\), \(v_Q \in V_Q\) such that \(v \mid v_Q\). Relative to the tensor product in the above definition of \(\log(\alpha \mathcal{F}_v)\), write

\[
\log(A, \alpha \mathcal{F}_v) \overset{\text{def}}{=} \log(\alpha \mathcal{F}_v) \otimes \left\{ \bigotimes_{\beta \in A \setminus \{\alpha\}} \log(\beta \mathcal{F}_{v_Q}) \right\} \subseteq \log(A \mathcal{F}_{v_Q})
\]

for the ind-topological submodule determined by the tensor product of the factors labeled by \(\beta \in A \setminus \{\alpha\}\) with the tensor product of the direct summand with subscript \(v\) of the factor labeled \(\alpha\). Then \(\log(A, \alpha \mathcal{F}_v)\) forms a direct summand of the ind-topological ring \(\log(A \mathcal{F}_{v_Q})\); \(\log(A, \alpha \mathcal{F}_v)\) may be regarded as an inductive limit of direct sums of ind-topological fields; such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on \(\log(A, \alpha \mathcal{F}_v)\). Moreover, by forming the tensor product with “1’s” in the factors labeled by \(\beta \in A \setminus \{\alpha\}\), one obtains a natural injective homomorphism of ind-topological rings

\[
\log(\alpha \mathcal{F}_v) \to \log(A, \alpha \mathcal{F}_v)
\]

that, for suitable choices [which are, in fact, cofinal] of objects appearing in the inductive limit descriptions given above for the domain and codomain, induces an isomorphism of such an object in the domain onto each of the direct summand ind-topological fields of the object in the codomain. In particular, the integral structure

\[
\overline{\Psi}_{\log(\alpha \mathcal{F}_v)} \overset{\text{def}}{=} \Psi_{\log(\alpha \mathcal{F}_v)} \cup \{0\} \subseteq \log(\alpha \mathcal{F}_v)
\]

[cf. the notation of Definition 1.1, (i), (ii)] determines integral structures on each of the direct summand ind-topological fields that appear in the inductive limit descriptions of \(\log(A, \alpha \mathcal{F}_v)\), \(\log(A \mathcal{F}_{v_Q})\).

Proof. The various assertions of Proposition 3.1 follow immediately from the definitions and the references quoted in the statements of these assertions [cf. also Remark 3.1.1. (i), below]. \(\Box\)

Remark 3.1.1.

(i) Let \(v \in V\). In the notation of [IUTchI], Definition 3.1, write \(k \overset{\text{def}}{=} K_v\); let \(\overline{k}\) be an algebraic closure of \(k\). Then, roughly speaking, in the notation of Proposition 3.1,

\[
\log(\alpha \mathcal{F}_v) \Rightarrow \overline{k}; \quad \overline{\Psi}_{\log(\alpha \mathcal{F}_v)} \Rightarrow \mathcal{O}_{\overline{k}};
\]
\[ \log^{(A, \alpha) \mathcal{F}_w} \xrightarrow{\sim} \bigotimes_k \mathcal{F}_w \xrightarrow{\sim} \lim_{\to} \bigoplus_k \mathcal{O}_k \supseteq \lim_{\to} \bigoplus \mathcal{O}_k \]

— i.e., one verifies immediately that each ind-topological field \( \log^{(A, \alpha) \mathcal{F}_w} \) is isomorphic to \( \mathcal{F}_w \); each \( \log^{(A, \alpha) \mathcal{F}_w} \) is a topological tensor product [say, over \( \mathbb{Q} \)] of copies of \( \mathcal{F}_w \), hence may be described as an inductive limit of direct sums of copies of \( \mathcal{F}_w \); each \( \mathcal{V}_{\log(v_{\alpha} \mathcal{F}_w)} \) is a copy of the set [i.e., a ring, when \( v \in \mathcal{V}_{\text{non}} \)] of integers \( \mathcal{O}_k \subseteq \mathcal{F}_w \).

In particular, the “integral structures” discussed in the final portion of Proposition 3.1, (ii), correspond to copies of \( \mathcal{O}_k \) contained in copies of \( \mathcal{F}_w \).

(ii) Ultimately, for \( v \in \mathcal{V} \), we shall be interested [cf. Proposition 3.9, (i), (ii), below] in considering log-volumes on the portion of \( \log^{(A, \alpha) \mathcal{F}_w} \) corresponding to \( K_w \). On the other hand, let us recall that we do not wish to consider all of the valuations in \( \mathcal{V}(K) \). That is to say, we wish to restrict ourselves to considering the subset \( \mathcal{V} \subseteq \mathcal{V}(K) \), equipped with the natural bijection \( \mathcal{V} \xrightarrow{\sim} \mathcal{V}_{\text{mod}} \) [cf. [IUTchI], Definition 3.1, (e)], which we wish to think of as a sort of “local analytic section” [cf. the discussion of [IUTchI], Remark 4.3.1, (i)] of the natural morphism \( \text{Spec}(K) \to \text{Spec}(F) \) [or, perhaps more precisely, \( \text{Spec}(K) \to \text{Spec}(F_{\text{mod}}) \)].

In particular, it will be necessary to consider these log-volumes on the portion of \( \log^{(A, \alpha) \mathcal{F}_w} \) corresponding to \( K_w \) relative to the weight \( \sum_{\mathcal{V} \supseteq v | v_q} (F_{\text{mod}})_v \). Here, we write \( v \in \mathcal{V}_{\text{mod}} \) for the element determined [via the natural bijection just discussed] by \( v \) [cf. the discussion of [IUTchI], Example 3.5, (i), (ii), (iii), where similar factors appear].

When, moreover, we consider direct sums over all \( v \in \mathcal{V} \) lying over a given \( v_q \in \mathcal{V}_Q \) as in the case of \( \log^{(A, \alpha) \mathcal{F}_w} \), it will be convenient to use the normalized weight

\[
\frac{1}{\sum_{\mathcal{V}} \prod_{\alpha} [K_{\mathcal{V}} : (F_{\text{mod}})_v]} \]

— i.e., normalized so that multiplication by \( p_{v_q} \) affects log-volumes by addition or subtraction [that is to say, depending on whether \( v_q \in \mathcal{V}_{\text{arc}} \) or \( v_q \in \mathcal{V}_{\text{non}} \)] of the quantity \( \log(p_{v_q}) \in \mathbb{R} \). In a similar vein, when we consider log-volumes on the portion of \( \log^{(A, \alpha) \mathcal{F}_w} \) corresponding to the tensor product of various \( K_w \), where \( \mathcal{V} \supseteq v_{\alpha} | v_q \), it will be necessary to consider these log-volumes relative to the weight

\[
\prod_{\alpha \in A} \frac{1}{[K_{\mathcal{V}_{\alpha}} : (F_{\text{mod}})_v]} \]

— where we write \( v_{\alpha} \in \mathcal{V}_{\text{mod}} \) for the element determined by \( v_{\alpha} \). When, moreover, we consider direct sums over all possible choices for the data \( \{v_{\alpha}\}_{\alpha \in A} \), it will be convenient to use the normalized weight

\[
\frac{1}{\sum_{\mathcal{V}} \prod_{\alpha} [K_{\mathcal{V}_{\alpha}} : (F_{\text{mod}})_v]} \cdot \left\{ \sum_{\{w_{\alpha}\}_{\alpha \in A}} \left( \prod_{\alpha \in A} [(F_{\text{mod}})_{w_{\alpha}} : Q_{v_q}] \right) \right\} \]

— where the sum is over all collections \( \{w_{\alpha}\}_{\alpha \in A} \) of [not necessarily distinct!] elements \( w_{\alpha} \in \mathcal{V}_{\text{mod}} \) lying over \( v_q \) and indexed by \( \alpha \in A \). Again, these normalized weights are normalized so that multiplication by \( p_{v_q} \) affects log-volumes by addition.
or subtraction [that is to say, depending on whether $v_Q \in V_Q^\text{arc}$ or $v_Q \in V_Q^\text{non}$] of the quantity $\log(p_{v_Q}) \in \mathbb{R}$.

(iii) In the discussion to follow, we shall, for simplicity, use the term “measure space” to refer to a locally compact Hausdorff topological space whose topology admits a countable basis, and which is equipped with a complete Borel measure in the sense of [Royden], Chapter 11, §1; [Royden], Chapter 14, §1. In particular, one may speak of the product measure space [cf. [Royden], Chapter 12, §4] of any finite nonempty collection of measure spaces. Then observe that care must be exercised when considering the various weighted sums of log-volumes discussed in (ii), since, unlike, for instance, the log-volumes discussed in [item (a) of] [AbsTopIII], Proposition 5.7, (i), (ii),

such weighted sums of log-volumes do not, in general, arise as some positive real multiple of the [natural] logarithm of a “volume” or “measure” in the usual sense of measure theory.

In particular, when considering direct sums of the sort that appear in the second or third displays of the statement of Proposition 3.1, although it is clear from the definitions how to compute a weighted sum of log-volumes of the sort discussed in (ii) in the case of a region that arises as a direct product of, say, compact subsets of positive measure in each of the direct summands [i.e., since the volume/measure of such a compact subset may be computed as the infimum of the volume/measure of the compact open subsets that contain it], it is not immediately clear from the definitions how to compute such a weighted sum of log-volumes in the case of more general regions. In the following, for ease of reference, let us refer to such a

region that arises as a direct product of compact subsets of positive measure in each of the direct summands as a direct product region

and to a

region that arises as a direct product of relatively compact subsets in each of the direct summands as a direct product pre-region.

Then we observe in the remainder of the present Remark 3.1.1 that although, in the present series of papers,

the regions that will actually be of interest in the development of the theory are, in fact, direct product [pre-]regions, in which case the computation of weighted sums of log-volumes is completely straightforward [cf. also the discussion of Remark 3.9.7, (ii), (iii), below],

in fact,

weighted sums of log-volumes of the sort discussed in (ii) may be computed for, say, arbitrary Borel sets by applying the elementary construction discussed in (iv) below.

Here, in the context of the situation discussed in the final portion of (ii), we note that this construction in (iv) below is applied relative to the following given data:

- the finite set “$V$” is taken to be the direct product

$$\prod_{\alpha \in A} V_{v_Q} \quad (\Rightarrow \prod_{\alpha \in A} (V_{\text{mod}})_{v_Q})$$
[where the subscript “\(v_Q\)” denotes the fiber over \(v_Q \in \mathbb{V}_Q\);]

· for “\(v \in V\)” the cardinality “\(N_v\)” is taken to be the product that appears in the discussion of (ii)

\[
\prod_{\alpha \in A} [K_{\Sigma_\alpha} : (F_{\text{mod}})_{\nu_\alpha}]
\]

[where we think of “\(v \in V\)” as a collection \(\{\Sigma_\alpha\}_{\alpha \in A}\) of elements of \(\mathbb{V}_v\) that lies over a collection \(\{v_\alpha\}_{\alpha \in A}\) of elements of \((\mathbb{V}_{\text{mod}})_{v_Q}\), while “\(M_v\)” is taken to be the \([\text{radial}, \text{ if } v_Q \in \mathbb{V}^\text{arc}\] portion of the direct summand in the third display of the statement of Proposition 3.1 indexed by \(v \in V\) that corresponds to the tensor product of the \(\{K_{\Sigma_\alpha}\}_{\alpha \in A}\).]

[By the “radial” portion of a topological tensor product of a finite collection of complex archimedean fields, we mean the direct product of the copies of \(\mathbb{R}_{>0}\) that arise by forming the quotients by the units [i.e., copies of \(\mathbb{S}^1\)] of each of the complex archimedean fields that appears in the direct sum of fields [cf. (i)] that arises from such a topological tensor product.] Then one verifies immediately that, in the case of “direct product regions” [as discussed above], the result of multiplying the [natural] logarithm of the “\(\mathcal{E}\)-weighted measure \(\mu_\mathcal{E}(-)\)” of (iv) by a suitable normalization factor [i.e., a suitable positive real number] yields the weighted sums of log-volumes discussed in (ii).

(iv) Let \(V\) a nonempty finite set; \(\mathcal{E} \overset{\text{def}}{=} \{E_v\}_{v \in V}\) a collection of nonempty finite sets; \(\mathcal{M} \overset{\text{def}}{=} \{(M_v, \mu_v)\}_{v \in V}\) a collection of nonempty measure spaces [cf. the discussion of (iii) above]. For \(v \in V\), write

\[
E \overset{\text{def}}{=} \prod_{v' \in V} E_{v'}; \quad E_{\not= v} \overset{\text{def}}{=} \prod_{v \not= v' \in V} E_{v'};
\]

\[
E \times V \to W \overset{\text{def}}{=} \prod_{v' \in V} E_{\not= v'} \times \{v'\} \to V
\]

— where the first arrow “\(\to\)” is defined by the condition that, for \(v' \in V\), it restricts to the natural projection \(E \times \{v'\} \to E_{\not= v'} \times \{v'\}\) on \(E \times \{v'\}\); the second arrow “\(\to\)” is defined by the condition that, for \(v' \in V\), it restricts to the natural projection \(E_{\not= v'} \times \{v'\} \to \{v'\}\) on \(E_{\not= v'} \times \{v'\}\). If \(W \ni w \mapsto v \in V\) via the natural surjection \(W \to V\) just discussed, then write \((M_w, \mu_w) \overset{\text{def}}{=} (M_v, \mu_v)\). If \(Z\) is a subset of \(W\) or \(V\), then we shall write

\[
M_Z \overset{\text{def}}{=} \prod_{z \in Z} M_z; \quad M_{E \times V} \overset{\text{def}}{=} \prod_{(e,v) \in E \times V} M_v = \prod_{e \in E} M_v;
\]

\[
(M_{E \times V} \ni) M_{E \times V} \overset{\text{def}}{=} \left\{ \{m_{e,v}\}_{(e,v) \in E \times V} \mid m_{e',v} = m_{e'',v}, \forall (e', e'') \in E \times E_{\not= v} E \subseteq E \times E \right\} \overset{\text{def}}{=} \sim M_W
\]

— where the bijection \(M_{E \times V} \sim M_W\) is the map induced by the various natural projections \(E \to E_{\not= v}\) that constitute the natural projection \(E \times V \to W\); this bijection \(M_{E \times V} \sim M_W\) is easily verified to be a homeomorphism. Thus, \(M_W\),
\(M_V\), and \(M_{E \times V}\) are equipped with natural product measure space structures; the bijection \(M_{E \times V} \sim M_W\), together with the measure space structure on \(M_W\), induces a measure space structure on \(M_{E \times V}\). In particular, if \(S \subseteq M_V\) is any Borel set, then the product

\[
\prod_{e \in E} S \subseteq M_{E \times V}
\]

is a Borel set of \(M_{E \times V}\); the intersection of this product with \(M_{E^* V}\)

\[
S_E \overset{\text{def}}{=} \left\{ \prod_{e \in E} S \right\} \cap M_{E^* V} \subseteq M_{E^* V}
\]

is a Borel set of \(M_{E^* V}\) (\(\sim M_W\)). Thus, in summary, for any Borel set \(S \subseteq M_V\), one may speak of the “\(E\)-weighted measure”

\[
\mu_E(S) = \mathbb{R}_{\geq 0} \bigcup \{+\infty\}
\]

of \(S\), i.e., the measure, relative to the measure space structure of \(M_{E^* V}\) (\(\sim M_W\)), of \(S_E\). Since, moreover, one verifies immediately that the above construction is functorial with respect to isomorphisms of the given data \((V, E, M)\), it follows immediately that, in fact, \(\mu_E(-)\) is completely determined by the cardinalities \(N \overset{\text{def}}{=} \{N_v\}_{v \in V}\) of the finite sets \(E = \{E_v\}_{v \in V}\), i.e., by the data \((V, N, M)\). Finally, we observe that when \(S \subseteq M_V\) is a “direct product region” [cf. the discussion of (iii)], i.e., a set of the form \(\prod_{v \in V} S_v\), where \(S_v \subseteq M_v\) is a compact subset of positive measure, then a straightforward computation reveals that

\[
\frac{1}{N_E} \cdot \mu_E^{\log}(S) = \sum_{v \in V} \frac{1}{N_v} \cdot \mu_v^{\log}(S_v)
\]

— where we write \(N_E = \prod_{v \in V} N_v\), and each superscript “\(\log\)” denotes the natural logarithm of the corresponding quantity without a superscript.

**Remark 3.1.2.** The constructions involving local holomorphic tensor packets given in Proposition 3.1 may be applied to the capsules that appear in the various \(F\)-prime-strip processions obtained by considering the evident \(F\)-prime-strip analogues [cf. [IUTchI], Remark 5.6.1; [IUTchI], Remark 6.12.1] of the holomorphic processions discussed in [IUTchI], Proposition 4.11, (i); [IUTchI], Proposition 6.9, (i).

**Proposition 3.2. (Local Mono-analytic Tensor Packets)** Let

\[
\{^\alpha D^+\}_{\alpha \in A} = \left\{ \{^\alpha D^+\}_{v \in V} \right\}_{\alpha \in A}
\]

be an \(n\)-capsule, with index set \(A\), of \(D^+\)-prime-strips [relative to the given initial \(\Theta\)-data — cf. [IUTchI], §0; [IUTchI], Definition 4.1, (iii)]. Then [cf. the notation of Proposition 1.2, (vi), (vii)] we shall refer to the correspondence

\[
\mathbb{V} \ni v q \mapsto \log\left(^\alpha D^+_{v q}\right) \overset{\text{def}}{=} \bigoplus_{V \ni v \mid v q} \log\left(^\alpha D^+_{v q}\right)
\]
as the \([1\text{-}]{\text{tensor packet}}\) associated to the \(D^{+}\)-prime-strip \(\alpha D^{+}\) and to the correspondence
\[
\forall Q \ni v_Q \mapsto \log(A D^{+}_{v_Q}) = \bigotimes_{\alpha \in A} \log(\alpha D^{+}_{v_Q})
\]
— where the tensor product is to be understood as a tensor product of ind-topological modules — as the \([n\text{-}]\text{tensor packet}\) associated to the collection of \(D^{+}\)-prime-strips \(\{\alpha D^{+}\}_{\alpha \in A}\). For \(\alpha \in A, v \in \mathbb{V}, v_Q \in \mathbb{V}_Q\) such that \(v \mid v_Q\), we shall write
\[
\log(A,\alpha D^{+}_{v_Q}) \subseteq \log(A D^{+}_{v_Q})
\]
for the ind-topological submodule determined by the tensor product of the factors labeled by \(\beta \in A \setminus \{\alpha\}\) with the tensor product of the direct summand with subscript \(v\) of the factor labeled \(\alpha\) [cf. Proposition 3.1, (ii)]. If the capsule of \(D^{+}\)-prime-strips \(\{\alpha D^{+}\}_{\alpha \in A}\) arises from a capsule of \(\mathcal{F}^{+\times \mu}\)-prime-strips
\[
\{\alpha \mathfrak{F}^{+\times \mu}\}_{\alpha \in A} = \left\{\{\alpha \mathfrak{F}^{+\times \mu}_{v}\}_{v \in \mathbb{V}}\right\}_{\alpha \in A}
\]
[relative to the given initial \(\Theta\)-data — cf. [IUTChI], §0; [IUTChII], Definition 4.9, (vii)], then we shall use similar notation to the notation just introduced concerning \(\{\alpha D^{+}\}_{\alpha \in A}\) to denote objects associated to \(\{\alpha \mathfrak{F}^{+\times \mu}\}_{\alpha \in A}\), i.e., by replacing \(\mathcal{D}^{+}\) in the above notational conventions by \(\mathcal{F}^{+\times \mu}\) [cf. also the notation of Proposition 1.2, (vi), (vii)]. Then:

(i) (Mono-analytic/Holomorphic Compatibility) Suppose that the capsule of \(D^{+}\)-prime-strips \(\{\alpha D^{+}\}_{\alpha \in A}\) arises from the capsule of \(\mathcal{F}\)-prime-strips \(\{\alpha \mathfrak{F}\}_{\alpha \in A}\) of Proposition 3.1; write \(\{\alpha \mathfrak{F}^{+\times \mu}\}_{\alpha \in A}\) for the capsule of \(\mathcal{F}^{+\times \mu}\)-prime-strips associated to \(\{\alpha \mathfrak{F}\}_{\alpha \in A}\). Then the poly-isomorphisms \(\log(\mathcal{I}^{\mathbb{D}^{+}_{v_Q}}) \supset \log(\mathcal{I}^{\mathcal{F}^{+\times \mu}_{v_Q}}) \supset \log(\mathcal{I}^{\mathcal{F}_{v_Q}})\)
\[
\mathcal{I}(\alpha D^{+}_{v_Q}) \subseteq \log(\alpha D^{+}_{v_Q}); \quad \mathcal{I}(A D^{+}_{v_Q}) \subseteq \log(A D^{+}_{v_Q}); \quad \mathcal{I}(\alpha \mathfrak{F}^{+\times \mu}_{v_Q}) \subseteq \log(\alpha \mathfrak{F}^{+\times \mu}_{v_Q}) \supset \log(A \mathfrak{F}_{v_Q})
\]

between the various “mono-analytic” tensor packets of the present Proposition 3.2 and the “holomorphic” tensor packets of Proposition 3.1.

(ii) (Integral Structures) If \(\forall \ni v \mid v_Q \in \mathbb{V}^{\mathrm{non}}_Q\), then the mono-analytic log-shells “\(\mathcal{I}^{\mathbb{D}^{+}_{v_Q}}\)” of Proposition 1.2, (vi), determine [i.e., by forming suitable direct sums and tensor products] topological submodules
\[
\mathcal{I}(\alpha D^{+}_{v_Q}) \subseteq \log(\alpha D^{+}_{v_Q}); \quad \mathcal{I}(A D^{+}_{v_Q}) \subseteq \log(A D^{+}_{v_Q}); \quad \mathcal{I}(\alpha \mathfrak{F}^{+\times \mu}_{v_Q}) \subseteq \log(\alpha \mathfrak{F}^{+\times \mu}_{v_Q})
\]
— which may be regarded as integral structures on the \(\mathbb{Q}\)-spans of these submodules. If \(\forall \ni v \mid v_Q \in \mathbb{V}^{\mathrm{res}}_Q\), then by regarding the mono-analytic log-shell “\(\mathcal{I}^{\mathbb{D}^{+}_{v_Q}}\)” of Proposition 1.2, (vii), as the “closed unit ball” of a Hermitian metric on “\(\log(\mathcal{I}^{\mathbb{D}^{+}_{v_Q}})\)” and considering the induced direct sum Hermitian metric on \(\log(\alpha D^{+}_{v_Q})\),
together with the induced tensor product Hermitian metric on \( \log(\mathcal{A}D_{v_Q}^+) \), one obtains Hermitian metrics on \( \log(\mathcal{A}D_{v_Q}^-) \), \( \log(\mathcal{A}D_{v_Q}^+ \mathcal{A}^\alpha \mathcal{D}_v^+) \), whose associated closed unit balls

\[ \mathcal{I}(\mathcal{A}D_{v_Q}^-) \subseteq \log(\mathcal{A}D_{v_Q}^+) ; \quad \mathcal{I}(\mathcal{A}D_{v_Q}^+) \subseteq \log(\mathcal{A}D_{v_Q}^+) ; \quad \mathcal{I}(\mathcal{A}^\alpha \mathcal{D}_v^+) \subseteq \log(\mathcal{A}^\alpha \mathcal{D}_v^+) \]

may be regarded as integral structures on \( \log(\mathcal{A}D_{v_Q}^-) \), \( \log(\mathcal{A}D_{v_Q}^+) \), and \( \log(\mathcal{A}^\alpha \mathcal{D}_v^+) \), respectively. For arbitrary \( \mathcal{V} \ni v \mid v_Q \in \mathcal{V}_Q \), we shall denote by “\( \mathcal{I} ((-) ) \)” the \( \mathcal{Q} \)-span of “\( \mathcal{I} ((-) ) \)”; also, we shall apply this notation involving “\( \mathcal{I} ((-) ) \)” with “\( \mathcal{D}^- \)” replaced by “\( \mathcal{F} \)” or “\( \mathcal{F} \times \mathcal{M} \)” for the various objects obtained from the “\( \mathcal{D}^- \)-versions” discussed above by applying the natural poly-isomorphisms of (i).

**Proof.** The various assertions of Proposition 3.2 follow immediately from the definitions and the references quoted in the statements of these assertions. \( \Box \)

**Remark 3.2.1.** The issue of estimating the discrepancy between the holomorphic integral structures of Proposition 3.1, (ii), and the mono-analytic integral structures of Proposition 3.2, (ii), will form one of the main topics to be discussed in [IUTchIV] — cf. also Remark 3.9.1 below.

**Remark 3.2.2.** The constructions involving local mono-analytic tensor packets given in Proposition 3.2 may be applied to the capsules that appear in the various \( \mathcal{D}^- \)-prime-strip processions — i.e., mono-analytic processions — discussed in [IUTchI], Proposition 4.11, (ii); [IUTchI], Proposition 6.9, (ii).

**Proposition 3.3. (Global Tensor Packets)** Let

\[ \dagger \mathcal{HT}^{\Theta_{\pm \text{ell}}} \]

be a \( \Theta_{\pm \text{ell}} \)-NF-Hodge theater [relative to the given initial \( \Theta \)-data] — cf. [IUTchI], Definition 6.13, (i). Thus, \( \dagger \mathcal{HT}^{\Theta_{\pm \text{ell}}} \) determines \( \Theta_{\text{NF}} \)- and \( \Theta_{\pm \text{ell}} \)-Hodge theaters \( \dagger \mathcal{HT}^{\Theta_{\text{NF}}} \), \( \dagger \mathcal{HT}^{\Theta_{\pm \text{ell}}} \) as in [IUTchII], Corollary 4.8. Let \( \{ \alpha \mathcal{F} \} \alpha \in A \) be an n-capsule of \( \mathcal{F} \)-prime-strips as in Proposition 3.1. Suppose, further, that \( A \) is a subset of the index set \( J \) that appears in the \( \Theta_{\text{NF}} \)-Hodge theater \( \dagger \mathcal{HT}^{\Theta_{\text{NF}}} \), and that, for each \( \alpha \in A \), we are given a log-link

\[ \alpha \mathcal{F} \xrightarrow{\log} \dagger \mathcal{F}_\alpha \]

— i.e., a poly-isomorphism of \( \mathcal{F} \)-prime-strips \( \log(\alpha \mathcal{F}) \xrightarrow{\sim} \dagger \mathcal{F}_\alpha \) [cf. Definition 1.1, (iii)]. Next, recall the field \( \dagger \mathcal{M}^{\text{mod}} \) discussed in [IUTchII], Corollary 4.8, (i); thus, one also has, for \( j \in J \), a labeled version \( \dagger \mathcal{M}^{\text{mod}}_j \) of this field [cf. [IUTchII], Corollary 4.8, (ii)]. We shall refer to

\[ (\dagger \mathcal{M}^{\text{mod}}_{\text{mod}})_A \overset{\text{def}}{=} \bigotimes_{\alpha \in A} (\dagger \mathcal{M}^{\text{mod}}_{\text{mod}})_{\alpha} \]
(i) (Ring Structures) The field structure on the various \((\hat{^\dagger M}_\text{mod})_\alpha\), for \(\alpha \in A\), determine a ring structure on \((\hat{^\dagger M}_\text{mod})_A\) with respect to which \((\hat{^\dagger M}_\text{mod})_A\) decomposes, uniquely, as a direct sum of number fields. Moreover, the various localization functors \(\mathcal{F}_j \rightarrow \hat{^\dagger}_j\) considered in [IUTchII], Corollary 4.8, (iii), determine, by composing with the given log-links, a natural injective localization ring homomorphism

\[
(\hat{^\dagger M}_\text{mod})_A \rightarrow \log^j(A_{\mathcal{V}_Q}) \overset{\text{def}}{=} \prod_{v_Q \in \mathcal{V}_Q} \log^j(A_{v_Q})
\]

to the product of the local holomorphic tensor packets considered in Proposition 3.1.

(ii) (Integral Structures) Fix an element \(\alpha \in A\). Then by forming the tensor product with \(\hat{^\dagger}'\)'s in the factors labeled by \(\beta \in A \setminus \{\alpha\}\), one obtains a natural ring homomorphism

\[
(\hat{^\dagger M}_\text{mod})_\alpha \rightarrow (\hat{^\dagger M}_\text{mod})_A
\]

that induces an isomorphism of the domain onto a subfield of each of the direct summand number fields of the codomain. For each \(v_Q \in \mathcal{V}_Q\), this homomorphism is compatible, in the evident sense, relative to the localization homomorphism of (i), with the natural homomorphism of ind-topological rings considered in Proposition 3.1, (ii). Moreover, for each \(v_Q \in \mathcal{V}_Q^{\text{non}}\), the composite of the above displayed homomorphism with the component at \(v_Q\) of the localization homomorphism of (i) maps the ring of integers of the number field \((\hat{^\dagger M}_\text{mod})_\alpha\) into the submodule constituted by the integral structure on \(\log^j(A_{\mathcal{V}_Q})\) considered in Proposition 3.1, (ii); for each \(v_Q \in \mathcal{V}_Q^{\text{arc}}\), the composite of the above displayed homomorphism with the component at \(v_Q\) of the localization homomorphism of (i) maps the set of archimedean integers [i.e., elements of absolute value \(\leq 1\) at all archimedean primes] of the number field \((\hat{^\dagger M}_\text{mod})_\alpha\) into the direct product of subsets constituted by the integral structures considered in Proposition 3.1, (ii), on the various direct summand ind-topological fields of \(\log^j(A_{\mathcal{V}_Q})\).

Proof. The various assertions of Proposition 3.3 follow immediately from the definitions and the references quoted in the statements of these assertions.

Remark 3.3.1. One may perform analogous constructions to the constructions of Proposition 3.3 for the fields \(\hat{^\dagger}_{\text{mod}}^{D,\Theta^\pm\text{ell}NF}_j\) of [IUTchII], Corollary 4.7, (ii) [cf. also the localization functors of [IUTchII], Corollary 4.7, (iii)], constructed from the associated \(D,\Theta^\pm\text{ell}NF\)-Hodge theater \(\hat{\mathcal{H}}^{D,\Theta^\pm\text{ell}NF}\). These constructions are compatible with the corresponding constructions of Proposition 3.3, in the evident sense, relative to the various labeled Kummer-theoretic isomorphisms of [IUTchII], Corollary 4.8, (ii). We leave the routine details to the reader.
Remark 3.3.2.

(i) One may consider the image of the localization homomorphism of Proposition 3.3, (i), in the case of the various local holomorphic tensor packets arising from processions, as discussed in Remark 3.1.2. Indeed, at the level of the labels involved, this is immediate in the case of the \( \mathbb{F}_l^* \)-processions of [IUTchI], Proposition 4.11, (i). On the other hand, in the case of the \( |\mathbb{F}_l| \)-processions of [IUTchI], Proposition 6.9, (i), this may be achieved by applying the identifying isomorphisms between the zero label \( 0 \in |\mathbb{F}_l| \) and the diagonal label \( \langle \mathbb{F}_l^* \rangle \) associated to \( \mathbb{F}_l^* \) discussed in [the final display of] [IUTchII], Corollary 4.6, (iii) [cf. also [IUTchII], Corollary 4.8, (ii)].

(ii) In a similar vein, one may compose the \( \mathcal{D}-\Theta^{\pm\ell} \) HF-Hodge theater version" discussed in Remark 3.3.1 of the localization homomorphism of Proposition 3.3, (i), with the product over \( v_Q \in V^Q \) of the inverses of the upper right-hand displayed isomorphisms at \( v_Q \) of Proposition 3.2, (i), and then consider the image of this composite morphism in the case of the various local mono-analytic tensor packets arising from processions, as discussed in Remark 3.2.2. Just as in the holomorphic case discussed in (i), in the case of the \( |\mathbb{F}_l| \)-processions of [IUTchI], Proposition 6.9, (ii), this obliges one to apply the identifying isomorphisms between the zero label \( 0 \in |\mathbb{F}_l| \) and the diagonal label \( \langle \mathbb{F}_l^* \rangle \) associated to \( \mathbb{F}_l^* \) discussed in [the final display of] [IUTchII], Corollary 4.5, (iii).

(iii) The various images of global tensor packets discussed in (i) and (ii) above may be identified — i.e., in light of the injectivity of the homomorphisms applied to construct these images — with the global tensor packets themselves. These local holomorphic/local mono-analytic global tensor packet images will play a central role in the development of the theory of the present §3 [cf., e.g., Proposition 3.7, below].

Remark 3.3.3. The log-shifted nature of the localization homomorphism of Proposition 3.3, (i), will play a crucial role in the development of the theory of present §3 — cf. the discussion of [IUTchII], Remark 4.8.2, (i), (iii).

\[
\begin{align*}
\mathbb{F}_l^* & \rightarrow \mathbb{F}_l^{(\ell)*} \\
\mathbb{F}_l^{(\ell)*} & \rightarrow \mathbb{F}_l^{(\ell)*} \\
\mathbb{F}_l^{(\ell)*} & \rightarrow \mathbb{F}_l^{(\ell)*} \\
\mathbb{F}_l^{(\ell)*} & \rightarrow \mathbb{F}_l^{(\ell)*}
\end{align*}
\]

\[
\begin{align*}
\mathbb{S}_1 & \rightarrow \mathbb{S}_{1+1} \\
\mathbb{S}_{1+1} & \rightarrow \mathbb{S}_{1+1} \\
\mathbb{S}_{1+1} & \rightarrow \mathbb{S}_{1+1} \\
\mathbb{S}_{1+1} & \rightarrow \mathbb{S}_{1+1}
\end{align*}
\]

Fig. 3.1: Splitting monoids of LGP-monoids acting on tensor packets

Proposition 3.4. (Local Packet-theoretic Frobenioids)

(i) (Single Packet Monoids) In the situation of Proposition 3.1, fix elements \( \alpha \in A, \nu \in \overline{V}, v_Q \in \mathbb{V}_Q \) such that \( \overline{\nu} \mid v_Q \). Then the operation of forming the image via the natural homomorphism \( \log(A,\alpha,F_2) \rightarrow \log(\alpha,F_2) \) [cf. Proposition 3.1, (ii)]
of the monoid \(\Psi_{\log}(\alpha F_v)\) [cf. the notation of Definition 1.1, (i), (ii)], together with its submonoid of units \(\Psi^X_{\log}(\alpha F_v)\) and realification \(\Psi^R_{\log}(\alpha F_v)\), determines monoids

\[
\Psi_{\log}(A,\alpha F_v), \quad \Psi^X_{\log}(A,\alpha F_v), \quad \Psi^R_{\log}(A,\alpha F_v)
\]

— which are equipped with \(G_v(\alpha)\)-actions when \(v \in \mathbb{V}^{\text{non}}\) and, in the case of the first displayed monoid, with a pair consisting of an \(\text{Aut}\)-holomorphic orbispace and a Kummer structure when \(v \in \mathbb{V}^{\text{arc}}\). We shall think of these monoids as [possibly realified] subquotients of

\[
\log(A,\alpha F_v)
\]

that act [multiplicatively] on suitable [possibly realified] subquotients of \(\log(A,\alpha F_v)\).

In particular, when \(v \in \mathbb{V}^{\text{non}}\), the first displayed monoid, together with its \(\alpha\Pi_v\)-action, determine a Frobenioid equipped with a natural isomorphism to \(\log(A,F_v)\); when \(v \in \mathbb{V}^{\text{arc}}\), the first displayed monoid, together with its Aut-holomorphic orbispace and Kummer structure, determine a collection of data equipped with a natural isomorphism to \(\log(A,F_v)\).

(ii) (Local Logarithmic Gaussian Procession Monoids) Let

\[
\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}} \xrightarrow{\log} \mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}}
\]

be a log-link of \(\Theta^{\pm\text{ell}}_{\text{NF}}\)-Hodge theaters as in Proposition 1.3, (i) [cf. also the situation of Proposition 3.3]. Consider the \(\mathcal{F}\)-prime-strip processations that arise as the \(\mathcal{F}\)-prime-strip analogues [cf. Remark 3.1.2; [IUTchI], Remark 6.12.1] of the holomorphic processions discussed in [IUTchI], Proposition 6.9, (i), when the functor of [IUTchI], Proposition 6.9, (i), is applied to the \(\Theta^{\pm}\)-bridges associated to \(\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}}, \mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}}\); we shall refer to such processions as \(\pm\frac{1}{2}\) or \(\pm\frac{3}{2}\) processions. Here, we recall that for \(j \in \{1,\ldots, l\}^\ast\), the index set of the \((j+1)\)-capsule that appears in such a procession is denoted \(S_{j+1}^\pm\). Then by applying the various constructions of “single packet monoids” given in (i) in the case of the various capsules of \(\mathcal{F}\)-prime-strips that appear in a holomorphic \(\pm\frac{1}{2}\)-procession — i.e., more precisely, in the case of the label \(j \in \{1,\ldots, l\}^\ast\) [which we shall occasionally identify with its image in \(\mathbb{F}_l^\ast \subseteq \vert \mathbb{F}_l \vert\)] that appears in the \((j+1)\)-capsule of the \(\pm\frac{1}{2}\)-procession — to the pull-backs, via the poly-isomorphisms that appear in the definition [cf. Definition 1.1, (iii)] of the given log-link, of the [collections of] monoids equipped with actions by topological groups when \(v \in V^{\text{non}}\) and splittings [up to torsion, when \(v \in V^{\text{bad}}\)] \(\Psi_{\log}(\mathfrak{F}_{\text{gau}})(\mathfrak{HT}^\Theta_{\text{gau}})_{\log} \Psi_{\log}(\mathfrak{F}_{\text{gau}})(\mathfrak{HT}^\Theta_{\text{gau}})_{\log}\) of [IUTchII], Corollary 4.6, (iv), for \(v \in V\), one obtains a functorial algorithm in the log-link of \(\Theta^{\pm\text{ell}}_{\text{NF}}\)-Hodge theaters \(\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}} \xrightarrow{\log} \mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}}\) for constructing [collections of] monoids equipped with actions by topological groups when \(v \in V^{\text{non}}\) and splittings [up to torsion, when \(v \in V^{\text{bad}}\)]

\[
\forall v \ni v \quad \Psi_{\log}(\mathfrak{F}_{\text{gau}})(\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}})_{\log} \quad \forall v \ni v \quad \Psi_{\log}(\mathfrak{F}_{\text{gau}})(\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}})_{\log}
\]

— which we refer to as “[local] LGP-monoids”, or “logarithmic Gaussian procession monoids” [cf. Fig. 3.1 above]. Here, we note that the notation “(\(\mathfrak{HT}^\Theta^{\pm\text{ell}}_{\text{NF}}\)”,
constitutes a slight abuse of notation. Also, we note that this functorial algorithm requires one to apply the compatibility of the given log-link with the $\mathbb{P}_t^{x\pm}$-symmetrizing isomorphisms involved [cf. Remark 1.3.2]. For $v \in \mathbb{V}^{\text{bad}}$, the component labeled $j \in \{1, \ldots, l^*\}$ of the submodule of Galois invariants [cf. (i)] of the entire LGP-monoid $\Psi_{\mathcal{F}_{\mathcal{LG}_P}}(\mathcal{H}_T^{G^{\pm,NF}})_v$ is a subset of

$$\mathcal{T}_Q(\mathbb{S}_{j+1}^{\pm,j+1} \mathcal{F}_v)$$

[i.e., where the notation “$;.$” denotes the result of applying the discussion of (i) to the case of $\mathcal{F}$-prime-strips labeled “$;.$”; cf. also the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on $\mathcal{T}_Q(\mathbb{S}_{j+1}^{\pm,j+1} \mathcal{F}_v)$ [cf. the constructions of [IUTchII], Corollary 3.6, (ii)]. For any $v \in \mathbb{V}$, the component labeled $j \in \{1, \ldots, l^*\}$ of the submodule of Galois invariants [cf. (i) when $v \in \mathbb{V}^{\text{mon}}$; this Galois action is trivial when $v \in \mathbb{V}^{\text{arc}}$] of the unit portion $\Psi_{\mathcal{F}_{\mathcal{LG}_P}}(\mathcal{H}_T^{G^{\pm,NF}})_v$ of such an LGP-monoid is a subset of

$$\mathcal{T}_Q(\mathbb{S}_{j+1}^{\pm,j+1} \mathcal{F}_v)$$

[cf. the discussion of (i); the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on $\mathcal{T}_Q(\mathbb{S}_{j+1}^{\pm,j+1} \mathcal{F}_v)$ [cf. the constructions of [IUTchII], Corollary 3.6, (ii); [IUTchII], Proposition 4.2, (iv); [IUTchII], Proposition 4.4, (iv)].

Proof. The various assertions of Proposition 3.4 follow immediately from the definitions and the references quoted in the statements of these assertions.

Proposition 3.5. (Kummer Theory and Upper Semi-compatibility for Vertically Coric Local LGP-Monoids) Let $\{n,m \mathcal{H}_T^{G^{\pm,NF}}\}_{n,m \in \mathbb{Z}}$ be a collection of distinct $G^{\pm,NF}$-Hodge theaters [relative to the given initial $G$-data] — which we think of as arising from a Gaussian log-theta-lattice [cf. Definition 1.4]. For each $n \in \mathbb{Z}$, write

$$n,^o \mathcal{H}_T^{D, G^{\pm,NF}}$$

for the $D-G^{\pm,NF}$-Hodge theater determined, up to isomorphism, by the various $n,m \mathcal{H}_T^{G^{\pm,NF}}$, where $m \in \mathbb{Z}$, via the vertical coricity of Theorem 1.5, (i).

(i) (Vertically Coric Local LGP-Monoids and Associated Kummer Theory) Write

$$\mathfrak{F}(^{n,^o,^o D})_t$$

for the $\mathcal{F}$-prime-strip associated [cf. [IUTchII], Remark 4.5.1, (i)] to the labeled collection of monoids “$\Psi_{\mathfrak{ms}}(^{n,^o,^o D})_t$” of [IUTchII], Corollary 4.5, (iii) [i.e., where we take “$;.$” to be “$;.$”]. Recall the constructions of Proposition 3.4, (ii), involving $\mathcal{F}$-prime-strip processions. Then by applying these constructions to the $\mathcal{F}$-prime-strips “$\mathfrak{F}(^{n,^o,^o D})_t$” and the various full log-links associated [cf. the discussion of Proposition 1.2, (ix)] to these $\mathcal{F}$-prime-strips — which we consider in a fashion compatible with the $\mathbb{P}_t^{x\pm}$-symmetries involved [cf. Remark 1.3.2; Proposition
3.4, (ii)] — we obtain a functorial algorithm in the \(D-\Theta^{\pm\text{ell}}\text{NF-Hodge theater} n,\circ \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}}\) for constructing [collections of] monoids

\[ \forall \nu \ni \nu \mapsto \Psi_{\text{LGP}}(n,\circ \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu}, \quad \forall \nu \ni \nu \mapsto \infty \Psi_{\text{LGP}}(n,\circ \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu} \]

equipped with actions by topological groups when \(\nu \in \mathcal{V}^{\text{non}}\) and splittings [up to torsion, when \(\nu \in \mathcal{V}^{\text{bad}}\)] — which we refer to as “vertically coric [local] LGP-monoids”. For each \(n,m \in \mathbb{Z}\), this functorial algorithm is compatible [in the evident sense] with the functorial algorithm of Proposition 3.4, (ii) — i.e., where we take “\(\dagger\)” to be “\(n,m\)” and “\(\ddagger\)” to be “\(n,m-1\)” — relative to the Kummer isomorphisms of labeled data

\[ \Psi_{\text{cns}}(n,m; \mathcal{F}_{\nu})_{t} \sim \rightarrow \Psi_{\text{cns}}(n,\circ \mathcal{D}_{\nu})_{t} \]

of [IUTchII], Corollary 4.6, (iii), and the evident identification, for \(m' = m, m-1\), of \(n,m; \mathcal{F}_{t}^{\nu}\) [i.e., the \(\mathcal{F}\)-prime-strip that appears in the associated \(\Theta^{\pm}\)-bridge] with the \(\mathcal{F}\)-prime-strip associated to \(\Psi_{\text{cns}}(n,m; \mathcal{F}_{t})_{\nu}\). In particular, for each \(n,m \in \mathbb{Z}\), we obtain Kummer isomorphisms of [collections of] monoids

\[ \Psi_{\mathcal{F}_{\text{LGP}}}(n,m; \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu} \sim \rightarrow \Psi_{\text{LGP}}(n,\circ \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu} \]

\[ \infty \Psi_{\mathcal{F}_{\text{LGP}}}(n,m; \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu} \sim \rightarrow \infty \Psi_{\text{LGP}}(n,\circ \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}})_{\nu} \]

equipped with actions by topological groups when \(\nu \in \mathcal{V}^{\text{non}}\) and splittings [up to torsion, when \(\nu \in \mathcal{V}^{\text{bad}}\)], for \(\nu \in \mathcal{V}\).

(ii) (Upper Semi-compatibility) The Kummer isomorphisms of the final two displays of (i) are “upper semi-compatible” — cf. the discussion of “upper semi-commutativity” in Remark 1.2.2, (iii) — with the various \(\text{log-links of} \ \Theta^{\pm\text{ell}}\text{NF-Hodge theaters} n,m-1; \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}} \longrightarrow \mathcal{H}^{D-\Theta^{\pm\text{ell}}\text{NF}}\) [where \(m \in \mathbb{Z}\)] of the Gaussian log-theta-lattice under consideration in the following sense. Let \(j \in \{0,1,\ldots,l\}\). Then:

(a) (Nonarchimedean Primes) For \(\nu_{Q} \in \mathcal{V}^{\text{non}}\), the topological module

\[ \mathcal{I}^{\oplus_{j+1}\mathcal{F}(n,\circ \mathcal{D}_{\nu})_{\nu_{Q}}} \]

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow “\(j\)” to be 0], in the vertically coric context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] — contains the images of the submodules of Galois invariants [where we recall the Galois actions that appear in the data of [IUTchII], Corollary 4.6, (i), (iii)] of the groups of units \((\Psi_{\text{cns}}(n,m; \mathcal{F}_{t})_{|t|})_{\nu_{Q}}, \text{ for } \forall \nu \ni \nu_{Q}\) and \(|t| \in \{0,\ldots,j\}\), via both

1. the tensor product, over such \(|t|\), of the [relevant] Kummer isomorphisms of (i), and

2. the tensor product, over such \(|t|\), of the pre-composite of these Kummer isomorphisms with the \(m'\)-th iterates [cf. Remark
1.1.1] of the \( \log \)-links, for \( m' \geq 1 \), of the \( n \)-th column of the Gaussian \( \log \)-theta-lattice under consideration [cf. the discussion of Remark 1.2.2, (i), (iii)].

(b) (Archimedean Primes) For \( v_Q \in \mathcal{V}^{arc} \), the closed unit ball

\[
\mathcal{I}(\mathcal{G}_j^{+} \mathcal{F}(n,0)\mathcal{D}_\nu)_{v_Q}
\]

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow “\( j \)” to be 0], in the \textit{vertically coric} context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] — contains the image, via the tensor product, over \( |t| \in \{0,\ldots,j\} \), of the \([relevant\) Kummer isomorphisms of (i), of both

(1) the groups of units \( (\Psi_{\text{cns}}(n,m)\mathcal{F}_\nu)_{\mathcal{V}} \), for \( \mathcal{V} \ni \nu \mid v_Q \), and

(2) the closed balls of radius \( \pi \) inside \( (\Psi_{\text{cns}}(n,m)\mathcal{F}_\nu)_{\mathcal{V}} \sim \mathbb{G}_\nu \) [cf. the

notational conventions of Definition 1.1], for \( \mathcal{V} \ni \nu \mid v_Q \).

Here, we recall from the discussion of Remark 1.2.2, (ii), (iii), that, if we regard each \( \log \)-link as a correspondence that only concerns the units that appear in its domain [cf. Remark 1.1.1], then a closed ball as in (2) contains, for each \( m' \geq 1 \), a subset that \textit{surjects}, via the \( m' \)-th iterate of the \( \log \)-link of the \( n \)-th column of the Gaussian \( \log \)-theta-lattice under consideration, onto the subset of the group of units \( (\Psi_{\text{cns}}(n,m-m')\mathcal{F}_\nu)_{\mathcal{V}} \times \) on which this iterate is defined.

(c) (Bad Primes) Let \( \nu \in \mathcal{V}^{\text{bad}} \); suppose that \( j \neq 0 \). Recall that the various monoids “\( \Psi_{\mathcal{F}_{\mathcal{LGP}}}(-)_{\nu} \)” \( \Psi_{\mathcal{F}_{\mathcal{LGP}}}(-)_{\nu} \)” constructed in Proposition 3.4, (ii), as well as the monoids “\( \Psi_{\mathcal{LGP}}(-)_{\nu} \)” \( \Psi_{\mathcal{LGP}}(-)_{\nu} \)” constructed in (i) above, are equipped with natural \textit{splittings up to torsion}. Write

\[
\Psi_{\mathcal{F}_{\mathcal{LGP}}}(-)_{\nu} \subseteq \Psi_{\mathcal{F}_{\mathcal{LGP}}}(-)_{\nu} \quad \Psi_{\mathcal{LGP}}(-)_{\nu} \subseteq \Psi_{\mathcal{LGP}}(-)_{\nu}
\]

for the submonoids corresponding to these splittings [cf. the submonoids “\( \mathcal{O}(-) \subseteq \mathcal{O}(\cdot) \)” discussed in Definition 2.4, (i), in the case of “\( \Psi(-) \); the notational conventions of Theorem 2.2, (ii), in the case of “\( \Psi(-) \).”]

[Thus, the subgroup of units of “\( \Psi(-) \)” consists of the \( 2\ell \)-torsion subgroup of “\( \Psi \”, while the subgroup of units of “\( \Psi(-) \)” contains the entire torsion subgroup of “\( \Psi \).”] Then, as \( m \) ranges over the elements of \( \mathbb{Z} \), the actions, via the \([relevant\) Kummer isomorphisms of (i), of the various monoids \( \Psi_{\mathcal{F}_{\mathcal{LGP}}}((n,m)\mathcal{H}(\Theta_\tau^{\pm} \mathcal{L})(\nu) \subseteq \Psi_{\mathcal{F}_{\mathcal{LGP}}}((n,m)\mathcal{H}(\Theta_\tau^{\pm} \mathcal{L})(\nu)) \) on the

ind-topological modules

\[
\mathcal{I}^{Q}(\mathcal{G}_{j+1}^{+} \mathcal{F}(n,0)\mathcal{D}_\nu)\mathcal{V} \subseteq \mathcal{I}^{Q}(\mathcal{G}_{j+1}^{+} \mathcal{F}(n,0)\mathcal{D}_\nu)\mathcal{V}
\]

[where \( j = 1,\ldots,l^* \)] — i.e., that arise from applying the constructions of Proposition 3.4, (ii), in the \textit{vertically coric} context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] — are mutually
compatible, relative to the log-links of the $n$-th column of the Gaussian log-theta-lattice under consideration, in the sense that the only portions of these actions that are possibly related to one another via these log-links are the indeterminacies with respect to multiplication by roots of unity in the domains of the log-links, that is to say, indeterminacies at $m$ that correspond, via the log-link, to “addition by zero” — i.e., to no indeterminacy! — at $m + 1$.

Now let us think of the submodules of Galois invariants [cf. the discussion of Proposition 3.4, (ii)] of the various groups of units, for $v \in \mathcal{V}$,

$$(\Psi_{\text{cns}}(n, m \mathfrak{F}_\Theta)_{|t|})^\times, \quad \Psi_{\mathcal{F}_{\text{LG}}}^{(n, m \mathcal{H} T \Theta^\pm \ell \mathfrak{F})|\mathcal{V}}$$

and the splitting monoids, for $v \in \mathcal{V}^{\text{bad}}$,

$$\Psi_{\mathcal{F}_{\text{LG}}}^{(n, m \mathcal{H} T \Theta^\pm \ell \mathfrak{F})|\mathcal{V}}$$

as acting on various portions of the modules, for $v_\mathcal{Q} \in \mathcal{V}_\mathcal{Q}$,

$$\mathcal{I}^{\mathcal{Q}}(n, o \mathcal{D}_\Theta |v_0)$$

not via a single Kummer isomorphism as in (i) — which fails to be compatible with the log-links of the Gaussian log-theta-lattice! — but rather via the totality of the various pre-composites of Kummer isomorphisms with iterates [cf. Remark 1.1.1] of the log-links of the Gaussian log-theta-lattice — i.e., precisely as was described in detail in (a), (b), (c) above [cf. also the discussion of Remark 3.11.4 below]. Thus, one obtains a sort of “log-Kummer correspondence” between the totality, as $m$ ranges over the elements of $\mathbb{Z}$, of the various groups of units and splitting monoids just discussed [i.e., which are labeled by \(n, m\)] and their actions [as just described] on the \(\mathcal{D}^{\mathbb{Q}}\) labeled by \(n, o\) which is invariant with respect to the translation symmetries [cf. Proposition 1.3, (iv)] of the $n$-th column of the Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].

Proof. The various assertions of Proposition 3.5 follow immediately from the definitions and the references quoted in the statements of these assertions. 

Example 3.6. Concrete Representations of Global Frobenioids. Before proceeding, we pause to take a closer look at the Frobenioid \(\mathcal{F}^{\mathbb{Q}}_{\text{mod}}\) of [IUTchI], Example 5.1, (iii), i.e., more concretely speaking, the Frobenioid of arithmetic line bundles on the stack “$S^\mathbb{Q}_{\text{mod}}$” of [IUTchI], Remark 3.1.5. Let us write

$$\mathcal{F}^{\mathbb{Q}}_{\text{mod}}$$

for the Frobenioid \(\mathcal{F}^{\mathbb{Q}}_{\text{mod}}\) of [IUTchI], Example 5.1, (iii), in the case where the data denoted by the label “\(\mathbb{Q}\)” arises [in the evident sense] from data as discussed in [IUTchI], Definition 3.1. In the following discussion, we shall use the notation of [IUTchI], Definition 3.1.
(i) (Rational Function Torsor Version) For each $v \in V$, the valuation on $K_v$ determined by $v$ determines a group homomorphism $\beta_v : F^\times_{\text{mod}} \to K_v^\times / \mathcal{O}_{K_v}^\times$ [cf. Remark 3.6.1 below]. Then let us define a category $\mathcal{F}^\otimes_{\text{MOD}}$ as follows. An object $\mathcal{T} = (T, \{t_v\}_{v \in V})$ of $\mathcal{F}^\otimes_{\text{MOD}}$ consists of a collection of data

(a) an $F^\times_{\text{mod}}$-torsor $T$;

(b) for each $v \in V$, a trivialization $t_v$ of the torsor $T_v$ obtained from $T$ by executing the "change of structure group" operation determined by the homomorphism $\beta_v$ subject to the condition that there exists an element $t \in T$ such that $t_v$ coincides with the trivialization of $T_v$ determined by $t$ for all but finitely many $v$. An elementary morphism $\mathcal{T}_1 = (T_1, \{t_{1,v}\}_{v \in V}) \to \mathcal{T}_2 = (T_2, \{t_{2,v}\}_{v \in V})$ between objects of $\mathcal{F}^\otimes_{\text{MOD}}$ is defined to be an isomorphism $T_1 \simeq T_2$ of $F^\times_{\text{mod}}$-torsors which is integral at each $v \in V$, i.e., maps the trivialization $t_{1,v}$ to an element of the $\mathcal{O}_{K_v}^\times$-orbit of $t_{2,v}$. There is an evident notion of composition of elementary morphisms, as well as an evident notion of tensor powers $\mathcal{T}^\otimes n$, for $n \in \mathbb{Z}$, of an object $\mathcal{T}$ of $\mathcal{F}^\otimes_{\text{MOD}}$. A morphism $\mathcal{T}_1 = (T_1, \{t_{1,v}\}_{v \in V}) \to \mathcal{T}_2 = (T_2, \{t_{2,v}\}_{v \in V})$ between objects of $\mathcal{F}^\otimes_{\text{MOD}}$ is defined to consist of a positive integer $n$ and an elementary morphism $(\mathcal{T}_1)^{\otimes n} \to \mathcal{T}_2$. There is an evident notion of composition of morphisms. Thus, $\mathcal{F}^\otimes_{\text{MOD}}$ forms a category. In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $\mathcal{F}^\otimes_{\text{MOD}}$ admits a natural Frobenioid structure [cf. [FrdI], Definition 1.3], for which the base category is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the linear morphisms, and the positive integer "$n" that appears in the definition of a morphism of $\mathcal{F}^\otimes_{\text{MOD}}$ is the Frobenius degree of the morphism. Moreover, by associating to an arithmetic line bundle on $S_{\text{mod}}$ the $F^\times_{\text{mod}}$-torsor determined by restricting the line bundle to the generic point of $S_{\text{mod}}$ and the local trivializations at $v \in V$ determined by the various local integral structures, one verifies immediately that there exists a natural isomorphism of Frobenioids

$$\mathcal{F}^\otimes_{\text{mod}} \simeq \mathcal{F}^\otimes_{\text{MOD}}$$

that induces the identity morphism $F^\times_{\text{mod}} \to F^\times_{\text{mod}}$ on the associated rational function monoids [cf. [FrdI], Corollary 4.10].

(ii) (Local Fractional Ideal Version) Let us define a category $\mathcal{F}^\otimes_{\text{mod}}$ as follows. An object

$$\mathcal{J} = \{J_v\}_{v \in V}$$

of $\mathcal{F}^\otimes_{\text{mod}}$ consists of a collection of "fractional ideals" $J_v \subseteq K_v$ for each $v \in V$ — i.e., a finitely generated nonzero $\mathcal{O}_{K_v}$-submodule of $K_v$ when $v \in V^{\text{non}}$, a positive real multiple of $\mathcal{O}_{K_v}$ defined as $\{\lambda \in K_v^\times \mid |\lambda| \leq 1\} \subseteq K_v$ when $v \in V^{\text{arc}}$ — such that $J_v = \mathcal{O}_{K_v}$ for all but finitely many $v$. If $\mathcal{J} = \{J_v\}_{v \in V}$ is an object of $\mathcal{F}^\otimes_{\text{mod}}$, then for any element $f \in F^\times_{\text{mod}}$, one obtains an object $f \cdot \mathcal{J} = \{f \cdot J_v\}_{v \in V}$ of $\mathcal{F}^\otimes_{\text{mod}}$ by multiplying each of the fractional ideals $J_v$ by $f$. Moreover, if $\mathcal{J} = \{J_v\}_{v \in V}$ is an object of $\mathcal{F}^\otimes_{\text{mod}}$, then for any $n \in \mathbb{Z}$, there is an evident notion of the $n$-th tensor power $\mathcal{J}^{\otimes n}$ of $\mathcal{J}$. An elementary morphism $\mathcal{J}_1 = \{J_{1,v}\}_{v \in V} \to \mathcal{J}_2 = \{J_{2,v}\}_{v \in V}$ between objects of $\mathcal{F}^\otimes_{\text{mod}}$ is...
defined to be an element $f \in F_{\text{mod}}^\times$ that is integral with respect to $J_1$ and $J_2$ in the sense that $f \cdot J_{1,v} \subseteq J_{2,v}$ for each $v \in \mathbb{V}$. There is an evident notion of composition of elementary morphisms. A morphism $J_1 = \{J_{1,v}\}_{v \in \mathbb{V}} \to J_2 = \{J_{2,v}\}_{v \in \mathbb{V}}$ between objects of $F_{\text{mod}}^\oplus$ is defined to consist of a positive integer $n$ and an elementary morphism $(J_1)^{\otimes n} \to J_2$. There is an evident notion of composition of morphisms. Thus, $F_{\text{mod}}^\oplus$ forms a category. In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $F_{\text{mod}}^\oplus$ admits a natural Frobenioid structure [cf. [FrdI], Definition 1.3], for which the base category is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the linear morphisms, and the positive integer “$n$” that appears in the definition of a morphism of $F_{\text{mod}}^\oplus$ is the Frobenius degree of the morphism. Moreover, by associating to an object $J = \{J_v\}_{v \in \mathbb{V}}$ of $F_{\text{mod}}^\oplus$ the arithmetic line bundle on $S_{\text{mod}}$ obtained from the trivial arithmetic line bundle on $S_{\text{mod}}$ by modifying the integral structure of the trivial line bundle at $v \in \mathbb{V}$ in the fashion prescribed by $J_v$, one verifies immediately that there exists a natural isomorphism of Frobenioids

$$F_{\text{mod}}^\oplus \xrightarrow{\sim} F_{\text{mod}}^\oplus$$

that induces the identity morphism $F_{\text{mod}}^\times \to F_{\text{mod}}^\times$ on the associated rational function monoids [cf. [FrdI], Corollary 4.10].

(iii) By composing the isomorphisms of Frobenioids of (i) and (ii), one thus obtains a natural isomorphism of Frobenioids

$$F_{\text{mod}}^\oplus \xrightarrow{\sim} F_{\text{MOD}}^\oplus$$

that induces the identity morphism $F_{\text{mod}}^\times \to F_{\text{mod}}^\times$ on the associated rational function monoids [cf. [FrdI], Corollary 4.10]. One verifies immediately that although the above isomorphism of Frobenioids is not necessarily determined by the condition that it induce the identity morphism on $F_{\text{mod}}^\times$, the induced isomorphism between the respective perfections [hence also on realifications] of $F_{\text{mod}}^\oplus$, $F_{\text{MOD}}^\oplus$ is completely determined by this condition.

Remark 3.6.1. Note that, as far as the various constructions of Example 3.6, (i), are concerned, the various homomorphisms $\beta_v$, for $v \in \mathbb{V}$, may be thought of, alternatively, as a collection of

subquotients of the perfection $(F_{\text{mod}}^\times)^{\text{pf}}$ of $F_{\text{mod}}^\times$

— each of which is equipped with a submonoid of “nonnegative elements” — that are completely determined by the ring structure of the field $F_{\text{mod}}$ [i.e., equipped with its structure as the field of moduli of $X_F$].

Remark 3.6.2.

(i) In the theory to be developed below, we shall be interested in relating certain Frobenioids — which will, in fact, be isomorphic to the realification of $F_{\text{mod}}^\oplus$ — that lie on opposite sides of [a certain enhanced version of] the $\Theta_{\text{gan}}^\times$-link to one another. In particular, at the level of objects of the Frobenioids involved, it only
makes sense to work with isomorphism classes of objects that are preserved by
the isomorphisms of Frobenioids that appear. Here, we note that the isomorphism
classes of the sort of Frobenioids that appear in this context are determined by the
divisor and rational function monoids of the [model] Frobenioid in question [cf.
the constructions given in [FrdI], Theorem 5.2, (i), (ii)]. In this context, we observe
that the rational function monoid $\mathcal{F}_\text{mod}^\times$ of $\mathcal{F}_\text{mod}^\circ$ satisfies the following fundamental
property:

[the union with $\{0\}$ of] $\mathcal{F}_\text{mod}^\times$ admits a natural additive structure.

In this context, we note that this property is not satisfied by

(a) the rational function monoids of the perfection or realification of $\mathcal{F}_\text{mod}^\circ$

(b) subgroups $\Gamma \subseteq \mathcal{F}_\text{mod}^\times$ — such as, for instance, the trivial subgroup $\{1\}$
or the subgroup of $S$-units, for $S \subseteq \mathcal{V}_\text{mod}$ a nonempty finite subset —
that do not arise as the multiplicative group of some subfield of $\mathcal{F}_\text{mod}$ [cf.
[AbsTopIII], Remark 5.10.2, (iv)].

The significance of this fundamental property is that it allows one to represent the
objects of $\mathcal{F}_\text{mod}^\circ$ additively, i.e., as modules — cf. the point of view of Example 3.6,
(ii). At a more concrete level, if, in the notation of (b), one considers the result of
“adding” two elements of a $\Gamma$-torsor [cf. the point of view of Example 3.6, (i)!],
then the resulting “sum” can only be rendered meaningful, relative to the given $\Gamma$-torsor,
if $\Gamma$ is additively closed. The additive representation of objects of $\mathcal{F}_\text{mod}^\circ$ will be
of crucial importance in the theory of the present series of papers since it will allow
us to relate objects of $\mathcal{F}_\text{mod}^\circ$ on opposite sides of [a certain enhanced version of]
the $\Theta^\times\mu_\text{gau}$-link to one another — which, a priori, are only related to one another at
the level of realifications in a multiplicative fashion — by means of [“additive”]
mono-analytic log-shells [cf. the discussion of [IUTchII], Remark 4.7.2].

(ii) One way to understand the content of the discussion of (i) is as follows:
whereas

the construction of $\mathcal{F}_\text{mod}^\circ$ depends on the additive structure of $\mathcal{F}_\text{mod}^\times$
in an essential way,

the construction of $\mathcal{F}_\text{MOD}^\circ$ is strictly multiplicative in nature.

Indeed, the construction of $\mathcal{F}_\text{MOD}^\circ$ given in Example 3.6, (i), is essentially the same
as the construction of $\mathcal{F}_\text{mod}^\circ$ given in [FrdI], Example 6.3 [i.e., in effect, in [FrdI],
Theorem 5.2, (i)]. From this point of view, it is natural to identify $\mathcal{F}_\text{MOD}^\circ$ with
$\mathcal{F}_\text{mod}^\circ$ via the natural isomorphism of Frobenioids of Example 3.6, (i). We shall
often do this in the theory to be developed below.

**Proposition 3.7.** (Global Packet-theoretic Frobenioids)

(i) (Single Packet Rational Function Torsor Version) In the notation
of Proposition 3.3: For each $\alpha \in A$, there is an algorithm for constructing, as
discussed in Example 3.6, (i) [cf. also Remark 3.6.1], from the [number] field given
by the image

$$(\downarrow \mathcal{M}_\text{MOD}^\circ)_\alpha$$
of the composite

$$(\dag \mathbb{M}_{\text{mod}})^{\alpha} \rightarrow (\dag \mathbb{M}_{\text{mod}})^{\alpha} \rightarrow \log^{(A\mathcal{F}_V)}$$

do the homomorphisms of Proposition 3.3, (i), (ii), a Frobenioid $(\dag \mathcal{F}_{\text{MOD}})^{\alpha}$, together with a natural isomorphism of Frobenioids

$$(\dag \mathcal{F}_{\text{mod}})^{\alpha} \sim (\dag \mathcal{F}_{\text{MOD}})^{\alpha}$$

[cf. the notation of [IUTchII], Corollary 4.8, (ii)] that induces the tautological isomorphism $(\dag \mathbb{M}_{\text{mod}})^{\alpha} \sim (\dag \mathbb{M}_{\text{MOD}})^{\alpha}$ on the associated rational function monoids [cf. Example 3.6, (i)]. We shall often use this isomorphism of Frobenioids to identify $(\dag \mathcal{F}_{\text{mod}})^{\alpha}$ with $(\dag \mathcal{F}_{\text{MOD}})^{\alpha}$ [cf. Remark 3.6.2, (ii)]. Write $(\dag \mathcal{F}_{\text{mod}}^{\alpha})$ for the realification of $(\dag \mathcal{F}_{\text{MOD}})^{\alpha}$.

(ii) (Single Packet Local Fractional Ideal Version) In the notation of Propositions 3.3, 3.4: For each $\alpha \in A$, there is an algorithm for constructing, as discussed in Example 3.6, (ii), from the [number] field $(\dag \mathbb{M}_{\text{mod}})^{\alpha} \overset{\text{def}}{=} (\dag \mathbb{M}_{\text{MOD}})^{\alpha}$ [cf. (i)] and the Galois invariants of the local monoids

$$\Psi_{\log(A^{\alpha}\mathcal{F}_V)} \subseteq \log(A^{\alpha}\mathcal{F}_V)$$

for $v \in \mathbb{V}$ of Proposition 3.4, (i) — i.e., so the corresponding local “fractional ideal $J_v$” of Example 3.6, (ii), is a subset [indeed a submodule when $v \in \mathbb{V}_{\text{mod}}$] of $\mathcal{T}^{(A^{\alpha}\mathcal{F}_V)}$ whose Q-span is equal to $\mathcal{T}^{(A^{\alpha}\mathcal{F}_V)}$ [cf. the notational conventions of Proposition 3.2, (ii)] — a Frobenioid $(\dag \mathcal{F}_{\text{mod}}^{\alpha})$, together with natural isomorphisms of Frobenioids

$$(\dag \mathcal{F}_{\text{mod}}^{\alpha}) \sim (\dag \mathcal{F}_{\text{mod}}^{\alpha}); \quad (\dag \mathcal{F}_{\text{mod}}^{\alpha}) \sim (\dag \mathcal{F}_{\text{MOD}}^{\alpha})$$

that induce the tautological isomorphisms $(\dag \mathbb{M}_{\text{mod}})^{\alpha} \sim (\dag \mathbb{M}_{\text{mod}})^{\alpha}$, $(\dag \mathbb{M}_{\text{mod}})^{\alpha} \sim (\dag \mathbb{M}_{\text{MOD}})^{\alpha}$, on the associated rational function monoids [cf. the natural isomorphism of Frobenioids of (i); Example 3.6, (ii), (iii)]. Write $(\dag \mathcal{F}_{\text{mod}}^{\alpha})$ for the realification of $(\dag \mathcal{F}_{\text{mod}}^{\alpha})$.

(iii) (Global Realified LGP-Frobenioids) In the notation of Proposition 3.4: By applying the composites of the isomorphisms of Frobenioids $^v\mathbb{C}_{j}$ $\sim$ $(\dag \mathcal{F}_{\text{mod}})$ ” of [IUTchII], Corollary 4.8, (iii), with the realifications “$(\dag \mathcal{F}_{\text{mod}}^{\alpha})$ $\sim$ $(\dag \mathcal{F}_{\text{MOD}}^{\alpha})$” of the isomorphisms of Frobenioids of (i) above to the global realified Frobenioid portion $^v\mathbb{C}_{\text{gau}}$ of the $\mathbb{F}_{\text{gau}}$-prime-strip $^v\mathbb{S}_{\text{gau}}$ of [IUTchII], Corollary 4.10, (ii) [cf. Remarks 1.5.3, (iii); 3.3.2, (i)], one obtains a functorial algorithm in the $\log$-link of $\Theta^{\mathbb{G}_{\text{ell}}}$ NF-Hodge theaters $^{v}\mathcal{H}\Theta^{\mathbb{G}_{\text{ell}}}$ $\log$ $^{v}\mathcal{H}\Theta^{\mathbb{G}_{\text{ell}}}$. Proposition 3.4, (ii), for constructing a Frobenoid

$$^v\mathbb{C}_{\text{LGP}}^{\mathbb{G}_{\text{ell}}}$$

— which we refer to as a “global realified LGP-Frobenioid”. Here, we note that the notation “$(\dag \mathcal{H}\Theta^{\mathbb{G}_{\text{ell}}})$” constitutes a slight abuse of notation. In particular, the global realified Frobenioid $^v\mathbb{C}_{\text{LGP}}^{\mathbb{G}_{\text{ell}}}$ $\overset{\text{def}}{=} \mathbb{C}_{\text{LGP}}^{\mathbb{G}_{\text{ell}}}((\dag \mathcal{H}\Theta^{\mathbb{G}_{\text{ell}}})), \text{ together with}
the collection of data $\Psi_{\mathcal{F}_{\text{LGP}}}^{\dag}(\mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}})$ constructed in Proposition 3.4, (ii), give rise, in a natural fashion, to an $\mathcal{F}^{\dag}$-prime-strip

$$\dag \mathfrak{d}^{\dag}_{\text{LGP}} = (\dag \mathcal{C}_{\text{LGP}}, \text{Prime}(\mathcal{C}_{\text{LGP}}^{\dag}) \sim \mathfrak{Y}, \dag \mathfrak{d}^{\dag}_{\text{LGP}}, \{\dag \rho_{\text{LGP},\mathfrak{X}}\}_{\mathfrak{X} \in \mathfrak{Y}})$$

— cf. the construction of the $\mathcal{F}^{\dag}$-prime-strip $\dag \mathfrak{d}^{\dag}_{\text{gau}}$ in [IUTchII], Corollary 4.10, (ii) — together with a natural isomorphism

$$\dag \mathfrak{d}^{\dag}_{\text{gau}} \sim \dag \mathfrak{d}^{\dag}_{\text{LGP}}$$

of $\mathcal{F}^{\dag}$-prime-strips [i.e., that arises tautologically from the construction of $\dag \mathfrak{d}^{\dag}_{\text{LGP}}$].

(iv) (Global Realified $\text{lgp}$-Frobenioids) In the situation of (iii) above, write $\Psi_{\mathcal{F}_{\text{lgp}}}^{\dag}(\mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}) \overset{\text{def}}{=} \Psi_{\mathcal{F}_{\text{LGP}}}^{\dag}(\mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}), \dag \mathfrak{c}^{\dag}_{\text{lgp}} \overset{\text{def}}{=} \dag \mathfrak{c}^{\dag}_{\text{LGP}}$. Then by replacing, in the construction of (iii), the isomorphisms “$\dag (\mathcal{F}_{\text{mod}}^{\otimes \otimes})_\alpha \sim (\mathcal{F}_{\text{mod}}^{\otimes \otimes})_\alpha$” by the natural isomorphisms “$\dag (\mathcal{F}_{\text{mod}}^{\otimes \otimes})_\alpha \sim (\mathcal{F}_{\text{mod}}^{\otimes \otimes})_\alpha$” [cf. (ii)], one obtains a functorial algorithm in the log-link of $\Theta^{\pm \text{ell}}_{\text{NF}}$-Hodge theaters $\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}} \overset{\text{log}}{\rightarrow} \dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}$ of Proposition 3.4, (ii), for constructing a Frobenoid

$$\mathcal{C}^{\dag}_{\text{lgp}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}})$$

— which we refer to as a “global realified $\text{lgp}$-Frobenioid” — as well as an $\mathcal{F}^{\dag}$-prime-strip

$$\dag \mathfrak{c}^{\dag}_{\text{lgp}} = (\dag \mathcal{C}_{\text{lgp}}, \text{Prime}(\mathcal{C}_{\text{lgp}}^{\dag}) \sim \mathfrak{Y}, \dag \mathfrak{c}^{\dag}_{\text{lgp}}, \{\dag \rho_{\text{lgp},\mathfrak{X}}\}_{\mathfrak{X} \in \mathfrak{Y}})$$

— where we write $\dag \mathcal{C}_{\text{lgp}}^{\dag} \overset{\text{def}}{=} \mathcal{C}^{\dag}_{\text{lgp}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}})$ — together with tautological isomorphisms

$$\dag \mathfrak{d}^{\dag}_{\text{gau}} \sim \dag \mathfrak{d}^{\dag}_{\text{LGP}} \sim \dag \mathfrak{d}^{\dag}_{\text{lgp}}$$

of $\mathcal{F}^{\dag}$-prime-strips [cf. (iii)].

(v) (Realified Product Embeddings and Non-realified Global Frobenioids) The constructions of $\mathcal{C}^{\dag}_{\text{LGP}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}), \mathcal{C}^{\dag}_{\text{lgp}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}})$ given in (iii) and (iv) above give rise to a commutative diagram of categories

$$\mathcal{C}^{\dag}_{\text{LGP}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}) \rightarrow \prod_{j \in \mathbb{F}^*_+} (\dag \mathcal{F}^{\otimes \otimes}_{\text{mod}})_j$$

$$\mathcal{C}^{\dag}_{\text{lgp}}(\dag \mathcal{H}^{\Theta^{\pm \text{ell}}_{\text{NF}}}) \rightarrow \prod_{j \in \mathbb{F}^*_+} (\dag \mathcal{F}^{\otimes \otimes}_{\text{mod}})_j$$

— where the horizontal arrows are embeddings that arise tautologically from the constructions of (iii) and (iv) [cf. [IUTchII], Remark 4.8.1, (i)]; the vertical arrows are isomorphisms; the left-hand vertical arrow arises from the second isomorphism that appears in the final display of (iv); the right-hand vertical arrow is the product of the realifications of copies of the inverse of the second isomorphism that appears in the final display of (ii). In particular, by applying the definition of $(\dag \mathcal{F}^{\otimes \otimes}_{\text{mod}})_j$ —
i.e., in terms of local fractional ideals [cf. (ii)] — together with the products of realification functors

\[
\prod_{j \in F^*_l} (\dagger \mathcal{F}_{\text{mod}}^\otimes)_j \to \prod_{j \in F^*_l} (\dagger \mathcal{F}_{\text{mod}}^\varnothing)_j
\]

[cf. [FrdI], Proposition 5.3], one obtains an algorithm for constructing, in a fashion compatible [in the evident sense] with the local isomorphisms \( \{\dagger \rho_{\text{lgp}, v}\}_{v \in \mathbb{V}} \), \( \{\dagger \rho_{\text{LGP}, v}\}_{v \in \mathbb{V}} \) of (iii) and (iv), objects of the [global] categories \( C^\dagger_{\text{lgp}}(\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}}) \), \( C^\dagger_{\text{LGP}}(\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}}) \) from the local fractional ideals generated by elements of the monoids [cf. (iv); Proposition 3.4, (ii)]

\[
\Psi_{\mathcal{F}_{\text{lgp}}}(\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}})_v
\]

for \( v \in \mathbb{V}^{\text{bad}} \).

Proof. The various assertions of Proposition 3.7 follow immediately from the definitions and the references quoted in the statements of these assertions. \( \square \)

Definition 3.8.

(i) In the situation of Proposition 3.7, (iv), (v), write \( \Psi_{\mathcal{F}_{\text{lgp}}}^\dagger \) \( (-)_v \) \( \text{def} \) \( \Psi_{\mathcal{F}_{\text{LGP}}}^\dagger \) \( (-)_v \), for \( v \in \mathbb{V}^{\text{bad}} \) [cf. the notation of Proposition 3.5, (ii), (c)]. Then we shall refer to the object of

\[
\prod_{j \in F^*_l} (\dagger \mathcal{F}_{\text{MOD}}^\otimes)_j \text{ or } \prod_{j \in F^*_l} (\dagger \mathcal{F}_{\text{mod}}^\varnothing)_j
\]

— as well as its realification, regarded as an object of \( \dagger C^\dagger_{\text{lgp}} = C^\dagger_{\text{LGP}}(\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}}) \) or \( \dagger C^\dagger_{\text{lgp}} = C^\dagger_{\text{lgp}}(\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}}) \) [cf. Proposition 3.7, (iii), (iv), (v)] — determined by any collection, indexed by \( v \in \mathbb{V}^{\text{bad}} \), of generators up to torsion of the monoids \( \Psi_{\mathcal{F}_{\text{lgp}}}^\dagger (\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}})_v \) as a \( \Theta \)-pilot object [cf. also Remark 3.8.1 below]. We shall refer to the object of the [global realified] Frobenioid

\[
\dagger C^\dagger_{\Delta}
\]

of [IUTchII], Corollary 4.10, (i), determined by any collection, indexed by \( v \in \mathbb{V}^{\text{bad}} \), of generators up to torsion of the splitting monoid associated to the split Frobenioid \( \dagger \mathcal{F}_{\Delta, v} \) [i.e., the data indexed by \( v \) of the \( \mathcal{F}^\dagger \)-prime-strip \( \dagger \mathcal{F}_{\Delta} \) of [IUTchII], Corollary 4.10, (i)] — that is to say, at a more concrete level, determined by the \( \xi^v \), for \( v \in \mathbb{V}^{\text{bad}} \) [cf. the notation of [IUTchI], Example 3.2, (iv)] — as a \( q \)-pilot object [cf. also Remark 3.8.1 below].

(ii) Let

\[
\dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}} \xrightarrow{\text{log}} \dagger \mathcal{H}\mathcal{T}^{\Theta_{\pm \text{ell}}\text{NF}}
\]
be a log-link of $\Theta^{\pm\ell} \text{-} \text{NF-Hodge theaters}$ and

$$* HT^{\Theta^{\pm\ell} \text{NF}}$$

a $\Theta^{\pm\ell} \text{-} \text{NF-Hodge theater}$ [all relative to the given initial $\Theta$-data]. Recall the $F^\dagger$-prime-strip

$$* S^\dagger_{\Delta}$$

constructed from $* HT^{\Theta^{\pm\ell} \text{NF}}$ in [IUTchII], Corollary 4.10, (i). Following the notational conventions of [IUTchII], Corollary 4.10, (iii), let us write $S_{\Delta}^\dagger \times \mu^{\dagger}$ (respectively, $S_{\Delta}^\dagger \times \mu^{\dagger}$) for the $F^\dagger \times \mu^{\dagger}$-prime-strip associated to the $F^\dagger$-prime-strip $S^\dagger_{\Delta}$ (respectively, $S_{\Delta}^{\dagger}$) [cf. Proposition 3.7, (iii), (iv); [IUTchII], Definition 4.9, (viii); the functorial algorithm described in [IUTchII], Definition 4.9, (vi)]. Then — in the style of [IUTchII], Corollary 4.10, (iii) — we shall refer to the full poly-isomorphism of $F^\dagger \times \mu^{\dagger}$-prime-strips $S_{\Delta}^{\dagger} \times \mu^{\dagger}$ as the $\Theta_{\text{LGP}}^{\times \mu^{\dagger}}$-link

$$\Theta_{\text{LGP}}^{\times \mu^{\dagger}} \longrightarrow * S^\dagger_{\Delta}$$

from $\Theta_{\text{LGP}}^{\times \mu^{\dagger}}$ to $* S^\dagger_{\Delta}$, relative to the log-link $\Theta_{\text{LGP}}^{\times \mu^{\dagger}} \longrightarrow \Theta_{\text{LGP}}^{\times \mu^{\dagger}}$ as the $\Theta_{\text{LGP}}^{\times \mu^{\dagger}}$-link

$$\Theta_{\text{LGP}}^{\times \mu^{\dagger}} \longrightarrow * S^\dagger_{\Delta}$$

from $\Theta_{\text{LGP}}^{\times \mu^{\dagger}}$ to $* S^\dagger_{\Delta}$, relative to the log-link $\Theta_{\text{LGP}}^{\times \mu^{\dagger}} \longrightarrow \Theta_{\text{LGP}}^{\times \mu^{\dagger}}$.

(iii) Let $\{n,m \mid HT^{\Theta^{\pm\ell} \text{NF}}\}_{n,m \in \mathbb{Z}}$ be a collection of distinct $\Theta^{\pm\ell} \text{-} \text{NF-Hodge theaters}$ [relative to the given initial $\Theta$-data] indexed by pairs of integers. Then we shall refer to the first (respectively, second) diagram

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\log & \log & \log & \\
\Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \\
\Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \\
\log & \log & \log & \\
\Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \\
\Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \Theta^\times_{\text{LGP}}^{\mu} & \\
\log & \log & \log & \\
\vdots & \vdots & \vdots & \\
\end{array}
\]
— where the vertical arrows are the full log-links, and the horizontal arrow of the first (respectively, second) diagram from \( n, m \mathrm{HT}^{\Theta \pm \text{ell} \text{NF}} \) to \( n+1, m+1 \mathrm{HT}^{\Theta \pm \text{ell} \text{NF}} \) is the \( \Theta_{\text{LGP}}^{\times \mu} \)- (respectively, \( \Theta_{\text{lgp}}^{\times \mu} \)-) link from \( n, m \mathrm{HT}^{\Theta \pm \text{ell} \text{NF}} \) to \( n+1, m+1 \mathrm{HT}^{\Theta \pm \text{ell} \text{NF}} \), relative to the full log-link \( n,m-1 \mathrm{HT}^{\Theta^{\pm \text{ell} \text{NF}}} \xrightarrow{\log} n,m \mathrm{HT}^{\Theta^{\pm \text{ell} \text{NF}}} \) [cf. (ii)] — as the [LGP-Gaussian] (respectively, [lgp-Gaussian]) log-theta-lattice. Thus, [cf. Definition 1.4] either of these diagrams may be represented symbolically by an oriented graph

\[
\begin{array}{c}
\vdots \\
\uparrow \\
\vdots \\
\downarrow \\
\vdots \\
\downarrow \\
\vdots \\
\downarrow \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\downarrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\uparrow \\
\vdots \\
\end{array}
\]

— where the “\( \bullet \)’s” correspond to the “\( n,m \mathrm{HT}^{\Theta \pm \text{ell} \text{NF}} \)”.

**Remark 3.8.1.** The LGP-Gaussian and lgp-Gaussian log-theta-lattices are, of course, closely related, but, in the theory to be developed below, we shall mainly be interested in the LGP-Gaussian log-theta-lattice [for reasons to be explained in Remark 3.10.1, (ii), below]. On the other hand, our computation of the \( \Theta_{\text{LGP}}^{\times \mu} \)-link will involve the \( \Theta_{\text{lgp}}^{\times \mu} \)-link, as well as related \( \Theta \)-pilot objects, in an essential way. Here, we note, for future reference, that both the \( \Theta_{\text{LGP}}^{\times \mu} \)- and the \( \Theta_{\text{lgp}}^{\times \mu} \)-link map \( \Theta \)-pilot objects to \( q \)-pilot objects. Also, we observe that this terminology of “\( \Theta \)-pilot/\( q \)-pilot objects” is consistent with the notion of a “pilot object” associated to a \( \mathcal{F}^\mu \)-prime-strip, as defined in [IUTchII], Definition 4.9, (viii).

**Remark 3.8.2.** One verifies immediately that the main results obtained so far concerning Gaussian log-theta-lattices — namely, Theorem 1.5, Proposition 2.1,
Corollary 2.3 [cf. also Remark 2.3.2], and Proposition 3.5 — generalize immediately [indeed, “formally”] to the case of LGP- or lgp-Gaussian log-theta-lattices. Indeed, the substantive content of these results concerns portions of the log-theta-lattices involved that are substantively unaffected by the transition from “Gaussian” to “LGP- or lgp-Gaussian”.

Remark 3.8.3. In the definition of the various horizontal arrows of the log-theta-lattices discussed in Definition 3.8, (iii), it may appear to the reader, at first glance, that, instead of working with \( F \top \times_{\mu} \times \mu \)-prime-strips, it might in fact be sufficient to replace the unit [i.e., \( F \top \times_{\mu} \times \mu \)-prime-strip] portions of these prime-strips by the associated log-shells [cf. Proposition 1.2, (vi), (vii)], on which, at nonarchimedean \( v \in \mathbb{V} \), the associated local Galois groups act trivially. In fact, however, this is not the case. That is to say, the nontrivial Galois action on the local unit portions of the \( F \top \times_{\mu} \times \mu \)-prime-strips involved is necessary in order to consider the Kummer theory [cf. Proposition 3.5, (i), (ii), as well as Proposition 3.10, (i), (iii); Theorem 3.11, (iii), (c), (d), below] of the various local and global objects for which the log-shells serve as “multiradial containers” [cf. the discussion of Remark 1.5.2]. Here, we recall that this Kummer theory plays a crucial role in the theory of the present series of papers in relating corresponding Frobenius-like and étale-like objects [cf. the discussion of Remark 1.5.4, (i)].

Proposition 3.9. (Log-volume for Packets and Processions)

(i) (Local Holomorphic Packets) In the situation of Proposition 3.2, (i), (ii): Suppose that \( V \ni v \mid v_Q \in V_{\text{non}}^\mathbb{Q}, \alpha \in A \). Then the \( p_v \)-adic log-volume on each of the direct summand \( p_v \)-adic fields of \( I_{\mathbb{Q}}(\alpha F_{v_Q}), I_{\mathbb{Q}}(A F_{v_Q}), \) and \( I_{\mathbb{Q}}(A,\alpha F_{v_Q}) \) — cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of normalized weights in Remark 3.1.1, (ii), (iii), (iv) — determines \( \mu_{\log}^{\alpha,v} : M(I_{\mathbb{Q}}(\alpha F_{v_Q})) \to \mathbb{R} \); \( \mu_{\log}^{A,v} : M(I_{\mathbb{Q}}(A F_{v_Q})) \to \mathbb{R} \); \( \mu_{A,\alpha,v}^{\log} : M(I_{\mathbb{Q}}(A,\alpha F_{v_Q})) \to \mathbb{R} \) — where we write “\( M(-) \)” for the set of nonempty compact open subsets of \( (-) \) — such that the log-volume of each of the “local holomorphic” integral structures of Proposition 3.1, (ii) — i.e., the elements

\[
O_{\alpha,F_{v_Q}} \subseteq I_{\mathbb{Q}}(\alpha F_{v_Q}); \quad O_{A,F_{v_Q}} \subseteq I_{\mathbb{Q}}(A F_{v_Q}); \quad O_{A,\alpha,F_{v_Q}} \subseteq I_{\mathbb{Q}}(A,\alpha F_{v_Q})
\]

of “\( M(-) \)” given by the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand \( p_v \)-adic fields — is equal to zero. Here, we assume that these log-volumes are normalized so that multiplication of an element of “\( M(-) \)” by \( p_v \) corresponds to adding the quantity \(-\log(p_v) \in \mathbb{R} \); we shall refer to this normalization as the packet-normalization. Suppose that \( V \ni v \mid v_Q \in V_{\text{arc}}^\mathbb{Q}, \alpha \in A \). Then the sum of the radial log-volumes on each of the direct summand complex archimedean fields of \( I_{\mathbb{Q}}(\alpha F_{v_Q}), I_{\mathbb{Q}}(A F_{v_Q}), \) and \( I_{\mathbb{Q}}(A,\alpha F_{v_Q}) \) — cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of normalized
weights in Remark 3.1.1, (ii), (iii), (iv) — determines [cf. [AbsTopIII], Proposition 5.7, (ii)] log-volumes

\[ \mu_{A,\alpha,v}^\log : \mathcal{M}(I^Q(A,\alpha)F_v) \to \mathbb{R} \]

— where we write \( \mathcal{M}(-) \) for the set of compact closures of nonempty open subsets of \((-)\) — such that the log-volume of each of the “local holomorphic” integral structures of Proposition 3.1, (ii) — i.e., the elements

\[ O_{A,F_v} \subseteq I^Q(A,F_v); \quad O_{A,\alpha,F_v} \subseteq I^Q(A,\alpha,F_v) \]

of \( \mathcal{M}(-) \) given by the products of the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand complex archimedean fields — is equal to zero. Here, we assume that these log-volumes are normalized so that multiplication of an element of \( \mathcal{M}(-) \) by \( e = 2.71828... \) corresponds to adding the quantity \( 1 = \log(e) \in \mathbb{R} \); we shall refer to this normalization as the packet-normalization. In both the nonarchimedean and archimedean cases, \( \mu_{A,v}^\log \) is invariant with respect to permutations of \( A \). Finally, when working with collections of capsules in a procession, as in Proposition 3.4, (ii), we obtain, in both the nonarchimedean and archimedean cases, log-volumes on the products of the \( \mathcal{M}(-) \) associated to the various capsules under consideration, which we normalize by taking the average, over the various capsules under consideration; we shall refer to this normalization as the procession-normalization [cf. Remark 3.9.3 below].

(ii) (Mono-analytic Compatibility) In the situation of Proposition 3.2, (i), (ii): Suppose that \( v \not\in \mathcal{V} \) for \( v \in \mathcal{V}_Q \). Then by applying the \( p_v \)-adic log-volume, when \( v \in \mathcal{V}^\text{non} \), or the radial log-volume, when \( v \in \mathcal{V}^\text{arc} \), on the mono-analytic log-shells \( I^{D_\mathbb{P}_v} \) of Proposition 1.2, (vi), (vii), (viii), and adjusting appropriately [cf. Remark 3.9.1 below for more details] to account for the discrepancy between the “local holomorphic” integral structures of Proposition 3.1, (ii), and the “mono-analytic” integral structures of Proposition 3.2, (ii), one obtains [by a slight abuse of notation] log-volumes

\[ \mu_{A,\alpha,v}^\log : \mathcal{M}(I^Q(A,\alpha)D_v) \to \mathbb{R} \]

— where \( \mathcal{M}(-) \) is as in (i) above — which are compatible with the log-volumes obtained in (i), relative to the natural poly-isomorphisms of Proposition 3.2, (i). In particular, these log-volumes may be constructed via a functorial algorithm from the \( D_v \)-prime-strips under consideration. If one considers the mono-analyticization [cf. [IUTchI], Proposition 6.9, (ii)] of a holomorphic procession as in Proposition 3.4, (ii), then taking the average, as in (i) above, of the packet-normalized log-volumes of the above display gives rise to procession-normalized log-volumes, which are compatible, relative to the natural poly-isomorphisms of Proposition 3.2, (i), with the procession-normalized log-volumes of (i). Finally, by replacing \( D_v^+ \) by \( F_v^+ \times \mu \) [cf. also the discussion of Proposition 1.2, (vi),
(vii), (viii)], one obtains a similar theory of log-volumes for the various objects associated to the mono-analytic log-shells \( \mathcal{D}_{F^{\times} \times F} \), which is compatible with the theory obtained for \( \mathcal{D}^\circ \) relative to the various natural poly-isomorphisms of Proposition 3.2, (i).

(iii) (Global Compatibility) In the situation of Proposition 3.7, (i), (ii):
Write
\[
\mathcal{I}^Q(A,F_{V_Q}) \overset{\text{def}}{=} \prod_{v_Q \in V_Q} \mathcal{I}^Q(A,F_{v_Q}) \subseteq \log(A,F_{V_Q}) = \prod_{v_Q \in V_Q} \log(A,F_{v_Q})
\]
and
\[
\mathcal{M}(\mathcal{I}^Q(A,F_{V_Q})) \subseteq \prod_{v_Q \in V_Q} \mathcal{M}(\mathcal{I}^Q(A,F_{v_Q}))
\]
for the subset of elements whose components, indexed by \( v_Q \in V_Q \), have zero log-volume [cf. (i)] for all but finitely many \( v_Q \in V_Q \). Then, by adding the log-volumes of (i) [all but finitely many of which are zero!] at the various \( v_Q \in V_Q \), one obtains a global log-volume
\[
\mu^{\log}_{A,V_Q} : \mathcal{M}(\mathcal{I}^Q(A,F_{V_Q})) \to \mathbb{R}
\]
which is invariant with respect to multiplication by elements of
\[
(\dag M_{\text{mod}})_{\alpha} = (\dag M_{\text{MOD}})_{\alpha} \subseteq \mathcal{I}^Q(A,F_{V_Q})
\]
as well as with respect to permutations of \( A \), and moreover, satisfies the following property concerning [the elements of “\( \mathcal{M}(\cdot) \)” determined by] objects \( \mathcal{J} = \{ J_v \}_{v \in V} \) of \( (\dag M_{\text{mod}})_{\alpha} \) [cf. Example 3.6, (ii); Proposition 3.7, (ii)]: the global log-volume \( \mu^{\log}_{A,V_Q}(\mathcal{J}) \) is equal to the degree of the arithmetic line bundle determined by \( \mathcal{J} \) [cf. the discussion of Example 3.6, (ii); the natural isomorphism \( (\dag F_{\text{mod}})_{\alpha} \overset{\sim}{\to} (\dag F_{\text{mod}})_{\alpha} \) of Proposition 3.7, (ii)], relative to a suitable normalization.

(iv) (Log-link Compatibility) Let \( \{ n,m \mathcal{H}T^{\Theta_{\text{ell}}} \}_{n,m \in \mathbb{Z}} \) be a collection of distinct \( \Theta_{\text{ell}} \)-Hodge theaters [relative to the given initial \( \Theta \)-data] — which we think of as arising from an LGP-Gaussian log-theta-lattice [cf. Definition 3.8, (iii)]. Then [cf. also the discussion of Remark 3.9.4 below]:

(a) For \( n,m \in \mathbb{Z} \), the log-volumes constructed in (i), (ii), (iii) above determine log-volumes on the various \( \mathcal{I}^Q((-)) \)” that appear in the construction of the local/global LGP-/lgp-monoids/Frobenioids that appear in the \( \mathcal{F}_{\text{LGP}} \)-prime-strips \( n,m \mathcal{S}_{\text{LGP}}, n,m \mathcal{S}_{\text{lgp}} \) constructed in Proposition 3.7, (iii), (iv), relative to the log-link \( n,m^{-1}\mathcal{H}T^{\Theta_{\text{ell}}} \overset{\log}{\to} n,m \mathcal{H}T^{\Theta_{\text{ell}}} \).

(b) At the level of the \( \mathbb{Q} \)-spans of log-shells \( \mathcal{I}^Q((-)) \)” that arise from the various \( \mathcal{F} \)-prime-strips involved, the log-volumes of (a) indexed by \( (n,m) \) are compatible — in the sense discussed in Propositions 1.2, (iii); 1.3, (iii) — with the corresponding log-volumes indexed by \( (n,m-1) \), relative to the log-link \( n,m^{-1}\mathcal{H}T^{\Theta_{\text{ell}}} \overset{\log}{\to} n,m \mathcal{H}T^{\Theta_{\text{ell}}} \).
Proof. The various assertions of Proposition 3.9 follow immediately from the definitions and the references quoted in the statements of these assertions. □

Remark 3.9.1. In the spirit of the explicit descriptions of Remark 3.1.1, (i) [cf. also Remark 1.2.2, (i), (ii)], we make the following observations.

(i) Suppose that $v_Q \in \mathcal{V}^{\text{non}}_Q$. Write $\{v_1, \ldots, v_{n_{v_Q}}\}$ for the [distinct!] elements of $\mathcal{V}$ that lie over $v_Q$. For each $i = 1, \ldots, n_{v_Q}$, set $k_i \overset{\text{def}}{=} K_{v_i}$; write $O_{k_i} \subseteq k_i$ for the ring of integers of $k_i$,

$$I_i \overset{\text{def}}{=} (p_{v_Q}^*)^{-1} \cdot \log_{k_i}(O_{k_i}^\times) \subseteq k_i$$

— where $p_{v_Q}^* = p_{v_Q}$ if $p_{v_Q}$ is odd, $p_{v_Q}^* = p_{v_Q}^2$ if $p_{v_Q}$ is even — cf. Remark 1.2.2, (i). Then, roughly speaking, in the notation of Proposition 3.9, (i), the mono-analytic integral structures of Proposition 3.2, (ii), in

$$\mathcal{I}^Q(\alpha F_{v_Q}) \overset{\sim}{\to} \bigoplus_{i=1}^{n_{v_Q}} I_i; \quad \mathcal{I}^Q(A F_{v_Q}) \overset{\sim}{\to} \bigotimes_{\alpha \in A} \mathcal{I}^Q(\alpha F_{v_Q})$$

are given by

$$\mathcal{I}(\alpha F_{v_Q}) \overset{\sim}{\to} \bigoplus_{i=1}^{n_{v_Q}} I_i; \quad \mathcal{I}(A F_{v_Q}) \overset{\sim}{\to} \bigotimes_{\alpha \in A} \mathcal{I}(\alpha F_{v_Q})$$

while the local holomorphic integral structures

$$\mathcal{O}_{\alpha, F_{v_Q}} \subseteq \mathcal{I}^Q(\alpha F_{v_Q}); \quad \mathcal{O}_{A, F_{v_Q}} \subseteq \mathcal{I}^Q(A F_{v_Q})$$

of Proposition 3.9, (i), in the ind-topological rings $\mathcal{I}^Q(\alpha F_{v_Q}), \mathcal{I}^Q(A F_{v_Q})$ — both of which are direct sums of finite extensions of $\mathcal{O}_{p_{v_Q}}$ — are given by the subrings of integers in $\mathcal{I}^Q(\alpha F_{v_Q}), \mathcal{I}^Q(A F_{v_Q})$. Thus, by applying the formula of the final display of [AbsTopIII], Proposition 5.8, (iii), for the log-volume of $I_i$, [one verifies easily that] one may compute the log-volumes

$$\mu_{\log, \alpha, v_Q}(\mathcal{I}(\alpha F_{v_Q})), \quad \mu_{\log, A, v_Q}(\mathcal{I}(A F_{v_Q}))$$

entirely in terms of the given initial $\Theta$-data. We leave the routine details to the reader.

(ii) Suppose that $v_Q \in \mathcal{V}^{\text{arc}}_Q$. Write $\{v_1, \ldots, v_{n_{v_Q}}\}$ for the [distinct!] elements of $\mathcal{V}$ that lie over $v_Q$. For each $i = 1, \ldots, n_{v_Q}$, set $k_i \overset{\text{def}}{=} K_{v_i}$; write $O_{k_i} \overset{\text{def}}{=} \{\lambda \in k_i \mid |\lambda| \leq 1\} \subseteq k_i$ for the “set of integers” of $k_i$,

$$I_i \overset{\text{def}}{=} \pi \cdot O_{k_i} \subseteq k_i$$

— cf. Remark 1.2.2, (ii). Then, roughly speaking, in the notation of Proposition 3.9, (i), the discrepancy between the mono-analytic integral structures of
Proposition 3.2, (ii), determined by the $\mathcal{I}(\hat{\mathcal{F}}_v) \sim \mathcal{I}_i \subseteq k_i$ and the local holomorphic integral structures

$$O_{\alpha \mathcal{F}_v} \subseteq \mathcal{I}_Q^Q(\alpha \mathcal{F}_v) \sim \bigoplus_{i=1}^{n_{\mathcal{F}_v}} k_i$$

$$O_{\mathcal{A} \mathcal{F}_v} \subseteq \mathcal{I}_Q^Q(\mathcal{A} \mathcal{F}_v) \sim \bigotimes_{\alpha \in A} \mathcal{I}_Q^Q(\alpha \mathcal{F}_v)$$

of Proposition 3.9, (i), in the topological rings $\mathcal{I}_Q^Q(\alpha \mathcal{F}_v)$, $\mathcal{I}_Q^Q(\mathcal{A} \mathcal{F}_v)$ — both of which are direct sums of complex archimedean fields — determined by taking the product [relative to this direct sum decomposition] of the respective “subsets of integers” may be computed entirely in terms of the given initial $\Theta$-data, by applying the following two [easily verified] observations:

(a) Equip $\mathbb{C}$ with its standard Hermitian metric, i.e., the metric determined by the complex norm. This metric on $\mathbb{C}$ determines a tensor product metric on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, as well as a direct sum metric on $\mathbb{C} \oplus \mathbb{C}$. Then, relative to these metrics, any isomorphism of topological rings [i.e., arising from the Chinese remainder theorem]

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \sim \mathbb{C} \oplus \mathbb{C}$$

is compatible with these metrics, up to a factor of 2, i.e., the metric on the right-hand side corresponds to 2 times the metric on the left-hand side.

(b) Relative to the notation of (a), the direct sum decomposition $\mathbb{C} \oplus \mathbb{C}$, together with its Hermitian metric, is preserved, relative to the displayed isomorphism of (a), by the operation of conjugation on either of the two copies of “$\mathbb{C}$” that appear in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, as well as by the operations of multiplying by $\pm 1$ or $\pm \sqrt{-1}$ via either of the two copies of “$\mathbb{C}$” that appear in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

We leave the routine details to the reader.

(iii) The computation of the discrepancy between local holomorphic and mono-analytic integral structures will be discussed in more detail in [IUTchIV], §1.

Remark 3.9.2. In the situation of Proposition 3.9, (iii), one may construct [“mono-analytic”] algorithms for recovering the subquotient of the perfection of $(\hat{\mathcal{M}}_{\mathcal{M}^{\oplus}_0})_{\alpha} = (\hat{\mathcal{M}}_{\mathcal{M}^{\oplus}_0})_{\alpha}$ associated to $w \in \mathcal{V}$ [cf. Remark 3.6.1], together with the submonoid of “nonnegative elements” of such a subquotient, by considering the effect of multiplication by elements of $(\hat{\mathcal{M}}_{\mathcal{M}^{\oplus}_0})_{\alpha} = (\hat{\mathcal{M}}_{\mathcal{M}^{\oplus}_0})_{\alpha}$ on the log-volumes defined on the various $\mathcal{I}_Q^Q(A, \alpha \mathcal{F}_v) \sim \mathcal{I}_Q^Q(A, \alpha \mathcal{D}_v)$ [cf. Proposition 3.9, (ii)].

Remark 3.9.3. With regard to the procession-normalizations discussed in Proposition 3.9, (i), (ii), the reader might wonder the following: Is it possible to work with
more general weighted averages, i.e., as opposed to just averages, in the usual sense, over the capsules that appear in the procession?

The answer to this question is “no”. Indeed, in the situation of Proposition 3.4, (ii), for \( j \in \{1, \ldots, l^* \} \), the packet-normalized log-volume corresponding to the capsule with index set \( S_{j+1}^\pm \) may be thought of as a log-volume that arises from “any one of the log-shells whose label \( \in \{0, 1, \ldots, j \} \)”. In particular, if \( j', j_1, j_2 \in \{1, \ldots, l^* \} \), and \( j' \leq j_1, j_2 \), then log-volumes corresponding to the same log-shell labeled \( j' \) might give rise to packet-normalized log-volumes corresponding to either of [the capsules with index sets] \( S_{j_1+1}^\pm, S_{j_2+1}^\pm \). That is to say, in order for the resulting notion of a procession-normalized log-volume to be compatible with the appearance of the component labeled \( j' \) in various distinct capsules of the procession — i.e., compatible with the various inclusion morphisms of the procession! — one has no choice but to assign the same weights to [the capsules with index sets] \( S_{j_1+1}^\pm, S_{j_2+1}^\pm \).

Remark 3.9.4. The log-link compatibility of log-volumes discussed in Proposition 3.9, (iv), may be formulated somewhat more explicitly by applying various elementary observations, as follows.

(i) Let \((M, \mu_M)\) be a measure space [i.e., in the sense of the discussion of Remark 3.1.1, (iii)]. We shall say that a subset \( S \subseteq M \) is pre-ample if \( S \) is a relatively compact Borel set, that a pre-ample subset \( S \subseteq M \) is ample if \( \mu_M(S) > 0 \), and that \((M, \mu_M)\) is ample if there exists an ample subset of \( M \). In the following, for the sake of simplicity, we assume that \((M, \mu_M)\) is ample. Also, to simplify the notation, we shall often denote the dependence of objects constructed from the pair \((M, \mu_M)\) by means of the notation “\((M)\)” [i.e., as opposed to “\((M, \mu_M)\)”]. Write

\[
\text{Sub}(M)
\]

for the set of pre-ample subsets of \( M \) and

\[
\text{Fn}(M)
\]

for the set of Borel measurable functions \( f : M \to \mathbb{R}_{\geq 0} \) such that the image \( f(M) \subseteq \mathbb{R}_{\geq 0} \) of \( f \) is a finite set, and, moreover, \( M \supseteq f^{-1}(\mathbb{R}_{>0}) \in \text{Sub}(M) \). Observe that \( \text{Fn}(M) \) is equipped with a natural monoid structure [induced by the natural monoid structure on \( \mathbb{R}_{\geq 0} \)], as well as a natural action by \( \mathbb{R}_{>0} \) [induced by the natural action by multiplication of \( \mathbb{R}_{>0} \) on \( \mathbb{R}_{\geq 0} \)]. By assigning to an element \( S \in \text{Sub}(M) \) the characteristic function \( \chi_S : M \to \mathbb{R}_{\geq 0} \) [i.e., which is \( = 1 \) on \( S \) and \( = 0 \) on \( M \setminus S \)], we shall regard \( \text{Sub}(M) \) as a subset of \( \text{Fn}(M) \). Note that integration over \( M \), relative to the measure \( \mu_M \), determines an \( \mathbb{R}_{>0} \)-equivariant surjection

\[
\int_M : \text{Fn}(M) \to \mathbb{R}_{\geq 0}
\]

whose restriction to \( \text{Sub}(M) \) maps \( \text{Sub}(M) \ni S \mapsto \mu_M(S) \in \mathbb{R}_{\geq 0} \). In particular, if we write \( \text{FnRss}_M : \text{Fn}(M) \to \text{Rss}(M) \) for the natural map to the quotient set of \( \text{Fn}(M) \) [i.e., the set of equivalence classes of elements of \( \text{Fn}(M) \)] determined by \( \int_M \)
so \( \mathbb{Rss}(M) \) also admits a natural monoid structure, as well as a natural action by \( \mathbb{R}_{\geq 0} \), then we obtain a natural \( \mathbb{R}_{\geq 0} \)-equivariant isomorphism of monoids

\[
\int_{M}^{\mathbb{Rss}} : \mathbb{Rss}(M) \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0}
\]

such that \( \int_{M} = \int_{M}^{\mathbb{Rss}} \circ \mathbb{FnRSs}_{M} \). Here,

we wish to think of integration \( \int_{M} \) — and hence of the quotient

\[
\mathbb{FnRSs}_{M} : \mathbb{Fn}(M) \twoheadrightarrow \mathbb{Rss}(M)
\]

— as a sort of “\textit{realified semi-simplification}” of \( (M, \mu_{M}) \)

[i.e., roughly in the spirit of the Grothendieck group associated to an additive category], that is to say, a quotient in the category of commutative monoids with \( \mathbb{R}_{\geq 0} \)-action, whose restriction to \( \text{Sub}(M) \subseteq \mathbb{Fn}(M) \)

\begin{itemize}
  \item identifies \( S_{1}, S_{2} \in \text{Sub}(M) \) such that \( \mu_{M}(S_{1}) = \mu_{M}(S_{2}) \) [such as, for instance, additive translates of an element \( S \in \text{Sub}(M) \) relative to an additive structure on \( M \) with respect to which \( \mu_{M} \) is invariant];
  \item maps \( S_{1} \cup S_{2} \subseteq \text{Sub}(M) \) to the sum [relative to the monoid structure on the quotient] of the images of \( S_{1}, S_{2} \subseteq \text{Sub}(M) \) whenever \( S_{1}, S_{2} \subseteq \text{Sub}(M) \) are disjoint [i.e., as subsets of \( M \)].
\end{itemize}

We shall refer to a subset \( E \subseteq \mathbb{Fn}(M) \) as ample if \( (\mathbb{R}_{\geq 0} \supseteq \mathbb{R}_{\geq 0} \cap \int_{M}(E) \neq \emptyset) \). Thus, if, for instance, \( S \subseteq \text{Sub}(M) \) is ample and compact, then the pair \( (S, \mu_{M}|_{S}) \) obtained by restricting \( \mu_{M} \) to \( S \) is an ample measure space that determines compatible natural inclusions \( \text{Sub}(S) \hookrightarrow \text{Sub}(M), \mathbb{Fn}(S) \hookrightarrow \mathbb{Fn}(M) \) [the latter of which is defined by extension by zero] — which we shall use to regard \( \text{Sub}(S), \mathbb{Fn}(S) \) as subsets of \( \text{Sub}(M), \mathbb{Fn}(M) \), respectively — such that the subsets \( \text{Sub}(S), \mathbb{Fn}(S) \subseteq \mathbb{Fn}(M) \) are ample. If \( E \subseteq \mathbb{Fn}(M) \) is ample, then the image of \( E \) in \( \mathbb{Rss}(M) \) determines a natural subset \( \mathbb{Rss}(E) \subseteq \mathbb{Rss}(M) \), whose \( \mathbb{R}_{\geq 0} \)-orbit \( \mathbb{R}_{\geq 0} \cdot \mathbb{Rss}(E) \) is equal to \( \mathbb{Rss}(M) \). In particular, if \( S \subseteq \text{Sub}(M) \) is ample and compact, then we obtain natural \( \mathbb{R}_{\geq 0} \)-equivariant isomorphisms of monoids

\[
\mathbb{Rss}(S) \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0} \cdot \mathbb{Rss}(\mathbb{Fn}(S)) \overset{\sim}{\rightarrow} \mathbb{Rss}(M)
\]

— where, the notation “\( \mathbb{R}_{\geq 0} \cdot \mathbb{Rss}(\mathbb{Fn}(S)) \)” is intended relative to the interpretation of \( \mathbb{Fn}(S) \) as a subset of \( \mathbb{Fn}(M) \) — such that the composite isomorphism \( \mathbb{Rss}(S) \overset{\sim}{\rightarrow} \mathbb{Rss}(M) \) is compatible with the isomorphisms \( \int_{S}^{\mathbb{Rss}} : \mathbb{Rss}(S) \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0} \), \( \int_{M}^{\mathbb{Rss}} : \mathbb{Rss}(M) \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0} \). Finally, we observe that if \( (M_{1}, \mu_{M_{1}}) \) and \( (M_{2}, \mu_{M_{2}}) \) are ample measure spaces, then the product measure space \( (M_{1} \times M_{2}, \mu_{M_{1} \times M_{2}}) \) is also an ample measure space; moreover, there is a natural map

\[
\text{Sub}(M_{1}) \times \text{Sub}(M_{2}) \rightarrow \text{Sub}(M_{1} \times M_{2})
\]

that maps \( (S_{1}, S_{2}) \mapsto S_{1} \times S_{2} \) and induces a natural \( \mathbb{R}_{\geq 0} \)-equivariant isomorphism of monoids

\[
\mathbb{Rss}(M_{1}) \otimes \mathbb{Rss}(M_{2}) \overset{\sim}{\rightarrow} \mathbb{Rss}(M_{1} \times M_{2})
\]
that is compatible with the isomorphisms \( \int_{M_1}^{\text{Rss}} \otimes \int_{M_2}^{\text{Rss}} : \text{Rss}(M_1) \otimes \text{Rss}(M_2) \sim \mathbb{R}_{\geq 0}, \)

\( \int_{M_1 \times M_2}^{\text{Rss}} : \text{Rss}(M_1 \times M_2) \sim \mathbb{R}_{\geq 0}. \) [Here, we observe that there is a natural notion of “tensor product of monoids isomorphic to \( \mathbb{R}_{\geq 0} \)” since such a monoid may be thought of, by passing to the groupification of such a monoid, as a one-dimensional \( \mathbb{R} \)-vector space equipped with a subset [which forms a \( \mathbb{R}_{>0} \)-torsor] of “positive elements”].

(ii) One very rough approach to understanding the log-link compatibility of log-volumes is the following. Suppose that instead of knowing this property, one only knows that

each application of the log-link has the effect of dilating volumes by a factor \( \lambda \in \mathbb{R}_{>0} \setminus \{1\}. \)

[Here, relative to the notation of (i), we observe that this sort of situation in which volumes are dilated in a nontrivial fashion may be seen in the following example:

Suppose that \( M \overset{\text{def}}{=} \mathbb{Q}_p \), for some prime number \( p \), equipped with the [additive] Haar measure \( \mu_{\mathbb{Q}_p} \) normalized so that \( \mathbb{Z}_p \subseteq \mathbb{Q}_p \) has measure 1, so \( (\mathbb{Q}_p, \mu_{\mathbb{Q}_p}) \) is an ample measure space in the sense of (i). Then multiplication by \( p \) induces a bijection \( \alpha_p : \mathbb{Q}_p \sim \mathbb{Q}_p. \) Moreover, \( \alpha_p \) induces compatible bijections \( \text{Sub}(\alpha_p) : \text{Sub}(\mathbb{Q}_p) \sim \text{Sub}(\mathbb{Q}_p), \text{Fn}(\alpha_p) : \text{Fn}(\mathbb{Q}_p) \sim \text{Fn}(\mathbb{Q}_p), \text{Rss}(\alpha_p) : \text{Rss}(\mathbb{Q}_p) \sim \text{Rss}(\mathbb{Q}_p). \) On the other hand, [unlike the situation discussed in (i) concerning the “composite isomorphism \( \text{Rss}(S) \sim \text{Rss}(M) \)”] in the present context, \( \text{Rss}(\alpha_p) \) is not compatible with the isomorphisms \( \int_{\mathbb{Q}_p}^{\text{Rss}} : \text{Rss}(\mathbb{Q}_p) \sim \mathbb{R}_{\geq 0} \) in the domain and codomain of \( \text{Rss}(\alpha_p) \), i.e., it is only compatible up to a factor \( p^{-1} (\neq 1)! \]

Then in order to compute log-volumes in a fashion that is consistent with the various arrows [i.e., both Kummer isomorphisms and log-links!] of the “systems” constituted by the log-Kummer correspondences discussed in Proposition 3.5, (ii), it would be necessary to regard the various “log-volumes” computed as only giving rise to well-defined elements [not \( \in \mathbb{R} \), but rather]

\[ \in \mathbb{R} / \mathbb{Z} \cdot \log(\lambda) \quad (\cong \mathbb{S}^1) \]

— a situation which is not acceptable, relative to the goal of obtaining log-volume estimates [i.e., as in Corollary 3.12 below] for the various objects for which log-shells serve as “multiradial containers” [cf. the discussion of Remark 1.5.2; the content of Theorem 3.11 below].

(iii) In the following discussion, we use the notation of Remark 1.2.2, (i). Thus, we regard \( k \) as being equipped with the [additive] Haar measure \( \mu_k \) normalized so that \( \mu_k(\mathcal{O}_k) = 1 \) [cf. [AbsTopIII], Proposition 5.7, (i)]. Then \( (k, \mu_k) \) is an ample measure space in the sense of (i); \( \mathcal{O}_k^\times \subseteq k \) is an ample subset; for any compact ample subset \( S \subseteq \mathcal{O}_k^\times \) on which \( \log_k : \mathcal{O}_k^\times \to k \) is injective, we have \( \mu_k(S) = \mu_k(\log_k(S)) \) [cf. [AbsTopIII], Proposition 5.7, (i), (c)]. In particular, by applying the formalism of realified semi-simplifications introduced in (i), we conclude that the diagram

\[
k \supseteq \mathcal{O}_k^\times \xrightarrow{\log_k} \log_k(\mathcal{O}_k^\times) \subseteq k
\]

\[
S \quad \sim \quad \log_k(S)
\]
induces a commutative diagram
\[
\begin{array}{cccc}
\mathbb{R}ss(k) & \xleftarrow{\sim} & \mathbb{R}ss(O_k^\times) & \xleftarrow{\sim} & \mathbb{R}ss(S) & \xrightarrow{\sim} & \mathbb{R}ss(\log_k(S)) \\
\downarrow_{\int_{k}^{Rss}} & & \downarrow_{\int_{O_k^\times}^{Rss}} & & \downarrow_{\int_{S}^{Rss}} & & \downarrow_{\int_{\log_k(S)}^{Rss}} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} \\
\mathbb{R}ss(\log_k(O_k^\times)) & \xrightarrow{\sim} & \mathbb{R}ss(k) & & \mathbb{R}ss(\log_k(O_k^\times)) & \xrightarrow{\sim} & \mathbb{R}ss(k) \\
\int_{\log_k(O_k^\times)}^{Rss} & & \int_{k}^{Rss} & & \int_{\log_k(O_k^\times)}^{Rss} & & \int_{k}^{Rss} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}
\]

— where the vertical arrows are \(\mathbb{R}_{>0}\)-equivariant isomorphisms of monoids, and the composite \([\mathbb{R}_{>0}\text{-equivariant}]\) isomorphism \([\text{of monoids}]\) \(\mathbb{R}ss(O_k^\times) \xrightarrow{\sim} \mathbb{R}ss(\log_k(O_k^\times))\) is easily verified to be independent of the choice of the compact ample subset \(S \subseteq O_k^\times\). [Also, we observe that it is easily verified that there exist compact ample subsets \(S \subseteq O_k^\times\) for which the induced map \(S \twoheadrightarrow \log_k(S)\) is injective.] One may then compose this diagram with the bijection
\[
\log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R} \cup \{-\infty\}
\]
determined by the natural logarithm and then multiply by a suitable normalization factor \(\in \mathbb{R}_{>0}\) to conclude that

the diagram
\[
k \supseteq O_k^\times \xrightarrow{\log_k} \log_k(O_k^\times) \subseteq k
\]
induces \(\mathbb{R}_{>0}\)-equivariant isomorphisms of monoids on the respective realified semi-simplifications \("\mathbb{R}ss(-)"\), all of which are compatible with the log-volume maps on each of the \("\mathbb{R}ss(-)\)'s", i.e., which restrict to the “usual log-volume maps” on the respective “\(\text{Sub}(-)\)'s”, relative to the natural maps “\(\text{Sub}(-) \rightarrow \mathbb{R}ss(-)\)”.

This is one way to formulate the log-link compatibility of log-volumes discussed in Proposition 3.9, (iv), in the case of \(\mathcal{U} \in \mathcal{Y}^{\text{non}}\). Finally, we observe that this log-link compatibility with log-volumes is itself compatible with passing to finite extensions of \(k\) [or, more generally, \(\mathbb{Q}_{p_\infty}\)], as follows. Let \(k_1 \subseteq k_2\) be finite field extensions of \(\mathbb{Q}_{p_\infty}\). We shall use analogous notation for objects associated to \(k_1\) and \(k_2\) to the notation that was used above for objects associated to \(k\). Then observe that since \(O_{k_2}\) is a finite free \(O_{k_1}\)-module of rank \([k_2 : k_1]\), it follows that the [additive] compact topological group \(O_{k_2}\) is isomorphic to the product of \([k_2 : k_1]\) copies of the [additive] compact topological group \(O_{k_1}\). In particular, since the Haar measure of a compact topological group is invariant with respect to arbitrary automorphisms of the topological group, we thus conclude [cf. the discussion of product measure spaces in (i)] that the inclusion of topological fields \(k_1 \hookrightarrow k_2\) induces natural \(\mathbb{R}_{>0}\)-equivariant isomorphisms of monoids
\[
\mathbb{R}ss(k_1)^{\otimes [k_2:k_1]} \xleftarrow{\sim} \mathbb{R}ss(O_{k_1})^{\otimes [k_2:k_1]} \xrightarrow{\sim} \mathbb{R}ss(O_{k_2}) \xrightarrow{\sim} \mathbb{R}ss(k_2)
\]
such that the composite isomorphism \(\mathbb{R}ss(k_1)^{\otimes [k_2:k_1]} \xrightarrow{\sim} \mathbb{R}ss(k_2)\) is compatible with the \(\mathbb{R}_{>0}\)-equivariant isomorphisms of monoids
\[
\left(\int_{k_1}^{\mathbb{R}ss}\right)^{\otimes [k_2:k_1]} : \mathbb{R}ss(k_1)^{\otimes [k_2:k_1]} \xrightarrow{\sim} \mathbb{R}_{\geq 0}
\]
and \( \int_{k_2}^{\mathbb{R}_{ss}} : \mathbb{R}_{ss}(k_2) \sim \mathbb{R}_{\geq 0} \).

(iv) In the notation of (iii), suppose that \( v \in \mathbb{V}^{\text{bad}} \); write \( q \overset{\text{def}}{=} q \cdot v \). Thus, we have a submonoid

\[
\mathcal{O}_k^\times \times q_N \subseteq k
\]

of the underlying multiplicative monoid of \( k \). Then the various arrows of the log-Kummer correspondence discussed in Proposition 3.5, (ii), may be thought of, from the point of view of a vertically coric étale holomorphic copy of “\( k \)” [i.e., a copy labeled “\( n, o \)”, as in Proposition 3.5, (i)], as corresponding to the operations

\[
k \leadsto \mathcal{O}_k^\times \times q_N (\subseteq k) \\
\leadsto \mathcal{O}_k^\times (\subseteq k) \\
\leadsto \log_k (\mathcal{O}_k^\times) (\subseteq k) \leadsto k
\]

— i.e., of passing first from \( k \) to the multiplicative submonoid \( \mathcal{O}_k^\times \times q_N \), then to the multiplicative submonoid \( \mathcal{O}_k^\times \), then applying \( \log_k \) to obtain an additive submonoid of \( k \), and finally passing from this submonoid back to \( k \) itself. Then the log-volume compatibility discussed in (iii) may be understood, in the context of the log-Kummer correspondence, as the statement that

the operations of the above display induce \( \mathbb{R}_{>0} \)-equivariant isomorphisms of monoids

\[
\mathbb{R}_{ss}(k) \sim \mathbb{R}_{ss}(\mathcal{O}_k^\times) \sim \mathbb{R}_{ss}(\log_k (\mathcal{O}_k^\times)) \sim \mathbb{R}_{ss}(k)
\]

that are compatible with the respective [normalized] log-volume maps to \( \mathbb{R} \cup \{ -\infty \} \) [cf. the discussion of (iii)], in such a way as to avoid any interference, up to multiplication by roots of unity, with the submonoid \( q_N \subseteq k \), which induces, by applying the [normalized] log-volume to the image of \( \mathcal{O}_k \subseteq k \) via multiplication by elements of this submonoid, an embedding

\[
\mathbb{N} \hookrightarrow \mathbb{R}_{ss}(k) \sim \mathbb{R} \cup \{ -\infty \}
\]

that maps \( \mathbb{N} \ni 1 \mapsto -\log(q) \in \mathbb{R} \)

[where we write \( \log(q) \overset{\text{def}}{=} \text{ord}_p(q) \cdot \log(p) \in \mathbb{R} \) — cf. the notation of Remark 2.4.2, (ii)]. A similar interpretation of log-volume compatibility in the context of the log-Kummer correspondence may be given in the case of \( v \in \mathbb{V}^{\text{good}} \cap \mathbb{V}^{\text{mon}} \) by simply omitting the portion of the above discussion concerning “\( q \)”.

(v) In the notation of Remark 3.9.1, (i), we observe that the discussion of (iii), (iv), may be extended to topological tensor products of the form

\[
k_{i_A} \overset{\text{def}}{=} \bigotimes_{\alpha \in A} k_{i_\alpha}
\]

— where \( i_\alpha \in \{ 1, \ldots, n_{v_\alpha} \} \), for each \( \alpha \in A \), and we regard \( k_{i_\alpha} \) as being equipped with the [additive] Haar measure normalized [cf. Proposition 3.9, (i)] so that the
ring of integers $\mathcal{O}_{k_i^A} \subseteq k_i^A$ [i.e., the integral structure discussed in Proposition 3.1, (ii)] has Haar measure $= 1$. Indeed, each of the direct summand fields of $k_i^A$ [cf. Proposition 3.1, (i)] may be taken to be a [finite extension of] “$k$” as in (iii), (iv). In particular, the measure space $k_i^A$ may be regarded as a product measure space of [finite extensions of] “$k$” as in (iii), (iv), so one may extend (iii), (iv) to $k_i^A$ by applying (iii), (iv) to each factor of this product measure space [cf. the discussion of product measure spaces in (i)]. [We leave the routine details to the reader.] On the other hand, in this context, it is also of interest to observe that it follows immediately from the discussion of compatibility with finite extensions in (iii), together with the discussion of product measure spaces in (i), that, for each $\alpha \in A$, the natural structure of $k_i^A$ as a $k_i^\alpha$-algebra determines natural $\mathbb{R}_{>0}$-equivariant isomorphisms of monoids

$$\mathbb{R}ss(\mathbb{Q}_{p_v}) \otimes^{d_A} \sim \mathbb{R}ss(k_i^\alpha) \otimes^{d_\alpha} \sim \mathbb{R}ss(k_i^A)$$

— where we write $d_\alpha \overset{\text{def}}{=} \prod_{\beta \in A \setminus \{\alpha\}} [k_i^\beta : \mathbb{Q}_{p_v}]$, $d_A \overset{\text{def}}{=} d_\alpha \cdot [k_i^\alpha : \mathbb{Q}_{p_v}]$. In particular,

the log-link compatibility of log-volumes [as discussed above] for the realified semi-simplification $\mathbb{R}ss(k_i^A)$ of the topological tensor product $k_i^A$ may be understood, for any $\alpha \in A$, as the [functorially induced!] $d_\alpha$-th tensor power of the log-link compatibility of log-volumes for the realified semi-simplification $\mathbb{R}ss(k_i^\alpha)$ of $k_i^\alpha$ or, alternatively/equivalently, as the [functorially induced!] $d_A$-th tensor power of the log-link compatibility of log-volumes for the realified semi-simplification $\mathbb{R}ss(\mathbb{Q}_{p_v})$

— where we note that the latter “alternative/equivalent” approach has the virtue of being independent of the choice of $\alpha \in A$.

(vi) In the following discussion, we use the notation of Remark 1.2.2, (ii). We regard the complex archimedean field $k$ as being equipped with the standard Euclidean metric [cf. the discussion of “metrics” in Remark 1.2.1, (ii)], with respect to which $\mathcal{O}_k^\times \subseteq k$ has length $2\pi$. This metric on $k$ thus determines measures $\mu_{|k|}$ on $|k| \overset{\text{def}}{=} k/\mathcal{O}_k^\times$ and $\mu_{\mathcal{O}_k^\times}$ on $\mathcal{O}_k^\times \subseteq k$ [cf. the situation discussed in [AbsTopIII], Proposition 5.7, (ii)] such that $(|k|, \mu_{|k|})$ and $(\mathcal{O}_k^\times, \mu_{\mathcal{O}_k^\times})$ are ample measure spaces in the sense of (i). Moreover, by

(a) thinking of $\mathcal{O}_k^\times$ as a union of closed arcs [i.e., whose interiors are disjoint] of measure $\mu_{\mathcal{O}_k^\times}(-) < \epsilon$, for some positive real number $\epsilon$,

(b) considering additive translates of such closed arcs that map one of the endpoints of the arc to $0 \in k$,
(c) projecting such additive translates via the natural surjection \( k \to |k| \), and
(d) passing to the limit \( \epsilon \to 0 \),

one verifies immediately that we obtain, by applying the formalism of realified semi-simplifications introduced in (i), a natural \( \mathbb{R}_{>0} \)-equivariant isomorphism of monoids \( \rho_k : \text{Rss}(\mathcal{O}_k^\times) \xrightarrow{\sim} \text{Rss}(|k|) \), together with a commutative diagram

\[
\begin{array}{cccc}
\text{Rss}(|k|) & \xleftarrow{\int_{|k|}} & \text{Rss}(\mathcal{O}_k^\times) & \xleftarrow{\int_{\mathcal{O}_k^\times}} & \text{Rss}(|k|) \\
\begin{array}{c}
\int_{\text{Rss}} \\
\int_{\text{Rss}} \\
\int_{\text{Rss}} \\
\int_{\text{Rss}}
\end{array} & & \begin{array}{c}
\int_{\mathcal{O}_k^\times} \\
\int_{\mathcal{O}_k^\times} \\
\int_{\log_k(\mathcal{O}_k^\times)} \\
\int_{|k|}
\end{array} & & \begin{array}{c}
\int_{\text{Rss}} \\
\int_{\text{Rss}} \\
\int_{\log_k(\mathcal{O}_k^\times)} \\
\int_{|k|}
\end{array}
\end{array}
\]

\[\mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0}\]

— where the vertical arrows are \( \mathbb{R}_{>0} \)-equivariant isomorphisms of monoids; the first "\( \xleftarrow{\sim} \)" is \( \rho_k \); we regard \( \log_k(\mathcal{O}_k^\times) \defeq \exp_k^{-1}(\mathcal{O}_k^\times) \) as being equipped with the measure \( \mu_{\log_k(\mathcal{O}_k^\times)} \) such that \( (\log_k(\mathcal{O}_k^\times), \mu_{\log_k(\mathcal{O}_k^\times)}) \) is an ample measure space] obtained by pulling back \( \mu_{|k|} \) via the homeomorphism \( \log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k| \) induced by restricting the natural surjection \( k \to |k| \) to \( \log_k(\mathcal{O}_k^\times) \subseteq k \); the second "\( \xleftarrow{\sim} \)" is the natural \( \mathbb{R}_{>0} \)-equivariant isomorphism of monoids naturally induced [i.e., by considering ample \( S \subseteq \log_k(\mathcal{O}_k^\times) \) that map bijectively to \( \exp_k(S) \subseteq \mathcal{O}_k^\times \)— cf. [AbsTopIII], Proposition 5.7, (ii), (c)] by the universal covering map \( \exp_k|_{\log_k(\mathcal{O}_k^\times)} : \log_k(\mathcal{O}_k^\times) \to \mathcal{O}_k^\times \); the "\( \xrightarrow{\sim} \)" is the natural \( \mathbb{R}_{>0} \)-equivariant isomorphism of monoids induced by the homeomorphism \( \log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k| \). One may then compose this diagram with the bijection \( \log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R} \cup \{-\infty\} \) determined by the natural logarithm and then multiply by a suitable normalization factor \( \in \mathbb{R}_{>0} \) to conclude that

the diagram

\[
|k| \xleftarrow{k} \mathcal{O}_k^\times \xrightarrow{\exp_k} \log_k(\mathcal{O}_k^\times) \subseteq k \xrightarrow{\sim} |k|
\]

induces \( \mathbb{R}_{>0} \)-equivariant isomorphisms of monoids on the respective realified semi-simplifications "\( \text{Rss}(\cdot) \)" of \( |k|, \mathcal{O}_k^\times, \log_k(\mathcal{O}_k^\times), \) and \( |k| \); each of these isomorphisms is compatible with the log-volume map on "\( \text{Rss}(\cdot) \)" i.e., which restricts to the "usual radial/angular log-volume map" on "\( \text{Sub}(\cdot) \)" [that is to say, the map uniquely determined by the radial/angular log-volume map of [AbsTopIII], Proposition 5.7, (ii), (a)] relative to the natural map "\( \text{Sub}(\cdot) \to \text{Rss}(\cdot) \)"

This is one way to formulate the log-link compatibility of log-volumes discussed in Proposition 3.9, (iv), in the case of \( v \in \mathbb{V}_{\text{arc}} \). One verifies immediately that one also has analogues for \( v \in \mathbb{V}_{\text{real}} \) of (iv), (v). [We leave the routine details to the reader.]

Remark 3.9.5. In situations that involve consideration of various sorts of regions [cf. the discussion of Remarks 3.1.1, (iii), (iv); 3.9.4] to which the log-volume may be applied, it is often of use to consider the notion of the holomorphic hull of a region.
(i) Suppose that we are in the situation of Proposition 3.9, (i). Let
\[ αU ⊆ I \mathbb{Q}((α F_\psi)) \] (respectively, \[ A U ⊆ I \mathbb{Q}(A α F_\psi) \] ; \[ A,α U ⊆ I \mathbb{Q}(A,α F_\psi) \]) be a subset that contains a relatively compact subset whose log-volume [cf. the discussion of Remark 3.1.1, (iii), (iv), as well as Remark 3.9.7, (ii), below] is finite [i.e., \( > -\infty \)]. If \( α U \) (respectively, \( A U \); \( A,α U \)) is relatively compact, then we define the holomorphic hull of \( α U \) (respectively, \( A U \); \( A,α U \)) to be the smallest subset of the form
\[ αH \overset{\text{def}}{=} λ \cdot O_α F_\psi \] (respectively, \( A H \overset{\text{def}}{=} λ \cdot O_{A,α} F_\psi \)) — where, relative to the direct sum decomposition of \( I \mathbb{Q}((-)) \) as a direct sum of fields [cf. the discussion of Proposition 3.9, (i)], \( λ \in I \mathbb{Q}((-)) \) is an element such that each component of \( λ \) [i.e., relative to this direct sum decomposition] is nonzero — that contains \( α U \) (respectively, \( A U \); \( A,α U \)). If \( α U \) (respectively, \( A U \); \( A,α U \)) is not relatively compact, then we define the holomorphic hull of \( α U \) (respectively, \( A U \); \( A,α U \)) to be \( I \mathbb{Q}(α F_\psi) \) (respectively, \( I \mathbb{Q}(A F_\psi) \); \( I \mathbb{Q}(A,α F_\psi) \)). One verifies immediately that the holomorphic hull is well-defined [under the conditions stated].

(ii) In the remainder of the discussion of the present Remark 3.9.5, for the sake of simplicity, we shall refer to “holomorphic hulls” as “hulls.” Write
\[ P \overset{\text{def}}{=} \{ P ⊆ I \mathbb{Q}((-)) \mid P \text{ is a direct product region} \} \]
\[ H \overset{\text{def}}{=} \{ H ⊆ I \mathbb{Q}((-)) \mid H \text{ is a hull} \} \]
— where the argument “\((-\))” is “\( α F_\psi \)”, “\( A F_\psi \)”, or “\( A,α F_\psi \)” [cf. (i)], and we observe that \( H ⊆ P \). Then it is essentially a tautology that the operation of forming the hull discussed in (i)
\[ □ U ↦ □ H \]
— where “□” is “\( α \)”, “\( A \)”, or “\( A,α \)” — determines a map
\[ φ : P → H \]
that may be characterized uniquely by the following properties

(P1) \( φ(H) = H \), for any \( H ∈ H \);
(P2) \( P ⊆ φ(P) \), for any \( P ∈ P \);
(P3) \( φ(P_1) ⊆ φ(P_2) \), for any \( P_1, P_2 ∈ P \) such that \( P_1 ⊆ P_2 \).

Indeed, since, as is easily verified, any intersection of elements of \( H \) which is of finite log-volume necessarily determines an element of \( H \), it follows formally from (P1), (P2), (P3) that
\[ φ(P) = \bigcap_{H ∈ H ⊇ P} H \]
for any \( P ∈ P \). Put another way, this map \( φ \) may be thought of as a sort of adjoint, or push forward in the opposite direction, of the inclusion \( H ⊆ P \). Alternatively, \( φ \) may be thought of as a sort of canonical splitting of the inclusion \( H ⊆ P \), or, in the spirit of the discussion of Remark 3.9.4, as a sort of integration operation. The compatibility [cf. (P2), (P3)] of \( φ \) with the pre-order structure on \( P \) determined
by inclusion of direct product regions will play an important role in the context of various log-volume estimates of regions.

(iii) Now we consider the various log-volumes $\mu_{(-)}^{\log}$ [where the argument “(−)” is “$\alpha, v_Q$”, “$A, v_Q$”, or “$A, \alpha, v$” — cf. (ii) of Proposition 3.9, (i) [cf. also Remark 3.1.1, (iii)], in the context of the discussion of (ii)]. In the following, for the sake of simplicity, we shall denote “$\mu_{(-)}^{\log}$” by $\mu^{\log}$. For $P \in \mathbb{P}$, write

$$\Phi(P) \overset{\text{def}}{=} \{ H \in \mathbb{H} \mid \phi(P) \supseteq H, (\mu^{\log}(\phi(P)) \geq \mu^{\log}(H) \geq \mu^{\log}(P) \} \subseteq \mathbb{H};$$

$$\Xi(P) \overset{\text{def}}{=} \{ H \in \mathbb{H} \mid \phi(P) \supseteq H, (\mu^{\log}(\phi(P)) \geq \mu^{\log}(H) = \mu^{\log}(P) \} \subseteq \Phi(P);$$

$$H_{\Phi(P)} \overset{\text{def}}{=} \bigcup_{H \in \Phi(P)} H \subseteq \phi(P); \quad H_{\Xi(P)} \overset{\text{def}}{=} \bigcup_{H \in \Xi(P)} H \subseteq H_{\Phi(P)} \subseteq \phi(P).$$

Thus, one may think of elements $\in \Phi(P)$ or $\in \Xi(P)$ as “log-volume approximations” of $P$ by means of hulls $\in \mathbb{H}$.

If one thinks of distinct elements $\in \Phi(P)$ or $\in \Xi(P)$ — i.e., of the issue of constructing a “log-volume hull-approximant” of $P$ — as a sort of indeterminacy [i.e., in the assignment to $P$ of a specific element $\in \mathbb{H}$!], then

this indeterminacy is compact, i.e., in the sense that all possible choices of an element $\in \Phi(P)$ or $\in \Xi(P)$ are contained in the compact set $\phi(P) \in \mathbb{H}$.

Indeed, developing the theory in such a way that

all the indeterminacies that occur in the theory are compact

is in some sense one important theme in the present series of papers. Note that this compactness would not be valid if, in the definition of $\Phi(−)$ or $\Xi(−)$, one omits the condition “$H \subseteq \phi(P)$”.

(iv) In the context of (iii), we observe that

$$\phi(P) \in \Phi(P), \text{ so } \phi(P) = H_{\Phi(P)},$$

but the issue of whether or not $\Xi(P) = \emptyset$ is not so immediate. Indeed:

(Ξ1) If either of the following conditions is satisfied, then it is easily verified that $\Xi(P) \neq \emptyset$:

(Ξnon1) if we write $K^{\text{cl}}$ for the Galois closure of $K$ over $Q$, then the residue field extension degree of each valuation $\in \mathbb{V}(K^{\text{cl}})$ that divides $v_Q \in \mathbb{V}_Q^{\text{non}}$ is $= 1$, and, moreover, $\mu^{\log}(P) = \mu^{\log}(Q)$, for some $Q \in \mathbb{P}$ which is a $\mathbb{Z}_{p_{\text{eq}}}^{\text{-submodule}}$ of $\mathcal{T}_Q^{\text{arc}}(−);$ (Ξarc1) $v_Q \in \mathbb{V}_Q^{\text{arc}}$.

(Ξ2) If one allows the $v_Q \in \mathbb{V}_Q$ in the present discussion to vary, and one considers global situations [i.e., which necessarily involve the unique valuation $\in \mathbb{V}_Q^{\text{arc}}$] as in Proposition 3.9, (iii), then it is easily verified that the global analogue of “$\Xi(P)$” is nonempty.
On the other hand, in general, it is not so clear whether or not \( \Xi(P) \neq \emptyset \). In this context, it is also of interest to observe that if \( P \in \mathbb{H} \), then

\[
\{ \phi(P) \} = \Phi(P) = \Xi(P), \text{ so } P = \phi(P) = H_{\Phi(P)} = H_{\Xi(P)},
\]

but in general, even in the situation of \((\Xi1)\), the inclusion \( P \subseteq \phi(P) = H_{\Phi(P)} \), as well as the induced inequality of log-volumes \( \mu^{\log}(P) \leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P)) \), is strict. Indeed, for instance,

\((\Xi3)\) in the situation of \((\Xi1)\), if \((V_{\text{mod}})_{v_0} \) [cf. the notational conventions of Remark 3.1.1, (iii)] and \( \Lambda \) are of cardinality \( \geq 2 \), then one verifies easily that there exist \( P \in \mathbb{P} \) for which \( \mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)}) \) (\( \leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P)) \)).

This sort of phenomenon may be seen in the following example:

Let \( p \) be a prime number. Write \( I \overset{\text{def}}{=} \mathbb{Q}_p \times \mathbb{Q}_p \);

\[
H_0 \overset{\text{def}}{=} \mathbb{Z}_p \times \mathbb{Z}_p \subseteq I; \quad H_1 \overset{\text{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times (p \cdot \mathbb{Z}_p) \subseteq I; \quad P \overset{\text{def}}{=} H_0 \cup \{(p^{-1}) \times \mathbb{Z}_p\} \subseteq I; \quad H_P \overset{\text{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times \mathbb{Z}_p \subseteq I
\]

— where we think of \( I \) as being equipped with the Haar measure \( \mu_I \) normalized so that \( \mu_I(H_0) = 1 \). Thus, \( p \) corresponds to \("p_{v_0}"\) such that \("(V_{\text{mod}})_{v_0}"\) is of cardinality \( 2 \); \( I \) corresponds to \("\mathbb{Z}^2(\mathfrak{F}_{v_0})"\); \( P \) corresponds to \("P"\); \( H_P \) corresponds to \("\phi(P)"\); \( H_0 \) and \( H_1 \) correspond to elements of \("\Xi(P)"\), so \( H_0 \cup H_1 \) corresponds to a subset of \("H_{\Xi(P)}"\), hence also a subset of \("H_{\Phi(P)} = \phi(P)"\). Then

\[
\mu_I(P) = \mu_I(H_0) = 1 < 2 - p^{-1} = \mu_I(H_0 \cup H_1)
\]

— i.e., the inequality of [log-]volumes in question is strict. In fact, by considering various translates of \( H_0, H_1 \) by automorphisms of the \( \mathbb{Z}_p \)-module \( H_P \), one verifies immediately that \( H_P \) corresponds not only to \("\phi(P) = H_{\Phi(P)}"\), but also to \("H_{\Xi(P)}"\). That is to say, this is a situation in which one has \("H_{\Xi(P)} = H_{\Phi(P)} = \phi(P)"\), hence also \("\mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)}) = \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))"\).

(v) Let \( E \) be a set, \( S \subseteq E \) a proper subset of \( E \) of cardinality \( \geq 2 \) [so \( S \neq \emptyset \neq E \setminus S \)]. Write

\[
(E \to) \quad E \overline{\setminus} S \overset{\text{def}}{=} (E \setminus S) \bigsqcup \{S\}
\]

[i.e., \("E \ \text{upper} \ S"\)] for the set-theoretic quotient of \( E \) by \( S \), i.e., the quotient of \( E \) obtained by identifying the elements of \( S \) and leaving \( E \setminus S \) unaffected. Write \( \overline{\setminus} S \overset{\text{def}}{=} \{S\} \subseteq E \overline{\setminus} S \). Then observe that

any set-theoretic map

\[
(E \supseteq) \quad S_1 \to S_2 (\subseteq E)
\]

between nonempty subsets \( S_1, S_2 \subseteq S (\subseteq E) \) induces, upon passing to the quotient \( E \to E \overline{\setminus} S \), the identity map

\[
(E \overline{\setminus} S \supseteq) \quad \overline{\setminus} S \to \overline{\setminus} S (\subseteq E \overline{\setminus} S)
\]
between the images [i.e., both of which are equal to $\mapsto S$] of $S_1$, $S_2$ in $E \mapsto S$, hence lies over the identity map $E \mapsto S \mapsto E \mapsto S$ on $E \mapsto S$.

Moreover, this map may be “extended” to the case where $S_i$ [for $i \in \{1, 2\}$] is empty if this $S_i$ is treated as a “formal intersection” [cf. our hypothesis that the cardinality of $S$ is $\geq 2$] — i.e., a “category-theoretic formal fiber product, or inverse system, over $E$” — of some collection of nonempty subsets of $S$. That is to say, such an inverse system induces, upon passing to the quotient $E \mapsto E \mapsto S$, a system that consists of identity maps between copies of $\mapsto S$. In particular,

if one thinks in terms of such formal inverse systems, then “formal empty sets” $\subseteq S$ (⊆ $E$) also map to $\mapsto S \subseteq E \mapsto S$.

Finally, we observe that the above discussion may be thought of as an

abstract set-theoretic formalization of the notions of upper semi-commutativity/semi-compatibility, as discussed in Remark 1.2.2, (iii); Remark 1.5.4, (iii); Proposition 3.5, (ii)

— i.e., where [cf. the notational conventions of Propositions 3.2, (ii); 3.5, (ii)] one takes the $S \subseteq E$ of the present discussion to be “$\mathcal{I}((-)) \subseteq \mathcal{I}^\mathbb{Q}((-))$”, and we observe that, in the context of upper semi-commutativity/semi-compatibility, the empty set always arises as an intersection between a nonempty set and the domain of definition [cf. the discussion of Remark 1.1.1] of the “set-theoretic logarithm map” under consideration.

(vi) Let us return to the discussion of (ii), (iii), (iv). Let $P \in \mathbb{P}$. Then let us observe that the abstract set-theoretic “$\mapsto S$-formalism” of (v) — i.e., where one takes “$S \subseteq E$” to be $\phi(P) \subseteq \mathcal{I}^\mathbb{Q}((-))$ — yields a convenient tool for identifying $P$ with its various log-volume hull-approximants $\in \Phi(P)$ or $\in \Xi(P)$ [all of which are nonempty subsets of $\phi(P) \in \mathbb{H}$ — cf. the discussion of (iii)], i.e., of passing to a quotient in which the indeterminacy discussed in (iii) is eliminated.

Moreover, one verifies easily that this identification is achieved in such a way that images of distinct $H_1, H_2 \in \mathbb{H}$ map to the same subset of $\mathcal{I}^\mathbb{Q}((-)) \mapsto \mathcal{I}^\mathbb{Q}((-)) \mapsto \phi(P)$ if and only if $H_1, H_2 \subseteq \phi(P)$. That is to say, the equivalence relation on $\mathbb{H}$ induced by the quotient map $\mathcal{I}^\mathbb{Q}((-)) \mapsto \mathcal{I}^\mathbb{Q}((-)) \mapsto \phi(P)$ is the “expected equivalence relation”

$\mathbb{H} \ni H_1 \sim H_2 \in \mathbb{H} \iff H_1, H_2 \subseteq \phi(P)$

on $\mathbb{H}$. Finally, we observe that the discussion of the present (vi) may be applied not only to single elements $P \in \mathbb{P}$, but also to bounded families of elements $P_B \overset{\text{def}}{=} \{P_\beta\}_{\beta \in B}$ indexed by some index set $B$ [i.e., collections of elements $P_\beta \in \mathbb{P}$ such that $\cup_{\beta \in B} P_\beta \subseteq \mathcal{I}^\mathbb{Q}((-))$ is relatively compact], by taking “$S \subseteq E$” in the discussion of (v) to be

$\phi(P_B) \overset{\text{def}}{=} \bigcap_{\mathbb{H} \ni H \supseteq P_\beta, \forall \beta \in B} H \subseteq \mathcal{I}^\mathbb{Q}((-))$

[cf. the representation of $\phi(P)$ as an intersection in (ii)].
(vii) The operation of forming the hull will play a crucial role in the context of Corollary 3.12 below, for the following reason:

the output of “possible images” [cf. the statement of Corollary 3.12] that arises from applying the multiradial algorithms of Theorem 3.11 below cannot be directly compared [i.e., at least in any a priori sense] to the objects in the local and global Frobenioids that appear in the codomain \( \mathcal{F}^\mathbf{p} \times^\mathbf{\mu} \)-prime-strip of the \( \Theta_{LGP} \)-link [cf. Definition 3.8, (i), (ii); [IUTchII], Definition 4.9, (viii)] determined by the arithmetic line bundle that gives rise to the \( q \)-pilot object.

The obstructions to performing such a comparison may be eliminated in the following way [cf., especially, the display of (Ob5)]:

(Ob1) \( \mathcal{O} \times \)-Indeterminacies acting on tensor packets of log-shells: The various “possible images” that occur as the output of the multiradial algorithms under consideration are regions — i.e., in essence, elements \( \in \mathbb{P} \) — contained in tensor packets of log-shells \( \mathcal{I}_k \) [where, for simplicity, we apply the notational conventions of Remark 1.2.2, (i), at nonarchimedean valuations]. By contrast, the arithmetic line bundle that gives rise to the \( q \)-pilot object arises, locally, as an ideal, i.e., an \( \mathcal{O}_k \)-submodule, contained in the \( \mathcal{O}_k \)-module \( \mathcal{O} \), which, to avoid confusion, we denote by \( \mathcal{O}_k^{mdl} \). Here, we observe that

unlike the ring \( \mathcal{O}_k \), the \( \mathcal{O}_k \)-module \( \mathcal{O}_k^{mdl} \) does not admit a canonical generator [i.e., a canonical element corresponding to the element \( 1 \in \mathcal{O}_k \)]; by contrast, \( \mathcal{I}_k \subseteq k \) can only be defined by using the ring structure of \( \mathcal{O}_k \) and is not, in general, stabilized by the natural action [via multiplication] by \( \mathcal{O}_k^{\times} \).

That is to say, \( \mathcal{O}_k^{mdl} \) only admits a “canonical generator” up to an indeterminacy given by multiplication by \( \mathcal{O}_k^{\times} \), i.e., an indeterminacy that does not stabilize \( \mathcal{I}_k \).

(Ob2) From arbitrary regions to arithmetic vector bundles, i.e., hulls: Thus, by passing from an arbitrary given region \( \in \mathbb{P} \) to the associated hull \( \phi(P) \in \mathbb{H} \), we obtain a region \( \phi(P) \in \mathbb{H} \) that is stabilized by the natural action of \( \mathcal{O}_k^{\times} \) [cf. (Ob1)] and, moreover, [unlike an arbitrary element \( \in \mathbb{P} \)] may be regarded as defining the local portion of a global arithmetic vector bundle relative to the ring structure labeled by some \( \alpha \in A \) [i.e., which is typically taken, when \( 0 \in A \subseteq |F| \), to be the zero label \( 0 \in |F| \)].

(Ob3) From arithmetic vector bundles to arithmetic line bundles via “det\( _{-\mathbb{M}}\)(-)”: Moreover, by forming the determinant of the arithmetic vector bundle constituted by a hull \( \in \mathbb{H} \), one obtains an arithmetic line bundle, i.e., which does indeed yield objects in the local and [by allowing \( v_\mathcal{Q} \in \mathcal{V}_\mathcal{Q} \), \( v \in \mathcal{V} \) to vary] global Frobenioids that appear in the codomain \( \mathcal{F}^\mathbf{p} \times^\mathbf{\mu} \)-prime-strip of the \( \Theta_{LGP} \)-link, hence may be compared, in a meaningful way, to the objects determined by the arithmetic line bundle that gives rise to the \( q \)-pilot object. Here, we observe that:

(Ob3-1) Weighted tensor powers/determinant: In fact, when forming such a “determinant”, it is necessary to perform the following
operations:

(Ob3-1-1) In order to obtain a “determinant” that is consistent with the computation of the log-volume by means of certain weighted sums [cf. the discussion of Remark 3.1.1], it is necessary to work with suitable positive tensor powers [i.e., corresponding to the weights — cf. the various products of “$N_v$’s” in the discussion of the final portion of Remark 3.1.1, (iv)] of the determinant line bundles corresponding to the various direct summands [as in the second and third displays of Proposition 3.1] of the tensor packet of log-shells “$T_Q^{\mathcal{O}((-\cdot))}$”.

(Ob3-1-2) In order to obtain a “determinant” that is consistent with the normalization of the log-volume given by “$\mathcal{O}(-\cdot)$” [cf. Proposition 3.9, (i)], it is necessary to tensor the “determinant” of (Ob3-1-1) with the inverse of the “determinant” [in the sense of (Ob3-1-1)] of the structure sheaf [i.e., “$\mathcal{O}(-\cdot)$”], which may be thought of as a sort of adjustment to take into account the ramification that occurs in the various local fields involved. [We leave the routine details to the reader.]

(Ob3-2) Positive tensor powers of the determinant: In the context of (Ob3-1), we observe that there is no particular reason to require that the various exponents [i.e., which correspond to weights — cf. the various products of “$N_v$’s” in the discussion of the final portion of Remark 3.1.1, (iv)] of these “suitable positive tensor powers” are necessarily relatively prime. In particular, the resulting “determinant” might in fact be more accurately described as a “determinant raised to some positive tensor power”. In the following, we shall denote this operation of forming the “determinant raised to some positive tensor power” by means of the notation

$$\text{“det}_{\otimes M}(-\cdot)$$

— where $M$ denotes the [uniquely determined] positive integer [cf. the positive integer “$N_F$” that appears in the final portion of the discussion of Remark 3.1.1, (iv)] such that this operation “$\text{det}_{\otimes M}(-\cdot)$” maps [the result of tensoring the “$\mathcal{O}(-\cdot)$” of Proposition 3.9, (i), with] an arithmetic line bundle to the $M$-th tensor power of the arithmetic line bundle. Thus, for instance, by taking $M$ to be sufficiently large [in the “multiplicative sense”, i.e., “sufficiently divisible”], we may, for the sake of simplicity, assume [cf. the “stack-theoretic twists” at $v \in \mathcal{V}^{\text{bad}}$, arising from the structure of the stack-theoretic quotient discussed in [IUTchI], Remark 3.1.5] that the localization at each $v \in \mathcal{V}$ of any arithmetic line bundle that appears as the output of the operation $\text{det}_{\otimes M}(-\cdot)$ is always trivial.

(Ob3-3) Determinants and log-volumes: Finally, we observe in passing that since [cf. the situation discussed in Proposition 3.9, (iii)]
the arithmetic degree of such an arithmetic line bundle may be interpreted, by working with suitable normalization factors, as the log-volume of the original arithmetic vector bundle [i.e., to which the operation $\det_{\mathbb{Q}_M}(-)$ was applied — cf. (Ob3-1), (Ob3-2); the discussion of Remark 3.1.1], this intermediate step of applying $\det_{\mathbb{Q}_M}(-)$ may be omitted in discussions in which one is only interested in computing log-volumes.

(Ob4) Positive tensor powers of arithmetic line bundles: From the point of view of the original goal [cf. the discussion at the beginning of the present (vii)] of obtaining objects that may be compared to the objects in the local and global Frobenioids that appear in the codomain $\mathcal{F}^{\times_\mu}$-prime-strip of the $\Theta_{\text{LGP}}^{\times_\mu}$-link determined by the arithmetic line bundle that gives rise to the $q$-pilot object, we thus conclude from (Ob3) that applying the operation $\det_{\mathbb{Q}_M}(-)$ yields objects that may indeed be compared to the objects in the local and global Frobenioids that appear in the codomain $\mathcal{F}^{\times_\mu}$-prime-strip of the $\Theta_{\text{LGP}}^{\times_\mu}$-link determined by the arithmetic line bundle that gives rise to the $M$-th tensor power of the $q$-pilot object.

Since, however, the internal structure of these local and global Frobenioids [as well of as the localization functors that relate local to global Frobenioids] remains unaffected, the latter “slightly modified goal” [i.e., of comparison with $M$-th tensor powers of objects that arise from the $q$-pilot object, as opposed to the “original goal” of comparison with objects that arise from the original $q$-pilot object] does not result in any substantive problems such as, for instance, an indeterminacy arising from confusion between a given arithmetic line bundle and its $M$-th tensor power [i.e., an indeterminacy analogous to the indeterminacy involving “$\mathbb{T}^{\text{ord}}$” discussed in Remark 2.3.3, (vi)]. One way to understand this situation is as follows:

(Ob4-1) From non-tensor-power to tensor-power Frobenius functors: One may think of the local and global Frobenioids [as well as of the localization functors that relate local to global Frobenioids] that appear in the “slightly modified goal” as “$M$-th tensor power versions” of the local and global Frobenioids that appear in the “original goal”. That is to say, one may think of these “tensor-power Frobenioids” as copies of the “non-tensor-power Frobenioids” obtained by applying the naive Frobenius functor of degree $M$ of [FrdI], Proposition 2.1, (i). In particular, we conclude [i.e., from [FrdII], Proposition 2.1, (i)] that the non-tensor-power Frobenioids completely determine the tensor-power Frobenioids.

(Ob4-2) From tensor-power to non-tensor-power Frobenioids via tensor power roots: Alternatively, one may think of the non-tensor-power Frobenioids [i.e., that appear in the “original goal”] as being obtained from the tensor-power Frobenioids [i.e., that appear in the “slightly modified goal”] by “extracting $M$-th power roots”. Since the rational function monoids [cf. [FrdII]],
Theorem 5.2, (ii)] that give rise to the various local Frobenioids under consideration [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] are not divisible at \( v \in \mathcal{X}^{\text{non}} \), the tensor-power Frobenioids only determine the non-tensor-power Frobenioids up to certain twists. Of course, these twists may be eliminated [cf. (Ob4-1)] simply by applying the naive Frobenius functor of degree \( M \).

(Ob4-3) Tensor-power-twist indeterminacies: In particular, if one thinks of the output of the crucial operation \( \det_{\otimes M}(-) \) [cf. (Ob3)] as lying in the tensor-power Frobenioids, then one may always “reconstruct” the non-tensor-power Frobenioids from the tensor-power Frobenioids simply by considering new copies of the tensor-power Frobenioids which are related to the given copies of tensor-power Frobenioids by applying the naive Frobenius functor of degree \( M \) whose domain is the new copies, and whose codomain is the given copies. On the other hand, these reconstructed non-tensor-power Frobenioids, though completely determined up to isomorphism, are only related to one another, when regarded over the given copies of tensor-power Frobenioids, up to certain twists — i.e., up to a “tensor-power-twist indeterminacy” — as discussed in (Ob4-2). Since, however, we shall ultimately [e.g., in the context of Corollary 3.12] only be interested in estimates of log-volumes, such tensor-power-twist indeterminacies will not have any substantive effect on our computations [i.e., of log-volumes — cf. the discussion of (Ob3-3)].

(Ob5) Independence of the “indeterminacy of possibilities”: The issue of selecting a specific element in some collection of “possible regions” \( \in \mathcal{P} \) that appears in the output of the multiradial algorithm is an issue that is internal to the algorithm. In particular, in order to compare, in a meaningful way, the output of the algorithm to some object — i.e., such as the arithmetic line bundle that gives rise to the \( q \)-pilot object — that is essentially external to the algorithm, it is necessary to work with objects that are independent of the choice of such a specific element/possibility. This may be achieved by working with the hull [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4)]

\[
\phi(P_B)
\]

associated to the [bounded] collection of possible regions \( P_B \overset{\text{def}}{=} \{P_\beta\}_{\beta \in \mathcal{B}} \) [cf. the discussion in the final portion of (vi)] that appears as the output of the multiradial algorithms under consideration and applying the abstract set-theoretic formalism of (v) [cf. also (vi)].

Here, we observe that this formalism of (v) may be applied not only to \( \phi(P_B) \) but also [cf. the discussion of (Ob3)] to \( \det_{\otimes M}(\phi(P_B)) \) and \( \mu^{\log}(\phi(P_B)) \) [and in a compatible fashion].

(Ob6) Hull-approximants for the log-volume of a given region: Since one is ultimately interested in estimating log-volumes [cf. the discussion of
(iii), (iv)], it is tempting to consider simply replacing a given region $P \in \mathbb{P}$ by $\mu^{\log}(P)$. On the other hand, in order to obtain objects comparable with the $q$-pilot object [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], one is obliged to work with hulls $\in \mathbb{H}$ [cf. also the discussion of (Ob7) below]. This state of affairs suggests working with, for instance, $\Xi(P) \subseteq \mathbb{H}$ [cf. also the discussion of (Ob7) below]. This already makes it more natural to consider $\Phi(P) \subseteq \mathbb{H}$ [cf. the discussion of (iii)], i.e., as opposed to $\Xi(P) \subseteq \mathbb{H}$. On the other hand, the issue of independence of the choice of a specific possibility internal to the algorithm under consideration [cf. (Ob5)] already means that one must consider $\mu^{\log}(\mathcal{H}\Phi(P))$ or $\mu^{\log}(\mathcal{H}\Xi(P))$, as opposed to $\mu^{\log}(P)$, which, in general, may be $> \mu^{\log}(\phi(P))$ [cf. the discussion of (iv)].

(Ob7) Compatibility with log-Kummer correspondences: In (Ob6), the discussion of the issue of simply replacing a given region $P \in \mathbb{P}$ by $\mu^{\log}(P)$ — i.e., put another way, of passing to the quotient [cf. the discussion of Remark 3.9.4, as well as of (viii) below] given by taking the log-volume — was subject to the constraint that one must construct, i.e., by working with hulls [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], objects that may be meaningfully compared to objects in the local and global Frobenioids that appear in the codomain $\mathcal{F}^{\mu}_{\times} \times \mu$-prime-strip of the $\Theta^{\times\mu}_{\LGP}$-link. This constraint prompts the following question:

Why is it that one cannot simply adopt log-volumes as the “ultimate stage for comparison” — that is to say, without passing through hulls or objects in the local and global Frobenioids referred to above?

At a more technical level, this question may be reformulated as follows:

Why is it that one cannot eliminate the $\mathcal{F}^{\mu}_{\times} \times \mu$-prime-strip portion [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $v \in \mathbb{V}_{\non}$,

the local Galois groups “$G_{\mathbb{V}}$” and units “$O^{\times\mu}_{\mathcal{F}_{\mathbb{V}}}$”

[cf. the notation of the discussion surrounding [IUTchI], Fig. II.2; here and in the following discussion, we regard “$O^{\times\mu}_{\mathcal{F}_{\mathbb{V}}}$” as being equipped with the auxiliary structure, i.e., a collection of submodules [cf. [IUTchII], Definition 4.9, (i)] or system of compatible surjections [cf. [IUTchII], Definition 4.9, (v)], with which it is equipped in the definition of an $\mathcal{F}^{\mu}_{\times} \times \mu$-prime-strip] — from the $\mathcal{F}^{\mu}_{\times} \times \mu$-prime-strips that appear in the $\Theta^{\times\mu}_{\LGP}$-link?

[Closely related issues are discussed in (ix), (x) below.] The essential reason for this may be understood as follows:

(Ob7-1) Local Galois groups: The local Galois groups “$G_{\mathbb{V}}$” [for, say, $v \in \mathbb{V}_{\non}$] satisfy the important property of being invariant, up to isomorphism, with respect to the transformations constituted
by the $\Theta^{x_{\mu}}_{\text{LGP}}$- and log-links — cf. the \textbf{vertical} and \textbf{horizontal coricity} properties discussed in Theorem 1.5, (i), (ii), as well as the discussion of [IUTchII], Remark 3.6.2, (ii). These coricity properties play a \textit{fundamental role} in the theory of the present paper, i.e., by allowing one to relate, via these coricity properties, objects on either side of the $\Theta^{x_{\mu}}_{\text{LGP}}$- and log-links which do \textit{not} satisfy such invariance properties. In particular, the theory of the present series of papers \textit{cannot} function properly if the local Galois groups are eliminated from the $F^{\times_{\mu}}$-prime-strips that appear in the $\Theta^{x_{\mu}}_{\text{LGP}}$-link.

(Ob7-2) \textbf{Units:} Thus, it remains to consider what happens if one eliminates the [Frobenius-like] units [but not the local Galois groups — cf. (Ob7-1)!] from the $F^{\times_{\mu}}$-prime-strips that appear in the $\Theta^{x_{\mu}}_{\text{LGP}}$-link. This amounts to replacing the $F^{\times_{\mu}}$-prime-strips that appear in the $\Theta^{x_{\mu}}_{\text{LGP}}$-link by the associated $F^{\times_{\mu}}$-prime-strips [cf. Definition 2.4, (iii)]. Of course, since one still has the local Galois groups, one can consider the \textit{étale-like units} “$O^{x_{\mu}}(G_v)$” [i.e., “$O^{x_{\mu}}(G)$”], in the case where one takes “$G$” to be $G_{\nu}$ of [IUTchII], Example 1.8, (iv). On the other hand, these \textit{étale-like units differ fundamentally} from their Frobenius-like counterparts in the following respect:

- The \textbf{Frobenius-like units} “$O^{x_{\mu}}_{F_{\nu}}$” in the $F^{\times_{\mu}}$-prime-strips that appear in the $\Theta^{x_{\mu}}_{\text{LGP}}$-link are \textit{tautologically} related \textbf{only} to the Frobenius-like units at the \textbf{same vertical coordinate} [i.e., in a vertical column of the log-theta-lattice] as the $\Theta^{x_{\mu}}_{\text{LGP}}$-link under consideration, i.e., \textbf{not} to the Frobenius-like units at other vertical coordinates in this vertical column. In particular, these Frobenius-like units arise from the \textbf{same underlying multiplicative} structure [i.e., of the ring structure determined, on various Frobenius-like multiplicative monoids, by the $\Theta^{x_{\mu}}_{\text{ell}}$-NF-Hodge theater to which they belong] as the local and global [Frobenius-like!] \textbf{value group} portion of the $F^{\times_{\mu}}$-prime-strip under consideration. Put another way, the \textbf{splittings} of unit group and value group portions that appear in the \textit{intrinsic structure} of the $F^{\times_{\mu}}$-prime-strips under consideration [cf. [IUTchII], Definition 4.9, (vi), (viii)] are \textbf{consistent} with the underlying multiplicative structure of the ring structure determined [on various Frobenius-like multiplicative monoids] by the $\Theta^{x_{\mu}}_{\text{ell}}$-NF-Hodge theater under consideration.

- By contrast, the \textbf{étale-like counterparts} of these Frobenius-like units are \textit{constrained} by their \textit{vertical coricity} [cf. Theorem 1.5, (i)] to be related, via the relevant log-Kummer correspondences, \textbf{simultaneously} to the corresponding Frobenius-like units at \textbf{every verti-}
cal coordinate in a vertical column of the log-theta-lattice as the $\Theta_{\text{LGP}}^{x,\mu}$-link under consideration. In particular, the relationship between these étale-like units and the corresponding Frobenius-like units at various vertical coordinates in the vertical column under consideration is subject to the action of arbitrary iterates of the log-link, hence to a complicated confusion between the unit group and value group portions at various vertical coordinates of this vertical column. This complicated confusion is inconsistent with the intrinsic structure of the $F^{x,\mu} \ni \text{-prime-strips}$ under consideration [cf. Definition 2.4, (iii)], that is to say, with treating the local and global value group portions of these $F^{x,\mu} \ni \text{-prime-strips}$ as objects that are not subject to any constraints in their relationship to the étale-like units, i.e., to the local Galois group portions of these $F^{x,\mu} \ni \text{-prime-strips}$. Put another way, if one regards the étale-like units as the sole access route, from the point of view of the Frobenius-like units in the codomain of the $\Theta_{\text{LGP}}^{x,\mu}$-link under consideration, to the Frobenius-like units in the domain of this $\Theta_{\text{LGP}}^{x,\mu}$-link, then one obtains a situation in which the data in the $F^{x,\mu} \ni \text{-prime-strips}$ [i.e., “non-mutually constrained local/global value group portions and local Galois groups”] is “over-constrained/over-determined.”

Thus, in summary, one cannot eliminate the $F^{x,\mu} \ni \text{-prime-strip portion}$ [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $u \in V^{\text{non}}$, the local Galois groups “$\mathcal{G}_u$” and units “$\mathcal{O}_{F_u}^{x,\mu}$” — from the $F^{x,\mu} \ni \text{-prime-strips}$ that appear in the $\Theta_{\text{LGP}}^{x,\mu}$-link. One important consequence of the fact that the local Galois group and unit portions are indeed included in the $F^{x,\mu} \ni \text{-prime-strips}$ that appear in the $\Theta_{\text{LGP}}^{x,\mu}$-link is the [“proper functioning”, as described in the present paper, of the] theory of log-Kummer correspondences and log-shells — which serve as “multiradial containers” [cf. Remarks 1.5.2, 2.3.3, 2.3.4, 3.8.3] — both of which play a central role in the present paper.

(Ob8) Vertical shifts in the output data: One important consequence of the theory of log-Kummer correspondences lies in the fact that it allows one to transport/relate [i.e., by applying the theory of log-Kummer correspondences!] the output of the multiradial algorithms under consideration to different vertical coordinates within a vertical column of the log-theta-lattice. In fact, this output — even if one works with hulls [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)]! — yields, a priori, objects in local and global Frobenioids that differ, strictly speaking, from the corresponding [multiplicative!] local and global Frobenioids that
appear in the input data of the algorithm [i.e., the codomain $F^\oplus \times \mu$-prime-strips of the $\Theta^\times \mu_{\text{LG}}$-link — cf. Definition 3.8, (i), (ii)] by a “vertical shift” in the log-theta-lattice, i.e., more concretely, by an application of the log-link [that is to say, which produces additive log-shells from the multiplicative “$O^\times \mu$’s” in the input data].

In particular, it is precisely by applying the theory of log-Kummer correspondences that we will ultimately be able to obtain objects [i.e., objects in local and global Frobenioids] arising from the output of the multiradial algorithms under consideration that may indeed be meaningfully compared with objects in the local and global Frobenioids that appear in the input data of the algorithm [cf. Step (xi-d) of the proof of Corollary 3.12 below]. On the other hand, in this context, it is important to note that since such comparable objects are obtained by applying the log-Kummer correspondence, the local and global Frobenioids to which these comparable objects belong are necessarily subject to the indeterminacies of the relevant log-Kummer correspondence, i.e., in more concrete terms, to arbitrary iterates of the log-link [cf. the discussion of the final portion of Remark 3.12.2, (v)].

(Ob9) Hulls in the context of the log-link and log-volumes: In the context of the discussion of the final portion of (Ob8), we observe that the operation of passing to realified semi-simplifications [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] in situations where one considers the log-link compatibility of the log-volume, is a quotient operation on both the domain and the codomain of the log-link that induces a natural bijection between log-volumes of hulls in the domain and codomain of the log-link. That is to say, the fact that this quotient operation [i.e., of passing to realified semi-simplifications] induces such a natural bijection is not affected — i.e., unlike the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) — by the fact that the operation of passing to realified semi-simplifications [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] involves, at various intermediate steps, the use of various regions which are not hulls. The fundamental qualitative difference between the present situation, on the one hand, and the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) [i.e., which required the formation of hulls!], on the other, may be understood as follows:

(Ob9-1) Formal indeterminacies acting on comparable objects:
Once the passage to comparable objects via $\det_{\otimes \mu}(-)$ of a suitable hull has been achieved [cf. the discussion of (Ob5)], the various formal, or stack-theoretic/diagram-theoretic, indeterminacies that arose from this passage to comparable objects — i.e.,

- the tensor-power-twist indeterminacies of (Ob4-3),
- the application of the $\bar{\alpha}$-formalism in (Ob5), and
- the indeterminacy with respect to application of arbitrary iterates of the log-link of (Ob8)

— have no effect on the comparability of the objects obtained.
in (Ob5). That is to say, these indeterminacies function solely as *compatibility conditions* that must be satisfied [e.g., by applying the theory of *realified semi-simplifications*, as developed in Remark 3.9.4, (iii), (iv), (v), (vi)] when passing to “coarse/set-theoretic invariants” such as the log-volume.

(Ob9-2) **Non-explicit relationships between comparable and non-comparable objects:** By contrast, the situation discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) was one in which — until the “final conclusion” of this discussion in (Ob5) — comparable objects had not yet been obtained. Put another way, prior to this “final conclusion”, the precise relationship between the non-comparable objects that occurred as the *a priori output* of the *multiradial algorithms* under consideration, on the one hand, and comparable objects, on the other, had *not* yet been *explicitly computed*.

Closely related issues are discussed in (ix) below.

(viii) In the context of (vi), (vii), it is of interest to observe that, just as in the case of the operations of

(sQ1) **Kummer-detachment**, i.e., passing from *Frobenius-like* [that is to say, strictly speaking, Frobenius-like structures that contain certain étale-like structures] to [“purely”] étale-like structures [cf. Remark 1.5.4, (i), as well as the *vertical arrows* of the commutative diagram of Remark 3.10.2 below], and

(sQ2) **Galois evaluation** [cf. [IUTchII], Remark 1.12.4, as well as the *horizontal arrows* of the commutative diagram of Remark 3.10.2 below],

the operations of

(sQ3) passing from more general regions to *positive tensor powers* of *determinants* of *hulls* and then applying the *abstract set-theoretic* $\tilde{\Lambda}$-*formalism* of (v) [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7)],

(sQ4) adjusting the *vertical shifts* [i.e., in the vertical column of the log-theta-lattice corresponding to the *codomain* of the $\Theta_{\text{LGP}}^{\mu}$-link under consideration] in the output of the multiradial algorithm by applying the *log-Kummer correspondence* [cf. the discussion of (Ob8)], as well as of

(sQ5) passing to *log-volumes* [cf. (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], via the formalism of *realified semi-simplifications* discussed in Remark 3.9.4, may be regarded as *intricate (sub)quotient* — or [cf. the discussion of (ii)] *push forward/splitting/integration* — operations. Indeed, from this point of view, the content of the *entire theory of the present series of papers* may be regarded as the development of

a suitable collection of *subquotient operation algorithms* for constructing a
relatively simple, concrete (sub)quotient

of the complicated apparatus constituted by the full log-theta-lattice.

The goal of this construction of (sub)quotient operation algorithms — i.e., of the entire theory of the present series of papers — may then be understood as

the computation of the projection, via the resulting relatively simple, concrete (sub)quotient, of

the “Θ-intertwining” [i.e., the structure on an abstract $\mathcal{F}^{+\times \mu}$-prime-strip as the $\mathcal{F}^{+\times \mu}$-prime-strip arising from the Θ-pilot object appearing in the domain of the $\Theta_{\text{LGP}}^{\times \mu}$-link of Definition 3.8, (ii)]

onto structures arising from the vertical column in the codomain of the $\Theta_{\text{LGP}}^{\times \mu}$-link, that is to say, where

the “q-intertwining” [i.e., the structure on an abstract $\mathcal{F}^{+\times \mu}$-prime-strip as the $\mathcal{F}^{+\times \mu}$-prime-strip arising from the q-pilot object appearing in the codomain of the $\Theta_{\text{LGP}}^{\times \mu}$-link of Definition 3.8, (ii)]

is in force [cf. the discussion of Remark 3.12.2, (ii), below].

This computation, when suitably interpreted, amounts, essentially tautologically, to the inequality of Corollary 3.12 below. Here, we observe that each of these (sub)quotient operations (sQ1), (sQ2), (sQ3), (sQ4), (sQ5) may be understood as an operation whose purpose is to simplify the quite complicated apparatus constituted by the full log-theta-lattice by allowing the introduction of various indeterminacies. Put another way, the nontriviality of these various (sub)quotient operations lies

in the very delicate balance between minimizing the indeterminacies that arise from passing to a quotient, while at the same time ensuring compatibility with the structures that exist prior to formation of the quotient.

Indeed:

- In the case of (sQ1), i.e., the case of Kummer-detachment indeterminacies, this delicate balance is discussed in detail in Remarks 1.5.4, 2.1.1, 2.2.1, 2.2.2, 2.3.3, as well as Remark 3.10.1, (ii), (iii), below.
- In the case of (sQ2), i.e., the case of Galois evaluation, the delicate issue of compatibility with Kummer theory is discussed in [IUTchII], Remark 1.12.4.
- In the case of (sQ3), i.e., the case of passing to hulls, various delicate issues — such as, for instance, the delicate issues of tensor-power-twist indeterminacies [cf. (Ob4-3)], the $\tilde{\alpha}$-formalism [cf. (Ob5)], and compatibility with log-Kummer correspondences [cf. (Ob7)] — are discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7) [cf. also (ix), (x) below].
- In the case of (sQ4), the adjustment of vertical shifts via log-Kummer correspondences results in an indeterminacy with respect to application
of arbitrary iterates of the log-link, i.e., in the vertical column of the log-theta-lattice corresponding to the codomain of the $\Theta^{x_\mu}_{\text{LGP}}$-link under consideration [cf. (Ob8)].

- In the case of (sQ5), i.e., the case of passing to log-volumes, various subtleties surrounding the compatibility of the log-volume with the log-link are discussed in detail in Remark 3.9.4, as well as in (vii) of the present Remark 3.9.5 [cf., especially, the discussion of (Ob9)].

Finally, in this context, we observe that, in light of the rigidity of étale-like structures [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)], i.e., at a more concrete level, of objects constructed via anabelian algorithms, the construction of suitable subquotients of the étale-like portion of the log-theta-lattice — that is to say, as in the case of (sQ2), (sQ3), (sQ4), (sQ5) — is a particularly nontrivial issue.

(ix) In the context of the discussion of (vii) [cf., especially, (Ob7), (Ob8), (Ob9)], (viii), it is important to observe that there is a fundamental qualitative difference between (sQ3), (sQ4), on the one hand, and (sQ5), on the other:

(cQ3) Compatibility of (sQ3) with $\mathcal{F}^{+\times_\mu}$-prime-strip data: The fact that [in the notation of (Ob1), (Ob2)] hulls $\in \mathbb{H}$ are stabilized by multiplication by elements of $\mathcal{O}_k$ implies that, by taking [a suitable positive tensor power of] the determinant [cf. (Ob3)], they determine objects [i.e., the “$\det_{\mathcal{O}_M}(\phi(P_B))$” of (Ob5)] in the local and [by allowing $v_0 \in V_Q, v \in V$ to vary] global Frobenioids that appear in the codomain $\mathcal{F}^{+\times_\mu}$-prime-strip of the $\Theta^{x_\mu}_{\text{LGP}}$-link. In particular, by considering suitable pull-back morphisms in these local Frobenioids [i.e., which correspond to base-change morphisms in conventional scheme theory — cf. [FrdI], Definition 1.3, (i)], one obtains objects equipped with natural faithful actions by the local Galois groups “$G_v$” and units “$\mathcal{O}^{x_\mu}_{F_v}$” [cf. the notation of (Ob7)], i.e., the data that corresponds to the $\mathcal{F}^{+\times_\mu}$-prime-strip portion of the $\mathcal{F}^{+\times_\mu}$-prime-strips that appear in the $\Theta^{x_\mu}_{\text{LGP}}$-link. Moreover, as discussed in (vi), the quotient induced on $\mathbb{H}$ by the set-theoretic $\wedge$-formalism of (v) [cf. the display of (Ob5)] may be understood as corresponding to the consideration of the “$\wedge$-category” consisting of

(\wedge^1_1) objects in the local Frobenoid under consideration equipped with a “structure poly-morphism” to the original object arising from a hull, i.e., the “$\det_{\mathcal{O}_M}(\phi(P_B))$” of (Ob5), given by the Aut($\det_{\mathcal{O}_M}(\phi(P_B))$)-orbit of a linear morphism in the local Frobenoid [cf. [FrdI], Definition 1.2, (i)] to $\det_{\mathcal{O}_M}(\phi(P_B))$ and

(\wedge^1_2) morphisms between such objects that are compatible with the structure poly-morphism.

[Alternatively, one could consider a slightly modified version of this “$\wedge$-category” by restricting the objects to be objects that arise from hull-approximants for the log-volume, i.e., as in the discussion of (Ob6).] By considering suitable pull-back morphisms in this $\wedge$-category, we again obtain objects equipped with mutually compatible [i.e., relative to varying the object within the $\wedge$-category] natural faithful actions by the local Galois groups “$G_v$” and units “$\mathcal{O}^{x_\mu}_{F_v}$” [cf. the notation of (Ob7)], i.e., the data
that corresponds to the $\mathcal{F}^\times \times \mu$-prime-strip portion of the $\mathcal{F}^\times \times \mu$-strips that appear in the $\Theta_{\text{LG}}^\times \mu$-link. Next, we observe that one may consider categories of “local-global $\tilde{\lambda}$-collections of objects”, i.e., categories whose objects are collections consisting of

$(\tilde{\lambda}^{1\text{-lc-gl}})$ a “local” object in the $\tilde{\lambda}$-category at each $v \in \mathbb{V}$,

$(\tilde{\lambda}^{2\text{-lc-gl}})$ a “global” object in the global realified Frobenioi of the $\mathcal{F}^\times \times \mu$-strip under consideration, and

$(\tilde{\lambda}^{3\text{-lc-gl}})$ localization isomorphisms between the image of the local object at each $v \in \mathbb{V}$ in the realification of the local Frobenioi at $v$ and the localization of the global object at the element $\in \text{Prime}(-)$ of the global realified Frobenioi corresponding to $v$ [and whose morphisms are compatible collections of morphisms between the respective portions of the data $\tilde{\lambda}^{1\text{-lc-gl}}$, $\tilde{\lambda}^{2\text{-lc-gl}}$, $\tilde{\lambda}^{3\text{-lc-gl}}$] — cf. the discussion of the [closely related] functors in the final displays of [FrdIII], Example 5.6, (iii), (iv). In particular, just as the tensor-power-twist indeterminacies of (sQ3) [cf. (Ob4-3)] and the indeterminacy with respect to application of arbitrary iterates of the log-link of (sQ4) [cf. (Ob8)] may be understood as “formal, or stack-theoretic/diagram-theoretic, quotients” [i.e., as opposed to “coarse/set-theoretic quotients” given by set-theoretic invariants such as the log-volume — cf. the discussion of (Ob9-1)], the pair consisting of

$(\tilde{\lambda}^{1\text{Q}})$ such a category of “local-global $\tilde{\lambda}$-collections” [cf. $\tilde{\lambda}^{1\text{-lc-gl}}$, $\tilde{\lambda}^{2\text{-lc-gl}}$, $\tilde{\lambda}^{3\text{-lc-gl}}$] and

$(\tilde{\lambda}^{2\text{Q}})$ the analogous category of “local-global collections”, i.e., where the “$\tilde{\lambda}$-category” at each $v \in \mathbb{V}$ [cf. $\tilde{\lambda}^{1\text{-lc-gl}}$] is replaced by the original local Frobenioi [portion of the $\mathcal{F}^\times \times \mu$-prime-strip under consideration] at each $v \in \mathbb{V}$,

may also be regarded as the “formal, or stack-theoretic, quotient” corresponding to the operation of considering “$\tilde{\lambda}^{2\text{Q}}$ modulo $\tilde{\lambda}^{1\text{Q}}$”.

(cQ4) Compatibility of (sQ4) with $\mathcal{F}^\times \times \mu$-prime-strip data: Since the adjustment of vertical shifts in (sQ4) is obtained precisely by applying the log-Kummer correspondence, this adjustment operation is tautologically compatible [cf. the vertical coricity of Theorem 1.5, (i)] with suitable isomorphisms between the local Galois groups “$G_v$” and the étale-like units “$O^\times \mu(G_v)$” [cf. the notation of (Ob7-1), (Ob7-2)] that appear. Alternatively, this adjustment operation is tautologically compatible with suitable isomorphisms between the local Galois groups “$G_v$” and the Frobenius-like units “$O^\times \mu_F$” [cf. the notation of (Ob7)], so long as one allows for an indeterminacy with respect to application of arbitrary iterates of the log-link [cf. the discussion of (Ob8)].

(iQ5) Incompatibility of (sQ5) with $\mathcal{F}^\times \times \mu$-prime-strip data: By contrast, unlike the situation with (sQ3), (sQ4), passing to log-volumes [i.e., (sQ5)] amounts precisely to forgetting the local Galois groups and Frobenius-like units, i.e., the data that corresponds to the $\mathcal{F}^\times \times \mu$-prime-strip portion...
of the $F^\times \times \mu$-prime-strips that appear in the $\Theta_{\text{LGP}}^\times \mu$-link [cf. the discussion of (Ob7)].

Here, we recall that (sQ3), (sQ4), (sQ5) all occur within the vertical column of the log-theta-lattice corresponding to the codomain of the $\Theta_{\text{LGP}}^\times \mu$-link under consideration. In particular, the various local Galois groups "$G_v$" are all equipped with rigidifications as quotients of [isomorphs of] "$\Pi_v$" [cf. the notation of the discussion surrounding [IUTchI], Fig. II.2]. Put another way [cf. also the discussion of (Ob9)]:

- One may think of the compatibility properties (cQ3), (cQ4) as a sort of arithmetic holomorphicity [relative to the vertical column under consideration] or, alternatively, as a sort of compatibility with the log-Kummer correspondence of this vertical column. This point of view is reminiscent of the use of the descriptive "holomorphic" in the term "holomorphic hull".

- Conversely, one may think of the incompatibility property (iQ5) as corresponding to the operation of forgetting this arithmetic holomorphic structure or, alternatively, as a sort of incompatibility with the log-Kummer correspondence of this vertical column.

From the point of view of the computation of the projection of the $\Theta$-intertwining onto the $q$-intertwining discussed in (viii), this fundamental qualitative difference — i.e., (cQ3), (cQ4) versus (iQ5) — has a very substantive consequence:

It is precisely by passing through (sQ3), (sQ4) — i.e., before applying (sQ5)! [cf. also the discussion of (Ob7)] — that the chain of poly-isomorphisms of $F^\times \times \mu$-prime-strips [i.e., including the $F^\times \times \mu$-prime-strip portion of these $F^\times \times \mu$-prime-strips!] that

- begins with the $F^\times \times \mu$-prime-strip arising from the $q$-pilot object in the codomain of the $\Theta_{\text{LGP}}^\times \mu$-link,

- passes through the $\Theta_{\text{LGP}}^\times \mu$-link to the domain of the $\Theta_{\text{LGP}}^\times \mu$-link,

- passes through the various poly-isomorphisms of $F^\times \times \mu$-prime-strips [cf. the diagram of Remark 3.10.2 below; the discussion of "IPL" in Remark 3.11.1, (iii), below] induced by (sQ1), (sQ2), and

- finally, passes through (sQ3), (sQ4), which satisfy the compatibility property with the log-Kummer correspondence discussed above [i.e., (cQ3), (cQ4)]

forms a closed loop, i.e., up to the introduction of the "formal quotient indeterminacies" discussed in (cQ3), (cQ4) [cf. also the discussion of (Ob9-1)].

In this context, we observe that a non-closed loop would yield a situation from which no nontrivial conclusions may be drawn, for essentially the same reason [that no nontrivial conclusions may be drawn] as in the case of the "distinct labels approach" of Remark 3.11.1, (vii), below [cf. also the discussion of (Ob9-2); Remark 3.12.2, (ii), (c1tw), (c1oy), below]. That it to say, it is only by constructing such a closed loop that one can complete the computation of the projection [that is to say, as discussed in (viii)] of the $\Theta$-intertwining onto the $q$-intertwining, i.e.,
complete the computation of the Θ-intertwining structure, up to suitable indeterminacies, on a $F^\uparrow \times \mu$-prime-strip that is constrained to be subject to the $q$-intertwining.

Here, we recall from the discussion of (Ob7) [cf. also (x) below for a discussion of a related topic] that the construction of this sort of mathematical structure — i.e.,

a $F^\uparrow \times \mu$-prime-strip that is simultaneously equipped with two intertwinings, namely, the Θ-intertwining, up to indeterminacies, and the $q$-intertwining

— cannot be achieved if one omits various subportions of the $F^\uparrow \times \mu$-prime-strip portion of the $F^\uparrow \times \mu$-prime-strip. It is this computation/construction that will allow us, in Corollary 3.12 below, to conclude nontrivial, albeit essentially tautological, consequences from the theory of the present series of papers, such as the inequality of Corollary 3.12 [cf. Substeps (xi-d), (xi-e), (xi-f), (xi-g) of the proof of Corollary 3.12; Fig. 3.8 below]. Put another way, if one attempts to skip either (sQ3) or (sQ4) and pass directly from (sQ2) to (sQ4) or (sQ5) [or from (sQ3) to (sQ5)], then the resulting chain of poly-isomorphisms of $F^\uparrow \times \mu$-prime-strips no longer forms a closed loop, and one can no longer conclude any nontrivial consequences from the theory of the present series of papers.

(x) In the context of the discussion of (vii), (viii), (ix), it is of interest to observe that it is not possible [at least in any immediate sense!] to work with regions $\in \mathbb{P}$ that do not necessarily belong to $\mathbb{H}$ — and hence avoid the operation of passing to the hull — by replacing the local and global Frobenioids [i.e., categories of local and global arithmetic line bundles] that appear in the definition of an $F^\uparrow \times \mu$-prime-strip [cf. [IUTchII], Definition 4.9, (viii)] by “more general categories of regions $\in \mathbb{P}$”.

Indeed, any sort of category of regions $\in \mathbb{P}$ necessarily requires consideration of the multi-dimensional underlying space of $\mathbb{T}^\mathbb{Q}((-))$ [cf. (ii)], i.e., in essence, an additive module of rank $> 1$. Put another way, the only natural way to relate various “lines” [i.e., rank 1 submodules] within this space to one another is by invoking the additive structure of this module. On the other hand, since the $\Theta^\times \mu$-link is not compatible with the additive structures in its domain and codomain, it is of crucial importance that the categories that are glued together via the $\Theta^\times \mu$-link be purely multiplicative in nature, i.e., independent, at least in an a priori sense, of the additive structures in the domain and codomain of the $\Theta^\times \mu$-link. In particular, one must, in effect, work with arithmetic line bundles [which — unlike arithmetic vector bundles of rank $> 1$! — may indeed be defined in a way that only uses the multiplicative structures of the rings involved] — cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4). Of course, instead of working [as we in fact do in the present series of papers] with arithmetic line bundles over $F_{\text{mod}}$, up to certain “stack-theoretic twists” at $\nu \in \mathbb{V}^\text{bad}$ [cf. [IUTchI], Remark 3.1.5], where we work with local arithmetic line bundles over $K_{\nu}$ [which are necessary in order to accommodate the use of various powers of $q$!] — cf. [IUTchI], Example 3.2, (iv)], one could instead consider working with arithmetic line bundles over $\mathbb{Q}$. Relative to the arithmetic line bundles over $F_{\text{mod}}$ or $K_{\nu}$, for $\nu \in \mathbb{V}^\text{bad}$, that in fact appear in
the present series of papers, working with arithmetic line bundles over \( \mathbb{Q} \) amounts, in effect, to applying some sort of

**norm**, or **determinant** operation, from \( F_{\text{mod}} \) down to \( \mathbb{Q} \) or, at \( v \in \mathbb{V}^{\text{bad}} \), from \( K_{\mathbf{u}} \) down to \( \mathbb{Q}_{p_{\mathbf{u}}} \)

[followed by tensor product with a certain fixed arithmetic line bundle on \( \mathbb{Q} \) or \( \mathbb{Q}_{p_{\mathbf{u}}} \), in order to take into account the ramification of \( F_{\text{mod}} \) over \( \mathbb{Q} \) or \( K_{\mathbf{u}} \) over \( \mathbb{Q}_{p_{\mathbf{u}}} \) — cf. the discussion of (Ob3-1-2)]. On the other hand, if we write \( G_{p_{\mathbf{u}}} \subseteq \text{Gal}(\overline{F}/\mathbb{Q}) \) for the unique decomposition group of \( p_{\mathbf{u}} \) that contains \( G_{\mathbf{u}} \), then one verifies immediately that the fact that \( G_{p_{\mathbf{u}}} \) does not admit a splitting

\[
G_{p_{\mathbf{u}}} \cong G_{\mathbf{u}} \times (G_{p_{\mathbf{u}}}/G_{\mathbf{u}})
\]

implies that this sort of norm operation from \( K_{\mathbf{u}} \) down to \( \mathbb{Q}_{p_{\mathbf{u}}} \) cannot be extended, in any meaningful sense, to any sort of \textbf{Galois-equivariant} [i.e., \( G_{\mathbf{u}} \)-equivariant] operation on algebraic closures of \( K_{\mathbf{u}} \) and \( \mathbb{Q}_{p_{\mathbf{u}}} \). Since the \textit{faithful action} of \( G_{\mathbf{u}} \) on the \textit{unit portion} of the local Frobenioids in an \( \mathcal{F}_{\mathbf{u}}^{\times \mu} \)-prime-strip plays a \textit{central role} [cf. the discussion of (Ob7)] in the theory of \textbf{log-Kummer correspondences} and \textbf{log-shells} [which play a \textit{central role} in the present paper!], the incompatibility of any sort of norm operation with the local Galois group \( G_{\mathbf{u}} \) makes such a norm operation \textit{fundamentally unsuited} for defining the \textit{gluings} that constitute the \( \Theta_{\text{LGPF}} \)-link.

**Remark 3.9.6.** In the context of Proposition 3.9, (iii), (iv) [cf. also Remark 3.9.4], we make the following observation. The \textbf{log-link compatibility} of Proposition 3.9, (iv) [cf. also Proposition 1.2, (iii); Proposition 1.3, (iii); Remark 3.9.4] amounts to a \textit{coincidence of log-volumes} — not of arbitrary regions that appear in the domain and codomain of the \textbf{log-link}, but rather — of certain \textit{types} of “sufficiently small” regions that appear in the domain and codomain of the \textbf{log-link}. On the other hand, the “product formula” — i.e., at a more concrete level, the “ratios of conversion” [cf. [IUTchI], Remark 3.5.1, (ii)] between \textit{log-volumes} at distinct \( v \in \mathbb{V} \) — may be formulated [without loss of generality!] in terms of such “sufficiently small” regions. Thus, in summary, we conclude that

the \textbf{log-link compatibility} of Proposition 3.9, (iv), implies a \textbf{compatibility} of “\textit{product formulas}”, i.e., of “\textit{ratios of conversion}” between \textit{log-volumes} at distinct \( v \in \mathbb{V} \), in the domain and codomain of the \textbf{log-link}.

In particular, in the context of Proposition 3.9, (iii), we conclude that Proposition 3.9, (iv), implies a \textbf{compatibility} between \textbf{global arithmetic degrees} in the domain and codomain of the \textbf{log-link}.

**Remark 3.9.7.** When computing \textit{log-volumes of various regions} of the sort considered in Proposition 3.9, it is useful to keep the following \textit{elementary observations} in mind:

(i) In the context of Proposition 3.9, (iii), the defining condition “\textit{zero log-volume} for all but finitely many \( v_{Q} \in \mathbb{V}_{Q} \)” for

\[
\mathcal{M}(\mathcal{I}_{Q}^{A}(\mathcal{F}_{v_{Q}})) \subseteq \prod_{v_{Q} \in \mathbb{V}_{Q}} \mathcal{M}(\mathcal{I}_{Q}^{A}(\mathcal{F}_{v_{Q}}))
\]
that is imposed on the various components indexed by $v_Q \in V_Q$ of the direct product of the above display may be satisfied by considering elements of this direct product such that for all but finitely many of the elements $w_Q \in V_Q^{\text{non}}$ for which $p_{w_Q}$ is unramified in $K$, the component at $w_Q$ is given by $I(A\mathcal{F}_{v_Q}) \subseteq I_Q(A\mathcal{F}_{v_Q})$. Indeed, for each such $w_Q \in V_Q^{\text{non}}$, the subset 

$$O(-) = I((-)) \subseteq I_Q((-))$$

[cf. the notation of Proposition 3.2, (ii); Proposition 3.9, (i); the final sentence of [AbsTopIII], Definition 5.4, (iii)] has zero log-volume. Finally, in the context of Proposition 3.9, (ii), we observe that, for each such $w_Q \in V_Q^{\text{non}}$, the subset $I((-)) \subseteq I_Q((-))$ is a mono-analytic invariant, which, moreover, [cf. Remark 3.9.5, (i)] is equal to its own holomorphic hull.

(ii) In the context of Proposition 3.9, (i), (ii), we observe that one may consider the log-volume of more general, say, relatively compact subsets $E \subseteq I_Q((-))$ [cf. the discussion of Remark 3.1.1, (iii)] than the sets which belong to $\mathfrak{M}(I_Q((-)))$, i.e.,

simply by defining the log-volume of $E$ to be the infimum of the log-volumes of the sets $E^* \in \mathfrak{M}(I_Q((-)))$ such that $E \subseteq E^*$.

This definition means that one must allow for the possibility that the log-volume of $E$ is $-\infty$. Alternatively [and essentially equivalently!], one can treat such $E$ by thinking of such an $E$ as corresponding to the

inverse system of $E^* \in \mathfrak{M}(I_Q((-)))$ such that $E \subseteq E^*$.

Here, when $E$ is a direct product pre-region, it is natural to consider instead the inverse system of direct product regions $E^* \in \mathfrak{M}(I_Q((-)))$ such that $E \subseteq E^*$ [cf. the discussion of Remark 3.1.1, (iii)]. This approach via inverse systems of regions each of which has finite log-volume has the advantage that it allows one to always work with finite log-volumes.

(iii) In a similar vein, in the context of Proposition 3.9, (iii), we observe that one may consider the log-volume of more general collections of relatively compact subsets [cf. the discussion of Remark 3.1.1, (iii)] than the collections of sets of the sort considered in the discussion of (i) above. Indeed, if

$$\{E_{v_Q} \subseteq I_Q(A\mathcal{F}_{v_Q})\}_{v_Q \in V_Q}$$

is a collection of subsets such that, for some collection of sets $\{E_{v_Q}^*\}_{v_Q}$ of the sort considered in the discussion of (i), it holds that $E_{v_Q} \subseteq E_{v_Q}^*$, for each $v_Q \in V_Q$, then one may

simply define the log-volume of $\{E_{v_Q}\}_{v_Q}$ to be the infimum of the log-volumes of the collections of sets $\{E_{v_Q}^*\}_{v_Q}$ of the sort considered in the discussion of (i) above such that $E_{v_Q} \subseteq E_{v_Q}^*$, for each $v_Q \in V_Q$ [in which case one must allow for the possibility that the log-volume of $E$ is $-\infty$]; alternatively [and essentially equivalently!], one may think of such a collection $\{E_{v_Q}\}_{v_Q}$ as corresponding to the
inverse system of collections \( \{ E^*_v \}_{v \in \mathbb{V}_Q} \) of the sort considered in the discussion of (i) above such that \( E^*_v \subseteq E^*_{vQ} \), for each \( vQ \in \mathbb{V}_Q \)

[an approach that has the advantage that it allows one to always work with finite log-volumes]. Here, in the case where each \( E^*_v \), for \( vQ \in \mathbb{V}_Q \), is a direct product pre-region, it is natural to consider instead inverse systems \( \{ E^*_v \}_{vQ} \) as above such that each \( E^*_v \), for \( vQ \in \mathbb{V}_Q \), is a direct product region [cf. the discussion of Remark 3.1.1, (iii)].

Proposition 3.10. (Global Kummer Theory and Non-interference with Local Integers) Let \( \{ n, m \mathcal{HT}_{\Theta^{\pm \text{ell}} \mathbb{NF}} \}_{n, m \in \mathbb{Z}} \) be a collection of distinct \( \Theta^{\pm \text{ell}} \mathbb{NF} \)-Hodge theaters [relative to the given initial \( \Theta \)-data] — which we think of as arising from an LGP-Gaussian log-theta-lattice [cf. Definition 3.8, (iii); Proposition 3.7; Remark 3.8.2]. For each \( n \in \mathbb{Z} \), write

\[
n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}}
\]

for the \( D, \Theta^{\pm \text{ell}} \mathbb{NF} \)-Hodge theater determined, up to isomorphism, by the various \( n, m \mathcal{HT}_{\Theta^{\pm \text{ell}} \mathbb{NF}} \), where \( m \in \mathbb{Z} \), via the vertical coricity of Theorem 1.5, (i) [cf. Remark 3.8.2].

(i) (Vertically Coric Global LGP-, \( \text{lgp} \)-Frobenioids and Associated Kummer Theory) Recall the constructions of various global Frobenioids in Proposition 3.7, (i), (ii), (iii), (iv), in the context of \( \mathcal{F} \)-prime-stripe processions. Then by applying these constructions to the \( \mathcal{F} \)-prime-strips \( \mathcal{F}^{(n, \circ \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})} \) [cf. the notation of Proposition 3.5, (i)] and the various full \( \mathcal{F} \)-links associated [cf. the discussion of Proposition 1.2, (ix)] to these \( \mathcal{F} \)-prime-strips — which we consider in a fashion compatible with the \( \mathcal{F}^{x \pm} \)-symmetries involved [cf. Remark 1.3.2; Proposition 3.4, (ii)] — we obtain functorial algorithms in the \( D, \Theta^{\pm \text{ell}} \mathbb{NF} \)-Hodge theater \( n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}} \) for constructing [number] fields, monoids, and Frobenioids equipped with natural isomorphisms

\[
\mathbb{M}_{\text{mod}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha} = \mathbb{M}_{\text{MOD}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha} \\
\supset \mathbb{M}_{\text{mod}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha} = \mathbb{M}_{\text{MOD}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha}
\]

\[
\mathcal{F}_{\text{mod}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha} \supset \mathcal{F}_{\text{mod}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha} \supset \mathcal{F}_{\text{MOD}}^{\circ}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\alpha}
\]

[cf. the number fields, monoids, and Frobenioids \( \mathbb{M}_{\text{mod}}^{\circ}(1 \mathcal{D}^{\circ})_{j} \supset \mathbb{M}_{\text{mod}}^{\circ}(1 \mathcal{D}^{\circ})_{j} \)] of [IUTchII], Corollary 4.7, (ii)] for \( \alpha \in A \), where \( A \) is a subset of \( J \) [cf. Proposition 3.3], as well as \( \mathcal{F}^{\pm} \)-prime-strips equipped with natural isomorphisms

\[
\mathcal{F}^{\pm}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\text{gau}} \supset \mathcal{F}^{\pm}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\text{LGP}} \supset \mathcal{F}^{\pm}(n^{\circ} \mathcal{HT}^{D, \Theta^{\pm \text{ell}} \mathbb{NF}})_{\text{lgp}}
\]

— [all of] which we shall refer to as being “vertically coric”. For each \( n, m \in \mathbb{Z} \), these functorial algorithms are compatible [in the evident sense] with the [“non-vertically coric”] functorial algorithms of Proposition 3.7, (i), (ii), (iii), (iv) —
i.e., where [in Proposition 3.7, (iii), (iv)] we take “\(\bar{\mathfrak{F}}\)” to be “\(n, m\)” and “\(\bar{\mathfrak{F}}\)” to be “\(n, m - 1\)” — relative to the Kummer isomorphisms of labeled data

\[
\Psi_{\text{cns}}(n, m')_{\mathfrak{F}} \sim \Psi_{\text{cns}}(n, \circ \mathfrak{F})_{\mathfrak{F}}
\]

\[
(n, m'm_{\mod})_j \sim M_{\mod}(n, \circ \mathfrak{F})_{\mathfrak{F}}; \quad (n, m'm_{\mod})_j \sim M_{\mod}(n, \circ \mathfrak{F})_{\mathfrak{F}}
\]

[cf. [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (ii)] and the evident identification, for \(m' = m, m - 1\), of \(n, m'm_{\mod}\), i.e., the \(F\)-prime-strip that appears in the associated \(\Theta^+\)-bridge] with the \(F\)-prime-strip associated to \(\Psi_{\text{cns}}(n, m')_{\mathfrak{F}}\) [cf. Proposition 3.5, (i)]. In particular, for each \(n, m \in \mathbb{Z}\), we obtain “Kummer isomorphisms” of fields, monoids, Frobenioids, and \(\mathcal{F}^\dagger\)-prime-strips

\[
(n, m'm_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}; \quad (n, m'm_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

\[
(n, m'm_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha} \quad (n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

\[
\mathfrak{F}_{\text{mod}} \sim \mathfrak{F}_{\text{mod}}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}; \quad (n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

\[
(n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha} \quad (n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

\[
\mathfrak{F}_{\text{mod}} \sim \mathfrak{F}_{\text{mod}}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha} \quad (n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

\[
(n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha} \quad (n, m's_{\mod})_{\alpha} \sim M_{\mod}(n, \circ \mathfrak{H}T^{D, \Theta^+\ell\ell\ell\ell_{\text{gau}}})_{\alpha}
\]

that are compatible with the various equalities, natural inclusions, and natural isomorphisms discussed above.

(ii) (Non-interference with Local Integers) In the notation of Propositions 3.2, (ii); 3.4, (i); 3.7, (i), (ii); 3.9, (iii), we have

\[
(\uparrow M_{\text{mod}}^\circ)_{\alpha} \bigcap \prod_{v \in \mathfrak{V}} \Psi_{\text{log}}(A, \alpha \mathfrak{F}_{\mathfrak{V}}) = (\uparrow M_{\text{mod}}^\circ)_{\alpha}
\]

\[
\left( \prod_{v \in \mathfrak{V}} \mathcal{I}^\mathbb{Q}(A, \alpha \mathfrak{F}_{\mathfrak{V}}) = \prod_{v \in \mathfrak{V}} \mathcal{I}^\mathbb{Q}(A \mathfrak{F}_{\mathfrak{V}}) = \mathcal{I}^\mathbb{Q}(A \mathfrak{F}_{\mathfrak{V}}) \right)
\]

— where we write \((\uparrow M_{\text{mod}}^\circ)_{\alpha} \subseteq (\uparrow M_{\text{mod}}^\circ)_{\alpha}\) for the [finite] subgroup of torsion elements, i.e., roots of unity; for \(v_0 \in \mathfrak{V}_{\mathbb{Q}}\), we identify the product \(\prod_{v \in \mathfrak{V}_{\mathbb{Q}}} \mathcal{I}^\mathbb{Q}(A, \alpha \mathfrak{F}_{\mathfrak{V}})\) with \(\mathcal{I}^\mathbb{Q}(A \mathfrak{F}_{\mathfrak{V}})\). Now let us think of the various groups

\[
(n, mM_{\text{mod}}^\circ)_{\mathfrak{F}}
\]

[of nonzero elements of a number field] as acting on various portions of the modules

\[
\mathcal{I}^\mathbb{Q}(S_{n+1}^+ \mathfrak{F}(n, \circ \mathfrak{F})_{\mathfrak{V}})
\]

[i.e., where the subscript “\(\mathfrak{V}_{\mathbb{Q}}\)” denote the direct product over \(v_0 \in \mathfrak{V}_{\mathbb{Q}}\) — cf. the notation of Proposition 3.5, (ii)] not via a single Kummer isomorphism as in (i), but rather via the totality of the various pre-composites of Kummer isomorphisms with iterates [cf. Remark 1.1.1] of the log-links of the LGP-Gaussian
log-theta-lattice — where we observe that these actions are mutually compatible up to [harmless!] “identity indeterminacies” at an adjacent “m”, precisely as a consequence of the equality of the first display of the present (ii) [cf. the discussion of Remark 1.2.3, (ii); the discussion of Definition 1.1, (ii), concerning quotients by “Ψv" at v ∈ Varc; the discussion of Definition 1.1, (iv), at v ∈ Vmon] — cf. also the discussion of Remark 3.11.4 below. Thus, one obtains a sort of “log-Kummer correspondence” between the totality, as m ranges over the elements of Z, of the various groups [of nonzero elements of a number field] just discussed [i.e., which are labeled by “n,m”] and their actions [as just described] on the “IQ” labeled by “n,o” which is invariant with respect to the translation symmetries [cf. Proposition 1.3, (iv)] of the n-th column of the LGP-Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].

(iii) (Frobenioid-theoretic log-Kummer Correspondences) The relevant Kummer isomorphisms of (i) induce, via the “log-Kummer correspondence” of (ii) [cf. also Proposition 3.7, (i); Remarks 3.6.1, 3.9.2], isomorphisms of Frobenioids

\[(n,m \mathcal{F}_{\text{MOD}}^\circ)_\alpha \cong \mathcal{F}_{\text{MOD}}^\circ(n,o \mathcal{HT}D.Θ^{+\ell}NF)_\alpha\]

\[(n,m \mathcal{F}_{\text{MOD}}^\circ \mathcal{R})_\alpha \cong \mathcal{F}_{\text{MOD}}^\circ R(n,o \mathcal{HT}D.Θ^{+\ell}NF)_\alpha\]

that are mutually compatible, as m varies over the elements of Z, with the log-links of the LGP-Gaussian log-theta-lattice. Moreover, these compatible isomorphisms of Frobenioids, together with the relevant Kummer isomorphisms of (i), induce, via the global “log-Kummer correspondence” of (ii) and the splitting monoid portion of the “log-Kummer correspondence” of Proposition 3.5, (ii), isomorphisms of associated \(F^{+\perp}\)-prime-strips [cf. Definition 2.4, (iii)]

\[n,m \mathcal{G}_{\text{LGP}}^{+\perp} \cong \mathcal{G}_{\text{LGP}}^{+\perp}(n,o \mathcal{HT}D.Θ^{+\ell}NF)\]

that are mutually compatible, as m varies over the elements of Z, with the log-links of the LGP-Gaussian log-theta-lattice.

Proof. The various assertions of Proposition 3.10 follow immediately from the definitions and the references quoted in the statements of these assertions. Here, we observe that the computation of the intersection of the first display of (ii) is an immediate consequence of the well-known fact that the set of nonzero elements of a number field that are integral at all of the places of the number field consists of the set of roots of unity contained in the number field [cf. the discussion of Remark 1.2.3, (ii); [Lang], p. 144, the proof of Theorem 5].

Remark 3.10.1.

(i) Note that the log-Kummer correspondence of Proposition 3.10, (ii), induces isomorphisms of Frobenioids as in the first display of Proposition 3.10, (iii), precisely because the construction of “(F_{\text{MOD}}^\circ)_\alpha” only involves the group “(\mathcal{M}_{\text{MOD}}^\circ)_\alpha”, together with the collection of subquotients of its perfection indexed by V [cf. Proposition 3.7, (i); Remarks 3.6.1, 3.9.2]. By contrast, the construction of “(F_{\text{MOD}}^\circ)_\alpha” also involves the local monoids “Ψ_{log(A,o F_{\text{arc}})} ⊆ log(A,o F_{\text{arc}})” in an essential way [cf.
Proposition 3.7, (ii)]. These local monoids are subject to a somewhat more complicated “log-Kummer correspondence” [cf. Proposition 3.5, (ii)] that revolves around “upper semi-compatibility”, i.e., in a word, one-sided inclusions, as opposed to precise equalities. The imprecise nature of such one-sided inclusions is incompatible with the construction of “$(\mathcal{F}_{\text{mod}}^{\otimes})_\alpha$”. In particular, one cannot construct log-link-compatible isomorphisms of Frobenioids for “$(\mathcal{F}_{\text{mod}}^{\otimes})_\alpha$” as in the first display of Proposition 3.10, (iii).

(ii) The precise compatibility of “$\mathcal{F}_{\text{MOD}}^{\otimes}$” with the log-links of the LGP-Gaussian log-theta-lattice [cf. the discussion of (i); the first “mutual compatibility” of Proposition 3.10, (iii)] makes it more suited [i.e., by comparison to “$\mathcal{F}_{\text{mod}}^{\otimes}$”] to the task of computing the Kummer-detachment indeterminacies [cf. Remark 1.5.4, (i), (iii)] that arise when one attempts to pass from the Frobenius-like structures constituted by the global portion of the domain of the $\Theta_{LGP}^{\times \mu}$-links of the LGP-Gaussian log-theta-lattice to corresponding étale-like structures. That is to say, the mutual compatibility of the isomorphisms

$$n,m \tilde{\delta}^{\otimes} \sim \tilde{\delta}^{\otimes} (n^\circ \mathcal{H}_{T,D-\Theta_{\text{ell}}^{\pm}^\text{ell}NF})_{LGP}$$

of the second display of Proposition 3.10, (iii), asserts, in effect, that such Kummer-detachment indeterminacies do not arise. This is precisely the reason why we wish to work with the LGP-, as opposed to the lgp-, Gaussian log-theta lattice [cf. Remark 3.8.1]. On the other hand, the essentially multiplicative nature of “$\mathcal{F}_{\text{MOD}}^{\otimes}$” [cf. Remark 3.6.2, (ii)] makes it ill-suited to the task of computing the étale-transport indeterminacies [cf. Remark 1.5.4, (i), (ii)] that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta_{LGP}^{\times \mu}$-link.

(iii) By contrast, whereas the additive nature of the local modules [i.e., local fractional ideals] that occur in the construction of “$\mathcal{F}_{\text{mod}}^{\otimes}$” renders “$\mathcal{F}_{\text{mod}}^{\otimes}$” ill-suited to the computation of Kummer-detachment indeterminacies [cf. the discussion of (i), (ii)], the close relationship [cf. Proposition 3.9, (i), (ii), (iii)] of these local modules to the mono-analytic log-shells that are coric with respect to the $\Theta_{LGP}^{\times \mu}$-link [cf. Theorem 1.5, (iv); Remark 3.8.2] renders “$\mathcal{F}_{\text{mod}}^{\otimes}$” well-suited to the computation of the étale-transport indeterminacies that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta_{LGP}^{\times \mu}$-link. That is to say, although various distortions of these local modules arise as a result of both [the Kummer-detachment indeterminacies constituted by] the local “upper semi-compatibility” of Proposition 3.5, (ii), and [the étale-transport indeterminacies constituted by] the discrepancy between local holomorphic and mono-analytic integral structures [cf. Remark 3.9.1, (i), (ii)], one may nevertheless compute — i.e., if one takes into account the various distortions that occur, “estimate” — the global arithmetic degrees of objects of “$\mathcal{F}_{\text{mod}}^{\otimes}$” by computing log-volumes [cf. Proposition 3.9, (iii)], which are bi-coric, i.e., coric with respect to both the $\Theta_{LGP}^{\times \mu}$-links [cf. Proposition 3.9, (ii)] and the log-links [cf. Proposition 3.9, (iv)] of the LGP-Gaussian log-theta-lattice. This computability is precisely the topic of Corollary 3.12 below. On the other hand, the issue of obtaining concrete estimates will be treated in [IUTchIV].
The various properties of $\mathcal{F}^\otimes_{\text{MOD}}$ and $\mathcal{F}^\otimes_{\text{mod}}$ discussed in (i), (ii), (iii) above are summarized in Fig. 3.2 above. In this context, it is of interest to observe that the natural isomorphisms of Frobenioids

$$\mathcal{F}^\otimes_{\text{mod}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}}) \alpha \sim \mathcal{F}^\otimes_{\text{MOD}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}}) \alpha$$

as well as the resulting isomorphisms of $\mathcal{F}^\dag$-prime-strips

$$\mathcal{F}^\dag_{\text{LGP}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}})_{\text{LGP}} \sim \mathcal{F}^\dag_{\text{lgp}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}})_{\text{lgp}}$$

of Proposition 3.10, (i), play the *highly nontrivial* role of relating [cf. the discussion of IUTechII, Remark 4.8.2, (i)] the “multiplicatively biased $\mathcal{F}^\otimes_{\text{MOD}}$” to the “additively biased $\mathcal{F}^\otimes_{\text{mod}}$” by means of the *global ring structure* of the number field $M^\otimes_{\text{mod}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}}) = M^\otimes_{\text{MOD}}(n, \alpha \circ \text{HT}_D \pm ell_{\text{NF}}).$ A similar statement holds concerning the tautological isomorphism of $\mathcal{F}^\dag$-prime-strips $\mathcal{F}^\dag_{\text{LGP}} \sim \mathcal{F}^\dag_{\text{lgp}}$ of Proposition 3.7, (iv).

**Remark 3.10.2.** In the context of the various *Kummer isomorphisms* discussed in the final display of Proposition 3.10, (i), it is useful to recall that the $\mathcal{F}^\dag\times^{\mu -} \text{prime-strips}$ $\mathcal{F}^\dag_{\text{LGP}} \times^{\mu -}$, $\mathcal{F}^\dag_{\text{lgp}} \times^{\mu -}$ that appear in the definition of the $\Theta_{\text{LGP}}^{\times^{\mu -}}$, $\Theta_{\text{lgp}}^{\times^{\mu -}}$ links in Definition 3.8, (ii), were constructed from the $\mathcal{F}^\dag\times^{\mu \circ} \text{prime-strip}$ $\mathcal{F}^\dag_{\text{env}} \times^{\mu \circ}$ [associated to the $\mathcal{F}^\dag$-prime-strip $\mathcal{F}^\dag_{\text{env}}$] of IUTechII, Corollary 4.10, (ii), in a
fashion that we review as follows. First, we remark that, in the present discussion, it is convenient for us to think of ourselves as working with objects arising from
the LGP-Gaussian log-theta-lattice of Definition 3.8, (iii) [so “†” will be replaced by
“(n, m)” or “(n, ◦)”]. Now recall, from the theory developed so far in the present
series of papers, that we have a commutative diagram of $F^{\oplus} \times \mu$-prime-strips

\[
\begin{array}{cccccc}
\mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{env}} & \xrightarrow{\sim} & \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{gau}} & \xrightarrow{\sim} & \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{LGP}} & \xrightarrow{\sim} & \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{lgp}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{env}} & \xrightarrow{\sim} & n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{gau}} & \xrightarrow{\sim} & n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{LGP}} & \xrightarrow{\sim} & n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{lgp}}
\end{array}
\]

— where

- for simplicity, we use the abbreviated version “\(n, \circ\)” of the notation “\(n, \circ \mathcal{H}\mathcal{T}\mathcal{D}_{\text{−N}}\)” of Proposition 3.10, (i);
- the first vertical arrow is the induced $F^{\oplus} \times \mu$-prime-strip version of
the Kummer isomorphism [whose codomain includes an argument “\(\mathcal{D}_{\rangle}\)”,
which we denote here by “\(n, \circ\)”] of the final display of Proposition 2.1, (ii)
[cf. also Proposition 2.1, (iii), (iv), (v)];
- the second, third, and fourth vertical arrows are the induced $F^{\oplus} \times \mu$
prime-strip versions of the Kummer isomorphisms of the final display of
Proposition 3.10, (i);
- the first lower horizontal arrow is the induced $F^{\oplus} \times \mu$-prime-strip version
of the evaluation isomorphism of the final display of [IUTchII], Corollary
4.10, (ii);
- the second and third lower horizontal arrows are the induced $F^{\oplus} \times \mu$
prime-strip versions of the tautological isomorphisms of the final displays
of Proposition 3.7, (iii), (iv);
- the first upper horizontal arrow is the induced $F^{\oplus} \times \mu$-prime-strip version
of the étale-like evaluation isomorphism implicit in the construction [via
[IUTchII], Corollary 4.6, (iv), (v)] of the evaluation isomorphism of the
final display of [IUTchII], Corollary 4.10, (ii);
- the second and third upper horizontal arrows are the induced $F^{\oplus} \times \mu$
prime-strip versions of the natural isomorphisms of the second display of
Proposition 3.10, (i).

That is to say, in summary,

the $F^{\oplus} \times \mu$-prime-strips $n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{LGP}}, n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{lgp}}$ that appear in the $\Theta^{\times \mu}_{\text{LGP}},$
\(\Theta^{\times \mu}_{\text{lgp}}\) links of Definition 3.8, (iii), were constructed from the $F^{\oplus} \times \mu$-prime-
strip $n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{env}}$ and related to this $F^{\oplus} \times \mu$-prime-
strip $n, m \mathfrak{D}^{\oplus} \times \mu (n, \circ)_{\text{env}}$ via the lower horizontal arrows of the above commutative diagram; moreover, each
of these lower horizontal arrows may be constructed by conjugating the
corresponding upper horizontal arrow by the relevant Kummer isomor-
phisms, i.e., by the vertical arrows in the diagram.
We are now ready to discuss the main theorem of the present series of papers.

**Theorem 3.11.** (Multiradial Algorithms via LGP-Monoids/Frobenioids) Fix a collection of initial Θ-data

\[(T/F, X_F, l, C_K, V, V_{mod, ϵ})\]

as in [IUTchI], Definition 3.1. Let

\[\{n,mHT^{±ellNF}\}_{n,m∈Z}\]

be a collection of distinct Θ±ellNF-Hodge theaters [relative to the given initial Θ-data] — which we think of as arising from an LGP-Gaussian log-theta-lattice [cf. Definition 3.8, (iii)]. For each \(n ∈ Z\), write

\[n,◦HT^D-Θ±ellNF\]

for the \(D-Θ±ellNF\)-Hodge theater determined, up to isomorphism, by the various \(HT^D-Θ±ellNF\), where \(m ∈ Z\), via the vertical coricity of Theorem 1.5, (i) [cf. Remark 3.8.2].

(i) (Multiradial Representation) Consider the procession of \(D^+\)-prime-strips \(\text{Proc}^{(n,◦D^+_T)}\)

\[\{n,◦D^+_0\} ↔ \{n,◦D^+_0, n,◦D^+_1\} ↔ \ldots ↔ \{n,◦D^+_0, n,◦D^+_1, \ldots, n,◦D^+_l\}\]

obtained by applying the natural functor of [IUTchI], Proposition 6.9, (ii), to [the \(D-Θ±ell\)-bridge associated to] \(n,◦HT^D-Θ±ellNF\). Consider also the following data:

(a) for \(V \ni v \mid v_0 ∈ V_Q, j ∈ |F|\), the topological modules and mon-analytic integral structures

\[I^Q(\sigma^±_{j+1,n,◦D^+_v}); I^Q(\sigma^±_{j+1,n,◦D^+_1};) \subseteq I^Q(\sigma^±_{j+1,n,◦D^+_v}); \]

where the notation “\(n,◦\)” denotes the result of applying the construction in question to the case of \(D^+\)-prime-strips labeled “\(n,◦\)” — of Proposition 3.2, (ii) [cf. also the notational conventions of Proposition 3.4, (ii)], which we regard as equipped with the procession-normalized mon-analytic log-volumes of Proposition 3.9, (ii);

(b) for \(V_{bad} \ni v\), the splitting monoid

\[Ψ_{LGP}((n,◦HT^D-Θ± ellNF)_v)\]

of Proposition 3.5, (ii), (c) [cf. also the notation of Proposition 3.5, (i)], which we regard — via the natural poly-isomorphisms

\[I^Q(\sigma^±_{j+1,n,◦D^+_1};) \sim I^Q(\sigma^±_{j+1,n,◦D^+_1}; F^{×,μ}(n,◦D^+_1);) \sim I^Q(\sigma^±_{j+1,n,◦D^+_1}; F(\sigma^±_{j+1,n,◦D^+_1};))\]
for \( j \in \mathbb{F}_l^* \) [cf. Proposition 3.2, (i), (ii)] — as a subset of

\[
\prod_{j \in \mathbb{F}_l^*} \mathcal{I}_Q(S_{j+1}^\pm; n, D_v^\perp)
\]

equipped with an (multiplicative) action on \( \prod_{j \in \mathbb{F}_l^*} \mathcal{I}_Q(S_{j+1}^\pm; n, D_v^\perp) \);

(c) for \( j \in \mathbb{F}_l^* \), the number field

\[
\mathbb{M}_\text{MOD}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j} = \mathbb{M}_\text{mod}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j} 
\subseteq \mathcal{I}_Q(S_{j+1}^\pm; n, D_v^\perp) \overset{\text{def}}{=} \prod_{v_Q \in V_Q} \mathcal{I}_Q(S_{j+1}^\pm; n, D_v^\perp)
\]

[cf. the natural poly-isomorphisms discussed in (b); Proposition 3.9, (iii); Proposition 3.10, (i)], together with natural isomorphisms between the associated global non-realified/realified Frobenioids

\[
\mathcal{F}_\text{MOD}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j} \sim \mathcal{F}_\text{mod}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j}
\]

\[
\mathcal{F}_\text{MOD}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j} \sim \mathcal{F}_\text{mod}^\oplus(n, D^\perp; \Theta^\pm; \nu; \nu^\perp)_{j}
\]

[cf. Proposition 3.10, (i)], whose associated “global degrees” may be computed by means of the log-volumes of (a) [cf. Proposition 3.9, (iii)].

Write

\[
n, \nu \in \mathfrak{LGP}_T
\]

for the collection of data (a), (b), (c) regarded up to indeterminacies of the following two types:

(Ind1) the indeterminacies induced by the automorphisms of the procession of \( D^\perp \)-prime-strips \( \text{Prc}(n, \nu^\perp) \);

(Ind2) for each \( v_Q \in V_{Q}^{\text{non}} \) (respectively, \( v_Q \in V_{Q}^{\text{arc}} \)), the indeterminacies induced by the action of independent copies of \( \text{Ism} \) [cf. Proposition 1.2, (vi)] (respectively, copies of each of the automorphisms of order 2 whose orbit constitutes the poly-automorphism discussed in Proposition 1.2, (vii)) on each of the direct summands of the \( j+1 \) factors appearing in the tensor product used to define \( \mathcal{I}_Q(S_{j+1}^\pm; n, D_v^\perp) \) [cf. (a) above; Proposition 3.2, (ii)] — where we recall that the cardinality of the collection of direct summands is equal to the cardinality of the set of \( v \in V \) that lie over \( v_Q \).

Then \( n, \nu \in \mathfrak{LGP}_T \) may be constructed via an algorithm in the procession of \( D^\perp \)-prime-strips \( \text{Prc}(n, \nu^\perp) \) that is functorial with respect to isomorphisms of processions of \( D^\perp \)-prime-strips. For \( n, n' \in \mathbb{Z} \), the permutation symmetries of the \( \text{étale-picture} \) discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce compatible poly-isomorphisms

\[
\text{Prc}(n, \nu^\perp) \sim \text{Prc}(n', \nu^\perp); \quad n, \nu \in \mathfrak{LGP}_T \quad \rightarrow \quad n', \nu \in \mathfrak{LGP}_T
\]
which are, moreover, compatible with the poly-isomorphisms

\[ n,\circ D^+_{0} \xrightarrow{\sim} n',\circ D^+_{0} \]

induced by the bi-coricity poly-isomorphisms of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].

(ii) (log-Kummer Correspondence) For \( n, m \in \mathbb{Z} \), the Kummer isomorphisms of labeled data

\[ \Psi_{\text{cns}}(n, m, D_{\nu})_t \sim \Psi_{\text{cns}}(n, m, D_{\nu})_t \]

\[ \{ \pi_{1, \text{sol}}(n, m, D_{\nu}) \} \sim \{ \pi_{1, \text{sol}}(n, m, D_{\nu}) \} \]

\[ (n, m, M^\oplus_{\text{mod}})_j \sim (n, m, D_{\nu})_j \]

— where \( t \in \text{LabCusp}^{\pm}(n, m, D_{\nu}) \) — of [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (i), (ii) [cf. also Propositions 3.5, (i); 3.10, (i), of the present paper] induce isomorphisms between the vertically coric data (a), (b), (c) of (i) [which we regard, in the present (ii), as data which has not yet been subjected to the indeterminacies (Ind1), (Ind2) discussed in (i)] and the corresponding data arising from each \( \Theta_{\pm\text{ell}}\text{NF-Hodge theater}^{n, m, D_{\nu}} \), i.e.:

(a) for \( \nu \ni v \mid v_Q \), \( j \in [F]_l \), isomorphisms with local mono-analytic tensor packets and their Q-spans

\[ \mathcal{I}(S^{\pm}_{j+1}; n, m, F_{v_Q}) \sim \mathcal{I}(S^{\pm}_{j+1}; n, m, F_{v_Q}) \]

\[ \mathcal{I}(S^{\pm}_{j+1}; n, m, F_{v_Q}) \sim \mathcal{I}(S^{\pm}_{j+1}; n, m, F_{v_Q}) \]

\[ \mathcal{I}(S^{\pm}_{j+1}; j; n, m, F_{v}) \sim \mathcal{I}(S^{\pm}_{j+1}; j; n, m, F_{v}) \]

\[ \mathcal{I}(S^{\pm}_{j+1}; j; n, m, F_{v}) \sim \mathcal{I}(S^{\pm}_{j+1}; j; n, m, F_{v}) \]

[cf. Propositions 3.2, (i), (ii); 3.4, (ii); 3.5, (i)], all of which are compatible with the respective log-volumes [cf. Proposition 3.9, (ii)];

(b) for \( \nu^{\text{bad}} \ni v \), isomorphisms of splitting monoids

\[ \Psi_{\text{LGP}}^{\pm}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}}) \xrightarrow{\sim} \Psi_{\text{LGP}}^{\pm}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}}) \]

[cf. Proposition 3.5, (i); Proposition 3.5, (ii), (c)];

(c) for \( j \in \mathbb{F}^*_l \), isomorphisms of number fields and global non-realified/reallified Frobenioids

\[ (n, m, M^\oplus_{\text{MOD}})_j \sim M^\oplus_{\text{MOD}}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}})_j \]

\[ (n, m, F^\oplus_{\text{MOD}})_j \sim F^\oplus_{\text{MOD}}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}})_j \]

\[ (n, m, F^\oplus_{\text{MOD}})_j \sim F^\oplus_{\text{MOD}}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}})_j \]

\[ (n, m, F^\oplus_{\text{MOD}})_j \sim F^\oplus_{\text{MOD}}(n, m, D_{\nu}^{\Theta_{\pm\text{ell}}\text{NF}})_j \]
which are compatible with the respective natural isomorphisms between "MOD"- and "mod"-subscripted versions [cf. Proposition 3.10, (i)]; here, the isomorphisms of the third line of the display induce isomorphisms of the global realified Frobenioid portions

\[ n,m \bigtriangleup \sim \mathcal{C}_{\text{LGP}}^{\max} \left( n,0 \mathcal{H}_{\text{D},\Theta}^{\pm \ell} \text{NF} \right) ; \quad n,m \bigtriangleup \sim \mathcal{C}_{\text{lp}}^{\max} \left( n,0 \mathcal{H}_{\text{D},\Theta}^{\pm \ell} \text{NF} \right) \]

of the \( F^\dag \)-prime-strips

\[ n,m \bigtriangleup \mathcal{D}_{\text{LGP}}, \quad \mathcal{D}_{\text{LGP}}^{\max} \left( n,0 \mathcal{H}_{\text{D},\Theta}^{\pm \ell} \text{NF} \right), \quad n,m \bigtriangleup \mathcal{D}_{\text{lp}}, \text{ and} \quad \mathcal{D}_{\text{lp}}^{\max} \left( n,0 \mathcal{H}_{\text{D},\Theta}^{\pm \ell} \text{NF} \right) \]

[cf. Propositions 3.7, (iii), (iv), (v); 3.10, (i)].

Moreover, as one varies \( m \in \mathbb{Z} \), the various isomorphisms of (b) and of the first line in the first display of (c) are mutually compatible with one another, relative to the log-links of the n-th column of the LGP-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the domains of these isomorphisms that are possibly related to one another via the log-links consist of roots of unity in the domains of the log-links [multiplication by which corresponds, via the log-link, to an “addition by zero” indeterminacy, i.e., to no indeterminacy!] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii). This mutual compatibility of the isomorphisms of the first line in the first display of (c) implies a corresponding mutual compatibility between the isomorphisms of the second and third lines in the first display of (c) that involve the subscript “MOD” [but not between the isomorphisms that involve the subscript “mod”!] — cf. Proposition 3.10, (iii); Remark 3.10.1]. On the other hand, the isomorphisms of (a) are subject to a certain “indeterminacy” as follows:

\[ \text{(Ind3)} \quad \text{as one varies } m \in \mathbb{Z}, \text{ the isomorphisms of (a) are “upper semi-compatible”, relative to the log-links of the n-th column of the LGP-Gaussian log-theta-lattice under consideration, in a sense that involves certain natural inclusions “\( \subseteq \)” at } v_Q \in \mathbb{V}_Q^{\text{non}} \text{ and certain natural surjections “\( \twoheadrightarrow \)” at } v_Q \in \mathbb{V}_Q^{\text{arc}} — \text{cf. Proposition 3.5, (ii), (a), (b), for more details.} \]

Finally, as one varies \( m \in \mathbb{Z} \), the isomorphisms of (a) are [precisely!] compatible, relative to the log-links of the n-th column of the LGP-Gaussian log-theta-lattice under consideration, with the respective log-volumes [cf. Proposition 3.9, (iv)].

\[ \text{(iii) (} \Theta^{x \mu}_{\text{LGP}} \text{-Link Compatibility)} \quad \text{The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various horizontal arrows — i.e., } \Theta^{x \mu}_{\text{LGP}} \text{-links — of the LGP-Gaussian log-theta-lattice under consideration as follows:} \]

\[ \text{(a) The first Kummer isomorphism of the first display of (ii) induces — by applying the } F^\dag \text{ symmetry of the } \Theta^{\pm \ell} \text{NF-Hodge theater,} \]

\[ n,m \bigtriangleup \mathcal{D}_{\triangle}^{x \mu} \rightarrow \mathcal{D}_{\triangle}^{x \mu} \left( n,0 \mathcal{D}_{\triangle}^{\max} \right) \]

[cf. Theorem 1.5, (iii)]. Relative to this Kummer isomorphism, the full poly-isomorphism of \( F^\dag \times \mu \)-prime-strips

\[ \mathcal{D}_{\triangle}^{x \mu} \left( n,0 \mathcal{D}_{\triangle}^{\max} \right) \rightarrow \mathcal{D}_{\triangle}^{x \mu} \left( n+1,0 \mathcal{D}_{\triangle}^{\max} \right) \]

is compatible with the full poly-isomorphism of \( F^\dag \times \mu \)-prime-strips

\[ n,m \bigtriangleup \mathcal{D}_{\triangle}^{x \mu} \rightarrow n+1,m \bigtriangleup \mathcal{D}_{\triangle}^{x \mu} \]
induced [cf. Theorem 1.5, (ii)] by the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration [cf. Theorem 1.5, (iii)].

(b) The $F^\triangleright$-prime-strips $n,m\,\tilde{\mathcal{F}}^\triangleright_{\text{env}}(n,\mathcal{D}_>)$ [cf. Proposition 2.1, (ii)] that appear implicitly in the construction of the $F^\triangleright$-prime-strips $n,m\,\tilde{\mathcal{F}}^\triangleright_{\text{LGP}}$, $\tilde{\mathcal{F}}^\triangleright(n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N)_{\text{LGP}}$, $n,m\,\tilde{\mathcal{F}}^\triangleright_{\text{LGP}}$, $\tilde{\mathcal{F}}^\triangleright(n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N)_{\text{LGP}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)] admit natural isomorphisms of associated $F^\times\mu$-prime-strips $n,m\,\tilde{\mathcal{T}}^\times\mu_{\triangle} \xrightarrow{\sim} n,m\,\tilde{\mathcal{T}}^\times\mu_{\text{env}}$, $\tilde{\mathcal{T}}^\times\mu(n,\mathcal{D}_>) \xrightarrow{\sim} \tilde{\mathcal{T}}^\times\mu(n,\mathcal{D}_>)$ [cf. Proposition 2.1, (vi)]. Relative to these natural isomorphisms and to the Kummer isomorphism discussed in (a) above, the full poly-isomorphism of $F^\times\mu$-prime-strips

$$\tilde{\mathcal{T}}^\times\mu(n,\mathcal{D}_>) \xrightarrow{\sim} \tilde{\mathcal{T}}^\times\mu(n+1,\mathcal{D}_>)$$

is compatible with the full poly-isomorphism of $F^\times\mu$-prime-strips

$$n,m\,\tilde{\mathcal{T}}^\times\mu_{\triangle} \xrightarrow{\sim} n+1,m\,\tilde{\mathcal{T}}^\times\mu_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration [cf. Corollary 2.3, (iii)].

(c) Recall the data $^{^{^{a,\mathcal{O}}\mathcal{R}}}$ [cf. Corollary 2.3, (ii)] associated to the $D$-$\Theta^{\pm\ell-\text{ell}}_N$-Hodge theater $n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N$ — data which appears implicitly in the construction of the $F^\triangleright$-prime-strips $n,m\,\tilde{\mathcal{F}}^\triangleright_{\text{LGP}}$, $\tilde{\mathcal{F}}^\triangleright(n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N)_{\text{LGP}}$, $n,m\,\tilde{\mathcal{F}}^\triangleright_{\text{LGP}}$, $\tilde{\mathcal{F}}^\triangleright(n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N)_{\text{LGP}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)]. This data that arises from $n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N$ is related to corresponding data that arises from the projective system of mono-theta environments associated to the tempered Frobenioids of the $\Theta^{\pm\ell-\text{ell}}_N$-Hodge theater $n,m\mathcal{H}^{D,\Theta^{\pm\ell-\text{ell}}}_N$ at $v \in \mathcal{V}^{\text{bad}}$ via the Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii) and Theorem 1.5, (iii) [cf. also (a), (b) above], (v). The algorithmic construction of these Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments, as well as of the poly-isomorphism

$$n,\mathcal{O}\,\mathcal{R} \xrightarrow{\sim} n+1,\mathcal{O}\,\mathcal{R}$$

induced by any permutation symmetry of the étale-picture [cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2] $n,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N \xrightarrow{\sim} n+1,\mathcal{O}^{D,\Theta^{\pm\ell-\text{ell}}}_N$ is compatible with the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration, e.g., with the full poly-isomorphism of $F^\times\mu$-prime-strips

$$n,m\,\tilde{\mathcal{T}}^\times\mu_{\triangle} \xrightarrow{\sim} n+1,m\,\tilde{\mathcal{T}}^\times\mu_{\triangle}$$
induced [cf. Theorem 1.5, (ii)] by these horizontal arrows [cf. Corollary 2.3, (iv)], in the sense that these constructions are stabilized/equivariant/functorial with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the LGP-Gaussian log-theta-lattice. Finally, the algorithmic construction of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various evaluation maps implicit in the portion of the log-Kummer correspondence discussed in (ii), (b), are compatible with the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies (Ind1), (Ind2), (Ind3) described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions are stabilized/equivariant/functorial with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the LGP-Gaussian log-theta-lattice.

(d) The algorithmic construction of the Kummer isomorphisms of the first display of (ii) [cf. also (a), (b) above; the gluing discussed in [IUTchII], Corollary 4.6, (iv); the Kummer compatibilities discussed in [IUTchII], Corollary 4.8, (iii); the relationship to the notation of [IUTchI], Definition 5.2, (vi), (viii), referred to in [IUTchII], Propositions 4.2, (i), and 4.4, (i)], as well as of the poly-isomorphisms between the data

\[
\left\{ \pi_1^{K-sol(n,\circ D^\oplus)} \right\} \hookrightarrow M_{\infty,\kappa}^\oplus(n,\circ D^\oplus)_j \\
\to M_{\infty,\kappa\times v}(n,\circ D^\oplus)_j \subseteq M_{\infty,\kappa\times v}(n,\circ D^\oplus)_j \quad \forall v \in V
\]

[i.e., of the second line of the first display of [IUTchII], Corollary 4.7, (iii)] induced by any permutation symmetry of the étale-picture [cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2] \( n,\circ \mathcal{H}^{D,-\Theta^{\pm \varepsilon}}\) are compatible [cf. the discussion of Remark 2.3.2] with the full poly-isomorphism of \( F^{\times \mu,\kappa}\)-prime-strips

\[
\mathfrak{S}_{\Delta}^{\Delta} \quad \supseteq \quad n+1,\kappa\mathfrak{S}_{\Delta}^{\Delta} \quad \supseteq \quad n+m,\mathfrak{S}_{\Delta}^{\Delta}\]

induced [cf. Theorem 1.5, (ii)] by the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration, in the sense that these constructions are stabilized/equivariant/functorial with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the LGP-Gaussian log-theta-lattice. Finally, the algorithmic construction of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various evaluation maps implicit in the portion of the log-Kummer correspondence discussed in (ii), (c), are compatible with the horizontal arrows of the LGP-Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies (Ind1), (Ind2), (Ind3) described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions
are stabilized/equivariant functorial with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the LGP-Gaussian log-theta-lattice.

Proof. The various assertions of Theorem 3.11 follow immediately from the definitions and the references quoted in the statements of these assertions — cf. also the various related observations of Remarks 3.11.1, 3.11.2, 3.11.3, 3.11.4 below.

Remark 3.11.1.

(i) One way to summarize the content of Theorem 3.11 is as follows:

Theorem 3.11 gives an algorithm for describing, up to certain relatively mild indeterminacies, the LGP-monoids [cf. Fig. 3.1] — i.e., in essence, the theta values

\[ \left\{ q_j^2 \right\}_{j=1,...,l^\ast} \]

— which are constructed relative to the scheme/ring structure, i.e., “arithmetic holomorphic structure”, associated to one vertical line [i.e., “(n, o)” for some fixed \( n \in \mathbb{Z} \)] in the LGP-Gaussian log-theta-lattice under consideration, in terms of the a priori alien arithmetic holomorphic structure of another vertical line [i.e., “(n + 1, o)”] in the LGP-Gaussian log-theta-lattice under consideration [cf., especially, the final portion of Theorem 3.11, (i), concerning functoriality and compatibility with the permutation symmetries of the étale-picture].

This point of view is consistent with the point of view of the discussion of Remark 1.5.4; [IUTchII], Remark 3.8.3, (iii).

(ii) Although the various versions of the \( \Theta \)-link are defined [cf. Definition 3.8, (ii)] as gluings of

the \( \mathcal{F}_{\bullet}^{\times \mu} \)-prime-strip whose associated pilot object [cf. [IUTchII], Definition 4.9, (viii)] is some sort of \( \Theta \)-pilot object in the domain of the \( \Theta \)-link to

the \( \mathcal{F}_{\bullet}^{\times \mu} \)-prime-strip whose associated pilot object is some sort of \( q \)-pilot object in the codomain of the \( \Theta \)-link,

in fact it is not difficult to see that the theory developed in the present series of papers remains essentially unaffected

even if one replaces this \( q \)-pilot \( \mathcal{F}_{\bullet}^{\times \mu} \)-prime-strip in the codomain of the \( \Theta \)-link by some other \( \mathcal{F}_{\bullet}^{\times \mu} \)-prime-strip

such as, for instance, the \( \mathcal{F}_{\bullet}^{\times \mu} \)-prime-strip whose associated pilot object is the \( q^\lambda \)-pilot object [i.e., the \( \lambda \)-th power of the \( q \)-pilot object, for some positive integer \( \lambda > 1 \)] — cf. the discussion of Remark 3.12.1, (ii), below. One way to formulate this observation is as follows: The \( \Theta \)-link compatibility described in Theorem 3.11, (iii), may be interpreted as an assertion to the effect that the functorial construction
algorithm for the $\Theta$-pilot object up to certain mild indeterminacies [i.e., (Ind1), (Ind2), (Ind3)] that is given in Theorem 3.11 may be regarded as

an algorithm whose input data is an $F^\nabla\times\mu$-prime-strip [i.e., the $F^\nabla\times\mu$-prime-strip that appears in the codomain of the $\Theta$-link], and whose functoriality is with respect to arbitrary isomorphisms of the $F^\nabla\times\mu$-prime-strips that appear as input data of the algorithm.

From the point of view of the gluing given by the $\Theta$-link, this functoriality in the input data given by an $F^\nabla\times\mu$-prime-strip may be interpreted in the following way:

this functoriality allows one to regard the functorial construction algorithm for the $\Theta$-pilot object up to certain mild indeterminacies that is given in Theorem 3.11 as an algorithm with respect to which the codomain $\Theta^{\pm\text{ell}}$NF-Hodge theater of the $\Theta$-link [together with the other $\Theta^{\pm\text{ell}}$NF-Hodge theaters in the same vertical line of the log-theta-lattice as this codomain $\Theta^{\pm\text{ell}}$NF-Hodge theater] — i.e., in effect, the q-pilot $F^\nabla\times\mu$-prime-strip, equipped with the rigidification determined by the arithmetic holomorphic structure constituted by this vertical line of $\Theta^{\pm\text{ell}}$NF-Hodge theaters — is "coric", i.e., "remains invariant" / "may be regarded as being held fixed" throughout the execution of the various operations of the algorithm.

This interpretation will play a crucial role in the application of Theorem 3.11 to Corollary 3.12 below.

(iii) On the other hand, the étale-picture permutation symmetries discussed in the final portion of Theorem 3.11, (i) [cf. also the references to these symmetries in Theorem 3.11, (iii), (c), (d)], may be interpreted as follows: The output data of the functorial construction algorithm of Theorem 3.11 consists of a representation of the data of Theorem 3.11, (i), (b), (c) [cf. also Theorem 3.11, (iii), (c), (d)], up to certain mild indeterminacies on the mono-analytic étale-like log-shells of Theorem 3.11, (i), (a), that satisfies the following properties:

- (Input prime-strip link (IPL)) This output data is constructed in such a way that it is linked/related, via full poly-isomorphisms of $F^\nabla\times\mu$-prime-strips induced by operations in the algorithm, to the input data prime-strip, i.e., the "coric"/"fixed" q-pilot $F^\nabla\times\mu$-prime-strip, equipped with its rigidifying arithmetic holomorphic structure [cf. the discussion of (ii)]. In particular, we note that each of these "intermediate" $F^\nabla\times\mu$-prime-strips that appears in the construction may itself be taken to be both

the input data of the functorial algorithm of Theorem 3.11 [cf. the discussion of (ii)]

and

[by applying the full poly-isomorphisms of $F^\nabla\times\mu$-prime-strips that link/re relate it to the q-pilot $F^\nabla\times\mu$-prime-strip] the input data for the Kummer theory surrounding the q-pilot object $F^\nabla\times\mu$-prime-strip in its rigidifying $\Theta^{\pm\text{ell}}$NF-Hodge theater [cf. the discussion of (ii)].
At a more explicit level, the linking isomorphisms of “intermediate” \( F^{×\mu} \)-prime-strips are given by composing

- the inverses of the first two lower horizontal arrows of the commutative diagram of Remark 3.10.2, followed by
- the first vertical arrow of this diagram — corresponding to the Kummer theory portion of Theorem 3.11, (iii), (c), (d) — followed by
- the three upper horizontal arrows of the diagram — corresponding to the evaluation map portion of Theorem 3.11, (iii), (c), (d).

Here, we observe that the final evaluation map portion of this composite involves a construction of the \( \Theta \)-pilot object up to certain indeterminacies [i.e., (Ind1), (Ind2), (Ind3)], which, by applying the discussion of Remark 2.4.2, (v), (vi), may be interpreted — provided that certain sign conditions [cf. the discussion of Remark 2.4.2, (iv), (vi)] are satisfied, and one takes into account the considerations discussed in Remarks 3.9.6 [concerning the product formula], 3.9.7 [concerning inverse systems of direct product regions] — as a construction of the global realified Frobenioid portion of an \( F^{×\mu} \)-prime-strip, together with various possibilities [corresponding to the indeterminacies] for the “further rigidification” determined by the pilot object.

- (Simultaneous holomorphic expressibility (SHE)) The construction of this output data, as well as the output data itself, is expressed in terms that are simultaneously valid/executable/well-defined relative to both

  the arithmetic holomorphic structure that gives rise to the \( \Theta \)-pilot object in the domain of the \( \Theta \)-link — i.e., in more technical language, in terms of/as a function of structures in the \( \Theta^{\pm}\text{ell} \)NF-Hodge theater in the domain of the \( \Theta \)-link —

  and

  the arithmetic holomorphic structure that gives rise to the input data prime-strip [i.e., such as the \( q \)-pilot \( F^{×\mu} \)-prime-strip, as discussed in (ii)] in the codomain of the \( \Theta \)-link — i.e., in more technical language, in terms of/as a function of structures in the \( \Theta^{\pm}\text{ell} \)NF-Hodge theater in the codomain of the \( \Theta \)-link.

In passing, we observe that this property “SHE” may be understood, in a slightly more concrete way, as corresponding to the fact that the chain of (sub)quotients considered in Remark 3.9.5, (viii), (ix), forms a closed loop.

These two fundamental properties of the output data of the algorithm of Theorem 3.11 will play a central role in the application of Theorem 3.11 to Corollary 3.12 below. In the context of these two fundamental properties, it is interesting to observe that, relative to the analogy between multiradiality and crystals/connections [cf. [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii), (iii)],
the distinction between abstract $F^{\uparrow \downarrow} \times \mu$-prime-strips and various specific realizations of such $F^{\uparrow \downarrow} \times \mu$-prime-strips [e.g., arising from the structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater] may be understood as corresponding to

the distinction between reduced characteristic $p$ schemes [where $p$ is a prime number] and thickenings of such schemes over $\mathbb{Z}_p$ in the context of $p$-adic crystals.

(iv) The SHE property discussed in (iii) may be thought of as a sort of “parallel transport” mechanism for the $\Theta$-pilot object [cf. the analogy between multiradiality and connections, as discussed in [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii)], up to certain mild indeterminacies, from the [arithmetic holomorphic structure represented by the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in the] domain of the $\Theta$-link to the [arithmetic holomorphic structure represented by the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in] codomain of the $\Theta$-link. On the other hand, in this context, it is important to observe that:

- (Algorithmic parallel transport (APT)) This parallel transport mechanism does not consist of a simple instance of transport of some set-theoretic region [such as the region in the tensor packet of log-shells determined by the $\Theta$-pilot object in the domain of the $\Theta$-link] via some set-theoretic function. Rather, it consists of a construction algorithm that is simultaneously valid/executable/well-defined with respect to the arithmetic holomorphic structures in the domain and codomain of the $\Theta$-link [cf. the discussion of (iii)].

[In this context, it is important to remember that although this construction algorithm may yield, as output, various “possible regions”, such possible regions cannot necessarily be directly compared with various structures in the codomain of the $\Theta$-link. That is to say, such comparisons typically require the application of further techniques, as discussed in Remark 3.9.5, (vii).] In particular, if one takes the point of view — as will be done in Corollary 3.12 below! — that one is only interested in considering the qualitative logical aspects/consequences of the construction algorithm of Theorem 3.11, then:

- (Hidden internal structures (HIS)) One may [and, indeed, it is often useful to] regard this construction algorithm of Theorem 3.11 as a construction algorithm for producing “some sort of output data” satisfying various properties [cf. (iii)] associated to “some sort of input data” [cf. (ii)] and forget that this construction algorithm of Theorem 3.11 has anything to do with theta functions [e.g., the theory of [EtTh]] or theta values [i.e., the $\left\{ \frac{q_j}{2} \right\}_{j=1,\ldots,\ell^*}$.]

That is to say, theta functions/theta values may be regarded as HIS of the construction algorithm of Theorem 3.11 — somewhat like the internal structure of the CPU or operating system of a computer! — i.e., internal structures whose technical details are [of course, of crucial importance from the point of view of the actual functioning of the construction algorithm, but nonetheless] irrelevant or uninteresting from the point of view of the “end user”, who is only interested in applying the
construction algorithm to certain input data to obtain certain output data. [Here, we observe in passing that, relative to this analogy with the internal structure of the CPU or operating system of a computer, the $\mathcal{F}_{\oplus}^{\times_{\mu}}$-prime-strips that occur in the $\Theta$-link may be thought of as a sort connecting cable, i.e., of the sort that is used to link distinct computers via the internet. That is to say, despite the fact that such a connecting cable may have a very simple internal structure by comparison to the computers that it connects, the connection that it furnishes has highly nontrivial consequences [e.g., as in the case of the internet] — cf. the discussion in (iii) of the input prime-strip link (IPL) and the analogy with crystals/connections.] On the other hand, we observe that, unlike Corollary 3.12 below, which only concerns qualitative logical aspects/consequences of the construction algorithm of Theorem 3.11, the explicit computation to be performed in [IUTchIV], §1, of the log-volumes that occur in the statement of Corollary 3.12 makes essential use of the way in which theta values occur in the construction algorithm of Theorem 3.11.

(v) Thus, in summary, the above discussion yields a slightly different, and in some sense more detailed, way [by comparison to (i)] to summarize the content of the construction algorithm of Theorem 3.11 [cf. also the discussion of Remark 3.12.2, (ii), below]: The functorial construction algorithm of Theorem 3.11 is an algorithm whose

- input data consists solely of an $\mathcal{F}_{\oplus}^{\times_{\mu}}$-prime-strip, regarded up to isomorphism [cf. (ii)], and whose
- output data consists of certain data that is linked/related, via full poly-isomorphisms of $\mathcal{F}_{\oplus}^{\times_{\mu}}$-prime-strips induced by operations in the algorithm, to the input data prime-strip, and, moreover, whose construction algorithm may be expressed in terms that are simultaneously valid/executable/well-defined relative to both the arithmetic holomorphic structure that gives rise to the $\Theta$-pilot object in the domain of the $\Theta$-link and the arithmetic holomorphic structure that gives rise to the input data prime-strip [i.e., such as the $q$-pilot $\mathcal{F}_{\oplus}^{\times_{\mu}}$-prime-strip].

This construction algorithm of Theorem 3.11 makes crucial use of certain HIS such as theta functions and theta values, but these HIS may be ignored, if one is only interested in the qualitative logical aspects/consequences of the input and output data of the algorithm.

(vi) In the context of the input prime-strip link (IPL) and simultaneous holomorphic expressibility (SHE) properties discussed in (iii), it is perhaps of interest to consider what happens in the case of the very simple, naive example discussed in Remark 2.2.2, (i). That is to say, suppose that one considers the “naive version" of the $\Theta$-link given by a correspondence of the form

$$q \mapsto q^\lambda$$  

— where $\lambda > 1$ is a positive integer — relative to a single arithmetic holomorphic structure, i.e., in effect, ring structure “$R$". [Here, we remark that, unlike the situation considered in the discussion of (ii), where “$q^\lambda$" appears in the codomain of some modified version of the $\Theta$-link, the “$q^\lambda$" in the present discussion appears in the domain of some modified version of the $\Theta$-link.] Then the very definition of
this naive version of the Θ-link yields an explicit construction algorithm for “$q^\lambda$”, namely, as the λ-th power of “$\hat{q}$”. That is to say, this [essentially tautological!] explicit construction algorithm for “$q^\lambda$” satisfies the SHE property considered in (iii) in the sense that

the tautological construction algorithm given by taking “the λ-th power of $\hat{q}$” may be regarded as simultaneously executable relative to both the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to “$\hat{q}$” and the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to “$q^\lambda$”.

On the other hand, we observe that this sort of [essentially tautological!] SHE property is achieved as the cost of sacrificing the establishment of the analogue of the IPL property of (iii), in the sense that

if one restricts oneself to considering “$\hat{q}$” and “$q^\lambda$” inside the fixed container constituted by the given arithmetic holomorphic structure [i.e., in effect, ring structure “$R$”] that gives rise to “$\hat{q}$”, then the tautological construction algorithm considered above does not induce any sort of identification between “$\hat{q}$” and “$q^\lambda$”.

(vii) We maintain the notation of (vi). One may then approach the issue of establishing the analogue of the IPL property of (iii) by introducing a formal symbol “$\ast$” [corresponding to the abstract $\mathcal{F}^{\uparrow\downarrow} \times \mu$-prime-strips that appear in the Θ-link] and then considering one of the following two approaches:

· (Distinct labels) It is essentially a tautology that in order to consider both of the assignments $\ast \mapsto \hat{q}$ and $\ast \mapsto \hat{q}^\lambda$ simultaneously [i.e., in order to establish the analogue of the IPL property of (iii)!], it is necessary to introduce distinct labels “$\downarrow$” and “$\uparrow$” for the arithmetic holomorphic structures [i.e., in effect, ring structures] that give rise to “$\hat{q}$” and “$q^\lambda$”, respectively. That is to say, it is a tautology that one may consider the assignments

\[ \ast \mapsto \downarrow \hat{q}^\lambda, \quad \ast \mapsto \uparrow q \]

simultaneously and without introducing any inconsistencies. On the other hand, this approach via the introduction of tautologically distinct labels — which may be summarized via the diagram

\[
\begin{array}{c}
\ast \mapsto \downarrow \hat{q}^\lambda \in \downarrow R \\
\| \quad ?? \quad \| \\
\ast \mapsto \uparrow q \in \uparrow R
\end{array}
\]

IPL: holds
SHE: ??

— has the drawback that it is by no means clear, at least in any a priori sense, how to establish the analogue of the SHE property of (iii), since it
is by no means clear, at least in any a priori sense, how to “compute” the relationship between the “†” and “‡” arithmetic holomorphic structures [i.e., in effect, ring structures].

(Forced identification of arithmetic holomorphic structures) Of course, one may then attempt to remedy the drawback that appeared in the distinct labels approach by simply arbitrarily identifying the “†” and “‡” arithmetic holomorphic structures [i.e., in effect, ring structures], that is to say, by simply deleting/forgetting the distinct labels “†” and “‡”. This approach — which may be summarized via the diagram

\[
egin{align*}
  * & \mapsto q^\lambda \in R \\
  \vdots & \\
  ? & \\
  \vdots & \\
  * & \mapsto q \in R
\end{align*}
\]

— allows one to apply the [tautological!] construction algorithm discussed in (vi). On the other hand, this approach has the drawback that, in order to consider the assignments

\[
egin{align*}
  * & \mapsto q^\lambda, \\
  * & \mapsto q
\end{align*}
\]

simultaneously and consistently [i.e., in order to establish the analogue of the IPL property of (iii)], one is led [at least in the absence of more sophisticated machinery!] to regard “q” as being only well-defined up to possible confusion with “q^\lambda”, for some indeterminate \( n \in \mathbb{Z} \). That is to say, in summary, this approach gives rise to a sort of “uninteresting/trivial multiradial representation” of “q^\lambda” via

\[
\{ q^\lambda \}_{n \in \mathbb{Z}}
\]

— which [despite being uninteresting/trivial!] does indeed satisfy the formal analogues of the IPL and SHE properties of (iii).

(viii) We conclude our discussion of the simple, naive examples discussed in (vi) and (vii) by considering the relationship between these simple, naive examples and the theory of the present series of papers. We begin by observing that the “trivial multiradial representation \( \{ q^\lambda \}_{n \in \mathbb{Z}} \)” discussed in (vii) is, on the one hand, of interest, in the context of the IPL and SHE properties of (iii), in that it constitutes a useful elementary “toy model” for considering the qualitative logical aspects of these fundamental properties satisfied by the multiradial construction algorithm of Theorem 3.11. On the other hand, this “trivial multiradial representation” is useless from the point of view of applications such as the log-volume estimates given in Corollary 3.12 below [cf. the discussion of the final portion of (iv)]
for the following reasons: This “trivial multiradial representation \( \{ q^{\lambda} \}_{n \in \mathbb{Z}} \)” is obtained by

\begin{itemize}
  \item allowing for indeterminacies in the value group portion [i.e., “\( q^{\mathbb{Z}} \)’s”]
  \item of the data under consideration,
  
  \item while the unit group portion [i.e., the \( \mathcal{O}^{\times} \)’s] associated to the local fields that appear] of the data under consideration is held rigid [i.e., not subject to indeterminacies];
  
  \item only working with the multiplicative structure constituted by the value group portion of the rings involved, and
  
  \item ignoring] issues related to the additive structure of the rings involved, especially, issues related to the intertwining between the additive and multiplicative structures of these rings [cf. the discussion of Remark 3.12.2, (ii), below].
\end{itemize}

By contrast, the log-volume estimates of Corollary 3.12 below rely, in an essential way, on the fact that in the multiradial construction algorithm of Theorem 3.11:

\begin{itemize}
  \item the value group portions of the data under consideration [i.e., the \( \mathcal{F}^{\times} \)-prime-strips associated to the \( \mathcal{F}^{\times} \times \mu \)-prime-strips that appear in the definition of the \( \Theta \)-link] are held rigid [i.e., are not subject to indeterminacies],
  
  \item while the unit group portions of the data under consideration [i.e., the \( \mathcal{F}^{+} \times \mu \)-prime-strips associated to the \( \mathcal{F}^{\times} \times \mu \)-prime-strips that appear in the definition of the \( \Theta \)-link] are subject to the indeterminacies (Ind1), (Ind2), (Ind3);
  
  \item the multiradial construction algorithm makes use, via the log-Kummer correspondence, of the structure of the intertwining between the additive and multiplicative structures of the rings involved [cf. the discussion of Remark 3.12.2, (ii), (iii), (iv), (v), below].
\end{itemize}

Finally, we observe that the technique of assigning distinct labels that appears in the distinct labels approach discussed in (vii) is formalized in the theory of the present series of papers by means of the notion of Frobenius-like structures, i.e., at a more concrete level, mathematical objects that, at least a priori, only make sense within the \( \Theta^{\pm} \)-NF-Hodge theater labeled “\((n,m)\)” [where \( n,m \in \mathbb{Z} \)] of the log-theta-lattice. The problem of relating objects arising from \( \Theta^{\pm} \)-NF-Hodge theaters with distinct labels “\((n,m)\)” is then resolved in the present series of papers — not by means of “forced identification” [i.e., in the style of the discussion of (vii)] of \( \Theta^{\pm} \)-NF-Hodge theaters with distinct labels, but rather — by considering the permutation symmetries [i.e., of the sort discussed in the final portion of Theorem 3.11, (i)] satisfied by \( \acute{\text{e}} \)tale-like structures. Here, it is perhaps useful to recall that the fundamental model for such permutation symmetries is, in the notation of [IUTchII], Example 1.8, (i),

\[ \Pi \rightarrow G \leftarrow \Pi \]

— where the arrows “\( \rightarrow \)” and “\( \leftarrow \)” denote the poly-morphism given by composing the natural surjection \( \Pi \rightarrow \Pi/\Delta \) with the full poly-isomorphism \( \Pi/\Delta \rightarrow G \),
and we observe that the diagram of this display admits a permutation symmetry that switches these two arrows “$\rightarrow$” and “$\leftarrow$”.

**Remark 3.11.2.**

(i) In Theorem 3.11, (i), we do not apply the formalism or language developed in [IUTchII], §1, for discussing multiradiality. Nevertheless, the approach taken in Theorem 3.11, (i) — i.e., by regarding the collection of data (a), (b), (c) up to the indeterminacies given by (Ind1), (Ind2) — to constructing “multiradial representations” amounts, in essence, to a special case of the tautological approach to constructing multiradial environments discussed in [IUTchII], Example 1.9, (ii). That is to say, this tautological approach is applied to the vertically coric constructions of Proposition 3.5, (i); 3.10, (i), which, a priori, are uniradial in the sense that they depend, in an essential way, on the arithmetic holomorphic structure constituted by a particular vertical line — i.e., “$(n, o)$” for some fixed $n \in \mathbb{Z}$ — in the LGP-Gaussian log-theta-lattice under consideration.

(ii) One important underlying aspect of the tautological approach to multiradiality discussed in (i) is the treatment of the various labels that occur in the multiplicative and additive combinatorial Teichmüller theory associated to the $D$-$\Theta^{\pm_{\text{ell}}}$NF-Hodge theater $n \circ H_{D}^{\pm_{\text{ell}}}$NF under consideration [cf. the theory of [IUTchI], §4, §6]. The various transitions between types of labels is illustrated in Fig. 3.3 below. Here, we recall that:

(a) the passage from the $F^\pm_{i}$-symmetry to labels $\in F_{i}$ forms the content of the associated $D$-$\Theta^{\pm_{\text{ell}}}$-Hodge theater [cf. [IUTchI], Remark 6.6.1];

(b) the passage from labels $\in F_{i}$ to labels $\in |F_{i}|$ forms the content of the functorial algorithm of [IUTchI], Proposition 6.7;

(c) the passage from labels $\in |F_{i}|$ to $\pm$-processions forms the content of [IUTchI], Proposition 6.9, (ii);

(d) the passage from the $F^\pm_{i}$-symmetry to labels $\in F^\pm_{i}$ forms the content of the associated $D$-$\Theta$NF-Hodge theater [cf. [IUTchI], Remark 4.7.2, (i)];

(e) the passage from labels $\in F^*_{i}$ to $\ast$-processions forms the content of [IUTchI], Proposition 4.11, (ii);

(f) the compatibility between $\ast$-processions and $\pm$-processions, relative to the natural inclusion of labels $F^*_{i} \hookrightarrow |F_{i}|$, forms the content of [IUTchI], Proposition 6.9, (iii).

Here, we observe in passing that, in order to perform these various transitions, it is absolutely necessary to work with all of the labels in $F_{i}$ or $|F_{i}|$, i.e., one does not have the option of “arbitrarily omitting certain of the labels” [cf. the discussion of [IUTchII], Remark 2.6.3; [IUTchII], Remark 3.5.2]. Also, in this context, it is important to note that there is a fundamental difference between the labels $\in F_{i}$, $|F_{i}|$, $F^*_{i}$ — which are essentially arithmetic holomorphic in the sense that they depend, in an essential way, on the various local and global arithmetic fundamental groups involved — and the index sets of the mono-analytic $\pm$-processions.
that appear in the multiradial representation of Theorem 3.11, (i). Indeed, these index sets are just "naked sets" which are determined, up to isomorphism, by their cardinality. In particular,

*the construction of these index sets is independent of the various arithmetic holomorphic structures involved.*

Indeed, it is precisely this property of these index sets that renders them suitable for use in the construction of the multiradial representations of Theorem 3.11, (i). As discussed in [IUTchI], Proposition 6.9, (i), for \( j \in \{0, \ldots, l^*\} \), there are precisely \( j + 1 \) possibilities for the "element labeled \( j \)" in the index set of cardinality \( j + 1 \); this leads to a total of \( (l^* + 1)! = l^! \) possibilities for the "label identification" of elements of index sets of capsules appearing in the mono-analytic ±-processions of Theorem 3.11, (i). Finally, in this context, it is of interest to recall that the "rounder approach to symmetrization" that arises when one works with mono-analytic processions is ["downward"] compatible with the finer arithmetically holomorphic approach to symmetrization that arises from the \( F^\pm_l \)-symmetry [cf. [IUTchII], Remark 3.5.3; IUTchII], Remark 4.5.2, (ii); IUTchII], Remark 4.5.3, (ii)].

\[
\begin{array}{cc}
F^\pm_l \text{-symmetry} & F^*_l \text{-symmetry} \\
\downarrow & \downarrow \\
\text{labels} \in F_l & \text{labels} \in |F| \\
\text{→} & \text{←} \\
\pm\text{-procession} & \#\text{-procession}
\end{array}
\]

Fig. 3.3: Transitions from symmetries to labels to processions in a \( \Theta^\pm_{\text{cell}}\text{NF-Hodge theater} \)

(iii) Observe that the "Kummer isomorphism of global realified Frobenioids" that appears in the theory of [IUTchII], §4 — i.e., more precisely, the various versions of the isomorphism of Frobenioids \( _a^i\mathcal{O}^\Rightarrow \mathcal{D}^\#(\mathcal{I}^\#)^\rangle \) discussed in [IUTchII], Corollary 4.6, (ii), (v) — is constructed by considering isomorphisms between local value groups obtained by forming the quotient of the multiplicative groups associated to the various local fields that appear by the subgroups of local units [cf. [IUTchII], Propositions 4.2, (ii); 4.4, (ii)]. In particular, such "Kummer isomorphisms" fail to give rise to a "log-Kummer correspondence", i.e., they fail to satisfy mutual compatibility properties of the sort discussed in the final portion of Theorem 3.11, (ii). Indeed, as discussed in Remark 1.2.3, (i) [cf. also [IUTchII], Remark 1.12.2, (iv)], at \( \nu \in \mathcal{V}_{\text{form}} \), the operation of forming a multiplicative quotient by local units corresponds, on the opposite side of the log-link, to
forming an additive quotient by the submodule obtained as the $p_v$-adic logarithm of these local units. This is precisely why, in the context of Theorem 3.11, (ii), we wish to work with the global non-realified/realified Frobenioids \( F^\ast_{\text{MOD}} \) that arise from copies of \( F_{\text{mod}} \) which satisfy a "lqg-Kummer correspondence", as described in the final portion of Theorem 3.11, (ii) [cf. the discussion of Remark 3.10.1]. On the other hand, the pathologies/indeterminacies that arise from working with global arithmetic line bundles by means of various local data at \( v \in \mathcal{V} \) in the context of the \( \text{lqg-link} \) are formalized via the theory of the global Frobenioids \( F^\ast_{\text{mod}} \), together with the "upper semi-compatibility" of local units discussed in the final portion of Theorem 3.11, (ii) [cf. also the discussion of Remark 3.10.1].

(iv) In the context of the discussion of global realified Frobenioids given in (iii), we observe that, in the case of the global realified Frobenioids [constructed by means of \( F^\ast_{\text{MOD}} \)] that appear in the \( \mathcal{F}^\ast \)-prime-strips \( n,m \mathfrak{D}_{\text{LGP}} \), \( \mathfrak{D}^\ast (n,\circ \mathcal{H}\mathcal{T}^D_\Theta^{\pm \ell \eta})_{\text{LGP}} \) [cf. Theorem 3.11, (ii), (c)], the various localization functors that appear [i.e., the various \( \mathfrak{f}_\rho_v \) of [IUTchI], Definition 5.2, (iv); cf. also the isomorphisms of the second display of [IUTchII], Corollary 4.6, (v)] may be reconstructed, in the spirit of the discussion of Remark 3.9.2, “by considering the effect of multiplication by elements of the [non-realified] global monoids under consideration on the log-volumes of the various local mono-analytic tensor packets that appear”. [We leave the routine details to the reader.] This reconstructibility, together with the mutual incompatibilities observed in (iii) above that arise when one attempts to work simultaneously with log-shells and with the splitting monoids of the \( \mathcal{F}^\ast \)-prime-strip \( n,m \mathfrak{D}_{\text{LGP}} \) at \( v \in \mathcal{V}_{\text{good}} \), are the primary reasons for our omission of explicit mention of the splitting monoids at \( v \in \mathcal{V}_{\text{good}} \) [which in fact appear as part of the data \( n,\circ \mathfrak{N} \) considered in the discussion of Theorem 3.11, (ii), (c)] from the statement of Theorem 3.11 [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b); Theorem 3.11, (iii), (c), in the case of \( v \in \mathcal{V}_{\text{bad}} \)].

**Remark 3.11.3.** Before proceeding, we pause to discuss the relationship between the lqg-Kummer correspondence of Theorem 3.11, (ii), and the \( \Theta^\times_{\text{LGP}} \)-link compatibility of Theorem 3.11, (iii).

(i) First, we recall [cf. Remarks 1.4.1, (i); 3.8.2] that the various squares that appear in the [LGP-Gaussian] log-theta-lattice are far from being \( 1 \)-commutative! On the other hand, the bi-coricity of \( \mathcal{F}^\ast \times \mu \)-prime-strips and mono-analytic log-shells discussed in Theorem 1.5, (iii), (iv), may be interpreted as the statement that the various squares that appear in the [LGP-Gaussian] log-theta-lattice are in fact \( 1 \)-commutative with respect to [the portion of the data associated to each \( \bullet \) in the log-theta-lattice that is constituted by] these bi-coric \( \mathcal{F}^\ast \times \mu \)-prime-strips and mono-analytic log-shells.

(ii) Next, let us observe that in order to relate both the unit and value group portions of the domain and codomain of the \( \Theta^\times_{\text{LGP}} \)-link corresponding to adjacent vertical lines — i.e., \( (n-1,*) \) and \( (n,*) \) — of the [LGP-Gaussian] log-theta-lattice to one another, it is necessary to relate these unit and value group portions to one another by means of a single \( \Theta^\times_{\text{LGP}} \)-link, i.e., from \((n-1,m) \) to \((n,m) \).
That is to say, from the point of view of constructing the various LGP-monoids that appear in the multiradial representation of Theorem 3.11, (i), one is tempted to work with correspondences between value groups on adjacent vertical lines that lie in a vertically once-shifted position — i.e., say, at \((n - 1, m)\) and \((n, m)\) — relative to the correspondence between unit groups on adjacent vertical lines, i.e., say, at \((n - 1, m - 1)\) and \((n, m - 1)\). On the other hand, such an approach fails, at least from an a priori point of view, precisely on account of the noncommutativity discussed in (i). Finally, we observe that in order to relate both unit and value groups by means of a single \(\Theta^{x\mu}_{\text{LGP}}\)-link,

it is necessary to avail oneself of the \(\Theta^{x\mu}_{\text{LGP}}\)-link compatibility properties discussed in Theorem 3.11, (iii) — i.e., of the theory of §2 and [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) — so as to insulate the cyclotomes that appear in the Kummer theory surrounding the étale theta function and \(\kappa\)-coric functions from the \(\text{Aut}_{F^+\times\mu}(-)\)-indeterminacies that act on the \(F^+\times\mu\)-prime-strips involved as a result of the application of the \(\Theta^{x\mu}_{\text{LGP}}\)-link

— cf. the discussion of Remarks 2.2.1, 2.3.2.

(iii) As discussed in (ii) above, a “vertically once-shifted” approach to relating units on adjacent vertical lines fails on account of the noncommutativity discussed in (i). Thus, one natural approach to treating the units in a “vertically once-shifted” fashion — which, we recall, is necessary in order to relate the LGP-monoids on adjacent vertical lines to one another! — is to apply the bi-coricity of mono-analytic log-shells discussed in (i). On the other hand, to take this approach means that one must work in a framework that allows one to relate [cf. the discussion of Remark 1.5.4, (i)] the “Frobenius-like” structure constituted by the Frobenioid-theoretic units [i.e., which occur in the domain and codomain of the \(\Theta^{x\mu}_{\text{LGP}}\)-link] to corresponding étale-like structures simultaneously via both

(a) the usual Kummer isomorphisms — i.e., so as to be compatible with the application of the compatibility properties of Theorem 3.11, (iii), as discussed in (ii) — and

(b) the composite of the usual Kummer isomorphisms with [a single iterate of] the log-link — i.e., so as to be compatible with the bi-coric treatment of mono-analytic log-shells [as well as the closely related construction of LGP-monoids] proposed above.

Such a framework may only be realized if one relates Frobenius-like structures to étale-like structures in a fashion that is invariant with respect to pre-composition with various iterates of the log-link [cf. the final portions of Propositions 3.5, (ii); 3.10, (ii)]. This is precisely what is achieved by the log-Kummer correspondences of the final portion of Theorem 3.11, (ii).

(iv) The discussion of (i), (ii), (iii) above may be summarized as follows: The log-Kummer correspondences of the final portion of Theorem 3.11, (ii), allow one to

(a) relate both the unit and the value group portions of the domain and codomain of the \(\Theta^{x\mu}_{\text{LGP}}\)-link corresponding to adjacent vertical lines of the [LGP-Gaussian] log-theta-lattice to one another, in a fashion that
(b) insulates the cyclotomes/Kummer theory surrounding the étale theta function and κ-coric functions involved from the Aut\(\mathcal{F}\times\mathbb{\mu}(\cdot)\)-indeterminacies that act on the \(\mathcal{F}\times\mathbb{\mu}\)-prime-strips involved as a result of the application of the \(\Theta\times\mathbb{\mu}\)-link [cf. Theorem 3.11, (iii)], and, moreover,

(c) is compatible with the bi-coricity of the mono-analytic log-shells [cf. Theorem 1.5, (iv)], hence also with the operation of relating the LGP-monoids that appear in the multiradial representation of Theorem 3.11, (i), corresponding to adjacent vertical lines of the [LGP-Gaussian] log-theta-lattice to one another.

These observations will play a key role in the proof of Corollary 3.12 below.

**Remark 3.11.4.** In the context of the compatibility discussed in the final portion of Theorem 3.11, (iii), (c), (d), we make the following observations.

(i) First of all, we observe that consideration of the log-Kummer correspondence in the context of the compatibility discussed in the final portion of Theorem 3.11, (iii), (c), (d), amounts precisely to forgetting the labels of the various Frobenius-like “•’s” [cf. the notation of the final display of Proposition 1.3, (iv)], i.e., to identifying data associated to these Frobenius-like “•’s” with the corresponding data associated to the étale-like “◦”. In particular, [cf. the discussion of Theorem 3.11, (ii), preceding the statement of (Ind3)] multiplication of the data considered in Theorem 3.11, (ii), (b), (c), by roots of unity must be “identified” with the identity automorphism. Put another way, this data of Theorem 3.11, (ii), (b), (c), may only be considered up to multiplication by roots of unity. Thus, for instance, it only makes sense to consider orbits of this data relative to multiplication by roots of unity [i.e., as opposed to specific elements within such orbits]. This does not cause any problems in the case of the theta values considered in Theorem 3.11, (ii), (b), precisely because the theory developed so far was formulated precisely in such a way as to be invariant with respect to such indeterminacies [i.e., multiplication of the theta values by 2\(l\)-th roots of unity — cf. the left-hand portion of Fig. 3.4 below]. In the case of the number fields [i.e., copies of \(F_{\text{mod}}\)] considered in Theorem 3.11, (ii), (c), the resulting indeterminacies do not cause any problems precisely because, in the theory of the present series of papers, ultimately one is only interested in the global Frobenioids [i.e., copies of \(\mathcal{F}_{\text{mod}}^\circ\) and \(F_{\text{mod}}^\circ\) and their realifications] associated to these number fields by means of constructions that only involve

- local data, together with
- the entire set — i.e., which, unlike specific elements of this set, is stabilized by multiplication by roots of unity of the number field [cf. the left-hand portion of Fig. 3.5 below] — constituted by the number field under consideration

[cf. the constructions of Example 3.6, (i), (ii); the discussion of Remark 3.9.2]. In this context, we recall from the discussion of Remark 2.3.3, (vi), that the operation of forgetting the labels of the various Frobenius-like “•’s” also gives rise to various indeterminacies in the cyclotomic rigidity isomorphisms applied in the log-Kummer correspondence. On the other hand, in the case of the theta values considered in Theorem 3.11, (ii), (b), we recall from this discussion of
Remark 2.3.3, (vi), that such indeterminacies are in fact trivial [cf. the right-hand portion of Fig. 3.4 below]. In the case of the number fields [i.e., copies of $F_{\text{mod}}$] considered in Theorem 3.11, (ii), (c), we recall from this discussion of Remark 2.3.3, (vi), that such cyclotomic rigidity isomorphism indeterminacies amount to a possible indeterminacy of multiplication by $\pm 1$ on copies of the multiplicative group $F_{\text{mod}}^{\times}$ [cf. the right-hand portion of Fig. 3.5 below], i.e., indeterminacies which do not cause any problems, again, precisely as a consequence of the fact that such indeterminacies stabilize the entire set [i.e., as opposed to specific elements of this set] constituted by the number field under consideration. Finally, in this context, we observe [cf. the discussion at the beginning of Remark 2.3.3, (viii)] that, in the case of the various local data at $\mathfrak{v} \in \mathcal{V}_{\text{non}}$ that appears in Theorem 3.11, (ii), (a), and gives rise to the holomorphic log-shells that serve as containers for the data considered in Theorem 3.11, (ii), (b), (c), the corresponding cyclotomic rigidity isomorphism indeterminacies are in fact trivial. Indeed, this triviality may be understood as a consequence of the fact the following observation: Unlike the case with the cyclotomic rigidity isomorphisms that are applied in the context of the geometric containers [cf. the discussion of Remark 2.3.3, (i)] that appear in the case of the data of Theorem 3.11, (ii), (b), (c), i.e., which give rise to “vicious circles”/“loops” consisting of identification morphisms that differ from the usual natural identification by multiplication by elements of the submonoid $\mathcal{I}_{\text{ord}} \subseteq \mathbb{Z}_{\geq 1}$ [cf. the discussion of Remark 2.3.3, (vi)],

the cyclotomic rigidity isomorphisms that are applied in the context of this local data — even when subject to the various identifications arising from forgetting the labels of the various Frobenius-like “•’s”! — only give rise to natural isomorphisms between “geometric” cyclotomes arising from the geometric fundamental group and “arithmetic” cyclotomes arising from copies of the absolute Galois group of the base [local] field [cf. [AbsTopIII], Corollary 1.10, (c); [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1].

That is to say, no “vicious circles”/“loops” arise since there is never any confusion between such “geometric” and “arithmetic” cyclotomes. [A similar phenomenon may be observed at $\mathfrak{v} \in \mathcal{V}_{\text{arc}}$ with regard to the Kummer structures considered in [IUTchI], Example 3.4, (i).] Thus, in summary,

the various indeterminacies that, a priori, might arise in the context of the portions of the log-Kummer correspondence that appear in the final portion of Theorem 3.11, (iii), (c), (d), are in fact “invisible”, i.e., they have no substantive effect on the objects under consideration [cf. also the discussion of (ii) below]. This is precisely the sense in which the “compatibility” stated in the final portion of Theorem 3.11, (iii), (c), (d), is to be understood.

(ii) In the context of the discussion of (i), we make the following observation:

the discussion in (i) of indeterminacies that, a priori, might arise in the context of the portions of the log-Kummer correspondence that appear in the final portion of Theorem 3.11, (iii), (c), (d), is complete, i.e., there are no further possible indeterminacies that might appear.
Indeed, this observation is a consequence of the “general nonsense” observation [cf., e.g., the discussion of [FrdII], Definition 2.1, (ii)] that, in general, “Kummer isomorphisms” are completely determined by the following data:

(a) isomorphisms between the respective cyclotomes under consideration;
(b) the Galois action on roots of elements of the monoid under consideration.

That is to say, the compatibility of all of the various constructions that appear with the actions of the relevant Galois groups [or arithmetic fundamental groups] is tautological, so there is no possibility that further indeterminacies might arise with respect to the data of (b). On the other hand, the effect of the indeterminacies that might arise with respect to the data of (a) was precisely the content of the latter portion of the discussion of (i) [i.e., of the discussion of Remark 2.3.3, (vi), (viii)].

(iii) In the context of the discussion of (i), we observe that the “invisible indeterminacies” discussed in (i) in the case of the data considered in Theorem 3.11, (ii), (b), (c), may be thought of as a sort of analogue for this data of the indeterminacy (Ind3) [cf. the discussion of the final portion of Theorem 3.11, (ii)] to which the data of Theorem 3.11, (ii), (a), is subject. By contrast, the multiradiality and radial/coric decoupling discussed in Remarks 2.3.2, 2.3.3 [cf. also Theorem 3.11, (iii), (c), (d)] may be understood as asserting precisely that the indeterminacies (Ind1), (Ind2) discussed in Theorem 3.11, (i), which act, essentially, on the data of Theorem 3.11, (ii), (a), have no effect on the geometric containers [cf. the discussion of Remark 2.3.3, (i)] that underlie [i.e., prior to execution of the relevant evaluation operations] the data considered in Theorem 3.11, (ii), (b), (c).

\[
\mu_2 \rightsquigarrow \left\{ \frac{\mu^2}{} \right\}_{j=1,\ldots,t^*} \cup \{1\} \quad (\subseteq \pm \mathbb{N}_{\geq 1})
\]

Fig. 3.4: Invisible indeterminacies acting on theta values

\[
\mu(F_{\text{mod}}^{\times}) \rightsquigarrow F_{\text{mod}}^{\times} \cup \{\pm 1\} \quad (\subseteq \pm \mathbb{N}_{\geq 1})
\]

Fig. 3.5: Invisible indeterminacies acting on copies of \(F_{\text{mod}}^{\times}\)

The following result may be thought of as a relatively concrete consequence of the somewhat abstract content of Theorem 3.11.

**Corollary 3.12. (Log-volume Estimates for \(\Theta\)-Pilot Objects)** Suppose that we are in the situation of Theorem 3.11. Write

\[
-|\log(\Theta)| \in \mathbb{R} \bigcup \{+\infty\}
\]

for the procession-normalized mono-analytic log-volume [i.e., where the average is taken over \(j \in \mathbb{F}_1^*\) — cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a)] of the holomorphic hull [cf. Remark 3.9.5, (i)] of the union of the possible images of a \(\Theta\)-pilot object [cf. Definition 3.8, (i)],
relative to the relevant Kummer isomorphisms [cf. Theorem 3.11, (ii)], in the multiradial representation of Theorem 3.11, (i), which we regard as subject to the indeterminacies (Ind1), (Ind2), (Ind3) described in Theorem 3.11, (i), (ii). Write

\[-|\log(q)| \in \mathbb{R}\]

for the procession-normalized mono-analytic log-volume of the image of a \(q\)-pilot object [cf. Definition 3.8, (i)], relative to the relevant Kummer isomorphisms [cf. Theorem 3.11, (ii)], in the multiradial representation of Theorem 3.11, (i), which we do not regard as subject to the indeterminacies (Ind1), (Ind2), (Ind3) described in Theorem 3.11, (i), (ii). Here, we recall the definition of the symbol “\(\triangle\)” as the result of identifying the labels “0” and “\((\mathbb{F}^*)\)” [cf. [IUTchII], Corollary 4.10, (i)]. In particular, \(|\log(q)| > 0\) is easily computed in terms of the various \(q\)-parameters of the elliptic curve \(E_F\) [cf. [IUTchI], Definition 3.1, (b)] at \(v \in V_{bad}^\dag (\neq \emptyset)\). Then it holds that \(-|\log(\Theta)| \in \mathbb{R}\), and

\[-|\log(\Theta)| \geq -|\log(q)|\]

\(- i.e., C_\Theta \geq -1\) for any real number \(C_\Theta \in \mathbb{R}\) such that \(-|\log(\Theta)| \leq C_\Theta \cdot |\log(q)|\).

Proof. We begin by observing that, since \(|\log(q)| > 0\), we may assume without loss of generality in the remainder of the proof that

\[-|\log(\Theta)| < 0\]

whenever \(-|\log(\Theta)| \in \mathbb{R}\) [i.e., since an inequality \(-|\log(\Theta)| \geq 0\) would imply that \(-|\log(\Theta)| \geq 0 > -|\log(q)|\)]. Now suppose that we are in the situation of Theorem 3.11. For \(n \in \mathbb{Z}\), write

\[n,^o U \overset{\text{def}}{=} \left\{ n,^o U_{j,v_Q} \right\}_{j \in |F_i|, v_Q \in V_Q} \subseteq n,^o U^Q \overset{\text{def}}{=} \left\{ n,^o U^Q_{j,v_Q} \right\}_{j \in |F_i|, v_Q \in V_Q}\]

[where we observe that the “\(\subseteq\)” constitutes a slight abuse of notation] for the collection of subsets \(n,^o U_{j,v_Q} \subseteq n,^o U^Q_{j,v_Q} \overset{\text{def}}{=} I_Q(\mathcal{D}_{j+1,n,^o D_{v_Q}})\) [cf. Theorem 3.11, (i), (a)] given by the various unions, for \(j \in |F_i|\) and \(v_Q \in V_Q\), of the possible images of a \(\Theta\)-pilot object [cf. Definition 3.8, (i)], relative to the relevant Kummer isomorphisms [cf. Theorem 3.11, (ii)], in the multiradial representation of Theorem 3.11, (i), (ii); which we regard as subject to the indeterminacies (Ind1), (Ind2), (Ind3) described in Theorem 3.11, (i), (ii);

\[n,^o \overline{U} = \left\{ n,^o \overline{U}_{j,v_Q} \right\}_{j \in |F_i|, v_Q \in V_Q} \subseteq n,^o \overline{U}^Q = \left\{ n,^o \overline{U}^Q_{j,v_Q} \right\}_{j \in |F_i|, v_Q \in V_Q}\]

[where we observe that the “\(\subseteq\)” constitutes a slight abuse of notation] for the collection of subsets \(n,^o \overline{U}_{j,v_Q} \subseteq n,^o \overline{U}^Q_{j,v_Q} = I_Q(\mathcal{D}_{j+1,n,^o D_{v_Q}})\) [cf. Theorem 3.11, (i), (a)] given by the various holomorphic hulls [cf. Remark 3.9.5, (i)] of the subsets
we may restrict our attention to possible images of a $\Theta$-pilot object that correspond to data [i.e., collections of regions] that may be interpreted as an $F^\uparrow\downarrow\rightarrow\mu$-prime-strip.

Now we proceed to review precisely what is achieved by the various portions of Theorem 3.11 and, indeed, by the theory developed thus far in the present series of papers. This review leads naturally to an interpretation of the theory that gives rise to the inequality asserted in the statement of Corollary 3.12. For ease of reference, we divide our discussion into steps, as follows.

(i) In the following discussion, we concentrate on a single arrow — i.e., a single $\Theta_{LGP}^\times\mu$-link

\[
0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}
\]

— of the [LGP-Gaussian] log-theta-lattice under consideration. This arrow consists of the full poly-isomorphism of $F^\uparrow\downarrow\rightarrow\mu$-prime-strips

\[
0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}
\]

[cf. Definition 3.8, (ii)]. This poly-isomorphism may be thought of as consisting of a "unit portion" constituted by the associated [full] poly-isomorphism of $F^\uparrow\downarrow\rightarrow\mu$-prime-strips

\[
0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}
\]

and a "value group portion" constituted by the associated [full] poly-isomorphism of $F^\uparrow\downarrow\rightarrow\mu$-prime-strips

\[
0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}
\]

[cf. Definition 2.4, (iii)]. This value group portion of the $\Theta_{LGP}^\times\mu$-link maps $\Theta$-pilot objects of $0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}$ to q-pilot objects of $1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}$ [cf. Remark 3.8.1].

(ii) Whereas the units of the Frobenoioids that appear in the $F^\uparrow\downarrow\rightarrow\mu$-prime-strip $0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}$ are subject to Aut$_{F^\uparrow\downarrow\rightarrow\mu}(-)$-indeterminacies [i.e., "(Ind1), (Ind2)" — cf. Theorem 3.11, (iii), (a), (b)], the cyclotomes that appear in the Kummer theory surrounding the étale theta function and $\kappa$-coric functions, i.e., which give rise to the "value group portion" $0.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}} \xrightarrow{\Theta_{LGP}^\times\mu} 1.0\mathcal{H}\Theta^{\pm\text{eall}\text{NF}}$, are insulated from these Aut$_{F^\uparrow\downarrow\rightarrow\mu}(-)$-indeterminacies — cf. Theorem 3.11, (iii), (c), (d); the discussion of Remark 3.11.3, (iv); Fig. 3.6 below. Here, we recall that in the case of the étale theta function, this follows from the theory of §2, i.e., in essence, from the cyclotomic rigidity of mono-theta environments, as discussed in [EtTh]. On the other hand, in the case of $\kappa$-coric functions, this follows from the algorithms discussed in [IUTchl], Example 5.1, (v); [IUTchl], Definition 5.2, (vi), (viii).
<table>
<thead>
<tr>
<th></th>
<th>$\Theta$-related objects</th>
<th>$NF$-related objects</th>
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</thead>
<tbody>
<tr>
<td>require</td>
<td>local LGP-monoids</td>
<td>copies of $F_{\text{mod}}$</td>
</tr>
<tr>
<td>mono-analytic containers,</td>
<td></td>
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<tr>
<td>Kummer theory</td>
<td></td>
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<tr>
<td>incompatible with $(\text{Ind}1), (\text{Ind}2)$</td>
<td>[cf. Proposition 3.4, (ii)]</td>
<td>[cf. Proposition 3.7, (i)]</td>
</tr>
<tr>
<td>independent of</td>
<td></td>
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<tr>
<td>mono-analytic containers,</td>
<td></td>
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<tr>
<td>Kummer theory</td>
<td></td>
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</tr>
<tr>
<td>compatible with $(\text{Ind}1), (\text{Ind}2)$</td>
<td>[cf. Corollary 2.3]</td>
<td>[cf. Remark 2.3.2]</td>
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</table>

Fig. 3.6: Relationship of theta- and number field-related objects to mono-analytic containers

(iii) In the following discussion, it will be of crucial importance to relate simultaneously both the unit and the value group portions of the $\Theta_{\text{LGP}}^{x\mu}$-link(s) involved on the 0-column [i.e., the vertical line indexed by 0] of the log-theta-lattice under consideration to the corresponding unit and value group portions on the 1-column [i.e., the vertical line indexed by 1] of the log-theta-lattice under consideration. On the other hand, if one attempts to relate the unit portions via one $\Theta_{\text{LGP}}^{x\mu}$-link [say, from $(0, m)$ to $(1, m)$] and the value group portions via another $\Theta_{\text{LGP}}^{x\mu}$-link [say, from $(0, m')$ to $(1, m')$, for $m' \neq m$], then the non-commutativity of the log-theta-lattice renders it practically impossible to obtain conclusions that require one to relate both the unit and the value group portions simultaneously [cf. the discussion of Remark 3.11.3, (i), (ii)]. This is precisely why we concentrate on a single $\Theta_{\text{LGP}}^{x\mu}$-link [cf. (i)].

(iv) The issue discussed in (iii) is relevant in the context of the present discussion for the following reason. Ultimately, we wish to apply the bi-coricity of the units [cf. Theorem 1.5, (iii), (iv)] in order to compute the 0-column $\Theta$-pilot object in terms of the arithmetic holomorphic structure of the 1-column. In order to do this, one must work with units that are vertically once-shifted [i.e., lie at $(n, m - 1)$] relative to the value group structures involved [i.e., which lie at $(n, m)$] — cf. the discussion of Remark 3.11.3, (ii). The solution to the problem of simultaneously accommodating these apparently contradictory requirements — i.e., “vertical shift” vs. “impossibility of vertical shift” [cf. (iii)] — is given precisely by working, on the 0-column, with structures that are invariant with respect to
vertical shifts [i.e., “(0, m) ↦ (0, m + 1)”] of the log-theta-lattice [cf. the discussion surrounding Remark 1.2.2, (iii), (a)] such as vertically coric structures [i.e., indexed by “(m, ◦)” that are related to the “Frobenius-like” structures which are not vertically coric by means of the log-Kummer correspondences of Theorem 3.11, (ii). Here, we note that this “solution” may be implemented only at the cost of admitting the “indeterminacy” constituted by the upper semi-compatibility of (Ind3).

(v) Thus, we begin our computation of the 0-column Θ-pilot object in terms of the arithmetic holomorphic structure of the 1-column by relating the units on the 0- and 1-columns by means of the unit portion

\[ 0, 0 \overset{\epsilon}{\rightarrow} \mu_{\text{LGP}} \rightarrow 1, 0 \overset{\epsilon}{\rightarrow} \mu_{\Delta} \]

of the \( \Theta^\times_{\text{LGP}} \)-link from (0, 0) to (1, 0) [cf. (i)] and then applying the bi-coricity of the units of Theorem 1.5, (iii), (iv). In particular, the mono-analytic log-shell interpretation of this bi-coricity given in Theorem 1.5, (iv), will be applied to regard these mono-analytic log-shells as “multiradial mono-analytic containers” [cf. the discussion of Remark 1.5.2, (i), (ii), (iii)] for the various [local and global] value group structures that constitute the Θ-pilot object on the 0-column — cf. Fig. 3.6 above. [Here, we observe that the parallel treatment of “theta-related” and “number field-related” objects is reminiscent of the discussion of [IUTchII], Remark 4.11.2, (iv).] That is to say, we will relate the various Frobenioid-theoretic [i.e., “Frobenius-like” — cf. Remark 1.5.4, (i)]

- local units at \( v \in \mathcal{V} \),
- splitting monoids at \( v \in \mathcal{V}_{\text{bad}} \), and
- global Frobenioids

indexed by \((0, m)\), for \( m \in \mathbb{Z}\), to the vertically coric [i.e., indexed by “(0, ◦)”] versions of these bi-coric mono-analytic containers by means of the log-Kummer correspondences of Theorem 3.11, (ii), (a), (b), (c) — i.e., by varying the “Kummer input index” \((0, m)\) along the 0-column.

(vi) In the context of (v), it is useful to recall that the log-Kummer correspondences of Theorem 3.11, (ii), (b), (c), are obtained precisely as a consequence of the splittings, up to roots of unity, of the relevant monoids into unit and value group portions constructed by applying the Galois evaluation operations discussed in Remarks 2.2.2, (iii) [in the case of Theorem 3.11, (ii), (b)], and 2.3.2 [in the case of Theorem 3.11, (ii), (c)]. Moreover, we recall that the Kummer theory surrounding the local LGP-monoids of Proposition 3.4, (ii), depends, in an essential way, on the theory of [IUTchII], §3 [cf., especially, [IUTchII], Corollaries 3.5, 3.6], which, in turn, depends, in an essential way, on the Kummer theory surrounding mono-theta environments established in [EtTh]. Thus, for instance, we recall that the discrete rigidity established in [EtTh] is applied so as to avoid working, in the tempered Frobenioids that occur, with \( \hat{\mathbb{Z}}\)-divisors/line bundles” [i.e., “\( \hat{\mathbb{Z}}\)-completions” of \( \mathbb{Z}\)-modules of divisors/line bundles], which are fundamentally incompatible with conventional notions of divisors/line bundles, hence, in particular, with mono-theta-theoretic cyclotomic rigidity [cf. Remark 2.1.1, (v)]. Also, we recall that “isomorphism class compatibility” — i.e., in the terminology of
“compatibility with the topology of the tempered fundamental group” [cf. the discussion at the beginning of Remark 2.1.1] — allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the ring-theoretic basepoints that occur on either side of the log-link [cf. Remarks 2.1.1, (ii), and 2.3.3, (vii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the log-Kummer correspondence for the splitting monoids of local LGP-monoids, whose construction depends, in an essential way [cf. the theory of [IUTchII], §3, especially, [IUTchII], Corollaries 3.5, 3.6], on the conjugate synchronization arising from the $F_t^{x^\pm}$-symmetry. That is to say,

it is precisely by establishing this conjugate synchronization arising from the $F_t^{x^\pm}$-symmetry relative to these basepoints that occur on either side of the log-link that one is able to conclude the crucial compatibility of this conjugate synchronization with the log-link [cf. Remark 1.3.2].

A similar observation may be made concerning the MLF-Galois pair approach to the cyclotomic rigidity isomorphism that is applied at $v \in V^\text{good} \cap V^\text{non}$ [cf. [IUTchII], Corollary 1.11, (a); [IUTchII], Remark 1.11.1, (i), (a); [IUTchII], Proposition 4.2, (i); [AbsTopIII], Proposition 3.2, (iv), as well as Remark 2.3.3, (viii), of the present paper], which amounts, in essence, to computations involving the Galois cohomology groups of various subquotients — such as torsion subgroups [i.e., roots of unity] and associated value groups — of the [multiplicative] module of nonzero elements of an algebraic closure of the mixed characteristic local field involved [cf. the proof of [AbsAnab], Proposition 1.2.1, (vii)] — i.e., algorithms that are manifestly compatible with the topology of the profinite groups involved [cf. the discussion of Remark 2.3.3, (viii)], in the sense that they do not require one to pass to Kummer towers [cf. the discussion of [IUTchII], Remark 3.6.4, (i)], which are fundamentally incompatible with the ring structure of the fields involved. Here, we note in passing that the corresponding property for $v \in V^\text{arc}$ [cf. [IUTchII], Proposition 4.4, (i)] holds as a consequence of the interpretation discussed in [IUTchII], Remark 3.4.2, of Kummer structures in terms of co-holomorphicizations. On the other hand, the approaches to cyclotomic rigidity just discussed for $v \in V^\text{bad}$ and $v \in V^\text{good}$ differ quite fundamentally from the approach to cyclotomic rigidity taken in the case of [global] number fields in the algorithms described in [IUTchII], Example 5.1, (v); [IUTchII], Definition 5.2, (vi), (viii), which depend, in an essential way, on the property

$$Q_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$$

— i.e., which is fundamentally incompatible with the topology of the profinite groups involved [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii)] in the sense that it clearly cannot be obtained as some sort of limit of corresponding properties of $(\mathbb{Z}/N\mathbb{Z})^\times$! Nevertheless, with regard to uni-/multi-radiality issues, this approach to cyclotomic rigidity in the case of the number fields resembles the theory of mono-theta-theoretic cyclotomic rigidity at $v \in V^\text{bad}$ in that it admits a natural multiradial formulation [cf. Theorem 3.11, (iii), (d); the discussion of Remarks 2.3.2, 3.11.3], in sharp contrast to the essentially uniradial nature of the approach to cyclotomic rigidity via MLF-Galois pairs at $v \in V^\text{good} \cap V^\text{non}$ [cf. the discussion of [IUTchII], Remark 1.11.3]. These observations are summarized in Fig. 3.7 below.
Finally, we recall that [one verifies immediately that] the various approaches to cyclotomic rigidity just discussed are *mutually compatible* in the sense that they yield the same cyclotomic rigidity isomorphism in any setting in which more than one of these approaches may be applied.

<table>
<thead>
<tr>
<th>Approach to cyclotomic rigidity</th>
<th>Uni-/multi-radiality</th>
<th>Compatibility with $F_l^{\times\pm}$-symmetry, profinite/tempered topologies, ring structures, log-link</th>
</tr>
</thead>
<tbody>
<tr>
<td>mono-theta environments</td>
<td>multiradial</td>
<td>compatible</td>
</tr>
<tr>
<td>MLF-Galois pairs, via Brauer groups</td>
<td>uniradial</td>
<td>compatible</td>
</tr>
<tr>
<td>number fields, via $\mathbb{Q}_{&gt;0} \cap \hat{\mathbb{Z}}^\times = {1}$</td>
<td>multiradial</td>
<td>incompatible</td>
</tr>
</tbody>
</table>

Fig. 3.7: Three approaches to cyclotomic rigidity

(vii) In the context of the discussion in the final portion of (vi), it is of interest to recall that the constructions underlying the crucial *bi-coricity* theory of Theorem 1.5, (iii), (iv), depend, in an essential way, on the *conjugate synchronization* arising from the $F_l^{\times\pm}$-symmetry, which allows one to relate the local monoids and Galois groups at distinct labels $\ell \in |F_l|$ to one another in a fashion that is simultaneously compatible both with

- the *vertically coric* structures and *Kummer theory* that give rise to the log-*Kummer correspondences* of Theorem 3.11, (ii),

and with

- the property of *distinguishing* [i.e., not identifying] data indexed by distinct labels $\ell \in |F_l|$ — cf. the discussion of Remark 1.5.1, (i), (ii). Since, moreover, this crucial conjugate synchronization is *fundamentally incompatible* with the $F_l^*$-symmetry, it is necessary to work with these two symmetries separately, as was done in [IUTchI], §4, §5, §6 [cf. [IUTchII], Remark 4.7.6]. Here, it is useful to recall that the $F_l^*$-symmetry also plays a crucial role, in that it allows one to “descend to $F_{mod}$” at the level of absolute Galois groups [cf. [IUTchII], Remark 4.7.6]. On the other hand, both the $F_l^{\times\pm}$- and $F_l^*$-symmetries share the property of being compatible with the *vertical coricity* and relevant *Kummer isomorphisms* of the 0-column — cf. the log-*Kummer correspondences* of Theorem 3.11, (ii), (b) [in the case of the
\( F_l^{\times \pm} \)-symmetry], (c) [in the case of the \( F_l^* \)-symmetry]. Here, we recall that the vertically coric versions of both the \( F_l^{\times \pm} \)- and the \( F_l^* \)-symmetries depend, in an essential way, on the arithmetic holomorphic structure of the 0-column, hence give rise to multiradial structures via the tautological approach to constructing such structures discussed in Remark 3.11.2, (i), (ii).

(viii) In the context of (vii), it is useful to recall that in order to construct the \( F_l^{\times \pm} \)-symmetry, it is necessary to make use of global \( \pm \)-synchronizations of various local \( \pm \)-indeterminacies. Since the local tempered fundamental groups at \( v \in V_{\text{bad}} \) do not extend to a “global tempered fundamental group”, these global \( \pm \)-synchronizations give rise to profinite conjugacy indeterminacies in the vertically coric construction of the LGP-monoids [i.e., the theta values at torsion points] given in [IUTchII], §2, which are resolved by applying the theory of [IUTchI], §2 — cf. the discussion of [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii).

(ix) In the context of (vii), it is also useful to recall the important role played, in the theory of the present series of papers, by the various “copies of \( F_{\text{mod}} \)”, i.e., more concretely, in the form of the various copies of the global Frobenioids “\( F_{\text{MOD}}^\otimes \)” and “\( F_{\text{mod}}^\otimes \)” and their realifications. That is to say, the ring structure of the global field \( F_{\text{mod}} \) allows one to bridge the gap — i.e., furnishes a translation apparatus — between the multiplicative structures constituted by the global realified Frobenioids related via the \( \Theta_{\text{LGP}}^{\times \mu} \)-link and the additive representations of these global Frobenioids that arise from the “mono-analytic containers” furnished by the mono-analytic log-shells [cf. (v)]. Here, the precise compatibility of the ingredients for “\( F_{\text{MOD}}^\otimes \)” with the log-Kummer correspondence renders “\( F_{\text{mod}}^\otimes \)” better suited to describing the relation to the \( \Theta_{\text{LGP}}^{\times \mu} \)-link [cf. Remark 3.10.1, (ii)]. On the other hand, the local portion of “\( F_{\text{mod}}^\otimes \)” — i.e., which is subject to “upper semi-compatibility” [cf. (Ind3)], hence only “approximately compatible” with the log-Kummer correspondence — renders it better suited to explicit estimates of global arithmetic degrees, by means of log-volumes [cf. Remark 3.10.1, (iii)].

(x) Thus, one may summarize the discussion thus far as follows. The theory of “Kummer-detachment” — cf. Remarks 1.5.4, (i); 2.1.1; 3.10.1, (ii), (iii) — furnished by Theorem 3.11, (ii), (iii) — allows one to relate the Frobenioid-theoretic [i.e., “Frobenius-like”] structures that appear in the domain [i.e., at (0, 0)] of the \( \Theta_{\text{LGP}}^{\times \mu} \)-link [cf. (i)] to the multiradial representation described in Theorem 3.11, (i), (a), (b), (c), but only at the cost of introducing the indeterminacies

(Ind1) — which may be thought of as arising from the requirement of compatibility with the permutation symmetries of the étale-picture [cf. Theorem 3.11, (i)];

(Ind2) — which may be thought of as arising from the requirement of compatibility with the Aut_{\( F_{\text{mod}}^{\times \pm} \)}(-)-indeterminacies that act on the domain/codomain of the \( \Theta_{\text{LGP}}^{\times \mu} \)-link [cf. (ii); Theorem 3.11, (i), (iii)], i.e., with the horizontal arrows of the log-theta-lattice;

(Ind3) — which may be thought of as arising from the requirement of compatibility with the log-Kummer correspondences of Theorem 3.11, (ii), i.e., with the vertical arrows of the log-theta-lattice.
The various indeterminacies (Ind1), (Ind2), (Ind3) to which the multiradial representation is subject may be thought of as data that describes some sort of “formal quotient”, like the “fine moduli spaces” that appear in algebraic geometry. In this context, the procession-normalized mono-analytic log-volumes [i.e., where the average is taken over $j \in \mathbb{F}_l^*$ of Theorem 3.11, (i), (a), (c), furnish a means of constructing a sort of associated “coarse space” or “inductive limit” [of the “inductive system” constituted by this “formal quotient”] — i.e., in the sense that [one verifies immediately — cf. Proposition 3.9, (ii)] that the resulting log-volumes $\in \mathbb{R}$ are invariant with respect to the indeterminacies (Ind1), (Ind2), and have the effect of converting the indeterminacy (Ind3) into an inequality [from above]. Moreover, the log-link compatibility of the various log-volumes that appear [cf. Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] ensures that these log-volumes are compatible with [the portion of the “formal quotient” / “inductive system” constituted by] the various arrows [i.e., Kummer isomorphisms and log-links] of the $\log$-Kummer correspondence of Theorem 3.11, (ii). Here, we note that the averages over $j \in \mathbb{F}_l^*$ that appear in the definition of the procession-normalized volumes involved may be thought of as a consequence of the $\mathbb{F}_l^*$-symmetry acting on the labels of the theta values that give rise to the LGP-monoids — cf. also the definition of the symbol “$\triangle$” in [IUTchII], Corollary 4.10, (i), via the identification of the symbols “0” and “($\mathbb{F}_l^*$)”; the discussion of Remark 3.9.3. Also, in this context, it is of interest to observe that the various tensor products that appear in the various local mono-analytic tensor packets that arise in the multiradial representation of Theorem 3.11, (i), (a), have the effect of identifying the operation of “multiplication by elements of $\mathbb{Z}$” — and hence also the effect on log-volumes of such multiplication operations! — at different labels $\in \mathbb{F}_l^*$.

(xi) For ease of reference, we divide this step into substeps, as follows.

(xi-a) Consider a $q$-pilot object at $(1,0)$, which we think of — relative to the relevant copy of “$\mathcal{F}_{\text{mob}}$” — in terms of the holomorphic log-shells constructed at $(1,0)$ [cf. the discussion of Remark 3.12.2, (iv), (v), below]. Then the $\Theta_{\text{LGP}}^{\mu}$-link from $(0,0)$ to $(1,0)$ may be interpreted as a sort of gluing isomorphism that relates the arithmetic holomorphic structure — i.e., the “conventional ring/scheme-theory” — at $(1,0)$ to the arithmetic holomorphic structure at $(0,0)$ in such a way that the $\Theta$-pilot object at $(0,0)$ [thought of as an object of the relevant global realified Frobenioi] corresponds to the $q$-pilot object at $(1,0)$ [cf. (i); the discussion of Remark 3.12.2, (ii), below].

(xi-b) On the other hand, the multiradial construction algorithm of Theorem 3.11, which was summarized in the discussion of (x), yields a construction of a collection of possibilities of output data contained in

$$(0,\varnothing U^{\mathcal{U}} \supseteq 0,\varnothing U \xrightarrow{\sim} 1,\varnothing U (\subseteq 1,\varnothing U^{\mathcal{U}}))$$

— where the isomorphism “$\xrightarrow{\sim}$” arises from the permutation symmetries discussed in the final portion of Theorem 3.11, (i) — that satisfies the input prime-stripe link (IPL) and simultaneous holomorphic expressibility (SHE) properties discussed in Remark 3.11.1, (iii), (iv), (v) [cf. also the discussion of “possible
images” at the beginning of the present proof. Here, with regard to (IPL), we observe that the \( F^{\uparrow} \)-prime-strip portion of the link/relationship of this collection of possibilities of output data to the input data \((F^{\uparrow} \times \mu-)\)prime-strip [cf. Remark 3.11.1, (ii)] consists precisely of (full poly-)isomorphisms of \( F^{\uparrow} \)-prime-strips, while the corresponding link/relationship for \( F^{\downarrow} \times \mu \)-prime-strips is somewhat more complicated, as a result of the indeterminacies (Ind1), (Ind2), (Ind3). Also, in this context, we observe that, although the multiradial construction algorithm of Theorem 3.11 in fact involves the \( \Theta \)-pilot object at \((0, 0)\), in the present discussion of Step (xi), we shall only be concerned with qualitative logical aspects/consequences of this construction algorithm, i.e., with the

- input prime-strip link (IPL),
- simultaneous holomorphic expressibility (SHE), and
- algorithmic parallel transport (APT)

properties discussed in Remark 3.11.1, (iii), (iv), (v). That is to say, we shall take the point of view of “temporarily forgetting” — cf. the discussion of hidden internal structures (HIS) in Remark 3.11.1, (iv) — the fact that the multiradial construction algorithm of Theorem 3.11 in fact involves \( \Theta \)-pilot objects, theta functions/values, mono-theta environments. Alternatively, in the discussion to follow, we shall, roughly speaking, think of the multiradial construction algorithm of Theorem 3.11 as

“some” algorithm that transforms a certain type of input data into a certain type of output data and, moreover, satisfies certain properties (IPL) and (SHE).

(xi-c) Thus, the discussion of the (IPL) and (SHE) properties in (xi-b) may be summarized as follows:

The multiradial construction algorithm of Theorem 3.11 yields a collection of possibilities of output data in \( 1, \circ \cup (\subseteq 1, \circ \cup Q) \) that are linked/related [cf. (IPL)], via isomorphisms of \( F^{\uparrow} \)-prime-strips, to the representation [via the log-Kummer correspondence in the 1-column] of the \( q \)-pilot object at \((1, 0)\) on \( 1, \circ \cup Q \), and, moreover, whose construction may be expressed entirely relative to the arithmetic holomorphic structure in the 1-column [cf. (SHE)].

Here, we recall that, in more concrete language, this “arithmetic holomorphic structure in the 1-column” amounts, in essence, to the ring structure labeled “1, \( \circ \)”. Moreover, by slightly enlarging the collection of possibilities of output data under consideration by working with the holomorphic hull \( 1, \circ \uparrow (\subseteq 1, \circ \uparrow) \), we obtain output data that is expressed — not in terms of regions contained in various tensor products of local fields labeled “1, \( \circ \)” [i.e., more concretely, various isomorphs of “\( K_{v} \)” for \( v \in \mathbb{V} \)], but rather — in terms of localizations of arithmetic vector bundles over certain local rings labeled “1, \( \circ \)” [i.e., more concretely, various isomorphs of “\( \mathcal{O}_{K_{v}} \)” for \( v \in \mathbb{V} \)] — cf. the discussion of Remarks 3.9.5, (vii), (Ob1), (Ob2), (Ob5); 3.12.2, (v), below. Such an expression in terms of “localizations of arithmetic vector bundles” is necessary in order to render the output data in a form that is comparable to the representation of the \( q \)-pilot object [i.e., which arises from a certain arithmetic line bundle] at \((1, 0)\) on \( 1, \circ \cup Q \).
(xi-d) The discussion of (xi-c) thus yields the following conclusion:

The multiradial construction algorithm of Theorem 3.11, followed by formation of the holomorphic hull, yields a collection of possibilities of output data in \( \mathcal{U}^{0.1} \) that are linked/related [cf. (IPL)], via isomorphisms of \( \mathcal{F}^{\beta \triangleright} \)-prime-strips, to the representation [via the log-Kummer correspondence in the 1-column] of the \( q \)-pilot object at \((1,0)\) on \( \mathcal{U}^{0.1} \), and, moreover, whose construction may be expressed entirely relative to localizations of arithmetic vector bundles over rings that arise in the arithmetic holomorphic structure in the 1-column [cf. (SHE)].

Here, we observe that these “localizations of arithmetic vector bundles” are [unlike the arithmetic line bundle that gives rise to the \( q \)-pilot object] of rank > 1. Moreover, the \( q \)-pilot object is defined at the level of realifications of Frobenioids of [global] arithmetic line bundles. Thus, it is only by forming [a suitable positive tensor power of] the determinant of the localizations of arithmetic vector bundles mentioned in the above display [cf. Remark 3.9.5, (vii), (Ob3), (Ob4)] and then applying the [suitably normalized, with respect to \( j \in \mathbb{F}_l \)] log-volume to various regions — i.e., the region \( \mathcal{U}^{0.1} \) and the region that arises from the representation of the \( q \)-pilot object at \((1,0)\) on \( \mathcal{U}^{0.1} \) [cf. Remark 3.9.5, (vii), (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], that we are able to obtain completely comparable objects [cf. Remarks 3.9.5, (vii), (Ob5), (Ob6), (Ob7), (Ob8), (Ob9); 3.9.5, (viii), (ix)], namely,

\[
\mathbb{R} \leq -|\log(q)| \quad \text{def} \quad \{ \lambda \in \mathbb{R} \mid \lambda \leq -|\log(q)| \} \subseteq \mathbb{R}; \quad -|\log(q)| \in \mathbb{R}
\]

— where we recall that, by definition, \( -|\log(q)| \) is the [negative — cf. the discussion of “possible images” at the beginning of the present proof] log-volume of \( \mathcal{U}^{0.1} \), while \( -|\log(q)| \) is the log-volume of the region that arises from the representation of the \( q \)-pilot object at \((1,0)\) on \( \mathcal{U}^{0.1} \). In this context, it is useful to recall from Proposition 3.9, (iii) [cf. also the discussion of Remarks 3.9.2, 3.9.6], that global arithmetic degrees of objects of global realified Frobenioids may be interpreted as log-volumes [cf. also the discussion of Remarks 1.5.2, (iii); 3.10.1, (iv), as well as of Remark 3.12.2, (v), below]. Finally, in this context, we observe [cf. the first display of the present (xi-d)] that it is of crucial importance to apply the log-Kummer correspondence in the 1-column [cf. the discussion of log-Kummer correspondences in Remark 3.9.5, (vii), (Ob7), (Ob8); Remark 3.9.5, (viii), (sQ4); Remark 3.9.5, (ix); the final portion of Remark 3.9.5, (x); the discussion of the final portion of Remark 3.12.2, (v), below], in order to rectify the vertical shift/mismatch [cf. the discussion of (iii), (iv) in the case of the 0-column] between the unit portion of \( \mathcal{G}^{\beta \triangleright} \times \mu^{-} \) and the log-shells arising from [the image via the relevant Kummer isomorphisms of] this unit portion, which give rise to the tensor packets of log-shells that constitute \( \mathcal{U}^{0.1} \).

(xi-e) Next, let us recall that the relationship, i.e., that arises by applying the log-volume to the pilot-object, between the \( \text{pilot-object log-volume} - |\log(q)| \in \mathbb{R} \) and the input data \((\mathcal{F}^{\beta \triangleright} \times \mu^{-})\text{prime-strip}\) is precisely the relationship prescribed/imposed by the arithmetic holomorphic structure in the 1-column, i.e., via the representation of the input data \((\mathcal{F}^{\beta \triangleright} \times \mu^{-})\text{prime-strip}\) on \( \mathcal{U}^{0.1} \) relative
to this 1-column arithmetic holomorphic structure. That is to say, “expressibility relative to the arithmetic holomorphic structure in the 1-column” [cf. (SHE)] amounts precisely to

“expressibility via operations that are valid/executable/well-defined even when subject to the condition that the pilot-object log-volume associated to the input data \( (F^\text{\#} \times \mu^-)\)-prime-strip [which is, of course, linked/related, via isomorphisms of \( F^\text{\#} \times \mu^- \)-prime-strips, to the possible output data \( F^\text{\#} \times \mu^- \)-prime-strips!] be equal to the fixed value \(-|\log(q)| \in \mathbb{R}\).”

In particular, the discussion of (xi-d) thus yields the following conclusion:

The multiradial construction algorithm of Theorem 3.11, followed by formation of the holomorphic hull and application of the log-volume, yields a collection of possible log-volumes of pilot-object output data

\[ \mathbb{R} \leq -|\log(\Theta)| \subseteq \mathbb{R} \]

that are linked/related [cf. (IPL)], via isomorphisms of \( F^\text{\#} \)-prime-strips, to the pilot-object log-volume

\[-|\log(q)| \in \mathbb{R}\]

of the input data \( (F^\text{\#} \times \mu^-)\)-prime-strip [cf. (SHE)].

(xi-f) Thus, we conclude from (xi-e) that

the construction of the subset \( \mathbb{R} \leq -|\log(\Theta)| \subseteq \mathbb{R} \) of possible pilot-object log-volumes of output data is subject to the condition that this construction of output data possibilities constitutes, in particular, a construction [perhaps only up to some sort of “approximation”, as a result of various indeterminacies] of the pilot-object log-volume of the input data \( (F^\text{\#} \times \mu^-)\)-prime-strip, namely, \(-|\log(q)| \in \mathbb{R}\).

The inclusion \(-|\log(q)| \in \mathbb{R} \leq -|\log(\Theta)|\), hence also the inequality

\[-|\log(q)| \leq -|\log(\Theta)| \in \mathbb{R}\]

— i.e., the conclusion that \( C_\Theta \geq -1 \) for any \( C_\Theta \in \mathbb{R} \) such that \(-|\log(\Theta)| \leq C_\Theta \cdot |\log(q)|\) — in the statement of Corollary 3.12, then follows formally.

(xi-g) Thus, in summary,

the multiradial construction algorithm of Theorem 3.11, followed by formation of the holomorphic hull and application of the log-volume, yields two tautologically equivalent ways to compute the log-volume of the q-pilot object at \((1, 0)\) — cf. Fig. 3.8 below.
### Multiradial Representation at 0-Column $(0, \circ)$

- **Kummer-detrachtment** via log-Kummer
- **com-pati-son** with $\Theta_{LGP}^{\times \mu}$-link

- **$\Theta$-pilot object** in $\Theta^{\pm \text{ell}}$NF-Hodge theater at $(0, 0)$

- $(\sim)_{\text{hol.}}$-portion,
- $(-)^{+ \times \mu}$-portion

- $\approx$ of $\Theta_{LGP}^{\times \mu}$-link

### Multiradial Representation at 1-Column $(1, \circ)$

- **com-pati-son** hol. pari-son
- **$q$-pilot object** in $\Theta^{\pm \text{ell}}$NF-Hodge theater at $(1, 0)$

- $\downarrow$ hull, log-vol.
- $\uparrow$ vol.

#### Fig. 3.8: Two tautologically equivalent ways to compute the log-volume of the $q$-pilot object at $(1, 0)$

(xi-h) In this context, it is useful to recall that the above argument depends, in an essential way [cf. the discussion of (ii), (vi)], on the theory of [EtTh], which does not admit any evident generalization to the case of $N$-th tensor powers of $\Theta$-pilot objects, for $N \geq 2$. That is to say, the log-volume of such an $N$-th tensor power of a $\Theta$-pilot object must always be computed as the result of multiplying the log-volume of the original $\Theta$-pilot object by $N$ — cf. Remark 2.1.1, (iv); [IUTchII], Remark 3.6.4, (iii), (iv). In particular, although the analogue of the above argument for such an $N$-th tensor power would lead to sharper inequalities than the inequalities obtained here, it is difficult to see how to obtain such sharper inequalities via a routine generalization of the above argument. In fact, as we shall see in [IUTchIV], these sharper inequalities are known to be false [cf. [IUTchIV], Remark 2.3.2, (ii)].

(xii) In the context of the argument of (xi), it is useful to observe the important role played by the global realified Frobenioids that appear in the $\Theta_{LGP}^{\times \mu}$-link. That is to say, since ultimately one is only concerned with the computation of log-volumes, it might appear, at first glance, that it is possible to dispense with the use of such global Frobenioids and instead work only with the various local Frobenioids, for $v \in V$, that are directly related to the computation of log-volumes. On the other hand, observe that since the isomorphism of [local or global!] Frobenioids arising from the $\Theta_{LGP}^{\times \mu}$-link only preserves isomorphism classes of objects of these Frobenioids [cf. the discussion of Remark 3.6.2, (i)], to work only with local Frobenioids means that one must contend with the indeterminacy of not knowing whether, for instance, such a local Frobenioid object at some $v \in V^{\text{non}}$ corresponds to a given open submodule of the log-shell at $\overline{v}$ or to, say, the $p^N_{\overline{v}}$-multiple of this submodule, for $N \in \mathbb{Z}$. Put another way, one must contend with the indeterminacy arising from the fact that, unlike the case with the global Frobenioids $\mathcal{F}_{\text{mod}}^{\geq \mathbb{R}}$, $\mathcal{F}_{\text{mod}}^{\leq \mathbb{R}}$, objects of the various local Frobenioids that arise admit endomorphisms
which are not automorphisms. This indeterminacy has the effect of rendering meaningless any attempt to perform a precise log-volume computation as in (xi).

Remark 3.12.1.

(i) In [IUTchIV], we shall be concerned with obtaining more explicit upper bounds on \( -|\log(\Theta)| \), i.e., estimates “\( C_\Theta \)” as in the statement of Corollary 3.12.

(ii) It is not difficult to verify that, for \( \lambda \in \mathbb{Q}_{>0} \), one may obtain a similar theory to the theory developed in the present series of papers [cf. the discussion of Remark 3.11.1, (ii)] for “generalized \( \Theta^{x,\mu}_{\text{LGP}}\)-links” of the form

\[
q^\lambda \quad \mapsto \quad \frac{1^2}{\cdots} \quad (t^*)^2
\]

— i.e., so the theory developed in the present series of papers corresponds to the case of \( \lambda = 1 \). This sort of “generalized \( \Theta^{x,\mu}_{\text{LGP}}\)-link” is roughly reminiscent of — but by no means equivalent to! — the sort of issues considered in the discussion of Remark 2.2.2, (i). Here, we observe that raising to the \( \lambda \)-th power on the “\( q \) side” differs quite fundamentally from raising to the \( \lambda \)-th power on the “\( q^{(1^2\cdots(t^*)^2)} \) side”, an issue that is discussed briefly [in the case of \( \lambda = N \)] in the final portion of Step (xi) of the proof of Corollary 3.12. That is to say, “generalized \( \Theta^{x,\mu}_{\text{LGP}}\)-links” as in the above display differ fundamentally both from the situation of Remark 2.2.2, (i), and the situation discussed in the final portion of Step (xi) of the proof of Corollary 3.12 in that the theory of the first power of the étale theta function is left unchanged [i.e., relative to the theory developed in the present series of papers] — cf. the discussion of Remark 2.2.2, (i); Step (xi) of the proof of Corollary 3.12. At any rate, in the case of “generalized \( \Theta^{x,\mu}_{\text{LGP}}\)-links” as in the above display, one may apply the same arguments as the arguments used to prove Corollary 3.12 to conclude the inequality

\[
C_\Theta \geq -\lambda
\]

— i.e., which is sharper, for \( \lambda < 1 \), than the inequality obtained in Corollary 3.12 in the case of \( \lambda = 1 \). In fact, however, such sharper inequalities will not be of interest to us, since, in [IUTchIV], our estimates for the upper bound \( C_\Theta \) will be sufficiently rough as to be unaffected by adding a constant of absolute value \( \leq 1 \).

(iii) In the context of the discussion of (ii) above, it is of interest to note that the multiradial theory of mono-theta-theoretic cyclotomic rigidity, and, in particular, the theory of the first power of the étale theta function, may be regarded as a theory that concerns a sort of “canonical profinite volume” on the elliptic curves under consideration associated to the first power of the ample line bundle corresponding to the étale theta function. This point of view is also of interest in the context of the discussion of various approaches to cyclotomic rigidity summarized in Fig. 3.7 [cf. also the discussion of Remark 2.3.3]. Indeed, the elementary fact “\( \mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\} \)”, which plays a key role in the multiradial algorithms for cyclotomic rigidity isomorphisms in the number field case
[cf. [IUTchI], Example 5.1, (v), as well as the discussion of Remarks 2.3.2, 2.3.3 of the present paper], may be regarded as an immediate consequence of an easy interpretation of the product formula in terms of the geometry of the domain in the archimedean completion of the number field \( \mathbb{Q} \) determined by the inequality “\( \leq 1 \)” i.e., a domain which may be thought of as a sort of concrete geometric representation of a “canonical unit of volume” of the number field \( \mathbb{Q} \).

**Remark 3.12.2.**

(i) One of the main themes of the present series of papers is the issue of dismantling the two underlying combinatorial dimensions of a number field — cf. Remarks 1.2.2, (vi), of the present paper, as well as [IUTchI], Remarks 3.9.3, 6.12.3, 6.12.6; [IUTchII], Remarks 4.7.5, 4.7.6, 4.11.2, 4.11.3, 4.11.4. The principle examples of this topic may be summarized as follows:

(a) splittings of various monoids into unit and value group portions;

(b) separating the “\( \mathbb{F}_l \)” arising from the \( l \)-torsion points of the elliptic curve — which may be thought of as a sort of “finite approximation” of \( \mathbb{Z} \!) — into a [multiplicative] \( \mathbb{F}_l^* \)-symmetry — which may also be thought of as corresponding to the global arithmetic portion of the arithmetic fundamental groups involved — and a(n) [additive] \( \mathbb{F}_l^{\pm} \)-symmetry — which may also be thought of as corresponding to the geometric portion of the arithmetic fundamental groups involved;

(c) separating the ring structures of the various global number fields that appear into their respective underlying additive structures — which may be related directly to the various log-shells that appear — and their respective underlying multiplicative structures — which may be related directly to the various Frobenioids that appear.

From the point of view of Theorem 3.11, example (a) may be seen in the “non-interference” properties that underlie the log-Kummer correspondences of Theorem 3.11, (ii), (b), (c), as well as in the \( \Theta_{LGF}^\mu \)-link compatibility properties discussed in Theorem 3.11, (ii), (c), (d).

(ii) On the other hand, another important theme of the present §3 consists of the issue of “reassembling” these two dismantled combinatorial dimensions by means of the multiradial mono-analytic containers furnished by the mono-analytic log-shells — cf. Fig. 3.6 — i.e., of exhibiting the extent to which these two dismantled combinatorial dimensions cannot be separated from one another, at least in the case of the \( \Theta \)-pilot object, by describing the “structure of the intertwining” between these two dimensions that existed prior to their separation. From this point of view, one may think of the multiradial representations discussed in Theorem 3.11, (i) [cf. also Theorem 3.11, (ii), (iii)], as the final output of this “reassembling procedure” for \( \Theta \)-pilot objects. From the point of view of example (a) of the discussion of (i), this “reassembling procedure” allows one to compute/estimate the value group portions of various monoids of arithmetic interest in terms of the unit group portions of these monoids. It is precisely these estimates that give rise to the inequality obtained in Corollary 3.12. That is to say, from the point of view of dismantling/reassembling the intertwining between
value group and unit group portions, the argument of the proof of Corollary 3.12 may be summarized as follows:

(a) When considered from the point of view of log-volumes of $\Theta$-pilot and $q$-pilot objects, the correspondence of the $\Theta_{LGP}^{x,\mu}$-link [i.e., that sends $\Theta$-pilot objects to $q$-pilot objects] may seem a bit “mysterious” or even, at first glance, “self-contradictory” to some readers.

(b) On the other hand, this correspondence of the $\Theta_{LGP}^{x,\mu}$-link is made possible by the fact that one works with $\Theta$-pilot or $q$-pilot objects in terms of “sufficiently weakened data” [namely, the $\mathcal{F}^{x,\mu}$-prime-strips that appear in the definition of the $\Theta_{LGP}^{x,\mu}$-link], i.e., data that is “sufficiently weak” that one can no longer distinguish between $\Theta$-pilot and $q$-pilot objects.

(c) Thus, if one thinks of the $\mathcal{F}^{x,\mu}$-prime-strips that appear in the domain and codomain of the $\Theta_{LGP}^{x,\mu}$-link as a “single abstract $\mathcal{F}^{x,\mu}$-prime-strip" that is regarded/only known up to isomorphism, then the issue of which log-volume such an abstract $\mathcal{F}^{x,\mu}$-prime-strip corresponds to [cf. (a)] is precisely the issue of “which intertwining between value group and unit group portions” one considers, i.e., the issue of “which arithmetic holomorphic structure” [of the arithmetic holomorphic structures that appear in the domain and codomain of the $\Theta_{LGP}^{x,\mu}$-link] that one works in. Put another way, it is essentially a tautological consequence of the fact that these two arithmetic holomorphic structures in the domain and codomain of the $\Theta_{LGP}^{x,\mu}$-link are distinguished from one another that the $\Theta_{LGP}^{x,\mu}$-link yields a situation in which both the $\Theta$-intertwining [i.e., the intertwining associated to the $\Theta$-pilot object in the domain of the $\Theta_{LGP}^{x,\mu}$-link] and the $q$-intertwining [i.e., the intertwining associated to the $q$-pilot object in the codomain of the $\Theta_{LGP}^{x,\mu}$-link] are simultaneously valid, i.e.,

\[ (q\text{-intertwining holds}) \land (\Theta\text{-intertwining holds}) \]

— cf. the discussion of the “distinct labels approach” in Remark 3.11.1, (vi).

(d) On the other hand, from the point of view of the analogy between multiradiality and the classical theory of parallel transport via connections [cf. IUTchII, Remark 1.7.1], the multiradial representation of Theorem 3.11 [cf. also the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)] asserts that, up to the relatively mild “monodromy” constituted by the indeterminacies (Ind1), (Ind2), (Ind3), one may “parallel transport” or “confuse” the $\Theta$-pilot object in the domain of the $\Theta_{LGP}^{x,\mu}$-link, i.e., the $\Theta$-pilot object represented relative to its “native intertwining/arithmetic holomorphic structure”, with the $\Theta$-pilot object represented relative to the “alien intertwining/arithmetic holomorphic structure” in the codomain of the $\Theta_{LGP}^{x,\mu}$-link.

(e) In particular, one may fix the arithmetic holomorphic structure of the codomain of the $\Theta_{LGP}^{x,\mu}$-link, i.e., the “native intertwining/arithmetic holomorphic structure” associated to the $q$-pilot object in the codomain of the
\(\Theta_{LGP}^{\mu}\)-link, and then, by applying \((d^{itw})\) and working up to the indeterminacies \((\text{Ind}1), (\text{Ind}2), (\text{Ind}3)\) [cf. also the subtleties discussed in (iv), (v) below; Remark 3.9.5, (vii), (viii), (ix)], construct the “native intertwining/arithmetic holomorphic structure” associated to the \(\Theta\)-pilot object in the domain of the \(\Theta_{LGP}^{\mu}\)-link as a mathematical structure that is intrinsically associated to the underlying structure of — hence, in particular, simultaneously with/without invalidating the conditions imposed by — the “native intertwining/arithmetic holomorphic structure” associated to the \(q\)-pilot object in the codomain of the \(\Theta_{LGP}^{\mu}\)-link [cf. the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)]. Indeed, this point of view is precisely the point of view that is taken in the proof of Corollary 3.12 [cf., especially, Step (xi)].

\((f^{itw})\) One way of summarizing the situation described in \((e^{itw})\) is in terms of logical relations as follows. The multiradial representation of Theorem 3.11 [cf. also the discussion of Remark 3.11.1] may be thought of [cf. the first “\(\implies\)” of the following display] as an algorithm for constructing, up to suitable indeterminacies [cf. the discussion of \((e^{itw})\)], the “\(\Theta\)-intertwining” as a mathematical structure that is intrinsically associated to the underlying structure of — hence, in particular, simultaneously with/without invalidating the conditions imposed by — the “\(q\)-intertwining”, while holding the “single abstract \(F^{\mu}\)-prime-strip” of the discussion of \((c^{itw})\), \((c^{itw})\) fixed, i.e., in symbols:

\[
(q^{itw.}) \implies (q^{itw.}) \land (\Theta-itw./\text{indets.}) \implies (\Theta-itw./\text{indets.})
\]

— where the second “\(\implies\)” of the above display is purely formal; “itw.” and “/indets.” are to be understood, respectively as abbreviations for “intertwining holds” and “up to suitable indeterminacies”. Here, we observe that

the “\(\land\)” of the above display may be regarded as the “image” of, hence, in particular, as a consequence of, the “\(\land\)” in the display of \((c^{itw})\), via the various \((\text{sub})\)quotient operations discussed in Remark 3.9.5, (viii), i.e., whose subtle compatibility properties allow one to conclude the “\(\land\)” of the above display from the “\(\land\)” in the display of \((c^{itw})\).

Thus, at the level of logical relations,

the \(q\)-intertwining, hence also the log-volume of the \(q\)-pilot object in the codomain of the \(\Theta_{LGP}^{\mu}\)-link, may be thought of as a special case of the \(\Theta\)-intertwining, i.e., at a more concrete level, of the log-volume of the \(\Theta\)-pilot object in the domain of the \(\Theta_{LGP}^{\mu}\)-link, regarded up to suitable indeterminacies.

Corollary 3.12 then follows, essentially formally.

Alternatively, from the point of view of “[very rough! toy models]”, i.e., whose goal lies solely in representing certain overall qualitative aspects of a situation, one may think of the discussion of \((a^{itw}) \sim (f^{itw})\) given above in the following terms:
Consider two distinct copies $q\mathbb{R}$ and $\Theta\mathbb{R}$ of the topological field of real numbers $\mathbb{R}$, equipped with labels “$q$” and “$\Theta$”, together with an abstract symbol “$*$” and assignments

$$
\lambda_q : * \mapsto q(-h) \in q\mathbb{R}, \quad \lambda_\Theta : * \mapsto \Theta(-2h) \in \Theta\mathbb{R},
$$

— where, in the present discussion, we shall write “$q(-)$”, “$\Theta(-)$” to denote the respective elements/subsets of $q\mathbb{R}$, $\Theta\mathbb{R}$ determined by an element/subset “(−)” of $\mathbb{R}$; $h \in \mathbb{R}_{>0}$ is a positive real number that we are interested in bounding from above. If one forgets the distinct labels “$q$” and “$\Theta$”, then these two assignments $\lambda_q$, $\lambda_\Theta$ are mutually incompatible and cannot be considered simultaneously, i.e., they contradict one another [in the sense that $\mathbb{R} \ni -h \neq -2h \in \mathbb{R}$].

One aspect of the situation of (a) that renders the simultaneous consideration of the two assignments $\lambda_q$, $\lambda_\Theta$ valid — i.e., at the level of logical relations,

$$
( * \mapsto q(-h) \in q\mathbb{R} ) \land ( * \mapsto \Theta(-2h) \in \Theta\mathbb{R} )
$$

— is the use of the abstract symbol “$*$”, i.e., which is, a priori, entirely unrelated to any copies of $\mathbb{R}$ [such as $q\mathbb{R}$, $\Theta\mathbb{R}$].

The other aspect of the situation of (a) that renders the simultaneous consideration of the two assignments $\lambda_q$, $\lambda_\Theta$ valid — i.e., at the level of logical relations,

$$
( * \mapsto q(-h) \in q\mathbb{R} ) \land ( * \mapsto \Theta(-2h) \in \Theta\mathbb{R} )
$$

— is the use of the distinct labels “$q$”, “$\Theta$” for the copies of $\mathbb{R}$ that appear in the assignments $\lambda_q$, $\lambda_\Theta$.

Now let us consider an alternative approach to constructing the assignment $\lambda_\Theta$: We construct $\lambda_\Theta$ as the “assignment with indeterminacies”

$$
\lambda_\Theta^{\text{Ind}} : * \mapsto \Theta\mathbb{R}_{\leq -2h + \epsilon} \subseteq \Theta\mathbb{R}
$$

— where $\mathbb{R}_{\leq -2h + \epsilon} \overset{\text{def}}{=} \{ x \in \mathbb{R} \mid x \leq -2h + \epsilon \} \subseteq \mathbb{R}$; $\epsilon \in \mathbb{R}_{>0}$ is some positive number.

Now suppose that one verifies that one may construct the “assignment with indeterminacies” $\lambda_\Theta^{\text{Ind}}$ of (d) as a mathematical structure that is intrinsically associated to the underlying structure of the assignment $\lambda_q$ — hence, in particular, simultaneously with/without invalidating the conditions imposed by — the assignment $\lambda_q$, even if one forgets the labels “$q$”, “$\Theta$” that were appended to copies of $\mathbb{R}$, i.e., even if one identifies $q\mathbb{R}$, $\Theta\mathbb{R}$, in the usual way, with $\mathbb{R}$ [cf. the properties (IPL), (SHE) of Remark 3.11.1, (iii)]. That is to say, we suppose that one can show that the assignments determined, respectively, by $\lambda_q$, $\lambda_\Theta^{\text{Ind}}$, by identifying copies of $\mathbb{R}$, namely,
where the latter assignment may be considered as the assignment that maps \( \ast \) to “some [undetermined] element \( \in \mathbb{R}_{\leq -2h+\epsilon} \)” — are such that one may construct the latter assignment as a mathematical structure that is \textbf{intrinsically associated} to — hence, in particular, \textbf{simultaneously with/without invalidating} the conditions imposed by — the former assignment. Here, we note that it is \textit{not particularly relevant} that \( \mathbb{R}_{\leq -2h+\epsilon} \) arose as some sort of “perturbation via indeterminacies of \( 2h \)” [cf. the property (HIS) of Remark 3.11.1, (iv)].

\( (\text{toy}) \) The discussion of \((\text{e}^{\text{toy}})\) may be summarized at the level of \textit{logical relations} [cf. the displays of \((\text{b}^{\text{toy}}), (\text{e}^{\text{toy}})\)] as follows:

\[
(\ast \mapsto -h) \implies (\ast \mapsto -h) \land (\ast \mapsto \mathbb{R}_{\leq -2h+\epsilon}) \implies (\ast \mapsto \mathbb{R}_{\leq -2h+\epsilon})
\]

— that is to say, “\( \ast \mapsto -h \)” may be regarded as a \textbf{special case} of “\( \ast \mapsto \mathbb{R}_{\leq -2h+\epsilon} \)”, which, in turn, may be regarded as a “\textit{version with indeterminacies}” of “\( \ast \mapsto -2h \)”. One then concludes formally that \( -h \in \mathbb{R}_{\leq -2h+\epsilon} \) and hence that

\[
-h \leq -2h + \epsilon, \quad \text{i.e.,} \quad h \leq \epsilon
\]

— that is to say, the \textit{desired upper bound on} \( h \).

(iii) One fundamental aspect of the theory that renders possible the “reassembling procedure” discussed in (ii) [cf. the discussion of Step (iv) of the proof of Corollary 3.12] is the “\textbf{juggling of } \boxplus, \boxminus” [cf. the discussion of Remark 1.2.2, (vi)] effected by the \textit{log-links}, i.e., the \textbf{vertical arrows} of the log-theta-lattice. This “juggling of \( \boxplus, \boxminus \)” may be thought of as a sort of combinatorial way of representing the \textbf{arithmetic holomorphic structure} associated to a \textbf{vertical line} of the log-theta-lattice. Indeed, at \textit{archimedean primes}, this juggling amounts essentially to \textit{multiplication by } \( \pm i \), which is a well-known method [cf. the notion of an “almost complex structure”!] for representing holomorphic structures in the classical theory of differential manifolds. On the other hand, it is important to recall in this context that this “juggling of \( \boxplus, \boxminus \)” is precisely what gives rise to the \textbf{upper semi-compatibility indeterminacy} (Ind3) [cf. Proposition 3.5, (ii); Remark 3.10.1, (i)].

(iv) In the context of the discussion of (ii), (iii), it is of interest to \textbf{compare}, in the cases of the \textbf{0- and 1-columns} of the log-theta-lattice, the way in which the theory of \textbf{log-Kummer correspondences} associated to a vertical column of the log-theta-lattice is applied in the proof of Corollary 3.12, especially in Steps (x) and (xi). We begin by observing that the \textit{vertical column} [i.e., 0- or 1-column] under consideration may be depicted [“horizontally”] in the fashion of the diagram of
the third display of Proposition 1.3, (iv)

\[
\bullet_0 \\
\| \\
... \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow ... \\
... \searrow \downarrow \swarrow \ldots \\
\circ
\]

— where the “\(\bullet_0\)” in the first line of the diagram denotes the portion with vertical coordinate \(0\) [i.e., the portion at \((0,0)\) or \((1,0)\)] of the vertical column under consideration. As discussed in Step (iii) of the proof of Corollary 3.12, since the \(\Theta^{x,\mu}_{LGP}\)-link is fundamentally incompatible with the distinct arithmetic holomorphic structures — i.e., ring structures — that exist in the 0- and 1-columns, one is obliged to work with the Frobenius-like versions of the unit group and value group portions of monoids arising from “\(\bullet_0\)” in the definition of the \(\Theta^{x,\mu}_{LGP}\)-link precisely in order to avoid the need to contend, in the definition of the \(\Theta^{x,\mu}_{LGP}\)-link, with the issue of describing the “structure of the intertwining” [cf. the discussion of (ii)] between these unit group and value group portions determined by the distinct arithmetic holomorphic structures — i.e., ring structures — that exist in the 0- and 1-columns. On the other hand, one is also obliged to work with the \(\acute{e}tale\)-like “\(\circ\)” versions of various objects since it is precisely these vertically coric versions that allow one to access, i.e., by serving as containers [cf. the discussion of (ii)] for, the other “\(\bullet\)’s” in the vertical column under consideration. That is to say, although the various Kummer isomorphisms that relate various portions of the Frobenius-like “\(\bullet_0\)” to the corresponding portions of the \(\acute{e}tale\)-like “\(\circ\)” may at first give the impression that either “\(\bullet_0\)” or “\(\circ\)” is superfluous or unnecessary in the theory, in fact

both “\(\bullet_0\)” and “\(\circ\)” play an essential and by no means superfluous role
in the theory of the vertical columns of the log-theta-lattice.

This aspect of the theory is essentially the same in the case of both the 0- and the 1-columns. The log-link compatibility of the various log-volumes that appear [cf. the discussion of Step (x) of the proof of Corollary 3.12; Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] is another aspect of the theory that is essentially the same in the case of both the 0- and the 1-columns. Also, although the discussion of the “non-interference” properties that underlie the log-Kummer correspondences of Theorem 3.11, (ii), (b), (c), was only given explicitly, in effect, in the case of the 0-column, i.e., concerning \(\Theta\)-pilot objects, entirely similar “non-interference” properties hold for \(q\)-pilot objects. [Indeed, this may be seen, for instance, by applying the same arguments as the arguments that were applied in the case of \(\Theta\)-pilot objects, or, for instance, by specializing the non-interference properties obtained for \(\Theta\)-pilot objects to the index “\(j = 1\)” as in the discussion of “pivotal distributions” in [IUTchI], Example 5.4, (vii).] These similarities between the 0- and 1-columns are summarized in the upper portion of Fig. 3.9 below.

(v) In the discussion of (iv), we highlighted various similarities between the 0- and 1-columns of the log-theta-lattice in the context of Steps (x), (xi) of the
proof of Corollary 3.12. By contrast, one significant difference between the theory of log-Kummer correspondences in the 0- and 1-columns is

the lack of analogues for \( q \)-pilot objects of the crucial multiradiality properties summarized in Theorem 3.11, (iii), (c).

<table>
<thead>
<tr>
<th>Aspect of the theory</th>
<th>0-column/ ( \Theta )-pilot objects</th>
<th>1-column/ ( q )-pilot objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>essential role of both “( \cdot_0 )” and “( \circ )”</td>
<td>similar</td>
<td>similar</td>
</tr>
<tr>
<td>log-link compatibility of log-volumes</td>
<td>similar</td>
<td>similar</td>
</tr>
<tr>
<td>“non-interference” properties of log-Kummer correspondences</td>
<td>similar</td>
<td>similar</td>
</tr>
<tr>
<td>multiradiality properties of ( \Theta )/( q )-pilot objects</td>
<td>hold</td>
<td>do not hold</td>
</tr>
<tr>
<td>treatment of log-shells/ unit group portions</td>
<td>used as mono-analytic containers for regions</td>
<td>tautological documenting device for logarithmic relationship betw. ring structures</td>
</tr>
<tr>
<td>resulting indeterminacies acting on log-shells</td>
<td>(Ind1), (Ind2), (Ind3)</td>
<td>absorbed by applying holomorphic hulls, log-volumes</td>
</tr>
</tbody>
</table>

Fig. 3.9: Similarities and differences, in the context of the \( \Theta \times_{\text{LGP}} \)-link, between the 0- and 1-columns of the log-theta-lattice

— i.e., in effect, the lack of an analogue for the \( q \)-pilot objects of the theory of rigidity properties developed in [EtTh] [cf. the discussion of Remark 2.2.2, (i)]. Another significant difference between the theory of log-Kummer correspondences
in the 0- and 1-columns lies in the way in which the associated \textit{vertically coric holomorphic log-shells} [cf. Proposition 1.2, (ix)] are treated in their relationship to the \textbf{unit group} portions of monoids that occur in the various “•’s” of the log-Kummer correspondence. That is to say, in the case of the 0-column, these log-shells are used as \textbf{containers} [cf. the discussion of (ii)] for the various \textbf{regions} [i.e., subsets] arising from these unit group portions via various composites of arrows in the log-Kummer correspondence. This approach has the \textit{advantage of admitting an \textit{interpretation} — i.e., in terms of subsets of \textbf{mono-analytic log-shells} — that makes sense even relative to the \textit{distinct arithmetic holomorphic structures} that appear in the 1-column of the log-theta-lattice [cf. Remark 3.11.1]. On the other hand, it has the \textbf{drawback} that it gives rise to the \textit{upper semi-compatibility indeterminacy} (Ind3) discussed in the final portion of Theorem 3.11, (ii). By contrast,

\begin{quote}

in the case of the 1-column, since the associated \textbf{arithmetic holomorphic structure} is held \textbf{fixed} and \textbf{regarded} [cf. the discussion of Step (xi) of the proof of Corollary 3.12] as the \textbf{standard} with respect to which constructions arising from the 0-column are to be \textbf{computed}, there is \textbf{no need} [i.e., in the case of the 1-column] to require that the constructions applied \textbf{admit} \textbf{mono-analytic interpretations}.
\end{quote}

That is to say, in the case of the 1-column, the various unit group portions of monoids at the various “•’s” simply serve as a means of \textit{documenting the “logarithmic” relationship} [cf. the definition of the log-link given in Definition 1.1, (i), (ii)!] between the \textbf{ring structures} in the domain and codomain of the log-link. These ring structures give rise to the local copies of sets of integral elements “\(\mathcal{O}\)” with respect to which the “\(\text{\textit{mod}}\)” \textbf{versions} [cf. Example 3.6, (ii)] of \textbf{categories of arithmetic line bundles} are defined at the various “•’s”. Since the objects of these categories of arithmetic line bundles are \textbf{not equipped} with \textbf{local trivializations} at the various \(v \in V\) [cf. the discussion of \textit{isomorphism classes of objects} of Frobenioids in Remark 3.6.2, (i)],

\begin{quote}

\textbf{regions} in log-shells may only be related to such categories of arithmetic line bundles at the expense of allowing for an \textbf{indeterminacy} with respect to “\(\mathcal{O}^{\times}\)”-\textbf{multiples} at each \(v \in V\).
\end{quote}

It is precisely this indeterminacy that necessitates the introduction, in Step (xi) of the proof of Corollary 3.12, of \textbf{holomorphic hulls}, i.e., which have the effect of \textit{absorbing} this indeterminacy [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Finally, in Step (xi) of the proof of Corollary 3.12,

\begin{quote}

the \textbf{indeterminacy} in the \textit{specification of a particular member} of the \textbf{collection} of ring structures just discussed — i.e., arising from the \textit{choice of a particular composite} of arrows in the log-Kummer correspondence that is used to specify a \textbf{particular ring structure} among its various “logarithmic conjugates” — is \textbf{absorbed} by passing to \textbf{log-volumes} — i.e., by applying the \textbf{log-link compatibility} [cf. (iv)] of the various log-volumes associated to these ring structures [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Thus, unlike the case of the 0-column, where the \textbf{mono-analytic} interpretation via \textbf{regions} of mono-analytic log-shells gives rise only to \textbf{upper bounds} on log-volumes, the approach just discussed in the case of the 1-column — i.e., which makes essential use of the \textbf{ring structures} that are available
as a consequence of the fact that the arithmetic holomorphic structure is held fixed — gives rise to precise equalities [i.e., not just inequalities!] concerning log-volumes. These differences between the 0- and 1-columns are summarized in the lower portion of Fig. 3.9.

**Remark 3.12.3.**

(i) Let $S$ be a hyperbolic Riemann surface of finite type of genus $g_S$ with $r_S$ punctures. Write $\chi_S \overset{\text{def}}{=} -(2g_S - 2 + r_S)$ for the Euler characteristic of $S$ and $d\mu_S$ for the Kähler metric on $S$ [i.e., the $(1,1)$-form] determined by the Poincaré metric on the upper half-plane. Recall the analogy discussed in [IUTchI], Remark 4.3.3, between the theory of log-shells, which plays a key role in the theory developed in the present series of papers, and the classical metric geometry of hyperbolic Riemann surfaces. Then, relative to this analogy, the inequality obtained in Corollary 3.12 may be regarded as corresponding to the inequality

$$\chi_S = -\int_S d\mu_S < 0$$

— i.e., in essence, a statement of the hyperbolicity of $S$ — arising from the classical Gauss-Bonnet formula, together with the positivity of $d\mu_S$. Relative to the analogy between real analytic Kähler metrics and ordinary Frobenius liftings discussed in [pOrd], Introduction, §2 [cf. also the discussion of [pTeich], Introduction, §0], the local property constituted by this positivity of $d\mu_S$ may be thought of as corresponding to the [local property constituted by the] Kodaira-Spencer isomorphism of an indigenous bundle — i.e., which gives rise to the ordinarity of the corresponding Frobenius lifting on the ordinary locus — in the $p$-adic theory. As discussed in [AbsTopIII], §15, these properties of indigenous bundles in the $p$-adic theory may be thought of as corresponding, in the theory of log-shells, to the “maximal incompatibility” between the various Kummer isomorphisms and the corically constructed data of the Frobenius-picture of Proposition 1.2, (x). On the other hand, it is just this “maximal incompatibility” that gives rise to the “upper semi-commutativity” discussed in Remark 1.2.2, (iii), i.e., [from the point of view of the theory of the present §3] the upper semi-compatibility indeterminacy (Ind3) of Theorem 3.11, (ii), that underlies the inequality of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12].

(ii) The “metric aspect” of Corollary 3.12 discussed in (i) is reminiscent of the analogy between the theory of the present series of papers and classical complex Teichmüller theory [cf. the discussion of [IUTchI], Remark 3.9.3] in the following sense:

Just as classical complex Teichmüller theory is concerned with relating distinct holomorphic structures in a sufficiently canonical way as to minimize the resulting conformality distortion, the canonical nature of the algorithms discussed in Theorem 3.11 for relating alien arithmetic holomorphic structures [cf. Remark 3.11.1] gives rise to a relatively strong estimate of the [log-]volume distortion [cf. Corollary 3.12] resulting from such a deformation of the arithmetic holomorphic structure.
Remark 3.12.4. In light of the discussion of Remark 3.12.3, it is of interest to reconsider the analogy between the theory of the present series of papers and the $p$-adic Teichmüller theory of $[p\text{Ord}]$, $[p\text{Teich}]$, in the context of Theorem 3.11, Corollary 3.12.

(i) First, we observe that the splitting monoids at $v \in \mathcal{V}^{\text{bad}}$ [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b)] may be regarded as analogous to the canonical coordinates of $p$-adic Teichmüller theory [cf., e.g., $[p\text{Teich}]$, Introduction, §0.9] that are constructed over the ordinary locus of a canonical curve. In particular, it is natural to regard the bad primes $\in \mathcal{V}^{\text{bad}}$ as corresponding to the ordinary locus of a canonical curve and the good primes $\in \mathcal{V}^{\text{good}}$ as corresponding to the supersingular locus of a canonical curve. This point of view is reminiscent of the discussion of [IUTchII], Remark 4.11.4, (iii).

(ii) On the other hand, the bi-coric mono-analytic log-shells — i.e., the various local “$O \times \mu$” — that appear in the tensor packets of Theorem 3.11, (i), (a); Theorem 3.11, (ii), (a), may be thought of as corresponding to the [multiplicative!] Teichmüller representatives associated to the various Witt rings that appear in $p$-adic Teichmüller theory. Within a fixed arithmetic holomorphic structure, these mono-analytic log-shells arise from “local holomorphic units” — i.e., “$O^{\times}$” — which are subject to the $F \rtimes \pm l$-symmetry. These “local holomorphic units” may be thought of as corresponding to the positive characteristic ring structures on [the positive characteristic reductions of] Teichmüller representatives. Here, the uniradial, i.e., “non-multiradial”, nature of these “local holomorphic units” [cf. the discussion of [IUTchII], Remark 4.7.4, (ii); [IUTchII], Figs. 4.1, 4.2] may be regarded as corresponding to the incompatibility of Teichmüller representatives with the additive structure of Witt rings.

(iii) The set $F_l^*$ of $l^*$ “theta value labels”, which plays an important role in the theory of the present series of papers, may be thought of as corresponding to the “factor of $p$” that appears in the “mod $p/p^2$ portion”, i.e., the gap separating the “mod $p$” and “mod $p^2$” portions, of the rings of Witt vectors that occur in the $p$-adic theory. From this point of view, one may think of the procession-normalized volumes obtained by taking averages over $j \in F_l^*$ [cf. Corollary 3.12] as corresponding to the operation of dividing by $p$ to relate the “mod $p/p^2$ portion” of the Witt vectors to the “mod $p$ portion” of the Witt vectors [i.e., the characteristic $p$ theory]. In this context, the multiradial representation of Theorem 3.11, (i), by means of mono-analytic log-shells labeled by elements of $F_l^*$ may be thought of as corresponding to the derivative of the canonical Frobenius lifting on a canonical curve in the $p$-adic theory [cf. the discussion of [AbsTopIII], §I5] in the sense that this multiradial representation may be regarded as a sort of comparison of the canonical splitting monoids discussed in (i) to the “absolute constants” [cf. the discussion of (ii)] constituted by the bi-coric mono-analytic log-shells. This “absolute comparison” is precisely what results in the indeterminacies (Ind1), (Ind2) of Theorem 3.11, (i).

(iv) In the context of the discussion of (iii), we note that the set of labels $F_l^*$ may, alternatively, be thought of as corresponding to the infinitesimal moduli of the positive characteristic curve under consideration in the $p$-adic theory [cf. the
discussion of [IUTchII], Remark 4.11.4, (iii), (d)]. That is to say, the “deformation dimension” constituted by the horizontal dimension of the log-theta-lattice in the theory of the present series of papers or by the deformations modulo various powers of \( p \) in the \( p \)-adic theory [cf. Remark 1.4.1, (iii); Fig. 1.3] is highly canonical in nature, hence may be thought of as being equipped with a natural isomorphism to the “absolute moduli” — i.e., so to speak, the “moduli over \( \mathbb{F}_1 \)” — of the given number field equipped with an elliptic curve, in the theory of the present series of papers, or of the given positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle, in \( p \)-adic Teichmüller theory.

<table>
<thead>
<tr>
<th>( \text{Inter-universal Teichmüller theory} )</th>
<th>( \text{( p )-adic Teichmüller theory} )</th>
</tr>
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<tr>
<td>splitting monoids at ( v \in \mathcal{V}^{\text{bad}} )</td>
<td>canonical coordinates on the ordinary locus</td>
</tr>
<tr>
<td>bad primes ( \in \mathcal{V}^{\text{bad}} )</td>
<td>ordinary locus of a can. curve</td>
</tr>
<tr>
<td>good primes ( \in \mathcal{V}^{\text{good}} )</td>
<td>supersing. locus of a can. curve</td>
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<tr>
<td>mono-analytic log-shells ( \mathcal{O}^{\times \mu} )</td>
<td>([\text{multiplicative!}] ) Teich. reps.</td>
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<tr>
<td>uniradial “local hol. units ( \mathcal{O}^{\times} )” subject to ( \mathbb{F}_l^{\times \pm} )-symmetry</td>
<td>pos. char. ring structures on ([\text{pos. char. reductions of}] ) Teich. reps.</td>
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<tr>
<td>set of “theta value labels” ( \mathbb{F}_l^{\ast} )</td>
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<td>multiradial rep. via ( \mathbb{F}_l^{\ast} )-labeled mono-analytic log-shells [cf. (Ind1), (Ind2), (Ind3)]</td>
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<tr>
<td>set of “theta value labels” ( \mathbb{F}_l^{\ast} )</td>
<td>implicit “absolute moduli/( \mathbb{F}_1 )”</td>
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<td>inequality arising from upper semi-compatibility [cf. (Ind3)]</td>
<td>inequality arising from interference between Frobenius conjugates</td>
</tr>
</tbody>
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Fig. 3.10: The analogy between inter-universal Teichmüller theory and \( p \)-adic Teichmüller theory
(v) Let $A$ be the ring of Witt vectors of a perfect field $k$ of positive characteristic $p$; $X$ a smooth, proper hyperbolic curve over $A$ of genus $g_X$ which is canonical in the sense of $p$-adic Teichmüller theory; $\hat{X}$ the $p$-adic formal scheme associated to $X$; $\hat{U} \subseteq \hat{X}$ the ordinary locus of $\hat{X}$. Write $\omega_{X_k}$ for the canonical bundle of $X_k \overset{def}{=} X \times_A k$. Then when [cf. the discussion of (iii)] one computes the derivative of the canonical Frobenius lifting $\Phi : \hat{U} \to \hat{U}$ on $\hat{U}$, one must contend with “interference phenomena” between the various copies of some positive characteristic algebraic geometry set-up — i.e., at a more concrete level, the various Frobenius conjugates “$t^p$” [where $t$ is a local coordinate on $X_k$] associated to various $n \in \mathbb{N}_{\geq 1}$.

In particular, this derivative only yields [upon dividing by $p$] an inclusion [i.e., not an isomorphism!] of line bundles

$$\omega_{X_k} \hookrightarrow \Phi^*\omega_{X_k}$$

— also known as the “[square] Hasse invariant” [cf. [pOrd], Chapter II, Proposition 2.6; the discussion of “generalities on ordinary Frobenius liftings” given in [pOrd], Chapter III, §1]. Thus, at the level of global degrees of line bundles, we obtain an inequality [i.e., not an equality!]

$$(1 - p)(2g_X - 2) \leq 0$$

— which may be thought of as being, in essence, a statement of the hyperbolicity of $X$ [cf. the inequality of the display of Remark 3.12.3, (i)]. Since the “Frobenius conjugate dimension” [i.e., the “$n$” that appears in “$t^p$”] in the $p$-adic theory corresponds to the vertical dimension of the log-theta-lattice in the theory of the present series of papers [cf. Remark 1.4.1, (iii); Fig. 1.3], we thus see that the inequality of the above display in the $p$-adic case arises from circumstances that are entirely analogous to the circumstances — i.e., the upper semi-compatibility indeterminacy (Ind3) of Theorem 3.11, (ii) — that underlie the inequality of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12; the discussion of Remark 3.12.3, (i)].

(vi) The analogies of the above discussion are summarized in Fig. 3.10 above.


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