INTER-UNIVERSAL TEICHMÜLLER THEORY III: CANONICAL SPLITTINGS OF THE LOG-THETA-LATTICE

Shinichi Mochizuki

November 2019

Abstract. The present paper constitutes the third paper in a series of four papers and may be regarded as the *culmination* of the *abstract conceptual* portion of the theory developed in the series. In the present paper, we study the theory surrounding the **log-theta-lattice**, a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*. Here, we recall that $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters were associated, in the first paper of the series, to certain data, called *initial* Θ*-data*, that includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Each *arrow* of the log-theta-lattice corresponds to a certain *gluing operation* between the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the domain and codomain of the arrow. The **horizontal arrows** of the log-theta-lattice are defined as certain versions of the *"*Θ*-link"* that was constructed, in the second paper of the series, by applying the theory of *Hodge-Arakelov-theoretic evaluation* — i.e., evaluation in the style of the **scheme-theoretic Hodge-Arakelov theory** established by the author in previous papers — of the [reciprocal of the l-th root of the] **theta function** at **l-torsion points**. In the present paper, we focus on the theory surrounding the $\mathfrak{log}$**-link** between $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters. The $\mathfrak{log}$-link is obtained, roughly speaking, by applying, at each [say, for simplicity, nonarchimedean] valuation of the number field under consideration, the *local* p*-adic logarithm*. The significance of the $\mathfrak{log}$-link lies in the fact that it allows one to construct **log-shells**, i.e., roughly speaking, slightly adjusted forms of the image of the local units at the valuation under consideration via the local p-adic logarithm. The theory of log-shells was studied extensively in a previous paper by the author. The **vertical arrows** of the log-theta-lattice are given by the $\mathfrak{log}$-link. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of logarithmic Gaussian procession monoids"**. Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. These logarithmic Gaussian procession monoids, or **LGP-monoids**, for short, may be thought of as the log-shell-theoretic versions of the *Gaussian monoids* that were studied in the second paper of the series. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain **estimates** for the **log-volume** of these LGP-monoids. Explicit computations of these estimates will be applied, in the fourth paper of the series, to derive various *diophantine results*.

Contents:

Introduction

In the following discussion, we shall continue to use the notation of the Introduction to the first paper of the present series of papers [cf. [IUTchI], §I1]. In particular, we assume that are given an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. In the first paper of the series, we introduced and studied the basic properties of $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, which may be thought of as miniature models of the conventional scheme theory surrounding the given elliptic curve E_F over the number field F. In the present paper, which forms the third paper of the series, we study the theory surrounding the **$\mathfrak{log}$-link** between $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters. The $\mathfrak{log}$-link induces an *isomorphism between the underlying* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* and, roughly speaking, is obtained by applying, at each [say, for simplicity, nonarchimedean] valuation $\underline{v} \in \underline{\mathbb{V}}$, the *local* $p_{\underline{v}}$*-adic logarithm* to the local units [cf. Proposition 1.3, (i)]. The significance of the $\mathfrak{log}$-link lies in the fact that it allows one to construct **log-shells**, i.e., roughly speaking, slightly adjusted forms of the image of the local units at $\underline{v} \in \underline{\mathbb{V}}$ via the local $p_{\underline{v}}$-adic logarithm. The theory of log-shells was studied extensively in [AbsTopIII]. The introduction of log-shells leads naturally to the construction of *new versions* — namely, the **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links** [cf. Definition 3.8, (ii)] — of the Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$*-links* studied in [IUTchI], [IUTchII]. The resulting [*highly non-commutative*!] diagram of iterates of the $\mathfrak{log}$- [i.e., the *vertical arrows*] and $\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the *horizontal arrows*] — which we refer to as the **log-theta-lattice** [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1 below, in the case of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link] — plays a *central role* in the theory of the present series of papers.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

Fig. I.1: The [LGP-Gaussian] log-theta-lattice

Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of logarithmic Gaussian procession monoids"** [cf. Theorem A below]. Here, we recall that "multiradial algorithms" [cf. the discussion of [IUTchII], Introduction] are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link. These logarithmic Gaussian procession monoids, or **LGP-monoids**, for short, may be thought of as the log-shell-theoretic versions of the *Gaussian monoids* that were studied in [IUTchII]. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain **estimates** for the **log-volume** of these LGP-monoids [cf. Theorem B below]. These estimates will be applied to verify various *diophantine results* in [IUTchIV].

Recall [cf. [IUTchI], §I1] the notion of an *$\mathcal{F}$-prime-strip*. An $\mathcal{F}$-prime-strip consists of data indexed by the valuations $\underline{v} \in \underline{\mathbb{V}}$; roughly speaking, the data at each $\underline{v}$ consists of a *Frobenioid*, i.e., in essence, a system of *monoids* over a *base category*. For instance, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, this data may be thought of as an isomorphic copy of the *monoid with Galois action*

$$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$$

— where we recall that $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$ denotes the multiplicative monoid of nonzero integral elements of the completion of an algebraic closure $\overline{F}$ of F at a valuation lying over $\underline{v}$ [cf. [IUTchI], §I1, for more details]. The *$p_{\underline{v}}$-adic logarithm* $\log_{\underline{v}} : \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \to \overline{F}_{\underline{v}}$ at $\underline{v}$ then defines a natural *$\Pi_{\underline{v}}$-equivariant* isomorphism of ind-topological modules

$$(\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q} \ \stackrel{\sim}{\to}\ ) \quad \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q} \ \stackrel{\sim}{\to}\ \overline{F}_{\underline{v}}$$

— where we recall the notation "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} = \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} / \mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" from the discussion of [IUTchI], §1 — which allows one to equip $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q}$ with the *field structure* arising from the field structure of $\overline{F}_{\underline{v}}$. The portion at $\underline{v}$ of the **$\mathfrak{log}$-link** associated to an $\mathcal{F}$-prime-strip [cf. Definition 1.1, (iii); Proposition 1.2] may be thought of as the correspondence

$$\left\{ \Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \right\} \quad \stackrel{\mathfrak{log}}{\longrightarrow} \quad \left\{ \Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \right\}$$

in which one thinks of the copy of "$\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" on the *right* as obtained from the field structure induced by the $p_{\underline{v}}$-adic logarithm on the tensor product with $\mathbb{Q}$ of the copy of the units "$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" on the *left*. Since this correspondence induces an *isomorphism of topological groups* between the copies of $\Pi_{\underline{v}}$ on either side, one may think of $\Pi_{\underline{v}}$ as *"immune to"/"neutral with respect to"* — or, in the terminology of the present series of papers, **"coric"** with respect to — the transformation constituted by the $\mathfrak{log}$-link. This situation is studied in detail in [AbsTopIII], §3, and reviewed in Proposition 1.2 of the present paper.

By applying various results from **absolute anabelian geometry**, one may algorithmically reconstruct a copy of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" from $\Pi_{\underline{v}}$. Moreover,

by applying *Kummer theory*, one obtains natural isomorphisms between this *"coric version"* of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" and the copies of this data that appear on either side of the $\mathfrak{log}$-*link*. On the other hand, one verifies immediately that these Kummer isomorphisms are **not compatible** with the **coricity** of the copy of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" algorithmically constructed from $\Pi_{\underline{v}}$. This phenomenon is, in some sense, the *central theme* of the theory of [AbsTopIII], §3, and is reviewed in Proposition 1.2, (iv), of the present paper.

The introduction of the $\mathfrak{log}$-link leads naturally to the construction of **log-shells** at each $\underline{v} \in \underline{\mathbb{V}}$. If, for simplicity, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then the log-shell at $\underline{v}$ is given, roughly speaking, by the *compact additive module*

$$\mathcal{I}_{\underline{v}} \overset{\mathrm{def}}{=} p_{\underline{v}}^{-1} \cdot \log_{\underline{v}}(\mathcal{O}^{\times}_{K_{\underline{v}}}) \subseteq K_{\underline{v}} \subseteq \overline{F}_{\underline{v}}$$

[cf. Definition 1.1, (i), (ii); Remark 1.2.2, (i), (ii)]. One has natural *functorial algorithms* for constructing various versions of the notion of a log-shell — i.e., **mono-analytic/holomorphic** and **étale-like/Frobenius-like** — from $\mathcal{D}^{\vdash}$-/$\mathcal{D}$-/$\mathcal{F}^{\vdash}$-/$\mathcal{F}$-prime-strips [cf. Proposition 1.2, (v), (vi), (vii), (viii), (ix)]. Although, as discussed above, the relevant Kummer isomorphisms are *not compatible* with the $\mathfrak{log}$-link *"at the level of elements"*, the log-shell $\mathcal{I}_{\underline{v}}$ at $\underline{v}$ satisfies the important property

$$\mathcal{O}^{\triangleright}_{K_{\underline{v}}} \subseteq \mathcal{I}_{\underline{v}}; \quad \log_{\underline{v}}(\mathcal{O}^{\times}_{K_{\underline{v}}}) \subseteq \mathcal{I}_{\underline{v}}$$

— i.e., it **contains** *the* **images** *of the Kummer isomorphisms* associated to both the domain and the codomain of the $\mathfrak{log}$-link [cf. Proposition 1.2, (v); Remark 1.2.2, (i), (ii)]. In light of the *compatibility* of the $\mathfrak{log}$-link with *log-volumes* [cf. Propositions 1.2, (iii); 3.9, (iv)], this property will ultimately lead to **upper bounds** — i.e., as opposed to *"precise equalities"* — in the computation of *log-volumes* in Corollary 3.12 [cf. Theorem B below]. Put another way, although iterates [cf. Remark 1.1.1] of the $\mathfrak{log}$-link *fail to be compatible* with the various Kummer isomorphisms that arise, one may nevertheless consider the *entire diagram* that results from considering such iterates of the $\mathfrak{log}$-link and related Kummer isomorphisms [cf. Proposition 1.2, (x)]. We shall refer to such diagrams

$$\begin{array}{ccccccccc} \dots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the *horizontal arrows* correspond to the $\mathfrak{log}$-links [that is to say, to the *vertical arrows* of the log-theta-lattice!]; the "$\bullet$'s" correspond to the Frobenioid-theoretic data within a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater; the "$\circ$" corresponds to the *coric version* of this data [that is to say, in the terminology discussed below, *vertically coric* data of the log-theta-lattice]; the vertical/diagonal arrows correspond to the various *Kummer isomorphisms* — as **$\mathfrak{log}$-Kummer correspondences** [cf. Theorem 3.11, (ii); Theorem A, (ii), below]. Then the inclusions of the above display may be interpreted as a sort of **"upper semi-commutativity"** of such diagrams [cf. Remark 1.2.2, (iii)], which we shall also refer to as the **"upper semi-compatibility"** *of the* $\mathfrak{log}$-*link with the relevant Kummer isomorphisms* — cf. the discussion of the **"indeterminacy"** (Ind3) in Theorem 3.11, (ii).

By considering the $\mathfrak{log}$-links associated to the various *$\mathcal{F}$-prime-strips* that occur in a $\Theta^{\pm \text{ell}}$*NF-Hodge theater*, one obtains the notion of a **$\mathfrak{log}$-link** *between* $\Theta^{\pm \text{ell}}$*NF-Hodge theaters*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \text{ell}}\text{NF}}$$

[cf. Proposition 1.3, (i)]. As discussed above, by considering the iterates of the $\mathfrak{log}$- [i.e., the *vertical arrows*] and Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the *horizontal arrows*], one obtains a diagram which we refer to as the **log-theta-lattice** [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1, in the case of the $\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-link]. As discussed above, this diagram is **highly noncommutative**, since the definition of the $\mathfrak{log}$-link depends, in an essential way, on both the *additive* and the *multiplicative* structures — i.e., on the *ring structure* — of the various local rings at $\underline{v} \in \underline{\mathbb{V}}$, structures which are *not preserved* by the Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [cf. Remark 1.4.1, (i)]. So far, in the Introductions to [IUTchI], [IUTchII], as well as in the present Introduction, we have discussed various *"coricity"* properties — i.e., properties of *invariance* with respect to various types of "transformations" — in the context of Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links, as well as in the context of $\mathfrak{log}$-links. In the context of the log-theta-lattice, it becomes necessary to distinguish between various types of coricity. That is to say, coricity with respect to $\mathfrak{log}$-links [i.e., the vertical arrows of the log-theta-lattice] will be referred to as **vertical coricity**, while coricity with respect to Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the horizontal arrows of the log-theta-lattice] will be referred to as **horizontal coricity**. On the other hand, coricity properties that hold with respect to *all* of the arrows of the log-theta-lattice will be referred to as **bi-coricity** properties.

Relative to the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [IUTchI], §I4], we recall that a $\Theta^{\pm \text{ell}}$*NF-Hodge theater*, which may be thought of as a miniature model of the *conventional scheme theory* surrounding the given elliptic curve E_F over the number field F, corresponds to the *positive characteristic scheme theory* surrounding a hyperbolic curve over a positive characteristic perfect field that is equipped with a nilpotent ordinary indigenous bundle [cf. Fig. I.2 below]. Then the **rotation**, or **"juggling"**, effected by the **$\mathfrak{log}$-link** of the **additive** and **multiplicative** structures of the conventional scheme theory represented by a $\Theta^{\pm \text{ell}}$NF-Hodge theater may be thought of as corresponding to the **Frobenius morphism** *in positive characteristic* [cf. the discussion of [AbsTopIII], §I1, §I3, §I5]. Thus, just as the Frobenius morphism is completely well-defined in *positive characteristic*, the $\mathfrak{log}$-link may be thought of as a phenomenon that occurs within a *single* **arithmetic holomorphic structure**, i.e., a *vertical* line of the log-theta-lattice. By contrast, the essentially *non-ring/scheme-theoretic* relationship between $\Theta^{\pm \text{ell}}$NF-Hodge theaters constituted by the Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links corresponds to the relationship between the *"mod p^n" and "mod p^{n+1}"* portions of the ring of Witt vectors, in the context of a *canonical lifting* of the original positive characteristic data [cf. the discussion of Remark 1.4.1, (iii); Fig. I.2 below]. Thus, the **log-theta-lattice**, taken as a whole, may be thought of as corresponding to the **canonical lifting** of the original positive characteristic data, equipped with a corresponding **canonical Frobenius action/lifting** [cf. Fig. I.2 below]. Finally, the **non-commutativity** of the log-theta-lattice may be thought of as corresponding to the complicated **"intertwining"** that occurs in the theory of Witt vectors and canonical liftings between the Frobenius morphism in positive

characteristic and the mixed characteristic nature of the ring of Witt vectors [cf. the discussion of Remark 1.4.1, (ii), (iii)].

One important consequence of this *"noncommutative intertwining"* of the two dimensions of the log-theta-lattice is the following. Since each *horizontal arrow* of the log-theta-lattice [i.e., the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link] may only be used to relate — i.e., via various *Frobenioids* — the *multiplicative* portions of the ring structures in the domain and codomain of the arrow, one natural approach to relating the *additive* portions of these ring structures is to apply the theory of **log-shells**. That is to say, since each horizontal arrow is compatible with the **canonical splittings** [up to roots of unity] discussed in [IUTchII], Introduction, of the *theta/Gaussian monoids* in the domain of the horizontal arrow into *unit group* and *value group* portions, it is natural to attempt to relate the ring structures on either side of the horizontal arrow by applying the canonical splittings to

· relate the **multiplicative** structures on either side of the horizontal arrow by means of the **value group** portions of the theta/Gaussian monoids;

· relate the **additive** structures on either side of the horizontal arrow by means of the **unit group** portions of the theta/Gaussian monoids, **shifted once** via a *vertical arrow*, i.e., the **$\mathfrak{log}$-link**, so as to *"render additive"* the [*a priori*] multiplicative structure of these unit group portions.

Indeed, this is the approach that will ultimately be taken in Theorem 3.11 [cf. Theorem A below] to relating the ring structures on either side of a horizontal arrow. On the other hand, in order to actually implement this approach, it will be necessary to overcome numerous *technical obstacles*. Perhaps the most immediately obvious such obstacle lies in the observation [cf. the discussion of Remark 1.4.1, (ii)] that, precisely because of the *"noncommutative intertwining"* nature of the log-theta-lattice,

> any sort of algorithmic construction concerning objects lying in the *domain* of a horizontal arrow that involves **vertical shifts** [e.g., such as the approach to relating additive structures in the fashion described above] **cannot be "translated"** in any immediate sense into an algorithm that makes sense from the point of view of the *codomain* of the horizontal arrow.

In a word, our approach to overcoming this technical obstacle consists of working with objects in the *vertical line* of the log-theta-lattice that contains the *domain* of the horizontal arrow under consideration that satisfy the crucial property of being

invariant with respect to **vertical shifts**

— i.e., **shifts** via iterates of the **$\mathfrak{log}$-link** [cf. the discussion of Remarks 1.2.2, (iii); 1.4.1, (ii)]. For instance, *étale-like* objects that are **vertically coric** satisfy this invariance property. On the other hand, as discussed in the beginning of [IUTchII], Introduction, in the theory of the present series of papers, it is of crucial importance to be able to relate *corresponding Frobenius-like* and *étale-like structures* to one another via *Kummer theory*. In particular, in order to obtain structures

that are *invariant* with respect to *vertical shifts*, it is necessary to consider **$\mathfrak{log}$-Kummer correspondences**, as discussed above. Moreover, in the context of such $\mathfrak{log}$-Kummer correspondences, typically, one may only obtain structures that are invariant with respect to vertical shifts if one is willing to admit some sort of **indeterminacy**, e.g., such as the **"upper semi-compatibility"** [cf. the discussion of the *"indeterminacy"* (Ind3) in Theorem 3.11, (ii)] discussed above.

Inter-universal Teichmüller theory	*p-adic Teichmüller theory*
number field F	**hyperbolic curve** C over a *positive characteristic perfect field*
[**once-punctured**] **elliptic curve** X over F	*nilpotent ordinary* **indigenous bundle** P over C
Θ-link arrows of the *log-theta-lattice*	**mixed characteristic extension** structure of a ring of *Witt vectors*
$\mathfrak{log}$-link arrows of the *log-theta-lattice*	the **Frobenius morphism** in *positive characteristic*
the entire **log-theta-lattice**	the resulting **canonical lifting** + **canonical Frobenius action**; **canonical Frobenius lifting** over the ordinary locus
relatively straightforward *original construction of* **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link**	relatively straightforward *original construction of* **canonical liftings**
highly nontrivial *description of* **alien arithmetic holomorphic structure** via *absolute anabelian geometry*	highly nontrivial *absolute anabelian reconstruction of* **canonical liftings**

Fig. I.2: Correspondence between inter-universal Teichmüller theory and p-adic Teichmüller theory

One important property of the **$\mathfrak{log}$-link**, and hence, in particular, of the construction of **log-shells**, is its **compatibility** with the **$\mathbb{F}_l^{\rtimes\pm}$-symmetry** discussed in the Introductions to [IUTchI], [IUTchII] — cf. Remark 1.3.2. Here, we recall from the discussion of [IUTchII], Introduction, that the $\mathbb{F}_l^{\rtimes\pm}$-symmetry allows one to relate the various *$\mathcal{F}$-prime-strips* — i.e., more concretely, the various copies of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$] — associated to the various **labels** $\in \mathbb{F}_l$ that appear in the *Hodge-Arakelov-theoretic evaluation* of [IUTchII] in a fashion that is **compatible** with

· the **distinct nature** of distinct labels $\in \mathbb{F}_l$;

· the **Kummer isomorphisms** used to relate *Frobenius-like* and *étale-like* versions of the $\mathcal{F}$-prime-strips that appear, i.e., more concretely, the various copies of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$];

· the structure of the **underlying $\mathcal{D}$-prime-strips** that appear, i.e., more concretely, the various copies of the *[arithmetic] tempered fundamental group* "$\Pi_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$]

— cf. the discussion of [IUTchII], Introduction; Remark 1.5.1; Step (vii) of the proof of Corollary 3.12 of the present paper. This compatibility with the $\mathbb{F}_l^{\rtimes\pm}$-symmetry gives rise to the construction of

· **vertically coric** *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips, log-shells* by means of the **arithmetic holomorphic** structures under consideration;

· **mono-analytic** *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips, log-shells* which are **bi-coric**

— cf. Theorem 1.5. These *bi-coric mono-analytic log-shells* play a central role in the theory of the present paper.

One notable aspect of the *compatibility* of the $\mathfrak{log}$-*link* with the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* in the context of the theory of *Hodge-Arakelov-theoretic evaluation* developed in [IUTchII] is the following. One important property of *mono-theta environments* is the property of **"isomorphism class compatibility"**, i.e., in the terminology of [EtTh], *"compatibility with the* **topology** *of the* **tempered fundamental group***"* [cf. the discussion of Remark 2.1.1]. This "isomorphism class compatibility" allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the **ring-theoretic basepoints** that occur on either side of the **$\mathfrak{log}$-link** [cf. Remark 2.1.1, (ii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the $\mathfrak{log}$-*Kummer correspondences* discussed above. Here, we recall that the significance of working with such "ring-theoretic basepoints" lies in the fact that the full *ring structure* of the local rings involved [i.e., as opposed to, say, just the multiplicative portion of this ring structure] is necessary in order to construct the $\mathfrak{log}$-link. That is to say, it is precisely by establishing the *conjugate synchronization* arising from the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* relative to *these basepoints* that occur on either side of the $\mathfrak{log}$-link that one is able to conclude the crucial **compatibility** *of this conjugate synchronization with the* $\mathfrak{log}$-*link* discussed in Remark 1.3.2. Thus, in

summary, one important consequence of the "isomorphism class compatibility" of mono-theta environments is the **simultaneous compatibility** of

- the **Kummer theory** of **mono-theta environments**;
- the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**;
- the construction of the $\mathfrak{log}$**-link**.

This simultaneous compatibility is necessary in order to perform the construction of the [crucial!] *splitting monoids of LGP-monoids* referred to above — cf. the discussion of Step (vi) of the proof of Corollary 3.12.

In §2 of the present paper, we continue our preparation for the *multiradial construction of splitting monoids of LGP-monoids* given in §3 [of the present paper] by presenting a **global formulation** of the essentially *local theory* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. [IUTchII], §1, §2, §3] concerning the interpretation, via the notion of **multiradiality**, of various **rigidity** properties of **mono-theta environments**. That is to say, although much of the [essentially routine!] task of formulating the local theory of [IUTchII], §1, §2, §3, in global terms was accomplished in [IUTchII], §4, the [again essentially routine!] task of formulating the portion of this local theory that concerns *multiradiality* was not explicitly addressed in [IUTchII], §4. One reason for this lies in the fact that, from the point of view of the theory to be developed in §3 of the present paper, this global formulation of multiradiality properties of the mono-theta environment may be presented most naturally in the framework developed in §1 of the present paper, involving the **log-theta-lattice** [cf. Theorem 2.2; Corollary 2.3]. Indeed, the **étale-like** versions of the mono-theta environment, as well as the various objects constructed from the mono-theta environment, may be interpreted, from the point of view of the log-theta-lattice, as **vertically coric** structures, and are **Kummer-theoretically** related to their **Frobenius-like** [i.e., Frobenioid-theoretic] counterparts, which arise from the [Frobenioid-theoretic portions of the] various $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters in a vertical line of the log-theta-lattice [cf. Theorem 2.2, (ii); Corollary 2.3, (ii), (iii), (iv)]. Moreover, it is precisely the **horizontal arrows** of the log-theta-lattice that give rise to the $\widehat{\mathbb{Z}}^\times$**-indeterminacies** acting on copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that play a prominent role in the local multiradiality theory developed in [IUTchII] [cf. the discussion of [IUTchII], Introduction]. In this context, it is useful to recall from the discussion of [IUTchII], Introduction [cf. also Remark 2.2.1 of the present paper], that the essential content of this local multiradiality theory consists of the *observation* [cf. Fig. I.3 below] that, since *mono-theta-theoretic cyclotomic* and *constant multiple rigidity* only require the use of the portion of $\mathcal{O}^\times_{\underline{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, given by the *torsion subgroup* $\mathcal{O}^{\boldsymbol{\mu}}_{\underline{F}_{\underline{v}}} \subseteq \mathcal{O}^\times_{\underline{F}_{\underline{v}}}$ [i.e., the roots of unity], the *triviality* of the composite of natural morphisms

$$\mathcal{O}^{\boldsymbol{\mu}}_{\underline{F}_{\underline{v}}} \hookrightarrow \mathcal{O}^\times_{\underline{F}_{\underline{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{F}_{\underline{v}}}$$

has the effect of **insulating** the **Kummer theory** of the **étale theta function** — i.e., via the theory of the mono-theta environments developed in [EtTh] — from the $\widehat{\mathbb{Z}}^\times$**-indeterminacies** that act on the copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that arise in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* that appear in the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link.

$$\begin{array}{ccc} \mathrm{id} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \curvearrowright \\ \boxed{\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}} & \rightarrow & \boxed{\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}} \end{array}$$

Fig. I.3: Insulation from $\widehat{\mathbb{Z}}^{\times}$-indeterminacies in the context of mono-theta-theoretic cyclotomic, constant multiple rigidity

In §3 of the present paper, which, in some sense, constitutes the *conclusion* of the theory developed thus far in the present series of papers, we present the construction of the [splitting monoids of] **LGP-monoids**, which may be thought of as a **multiradial** version of the [splitting monoids of] **Gaussian monoids** that were constructed via the theory of *Hodge-Arakelov-theoretic evaluation* developed in [IUTchII]. In order to achieve this multiradiality, it is necessary to "multiradialize" the various components of the construction of the Gaussian monoids given in [IUTchII]. The first step in this process of "multiradialization" concerns the **labels** $j \in \mathbb{F}_l^{*}$ that occur in the Hodge-Arakelov-theoretic evaluation performed in [IUTchII]. That is to say, the construction of these labels, together with the closely related theory of $\mathbb{F}_l^{*}$**-symmetry**, depend, in an essential way, on the *full arithmetic tempered fundamental groups* "$\Pi_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, i.e., on the portion of the *arithmetic holomorphic structure* within a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater which is *not shared* by an **alien** *arithmetic holomorphic structure* [i.e., an arithmetic holomorphic structure related to the original arithmetic holomorphic structure via a horizontal arrow of the log-theta-lattice]. One naive approach to remedying this state of affairs is to simply consider the *underlying set*, of cardinality l^{*}, associated to $\mathbb{F}_l^{*}$, which we regard as being equipped with the full set of *symmetries* given by arbitrary permutation automorphisms of this underlying set. The problem with this approach is that it yields a situation in which, for each label $j \in \mathbb{F}_l^{*}$, one must contend with an *indeterminacy of* l^{*} *possibilities* for the element of this underlying set that corresponds to j [cf. [IUTchI], Propositions 4.11, (i); 6.9, (i)]. From the point of view of the *log-volume computations* to be performed in [IUTchIV], this degree of indeterminacy gives rise to log-volumes which are *"too large"*, i.e., to estimates that are not sufficient for deriving the various *diophantine results* obtained in [IUTchIV]. Thus, we consider the following alternative approach, via **processions** [cf. [IUTchI], Propositions, 4.11, 6.9]. Instead of working just with the underlying set associated to $\mathbb{F}_l^{*}$, we consider the *diagram of inclusions* of finite sets

$$\mathbb{S}^{\pm}_{1} \quad \hookrightarrow \quad \mathbb{S}^{\pm}_{1+1=2} \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad \mathbb{S}^{\pm}_{j+1} \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad \mathbb{S}^{\pm}_{1+l^{*}=l^{\pm}}$$

— where we write $\mathbb{S}^{\pm}_{j+1} \stackrel{\mathrm{def}}{=} \{0, 1, \dots, j\}$, for $j = 0, \dots, l^{*}$, and we think of each of these finite sets as being subject to arbitrary permutation automorphisms. That is to say, we think of the set $\mathbb{S}^{\pm}_{j+1}$ as a **container** for the labels $0, 1, \dots, j$. Thus, for each j, one need only contend with an *indeterminacy of* $j+1$ *possibilities* for the element of this container that corresponds to j. In particular, if one allows $j = 0, \dots, l^{*}$ to vary, then this approach allows one to *reduce* the resulting label indeterminacy from a total of $(l^{\pm})^{l^{\pm}}$ possibilities [where we write $l^{\pm} = 1 + l^{*} =$

$(l+1)/2]$ to a total of $l^{\pm}!$ possibilities. It turns out that this reduction will yield just the right estimates in the log-volume computations to be performed in [IUTchIV]. Moreover, this approach satisfies the important property of *insulating the* **"core label 0"** *from the various label indeterminacies* that occur.

Each element of each of the containers $\mathbb{S}^{\pm}_{j+1}$ may be thought of as parametrizing an *$\mathcal{F}$- or $\mathcal{D}$-prime-strip* that occurs in the *Hodge-Arakelov-theoretic evaluation* of [IUTchII]. In order to render the construction multiradial, it is necessary to replace such *holomorphic* $\mathcal{F}$-/$\mathcal{D}$-prime-strips by *mono-analytic* $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strips. In particular, as discussed above, one may construct, for each such $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strip, a collection of **log-shells** associated to the various $\underline{v} \in \underline{\mathbb{V}}$. Write $\mathbb{V}_{\mathbb{Q}}$ for the set of valuations of $\mathbb{Q}$. Then, in order to obtain objects that are *immune* to the various label indeterminacies discussed above, we consider, for each element $* \in \mathbb{S}^{\pm}_{j+1}$, and for each [say, for simplicity, *nonarchimedean*] $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

· the *direct sum of the* **log-shells** associated to the prime-strip labeled by the given element $* \in \mathbb{S}^{\pm}_{j+1}$ at the $\underline{v} \in \underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$;

we then form

· the **tensor product**, over the elements $* \in \mathbb{S}^{\pm}_{j+1}$, of these *direct sums.*

This collection of tensor products associated to $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ will be referred to as the **tensor packet** associated to the collection of prime-strips indexed by elements of $\mathbb{S}^{\pm}_{j+1}$. One may carry out this construction of the tensor packet either for *holomorphic* $\mathcal{F}$-/$\mathcal{D}$-prime-strips [cf. Proposition 3.1] or for *mono-analytic* $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strips [cf. Proposition 3.2].

The tensor packets associated to $\mathcal{D}^{\vdash}$-prime-strips will play a crucial role in the theory of §3, as **"multiradial mono-analytic containers"** for the principal objects of interest [cf. the discussion of Remark 3.12.2, (ii)], namely,

· the action of the **splitting monoids** of the **LGP-monoids** — i.e., the monoids generated by the **theta values** $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}$ — on the portion of the *tensor packets* just defined at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. Fig. I.4 below; Propositions 3.4, 3.5; the discussion of [IUTchII], Introduction];

· the action of copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} labeled by $j = 1, \dots, l^*$ on the product, over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, of the portion of the *tensor packets* just defined at $v_{\mathbb{Q}}$ [cf. Fig. I.5 below; Propositions 3.3, 3.7, 3.10].

$$\begin{array}{ccccccccccccc}
 & & \underline{\underline{q}}^{1} \curvearrowright & & & & \underline{\underline{q}}^{j^2} \curvearrowright & & & & \underline{\underline{q}}^{(l^*)^2} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots /^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots \dots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}$$

Fig. I.4: Splitting monoids of LGP-monoids acting on tensor packets

$$\begin{array}{ccccccccccc}
 & & (F^{\times}_{\mathrm{mod}})_1 \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_j \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_{l^*} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \ \dots \ /^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \ \dots \ \dots \ /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}$$

Fig. I.5: Copies of $F^{\times}_{\mathrm{mod}}$ acting on tensor packets

Indeed, these [splitting monoids of] **LGP-monoids** and copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} admit *natural embeddings into/actions on* the various *tensor packets* associated to *labeled $\mathcal{F}$-prime-strips* in each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ of the log-theta-lattice. One then obtains **vertically coric** versions of these splitting monoids of LGP-monoids and labeled copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the number field F_{mod} by applying suitable **Kummer isomorphisms** between

- *log-shells/tensor packets associated to [labeled] $\mathcal{F}$-prime-strips* and
- *log-shells/tensor packets associated to [labeled] $\mathcal{D}$-prime-strips.*

Finally, by passing to the

- *log-shells/tensor packets associated to [labeled] $\mathcal{D}^{\vdash}$-prime-strips*

— i.e., by *forgetting the* **arithmetic holomorphic structure** associated to a *specific vertical line* of the log-theta-lattice — one obtains the desired **multiradial representation**, i.e., description in terms that make sense from the point of view of an **alien** *arithmetic holomorphic structure*, of the **splitting monoids of LGP-monoids** and *labeled copies of the* **number field** F_{mod} discussed above. This passage to the multiradial representation is obtained by admitting the following *three types of* **indeterminacy**:

(Ind1): This is the indeterminacy that arises from the *automorphisms of* **processions of $\mathcal{D}^{\vdash}$-prime-strips** that appear in the multiradial representation — i.e., more concretely, from *permutation automorphisms* of the label sets $\mathbb{S}^{\pm}_{j+1}$ that appear in the processions discussed above, as well as from the *automorphisms of the $\mathcal{D}^{\vdash}$-prime-strips* that appear in these processions.

(Ind2): This is the ["non-(Ind1) portion" of the] indeterminacy that arises from the *automorphisms of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips* that appear in the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link — i.e., in particular, at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** acting on local copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [cf. the above discussion].

(Ind3): This is the indeterminacy that arises from the **upper semi-compatibility** *of the $\mathfrak{log}$-Kummer correspondences* associated to the specific vertical line of the log-theta-lattice under consideration [cf. the above discussion].

A detailed description of this multiradial representation, together with the indeterminacies (Ind1), (Ind2) is given in Theorem 3.11, (i) [and summarized in Theorem A, (i), below; cf. also Fig. I.6 below].

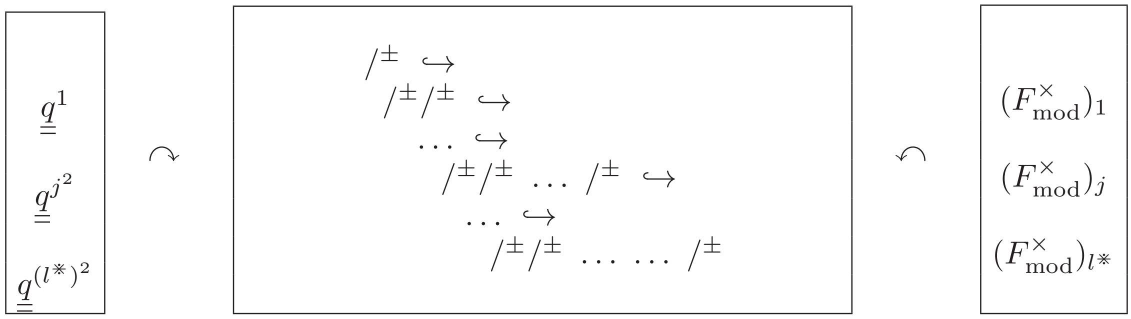

Fig. I.6: The full multiradial representation

One important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, *log-shells*, *splitting monoids of LGP-monoids*, and *number fields* — of this multiradial representation and the **$\mathfrak{log}$-Kummer correspondence** of the specific *vertical line* of the log-theta-lattice under consideration. This property — which may be thought of as a sort of **"non-interference"**, or **"mutual compatibility"**, property — asserts that the multiplicative monoids constituted by the splitting monoids of LGP-monoids and copies of $F^{\times}_{\mathrm{mod}}$ *"do not interfere"*, relative to the various arrows that occur in the $\mathfrak{log}$-*Kummer correspondence*, with the *local units* at $\underline{v} \in \underline{\mathbb{V}}$ that give rise to the *log-shells*. In the case of splitting monoids of LGP-monoids, this *non-interference/mutual compatibility* property is, in essence, a formal consequence of the existence of the **canonical splittings** [up to roots of unity] of the *theta/Gaussian monoids* that appear into *unit group* and *value group* portions [cf. the discussion of [IUTchII], Introduction]. Here, we recall that, in the case of the theta monoids, these canonical splittings are, in essence, a formal consequence of the **constant multiple rigidity** property of mono-theta environments reviewed above. In the case of copies of F_{mod}, this *non-interference/mutual compatibility* property is, in essence, a formal consequence of the well-known fact in elementary algebraic number theory that any nonzero element of a number field that is **integral** at every valuation of the number field is necessarily a **root of unity**. These mutual compatibility properties are described in detail in Theorem 3.11, (ii), and summarized in Theorem A, (ii), below.

Another important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, *log-shells*, *splitting monoids of LGP-monoids*, and *number fields* — of this multiradial representation and the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links**, i.e., the *horizontal arrows* of the log-theta-lattice under consideration. This property — which may be thought of as a property of **compatibility** with the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link — asserts that the *cyclotomic rigidity isomorphisms* that appear in the Kummer theory surrounding the splitting monoids of LGP-monoids and copies of $F^{\times}_{\mathrm{mod}}$ are *immune* to the $\widehat{\mathbb{Z}}^{\times}$-*indeterminacies* that act on the copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that arise in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In the case of splitting monoids of LGP-monoids, this property amounts precisely to the *multiradiality* theory developed in §2 [cf. the above

discussion], i.e., in essence, to the **mono-theta-theoretic cyclotomic rigidity** property reviewed in the above discussion. In the case of copies of $F^{\times}_{\mathrm{mod}}$, this property follows from the theory surrounding the construction of the cyclotomic rigidity isomorphisms discussed in [IUTchI], Example 5.1, (v). These compatibility properties are described in detail in Theorem 3.11, (iii), and summarized in Theorem A, (iii), below.

At this point, we pause to observe that although considerable attention has been devoted so far in the present series of papers, especially in [IUTchII], to the theory of *Gaussian monoids*, not so much attention has been devoted [i.e., outside of [IUTchI], §5; [IUTchII], Corollaries 4.7, 4.8] to [the multiplicative monoids constituted by] copies of $F^{\times}_{\mathrm{mod}}$. These copies of $F^{\times}_{\mathrm{mod}}$ enter into the theory of the *multiradial representation* discussed above in the form of various types of *global Frobenioids* in the following way. If one starts from the *number field* F_{mod}, one natural Frobenioid that can be associated to F_{mod} is the Frobenioid $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ of [stack-theoretic] *arithmetic line bundles* on [the spectrum of the ring of integers of] F_{mod} discussed in [IUTchI], Example 5.1, (iii) [cf. also Example 3.6 of the present paper]. From the point of view of the theory surrounding the *multiradial representation* discussed above, there are *two natural ways* to approach the construction of "$\mathcal{F}^{\circledast}_{\mathrm{mod}}$":

($\circledast_{\mathrm{MOD}}$) **(Rational Function Torsor Version)**: This approach consists of considering the category $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ of $F^{\times}_{\mathrm{mod}}$-*torsors* equipped with *trivializations* at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. Example 3.6, (i), for more details].

($\circledast_{\mathfrak{mod}}$) **(Local Fractional Ideal Version)**: This approach consists of considering the category $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ of collections of *integral structures* on the various completions $K_{\underline{v}}$ at $\underline{v} \in \underline{\mathbb{V}}$ and morphisms between such collections of integral structures that arise from multiplication by elements of $F^{\times}_{\mathrm{mod}}$ [cf. Example 3.6, (ii), for more details].

Then one has *natural isomorphisms of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}} \quad \stackrel{\sim}{\to} \quad \mathcal{F}^{\circledast}_{\mathrm{MOD}} \quad \stackrel{\sim}{\to} \quad \mathcal{F}^{\circledast}_{\mathfrak{mod}}$$

that induce the respective *identity morphisms* $F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10]. In particular, at first glance, $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ and $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ appear to be "essentially equivalent" objects.

On the other hand, when regarded from the point of view of the *multiradial representations* discussed above, these two constructions exhibit a number of significant differences — cf. Fig. I.7 below; the discussion of Remarks 3.6.2, 3.10.1. For instance, whereas the construction of ($\circledast_{\mathrm{MOD}}$) depends only on the **multiplicative** structure of $F^{\times}_{\mathrm{mod}}$, the construction of ($\circledast_{\mathfrak{mod}}$) involves the *module*, i.e., the **additive**, structure of the localizations $K_{\underline{v}}$. The global portion of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* (respectively, the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-*link*) is, by definition [cf. Definition 3.8, (ii)], constructed by means of the *realification* of the Frobenioid that appears in the construction of ($\circledast_{\mathrm{MOD}}$) (respectively, ($\circledast_{\mathfrak{mod}}$)). This means that the construction of the global portion of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link — which is the version of the Θ-link that is in fact ultimately used in the theory of the multiradial representation — depends only on the *multiplicative* monoid structure of a copy of $F^{\times}_{\mathrm{mod}}$, together with the various valuation

homomorphisms $F^{\times}_{\mathrm{mod}} \to \mathbb{R}$ associated to $\underline{v} \in \underline{\mathbb{V}}$. Thus, the *mutual compatibility* [discussed above] of copies of $F^{\times}_{\mathrm{mod}}$ with the $\mathfrak{log}$-*Kummer correspondence* implies that one may perform this construction of the global portion of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link in a fashion that is *immune* to the *"upper semi-compatibility" indeterminacy* (Ind3) [discussed above]. By contrast, the construction of $(\circledast_{\mathfrak{mod}})$ involves integral structures on the underlying local *additive* modules "$K_{\underline{v}}$", i.e., from the point of view of the multiradial representation, integral structures on *log-shells* and *tensor packets* of log-shells, which *are* subject to the "upper semi-compatibility" indeterminacy (Ind3) [discussed above]. In particular, the $\mathfrak{log}$-Kummer correspondence subjects the construction of $(\circledast_{\mathfrak{mod}})$ to *"substantial distortion"*. On the other hand, the essential role played by local integral structures in the construction of $(\circledast_{\mathfrak{mod}})$ enables one to compute the *global arithmetic degree* of the arithmetic line bundles constituted by objects of the category "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" in terms of **log-volumes** on **log-shells** and **tensor packets** of log-shells [cf. Proposition 3.9, (iii)]. This property of the construction of $(\circledast_{\mathfrak{mod}})$ will play a *crucial role* in deriving the **explicit estimates** for such log-volumes that are obtained in Corollary 3.12 [cf. Theorem B below].

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$	$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$
biased toward **multiplicative** structures	biased toward **additive** structures
easily related to **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link	easily related to **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, i.e., **mono-analytic log-shells**
admits **precise $\mathfrak{log}$-Kummer correspondence**	only admits **"upper semi-compatible" $\mathfrak{log}$-Kummer correspondence**
rigid, but not suited to explicit computation	subject to **substantial distortion**, but suited to **explicit estimates**

Fig. I.7: $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ versus $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$

Thus, in summary, the natural isomorphism $\mathcal{F}^{\circledast}_{\mathrm{MOD}} \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}$ discussed above plays the important role, in the context of the *multiradial representation* discussed above, of *relating*

- the **multiplicative** structure of the global number field F_{mod} to the **additive** structure of F_{mod},

· the **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link to the **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

Finally, in Corollary 3.12 [cf. also Theorem B below], we apply the *multiradial representation* discussed above to estimate certain *log-volumes* as follows. We begin by introducing some terminology [cf. Definition 3.8, (i)]. We shall refer to the object that arises in any of the versions [including *realifications*] of the global Frobenioid "$\mathcal{F}^{\circledast}_{\mathrm{mod}}$" discussed above — such as, for instance, the global realified Frobenioid that occurs in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link — by considering the arithmetic divisor determined by the zero locus of the elements "$\underline{\underline{q}}_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ as a *q-pilot object*. The **log-volume** of the ***q*-pilot object** will be denoted by

$$- \, |\log(\underline{\underline{q}})| \;\in\; \mathbb{R}$$

— so $|\log(\underline{\underline{q}})| > 0$ [cf. Corollary 3.12; Theorem B]. In a similar vein, we shall refer to the object that arises in the global realified Frobenioid that occurs in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link by considering the arithmetic divisor determined by the zero locus of the collection of *theta values* "$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ as a Θ-*pilot object*. The **log-volume** of the **holomorphic hull** — cf. Remark 3.9.5, (i); Step (xi) of the proof of Corollary 3.12 — of the **union** of the collection of **possible images** of the **Θ-pilot object** in the **multiradial representation** — i.e., where we recall that these "possible images" are subject to the **indeterminacies** (Ind1), (Ind2), (Ind3) — will be denoted by

$$- \, |\log(\underline{\underline{\Theta}})| \;\in\; \mathbb{R} \bigcup \{+\infty\}$$

[cf. Corollary 3.12; Theorem B]. Here, the reader might find the use of the notation "$-$" and "$|\dots|$" confusing [i.e., since this notation suggests that $-\,|\log(\underline{\underline{\Theta}})|$ is a *non-positive real number*, which would appear to imply that the possibility that $-\,|\log(\underline{\underline{\Theta}})| = +\infty$ may be excluded from the outset]. The reason for the use of this notation, however, is to express the point of view that $-\,|\log(\underline{\underline{\Theta}})|$ should be regarded as a *positive real multiple* of $-\,|\log(\underline{\underline{q}})|$ [i.e., which is indeed a *negative real number*!] plus a *possible error term*, which [*a priori*!] might be equal to $+\infty$. Then the content of Corollary 3.12, Theorem B may be summarized, roughly speaking [cf. Remark 3.12.1, (ii)], as a result concerning the

negativity *of the* **Θ-pilot log-volume** $|\log(\underline{\underline{\Theta}})|$

— i.e., where we write $|\log(\underline{\underline{\Theta}})| \stackrel{\mathrm{def}}{=} -(-\,|\log(\underline{\underline{\Theta}})|) \;\in\; \mathbb{R} \bigcup \{-\infty\}$. Relative to the analogy between the theory of the present series of papers and *complex/p-adic Teichmüller theory* [cf. [IUTchI], §I4], this result may be thought of as a statement to the effect that

> *"the pair consisting of a number field equipped with an elliptic curve is* **metrically hyperbolic***, i.e., has* **negative curvature***".*

That is to say, it may be thought of as a sort of analogue of the inequality

$$\chi_S \;=\; -\int_S d\mu_S \;<\; 0$$

arising from the classical **Gauss-Bonnet formula** on a hyperbolic Riemann surface of finite type S [where we write χ_S for the *Euler characteristic* of S and $d\mu_S$ for the Kähler metric on S determined by the *Poincaré metric* on the upper half-plane — cf. the discussion of Remark 3.12.3], or, alternatively, of the inequality

$$(1-p)(2g_X - 2) \ \leq \ 0$$

that arises by computing *global degrees of line bundles* in the context of the **Hasse invariant** that arises in p-adic Teichmüller theory [where X is a *smooth, proper hyperbolic curve* of genus g_X over the ring of Witt vectors of a perfect field of characteristic p which is *canonical* in the sense of p-adic Teichmüller theory — cf. the discussion of Remark 3.12.4, (v)].

The proof of Corollary 3.12 [i.e., Theorem B] is based on the following *fundamental observation*: the **multiradial representation** discussed above yields

two tautologically equivalent *ways to compute*
the **q-pilot log-volume** $-\ |\log(\underline{\underline{q}})|$

— cf. Fig. I.8 below; Step (xi) of the proof of Corollary 3.12. That is to say, suppose that one starts with the **q-pilot object** in the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ at $(1,0)$, which we think of as being represented, via the approach of $(\circledast_{\mathfrak{mod}})$, by means of the action of the various $\underline{\underline{q}}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, on the **log-shells** that arise, via the **$\mathfrak{log}$-link** ${}^{1,-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, from the various local "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" in the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{1,-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ at $(1,-1)$. Thus, if one considers the *value group* "$(-)^{\Vdash\blacktriangleright}$" and *unit group* "$(-)^{\vdash\times\boldsymbol{\mu}}$" portions of the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link ${}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ in the context of the *arithmetic holomorphic structure* of the vertical line $(1,\circ)$, this action on log-shells may be thought of as a somewhat **intricate "intertwining"** between these value group and unit group portions [cf. Remark 3.12.2, (ii)]. On the other hand, the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link ${}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ constitutes a sort of **gluing isomorphism** between the *arithmetic holomorphic structures* associated to the vertical lines $(0,\circ)$ and $(1,\circ)$ that is based on

> **forgetting this intricate intertwining**, i.e., by working solely with *abstract isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips.*

Thus, in order to relate the arithmetic holomorphic structures, say, at $(0,0)$ and $(1,0)$, one must apply the *multiradial representation* discussed above. That is to say, one starts by applying the theory of **bi-coric mono-analytic log-shells** given in Theorem 1.5. One then applies the **Kummer theory** surrounding the **splitting monoids of theta/Gaussian monoids** and copies of the **number field** F_{mod}, which allows one to pass from the **Frobenius-like** versions of various objects that appear in — i.e., that are necessary in order to consider — the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* to the corresponding **étale-like** versions of these objects that appear in the *multiradial representation*. This passage from Frobenius-like versions to étale-like versions is referred to as the operation of **Kummer-detachment** [cf. Fig. I.8; Remark 1.5.4, (i)]. As discussed above, this operation of Kummer-detachment is possible precisely

as a consequence of the **compatibility** of the multiradial representation with the *indeterminacies* (Ind1), (Ind2), (Ind3), hence, in particular, with the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**. Here, we recall that since the log-theta-lattice is, as discussed above, *far from commutative*, in order to represent the various *"*$\mathfrak{log}$*-link-conjugates"* at $(0, m)$ [for $m \in \mathbb{Z}$] in terms that may be understood from the point of view of the arithmetic holomorphic structure at $(1, 0)$, one must work [not only with the Kummer isomorphisms at a *single* $(0, m)$, but rather] with the **entire** $\mathfrak{log}$**-Kummer correspondence**. In particular, one must take into account the *indeterminacy* (Ind3). Once one completes the operation of Kummer-detachment so as to obtain *vertically coric* versions of objects on the vertical line $(0, \circ)$, one then passes to *multiradial objects*, i.e., to the "final form" of the *multiradial representation*, by taking into account [once again] the *indeterminacy* (Ind1), i.e., that arises from working with *[mono-analytic!]* $\mathcal{D}^{\vdash}$- [as opposed to $\mathcal{D}$-!] *prime-strips*. Finally, one computes the **log-volume** of the holomorphic hull of this "final form" multiradial representation of the Θ-pilot object — i.e., subject to the *indeterminacies* (Ind1), (Ind2), (Ind3)! — and concludes the desired estimates from the *tautological observation* that

> *the* **log-theta-lattice** *— and, in particular, the* **"gluing isomorphism"** *constituted by the* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** *— were* **constructed** *precisely in such a way as to ensure that the computation of the log-volume of the holomorphic hull of the union of the collection of possible images of the* Θ*-pilot object [cf. the definition of* $|\log(\underline{\underline{\Theta}})|$*] necessarily* **amounts to a computation of** *[an upper bound for]* $|\log(\underline{\underline{q}})|$

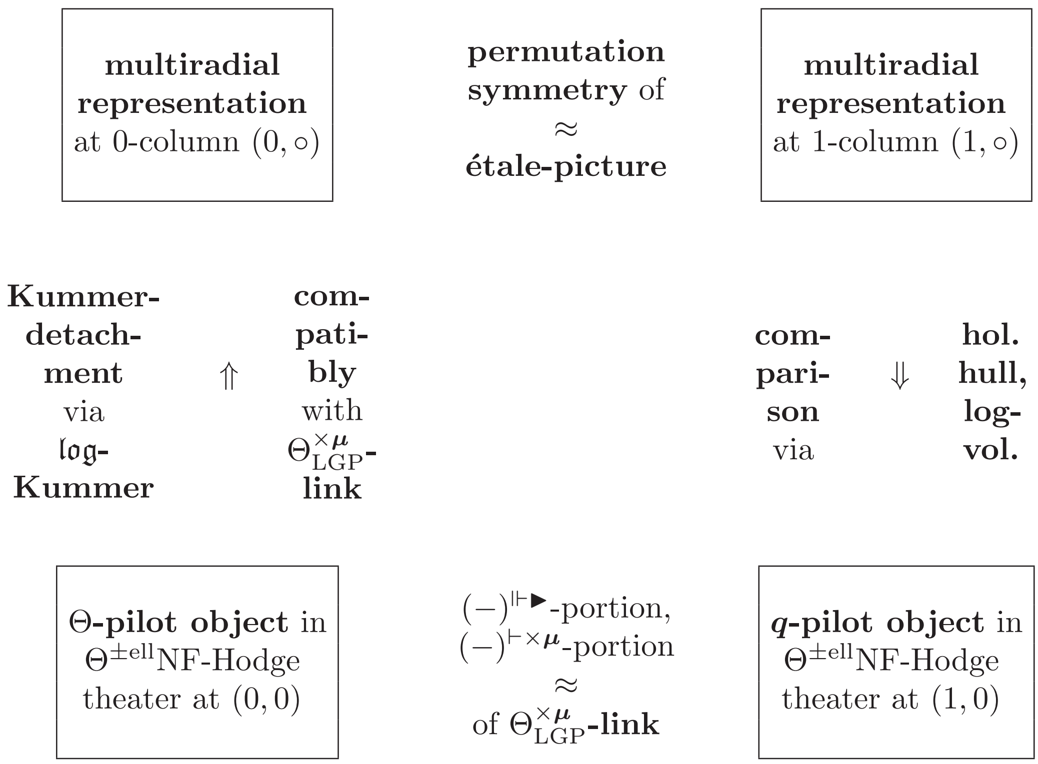

Fig. I.8: Two tautologically equivalent ways to compute the log-volume of the q-pilot object at $(1, 0)$

— cf. Fig. I.8; Step (xi) of the proof of Corollary 3.12. That is to say, the "gluing isomorphism" constituted by the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link relates two distinct *"arithmetic holomorphic structures"*, i.e., two distinct *copies* of conventional ring/scheme theory, that are glued together precisely by means of a relation that identifies the Θ-*pilot object* in the *domain* of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link with the q-*pilot object* in the *codomain* of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Thus, once one sets up such an *apparatus*, the computation of the log-volume of the holomorphic hull of the union of possible images of the Θ-pilot object in the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link in terms of the q-pilot object in the codomain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link amounts — *tautologically*! — to the computation of the log-volume of the q-pilot object [in the codomain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link] in terms of *itself*, i.e., to a computation that reflects certain *intrinsic properties* of this q-pilot object. This is the content of Corollary 3.12 [i.e., Theorem B]. As discussed above, this sort of *"computation of intrinsic properties"* in the present context of a number field equipped with an elliptic curve may be regarded as analogous to the "computations of intrinsic properties" reviewed above in the classical complex and p-adic cases.

We conclude the present Introduction with the following summaries of the *main results* of the present paper.

Theorem A. (Multiradial Algorithms for Logarithmic Gaussian Procession Monoids) *Fix a collection of* **initial Θ-data** $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ *as in [IUTchI], Definition 3.1. Let*

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$$

be a **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from a* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. For each $n \in \mathbb{Z}$, write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater determined, up to isomorphism, by the various ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, where $m \in \mathbb{Z}$, via the **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Multiradial Representation)** *Write*

$${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$$

for the **collection of data** *consisting of*

(a) **tensor packets** *of* **log-shells***;*

(b) **splitting monoids** *of* **LGP-monoids** *acting on the tensor packets of (a);*

(c) copies, labeled by $j \in \mathbb{F}^{\circledast}_l$, of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} *acting on the tensor packets of (a)*

[cf. Theorem 3.11, (i), (a), (b), (c), for more details] regarded up to **indeterminacies** *of the following two types:*

(Ind1) *the indeterminacies induced by the* **automorphisms** *of the* **procession of $\mathcal{D}^{\vdash}$-prime-strips** $\text{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$ *that gives rise to the tensor packets of (a);*

(Ind2) *the [“non-*(Ind1) *portion” of the] indeterminacies that arise from the* **automorphisms of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips** *that appear in the* $\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$*-link, i.e., in particular, at [for simplicity]* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$*, the* $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** *acting on local copies of “$\mathcal{O}^{\times\boldsymbol{\mu}}$”*

— cf. Theorem 3.11, (i), for more details. Then ${}^{n,\circ}\mathfrak{R}^{\text{LGP}}$ *may be constructed via an* **algorithm** *in the procession of $\mathcal{D}^{\vdash}$-prime-strips* $\text{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*, which is* **functorial** *with respect to isomorphisms of processions of $\mathcal{D}^{\vdash}$-prime-strips. For* $n, n' \in \mathbb{Z}$*, the* **permutation symmetries** *of the* **étale-picture** *discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce* **compatible poly-isomorphisms**

$$\text{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T) \ \stackrel{\sim}{\rightarrow}\ \text{Prc}({}^{n',\circ}\mathfrak{D}^{\vdash}_T); \quad {}^{n,\circ}\mathfrak{R}^{\text{LGP}} \ \stackrel{\sim}{\rightarrow}\ {}^{n',\circ}\mathfrak{R}^{\text{LGP}}$$

which are, moreover, compatible with the **bi-coricity** *poly-isomorphisms*

$${}^{n,\circ}\mathcal{D}^{\vdash}_0 \ \stackrel{\sim}{\rightarrow}\ {}^{n',\circ}\mathcal{D}^{\vdash}_0$$

of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].

(ii) **(log-Kummer Correspondence)** *For* $n, m \in \mathbb{Z}$*, the inverses of the* **Kummer isomorphisms** *associated to the various* **$\mathcal{F}$-prime-strips** *and* **NF-bridges** *that appear in the* $\Theta^{\pm\text{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ *induce “inverse Kummer”* **isomorphisms** *between the* **vertically coric** *data (a), (b), (c) of (i) and the corresponding* **Frobenioid-theoretic** *data arising from each* $\Theta^{\pm\text{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ *[cf. Theorem 3.11, (ii), (a), (b), (c), for more details]. Moreover, as one varies* $m \in \mathbb{Z}$*, the corresponding* **Kummer isomorphisms** *[i.e., inverses of “inverse Kummer” isomorphisms] of* **splitting monoids** *of* **LGP-monoids** *[cf. (i), (b)] and labeled copies of the* **number field** F_{mod} *[cf. (i), (c)] are* **mutually compatible***, relative to the* **log-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the [Frobenioid-theoretic] domains of these Kummer isomorphisms that are possibly related to one another via the log-links consist of* **roots of unity** *in the domains of the log-links [multiplication by which corresponds, via the log-link, to an* **“addition by zero”** *indeterminacy, i.e., to* **no indeterminacy***!] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii); Theorem 3.11, (ii), for more details. On the other hand, the Kummer isomorphisms of* **tensor packets** *of* **log-shells** *[cf. (i), (a)] are subject to a certain* **“indeterminacy”** *as follows:*

(Ind3) *as one varies* $m \in \mathbb{Z}$*, these Kummer isomorphisms of tensor packets of log-shells are* **“upper semi-compatible”***, relative to the* **log-links** *of the*

n-th column of the LGP*-Gaussian log-theta-lattice under consideration, in a sense that involves certain* **natural inclusions "$\subseteq$"** *at $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$ and certain* **natural surjections "$\twoheadrightarrow$"** *at $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ — cf. Proposition 3.5, (ii), (a), (b); Theorem 3.11, (ii), for more details.*

Finally, as one varies $m \in \mathbb{Z}$, these Kummer isomorphisms of tensor packets of log-shells are [precisely!] **compatible***, relative to the* **log-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, with the respective* **log-volumes** *[cf. Proposition 3.9, (iv)].*

(iii) **($\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-Link Compatibility)** *The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various* **horizontal arrows** *— i.e., $\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-links — of the* LGP*-Gaussian log-theta-lattice under consideration as follows: The* **tensor packets** *of* **log-shells** *[cf. (i), (a)] are compatible, relative to the relevant Kummer isomorphisms, with [the unit group portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of] the $\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-link [cf. the indeterminacy "*(Ind2)*" of (i)]; we refer to Theorem 3.11, (iii), (a), (b), for more details. The identity automorphism on the objects that appear in the construction of the* **splitting monoids** *of* **LGP-monoids** *via mono-theta environments [cf. (i), (b)] is compatible, relative to the relevant Kummer isomorphisms and isomorphisms of mono-theta environments, with the $\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-link [cf. the indeterminacy "*(Ind2)*" of (i)]; we refer to Theorem 3.11, (iii), (c), for more details. The identity automorphism on the objects that appear in the construction of the labeled copies of the* **number field** *F_{mod} [cf. (i), (c)] is compatible, relative to the relevant Kummer isomorphisms and cyclotomic rigidity isomorphisms [cf. the discussion of Remark 2.3.2; the constructions of [IUTchI], Example 5.1, (v)], with the $\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-link [cf. the indeterminacy "*(Ind2)*" of (i)]; we refer to Theorem 3.11, (iii), (d), for more details.*

Theorem B. (Log-volume Estimates for Multiradially Represented Splitting Monoids of Logarithmic Gaussian Procession Monoids) *Suppose that we are in the situation of Theorem A. Write*

$$-\,|\log(\underline{\underline{\Theta}})| \;\in\; \mathbb{R} \bigcup \{+\infty\}$$

for the **procession-normalized mono-analytic log-volume** *[where the average is taken over $j \in \mathbb{F}_l^{\circledast}$ — cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a), for more details] of the* **holomorphic hull** *[cf. Remark 3.9.5, (i)] of the* **union** *of the* **possible images** *of a* **Θ-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorems A, (ii); 3.11, (ii)], in the* **multiradial representation** *of Theorems A, (i); 3.11, (i), which we regard as* **subject** *to the* **indeterminacies** (Ind1), (Ind2), (Ind3) *described in Theorems A, (i), (ii); 3.11, (i), (ii). Write*

$$-\,|\log(\underline{\underline{q}})| \;\in\; \mathbb{R}$$

for the **procession-normalized mono-analytic log-volume** *of the image of a* **$\boldsymbol{q}$-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorems A, (ii); 3.11, (ii)], in the* **multiradial representation** *of*

Theorems A, (i); 3.11, (i), which we do **not** *regard as subject to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in Theorems A, (i), (ii); 3.11, (i), (ii). Here, we recall the definition of the symbol "$\triangle$" as the result of identifying the* **labels**

$$\text{“}0\text{” and “}\langle \mathbb{F}_l^{\circledast} \rangle\text{”}$$

[cf. [IUTchII], Corollary 4.10, (i)]. In particular, $|\log(\underline{\underline{q}})| > 0$ *is easily computed in terms of the various* ***q*-parameters** *of the elliptic curve* E_F *[cf. [IUTchI], Definition 3.1, (b)] at* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ $(\neq \emptyset)$*. Then it holds that* $-|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$*, and*

$$-\ |\log(\underline{\underline{\Theta}})| \ \geq \ -\ |\log(\underline{\underline{q}})|$$

— i.e., $C_\Theta \geq -1$ *for any real number* $C_\Theta \in \mathbb{R}$ *such that* $-|\log(\underline{\underline{\Theta}})| \ \leq \ C_\Theta \cdot |\log(\underline{\underline{q}})|$.

Acknowledgements:

The research discussed in the present paper profited enormously from the generous support that the author received from the *Research Institute for Mathematical Sciences*, a Joint Usage/Research Center located in Kyoto University. At a personal level, I would like to thank *Fumiharu Kato*, *Akio Tamagawa*, *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, *Ivan Fesenko*, *Fucheng Tan*, *Emmanuel Lepage*, *Arata Minamide*, and *Wojciech Porowski* for many stimulating discussions concerning the material presented in this paper. Also, I feel deeply indebted to *Go Yamashita*, *Mohamed Saïdi*, and *Yuichiro Hoshi* for their meticulous reading of and numerous comments concerning the present paper. Finally, I would like to express my deep gratitude to *Ivan Fesenko* for his quite substantial efforts to disseminate — for instance, in the form of a survey that he wrote — the theory discussed in the present series of papers.

Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: The Log-theta-lattice

In the present §1, we discuss various enhancements to the theory of **log-shells**, as developed in [AbsTopIII]. In particular, we develop the theory of the **$\mathfrak{log}$-link** [cf. Definition 1.1; Propositions 1.2, 1.3], which, together with the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-links of [IUTchII], Corollary 4.10, (iii), leads naturally to the construction of the **log-theta-lattice**, an apparatus that is *central* to the theory of the present series of papers. We conclude the present §1 with a discussion of various **coric structures** associated to the log-theta-lattice [cf. Theorem 1.5].

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. We begin by reviewing various aspects of the theory of *log-shells* developed in [AbsTopIII].

Definition 1.1. Let

$$^{\dagger}\mathfrak{F} = \{^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}$*-prime-strip* [relative to the given initial Θ-data — cf. [IUTchI], Definition 5.2, (i)]. Write

$$^{\dagger}\mathfrak{F}^{\vdash} = \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad ^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}} = \{^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad ^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the associated $\mathcal{F}^{\vdash}$*-*, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-*, $\mathcal{D}$*-prime-strips* [cf. [IUTchI], Remark 5.2.1, (ii); [IUTchII], Definition 4.9, (vi), (vii); [IUTchI], Remark 5.2.1, (i)]. Recall the *functorial algorithm* of [IUTchII], Corollary 4.6, (i), in the $\mathcal{F}$-prime-strip $^{\dagger}\mathfrak{F}$ for constructing the assignment $\Psi_{\text{cns}}(^{\dagger}\mathfrak{F})$ given by

$$\underline{\mathbb{V}}^{\text{non}} \ni \underline{v} \ \mapsto \ \Psi_{\text{cns}}(^{\dagger}\mathfrak{F})_{\underline{v}} \ \stackrel{\text{def}}{=} \ \left\{ G_{\underline{v}}(^{\dagger}\Pi_{\underline{v}}) \ \curvearrowright \ \Psi_{^{\dagger}\mathcal{F}_{\underline{v}}} \right\}$$
$$\underline{\mathbb{V}}^{\text{arc}} \ni \underline{v} \ \mapsto \ \Psi_{\text{cns}}(^{\dagger}\mathfrak{F})_{\underline{v}} \ \stackrel{\text{def}}{=} \ \Psi_{^{\dagger}\mathcal{F}_{\underline{v}}}$$

— where the data in brackets "$\{-\}$" is to be regarded as being well-defined only up to a $^{\dagger}\Pi_{\underline{v}}$*-conjugacy indeterminacy* [cf. [IUTchII], Corollary 4.6, (i), for more details]. In the following, we shall write

$$(-)^{\underline{\text{gp}}} \ \stackrel{\text{def}}{=} \ (-)^{\text{gp}} \bigcup \{0\}$$

for the *formal union* with $\{0\}$ of the groupification $(-)^{\text{gp}}$ of a [multiplicatlvey written] monoid "$(-)$". Thus, by setting the product of all elements of $(-)^{\underline{\text{gp}}}$ with 0 to be equal to 0, one obtains a natural monoid structure on $(-)^{\underline{\text{gp}}}$.

(i) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Write

$$(\Psi_{^{\dagger}\mathcal{F}_{\underline{v}}} \ \supseteq \ \Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}} \ \twoheadrightarrow) \quad \Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}} \ \stackrel{\text{def}}{=} \ (\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}})^{\text{pf}}$$

for the *perfection* $(\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}})^{\text{pf}}$ of the submonoid of units $\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ of $\Psi_{^{\dagger}\mathcal{F}_{\underline{v}}}$. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 3.1, (iv); [AbsTopIII], Proposition 3.2, (iii), (v)] that the natural, algorithmically constructible

ind-topological field structure on $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ allows one to define a $p_{\underline{v}}$*-adic logarithm* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, which, in turn, yields a *functorial algorithm* in the Frobenioid ${}^\dagger\mathcal{F}_{\underline{v}}$ for constructing an *ind-topological field structure* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. Write

$$\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$$

for the resulting *multiplicative monoid* of nonzero integers. Here, we observe that the resulting *diagram*

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \rightarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

is *compatible* with the various *natural actions* of ${}^\dagger\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$ on each of the [four] "Ψ's" appearing in this diagram. The pair $\{{}^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}\}$ now determines a *Frobenioid*

$$\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

[cf. [AbsTopIII], Remark 3.1.1; [IUTchI], Remark 3.3.2] — which is, in fact, *naturally isomorphic* to the Frobenioid ${}^\dagger\mathcal{F}_{\underline{v}}$, but which we wish to think of as being related to ${}^\dagger\mathcal{F}_{\underline{v}}$ via the *above diagram*. We shall denote this diagram by means of the notation

$${}^\dagger\mathcal{F}_{\underline{v}} \overset{\mathfrak{log}}{\longrightarrow} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

and refer to this relationship between ${}^\dagger\mathcal{F}_{\underline{v}}$ and $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^\dagger\mathcal{F}_{\underline{v}}$ [or, when ${}^\dagger\mathfrak{F}$ is fixed, *at* $\underline{v}$]. If $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \overset{\sim}{\rightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$ is any [poly-]isomorphism of Frobenioids, then we shall write

$${}^\dagger\mathcal{F}_{\underline{v}} \overset{\mathfrak{log}}{\longrightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to ${}^\dagger\mathcal{F}_{\underline{v}}$ with the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \overset{\sim}{\rightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$ and refer to this relationship between ${}^\dagger\mathcal{F}_{\underline{v}}$ and ${}^\ddagger\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** from ${}^\dagger\mathcal{F}_{\underline{v}}$ to ${}^\ddagger\mathcal{F}_{\underline{v}}$; when the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \overset{\sim}{\rightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full* $\mathfrak{log}$*-link* from ${}^\dagger\mathcal{F}_{\underline{v}}$ to ${}^\ddagger\mathcal{F}_{\underline{v}}$. Finally, we recall from [AbsTopIII], Definition 3.1, (iv), that the image in $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ of the submonoid of $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-invariants* of $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ constitutes a *compact topological module*, which we shall refer to as the *pre-log-shell*. Write $p^*_{\underline{v}} \overset{\mathrm{def}}{=} p_{\underline{v}}$ when $p_{\underline{v}}$ is *odd* and $p^*_{\underline{v}} \overset{\mathrm{def}}{=} p^2_{\underline{v}}$ when $p_{\underline{v}}$ is *even*. Then we shall refer to the result of multiplying the pre-log-shell by the factor $(p^*_{\underline{v}})^{-1}$ as the **log-shell**

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

[cf. [AbsTopIII], Definition 5.4, (iii)]. In particular, by applying the natural, algorithmically constructible ind-topological field structure on $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$ [cf. [AbsTopIII], Proposition 3.2, (iii)], it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$*-link*

$${}^\dagger\mathcal{F}_{\underline{v}} \overset{\mathfrak{log}}{\longrightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$$

— i.e., an object that is determined by the image of a certain portion [namely, the $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$-*invariants* of $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}$] of the *domain* of this $\mathfrak{log}$-link.

(ii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. For $N \in \mathbb{N}_{\geq 1}$, write $\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ for the subgroup of *N-th roots of unity* and $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ for the *[pointed] universal covering* of the topological group determined by the *groupification* $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ of the topological monoid $\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}}$. Then one verifies immediately that one may think of the composite covering of topological groups

$$\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$$

— where the second "$\twoheadrightarrow$" is the natural surjection — as a *[pointed] universal covering* of $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. That is to say, one may think of $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ as an *object constructed from* $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ [cf. also Remark 1.2.1, (i), below]. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 4.1, (iv); [AbsTopIII], Proposition 4.2, (i), (ii)] that the natural, algorithmically constructible [i.e., starting from the collection of data ${}^\dagger\mathcal{F}_{\underline{v}}$ — cf. [IUTchI], Definition 5.2, (i), (b)] *topological field structure* on $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ allows one to define a *[complex archimedean] logarithm* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, which, in turn, yields a *functorial algorithm* in the collection of data ${}^\dagger\mathcal{F}_{\underline{v}}$ [cf. [IUTchI], Definition 5.2, (i), (b)] for constructing a *topological field structure* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, together with a $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$-*Kummer structure* on ${}^\dagger\mathbb{U}_{\underline{v}} \stackrel{\mathrm{def}}{=} {}^\dagger\mathcal{D}_{\underline{v}}$ [cf. [AbsTopIII], Definition 4.1, (iv); [IUTchII], Proposition 4.4, (i)]. Write

$$\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$$

for the resulting *multiplicative monoid* of nonzero integral elements [i.e., elements of norm ≤ 1]. Here, we observe that the resulting *diagram*

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadleftarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

is *compatible* [cf. the discussion of [AbsTopIII], Definition 4.1, (iv)] with the *co-holomorphicizations* determined by the *natural* $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$-*Kummer* [cf. [IUTchII], Proposition 4.4, (i)] and $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$-*Kummer* [cf. the above discussion] *structures* on ${}^\dagger\mathbb{U}_{\underline{v}}$. The triple of data consisting of the topological monoid $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$, the Aut-holomorphic space ${}^\dagger\mathbb{U}_{\underline{v}}$, and the $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$-Kummer structure on ${}^\dagger\mathbb{U}_{\underline{v}}$ discussed above determines a *collection of data* [i.e., as in [IUTchI], Definition 5.2, (i), (b)]

$$\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

which is, in fact, *naturally isomorphic* to the collection of data ${}^\dagger\mathcal{F}_{\underline{v}}$, but which we wish to think of as being related to ${}^\dagger\mathcal{F}_{\underline{v}}$ via the *above diagram.* We shall denote this diagram by means of the notation

$${}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

and refer to this relationship between ${}^\dagger\mathcal{F}_{\underline{v}}$ and $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^\dagger\mathcal{F}_{\underline{v}}$ [or, when ${}^\dagger\mathfrak{F}$ is fixed, *at* $\underline{v}$]. If $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \stackrel{\sim}{\rightarrow} {}^\ddagger\mathcal{F}_{\underline{v}}$ is any

[poly-]isomorphism of collections of data [i.e., as in [IUTchI], Definition 5.2, (i), (b)], then we shall write

$$^{\dagger}\mathcal{F}_{\underline{v}} \stackrel{\mathfrak{log}}{\longrightarrow} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to ${}^{\dagger}\mathcal{F}_{\underline{v}}$ with the given [poly-]isomorphism $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ and refer to this relationship between ${}^{\dagger}\mathcal{F}_{\underline{v}}$ and ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** from ${}^{\dagger}\mathcal{F}_{\underline{v}}$ to ${}^{\ddagger}\mathcal{F}_{\underline{v}}$; when the given [poly-]isomorphism $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full* $\mathfrak{log}$*-link* from ${}^{\dagger}\mathcal{F}_{\underline{v}}$ to ${}^{\ddagger}\mathcal{F}_{\underline{v}}$. Finally, we recall from [AbsTopIII], Definition 4.1, (iv), that the submonoid of units $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ determines a *compact topological subquotient* of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, which we shall refer to as the *pre-log-shell*. We shall refer to the $\Psi^{\times}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the [uniquely determined] closed line segment of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ which is *preserved by multiplication by* ± 1 and whose endpoints differ by a *generator* of the kernel of the natural surjection $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ — or, equivalently, the $\Psi^{\times}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the result of *multiplying by* N the [uniquely determined] closed line segment of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ which is *preserved by multiplication by* ± 1 and whose endpoints differ by a *generator* of the kernel of the natural surjection $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ — as the **log-shell**

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$$

[cf. [AbsTopIII], Definition 5.4, (v)]. Thus, one may think of the *log-shell* as an *object constructed from* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. Moreover, by applying the natural, algorithmically constructible topological field structure on $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ $(= \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$, it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$*-link*

$$^{\dagger}\mathcal{F}_{\underline{v}} \stackrel{\mathfrak{log}}{\longrightarrow} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

— i.e., an object that is determined by the image of a certain portion [namely, the *subquotient* $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$] of the *domain* of this $\mathfrak{log}$-link.

(iii) Write

$$\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) \stackrel{\mathrm{def}}{=} \left\{ \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \right\}_{\underline{v} \in \underline{\mathbb{V}}}$$

for the collection of ind-topological modules constructed in (i), (ii) above indexed by $\underline{v} \in \underline{\mathbb{V}}$ — where the group structure arises from the *additive* portion of the field structures on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ discussed in (i), (ii); for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we regard $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ as equipped with its natural $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-action*. Write

$$\mathfrak{log}({}^{\dagger}\mathfrak{F}) \stackrel{\mathrm{def}}{=} \{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v} \in \underline{\mathbb{V}}}$$

for the $\mathcal{F}$-prime-strip determined by the data $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})$ constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$. We shall denote by

$$^{\dagger}\mathfrak{F} \stackrel{\mathfrak{log}}{\longrightarrow} \mathfrak{log}({}^{\dagger}\mathfrak{F})$$

the collection of diagrams $\{{}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$ constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$ and refer to this relationship between ${}^\dagger\mathfrak{F}$ and $\mathfrak{log}({}^\dagger\mathfrak{F})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^\dagger\mathfrak{F}$. If $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ is any [poly-]isomorphism of $\mathcal{F}$-prime-strips, then we shall write

$$ {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} $$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to ${}^\dagger\mathfrak{F}$ with the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ and refer to this relationship between ${}^\dagger\mathfrak{F}$ and ${}^\ddagger\mathfrak{F}$ as a **$\mathfrak{log}$-link** from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$; when the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full $\mathfrak{log}$-link* from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$. Finally, we shall write

$$ \mathcal{I}_{{}^\dagger\mathfrak{F}} \stackrel{\text{def}}{=} \{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

for the collection of log-shells constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$ and refer to this collection as the **log-shell** *associated to* ${}^\dagger\mathfrak{F}$ and [by a slight abuse of notation]

$$ \mathcal{I}_{{}^\dagger\mathfrak{F}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathfrak{F}) $$

for the collection of natural inclusions indexed by $\underline{v} \in \underline{\mathbb{V}}$. In particular, [cf. the discussion of (i), (ii)], it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$-*link*

$$ {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} $$

— i.e., an object that is determined by the image of a certain portion [cf. the discussion of (i), (ii)] of the *domain* of this $\mathfrak{log}$-link.

(iv) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Then observe that it follows immediately from the constructions of (i) that the *ind-topological modules with $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$-action* $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}})$ may be constructed *solely* from the collection of data ${}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ [i.e., the portion of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip ${}^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ labeled by $\underline{v}$]. That is to say, in light of the definition of a *$\times\boldsymbol{\mu}$-Kummer structure* [cf. [IUTchII], Definition 4.9, (i), (ii), (iv), (vi), (vii)], these constructions only require the *perfection "$(-)^{\text{pf}}$" of the units* and are *manifestly unaffected* by the operation of forming the quotient by a torsion subgroup of the units. Write

$$ \mathcal{I}_{{}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) $$

for the resulting ind-topological modules with $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$-action, *regarded as objects constructed from* ${}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$.

(v) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. Then by applying the algorithms for constructing "$k^\sim(G)$", "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (v), to the [object of the category "$\mathbb{TM}^\vdash$" of split topological monoids discussed in [IUTchI], Example 3.4, (ii), determined by the] *split Frobenioid* portion of the collection of data ${}^\dagger\mathcal{F}^\vdash_{\underline{v}}$, one obtains a *functorial algorithm* in the collection of data ${}^\dagger\mathcal{F}^\vdash_{\underline{v}}$ for constructing a *topological module* $\underline{\mathfrak{log}}({}^\dagger\mathcal{F}^\vdash_{\underline{v}})$ [i.e., corresponding to "$k^\sim(G)$"] and a *topological subspace* $\mathcal{I}_{{}^\dagger\mathcal{F}^\vdash_{\underline{v}}}$ [i.e., corresponding to "$\mathcal{I}(G)$"]. In fact, this functorial algorithm only makes use of the *unit portion* of this split Frobenioid, together with a *pointed universal covering*

of this unit portion. Moreover, by arguing as in (ii), one may in fact regard this functorial algorithm as an algorithm that only makes use of the *quotient of this unit portion by its N-torsion subgroup*, for $N \in \mathbb{N}_{\geq 1}$, together with a *pointed universal covering* of this quotient. That is to say, this functorial algorithm may, in fact, be regarded as a *functorial algorithm* in the collection of data ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ [cf. Remark 1.2.1, (i), below; [IUTchII], Definition 4.9, (v), (vi), (vii)]. Write

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}} \quad \subseteq \quad \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$$

for the resulting topological module equipped with a closed subspace, *regarded as objects constructed from* ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$.

(vi) Finally, just as in (iii), we shall write

$$\mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \quad \stackrel{\mathrm{def}}{=} \quad \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \quad \subseteq \quad \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \quad \stackrel{\mathrm{def}}{=} \quad \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the resulting collections of data constructed *solely* from the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ [i.e., which we do *not* regard as objects constructed from ${}^{\dagger}\mathfrak{F}$!].

Remark 1.1.1.

(i) Thus, $\mathfrak{log}$-links may be thought of as **correspondences** between *certain portions* of the ind-topological monoids in the *domain* of the $\mathfrak{log}$-link and *certain portions* of the ind-topological monoids in the *codomain* of the $\mathfrak{log}$-link. Frequently, in the theory of the present paper, we shall have occasion to consider **"iterates"** of $\mathfrak{log}$-links. The $\mathfrak{log}$-links — i.e., correspondences between certain portions of the ind-topological monoids in the domains and codomains of the $\mathfrak{log}$-links — that appear in such iterates are to be understood as being *defined only on the [local]* **units** [cf. also (ii) below, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$] that appear in the *domains of these* $\mathfrak{log}$*-links*. Thus, for instance, when considering [the nonzero elements of] a *global number field* embedded in an "idèlic" product [indexed by the set of all valuations of the number field] of localizations, we shall regard the $\mathfrak{log}$-links that appear as being *defined only on the product* [indexed by the set of all valuations of the number field] *of the groups of local units* that appear in the domains of these $\mathfrak{log}$-links. Indeed, in the theory of the present paper, the *only reason* for the introduction of $\mathfrak{log}$-links is to render possible

the *construction of the* **log-shells** *from the various [local]* **units**.

That is to say, the construction of log-shells does not require the use of the "non-unit" — i.e., the *local* and *global* **"value group"** — portions of the various monoids in the domain. Thus, when considering the effect of applying various iterates of $\mathfrak{log}$-links, it suffices, from the point of view of computing the effect of the construction of the log-shells from the local units, to consider the effect of such iterates on the various groups of local units that appear.

(ii) Suppose that we are in the situation of the discussion of *[local]* **units** in (i), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then, when thinking of **Kummer structures** in terms of Aut-**holomorphic structures** and **co-holomorphicizations**, as in the

discussion of [IUTchI], Remark 3.4.2, it is natural to regard the *[local] units* that appear as being, in fact, **"Aut-holomorphic semi-germs"**, that is to say,

- *projective systems of arbitrarily small neighborhoods* of the *[local]* **units** [i.e., of "$\mathbb{S}^1$" in "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\times(\mathcal{C}_{\underline{v}})$" in "$\mathcal{O}^\triangleright(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"], equipped with
- the Aut-*holomorphic structures* induced by resticting the ambient Aut-holomorphic structure [i.e., of "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\triangleright(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"],
- the *group structure [germ]* induced by resticting the ambient group structure [i.e., of "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\triangleright(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"], and
- a *choice of one of the two connected components* of the complement of the units in a sufficiently small neighborhood [i.e., determined by "$\mathcal{O}^\triangleright_{\mathbb{C}} \setminus \mathbb{S}^1 \subseteq \mathbb{C}^\times \setminus \mathbb{S}^1$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, by "$\mathcal{O}^\triangleright(\mathcal{C}_{\underline{v}}) \setminus \mathcal{O}^\times(\mathcal{C}_{\underline{v}}) \subseteq \mathcal{O}^\triangleright(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \setminus \mathcal{O}^\times(\mathcal{C}_{\underline{v}})$"].

Indeed, one verifies immediately that such *"Aut-holomorphic semi-germs"* are rigid in the sense that they *do not admit any nontrivial holomorphic automorphisms*. In particular, by thinking of the *[local] units* as *"Aut-holomorphic semi-germs"* in this way, the approach to **Kummer structures** in terms of Aut-**holomorphic structures** and **co-holomorphicizations** discussed in [IUTchI], Remark 3.4.2, carries over without change [cf. [AbsTopIII], Corollary 2.3, (i)]. Moreover, in light of the well-known *discreteness* of the image of the *units of a number field* via the logarithms of the various archimedean valuations of the number field [cf., e.g., [Lang], p. 144, Theorem 5], it follows that the "idèlic" aspects discussed in (i) are also unaffected by thinking in terms of Aut-holomorphic semi-germs.

Remark 1.1.2.

(i) In the notation of Definition 1.1, (i), we observe that the **tautological $\mathfrak{log}$-link**

$$^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})$$

satisfies the property that there is a **taulological identification** between the $^\dagger\Pi_{\underline{v}}$'s that appear in the data that gives rise to the *domain* [i.e., $\{^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{^\dagger\mathcal{F}_{\underline{v}}}]$ and the data that gives rise to the *codomain* [i.e., $\{^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})}\}]$ of the tautological $\mathfrak{log}$-link. By contrast, the $^\dagger\Pi_{\underline{v}}$ that appears in the data that gives rise to the *domain* of the **full $\mathfrak{log}$-link**

$$^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{F}_{\underline{v}}$$

is related to the $^\ddagger\Pi_{\underline{v}}$ [where we use analogous notational conventions for objects associated to $^\ddagger\mathcal{F}$ to the notational conventions already in force for objects associated to $^\dagger\mathcal{F}$] that appears in the data that gives rise to the *codomain* of the full $\mathfrak{log}$-link by means of a **full poly-isomorphism** $^\dagger\Pi_{\underline{v}} \xrightarrow{\sim} {}^\ddagger\Pi_{\underline{v}}$. In this situation,

> the *specific isomorphism* between the $^\dagger\Pi_{\underline{v}}$'s in the domain and codomain of the tautological $\mathfrak{log}$-link may be thought of as a sort of **specific "rigidifying path"** between the $^\dagger\Pi_{\underline{v}}$'s in the domain and codomain of the tautological $\mathfrak{log}$-link that is constructed precisely by using [in an essential

way!] *Frobenius-like monoids* that are related via the $p_{\underline{v}}$*-adic logarithm* [cf. the construction of Definition 1.1, (i)], i.e., by applying the *Galois-equivariance* of the power series defining the $p_{\underline{v}}$-adic logarithm to relate automorphisms of the monoid $\Psi_{^\dagger\mathcal{F}_{\underline{v}}}$ to [induced!] automorphisms of the monoid $\Psi^{\sim}_{^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})}$.

Here, the use of the term "path" is intended to be in the spirit of the notion of a *path in the étale groupoid* [i.e., in the context of the classical theory of the *étale fundamental group*], except that, in the present context, we allow *arbitrary automorphism indeterminacies*, instead of just inner automorphism indeterminacies. In the present paper, we shall work mainly with the **full $\mathfrak{log}$-link** [i.e., not with the tautological $\mathfrak{log}$-link!] since, in the context of the *multiradial algorithms* to be developed in §3 below, it will be of *crucial importance* to be able to

express the relationship between the *étale-like* $^{(-)}\Pi_{\underline{v}}$'s in the *domain* and *codomain* of the $\mathfrak{log}$-links that appear in *purely étale-like terms*, i.e., in a fashion that is [unlike the *specific "rigidifying path"* discussed above!] **free** of **any dependence** on the **Frobenius-like monoids** involved.

This is precisely what is achieved by working with the *"tautologically indeterminate isomorphism"* between $^{(-)}\Pi_{\underline{v}}$'s that underlies the *full* $\mathfrak{log}$*-link*.

(ii) An analogous discussion to that of (i) may be given in the situation of Definition 1.1, (ii), i.e., in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We leave the routine details to the reader.

From the point of view of the present series of papers, the theory of [AbsTopIII] may be summarized as follows.

Proposition 1.2. **($\mathfrak{log}$-links Between $\mathcal{F}$-prime-strips)** *Let*

$$^\dagger\mathfrak{F} = \{^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad ^\ddagger\mathfrak{F} = \{^\ddagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be $\mathcal{F}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], Definition 5.2, (i)] and

$$^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F}$$

a $\mathfrak{log}$-link from $^\dagger\mathfrak{F}$ to $^\ddagger\mathfrak{F}$. Write $^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$, $^\ddagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ for the associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii)]; $^\dagger\mathfrak{D}$, $^\ddagger\mathfrak{D}$ for the associated $\mathcal{D}$-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)]; $^\dagger\mathfrak{D}^{\vdash}$, $^\ddagger\mathfrak{D}^{\vdash}$ for the associated $\mathcal{D}^{\vdash}$-prime-strips [cf. [IUTchI], Definition 4.1, (iv)]. Also, let us recall the **diagrams**

$$\Psi_{^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}} \to \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}}) = \Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})} \xrightarrow{\sim} \Psi^{\mathrm{gp}}_{^\ddagger\mathcal{F}_{\underline{v}}} \qquad (*_{\mathrm{non}})$$

$$\Psi_{^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{^\dagger\mathcal{F}_{\underline{v}}} \twoheadleftarrow \underline{\mathfrak{log}}(^\dagger\mathcal{F}_{\underline{v}}) = \Psi^{\mathrm{gp}}_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})} \xrightarrow{\sim} \Psi^{\mathrm{gp}}_{^\ddagger\mathcal{F}_{\underline{v}}} \qquad (*_{\mathrm{arc}})$$

— where the $\underline{v}$ of $(*_{\mathrm{non}})$ (respectively, $(*_{\mathrm{arc}})$) belongs to $\underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{\mathbb{V}}^{\mathrm{arc}}$), and the [poly-]isomorphisms on the right are induced by the "$\xrightarrow{\mathfrak{log}}$" — of Definition 1.1, (i), (ii).

(i) **(Coricity of Associated $\mathcal{D}$-Prime-Strips)** *The* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$ *induces* **[poly-]isomorphisms**

$$ {}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}; \quad {}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash} $$

between the associated $\mathcal{D}$*- and* $\mathcal{D}^{\vdash}$*-prime-strips. In particular, the [poly-]isomorphism* ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$ *induced by* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$ *induces a [poly-]isomorphism*

$$ \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}) $$

between the collections of monoids equipped with auxiliary data of [IUTchII], Corollary 4.5, (i).

(ii) **(Simultaneous Compatibility with Ring Structures)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the* **natural** ${}^{\dagger}\Pi_{\underline{v}}$**-actions** *on the "*$\Psi$*'s" appearing in the diagram* $(*_{\mathrm{non}})$ *are compatible with the ind-topological* **ring structures** *on* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*. At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the* **co-holomorphicizations** *determined by the* **natural** $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$**-** *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ $(= \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$**-Kummer structures** *on* ${}^{\dagger}\mathbb{U}_{\underline{v}}$ *— which [cf. the discussion of Definition 1.1, (ii)] are compatible with the diagram* $(*_{\mathrm{arc}})$ *— are compatible with the topological* **ring structures** *on* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*.*

(iii) **(Simultaneous Compatibility with Log-volumes)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the diagram* $(*_{\mathrm{non}})$ *is compatible with the* **natural** $p_{\underline{v}}$**-adic log-volumes** *[cf. [AbsTopIII], Proposition 5.7, (i), (c); [AbsTopIII], Corollary 5.10, (ii)] on the subsets of* ${}^{\dagger}\Pi_{\underline{v}}$*-invariants of* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*. At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the diagram* $(*_{\mathrm{arc}})$ *is compatible with the* **natural angular log-volume** *[cf. Remark 1.2.1, (i), below; [AbsTopIII], Proposition 5.7, (ii); [AbsTopIII], Corollary 5.10, (ii)] on* $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and the* **natural radial log-volume** *[cf. [AbsTopIII], Proposition 5.7, (ii), (c); [AbsTopIII], Corollary 5.10, (ii)] on* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ *— cf. also Remark 1.2.1, (ii), below.*

(iv) **(Kummer theory)** *The* **Kummer isomorphisms**

$$ \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}); \qquad \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}) $$

of [IUTchII], Corollary 4.6, (i), **fail** *to be* **compatible** *with the [poly-]isomorphism* $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$ *of (i), relative to the diagrams* $(*_{\mathrm{non}})$*,* $(*_{\mathrm{arc}})$ *[and the notational conventions of Definition 1.1] — cf. [AbsTopIII], Corollary 5.5, (iv). [Here, we regard the diagrams* $(*_{\mathrm{non}})$*,* $(*_{\mathrm{arc}})$ *as diagrams that relate* $\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}$*, via the [poly-]isomorphism* $\mathfrak{log}({}^{\dagger}\mathfrak{F}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}$ *that determines the* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$*.]*

(v) **(Holomorphic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the* **log-shell**

$$ \mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \quad \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}) $$

satisfies the following properties: $(\mathrm{a}_{\mathrm{non}})$ $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *is* **compact***, hence of* **finite log-volume** *[cf. [AbsTopIII], Corollary 5.10, (i)];* $(\mathrm{b}_{\mathrm{non}})$ $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *contains the submonoid*

of ${}^{\dagger}\Pi_{\underline{v}}$*-invariants of* $\Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ *[cf. [AbsTopIII], Definition 5.4, (iii)]; (c_{non})* $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *contains the image of the submonoid of* ${}^{\dagger}\Pi_{\underline{v}}$*-invariants of* $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$*. At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the* **log-shell**

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \quad \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})$$

satisfies the following properties: (a_{arc}) $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *is* **compact***, hence of* **finite radial log-volume** *[cf. [AbsTopIII], Corollary 5.10, (i)]; (b_{arc})* $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *contains* $\Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ *[cf. [AbsTopIII], Definition 5.4, (v)]; (c_{arc}) the image of* $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *in* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *contains* $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *[i.e., in essence, the pre-log-shell].*

(vi) **(Nonarchimedean Mono-analytic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, if we write* ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} = \mathcal{B}({}^{\dagger}G_{\underline{v}})^0$ *for the portion of* ${}^{\dagger}\mathfrak{D}^{\vdash}$ *indexed by* $\underline{v}$ *[cf. the notation of [IUTchII], Corollary 4.5], then the algorithms for constructing "$k^{\sim}(G)$", "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (ii), yield a* **functorial algorithm** *in the category* ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ *for constructing an ind-topological module equipped with a continuous* ${}^{\dagger}G_{\underline{v}}$*-action*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{{}^{\dagger}G_{\underline{v}} \curvearrowright k^{\sim}({}^{\dagger}G_{\underline{v}})\right\}$$

and a topological submodule — i.e., a **"mono-analytic log-shell"** *—*

$$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{I}({}^{\dagger}G_{\underline{v}}) \subseteq k^{\sim}({}^{\dagger}G_{\underline{v}})$$

equipped with a $p_{\underline{v}}$*-adic log-volume [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural* **functorial algorithm** *[cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data* ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ *[i.e., the portion of* ${}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ *labeled by* $\underline{v}$*] for constructing an* Ism**-orbit of isomorphisms** *[cf. [IUTchII], Example 1.8, (iv); [IUTchII], Definition 4.9, (i), (vii)]*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$$

of ind-topological modules [cf. Definition 1.1, (iv)], as well as a **functorial algorithm** *[cf. [AbsTopIII], Corollary 5.10, (iv), (c), (d); the fourth display of [IUTchII], Corollary 4.5, (ii); the final display of [IUTchII], Corollary 4.6, (i)] in the collection of data* ${}^{\dagger}\mathcal{F}_{\underline{v}}$ *for constructing* **isomorphisms**

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \quad \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})$$

of ind-topological modules. The various isomorphisms of the last two displays are **compatible** *with one another, as well as with the respective* ${}^{\dagger}G_{\underline{v}}$*- and* $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-actions [relative to the natural identification* ${}^{\dagger}G_{\underline{v}} = G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$ *that arises from regarding* ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ *as an object constructed from* ${}^{\dagger}\mathcal{D}_{\underline{v}}$*], the respective* **log-shells***, and the respective* **log-volumes** *on these log-shells.*

(vii) **(Archimedean Mono-analytic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the algorithms for constructing "$k^{\sim}(G)$", "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (v), yield a* **functorial algorithm** *in* ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ *[regarded as an object of the category*

"$\mathbb{TM}^{\vdash}$" of split topological monoids discussed in [IUTchI], Example 3.4, (ii)] for constructing a topological module

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\mathrm{def}}{=} k^{\sim}({}^{\dagger}G_{\underline{v}})$$

and a topological subspace — i.e., a **"mono-analytic log-shell"** *—*

$$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{I}({}^{\dagger}G_{\underline{v}}) \subseteq k^{\sim}({}^{\dagger}G_{\underline{v}})$$

equipped with angular and radial log-volumes [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural **functorial algorithm** *[cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ for constructing a* **poly-isomorphism** *[i.e., an orbit of isomorphisms with respect to the* **independent** *actions of $\{\pm 1\}$ on each of the direct factors that occur in the construction of [AbsTopIII], Proposition 5.8, (v)]*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$$

of topological modules [cf. Definition 1.1, (v)], as well as a **functorial algorithm** *[cf. [AbsTopIII], Corollary 5.10, (iv), (c), (d); the fourth display of [IUTchII], Corollary 4.5, (ii); the final display of [IUTchII], Corollary 4.6, (i)] in the collection of data ${}^{\dagger}\mathcal{F}_{\underline{v}}$ for constructing* **poly-isomorphisms** *[i.e., orbits of isomorphisms with respect to the* **independent** *actions of $\{\pm 1\}$ on each of the direct factors that occur in the construction of [AbsTopIII], Proposition 5.8, (v)]*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\stackrel{\sim}{\rightarrow} \quad \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})$$

of topological modules. The various isomorphisms of the last two displays are **compatible** *with one another, as well as with the respective* **log-shells** *and the respective* **angular** *and* **radial log-volumes** *on these log-shells.*

(viii) **(Mono-analytic Log-shells)** *The various [poly-]isomorphisms of (vi), (vii) [cf. also Definition 1.1, (iii), (vi)] yield collections of [poly-]isomorphisms indexed by $\underline{v} \in \underline{\mathbb{V}}$*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) \stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\mathcal{I}_{{}^{\dagger}\mathfrak{D}^{\vdash}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \stackrel{\sim}{\rightarrow} \mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\begin{aligned} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \stackrel{\sim}{\rightarrow} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) &\stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \\ \Big(\stackrel{\sim}{\rightarrow} \quad \Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) &\stackrel{\mathrm{def}}{=} \{\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \Big) \end{aligned}$$

$$\mathcal{I}_{{}^{\dagger}\mathfrak{D}^{\vdash}} \stackrel{\sim}{\rightarrow} \mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \stackrel{\sim}{\rightarrow} \mathcal{I}_{{}^{\dagger}\mathfrak{F}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

— where, in the definition of "$\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F})$", we regard each $\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, as being equipped with its natural $G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}})$-action [cf. the discussion at the beginning of Definition 1.1].

(ix) **(Coric Holomorphic Log-shells)** *Let* $^*\mathfrak{D}$ *be a* $\mathcal{D}$*-prime-strip; write*

$$\mathfrak{F}(^*\mathfrak{D})$$

for the $\mathcal{F}$**-prime-strip** *naturally determined by* $\Psi_{\mathrm{cns}}(^*\mathfrak{D})$ *[cf. [IUTchII], Remark 4.5.1, (i)]. Suppose that* $^\dagger\mathfrak{F} = {}^\ddagger\mathfrak{F} = \mathfrak{F}(^*\mathfrak{D})$*, and that the given* $\mathfrak{log}$*-link* $\mathfrak{F}(^*\mathfrak{D}) = {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} = \mathfrak{F}(^*\mathfrak{D})$ *is the* **full** $\mathfrak{log}$*-link. Then there exists a* **functorial algorithm** *in the* $\mathcal{D}$*-prime-strip* $^*\mathfrak{D}$ *for constructing a collection of topological subspaces — i.e., a collection of* **"coric holomorphic log-shells"** *—*

$$\mathcal{I}_{^*\mathfrak{D}} \stackrel{\mathrm{def}}{=} \mathcal{I}_{^\dagger\mathfrak{F}}$$

of the collection $\Psi^{\mathrm{gp}}_{\mathrm{cns}}(^*\mathfrak{D})$*, which may be naturally identified with* $\Psi^{\mathrm{gp}}_{\mathrm{cns}}(^\ddagger\mathfrak{F})$*, together with a collection of natural isomorphisms [cf. (viii); the fourth display of [IUTchII], Corollary 4.5, (ii)]*

$$\mathcal{I}_{^*\mathfrak{D}^{\vdash}} \xrightarrow{\sim} \mathcal{I}_{^*\mathfrak{D}}$$

— where we write $^*\mathfrak{D}^{\vdash}$ *for the* $\mathcal{D}^{\vdash}$*-prime-strip determined by* $^*\mathfrak{D}$*.*

(x) **(Frobenius-picture)** *Let* $\{^n\mathfrak{F}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\mathcal{F}$**-prime-strips** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 5.2, (i)] indexed by the integers. Write* $\{^n\mathfrak{D}\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}$*-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)] and* $\{^n\mathfrak{D}^{\vdash}\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}^{\vdash}$*-prime-strips [cf. [IUTchI], Definition 4.1, (iv)]. Then the* **full** $\mathfrak{log}$**-links** $^n\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathfrak{F}$*, for* $n \in \mathbb{Z}$*, give rise to an* **infinite chain**

$$\dots \xrightarrow{\mathfrak{log}} {}^{(n-1)}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{n}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathfrak{F} \xrightarrow{\mathfrak{log}} \dots$$

of $\mathfrak{log}$**-linked** $\mathcal{F}$**-prime-strips** *which induces chains of full poly-isomorphisms*

$$\dots \xrightarrow{\sim} {}^n\mathfrak{D} \xrightarrow{\sim} {}^{(n+1)}\mathfrak{D} \xrightarrow{\sim} \dots \qquad \textit{and} \qquad \dots \xrightarrow{\sim} {}^n\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{(n+1)}\mathfrak{D}^{\vdash} \xrightarrow{\sim} \dots$$

on the associated $\mathcal{D}$*- and* $\mathcal{D}^{\vdash}$*-prime-strips [cf. (i)]. These chains may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\begin{array}{ccccccccc} \dots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the horizontal arrows correspond to the " $\xrightarrow{\mathfrak{log}}$ *'s"; the "*$\bullet$*'s" correspond to the "*$^n\mathfrak{F}$*"; the "*$\circ$*" corresponds to the "*$^n\mathfrak{D}$*", identified up to isomorphism; the vertical/diagonal arrows correspond to the* **Kummer isomorphisms** *of (iv). This oriented graph* $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *[cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a* **translation symmetry** *— that fixes the* **"core"** $\circ$*, but it does* **not admit arbitrary permutation symmetries***. For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.*

Proof. The various assertions of Proposition 1.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 1.2.1.

(i) Suppose that we are in the situation of Definition 1.1, (ii). Then at the level of *metrics* — i.e., which give rise to **angular log-volumes** as in Proposition 1.2, (iii) — we suppose that $\Psi^{\mathrm{gp}}_{\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{\dagger\mathcal{F}_{\underline{v}}}$ is equipped with the metric obtained by descending the metric of $\Psi^{\mathrm{gp}}_{\dagger\mathcal{F}_{\underline{v}}}$, but we regard the object

$\Psi^{\mathrm{gp}}_{\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{\dagger\mathcal{F}_{\underline{v}}}$ [or $\Psi^{\times}_{\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{\dagger\mathcal{F}_{\underline{v}}}$] as being equipped with a **"weight N"**

— i.e., which has the effect of ensuring that the **log-volume** of $\Psi^{\times}_{\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{\dagger\mathcal{F}_{\underline{v}}}$ is **equal** to that of $\Psi^{\times}_{\dagger\mathcal{F}_{\underline{v}}}$. That is to say, this convention concerning "weights" ensures that working with $\Psi^{\mathrm{gp}}_{\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{\dagger\mathcal{F}_{\underline{v}}}$ does not have any effect on various computations of log-volume.

(ii) Although, at first glance, the *compatibility* with *archimedean log-volumes* discussed in Proposition 1.2, (iii), appears to relate *"different objects"* — i.e., *angular* versus *radial* log-volumes — in the domain and codomain of the $\mathfrak{log}$-link under consideration, in fact, this compatibility property may be regarded as an **invariance** property — i.e., that relates *"similar objects"* in the domain and codomain of the $\mathfrak{log}$-link under consideration — by reasoning as follows. Let k be a *complex archimedean field*. Write $\mathcal{O}^{\times}_{k} \subseteq k$ for the group of elements of absolute value $= 1$ and $k^{\times} \subseteq k$ for the group of nonzero elements. In the following, we shall use the term *"metric on k"* to refer to a Riemannian metric on the real analytic manifold determined by k that is *compatible* with the two natural *almost complex structures* on this real analytic manifold and, moreover, is *invariant* with respect to arbitrary *additive translation* automorphisms of k. In passing, we note that any metric on k is also *invariant* with respect to *multiplication* by elements $\in \mathcal{O}^{\times}_{k}$. Next, let us observe that the metrics on k naturally form a *torsor* over $\mathbb{R}_{>0}$. In particular, if we write $k^{\times} \cong \mathcal{O}^{\times}_{k} \times \mathbb{R}_{>0}$ for the natural direct product decomposition, then one verifies immediately that

any metric on k is **uniquely determined** *either by its restriction to $\mathcal{O}^{\times}_{k} \subseteq k$ or by its restriction to $\mathbb{R}_{>0} \subseteq k$.*

Thus, if one regards the *compatibility* property concerning angular and radial log-volumes discussed in Proposition 1.2, (iii), as a property concerning the *respective restrictions* of the corresponding *uniquely determined* **metrics** [i.e., the metrics corresponding to the respective standard norms on the complex archimedean fields under consideration — cf. [AbsTopIII], Proposition 5.7, (ii), (a)], then this compatibility property discussed in Proposition 1.2, (iii), may be regarded as a property that asserts the **invariance** of the respective natural metrics with respect to the "transformation" constituted by the $\mathfrak{log}$-link.

Remark 1.2.2. Before proceeding, we pause to consider the significance of the various properties discussed in Proposition 1.2, (v). For simplicity, we suppose

that "$^{\dagger}\mathfrak{F}$" is the $\mathcal{F}$-prime-strip that arises from the data constructed in [IUTchI], Examples 3.2, (iii); 3.3, (i); 3.4, (i) [cf. [IUTchI], Definition 5.2, (i)].

(i) Suppose that $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Thus, $K_{\underline{v}}$ [cf. the notation of [IUTchI], Definition 3.1, (e)] is a **mixed-characteristic nonarchimedean local field**. Write $k \stackrel{\text{def}}{=} K_{\underline{v}}$, $\mathcal{O}_k \subseteq k$ for the ring of integers of k, $\mathcal{O}_k^{\times} \subseteq \mathcal{O}_k$ for the group of units, and $\log_k : \mathcal{O}_k^{\times} \to k$ for the $p_{\underline{v}}$-*adic logarithm.* Then, at a more concrete level — i.e., relative to the notation of the present discussion — the **log-shell** "$\mathcal{I}_{^{\dagger}\mathcal{F}_{\underline{v}}}$" corresponds to the submodule

$$\mathcal{I}_k \stackrel{\text{def}}{=} (p_{\underline{v}}^*)^{-1} \cdot \log_k(\mathcal{O}_k^{\times}) \ \subseteq \ k$$

— where $p_{\underline{v}}^* = p_{\underline{v}}$ if $p_{\underline{v}}$ is *odd*, $p_{\underline{v}}^* = p_{\underline{v}}^2$ if $p_{\underline{v}}$ is *even* — while the properties (b_{non}), (c_{non}) of Proposition 1.2, (v), correspond, respectively, to the evident **inclusions**

$$\mathcal{O}_k^{\triangleright} \ \stackrel{\text{def}}{=} \ \mathcal{O}_k \setminus \{0\} \ \subseteq \ \mathcal{O}_k \ \subseteq \ \mathcal{I}_k; \quad \log_k(\mathcal{O}_k^{\times}) \ \subseteq \ \mathcal{I}_k$$

of subsets of k.

(ii) Suppose that $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. Thus, $K_{\underline{v}}$ [cf. the notation of [IUTchI], Definition 3.1, (e)] is a **complex archimedean field**. Write $k \stackrel{\text{def}}{=} K_{\underline{v}}$, $\mathcal{O}_k \subseteq k$ for the subset of elements of absolute value ≤ 1, $\mathcal{O}_k^{\times} \subseteq \mathcal{O}_k$ for the group of elements of absolute value $= 1$, and $\exp_k : k \to k^{\times}$ for the *exponential map.* Then, at a more concrete level — i.e., relative to the notation of the present discussion — the **log-shell** "$\mathcal{I}_{^{\dagger}\mathcal{F}_{\underline{v}}}$" corresponds to the subset

$$\mathcal{I}_k \ \stackrel{\text{def}}{=} \ \{a \in k \mid |a| \leq \pi\} \ \subseteq \ k$$

of elements of absolute value $\leq \pi$, while the properties (b_{arc}), (c_{arc}) of Proposition 1.2, (v), correspond, respectively, to the evident **inclusions**

$$\mathcal{O}_k^{\triangleright} \ \stackrel{\text{def}}{=} \ \mathcal{O}_k \setminus \{0\} \ \subseteq \ \mathcal{O}_k \ \subseteq \ \mathcal{I}_k; \quad \mathcal{O}_k^{\times} \ \subseteq \ \exp_k(\mathcal{I}_k)$$

— where we note the slightly different roles played, in the archimedean [cf. the present (ii)] and nonarchimedean [cf. (i)] cases, by the exponential and logarithmic functions, respectively [cf. [AbsTopIII], Remark 4.5.2].

(iii) The diagram represented by the oriented graph $\vec{\Gamma}$ of Proposition 1.2, (x), is, of course, **far from commutative** [cf. Proposition 1.2, (iv)]! Ultimately, however, [cf. the discussion of Remark 1.4.1, (ii), below] we shall be interested in

(a) constructing **invariants** *with respect to the* $\mathbb{Z}$*-action on* $\vec{\Gamma}$ — i.e., in effect, constructing objects via functorial algorithms in the **coric** $\mathcal{D}$-prime-strips "$^{n}\mathfrak{D}$" —

while, at the same time,

(b) *relating the corically constructed objects* of (a) to the non-coric "$^{n}\mathfrak{F}$" via the various **Kummer isomorphisms** of Proposition 1.2, (iv).

That is to say, from the point of view of (a), (b), the content of the *inclusions* discussed in (i) and (ii) above may be interpreted, at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, as follows:

> the **coric holomorphic log-shells** of Proposition 1.2, (ix), contain *not only* the images, via the Kummer isomorphisms [i.e., the vertical/diagonal arrows of $\vec{\Gamma}$], of the various "$\mathcal{O}^{\triangleright}$" at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, *but also the* **images**, via the composite of the Kummer isomorphisms with the *various* **iterates** [cf. Remark 1.1.1] *of the* $\mathfrak{log}$*-link* [i.e., the horizontal arrows of $\vec{\Gamma}$], of the portions of the various "$\mathcal{O}^{\triangleright}$" at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ on which these iterates are defined.

An analogous statement in the case of $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ may be formulated by adjusting the wording appropriately so as to accommodate the latter portion of this statement, which corresponds to a certain *surjectivity* — we leave the routine details to the reader. Thus, although the diagram [corresponding to] $\vec{\Gamma}$ fails to be *commutative*,

> the *coric holomorphic log-shells* involved exhibit a sort of **"upper semi-commutativity"** with respect to containing/surjecting onto *the various images arising from composites of arrows in* $\vec{\Gamma}$.

(iv) Note that although the diagram $\vec{\Gamma}$ admits a natural "upper semi-commutativity" interpretation as discussed in (iii) above, it **fails** *to admit a corresponding "***lower** *semi-commutativity" interpretation*. Indeed, such a "lower semi-commutativity" interpretation would amount to the existence of some sort of *collection of portions of the various "$\mathcal{O}^{\triangleright}$'s" involved* [cf. the discussion of (i), (ii) above] — i.e., a sort of **"core"** — that are mapped to one another *isomorphically* by the various maps "$\log_k$"/"$\exp_k$" [cf. the discussion of (i), (ii) above] in a fashion that is **compatible** with the various **Kummer isomorphisms** that appear in the diagram $\vec{\Gamma}$. On the other hand, it is difficult to see how to construct such a collection of portions of the various "$\mathcal{O}^{\triangleright}$'s" involved.

(v) Proposition 1.2, (iii), may be interpreted in the spirit of the discussion of (iii) above. That is to say, although the diagram corresponding to $\vec{\Gamma}$ fails to be *commutative*, it is nevertheless *"***commutative** *with respect to* **log-volumes***"*, in the sense discussed in Proposition 1.2, (iii). This "commutativity with respect to log-volumes" allows one to work with log-volumes in a fashion that is *consistent with all composites of the various arrows of* $\vec{\Gamma}$. Log-volumes will play an important role in the theory of §3, below, as a sort of *mono-analytic version* of the notion of the *degree of a global arithmetic line bundle* [cf. the theory of [AbsTopIII], §5].

(vi) As discussed in [AbsTopIII], §I3, the $\mathfrak{log}$-links of $\vec{\Gamma}$ may be thought of as a sort of **"juggling of $\boxplus$, $\boxtimes$"** [i.e., of the two combinatorial dimensions of the ring structure constituted by addition and multiplication]. The **"arithmetic holomorphic structure"** constituted by the coric $\mathcal{D}$-prime-strips is *immune to this juggling*, and hence may be thought as representing a sort of **quotient** *of the horizontal arrow portion of* $\vec{\Gamma}$ *by the action of* $\mathbb{Z}$ [cf. (iii), (a)] — i.e., at the level of abstract oriented graphs, as a sort of **"oriented copy of $\mathbb{S}^1$"**. That is to say, the horizontal arrow portion of $\vec{\Gamma}$ may be thought of as a sort of **"unraveling"** of

this "oriented copy of $\mathbb{S}^1$", which is subject to the *"juggling of* $\boxplus$, $\boxtimes$*"* constituted by the $\mathbb{Z}$-action. Here, it is useful to recall that

(a) the *Frobenius-like structures* constituted by the monoids that appear in the horizontal arrow portion of $\vec{\Gamma}$ play the *crucial* role in the theory of the present series of papers of allowing one to construct such **"non-ring/scheme-theoretic filters"** as the **Θ-link** [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

By contrast,

(b) the *étale-like structures* constituted by the coric $\mathcal{D}$-prime-strips play the *crucial role* in the theory of the present series of papers of allowing one to construct objects that are capable of **"functorially permeating"** such non-ring/scheme-theoretic filters as the Θ-link [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

Finally, in order to *relate* the theory of (a) to the theory of (b), one must avail oneself of **Kummer theory** [cf. (iii), (b), above].

mono-anabelian coric *étale-like* structures	*invariant differential* $d\theta$ on $\mathbb{S}^1$
post-anabelian *Frobenius-like* structures	*coordinate functions* $\int_{\bullet} d\theta$ on $\vec{\Gamma}$

Fig. 1.1: Analogy with the differential geometry of $\mathbb{S}^1$

(vii) From the point of view of the discussion in (vi) above of the *"oriented copy of* $\mathbb{S}^1$*"* obtained by forming the quotient of the horizontal arrow portion of $\vec{\Gamma}$ by $\mathbb{Z}$, one may think of the *coric étale-like structures* of Proposition 1.2, (i) — as well as the various objects constructed from these coric étale-like structures via the various *mono-anabelian algorithms* discussed in [AbsTopIII] — as corresponding to the *"canonical invariant differential* $d\theta$*"* on $\mathbb{S}^1$ [which is, in particular, *invariant* with respect to the action of $\mathbb{Z}!$]. On the other hand, the various *post-anabelian Frobenius-like structures* obtained by *forgetting* the mono-anabelian algorithms applied to construct these objects — cf., e.g.., the "$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F})$" that appear in the *Kummer isomorphisms* of Proposition 1.2, (iv) — may be thought of as *coordinate functions* on the horizontal arrow portion of $\vec{\Gamma}$ [which are *not* invariant with respect to the action of $\mathbb{Z}!$] of the form "$\int_{\bullet} d\theta$" obtained by integrating the invariant differential $d\theta$ along various paths of $\vec{\Gamma}$ that emanate from some fixed vertex "$\bullet$" of $\vec{\Gamma}$. This point of view is summarized in Fig. 1.1 above. Finally, we observe that this point of view is reminiscent of the discussion of [AbsTopIII], §I5, concerning the analogy between the theory of [AbsTopIII] and the construction of canonical

coordinates via integration of Frobenius-invariant differentials in the classical p-adic theory.

Remark 1.2.3.

(i) Observe that, relative to the notation of Remark 1.2.2, (i), any **multiplicative indeterminacy** with respect to the action on $\mathcal{O}_k^{\triangleright}$ of some *subgroup* $H \subseteq \mathcal{O}_k^{\times}$ at some "$\bullet$" of the diagram $\vec{\Gamma}$ gives rise to an **additive indeterminacy** with respect to the action of $\log_k(H)$ on the copy of "$\mathcal{O}_k$" that corresponds to the subsequent "$\bullet$" of the diagram $\vec{\Gamma}$. In particular, if H consists of *roots of unity*, then $\log_k(H) = \{0\}$, so *the resulting additive indeterminacy ceases to exist*. This observation will play a *crucial role* in the theory of §3, below, when it is applied in the context of the *constant multiple rigidity* properties constituted by the *canonical splittings of theta and Gaussian monoids* discussed in [IUTchII], Proposition 3.3, (i); [IUTchII], Corollary 3.6, (iii) [cf. also [IUTchII], Corollary 1.12, (ii); the discussion of [IUTchII], Remark 1.12.2, (iv)].

(ii) In the theory of §3, below, we shall consider *global arithmetic line bundles*. This amounts, in effect, to considering **multiplicative translates** by $f \in F_{\text{mod}}^{\times}$ of the product of the various "$\mathcal{O}_k^{\times}$" of Remark 1.2.2, (i), (ii), as $\underline{v}$ ranges over the elements of $\underline{\mathbb{V}}$. Such translates are *disjoint* from one another, *except* in the case where f is a *unit* at all $\underline{v} \in \underline{\mathbb{V}}$. By elementary algebraic number theory [cf., e.g., [Lang], p. 144, the proof of Theorem 5], this corresponds precisely to the case where f is a *root of unity*. In particular, to consider quotients by this *multiplicative* action by $F_{\text{mod}}^{\times}$ at one "$\bullet$" of the diagram $\vec{\Gamma}$ [where we allow $\underline{v}$ to range over the elements of $\underline{\mathbb{V}}$] gives rise to an **additive indeterminacy** by *"logarithms of roots of unity"* at the subsequent "$\bullet$" of the diagram $\vec{\Gamma}$. In particular, at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, *the resulting additive indeterminacy ceases to exist* [cf. the discussion of (i); Definition 1.1, (iv)]; at $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, the resulting indeterminacy corresponds to considering certain quotients of the copies of "$\mathcal{O}_k^{\times}$" — i.e., of "$\mathbb{S}^1$" — that appear by *some finite subgroup* [cf. the discussion of Definition 1.1, (ii)]. These observations will be of use in the development of the theory of §3, below.

Remark 1.2.4.

(i) At this point, we pause to recall the important observation that the $\mathfrak{log}$-link is **incompatible** with the **ring structures** of $\Psi^{\text{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ and $\Psi^{\text{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ [cf. the notation of Proposition 1.2, (ii)], in the sense that it does not arise from a *ring homomorphism* between these two rings. The barrier constituted by this incompatibility between the ring structures on either side of the $\mathfrak{log}$-link is precisely what is referred to as the **"$\mathfrak{log}$-wall"** in the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I4]. This incompatibility with the respective ring structures implies that it is not possible, *a priori*, to transport objects whose structure depends on these ring structures via the $\mathfrak{log}$-link by invoking the principle of "transport of structure". From the point of view of the theory of the present series of papers, this means, in particular, that

> the $\mathfrak{log}$-wall is **incompatible** with **conventional scheme-theoretic basepoints**, which are defined by means of geometric points [i.e., *ring homomorphisms* of a certain type]

— cf. the discussion of [IUTchII], Remark 3.6.3, (i); [AbsTopIII], Remark 3.7.7, (i). In this context, it is useful to recall that étale fundamental groups — i.e., Galois groups — are defined as certain *automorphism groups of fields/rings*; in particular, the definition of such a Galois group "as a certain automorphism group of some ring structure" is *incompatible*, in a quite essential way, with the $\mathfrak{log}$-wall. In a similar vein, **Kummer theory**, which depends on the *multiplicative structure* of the ring under consideration, is also *incompatible*, in a quite essential way, with the $\mathfrak{log}$-wall [cf. Proposition 1.2, (iv)]. That is to say, in the context of the $\mathfrak{log}$-link,

> the only structure of interest that is manifestly **compatible** with the $\mathfrak{log}$-link [cf. Proposition 1.2, (i), (ii)] is the associated **$\mathcal{D}$-prime-strip**

— i.e., the *abstract topological groups* [isomorphic to "$\Pi_{\underline{v}}$" — cf. the notation of [IUTchI], Definition 3.1, (e), (f)] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ and *abstract* Aut-*holomorphic spaces* [isomorphic to "$\mathbb{U}_{\underline{v}}$" — cf. the notation of [IUTchII], Proposition 4.3] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Indeed, this observation is precisely the *starting point* of the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I1, §I4].

(ii) Other important examples of structures which are *incompatible with the* $\mathfrak{log}$*-wall* include

(a) the *additive structure* on the image of the Kummer map [cf. the discussion of [AbsTopIII], Remark 3.7.5];

(b) in the *"birational"* situation — i.e., where one replaces "$\Pi_{\underline{v}}$" by the absolute Galois group $\Pi_{\underline{v}}^{\mathrm{birat}}$ of the *function field* of the *affine curve* that gave rise to $\Pi_{\underline{v}}$ — the datum of the *collection of closed points* that determines the affine curve [cf. [AbsTopIII], Remark 3.7.7, (ii)].

Note, for instance in the case of (b), when, say, for simplicity, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, that one may think of the additional datum under consideration as consisting of the natural outer surjection $\Pi_{\underline{v}}^{\mathrm{birat}} \twoheadrightarrow \Pi_{\underline{v}}$ that arises from the *scheme-theoretic morphism* from the spectrum of the function field to the given affine curve. On the other hand, just as in the case of the discussion of scheme-theoretic basepoints in (i), the construction of such an object $\Pi_{\underline{v}}^{\mathrm{birat}} \twoheadrightarrow \Pi_{\underline{v}}$ whose structure depends, in an essential way, on the scheme [i.e., ring!] structures involved necessarily *fails to be compatible with the* $\mathfrak{log}$*-link* [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)].

(iii) One way to understand the *incompatibility* discussed in (ii), (b), is as follows. Write $\Delta_{\underline{v}}^{\mathrm{birat}}$, $\Delta_{\underline{v}}$ for the respective kernels of the natural surjections $\Pi_{\underline{v}}^{\mathrm{birat}} \twoheadrightarrow G_{\underline{v}}$, $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$. Then if one forgets about the *scheme-theoretic* basepoints discussed in (i), $G_{\underline{v}}$, $\Delta_{\underline{v}}^{\mathrm{birat}}$, and $\Delta_{\underline{v}}$ may be understood on *both* sides of the $\mathfrak{log}$-wall as *"some topological group"*, and each of the topological groups $\Delta_{\underline{v}}^{\mathrm{birat}}$, $\Delta_{\underline{v}}$ may be understood on *both* sides of the $\mathfrak{log}$-wall as being equipped with *"some outer $G_{\underline{v}}$-action"* — cf. the two diagonal arrows of Fig. 1.2 below. On the other hand, the datum of a *particular outer surjection* $\Delta_{\underline{v}}^{\mathrm{birat}} \twoheadrightarrow \Delta_{\underline{v}}$ [cf. the dotted line in Fig. 1.2] relating these two diagonal arrows — which depends, in an essential way, on the scheme [i.e., ring] structures involved! — necessarily *fails to be compatible with the* $\mathfrak{log}$*-link* [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)]. This issue

of *"triangular compatibility between independent indeterminacies"* is formally reminiscent of the issue of compatibility of outer homomorphisms discussed in [IUTchI], Remark 4.5.1, (i) [cf. also [IUTchII], Remark 2.5.2, (ii)].

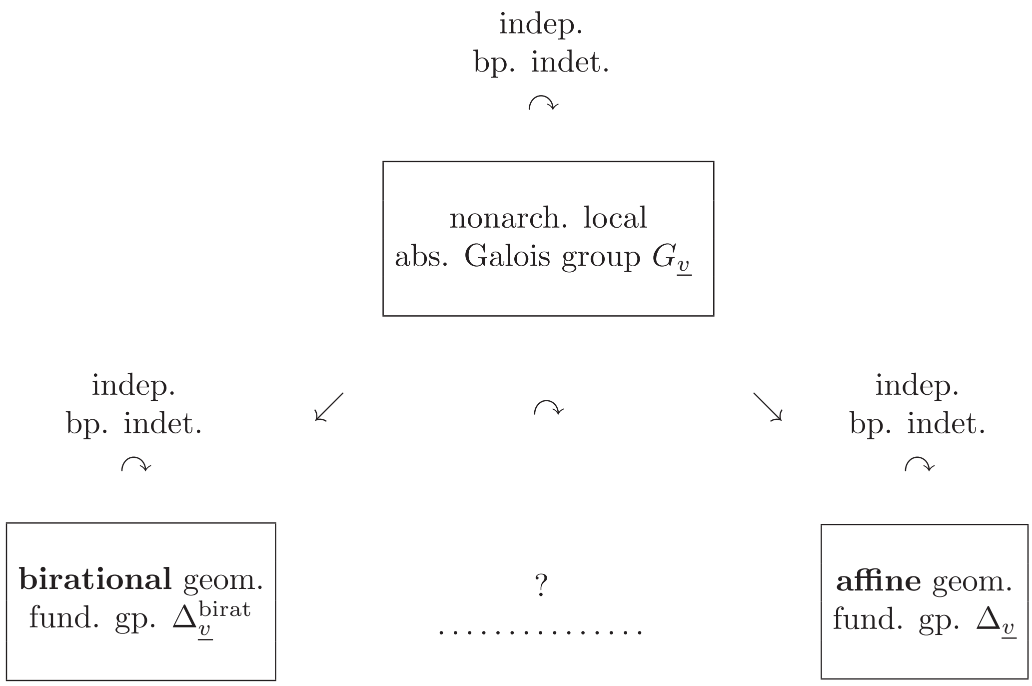

Fig. 1.2: Independent basepoint indeterminacies obstruct relationship between birational and affine geometric fundamental groups

Remark 1.2.5. The discussion in Remark 1.2.4 of the *incompatibility* of the $\mathfrak{log}$-*wall* with various structures that arise from *ring/scheme-theory* is closely related to the issue of avoiding the use of **fixed ring/scheme-theoretic reference models** in **mono-anabelian** construction algorithms [cf. the discussion of [IUTchI], Remark 3.2.1, (i); [AbsTopIII], §I4]. Put another way, at least in the context of the $\mathfrak{log}$-link [i.e., situations of the sort considered in [AbsTopIII], as well as in the present paper], mono-anabelian construction algorithms may be understood as

> algorithms whose **dependence** on data arising from such *fixed ring/scheme-theoretic reference models* is **"invariant"**, or **"coric"**, with respect to the action of $\mathfrak{log}$ on such models.

A substantial portion of [AbsTopIII], §3, is devoted precisely to the task of giving a *precise formulation* of this concept of *"invariance"* by means of such notions as *observables*, *families of homotopies*, and *telecores*. For instance, one approach to formulating the *failure* of the *ring structure of a fixed reference model* to be *"coric"* with respect to $\mathfrak{log}$ may be seen in [AbsTopIII], Corollary 3.6, (iv); [AbsTopIII], Corollary 3.7, (iv).

Proposition 1.3. **($\mathfrak{log}$-links Between $\Theta^{\pm\text{ell}}$NF-Hodge Theaters)** *Let*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}; \quad ^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$$

be $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — cf. [IUTchI], Definition 6.13, (i). Write* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *for the associated* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** *— cf. [IUTchI], Definition 6.13, (ii). Then:*

(i) **(Construction of the** $\mathfrak{log}$**-Link)** *Fix an isomorphism*

$$\Xi : {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters. Let* ${}^{\dagger}\mathfrak{F}_{\square}$ *be one of the* $\mathcal{F}$*-prime-strips that appear in the* Θ*- and* $\Theta^{\pm}$*-bridges that constitute* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *— i.e., either one of the* $\mathcal{F}$**-prime-strips**

$${}^{\dagger}\mathfrak{F}_{>}, \quad {}^{\dagger}\mathfrak{F}_{\succ}$$

or one of the **constituent** $\mathcal{F}$**-prime-strips** *of the capsules*

$${}^{\dagger}\mathfrak{F}_{J}, \quad {}^{\dagger}\mathfrak{F}_{T}$$

[cf. [IUTchI], Definition 5.5, (ii); [IUTchI], Definition 6.11, (i)]. Write ${}^{\ddagger}\mathfrak{F}_{\square}$ *for the corresponding* $\mathcal{F}$*-prime-strip of* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*. Then the poly-isomorphism determined by* Ξ *between the* $\mathcal{D}$*-prime-strips associated to* ${}^{\dagger}\mathfrak{F}_{\square}$, ${}^{\ddagger}\mathfrak{F}_{\square}$ *uniquely determines a poly-isomorphism* $\mathfrak{log}({}^{\dagger}\mathfrak{F}_{\square}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\square}$ *[cf. Definition 1.1, (iii); [IUTchI], Corollary 5.3, (ii)], hence a* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}$ *[cf. Definition 1.1, (iii)]. We shall denote by*

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

and refer to as a $\mathfrak{log}$**-link** *from* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *the collection of data consisting of* Ξ*, together with the collection of* $\mathfrak{log}$*-links* ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}$*, as "*$\square$*" ranges over all possibilities for the* $\mathcal{F}$*-prime-strips in question. When* Ξ *is replaced by a poly-isomorphism* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*, we shall also refer to the resulting collection of* $\mathfrak{log}$*-links [i.e., corresponding to each constituent isomorphism of the poly-isomorphism* Ξ*] as a* $\mathfrak{log}$**-link** *from* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*. When* Ξ *is the full poly-isomorphism, we shall refer to the resulting* $\mathfrak{log}$*-link as the* **full** $\mathfrak{log}$**-link**.

(ii) **(Coricity)** *Any* $\mathfrak{log}$*-link* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *induces [and may be thought of as "lying over"] a* **[poly-]isomorphism**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters [and indeed coincides with the* $\mathfrak{log}$*-link constructed in (i) from this [poly-]isomorphism of* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters].*

(iii) **(Further Properties of the** $\mathfrak{log}$**-Link)** *In the notation of (i), any* $\mathfrak{log}$*-link* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *satisfies, for each* $\mathcal{F}$*-prime-strip* ${}^{\dagger}\mathfrak{F}_{\square}$*, properties corresponding to the properties of Proposition 1.2, (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix), i.e., concerning* **simultaneous compatibility** *with* **ring structures** *and* **log-volumes**, **Kummer theory**, *and* **log-shells**.

(iv) **(Frobenius-picture)** *Let* $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] indexed by the integers. Write* $\{{}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters. Then the* **full** $\mathfrak{log}$**-links** ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, for* $n \in \mathbb{Z}$*, give rise to an* **infinite chain**

$$\ldots \xrightarrow{\mathfrak{log}} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} \ldots$$

of $\mathfrak{log}$**-linked** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *which induces a chain of full poly-isomorphisms*

$$\ldots \stackrel{\sim}{\rightarrow} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\sim}{\rightarrow} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\sim}{\rightarrow} \ldots$$

on the associated $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters. These chains may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\begin{array}{ccccccccc} \ldots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \ldots \\ & \ldots & & \searrow & \downarrow & \swarrow & & \ldots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the horizontal arrows correspond to the " $\xrightarrow{\mathfrak{log}}$ *'s"; the "*$\bullet$*'s" correspond to the "*${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*"; the "*$\circ$*" corresponds to the "*${}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*", identified up to isomorphism; the vertical/diagonal arrows correspond to the* **Kummer isomorphisms** *implicit in the statement of (iii). This oriented graph* $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *[cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a* **translation symmetry** *— that fixes the* **"core"** $\circ$*, but it does* **not admit arbitrary permutation symmetries**. *For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.*

Proof. The various assertions of Proposition 1.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 1.3.1. Note that in Proposition 1.3, (i), it was necessary to carry out the given construction of the $\mathfrak{log}$-link first for a **single** Ξ [i.e., as opposed to a poly-isomorphism Ξ], in order to maintain *compatibility* with the crucial **"±-synchronization"** [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] inherent in the structure of a $\Theta^{\pm\mathrm{ell}}$-Hodge theater.

Remark 1.3.2. In the construction of Proposition 1.3, (i), the constituent $\mathcal{F}$-prime-strips ${}^{\dagger}\mathfrak{F}_t$, for $t \in T$, of the capsule ${}^{\dagger}\mathfrak{F}_T$ are considered without regard to the $\mathbb{F}_l^{\rtimes\pm}$*-symmetries* discussed in [IUTchII], Corollary 4.6, (iii). On the other hand, one verifies immediately that the $\mathfrak{log}$**-links** associated, in the construction of Proposition 1.3, (i), to these $\mathcal{F}$-prime-strips ${}^{\dagger}\mathfrak{F}_t$, for $t \in T$ — i.e., more precisely, associated to the **labeled collections of monoids** $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_t$ of [IUTchII], Corollary 4.6, (iii) — are in fact **compatible** with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** discussed in [IUTchII], Corollary 4.6, (iii), hence also with the **conjugate synchronization** determined by these $\mathbb{F}_l^{\rtimes\pm}$-symmetrizing isomorphisms — cf. the discussion of Step

(vi) of the proof of Corollary 3.12 of §3 below. We leave the routine details to the reader.

Remark 1.3.3.

(i) In the context of Proposition 1.3 [cf. also the discussion of Remarks 1.2.4, 1.3.1, 1.3.2], it is of interest to *observe* that the relationship between the various **Frobenioid-theoretic** [i.e., *Frobenius-like*!] portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the *domain* and *codomain* of the **$\mathfrak{log}$-link** of Proposition 1.3, (i),

> *does* **not** *include any data — i.e., of the sort discussed in Remark 1.2.4, (ii), (a), (b); Remark 1.2.4, (iii) — that is* **incompatible***, relative to the relevant* **Kummer isomorphisms***, with the* **coricity** *property for étale-like structures given in Proposition 1.3, (ii).*

Indeed, this *observation* may be understood as a consequence of the fact [cf. Remarks 1.3.1, 1.3.2; [IUTchI], Corollary 5.3, (i), (ii), (iv); [IUTchI], Corollary 5.6, (i), (ii), (iii)] that these Frobenioid-theoretic portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration are completely [i.e., *fully faithfully*!] *controlled* [cf. the discussion of (ii) below for more details], via **functorial algorithms**, by the corresponding *étale-like* structures, i.e., structures that appear in the associated $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, which satisfy the crucial **coricity** property of Proposition 1.3, (ii).

(ii) In the context of (i), it is of interest to recall that the global portion of the underlying Θ^{ell}-bridges is defined [cf. [IUTchI], Definition 6.11, (ii)] in such a way that it does not contain any global Frobenioid-theoretic data! In particular, the issue discussed in (i) concerns only the Frobenioid-theoretic portions of the following:

(a) the various **$\mathcal{F}$-prime-strips** that appear;
(b) the underlying **Θ-Hodge theaters** of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration;
(c) the global portion of the underlying **NF-bridges** of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration.

Here, the Frobenioid-theoretic data of (c) gives rise to **independent** [i.e., for corresponding portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the *domain* and *codomain* of the $\mathfrak{log}$-*link*] **basepoints** with respect to the **$\mathbb{F}_l^{\circledast}$-symmetry** [cf. [IUTchI], Corollary 5.6, (iii); [IUTchI], Remark 6.12.6, (iii); [IUTchII], Remark 4.7.6]. On the other hand, the independent basepoints that arise from the Frobenioid-theoretic data of (b), as well as of the portion of (a) that lies in the underlying ΘNF-Hodge theater, do not cause any problems [i.e., from the point of view of the sort of *incompatibility* discussed in (i)] since this data is only subject to relationships *defined* by means of *full poly-isomorphisms* [cf. [IUTchI], Examples 4.3, 4.4]. That is to say, the $\mathcal{F}$-prime-strips that lie in the underlying $\Theta^{\pm \mathrm{ell}}$-Hodge theater constitute the most *delicate* [i.e., relative to the issue of independent basepoints!] portion of the Frobenioid-theoretic data of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater. This delicacy revolves

around the *global synchronization of* $\pm$*-indeterminacies* in the underlying $\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)]. On the other hand, this delicacy does not in fact cause any problems [i.e., from the point of view of the sort of *incompatibility* discussed in (i)] since [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] the synchronizations of $\pm$-indeterminacies in the underlying $\Theta^{\pm\mathrm{ell}}$-Hodge theater are *defined* [not by means of *scheme-theoretic* relationships, but rather] by applying the **intrinsic structure** *of the underlying* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater*, which satisfies the crucial **coricity** property of Proposition 1.3, (ii) [cf. the discussion of (i); Remarks 1.3.1, 1.3.2].

The diagrams discussed in the following Definition 1.4 will play a *central role* in the theory of the present series of papers.

Definition 1.4. We maintain the notation of Proposition 1.3 [cf. also [IUTchII], Corollary 4.10, (iii)]. Let $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers. Then we shall refer to either of the diagrams

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

— where the *vertical* arrows are the *full* $\mathfrak{log}$*-links*, and the *horizontal* arrows are the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$*-links* of [IUTchII], Corollary 4.10, (iii) — as the *log-theta-lattice*. We shall refer to the log-theta-lattice that involves the $\Theta^{\times\boldsymbol{\mu}}$- (respectively, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-)

links as *non-Gaussian* (respectively, *Gaussian*). Thus, either of these diagrams may be represented symbolically by an *oriented graph*

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

— where the "$\bullet$'s" correspond to the "${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$".

Remark 1.4.1.

(i) One fundamental property of the log-theta-lattices discussed in Definition 1.4 is the following:

> the various squares that appear in each of the log-theta-lattices discussed in Definition 1.4 are **far** *from being* [**1-**]**commutative**!

Indeed, whereas the *vertical* arrows in each log-theta-lattice are constructed by applying the various *logarithms* at $\underline{v} \in \underline{\mathbb{V}}$ — i.e., which are defined by means of power series that depend, in an essential way, on the *local ring structures* at $\underline{v} \in \underline{\mathbb{V}}$ — the *horizontal arrows* in each log-theta-lattice [i.e., the $\Theta^{\times \boldsymbol{\mu}}$-, $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-links] are *incompatible* with these local ring structures at $\underline{v} \in \underline{\mathbb{V}}$ in an essential way [cf. [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) Whereas the horizontal arrows in each log-theta-lattice [i.e., the $\Theta^{\times \boldsymbol{\mu}}$-, $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-links] allow one, roughly speaking, to **identify** the respective "$\mathcal{O}^{\times \boldsymbol{\mu}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ on either side of the horizontal arrow [cf. [IUTchII], Corollary 4.10, (iv)], in order to avail oneself of the theory of **log-shells** — which will play an essential role in the *multiradial representation of the Gaussian monoids* to be developed in §3 below — it is necessary for the "$\bullet$" [i.e., $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater] in which one operates to appear as the *codomain* of a $\mathfrak{log}$-link, i.e., of a vertical arrow of the log-theta-lattice [cf. the discussion of [AbsTopIII], Remark 5.10.2, (iii)]. That is to say, from the point of view of the goal of constructing the multiradial representation of the Gaussian monoids that is to be developed in §3 below,

> each execution of a *horizontal arrow* of the log-theta-lattice necessarily obligates a subsequent execution of a *vertical arrow* of the log-theta-lattice.

On the other hand, in light of the *noncommutativity* observed in (i), this **"intertwining"** *of the horizontal and vertical arrows of the log-theta-lattice* means

that the desired **multiradiality** — i.e., **simultaneous compatibility** with the arithmetic holomorphic structures on *both sides of a horizontal arrow* of the log-theta-lattice — can only be realized [cf. the discussion of Remark 1.2.2, (iii)] if one works with objects that are **invariant** *with respect to the vertical arrows* [i.e., with respect to the action of $\mathbb{Z}$ discussed in Proposition 1.3, (iv)], that is to say, with **"vertical cores"**, of the log-theta-lattice.

(iii) From the point of view of the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [AbsTopIII], §I5], the *vertical arrows* of the log-theta-lattice correspond to the *Frobenius morphism in positive characteristic*, whereas the *horizontal arrows* of the log-theta-lattice correspond to the *"transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$"*, i.e., the *mixed characteristic extension structure of a ring of Witt vectors* [cf. [IUTchI], Remark 3.9.3, (i)]. These correspondences are summarized in Fig. 1.3 below. In particular, the "intertwining of horizontal and vertical arrows of the log-theta-lattice" discussed in (ii) above may be thought of as the analogue, in the context of the theory of the present series of papers, of the well-known "intertwining between the mixed characteristic extension structure of a ring of Witt vectors and the Frobenius morphism in positive characteristic" that appears in the classical p-adic theory.

horizontal arrows of the *log-theta-lattice*	**mixed characteristic** extension structure of a ring of *Witt vectors*
vertical arrows of the *log-theta-lattice*	the **Frobenius** morphism in *positive characteristic*

Fig. 1.3: Analogy between the log-theta-lattice and p-adic Teichmüller theory

Remark 1.4.2.

(i) The horizontal and vertical arrows of the log-theta-lattices discussed in Definition 1.4 share the common property of being *incompatible with the local ring structures*, hence, in particular, with the *conventional scheme-theoretic basepoints* on either side of the arrow in question [cf. the discussion of [IUTchII], Remark 3.6.3, (i)]. On the other hand, whereas the linking data of the *vertical arrows* [i.e., the $\mathfrak{log}$-link] is **rigid** and corresponds to a **single fixed, rigid arithmetic holomorphic structure** in which *addition* and *multiplication* are subject to *"rotations"* [cf. the discussion of [AbsTopIII], §I3], the linking data of the *horizontal arrows* [i.e., the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-links] — i.e., more concretely, the "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ — is subject to a $\widehat{\mathbb{Z}}^\times$**-indeterminacy**, which has the effect of *obliterating the arithmetic holomorphic structure associated to a vertical line* of the log-theta-lattice [cf. the discussion of [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) If, in the spirit of the discussion of [IUTchII], Remark 1.11.2, (ii), one attempts to *"force"* the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow by

declaring — in the style of the $\mathfrak{log}$-link! — that these horizontal arrows induce an isomorphism of the respective "$\Pi_{\underline{v}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\rm non}$, then one must contend with a situation in which the "common arithmetic holomorphic structure rigidified by the isomorphic copies of $\Pi_{\underline{v}}$" is obliterated each time one takes into account the action of a nontrivial element of $\widehat{\mathbb{Z}}^\times$ [i.e., that arises from the $\widehat{\mathbb{Z}}^\times$-*indeterminacy* involved] on the corresponding "$\mathcal{O}^{\times\boldsymbol{\mu}}$". In particular, in order to keep track of the arithmetic holomorphic structure currently under consideration, one must, in effect, consider **paths** that record the sequence of "$\Pi_{\underline{v}}$-rigidifying" and "$\widehat{\mathbb{Z}}^\times$-indeterminacy" operations that one invokes. On the other hand, the horizontal lines of the log-theta-lattices given in Definition 1.4 amount, in effect, to *universal covering spaces* of the loops — i.e., *"unraveling paths of the loops"* [cf. the discussion of Remark 1.2.2, (vi)] — that occur as one invokes various series of "$\Pi_{\underline{v}}$-rigidifying" and "$\widehat{\mathbb{Z}}^\times$-indeterminacy" operations. Thus, in summary, any attempt as described above to *"force"* the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow does not result in any substantive simplification of the theory of the present series of papers. We refer the reader to [IUTchIV], Remark 3.6.3, for a discussion of a related topic.

We are now ready to state the *main result* of the present §1.

Theorem 1.5. **(Bi-cores of the Log-theta-lattice)** *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\rm bad}_{\rm mod},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Then any **Gaussian log-theta-lattice** *corresponding to this collection of initial Θ-data [cf. Definition 1.4] satisfies the following properties:*

(i) **(Vertical Coricity)** *The vertical arrows of the Gaussian log-theta-lattice induce* **full poly-isomorphisms** *between the respective associated $\mathcal{D}$-$\Theta^{\pm \rm ell}$NF-Hodge theaters*

$$\dots \ \stackrel{\sim}{\rightarrow}\ {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF} \ \stackrel{\sim}{\rightarrow}\ {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF} \ \stackrel{\sim}{\rightarrow}\ \dots$$

[cf. Proposition 1.3, (ii)]. Here, $n \in \mathbb{Z}$ is held fixed, while $m \in \mathbb{Z}$ is allowed to vary.

(ii) **(Horizontal Coricity)** *The horizontal arrows of the Gaussian log-theta-lattice induce* **full poly-isomorphisms** *between the respective associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\dots \ \stackrel{\sim}{\rightarrow}\ {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \stackrel{\sim}{\rightarrow}\ {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \stackrel{\sim}{\rightarrow}\ \dots$$

[cf. [IUTchII], Corollary 4.10, (iv)]. Here, $m \in \mathbb{Z}$ is held fixed, while $n \in \mathbb{Z}$ is allowed to vary.

(iii) **(Bi-coric $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-Prime-Strips)** *For $n, m \in \mathbb{Z}$, write ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$ for the $\mathcal{D}^{\vdash}$-prime-strip associated to the $\mathcal{F}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{F}^{\vdash}_{\triangle}$ labeled "$\triangle$" of the $\Theta^{\pm \rm ell}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \rm ell}\rm NF}$ [cf. [IUTchII], Corollary 4.10, (i)]; ${}^{n,m}\mathfrak{D}_{\succ}$ for the $\mathcal{D}$-prime-strip labeled "$\succ$" of the $\Theta^{\pm \rm ell}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \rm ell}\rm NF}$ [cf. [IUTchI],*

Definitions 6.11, (i), (iii); 6.13, (i)]. Let us **identify** *[cf. [IUTchII], Corollary 4.10, (i)] the collections of data*

$$\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_0 \text{ and } \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_{\langle \mathbb{F}_l^{*} \rangle}$$

via the **isomorphism** *of the final display of [IUTchII], Corollary 4.5, (iii), and denote by*

$$\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$$

the resulting $\mathcal{F}^{\vdash}$-prime-strip. [Thus, it follows immediately from the constructions involved — cf. the discussion of [IUTchII], Corollary 4.10, (i) — that there is a **natural identification isomorphism** $\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$*, where we write $\mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$ for the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{>})$.] Write*

$$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}), \quad \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$$

for the $\mathcal{F}^{\vdash\times}$-, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips determined by $\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$ [cf. [IUTchII], Definition 4.9, (vi), (vii)]. Thus, by applying the isomorphisms "$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}$, of [IUTchII], Corollary 4.5, (ii), [it follows immediately from the definitions that] there exists a **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$ for constructing an $\mathcal{F}^{\vdash\times}$-prime-strip $\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$, together with a* **functorial algorithm** *in the $\mathcal{D}$-prime-strip ${}^{n,m}\mathfrak{D}_{\succ}$ for constructing a* **natural isomorphism**

$$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$$

— i.e., in more intuitive terms, "$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$", hence also the associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$", may be naturally regarded, up to isomorphism, as objects constructed from ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$. Then the poly-isomorphisms of (i) [cf. Remark 1.3.2], (ii) induce, respectively, poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$$\ldots \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m+1}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \ldots$$

$$\ldots \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n+1,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \ldots$$

— where we note that, relative to the natural isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$ discussed above, the collection of isomorphisms that constitute the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the first line of the display is, in general, **strictly smaller** *than the collection of isomorphisms that constitute the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the second line of the display [cf. the existence of non-scheme-theoretic automorphisms of absolute Galois groups of MLF's, as discussed in [AbsTopIII], §I3]; the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the second line of the display are* **not full** *[cf. [IUTchII], Remark 1.8.1]. In particular, by composing these isomorphisms, one obtains* **poly-isomorphisms** *of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})$$

for arbitrary $n', m' \in \mathbb{Z}$. That is to say, in more intuitive terms, the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip "${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$", regarded up to a certain class of isomorphisms, is an

invariant — which we shall refer to as **"bi-coric"** *— of both the horizontal and the vertical arrows of the Gaussian log-theta-lattice. Finally, the Kummer isomorphisms "*$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$*" of [IUTchII], Corollary 4.6, (i), determine* **Kummer isomorphisms**

$$ {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) $$

which are **compatible** *with the poly-isomorphisms of (ii), as well as with the* $\times\boldsymbol{\mu}$*-Kummer structures at the* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *of the various* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips involved [cf. [IUTchII], Definition 4.9, (vi), (vii)]; a similar compatibility holds for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ *[cf. the discussion of the final portion of [IUTchII], Definition 4.9, (v)].*

(iv) **(Bi-coric Mono-analytic Log-shells)** *The poly-isomorphisms that constitute the bi-coricity property discussed in (iii) induce* **poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})\right\} $$

$$ \left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}))\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}))\right\} $$

for arbitrary $n, m, n', m' \in \mathbb{Z}$ *that are* **compatible** *with the* **natural poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}))\right\} $$

of Proposition 1.2, (viii). On the other hand, by applying the constructions of Definition 1.1, (i), (ii), to the collections of data "$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0$*" and "*$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle \mathbb{F}^{\circledast}_{l}\rangle}$*" used in [IUTchII], Corollary 4.10, (i), to construct* ${}^{n,m}\mathfrak{F}^{\vdash}_{\triangle}$ *[cf. Remark 1.3.2], one obtains a* **["holomorphic"] log-shell***, together with an enveloping "*$\underline{\mathfrak{log}}(-)$*" [cf. the pair "*$\mathcal{I}_{{}^{\dagger}\mathfrak{F}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F})$*" of Definition 1.1, (iii)], which we denote by*

$$ \mathcal{I}_{{}^{n,m}\mathfrak{F}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\triangle}) $$

[by means of a slight abuse of notation, since no $\mathcal{F}$*-prime-strip "*${}^{n,m}\mathfrak{F}_{\triangle}$*" has been defined!]. Then one has* **natural poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle})\right\} $$
$$ \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\triangle})\right\} $$

[cf. the poly-isomorphisms obtained in Proposition 1.2, (viii)]; here, the first " $\xrightarrow{\sim}$ *" may be regarded as being induced by the Kummer isomorphisms of (iii) and is* **compatible** *with the poly-isomorphisms induced by the poly-isomorphisms of (ii).*

(v) **(Bi-coric Mono-analytic Global Realified Frobenioids)** *Let* $n, m, n', m' \in \mathbb{Z}$*. Then the poly-isomorphisms of* $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}$ *induced by the full poly-isomorphisms of (i), (ii) induce [cf. [IUTchII], Corollaries 4.5, (ii); 4.10, (v)] an isomorphism of collections of data*

$$ (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$
$$ \xrightarrow{\sim} (\mathcal{D}^{\Vdash}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n',m'}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— *i.e., consisting of a Frobenioid, a bijection, and a collection of isomorphisms of topological monoids indexed by* $\underline{\mathbb{V}}$*. Moreover, this isomorphism of collections of data is* **compatible***, relative to the horizontal arrows of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of (ii)], with the* $\mathbb{R}_{>0}$**-orbits** *of the isomorphisms of collections of data*

$$({}^{n,m}\mathcal{C}^{\Vdash}_{\triangle}, \ \mathrm{Prime}({}^{n,m}\mathcal{C}^{\Vdash}_{\triangle}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ \{{}^{n,m}\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$
$$\xrightarrow{\sim} \quad (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}), \ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}}, \ \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

obtained by applying the functorial algorithm discussed in the final portion of [IUTchII], Corollary 4.6, (ii) [cf. also the latter portions of [IUTchII], Corollary 4.10, (i), (v)].

Proof. The various assertions of Theorem 1.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 1.5.1.

(i) Note that the theory of **conjugate synchronization** developed in [IUTchII] [cf., especially, [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)] plays an essential role in establishing the **bi-coricity** properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., at a more technical level, in constructing the objects equipped with a subscript "$\triangle$" that appear in Theorem 1.5, (iii); [IUTchII], Corollary 4.10, (i). That is to say, the conjugate synchronization determined by the various symmetrizing isomorphisms of [IUTchII], Corollaries 4.5, (iii); 4.6, (iii), may be thought of as a sort of **descent** mechanism that allows one to descend data that, *a priori*, is **label-dependent** [i.e., depends on the labels "$t \in \mathrm{LabCusp}^{\pm}(-)$"] to data that is **label-independent**. Here, it is important to recall that these labels depend, in an essential way, on the **"arithmetic holomorphic structures"** involved — i.e., at a more technical level, on the *geometric fundamental groups* involved — hence only make sense within a *vertical line* of the log-theta-lattice. That is to say, the significance of this transition from label-dependence to label-independence lies in the fact that this transition is precisely what allows one to construct objects that make sense in *horizontally adjacent* "•'s" of the log-theta-lattice, i.e., to construct *horizontally coric* objects [cf. Theorem 1.5, (ii); the second line of the fifth display of Theorem 1.5, (iii)]. On the other hand, in order to construct the horizontal arrows of the log-theta-lattice, it is necessary to work with **Frobenius-like structures** [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)]. In particular, in order to construct *vertically coric* objects [cf. the first line of the fifth display of Theorem 1.5, (iii)], it is necessary to pass to **étale-like** structures [cf. the discussion of Remark 1.2.4, (i)] by means of **Kummer isomorphisms** [cf. the final display of Theorem 1.5, (iii)]. Thus, in summary,

> the **bi-coricity** properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., roughly speaking, the bi-coricity of the various "$\mathcal{O}^{\times\boldsymbol{\mu}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — may be thought of as a consequence of the *intricate interplay of various aspects* of the theory of **Kummer-compatible conjugate synchronization** established in [IUTchII], Corollaries 4.5, (iii); 4.6, (iii).

(ii) In light of the central role played by the theory of conjugate synchronization in the constructions that underlie Theorem 1.5 [cf. the discussion of (i)], it is of interest to examine in more detail to what extent the highly technically nontrivial theory of conjugate synchronization may be replaced by a simpler apparatus. One naive approach to this problem is the following. Let G be a *topological group* [such as one of the absolute Galois groups $G_{\underline{v}}$ associated to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$]. Then one way to attempt to avoid the application of the theory of conjugate synchronization — which amounts, in essence, to the construction of a **diagonal embedding**

$$G \quad \hookrightarrow \quad G \times \ldots \times G$$

[cf. the notation "$\langle|\mathbb{F}_l|\rangle$", "$\langle\mathbb{F}_l^{\circledast}\rangle$" that appears in [IUTchII], Corollaries 3.5, 3.6, 4.5, 4.6] in a product of copies of G that, *a priori*, may only be identified with one another *up to conjugacy* [i.e., up to composition with an inner automorphism] — is to try to work, instead, with the $(G \times \ldots \times G)$**-conjugacy class** *of such a* **diagonal**. Here, to simplify the notation, let us assume that the above products of copies of G are, in fact, products of *two copies* of G. Then to *identify* the diagonal embedding $G \hookrightarrow G \times G$ with its $(G \times G)$-conjugates implies that one must consider **identifications**

$$(g, g) \quad \sim \quad (g, hgh^{-1}) \;=\; (g, [h, g] \cdot g)$$

[where $g, h \in G$] — i.e., one must identify (g, g) with the product of (g, g) with $(1, [h, g])$. On the other hand, the original purpose of working with distinct copies of G lies in considering **distinct Galois-theoretic Kummer classes** — corresponding to **distinct theta values** [cf. [IUTchII], Corollaries 3.5, 3.6] — at distinct components. That is to say, to identify elements of $G \times G$ that differ by a factor of $(1, [h, g])$ is **incompatible**, in an essential way, with the convention that such a factor $(1, [h, g])$ should correspond to distinct elements [i.e., "1" and "$[h, g]$"] at distinct components [cf. the discussion of Remark 1.5.3, (ii), below]. Here, we note that this incompatibility may be thought of as an essential consequence of the *highly nonabelian nature* of G, e.g., when G is taken to be a copy of $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Thus, in summary, this naive approach to replacing the theory of conjugate synchronization by a simpler apparatus is *inadequate* from the point of view of the theory of the present series of papers.

(iii) At a purely combinatorial level, the notion of conjugate synchronization is reminiscent of the **label synchronization** discussed in [IUTchI], Remark 4.9.2, (i), (ii). Indeed, both conjugate and label synchronization may be thought of as a sort of **combinatorial** *representation of the* **arithmetic holomorphic structure** associated to a single vertical line of the log-theta-lattice [cf. the discussion of [IUTchI], Remark 4.9.2, (iv)].

Remark 1.5.2.

(i) Recall that unlike the case with the action of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry on the various labeled copies of the absolute Galois group $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ [cf. [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)], it is *not* possible to establish an analogous theory of **conjugate synchronization** in the case of the $\mathbb{F}_l^{\circledast}$-symmetry for *labeled copies of* $\overline{F}$ [cf. [IUTchII], Remark 4.7.2]. This is to say, the closest analogue of the conjugate synchronization obtained in the local case relative to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is the

action of the $\mathbb{F}_l^{\circledast}$-symmetry on *labeled copies of the subfields* $F_{\mathrm{mod}} \subseteq F_{\mathrm{sol}} \subseteq \overline{F}$ and the *pseudo-monoid of* ${}_\infty\kappa$*-coric rational functions*, i.e., as discussed in [IUTchII], Corollaries 4.7, (ii); 4.8, (ii). One consequence of this incompatibility of the $\mathbb{F}_l^{\circledast}$-symmetry with the full algebraic closure $\overline{F}$ of F_{mod} is that, as discussed in [IUTchI], Remark 5.1.5, the reconstruction of the **ring structure** on labeled copies of the subfield $F_{\mathrm{sol}} \subseteq \overline{F}$ subject to the $\mathbb{F}_l^{\circledast}$-symmetry [cf. [IUTchII], Corollaries 4.7, (ii); 4.8, (ii)], **fails** to be **compatible** with the various **localization** operations that occur in the structure of a **$\mathcal{D}$-ΘNF-Hodge theater**. This is one quite essential reason why it is not possible to establish **bi-coricity** properties for, say, "$F_{\mathrm{sol}}^{\times}$" [which we regard as being equipped with the *ring structure* on the union of "$F_{\mathrm{sol}}^{\times}$" with $\{0\}$ — without which the abstract pair "$\mathrm{Gal}(F_{\mathrm{sol}}/F_{\mathrm{mod}}) \curvearrowright F_{\mathrm{sol}}^{\times}$" consisting of an abstract module equipped with the action of an abstract topological group is *not very interesting*] that are analogous to the bi-coricity properties established in Theorem 1.5, (iii), for "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [cf. the discussion of Remark 1.5.1, (i)]. From this point of view,

> the **bi-coric mono-analytic global realified Frobenioids** of Theorem 1.5, (v) — i.e., in essence, the notion of **"log-volume"** [cf. the point of view of Remark 1.2.2, (v)] — may be thought of as a sort of **"closest possible approximation"** to such a "bi-coric $F_{\mathrm{sol}}^{\times}$" [i.e., which does not exist].

Alternatively, from the point of view of the theory to be developed in §3 below,

> we shall apply the **bi-coric "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s"** of Theorem 1.5, (iii) — i.e., in the form of the **bi-coric mono-analytic log-shells** of Theorem 1.5, (iv) — to construct **"multiradial containers"** for the labeled copies of F_{mod} discussed above by applying the **localization functors** discussed in [IUTchII], Corollaries 4.7, (iii); 4.8, (iii).

That is to say, such "multiradial containers" will play the role of a **transportation mechanism** for "$F_{\mathrm{mod}}^{\times}$" — up to *certain indeterminacies*! — between *distinct arithmetic holomorphic structures* [i.e., distinct vertical lines of the log-theta-lattice].

(ii) In the context of the discussion of *"multiradial containers"* in (i) above, we recall [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)] that, in general, **Kummer theory** plays a *crucial role* precisely in situations in which one performs constructions — such as, for instance, the construction of the Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links — that are **"not bound to conventional scheme theory"**. That is to say, in the case of the labeled copies of "F_{mod}" discussed in (i), the **incompatibility** of **"solvable reconstructions"** of the **ring structure** with the **localization** operations that occur in a $\mathcal{D}$-ΘNF-Hodge theater [cf. [IUTchI], Remark 5.1.5] may be thought of as a reflection of the **dismantling** of the **global prime-tree structure** of a number field [cf. the discussion of [IUTchII], Remark 4.11.2, (iv)] that underlies the *construction of the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* performed in [IUTchI], [IUTchII], hence, in particular, as a reflection of the requirement of establishing a **Kummer-compatible** theory of **conjugate synchronization** relative to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry [cf. the discussion of Remark 1.5.1, (i)].

(iii) Despite the failure of labeled copies of "$F_{\mathrm{mod}}^{\times}$" to admit a natural *bi-coric structure* — a state of affairs that forces one to resort to the use of *"multiradial*

containers" in order to transport such labeled copies of "$F^{\times}_{\rm mod}$" to alien arithmetic holomorphic structures [cf. the discussion of (i) above] — the global Frobenioids associated to copies of "$F^{\times}_{\rm mod}$" nevertheless possess important properties that are *not* satisfied, for instance, by the bi-coric global realified Frobenioids discussed in Theorem 1.5, (v) [cf. also [IUTchI], Definition 5.2, (iv); [IUTchII], Corollary 4.5, (ii); [IUTchII], Corollary 4.6, (ii)]. Indeed, unlike the objects contained in the *realified* global Frobenioids that appear in Theorem 1.5, (v), the objects contained in the global Frobenioids associated to copies of "$F^{\times}_{\rm mod}$" correspond to *genuine* **"conventional arithmetic line bundles"**. In particular, by applying the **ring structure** of the copies of "$F_{\rm mod}$" under consideration, one can *push forward* such arithmetic line bundles so as to obtain *arithmetic vector bundles* over [the ring of rational integers] $\mathbb{Z}$ and then form *tensor products* of such arithmetic vector bundles. Such operations will play a key role in the theory of §3 below, as well as in the theory to be developed in [IUTchIV].

Remark 1.5.3.

(i) In [QuCnf] [cf. also [AbsTopIII], Proposition 2.6; [AbsTopIII], Corollary 2.7], a theory was developed concerning deformations of holomorphic structures on Riemann surfaces in which holomorphic structures are represented by means of **squares** or **rectangles** on the surface, while quasiconformal Teichmüller deformations of holomorphic structures are represented by **parallelograms** on the surface. That is to say, relative to suitable choices of local coordinates, quasiconformal Teichmüller deformations may be thought of as affine linear deformations in which one of the two underlying real dimensions of the Riemann surface is *dilated* by some factor $\in \mathbb{R}_{>0}$, while the other underlying real dimensions is *left undeformed*. From this point of view, the theory of **conjugate synchronization** — which may be regarded as a sort of **rigidity** that represents the *arithmetic holomorphic structure* associated to a vertical line of the log-theta-lattice [cf. the discussion given in [IUTchII], Remarks 4.7.3, 4.7.4, of the *uniradiality* of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry that underlies the phenomenon of conjugate synchronization] — may be thought of as a sort of **nonarchimedean arithmetic analogue** of the representation of holomorphic structures by means of squares/rectangles referred to above. That is to say, the *right angles* which are characteristic of squares/rectangles may be thought of as a sort of *synchronization* between the metrics of the two underlying real dimensions of a Riemann surface [i.e., metrics which, *a priori*, may differ by some *dilating* factor] — cf. Fig. 1.4 below. Here, we mention in passing that this point of view is reminiscent of the discussion of [IUTchII], Remark 3.6.5, (ii), in which the point of view is taken that the phenomenon of conjugate synchronization may be thought of as a reflection of the **coherence** of the **arithmetic holomorphic structures** involved.

(ii) Relative to the point of view discussed in (i), the approach described in Remark 1.5.1, (ii), to "avoiding conjugate synchronization by identifying the various conjugates of the diagonal embedding" corresponds — in light of the *highly nonabelian* nature of the groups involved! [cf. the discussion of Remark 1.5.1, (ii)] — to thinking of a holomorphic structure on a Riemann surface as an "equivalence class of holomorphic structures in the usual sense relative to the equivalence relation of differing by a Teichmüller deformation"! That is to say, such an [unconventional!]

approach to the definition of a holomorphic structure allows one to circumvent the issue of *rigidifying* the relationship between the metrics of the two underlying real dimensions of the Riemann surface — but only at the cost of rendering unfeasible any meaningful theory of "deformations of a holomorphic structure"!

(iii) The analogy discussed in (i) between conjugate synchronization [which arises from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry!] and the representation of a complex holomorphic structure by means of squares/rectangles may also be applied to the **"κ-sol-conjugate synchronization"** [cf. the discussion of [IUTchI], Remark 5.1.5] given in [IUTchII], Corollary 4.7, (ii); [IUTchII], Corollary 4.8, (ii), between, for instance, the various labeled non-realified and realified global Frobenioids by means of the $\mathbb{F}_l^{\circledast}$**-symmetry**. Indeed, this analogy is all the more apparent in the case of the *realified* global Frobenioids — which admit a natural $\mathbb{R}_{>0}$*-action*. Here, we observe in passing that, just as the theory of conjugate synchronization [via the $\mathbb{F}_l^{\rtimes\pm}$-symmetry] plays an essential role in the construction of the *local portions* of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta_{\text{gau}}^{\times\boldsymbol{\mu}}$-links given in [IUTchII], Corollary 4.10, (i), (ii), (iii),

> the **synchronization** of **global realified Frobenioids** by means of the $\mathbb{F}_l^{\circledast}$**-symmetry** may be related — via the isomorphisms of Frobenioids of the second displays of [IUTchII], Corollary 4.7, (iii); [IUTchII], Corollary 4.8, (iii) [cf. also the discussion of [IUTchII], Remark 4.8.1] — to the construction of the *global realified Frobenioid portion* of the $\Theta_{\text{gau}}^{\times\boldsymbol{\mu}}$**-link** given in [IUTchII], Corollary 4.10, (ii).

On the other hand, the synchronization involving the *non-realified* global Frobenioids may be thought of as a sort of *further rigidification* of the global realified Frobenioids. As discussed in Remark 1.5.2, (iii), this "further rigidification" will play an important role in the theory of §3 below.

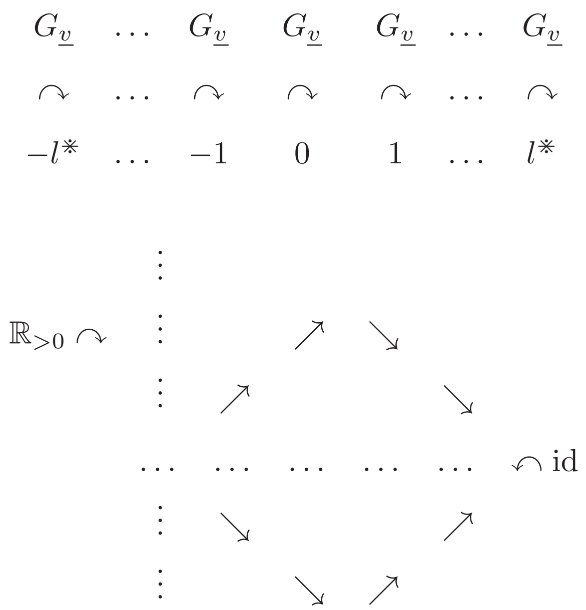

Fig. 1.4: Analogy between conjugate synchronization and the representation of complex holomorphic structures via squares/rectangles

Remark 1.5.4.

(i) As discussed in [IUTchII], Remark 3.8.3, (iii), one of the main themes of the present series of papers is the goal of giving an **explicit description** of what **one** *arithmetic holomorphic structure* — i.e., one *vertical line of the log-theta-lattice* — looks like from the point of view of a **distinct** *arithmetic holomorphic structure* — i.e., another vertical line of the log-theta-lattice — that is only related to the original arithmetic holomorphic structure via some mono-analytic core, e.g., the various bi-coric structures discussed in Theorem 1.5, (iii), (iv), (v). Typically, the objects of interest that are constructed within the original arithmetic holomorphic structure are **Frobenius-like** structures [cf. the discussion of [IUTchII], Remark 3.6.2], which, as we recall from the discussion of Remark 1.5.2, (ii) [cf. also the discussion of [IUTchII], Remark 3.6.2, (ii)], are necessary in order to perform constructions — such as, for instance, the construction of the Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links — that are **"not bound to conventional scheme theory"**. Indeed, the main example of such an object of interest consists precisely of the **Gaussian monoids** discussed in [IUTchII], §3, §4. Thus, the operation of describing such an object of interest from the point of view of a *distinct arithmetic holomorphic structure* may be broken down into *two steps*:

(a) passing from *Frobenius-like structures* to *étale-like structures* via various **Kummer isomorphisms**;

(b) transporting the resulting *étale-like structures* from one arithmetic holomorphic structure to another by means of various **multiradiality properties**.

In particular, the computation of what the object of interest looks like from the point of view of a distinct arithmetic holomorphic structure may be broken down into the computation of the **indeterminacies** or *"departures from rigidity"* that arise — i.e., the computation of *"what sort of* **damage** *is incurred to the object of interest"* — during the execution of each of these two steps (a), (b). We shall refer to the indeterminacies that arise from (a) as **Kummer-detachment indeterminacies** and to the indeterminacies that arise from (b) as **étale-transport indeterminacies**.

(ii) *Étale-transport indeterminacies* typically amount to the indeterminacies that occur as a result of the execution of various *"anabelian" or "group-theoretic" algorithms*. One fundamental example of such indeterminacies is constituted by the indeterminacies that occur in the context of Theorem 1.5, (iii), (iv), as a result of the existence of **automorphisms** of the various [copies of] local absolute Galois groups $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, which are *not of scheme-theoretic origin* [cf. the discussion of [AbsTopIII], §I3].

(iii) On the other hand, one important example, from the point of view of the theory of the present series of papers, of a *Kummer-detachment indeterminacy* is constituted by the **Frobenius-picture diagrams** given in Propositions 1.2, (x); 1.3, (iv) — i.e., the issue of *which path* one is to take from a particular "•" to the coric "◦". That is to say, despite the fact that these diagrams *fail to be commutative*, the **"upper semi-commutativity"** property satisfied by the **coric holomorphic**

log-shells involved [cf. the discussion of Remark 1.2.2, (iii)] may be regarded as a sort of computation, in the form of an *upper estimate*, of the Kummer-detachment indeterminacy in question. Another important example, from the point of view of the theory of the present series of papers, of a *Kummer-detachment indeterminacy* is given by the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** discussed in Remark 1.4.2 [cf. also the *Kummer isomorphisms* of the final display of Theorem 1.5, (iii)].

Section 2: Multiradial Theta Monoids

In the present §2, we **globalize** the **multiradial** portion of the local theory of **theta monoids** developed in [IUTchII], §1, §3, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [cf., especially, [IUTchII], Corollary 1.12; [IUTchII], Proposition 3.4] so as to cover the theta monoids/Frobenioids of [IUTchII], Corollaries 4.5, (iv), (v); 4.6, (iv), (v), and explain how the resulting theory may be fit into the framework of the **log-theta-lattice** developed in §1.

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. Let ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ be a $\Theta^{\pm\text{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data — cf. [IUTchI], Definition 6.13, (i)] and

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}\}_{n,m\in\mathbb{Z}}$$

a *collection of distinct* $\Theta^{\pm\text{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers, which we think of as arising from a *Gaussian log-theta-lattice*, as in Definition 1.4. We begin by reviewing the theory of theta monoids developed in [IUTchII].

Proposition 2.1. **(Vertical Coricity and Kummer Theory of Theta Monoids)** *We maintain the notation introduced above. Also, we shall use the notation* $\text{Aut}_{\mathcal{F}^{\Vdash}}(-)$ *to denote the group of automorphisms of the* $\mathcal{F}^{\Vdash}$*-prime-strip in parentheses. Then:*

(i) **(Vertically Coric Theta Monoids)** *In the notation of [IUTchII], Corollary 4.5, (iv), (v) [cf. also the assignment "0,* $\succ\ \mapsto\ >$*" of [IUTchI], Proposition 6.7], there are* **functorial algorithms** *in the* $\mathcal{D}$*- and* $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{D}_{>}$, ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ *associated to the* $\Theta^{\pm\text{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ *for constructing collections of data indexed by* $\underline{\mathbb{V}}$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto\ \Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}; \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto\ {}_{\infty}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$$

as well as a global realified Frobenioid

$$\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$$

equipped with a **bijection** $\text{Prime}(\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \overset{\sim}{\to} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$*, as described in detail in [IUTchII], Corollary 4.5, (v). In particular, each isomorphism of the full poly-isomorphism induced [cf. Theorem 1.5, (i)] by a* **vertical** *arrow of the* **Gaussian log-theta-lattice** *under consideration induces a compatible collection of isomorphisms*

$$\Psi_{\text{env}}({}^{n,m}\mathfrak{D}_{>}) \ \overset{\sim}{\to}\ \Psi_{\text{env}}({}^{n,m+1}\mathfrak{D}_{>}); \quad {}_{\infty}\Psi_{\text{env}}({}^{n,m}\mathfrak{D}_{>}) \ \overset{\sim}{\to}\ {}_{\infty}\Psi_{\text{env}}({}^{n,m+1}\mathfrak{D}_{>})$$

$$\mathcal{D}^{\Vdash}_{\text{env}}({}^{n,m}\mathfrak{D}^{\vdash}_{>}) \ \overset{\sim}{\to}\ \mathcal{D}^{\Vdash}_{\text{env}}({}^{n,m+1}\mathfrak{D}^{\vdash}_{>})$$

— *where the final isomorphism of Frobenioids is compatible with the respective bijections involving* "Prime$(-)$", *as well as with the respective local isomorphisms for each* $\underline{v} \in \underline{\mathbb{V}}$.

(ii) **(Kummer Isomorphisms)** *In the notation of [IUTchII], Corollary 4.6, (iv), (v), there are* **functorial algorithms** *in the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing collections of data indexed by* $\underline{\mathbb{V}}$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}; \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$$

as well as a global realified Frobenioid

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

equipped with a **bijection** $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \stackrel{\sim}{\to} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$, *as described in detail in [IUTchII], Corollary 4.6, (v). Moreover, there are* **functorial algorithms** *in* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing* **Kummer isomorphisms**

$$\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \ \stackrel{\sim}{\to} \ \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}); \quad {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \ \stackrel{\sim}{\to} \ {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$$

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \stackrel{\sim}{\to} \quad \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$$

— *where the final isomorphism of Frobenioids is compatible with the respective bijections involving* "Prime$(-)$", *as well as with the respective local isomorphisms for each* $\underline{v} \in \underline{\mathbb{V}}$ — *with the data discussed in (i) [cf. [IUTchII], Corollary 4.6, (iv), (v)]. Finally, the collection of data* $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *gives rise, in a natural fashion, to an* $\mathcal{F}^{\vdash}$*-prime-strip* $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *[cf. the* $\mathcal{F}^{\vdash}$*-prime-strip* "${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$" *of [IUTchII], Corollary 4.10, (ii)]; the global realified Frobenioid* $\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$, *equipped with the bijection* $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \stackrel{\sim}{\to} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$, *reviewed in (i), together with the* $\mathcal{F}^{\vdash}$*-prime-strip* $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$, *determine an* $\mathcal{F}^{\Vdash}$*-prime-strip* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *[cf. the* $\mathcal{F}^{\Vdash}$*-prime-strip* "${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$" *of [IUTchII], Corollary 4.10, (ii)]. In particular, the first and third Kummer isomorphisms of the above display may be interpreted as [compatible] isomorphisms*

$${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}} \ \stackrel{\sim}{\to} \ \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}); \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \ \stackrel{\sim}{\to} \ \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$$

of $\mathcal{F}^{\vdash}$*-,* $\mathcal{F}^{\Vdash}$*-prime-strips.*

(iii) **(Kummer Theory at Bad Primes)** *The portion at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *of the Kummer isomorphisms of (ii) is obtained by composing the Kummer isomorphisms of [IUTchII], Proposition 3.3, (i) — which, we recall, were defined by forming* **Kummer classes** *in the context of* **mono-theta environments** *that arise from* **tempered Frobenioids** — *with the isomorphisms on cohomology classes induced [cf. the upper left-hand portion of the first display of [IUTchII], Proposition 3.4, (i)] by the* **full poly-isomorphism of projective systems of mono-theta environments** "$\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}) \ \stackrel{\sim}{\to} \ \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$" *[cf. [IUTchII], Proposition 3.4; [IUTchII], Remark 4.2.1, (iv)] between projective systems of mono-theta environments that arise from tempered Frobenioids [i.e.,* "${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$"*] and projective systems of mono-theta*

environments that arise from the tempered fundamental group [i.e., "$^{\dagger}\mathcal{D}_{>,\underline{v}}$"] — cf. the left-hand portion of the third display of [IUTchII], Corollary 3.6, (ii), in the context of the discussion of [IUTchII], Remark 3.6.2, (i). Here, each "isomorphism on cohomology classes" is induced by the isomorphism on **exterior cyclotomes**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\mathcal{D}_{>,\underline{v}})) \quad \stackrel{\sim}{\to} \quad \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

determined by each of the isomorphisms that constitutes the full poly-isomorphism of projective systems of mono-theta environments discussed above. In particular, the **composite map**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\mathcal{D}_{>,\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \quad \to \quad (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times \boldsymbol{\mu}}$$

obtained by composing the result of applying "$\otimes\ \mathbb{Q}/\mathbb{Z}$" to this isomorphism on exterior cyclotomes with the **natural inclusion**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \quad \hookrightarrow \quad (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}$$

[cf. the notation of [IUTchII], Proposition 3.4, (i); the description given in [IUTchII], Proposition 1.3, (i), of the exterior cyclotome of a mono-theta environment that arises from a tempered Frobenioid] and the natural projection $(\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times} \twoheadrightarrow (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times \boldsymbol{\mu}}$ is equal to the **zero map***.*

(iv) **(Kummer Theory at Good Nonarchimedean Primes)** *The* **unit** *portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the Kummer isomorphisms of (ii) is obtained [cf. [IUTchII], Proposition 4.2, (iv)] as the unit portion of a "labeled version" of the* **isomorphism of ind-topological monoids** *equipped with a* **topological group action** *— i.e., in the language of [AbsTopIII], Definition 3.1, (ii), the isomorphism of "***MLF-Galois $\mathbb{TM}$-pairs***" — discussed in [IUTchII], Proposition 4.2, (i) [cf. also [IUTchII], Remark 1.11.1, (i), (a); [AbsTopIII], Proposition 3.2, (iv)]. In particular, the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the $\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}})$-***orbit** *of the second isomorphism of the final display of (ii) may be obtained as a "labeled version" of the* **"Kummer poly-isomorphism of semi-simplifications"** *given in the final display of [IUTchII], Proposition 4.2, (ii).*

(v) **(Kummer Theory at Archimedean Primes)** *The* **unit** *portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of the Kummer isomorphisms of (ii) is obtained [cf. [IUTchII], Proposition 4.4, (iv)] as the unit portion of a "labeled version" of the* **isomorphism of topological monoids** *discussed in [IUTchII], Proposition 4.4, (i). In particular, the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of the $\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}})$-***orbit** *of the second isomorphism of the final display of (ii) may be obtained as a "labeled version" of the* **"Kummer poly-isomorphism of semi-simplifications"** *given in the final display of [IUTchII], Proposition 4.4, (ii) [cf. also [IUTchII], Remark 4.6.1].*

(vi) **(Compatibility with Constant Monoids)** *The definition of the* **unit portion** *of the* **theta monoids** *involved [cf. [IUTchII], Corollary 4.10, (iv)] gives rise to* **natural isomorphisms**

$$^{\dagger}\mathfrak{F}^{\vdash\times}_{\triangle} \ \stackrel{\sim}{\to}\ ^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{env}}; \quad \mathfrak{F}^{\vdash\times}_{\triangle}(^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}) \ \stackrel{\sim}{\to}\ \mathfrak{F}^{\vdash\times}_{\mathrm{env}}(^{\dagger}\mathfrak{D}_{>})$$

— *i.e., where the morphism induced on* $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$*-prime-strips by the first displayed isomorphism is precisely the isomorphism of the first display of [IUTchII], Corollary 4.10, (iv) — of the respective associated* $\mathcal{F}^{\vdash \times}$*-prime-strips [cf. the notation of Theorem 1.5, (iii), where the label "*n, m*" is replaced by the label "*$\dagger$*"]. Moreover, these natural isomorphisms are* **compatible** *with the* **Kummer isomorphisms** *of (ii) above and Theorem 1.5, (iii).*

Proof. The various assertions of Proposition 2.1 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.1.1. The theory of **mono-theta environments** [cf. Proposition 2.1, (iii)] will play a crucial role in the theory of the present §2 [cf. Theorem 2.2, (ii); Corollary 2.3, (iv), below] in the *passage from* **Frobenius-like** *to* **étale-like** *structures* [cf. Remark 1.5.4, (i), (a)] at *bad primes*. In particular, the various **rigidity** properties of mono-theta environments established in [EtTh] play a fundamental role in ensuring that the resulting *"Kummer-detachment indeterminacies"* [cf. the discussion of Remark 1.5.4, (i)] are sufficiently mild so as to allow the establishment of the various reconstruction algorithms of interest. For this reason, we pause to review the *main properties* of mono-theta environments established in [EtTh] [cf. [EtTh], Introduction] — namely,

(a) **cyclotomic rigidity**
(b) **discrete rigidity**
(c) **constant multiple rigidity**
(d) **isomorphism class compatibility**
(e) **Frobenioid structure compatibility**

— and the roles played by these main properties in the theory of the present series of papers. Here, we remark that "isomorphism class compatibility" [i.e., (d)] refers to compatibility with the convention that various objects of the tempered Frobenioids [and their associated base categories] under consideration are known only *up to isomorphism* [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. In the Introduction to [EtTh], instead of referring to (d) in this form, we referred to the property of compatibility with the *topology of the tempered fundamental group*. In fact, however, this compatibility with the topology of the tempered fundamental group is a consequence of (d) [cf. [EtTh], Remarks 5.12.1, 5.12.2]. On the other hand, from the point of view of the present series of papers, the essential property of interest in this context is best understood as being the property (d).

(i) First, we recall that the significance, in the context of the theory of the present series of papers, of the *compatibility with the Frobenioid structure* of the tempered Frobenioids under consideration [i.e., (e)] — i.e., in particular, with the *monoidal portion*, equipped with its natural *Galois action*, of these Frobenioids — lies in the role played by this *"Frobenius-like"* monoidal portion in performing constructions — such as, for instance, the construction of the $\mathfrak{log}$-, Θ-, $\Theta^{\times \boldsymbol{\mu}}$-, or $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-links — that are **"not bound to conventional scheme theory"**, but may be related, via **Kummer theory**, to various *étale-like structures* [cf. the discussions of Remark 1.5.4, (i); [IUTchII], Remark 3.6.2, (ii); [IUTchII], Remark 3.6.4, (ii), (v)].

(ii) Next, we consider *isomorphism class compatibility* [i.e., (d)]. As discussed above, this compatibility corresponds to regarding each of the various objects of the tempered Frobenioids [and their associated base categories] under consideration as being known only *up to isomorphism* [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. As discussed in [IUTchII], Remark 3.6.4, (i), the significance of this property (d) in the context of the present series of papers lies in the fact that — unlike the case with the *projective systems* constituted by *Kummer towers* constructed from N-th power morphisms, which are compatible with only the *multiplicative*, but *not the additive structures* of the $p_{\underline{v}}$-adic local fields involved — *each individual object* in such a Kummer tower corresponds to a *single* field [i.e., as opposed to a projective system of multiplicative groups of fields]. This **field/ring structure** is necessary in order to apply the theory of the **$\mathfrak{log}$-link** developed in §1 — cf. the *vertical coricity* discussed in Proposition 2.1, (i). Note, moreover, that, unlike the $\mathfrak{log}$-, Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links, the N-th power morphisms that appear in a Kummer tower are **"algebraic"**, hence compatible with the conventional scheme theory surrounding the étale [or tempered] fundamental group. In particular, since the tempered Frobenioids under consideration may be constructed from such scheme-theoretic categories, the fundamental groups on either side of such an N-th power morphism may be related *up to an indeterminacy arising from an* **inner automorphism** of the tempered fundamental group [i.e., the "fundamental group" of the base category] under consideration — cf. the discussion of [IUTchII], Remark 3.6.3, (ii). On the other hand, the objects that appear in these Kummer towers necessarily arise from *nontrivial line bundles* [indeed, line bundles all of whose positive tensor powers are nontrivial!] on tempered coverings of a Tate curve — cf. the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9]; the crucial role played by the *commutator "$[-,-]$"* in the theory of *cyclotomic rigidity* [i.e., (a)] reviewed in (iv) below. In particular, the extraction of various N-th roots in a Kummer tower necessarily leads to *mutually non-isomorphic line bundles*, i.e., mutually non-isomorphic objects in the Kummer tower. From the point of view of *reconstruction algorithms*, such non-isomorphic objects may be **naturally** — i.e., **algorithmically** — related to another only via **indeterminate isomorphisms** [cf. (d)!]. This point of view is precisely the starting point of the discussion of — for instance, *"constant multiple indeterminacy"* in — [EtTh], Remarks 5.12.2, 5.12.3.

(iii) Next, we recall that the significance of *constant multiple rigidity* [i.e., (c)] in the context of the present series of papers lies in the construction of the **canonical splittings of theta monoids** via **restriction to the zero section** discussed, for instance, in [IUTchII], Corollary 1.12, (ii); [IUTchII], Proposition 3.3, (i); [IUTchII], Remark 1.12.2, (iv) [cf. also Remark 1.2.3, (i), of the present paper].

(iv) Next, we review the significance of *cyclotomic rigidity* [i.e., (a)] in the context of the present series of papers. First, we recall that this cyclotomic rigidity is essentially a consequence of the *nondegenerate* nature of the *commutator "$[-,-]$"* of the theta groups involved [cf. the discussion of [EtTh], Introduction; [EtTh], Remark 2.19.2]. Put another way, since this commutator is quadratic in nature, one may think of this nondegenerate nature of the commutator as a statement to the effect that *"the degree of the commutator is precisely* 2*"*. At a more concrete level, the cyclotomic rigidity arising from a mono-theta environment consists of

a certain specific isomorphism between the *interior* and *exterior cyclotomes* [cf. the discussion of [IUTchII], Definition 1.1, (ii); [IUTchII], Remark 1.1.1]. Put another way, one may think of this cyclotomic rigidity isomorphism as a sort of rigidification of a certain *"projective line of cyclotomes"*, i.e., the projectivization of the direct sum of the interior and exterior cyclotomes [cf. the computations that underlie [EtTh], Proposition 2.12]. In particular, this rigidification is fundamentally *nonlinear* in nature. Indeed, if one attempts to compose it with an N-th power morphism, then one is obliged to sacrifice constant multiple rigidity [i.e., (c)] — cf. the discussion of [EtTh], Remark 5.12.3. That is to say, the *distinguished nature* of the **"first power"** of the cyclotomic rigidity isomorphism is an important theme in the theory of [EtTh] [cf. the discussion of [EtTh], Remark 5.12.5; [IUTchII], Remark 3.6.4, (iii), (iv)]. The **multiradiality** of mono-theta-theoretic cyclotomic rigidity [cf. [IUTchII], Corollary 1.10] — which lies in stark contrast with the indeterminacies that arise when one attempts to give a multiradial formulation [cf. [IUTchII], Corollary 1.11; the discussion of [IUTchII], Remark 1.11.3] of the more classical *"MLF-Galois pair cyclotomic rigidity"* arising from local class field theory — will play a *central role* in the theory of the present §2 [cf. Theorem 2.2, Corollary 2.3 below].

(v) Finally, we review the significance of *discrete rigidity* [i.e., (b)] in the context of the present series of papers. First, we recall that, at a technical level, whereas cyclotomic rigidity may be regarded [cf. the discussion of (iv)] as a consequence of the fact that "the degree of the commutator is precisely 2", discrete rigidity may be regarded as a consequence of the fact that *"the degree of the commutator is ≤ 2"* [cf. the statements and proofs of [EtTh], Proposition 2.14, (ii), (iii)]. At a more concrete level, discrete rigidity assures one that one may restrict one's attentions to **$\mathbb{Z}$-multiples/powers** — as opposed to **$\widehat{\mathbb{Z}}$-multiples/powers** — of divisors, line bundles, and rational functions [such as, for instance, the *q-parameter*!] on the tempered coverings of a Tate curve that occur in the theory of [EtTh] [cf. [EtTh], Remark 2.19.4]. This prompts the following question:

> Can one develop a theory of *$\widehat{\mathbb{Z}}$-divisors/line bundles/rational functions* in, for instance, a parallel fashion to the way in which one considers *perfections* and *realifications* of Frobenioids in the theory of [FrdI]?

As far as the author can see at the time of writing, the answer to this question is *"no"*. Indeed, unlike the case with $\mathbb{Q}$ or $\mathbb{R}$, there is no notion of **positivity** [or negativity] in $\widehat{\mathbb{Z}}$. For instance, $-1 \in \widehat{\mathbb{Z}}$ may be obtained as a limit of positive integers. In particular, if one had a theory of $\widehat{\mathbb{Z}}$-divisors/line bundles/rational functions, then such a theory would necessarily require one to "confuse" positive [i.e., effective] and negative divisors, hence to work *birationally.* But to work birationally means, in particular, that one must sacrifice the conventional structure of isomorphisms [e.g., automorphisms] between line bundles — which plays an indispensable role, for instance, in the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9; the crucial role played by the *commutator* *"$[-,-]$"* in the theory of *cyclotomic rigidity* [i.e., (a)] reviewed in (iv) above].

Remark 2.1.2.

(i) In the context of the discussion of Remark 2.1.1, (v), it is of interest to recall [cf. [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii)] that the essential role played, in the context of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, by the *"global bookkeeping operations"* involving the **labels** of the evaluation points gives rise, in light of the **profinite** nature of the global étale fundamental groups involved, to a situation in which one must apply the "complements on tempered coverings" developed in [IUTchI], §2. That is to say, in the notation of the discussion given in [IUTchII], Remark 2.1.1, (i), of the various tempered coverings that occur at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, these "complements on tempered coverings" are applied precisely so as to allow one to restrict one's attention to the [*discrete*!] $\underline{\mathbb{Z}}$**-conjugates** — i.e., as opposed to [*profinite*!] $\widehat{\underline{\mathbb{Z}}}$-conjugates [where we write $\widehat{\underline{\mathbb{Z}}}$ for the profinite completion of $\underline{\mathbb{Z}}$] — of the theta functions involved. In particular, although such "evaluation-related issues", which will become relevant in the context of the theory of §3 below, do not play a role in the theory of the present §2, the role played by the theory of [IUTchI], §2, in the theory of the present series of papers may also be thought of as a sort of "discrete rigidity" — which we shall refer to as **"evaluation discrete rigidity"** — i.e., a sort of rigidity that is concerned with similar issues to the issues discussed in the case of *"mono-theta-theoretic discrete rigidity"* in Remark 2.1.1, (v), above.

(ii) Next, let us suppose that we are in the situation discussed in [IUTchII], Proposition 2.1. Fix $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Write $\Pi \stackrel{\mathrm{def}}{=} \Pi_{\underline{v}}$; $\widehat{\Pi}$ for the profinite completion of Π. Thus, we have *natural surjections* $\Pi \twoheadrightarrow l \cdot \underline{\mathbb{Z}}$ $(\subseteq \underline{\mathbb{Z}})$, $\widehat{\Pi} \twoheadrightarrow l \cdot \widehat{\underline{\mathbb{Z}}}$ $(\subseteq \widehat{\underline{\mathbb{Z}}})$. Write $\Pi^\dagger \stackrel{\mathrm{def}}{=} \widehat{\Pi} \times_{\widehat{\underline{\mathbb{Z}}}} \underline{\mathbb{Z}} \subseteq \widehat{\Pi}$. Next, we observe that from the point of view of the *evaluation points*, the evaluation discrete rigidity discussed in (i) corresponds to the issue of whether, relative to some arbitrarily chosen basepoint, the **"coordinates"** [i.e., element of the "torsor over $\underline{\mathbb{Z}}$" discussed in [IUTchII], Remark 2.1.1, (i)] of the evaluation point lie $\in \underline{\mathbb{Z}}$ or $\in \widehat{\underline{\mathbb{Z}}}$. Thus, if one is only concerned with the issue of arranging for these coordinates to lie $\in \underline{\mathbb{Z}}$, then one is led to pose the following question:

> Is it possible to simply use the *"partially tempered fundamental group"* $\Pi^\dagger$ instead of the "full" tempered fundamental group Π in the theory of the present series of papers?

The answer to this question is *"no"*. One way to see this is to consider the [easily verified] natural isomorphism

$$N_{\widehat{\Pi}}(\Pi^\dagger)/\Pi^\dagger \ \stackrel{\sim}{\to}\ \widehat{\underline{\mathbb{Z}}}/\underline{\mathbb{Z}}$$

involving the *normalizer* $N_{\widehat{\Pi}}(\Pi^\dagger)$ of $\Pi^\dagger$ in $\widehat{\Pi}$. One consequence of this isomorphism is that — unlike the tempered fundamental group Π [cf., e.g., [SemiAnbd], Theorems 6.6, 6.8] — the topological group $\Pi^\dagger$ *fails to satisfy various fundamental* **absolute anabelian** *properties* which play a *crucial role* in the theory of [EtTh], as well as in the present series of papers [cf., e.g., the theory of [IUTchII], §2]. At a more concrete level, unlike the case with the tempered fundamental group Π, the *profinite conjugacy indeterminacies* that act on $\Pi^\dagger$ give rise to $\widehat{\underline{\mathbb{Z}}}$*-translation*

indeterminacies acting on the coordinates of the evaluation points involved. That is to say, in the case of Π, such $\widehat{\underline{\mathbb{Z}}}$-translation indeterminacies are avoided precisely by applying the "complements on tempered coverings" developed in [IUTchI], §2 — i.e., in a word, as a consequence of the *"highly anabelian nature"* of the [full!] tempered fundamental group Π.

Theorem 2.2. (Kummer-compatible Multiradiality of Theta Monoids) *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Let ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *be a* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 6.13, (i)]. For* $\Box \in \{\Vdash, \Vdash\blacktriangleright\times\boldsymbol{\mu}, \vdash\times\boldsymbol{\mu}\}$, *write* $\mathrm{Aut}_{\mathcal{F}^{\Box}}(-)$ *for the group of automorphisms of the* $\mathcal{F}^{\Box}$*-prime-strip in parentheses [cf. [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (vi), (vii), (viii)].*

(i) **(Automorphisms of Prime-strips)** *The natural functors determined by assigning to an* $\mathcal{F}^{\Vdash}$*-prime-strip the associated* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*- and* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] and then composing with the natural isomorphisms of Proposition 2.1, (vi), determine natural homomorphisms*

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \ \to\ \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}(\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \ \twoheadrightarrow\ \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}))$$

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}) \ \to\ \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}) \ \twoheadrightarrow\ \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle})$$

— where the second arrows in each line are surjections — that are **compatible** *with the* **Kummer isomorphisms** *of Proposition 2.1, (ii), and Theorem 1.5, (iii) [cf. the final portions of Proposition 2.1, (iv), (v), (vi)].*

(ii) **(Kummer Aspects of Multiradiality at Bad Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. *Write*

$$_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \ \subseteq\ _{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}; \quad _{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \ \subseteq\ _{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$$

for the submonoids corresponding to the respective **splittings** *[cf. [IUTchII], Corollaries 3.5, (iii); 3.6, (iii)], i.e., the submonoids generated by "*$_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$*" [cf. the notation of [IUTchII], Proposition 3.1, (i)] and the respective* **torsion subgroups**. *Now consider the commutative diagram*

$$\begin{array}{ccccc}
_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} & \supseteq & _{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}} & \subseteq & _{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times}_{\underline{v}} \\
\downarrow & & \downarrow & & \downarrow \\
_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} & \supseteq & _{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\boldsymbol{\mu}}_{\underline{v}} & \subseteq & _{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times}_{\underline{v}}
\end{array}$$

$$\begin{array}{ccccc}
\twoheadrightarrow & _{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times\boldsymbol{\mu}}_{\underline{v}} & \stackrel{\sim}{\to} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}} \\
 & \downarrow & & \downarrow \\
\twoheadrightarrow & _{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times\boldsymbol{\mu}}_{\underline{v}} & \stackrel{\sim}{\to} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}}
\end{array}$$

— *where the inclusions "$\supseteq$", "$\subseteq$" are the natural inclusions; the surjections "$\twoheadrightarrow$" are the natural surjections; the superscript "$\boldsymbol{\mu}$" denotes the torsion subgroup; the superscript "$\times$" denotes the group of units; the superscript "$\times\boldsymbol{\mu}$" denotes the quotient "$(-)^{\times}/(-)^{\boldsymbol{\mu}}$"; the first four vertical arrows are the isomorphisms determined by the inverse of the second* **Kummer isomorphism** *of the third display of Proposition 2.1, (ii); ${}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}$ is as discussed in Theorem 1.5, (iii); ${}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle}$ is as discussed in [IUTchII], Corollary 4.10, (i); the final vertical arrow is the inverse of the* **"Kummer poly-isomorphism"** *determined by the second displayed isomorphism of [IUTchII], Corollary 4.6, (ii); the final upper horizontal arrow is the* **poly-isomorphism** *determined by composing the isomorphism determined by the inverse of the second displayed natural isomorphism of Proposition 2.1, (vi), with the poly-automorphism of $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}}$ induced by the* **full poly-automorphism** *of the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}$; the final lower horizontal arrow is the poly-automorphism determined by the condition that the final square be commutative. This commutative diagram is compatible with the various group actions involved relative to the following diagram*

$$\begin{array}{ccccc}\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) \\ & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) & \stackrel{\sim}{\to} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))\end{array}$$

[cf. the notation of [IUTchII], Proposition 3.1; [IUTchII], Remark 4.2.1, (iv); [IUTchII], Corollary 4.5, (iv)] — where "$\twoheadrightarrow$" denotes the natural surjection; "$\stackrel{\sim}{\to}$" denotes the full poly-automorphism of $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$. Finally, each of the various composite maps

$$_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}} \to \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}}$$

is equal to the **zero map** *[cf. ($b_{\underline{v}}$) below; the final portion of Proposition 2.1, (iii)]. In particular, the* **identity** *automorphism on the following objects is* **compatible***, relative to the various natural morphisms involved [cf. the above commutative diagram], with the collection of automorphisms of $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}}$ induced by* **arbitrary automorphisms** $\in \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle})$ *[cf. [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i)]:*

($a_{\underline{v}}$) $_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \supseteq {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}}$;

($b_{\underline{v}}$) $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z}$ *[cf. the discussion of Proposition 2.1, (iii)], relative to the natural isomorphism* $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z} \stackrel{\sim}{\to} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}}$ *of [IUTchII], Remark 1.5.2 [cf. ($a_{\underline{v}}$)];*

($c_{\underline{v}}$) the projective system of **mono-theta environments** $\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})$ *[cf. ($b_{\underline{v}}$)];*

($d_{\underline{v}}$) the splittings $_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \twoheadrightarrow {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}}$ *[cf. ($a_{\underline{v}}$)] by means of restriction to* **zero-labeled evaluation points** *[cf. [IUTchII], Proposition 3.1, (i)].*

Proof. The various assertions of Theorem 2.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 2.2.1. In light of the *central importance* of Theorem 2.2, (ii), in the theory of the present §2, we pause to examine the significance of Theorem 2.2, (ii), in more conceptual terms.

(i) In the situation of Theorem 2.2, (ii), let us write [for simplicity] $\Pi_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$, $G_{\underline{v}} \stackrel{\mathrm{def}}{=} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$, $\Pi_{\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$ [cf. (b$_{\underline{v}}$)]. Also, for simplicity, we write $(l \cdot \Delta_{\Theta}) \stackrel{\mathrm{def}}{=} (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$ [cf. [IUTchII], Proposition 1.5, (iii)]. Here, we recall that in fact, $(l \cdot \Delta_{\Theta})$ may be thought of as an object *constructed from* $\Pi_{\underline{v}}$ [cf. [IUTchII], Proposition 1.4]. Then the projective system of mono-theta environments $\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})$ [cf. (c$_{\underline{v}}$)] may be thought of as a sort of *"amalgamation of* $\Pi_{\underline{v}}$ *and* $\Pi_{\boldsymbol{\mu}}$*"*, where the amalgamation is such that it allows the *reconstruction of the* **mono-theta-theoretic cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta}) \ \stackrel{\sim}{\to}\ \Pi_{\boldsymbol{\mu}}$$

[cf. [IUTchII], Proposition 1.5, (iii)] — i.e., not just the $\widehat{\mathbb{Z}}^{\times}$*-orbit* of this isomorphism!

(ii) Now, in the notation of (i), the *Kummer classes* $\in {}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$ [cf. (a$_{\underline{v}}$)] constituted by the various *étale theta functions* may be thought of, for a suitable characteristic open subgroup $H \subseteq \Pi_{\underline{v}}$, as *twisted homomorphisms*

$$(\Pi_{\underline{v}} \supseteq)\ H \ \to\ \Pi_{\boldsymbol{\mu}}$$

whose restriction to $(l \cdot \Delta_{\Theta})$ coincides with the cyclotomic rigidity isomorphism $(l \cdot \Delta_{\Theta}) \stackrel{\sim}{\to} \Pi_{\boldsymbol{\mu}}$ discussed in (i). Then the essential content of Theorem 2.2, (ii), lies in the observation that

> since the **Kummer-theoretic link** between étale-like data and Frobenius-like data at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ is established by means of projective systems of **mono-theta environments** [cf. the discussion of Proposition 2.1, (iii)] — i.e., which do *not* involve the various monoids "$(-)^{\times\boldsymbol{\mu}}$"! — the **mono-theta-theoretic cyclotomic rigidity isomorphism** [i.e., *not just the* $\widehat{\mathbb{Z}}^{\times}$*-orbit* of this isomorphism!] is **immune** to the various automorphisms of the monoids "$(-)^{\times\boldsymbol{\mu}}$" which, from the point of view of the **multiradial formulation** to be discussed in Corollary 2.3 below, arise from isomorphisms of *coric data*.

Put another way, this "immunity" may be thought of as a sort of **decoupling** of the *"geometric"* [i.e., in the sense of the geometric fundamental group $\Delta_{\underline{v}} \subseteq \Pi_{\underline{v}}$] and *"base-field-theoretic"* [i.e., associated to the local absolute Galois group $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$] data which allows one to treat the exterior cyclotome $\Pi_{\boldsymbol{\mu}}$ — which, *a priori*, "looks base-field-theoretic" — as being part of the "geometric" data. From the point of view of the multiradial formulation to be discussed in Corollary 2.3 below [cf. also the discussion of [IUTchII], Remark 1.12.2, (vi)], this decoupling may be thought of as a sort of **splitting** into **purely radial** and **purely coric** components — i.e., with respect to which $\Pi_{\boldsymbol{\mu}}$ is *"purely radial"*, while the various monoids "$(-)^{\times\boldsymbol{\mu}}$" are *"purely coric"*.

(iii) Note that the immunity to automorphisms of the monoids "$(-)^{\times\boldsymbol{\mu}}$" discussed in (ii) lies in *stark contrast* to the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** that arise in the case of the cyclotomic rigidity isomorphisms constructed from **MLF-Galois pairs** in a fashion that makes *essential use of the monoids "$(-)^{\times\boldsymbol{\mu}}$"*, as discussed in [IUTchII], Corollary 1.11; [IUTchII], Remark 1.11.3. In the following discussion, let us write "$\mathcal{O}^{\times\boldsymbol{\mu}}$" for the various monoids "$(-)^{\times\boldsymbol{\mu}}$" that occur in the situation of Theorem 2.2; also, we shall use similar notation "$\mathcal{O}^{\boldsymbol{\mu}}$", "$\mathcal{O}^{\times}$", "$\mathcal{O}^{\rhd}$", "$\mathcal{O}^{\mathrm{gp}}$", "$\mathcal{O}^{\widehat{\mathrm{gp}}}$" [cf. the notational conventions of [IUTchII], Example 1.8, (ii), (iii), (iv), (vii)]. Thus, we have a diagram

$$\begin{array}{ccccccccc} \mathcal{O}^{\boldsymbol{\mu}} & \subseteq & \mathcal{O}^{\times} & \subseteq & \mathcal{O}^{\rhd} & \subseteq & \mathcal{O}^{\mathrm{gp}} & \subseteq & \mathcal{O}^{\widehat{\mathrm{gp}}} \\ & \searrow & \downarrow & & & & & & \\ & & \mathcal{O}^{\times\boldsymbol{\mu}} & & & & & & \end{array}$$

of natural morphisms between monoids equipped with $\Pi_{\underline{v}}$-actions. Relative to this notation, the essential *input data* for the cyclotomic rigidity isomorphism constructed from an MLF-Galois pair is given by "$\mathcal{O}^{\rhd}$" [cf. [IUTchII], Corollary 1.11, (a)]. On the other hand — unlike the case with $\mathcal{O}^{\boldsymbol{\mu}}$ — a $\widehat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times\boldsymbol{\mu}}$ does not lie under an *identity action* on $\mathcal{O}^{\times}$! That is to say, a $\widehat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times\boldsymbol{\mu}}$ can only be *lifted naturally* to $\widehat{\mathbb{Z}}^{\times}$-indeterminacies on $\mathcal{O}^{\times}$, $\mathcal{O}^{\widehat{\mathrm{gp}}}$ [cf. Fig. 2.1 below; [IUTchII], Corollary 1.11, (a), in the case where one takes "Γ" to be $\widehat{\mathbb{Z}}^{\times}$; [IUTchII], Remark 1.11.3, (ii)]. In the presence of such $\widehat{\mathbb{Z}}^{\times}$-indeterminacies, one can only recover the $\widehat{\mathbb{Z}}^{\times}$*-orbit* of the MLF-Galois-pair-theoretic cyclotomic rigidity isomorphism.

$$\begin{array}{c} \widehat{\mathbb{Z}}^{\times} \curvearrowright \qquad \widehat{\mathbb{Z}}^{\times} \curvearrowright \qquad \widehat{\mathbb{Z}}^{\times} \curvearrowright \\ \boxed{\begin{array}{ccccccccc} \mathcal{O}^{\times\boldsymbol{\mu}} & \twoheadleftarrow & \mathcal{O}^{\times} & \subseteq & \mathcal{O}^{\rhd} & \subseteq & \mathcal{O}^{\mathrm{gp}} & \subseteq & \mathcal{O}^{\widehat{\mathrm{gp}}} \\ & & (\supseteq \mathcal{O}^{\boldsymbol{\mu}}) & & & & & & \end{array}} \end{array}$$

Fig. 2.1: *Induced* $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* in the case of MLF-Galois pair cyclotomic rigidity

$$\begin{array}{ccc} \mathrm{id} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \curvearrowright \\ \boxed{\Pi_{\boldsymbol{\mu}} \stackrel{\sim}{\to} \mathcal{O}^{\boldsymbol{\mu}}} & \to & \boxed{\mathcal{O}^{\times\boldsymbol{\mu}}} \end{array}$$

Fig. 2.2: *Insulation from* $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* in the case of mono-theta-theoretic cyclotomic rigidity

(iv) Thus, in summary, [cf. Fig. 2.2 above]

> **mono-theta-theoretic cyclotomic rigidity** plays an essential role in the theory of the present §2 — and, indeed, in the theory of the present series of papers! — in that it serves to **insulate** the **étale theta function** from the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** which act on the **coric log-shells** [i.e., the various monoids "$(-)^{\times\boldsymbol{\mu}}$"].

The techniques that underlie the resulting *multiradiality of theta monoids* [cf. Corollary 2.3 below], cannot, however, be applied immediately to the case of *Gaussian monoids*. That is to say, the corresponding multiradiality of Gaussian monoids, to be discussed in §3 below, requires one to apply the theory of *log-shells* developed in §1 [cf. [IUTchII], Remark 2.9.1, (iii); [IUTchII], Remark 3.4.1, (ii); [IUTchII], Remark 3.7.1]. On the other hand, as we shall see in §3 below, the multiradiality of Gaussian monoids **depends** *in an essential way* on the multiradiality of theta monoids discussed in the present §2 as a sort of *"essential first step"* constituted by the *decoupling* discussed in (ii) above. Indeed, if one tries to consider the **Kummer theory** of the **theta values** [i.e., the "$\underline{\underline{q}}_{v}^{j^2}$" — cf. [IUTchII], Remark 2.5.1, (i)] just as elements of the *base field* — i.e., *without availing oneself of the theory of the étale theta function* — then it is difficult to see how to *rigidify the cyclotomes* involved by any means other than the theory of *MLF-Galois pairs* discussed in (iii) above. But, as discussed in (iii) above, this approach to cyclotomic rigidity gives rise to $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* — i.e., to *confusion* between the theta values "$\underline{\underline{q}}_{v}^{j^2}$" and their $\widehat{\mathbb{Z}}^{\times}$*-powers*, which is unacceptable from the point of view of the theory of the present series of papers! For another approach to understanding the indispensability of the multiradiality of theta monoids, we refer to Remark 2.2.2 below.

Remark 2.2.2.

(i) One way to understand the very *special role* played by the **theta values** [i.e., the values of the theta function] in the theory of the present series of papers is to consider the following naive question:

> Can one develop a similar theory to the theory of the present series of papers in which one replaces the $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-link
>
> $$\underline{\underline{q}} \quad \mapsto \quad \underline{\underline{q}}^{\left(\begin{smallmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{smallmatrix}\right)}$$
>
> [cf. [IUTchII], Remark 4.11.1] by a correspondence of the form
>
> $$\underline{\underline{q}} \ \mapsto \ \underline{\underline{q}}^{\lambda}$$
>
> — where λ is some *arbitrary positive integer*?

The answer to this question is *"no"*. Indeed, such a correspondence does not come equipped with the extensive **multiradiality** machinery — such as **mono-theta-theoretic cyclotomic rigidity** and the **splittings** determined by **zero-labeled**

evaluation points — that has been developed for the étale theta function [cf. the discussion of Step (vi) of the proof of Corollary 3.12 of §3 below]. For instance, the lack of mono-theta-theoretic cyclotomic rigidity means that one does not have an apparatus for **insulating** the **Kummer classes** of such a correspondence from the $\widehat{\mathbb{Z}}^{\times}$-indeterminacies that act on the various monoids "$(-)^{\times\boldsymbol{\mu}}$" [cf. the discussion of Remark 2.2.1, (iv)]. The splittings determined by zero-labeled evaluation points also play an essential role in **decoupling** these monoids "$(-)^{\times\boldsymbol{\mu}}$" — i.e., the **coric log-shells** — from the **"purely radial"** [or, put another way, "value group"] portion of such a correspondence "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$" [cf. the discussion of (iii) below; Remark 2.2.1, (ii); [IUTchII], Remark 1.12.2, (vi)]. Note, moreover, that if one tries to realize such a multiradial splitting via *evaluation* — i.e., in accordance with the principle of **"Galois evaluation"** [cf. the discussion of [IUTchII], Remark 1.12.4] — for a correspondence "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$" by, for instance, taking λ to be *one of the "j^2"* [where j is a positive integer] that appears as a value of the étale theta function, then one must contend with issues of **symmetry** between the zero-labeled evaluation point and the evaluation point corresponding to λ — i.e., symmetry issues that are resolved in the theory of the present series of papers by means of the theory surrounding the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the discussion of [IUTchII], Remarks 2.6.2, 3.5.2]. As discussed in [IUTchII], Remark 2.6.3, this sort of situation leads to numerous *conditions on the collection of evaluation points* under consideration. In particular, ultimately, it is difficult to see how to construct a theory as in the present series of papers for any collection of evaluation points other than the collection that is in fact adopted in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-link.

(ii) As discussed in Remark 2.2.1, (iv), we shall be concerned, in §3 below, with developing multiradial formulations for Gaussian monoids. These multiradial formulations will be subject to certain *indeterminacies*, which — although *sufficiently mild* to allow the execution of the *volume computations* that will be the subject of [IUTchIV] — are, nevertheless, *substantially more severe* than the indeterminacies that occur in the multiradial formulation given for theta monoids in the present §2 [cf. Corollary 2.3 below]. Indeed, the indeterminacies in the multiradial formulation given for theta monoids in the present §2 — which essentially consist of *multiplication by roots of unity* [cf. [IUTchII], Proposition 3.1, (i)] — are *essentially negligible* and may be regarded as a consequence of the highly nontrivial **Kummer theory** surrounding **mono-theta environments** [cf. Proposition 2.1, (iii); Theorem 2.2, (ii)], which, as discussed in Remark 2.2.1, (iv), cannot be mimicked for "theta values regarded just as elements of the base field". That is to say, the quite **exact** nature of the multiradial formulation for theta monoids — i.e., which contrasts sharply with the somewhat **approximate** nature of the multiradial formulation for Gaussian monoids to be developed in §3 — constitutes another *important ingredient* of the theory of the present paper that one must sacrifice if one attempts to work with correspondences $\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$ as discussed in (i), i.e., correspondences which do not come equipped with the extensive multiradiality machinery that arises as a consequence of the theory of the *étale theta function* developed in [EtTh].

(iii) One way to understand the significance, in the context of the discussions of (i) and (ii) above, of the **multiradial coric/radial decouplings** furnished by the splittings determined by the zero-labeled evaluation points is as follows. Ultimately, in order to establish, in §3 below, multiradial formulations for Gaussian

monoids, it will be of crucial importance to pass from the **Frobenius-like theta monoids** that appear in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link to **vertically coric étale-like** objects by means of **Kummer theory** [cf. the discussions of Remarks 1.2.4, (i); 1.5.4, (i), (iii)], in the context of the relevant **$\mathfrak{log}$-Kummer correspondences**, as discussed, for instance, in Remark 3.12.2, (iv), (v), below [cf. also [IUTchII], Remark 1.12.2, (iv)]. On the other hand, in order to obtain formulations expressed in terms that are meaningful from the point of view of the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link, it is necessary [cf. the discussion of Remark 3.12.2, (iv), (v), below] to relate this **Kummer theory** of **theta monoids** in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link to the Kummer theory constituted by the $\times\boldsymbol{\mu}$-Kummer structures that appear in the **horizontally coric** portion of the data that constitutes the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link [cf. Theorem 1.5, (ii)]. This is precisely what is achieved by the **Kummer-compatibility** of the multiradial splitting via *evaluation* — i.e., in accordance with the principle of **"Galois evaluation"** [cf. the discussion of [IUTchII], Remark 1.12.4]. This state of affairs [cf., especially, the two displays of [IUTchII], Corollary 1.12, (ii); the final arrow of the diagram "$(\dagger_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$" of [IUTchII], Corollary 1.12, (iii)] is illustrated in Fig. 2.3 below.

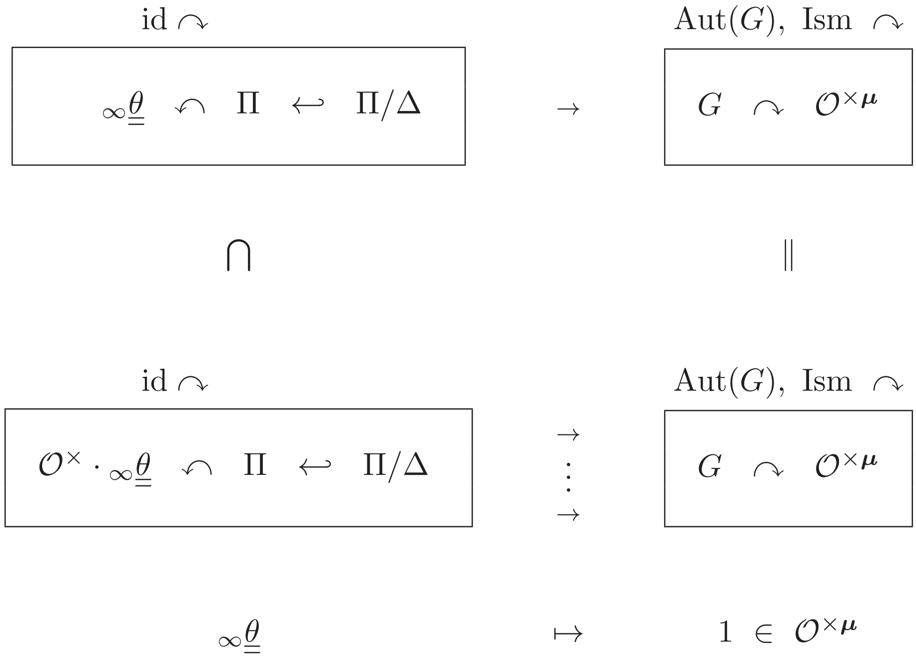

Fig. 2.3: *Kummer-compatible* splittings via *evaluation* at zero-labeled evaluation points [i.e., "$\Pi \hookleftarrow \Pi/\Delta$"]

Here, the *multiple arrows* [i.e., indicated by means of the "$\rightarrow$'s" separated by vertical dots] in the *lower portion* of the diagram correspond to the fact that the "$\mathcal{O}^{\times}$" on the left-hand side of this lower portion is related to the "$\mathcal{O}^{\times\boldsymbol{\mu}}$" on the right-hand side via an Ism-*orbit* of morphisms; the analogous arrow in the *upper portion* of the diagram consists of a *single arrow* [i.e., "$\rightarrow$"] and corresponds to the fact that the restriction of the multiple arrows in the lower portion of the diagram to "${}_{\infty}\underline{\underline{\theta}}$" amounts to a single arrow, i.e., precisely as a consequence of the fact that ${}_{\infty}\underline{\underline{\theta}} \mapsto 1 \in \mathcal{O}^{\times\boldsymbol{\mu}}$ [cf. the situation illustrated in Fig. 2.2]. On the other hand, the "Π/Δ's" on the left-hand side of both the upper and the lower portions of the

diagram are related to the "G's" on the right-hand side via the unique tautological $\mathrm{Aut}(G)$-orbit of isomorphisms. Thus, from the point of view of Fig. 2.3, the crucial **Kummer-compatibility** discussed above may be understood as the statement that

> the **multiradial** structure [cf. the lower portion of Fig. 2.3] on the "theta monoid $\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}$" furnished by the **splittings** via **Galois evaluation** into **coric/radial** components is **compatible** with the relationship between the respective **Kummer theories** of the "$\mathcal{O}^{\times}$" portion of "$\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}$" [on the left] and the coric "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [on the right].

This state of affairs lies in *stark contrast* to the situation that arises in the case of a naive correspondence of the form "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$" as discussed in (i): That is to say, in the case of such a naive correspondence, the corresponding arrows "$\rightarrow$" of the analogue of Fig. 2.3 map

$$\underline{\underline{q}}^{\lambda} \quad \mapsto \quad 1 \ \in \ \mathcal{O}^{\times\boldsymbol{\mu}}$$

and hence are **fundamentally incompatible** with passage to **Kummer classes**, i.e., since the Kummer class of $\underline{\underline{q}}^{\lambda}$ in a suitable cohomology group of Π/Δ is *by no means* mapped, via the poly-isomorphism $\Pi/\Delta \overset{\sim}{\rightarrow} G$, to the *trivial element* of the relevant cohomology group of G.

We conclude the present §2 with the following **multiradial** interpretation [cf. [IUTchII], Remark 4.1.1, (iii); [IUTchII], Remark 4.3.1] — in the spirit of the *étale-picture of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters* of [IUTchII], Corollary 4.11, (ii) — of the theory surrounding Theorem 2.2.

Corollary 2.3. **(Étale-picture of Multiradial Theta Monoids)** *In the notation of Theorem 2.2, let*

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$$

be a **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from a* **Gaussian log-theta-lattice** *[cf. Definition 1.4]. Write ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater associated to ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. Consider the* **radial environment** *[cf. [IUTchII], Example 1.7, (ii)] defined as follows. We define a collection of* **radial data**

$$^{\dagger}\mathfrak{R} \ = \ ({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}, \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{bad}}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}), \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \overset{\sim}{\rightarrow} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}))$$

to consist of

($a_{\mathfrak{R}}$) a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$;

($b_{\mathfrak{R}}$) the $\mathcal{F}^{\Vdash}$-prime-strip $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ associated to ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Proposition 2.1, (ii)];

($c_{\mathfrak{R}}$) the data ($a_{\underline{v}}$), ($b_{\underline{v}}$), ($c_{\underline{v}}$), ($d_{\underline{v}}$) of Theorem 2.2, (ii), for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, which we denote by ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}}$;

($d_{\mathfrak{R}}$) the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$ associated to ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Theorem 1.5, (iii)];

($e_{\mathfrak{R}}$) the **full poly-isomorphism** *of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$.*

We define a morphism between two collections of radial data ${}^{\dagger}\mathfrak{R} \rightarrow {}^{\ddagger}\mathfrak{R}$ [where we apply the evident notational conventions with respect to "†" and "‡"] to consist of data as follows:

($a_{\mathrm{Mor}_{\mathfrak{R}}}$) an isomorphism of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$;

($b_{\mathrm{Mor}_{\mathfrak{R}}}$) the isomorphism of $\mathcal{F}^{\Vdash}$-prime-strips $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{D}_{>})$ induced by the isomorphism of ($a_{\mathrm{Mor}_{\mathfrak{R}}}$);

($c_{\mathrm{Mor}_{\mathfrak{R}}}$) the isomorphism between collections of data ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}^{\mathrm{bad}}$ induced by the isomorphism of ($a_{\mathrm{Mor}_{\mathfrak{R}}}$);

($d_{\mathrm{Mor}_{\mathfrak{R}}}$) an isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle})$;

($e_{\mathrm{Mor}_{\mathfrak{R}}}$) we observe that the isomorphisms of ($b_{\mathrm{Mor}_{\mathfrak{R}}}$) and ($d_{\mathrm{Mor}_{\mathfrak{R}}}$) are necessarily compatible with the poly-isomorphisms of ($e_{\mathfrak{R}}$) for "†", "‡".

We define a collection of **coric data**

$$ {}^{\dagger}\mathfrak{C} \;=\; ({}^{\dagger}\mathfrak{D}^{\vdash}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash})) $$

to consist of

($a_{\mathfrak{C}}$) a $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}$;

($b_{\mathfrak{C}}$) the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash})$ associated to ${}^{\dagger}\mathfrak{D}^{\vdash}$ [cf. [IUTchII], Corollary 4.5, (ii); [IUTchII], Definition 4.9, (vi), (vii)].

We define a morphism between two collections of coric data ${}^{\dagger}\mathfrak{C} \rightarrow {}^{\ddagger}\mathfrak{C}$ [where we apply the evident notational conventions with respect to "†" and "‡"] to consist of data as follows:

($a_{\mathrm{Mor}_{\mathfrak{C}}}$) an isomorphism of $\mathcal{D}^{\vdash}$-prime-strips ${}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$;

($b_{\mathrm{Mor}_{\mathfrak{C}}}$) an isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ that induces the isomorphism ${}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$ on associated $\mathcal{D}^{\vdash}$-prime-strips of ($a_{\mathrm{Mor}_{\mathfrak{C}}}$).

The **radial algorithm** *is given by the assignment*

$$ {}^{\dagger}\mathfrak{R} \;=\; ({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}, \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{bad}}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}), \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) $$
$$ \mapsto \;\; {}^{\dagger}\mathfrak{C} \;=\; ({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) $$

— *together with the assignment on morphisms determined by the data of* $(d_{\mathrm{Mor}_{\mathfrak{R}}})$. *Then:*

(i) The functor associated to the radial algorithm defined above is **full** *and* **essentially surjective**. *In particular, the radial environment defined above is* **multiradial**.

(ii) Each $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, *for* $n, m \in \mathbb{Z}$, *defines, in an evident way, an associated collection of radial data* ${}^{n,m}\mathfrak{R}$. *The poly-isomorphisms induced by the* **vertical** *arrows of the* **Gaussian log-theta-lattice** *under consideration [cf. Theorem 1.5, (i)] induce poly-isomorphisms of radial data* $\ldots \xrightarrow{\sim} {}^{n,m}\mathfrak{R} \xrightarrow{\sim} {}^{n,m+1}\mathfrak{R} \xrightarrow{\sim} \ldots$. *Write*

$$ {}^{n,\circ}\mathfrak{R} $$

for the collection of radial data obtained by identifying the various ${}^{n,m}\mathfrak{R}$, *for* $m \in \mathbb{Z}$, *via these poly-isomorphisms and* ${}^{n,\circ}\mathfrak{C}$ *for the collection of coric data associated, via the radial algorithm defined above, to the radial data* ${}^{n,\circ}\mathfrak{R}$. *In a similar vein, the* **horizontal** *arrows of the Gaussian log-theta-lattice under consideration induce full poly-isomorphisms* $\ldots \xrightarrow{\sim} {}^{n,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} \ldots$ *of* $\mathcal{D}^{\vdash}$*-prime-strips [cf. Theorem 1.5, (ii)]. Write*

$$ {}^{\circ,\circ}\mathfrak{C} $$

for the collection of coric data obtained by identifying the various ${}^{n,\circ}\mathfrak{C}$, *for* $n \in \mathbb{Z}$, *via these poly-isomorphisms. Thus, by applying the radial algorithm defined above to each* ${}^{n,\circ}\mathfrak{R}$, *for* $n \in \mathbb{Z}$, *we obtain a diagram — i.e., an* **étale-picture of radial data** *— as in Fig. 2.4 below. This diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., the labels* $n \in \mathbb{Z}$*] and is* **compatible**, *in the evident sense, with the étale-picture of* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters of [IUTchII], Corollary 4.11, (ii).*

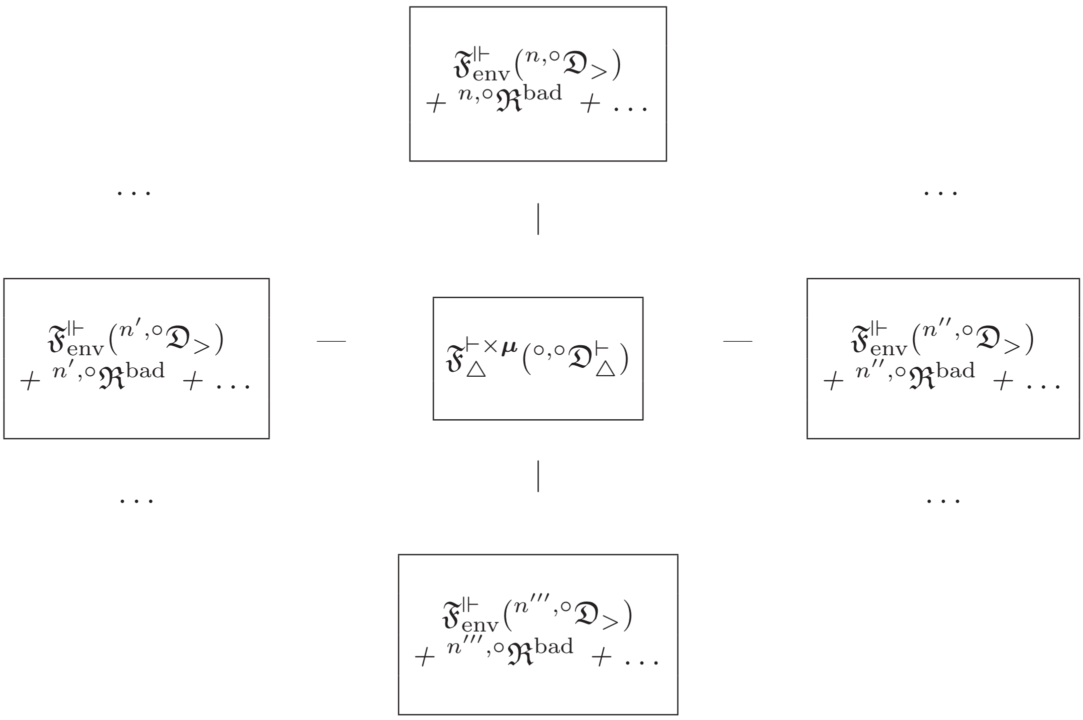

Fig. 2.4: Étale-picture of radial data

(iii) The [poly-]isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips of/induced by* $(e_{\mathfrak{R}})$, $(b_{\mathrm{Mor}_{\mathfrak{R}}})$, $(d_{\mathrm{Mor}_{\mathfrak{R}}})$ *[cf. also* $(e_{\mathrm{Mor}_{\mathfrak{R}}})$*] are* **compatible**, *relative to the* **Kummer isomorphisms** *of Proposition 2.1, (ii) [cf. also Proposition 2.1, (vi)], and Theorem 1.5, (iii), with the poly-isomorphisms — arising from the* **horizontal arrows** *of the Gaussian log-theta-lattice — of Theorem 1.5, (ii).*

(iv) The **algorithmic construction** *of the isomorphisms* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{D}_{>})$, ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}^{\mathrm{bad}}$ *of* $(b_{\mathrm{Mor}_{\mathfrak{R}}})$, $(c_{\mathrm{Mor}_{\mathfrak{R}}})$, *as well as of the* **Kummer isomorphisms** *and* **poly-isomorphisms of projective systems of mono-theta environments** *discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii)], and Theorem 1.5, (iii), (v), are* **compatible** *[cf. the final portions of Theorems 1.5, (v); 2.2, (ii)] with the* **horizontal arrows** *of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of Theorem 1.5, (ii)], in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the Gaussian log-theta-lattice.*

Proof. The various assertions of Corollary 2.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.3.1.

(i) In the context of the étale-picture of Fig. 2.4, it is of interest to recall the point of view of the discussion of [IUTchII], 1.12.5, (i), (ii), concerning the analogy between **étale-pictures** in the theory of the present series of papers and the **polar coordinate representation** of the **classical Gaussian integral**.

(ii) The étale-picture discussed in Corollary 2.3, (ii), may be thought of as a sort of **canonical splitting** of the portion of the **Gaussian log-theta-lattice** under consideration that involves **theta monoids** [cf. the discussion of [IUTchI], §I1, preceding Theorem A].

(iii) The portion of the **multiradiality** discussed in Corollary 2.3, (iv), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ corresponds, in essence, to the multiradiality discussed in [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i).

Remark 2.3.2. A similar result to Corollary 2.3 may be formulated concerning the **multiradiality** properties satisfied by the **Kummer theory** of ${}_{\infty}\kappa$**-coric structures** as discussed in [IUTchII], Corollary 4.8. That is to say, the Kummer theory of the **localization poly-morphisms**

$$\Big\{ \{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \quad \to \quad {}^{\dagger}\mathbb{M}_{\infty\kappa v_j} \subseteq {}^{\dagger}\mathbb{M}_{\infty\kappa\times v_j} \Big\}_{\underline{v}\in\underline{\mathbb{V}}}$$

discussed in [IUTchII], Corollary 4.8, (iii), is based on the **cyclotomic rigidity isomorphisms** for ${}_{\infty}\kappa$**-coric structures** discussed in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) [cf. also the discussion of [IUTchII], Corollary 4.8, (i)], which satisfy **"insulation"** properties analogous to the properties discussed in Remark 2.2.1 in the case of *mono-theta-theoretic cyclotomic rigidity.*

Moreover, the reconstruction of ${}_{\infty}\kappa$**-coric structures** from ${}_{\infty}\kappa\times$**-structures** via restriction of Kummer classes

$$^{\ddagger}\mathbb{M}_{\infty\kappa v_j} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v_j} \rightarrow {}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa\times v_j} \stackrel{\sim}{\rightarrow} {}^{\ddagger}\mathbb{M}^{\times}_{v_j}$$

as discussed in [IUTchI], Definition 5.2, (vi), (viii) — i.e., a reconstruction in accordance with the principle of **Galois evaluation** [cf. [IUTchII], Remark 1.12.4] — may be regarded as a **decoupling** into

- **radial** [i.e., $\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j$; ${}^{\dagger}\mathbb{M}_{\infty\kappa v_j}$; ${}^{\ddagger}\mathbb{M}_{\infty\kappa v_j}$] and
- **coric** [i.e., the quotient of ${}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa\times v_j} \stackrel{\sim}{\rightarrow} {}^{\ddagger}\mathbb{M}^{\times}_{v_j}$ by its torsion subgroup]

components, i.e., in an entirely analogous fashion to the *mono-theta-theoretic* case discussed in Remark 2.2.2, (iii). The **Galois evaluation** that gives rise to the *theta values* "$\underline{\underline{q}}_{v}^{j^2}$" in the case of theta monoids corresponds to the construction via Galois evaluation of the **monoids** "${}^{\dagger}\mathbb{M}^{\circledast}_{\text{mod}}$", i.e., via the operation of **restricting Kummer classes** associated to elements of ${}_{\infty}\kappa$**-coric structures**, as discussed in [IUTchI], Example 5.1, (v); [IUTchII], Corollary 4.8, (i) [cf. also [IUTchI], Definition 5.2, (vi), (viii)]. We leave the routine details of giving a formulation in the style of Corollary 2.3 to the reader.

Remark 2.3.3. In the context of Remark 2.3.2, it is of interest to compare and contrast the **multiradiality** properties that hold in the **theta** [cf. Remarks 2.2.1, 2.2.2; Corollary 2.3] and **number field** [cf. Remark 2.3.2] cases, as follows.

(i) One important *similarity* between the *theta* and *number field* cases lies in the establishment of **multiradiality** properties, i.e., such as the **radial/coric decoupling** discussed in Remarks 2.2.2, (iii); 2.3.2, by using the **geometric dimension** of the elliptic curve under consideration as a sort of

> **"multiradial geometric container"** for the **radial arithmetic data** of interest, i.e., **theta values** "$\underline{\underline{q}}_{v}^{j^2}$" or copies of the **number field** "F_{mod}".

That is to say, in the theta case, the theory of **theta functions** on **Tate curves** as developed in [EtTh] furnishes such a geometric container for the theta values, while in the number field case, the absolute anabelian interpretation developed in [AbsTopIII] of the theory of **Belyi maps** as **Belyi cuspidalizations** [cf. [IUTchI], Remark 5.1.4] furnishes such a geometric container for copies of F_{mod}. In this context, another important similarity is the passage from such a geometric container to the radial arithmetic data of interest by means of **Galois evaluation** [cf. Remark 2.2.2, (i), (iii); Remark 2.3.2].

(ii) One important theme of the present series of papers is the point of view of *dismantling the* **two underlying combinatorial dimensions** of [the ring of integers of] a number field — cf. the discussion of Remark 3.12.2 below. As discussed in [IUTchI], Remark 6.12.3 [cf. also [IUTchI], Remark 6.12.6], this dismantling may be compared to the dismantling of the **single complex holomorphic dimension** of the **upper half-plane** into **two underlying real dimensions**. If one considers this dismantling from such a classical point of view, then one is tempted to attempt to understand the dismantling into two underlying real dimensions, by, in effect,

base-changing from $\mathbb{R}$ to $\mathbb{C}$, so as to obtain **two-dimensional complex holomorphic objects**, which we regard as being equipped with some sort of **descent data** arising from the base-change from $\mathbb{R}$ to $\mathbb{C}$.

Translating this approach back into the case of number fields, one obtains a situation in which one attempts to understand the dismantling of the two underlying combinatorial dimensions of [the ring of integers of] a number field by working with *two-dimensional scheme-theoretic data* — i.e., such as an **elliptic curve** over [a suitable localization of the ring of integers of] a **number field** — equipped with *"suitable descent data"*. From this point of view, one may think of

> the **"multiradial geometric containers"** discussed in (i) as a sort of *realization* of such **two-dimensional scheme-theoretic data**,

and of

> the accompanying **Galois evaluation** operations, i.e., the **multiradial representations** up to certain **mild indeterminacies** obtained in Theorem 3.11, below [cf. also the discussion of Remark 3.12.2, below], as a sort of *realization* of the corresponding **"suitable descent data"**.

This sort of interpretation is reminiscent of the interpretation of **multiradiality** in terms of **parallel transport** via a **connection** as discussed in [IUTchII], Remark 1.7.1, and the closely related interpretation given in the discussion of [IUTchII], Remark 1.9.2, (iii), of the **tautological approach** to multiradiality in terms of **PD-envelopes** in the style of the p-adic theory of the crystalline site.

(iii) Another fundamental *similarity* between the *theta* and *number field* cases may be seen in the fact that the associated **Galois evaluation** operations — i.e., that give rise to the theta values "$\underline{\underline{q}}_{v}^{j^2}$" [cf. [IUTchII], Corollary 3.6] or copies of the number field "F_{mod}" [cf. [IUTchII], Corollary 4.8, (i), (ii)] — are performed in the context of the **log-link**, which depends, in a quite essential way, on the **arithmetic holomorphic** [i.e., **ring!**] structures of the various local fields involved — cf., for instance, the discussion of the relevant **log-Kummer correspondences** in Remark 3.12.2, (iv), (v), below. On the other hand, one fundamental *difference* between the *theta* and *number field* cases may be observed in the fact that whereas

- · the output data in the theta case — i.e., the **theta values** "$\underline{\underline{q}}_{v}^{j^2}$" — **depends**, in an essential way, on the **labels** $j \in \mathbb{F}_l^{*}$,
- · the output data in the number field case — i.e., the copies of the **number field** "F_{mod}" — is **independent** of these labels $j \in \mathbb{F}_l^{*}$.

In this context, let us recall that these labels $j \in \mathbb{F}_l^{*}$ correspond, in essence, to collections of *cuspidal inertia groups* [cf. [IUTchI], Definition 4.1, (ii)] of the *local geometric fundamental groups* that appear [i.e., in the notation of the discussion of Remark 2.2.2, (iii), the subgroup "Δ ($\subseteq \Pi$)" of the local arithmetic fundamental group Π]. On the other hand, let us recall that, in the context of these local arithmetic fundamental groups Π, the **arithmetic holomorphic structure** also depends, in an essential way, on the geometric fundamental group portion [i.e.,

"$\Delta \subseteq \Pi$"] of Π [cf., e.g., the discussion of [AbsTopIII], Theorem 1.9, in [IUTchI], Remark 3.1.2, (ii); the discussion of [AbsTopIII], §I3]. In particular, it is a quite *nontrivial fact* that

> the **Galois evaluation** and **Kummer theory** in the **theta** case may be performed [cf. [IUTchII], Corollary 3.6] in a consistent fashion that is **compatible** with **both** the **labels** $j \in \mathbb{F}_l^{*}$ [cf. also the associated **symmetries** discussed in [IUTchII], Corollary 3.6, (i)] and the **arithmetic holomorphic structures** involved

— i.e., both of which depend on "Δ" in an essential way. By contrast,

> the corresponding **Galois evaluation** and **Kummer theory** operations in the **number field** case are performed [cf. [IUTchII], Corollary 4.8, (i), (ii)] in a way that is **compatible** with the **arithmetic holomorphic structures** involved, but yields output data [i.e., copies of the number field "F_{mod}"] that is **free** of any **dependence** on the *labels* $j \in \mathbb{F}_l^{*}$.

Of course, the **global realified Gaussian Frobenioids** constructed in [IUTchII], Corollary 4.6, (v), which also play an important role in the theory of the present series of papers, involve global data that **depends**, in an essential way, on the **labels** $j \in \mathbb{F}_l^{*}$, but this dependence occurs only in the context of **global realified Frobenioids**, i.e., which [cf. the notation "$\Vdash$" as it is used in [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (viii), as well as in Definition 2.4, (iii), below] are **mono-analytic** in nature [i.e., do *not* depend on the *arithmetic holomorphic structure* of copies of the number field "F_{mod}"].

(iv) In the context of the observations of (iii), we make the further observation that it is a *highly nontrivial* fact that the construction algorithm for the **mono-theta-theoretic cyclotomic rigidity isomorphism** applied in the theta case admits $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** [cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchII], Corollary 3.6, (i)] in a fashion that is **consistent** with the **dependence** of the **theta values** on the labels $j \in \mathbb{F}_l^{*}$. As discussed in [IUTchII], Remark 1.1.1, (v), this state of affairs differs quite substantially from the state of affairs that arises in the case of the approach to cyclotomic rigidity taken in [IUTchI], Example 5.1, (v), which is based on a rather *"straightforward"* or *"naive"* utilization of the **Kummer classes** of **rational functions**. That is to say, the "highly nontrivial" fact just observed in the theta case would amount, from the point of view of this "naive Kummer approach" to cyclotomic rigidity, to the existence of a **rational function** [or, alternatively, a collection of rational functions *without "labels"*] that is **invariant** [up to, say, multiples by roots of unity] with respect to the $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** that appear, but nevertheless attains values on some $\mathbb{F}_l^{\rtimes\pm}$-orbit of points that have **distinct valuations** at **distinct points** — a situation that is clearly **self-contradictory**!

(v) One way to appreciate the *nontriviality* of the "highly nontrivial" fact observed in (iv) is as follows. One possible approach to realizing the apparently "self-contradictory" state of affairs constituted by a "*symmetric* rational function with *non-symmetric* values" consists of replacing the local arithmetic fundamental group "Π" [cf. the notation of the discussion of (iii)] by some suitable **closed subgroup of infinite index** of Π. That is to say, if one works with such infinite index closed

subgroups of Π, then the possibility arises that the Kummer classes of those rational functions that constitute the *obstruction to symmetry* in the case of some given rational function of interest [i.e., at a more concrete level, the rational functions that arise as *quotients* of the given rational function by its $\mathbb{F}_l^{\rtimes\pm}$-*conjugates*] *vanish* upon *restriction* to such infinite index closed subgroups of Π. On the other hand, this approach has the following *"fundamental deficiencies"*, both of which relate to an apparently **fatal lack of compatibility** with the **arithmetic holomorphic structures** involved:

· It is not clear that the **absolute anabelian** results of [AbsTopIII], §1 — i.e., which play a *fundamental role* in the theory of the present series of papers — admit generalizations to the case of such infinite index closed subgroups of Π.

· The vanishing of **Kummer classes** of certain rational functions that occurs when one *restricts* to such infinite index closed subgroups of Π will not, in general, be compatible with the **ring structures** involved [i.e., of the rings/fields of rational functions that appear].

In particular, this approach does not appear to be likely to give rise to a meaningful theory.

(vi) Another possible approach to realizing the apparently "self-contradictory" state of affairs constituted by a "*symmetric* rational function with *non-symmetric* values" consists of working with **distinct rational functions**, i.e., one **symmetric** rational function [or collection of rational functions] for constructing *cyclotomic rigidity isomorphisms* via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), and one **non-symmetric** rational function to which one applies *Galois evaluation* operations to construct the analogue of "theta values". On the other hand, this approach has the following *"fundamental deficiency"*, which again relates to a sort of **fatal lack of compatibility** with the **arithmetic holomorphic structures** involved: The crucial **absolute anabelian** results of [AbsTopIII], §1 [cf. also the discussion of [IUTchI], Remark 3.1.2, (ii), (iii)], depend, in an essential way, on the use of **numerous cyclotomes** [i.e., copies of "$\widehat{\mathbb{Z}}(1)$"] — which, for simplicity, we shall denote by

$$\boldsymbol{\mu}_{\text{et}}^{*}$$

in the present discussion — that arise from the various **cuspidal inertia groups** at the **cusps** "$*$" of [the various cuspidalizations associated to] the hyperbolic curve under consideration. These cyclotomes "$\boldsymbol{\mu}_{\text{et}}^{*}$" [i.e., for various cusps "$*$"] may be **naturally identified** with one another, i.e., via the *natural isomorphisms* of [AbsTopIII], Proposition 1.4, (ii); write

$$\boldsymbol{\mu}_{\text{et}}^{\forall}$$

for the cyclotome resulting from this natural identification. Moreover, since the various [pseudo-]monoids constructed by applying these anabelian results are constructed as sub[-pseudo-]monoids of first [group] cohomology modules with coefficients in the *cyclotome* $\boldsymbol{\mu}_{\text{et}}^{\forall}$, it follows [cf. the discussion of [IUTchII], Remark 1.5.2] that the *cyclotome*

$$\boldsymbol{\mu}_{\text{Fr}}$$

determined by [i.e., the cyclotome obtained by applying $\text{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ to the *torsion subgroup* of] such a [pseudo-]monoid may be **tautologically identified** — i.e.,

whenever the [pseudo-]monoid under consideration is regarded [not just as an abstract *"Frobenius-like"* [pseudo-]monoid, but rather] as the *"étale-like"* output data of an **anabelian construction** of the sort just discussed — with the cyclotome $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. In the context of the relevant $\mathfrak{log}$**-Kummer correspondences** [i.e., as discussed in Remark 3.12.2, (iv), (v), below; Theorem 3.11, (ii), below], we shall work with various **Kummer isomorphisms** between such Frobenius-like and étale-like versions of various [pseudo-]monoids, i.e., in the notation of the final display of Proposition 1.3, (iv), between various objects associated to the **Frobenius-like** "$\bullet$'s" and corresponding objects associated to the **étale-like** "$\circ$". Now so long as one regards these various Frobenius-like "$\bullet$'s" and the étale-like "$\circ$" as **distinct labels** for corresponding objects, the diagram constituted by the relevant $\mathfrak{log}$-Kummer correspondence does **not** *result in any "vicious circles" or* **"loops"**. On the other hand, ultimately in the theory of §3 [cf., especially, the final portion of Theorem 3.11, (iii), (c), (d), below; the proof of Corollary 3.12 below], we shall be interested in applying the theory to the task of constructing algorithms to describe objects of interest of *one arithmetic holomorphic structure* in terms of some **alien** *arithmetic holomorphic structure* [cf. Remark 3.11.1] by means of **"multiradial containers"** [cf. Remark 3.12.2, (ii)]. These multiradial containers arise from *étale-like* versions of objects, but are ultimately applied as *containers* for *Frobenius-like* versions of objects. That is to say,

> in order for *such multiradial containers to function as* **containers**, it is necessary to contend with the consequences of **identifying** the **Frobenius-like** and **étale-like** versions of various objects under consideration, e.g., in the context of the above discussion, of identifying $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$.

On the other hand, let us recall that the approach to constructing *cyclotomic rigidity isomorphisms* associated to rational functions via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), amounts in effect [i.e., in the context of the above discussion], to **"identifying"** various "$\boldsymbol{\mu}^{*}_{\mathrm{et}}$'s" with various "sub-cyclotomes" of "$\boldsymbol{\mu}_{\mathrm{Fr}}$" via morphisms that differ from the usual *natural identification* precisely by *multiplication* by the **order** [$\in \mathbb{Z}$] at "$*$" of the **zeroes/poles** of the rational function under consideration. That is to say,

> to execute such a **cyclotomic rigidity isomorphism construction** in a situation *subject to the* **further identification** *of* $\boldsymbol{\mu}_{\mathrm{Fr}}$ *with* $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$ [which, we recall, was obtained by identifying the various "$\boldsymbol{\mu}^{*}_{\mathrm{et}}$'s"!] does indeed result — at least in an *a priori* sense! — in **"vicious circles"/"loops"**

[cf. the discussion of [IUTchIV], Remark 3.3.1, (i); the reference to this discussion in [IUTchI], Remark 4.3.1, (ii)]. That is to say, in order to *avoid* any possible *contradictions* that might arise from such "vicious circles"/"loops", it is necessary to work with objects that are **"invariant"**, or **"coric"**, with respect to such "vicious circles"/"loops", i.e., to regard

> the *cyclotome* $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$ as being **subject** to **indeterminacies** with respect to **multiplication** by elements of the *submonoid*
> $$\mathbb{I}^{\mathrm{ord}} \quad \subseteq \quad \pm\mathbb{N}_{\geq 1} \quad \stackrel{\mathrm{def}}{=} \quad \mathbb{N}_{\geq 1} \times \{\pm 1\}$$
> generated by the **orders** [$\in \mathbb{Z}$] of the **zeroes/poles** of the rational function(s) that appear in the cyclotomic rigidity isomorphism construction under consideration.

In the following discussion, we shall also write $\mathbb{I}^{\mathrm{ord}}_{\geq 1} \subseteq \mathbb{N}_{\geq 1}$, $\mathbb{I}^{\mathrm{ord}}_{\pm} \subseteq \{\pm 1\}$ for the respective images of $\mathbb{I}^{\mathrm{ord}}$ via the natural projections to $\mathbb{N}_{\geq 1}$, $\{\pm 1\}$. This sort of indeterminacy is **fundamentally incompatible**, for *numerous reasons*, with any sort of construction that purports to be analogous to the construction of the "theta values" in the theory of the present series of papers, i.e., at least whenever the resulting *indeterminacy submonoid* $\mathbb{I}^{\mathrm{ord}} \subseteq \pm\mathbb{N}_{\geq 1}$ is *nontrivial*. For instance, it follows immediately, by considering the effect of *independent indeterminacies* of this type on *valuations* at *distinct* $\underline{v} \in \underline{\mathbb{V}}$, that such independent indeterminacies are **incompatible** with the **"product formula"** [i.e., with the structure of the *global realified Frobenioids* involved — cf. [IUTchI], Remark 3.5.1, (ii)]. Here, we observe that *this sort of indeterminacy does not occur in the* **theta** *case* [cf. Fig. 2.5 below] — i.e., the resulting *indeterminacy submonoid*

$$(\pm\mathbb{N}_{\geq 1} \supseteq) \quad \mathbb{I}^{\mathrm{ord}} \quad = \quad \{1\}$$

— precisely as a consequence of the fact [which is closely related to the *symmetry* properties discussed in [IUTchII], Remark 1.1.1, (v)] that

> *the order [$\in \mathbb{Z}$] of the* **zeroes/poles** *of the* **theta function** *at* **every cusp** *is* **equal to 1**

[cf. [EtTh], Proposition 1.4, (i); [IUTchI], Remark 3.1.2, (ii), (iii)] — a state of affairs that can *never* occur in the case of an *algebraic rational function* [i.e., since the *sum* of the orders [$\in \mathbb{Z}$] of the zeroes/poles of an algebraic rational function is always equal to 0]! On the other hand, in the **number field** case [cf. Fig. 2.6 below], the portion of the indeterminacy under consideration that is *constituted by* $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$ is *avoided* precisely [cf. the discussion of [IUTchI], Example 5.1, (v)] by applying the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

[cf. also the discussion of (vii) below!], which has the effect of **isolating** the $\widehat{\mathbb{Z}}^{\times}$-*torsor of interest* [i.e., some *specific* isomorphism between cyclotomes] from the subgroup of $\mathbb{Q}_{>0}$ generated by $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$. This technique for avoiding the indeterminacy constituted by $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$ **remains valid** even after the identification discussed above of $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. By contrast, the portion of the indeterminacy under consideration that is *constituted by* $\mathbb{I}^{\mathrm{ord}}_{\pm}$ is *avoided* in the construction of [IUTchI], Example 5.1, (v), precisely by applying the fact that the *inverse* of a nonconstant *κ-coric rational function* is *never κ-coric* [cf. the discussion of [IUTchI], Remark 3.1.7, (i)] — a technique that **depends**, in an essential way, on **distinguishing** cusps "$*$" at which the orders [$\in \mathbb{Z}$] of the zeroes/poles of the rational function(s) under consideration are **distinct**. In particular, this technique is **fundamentally incompatible** with the identification discussed above of $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. That is to say, in summary,

> in the **number field** case, in order to regard **étale-like** versions of objects as **containers** for **Frobenius-like** versions of objects, it is necessary to regard the relevant **cyclotomic rigidity isomorphisms** — hence also the **output data** of interest in the **number field** case, i.e., copies of [the union with $\{0\}$ of] the group "$F^{\times}_{\mathrm{mod}}$" — as being **subject** to an **indeterminacy** constituted by [possible] multiplication by $\{\pm 1\}$.

This does not result in any additional technical obstacles, however, since

the **output data** of interest in the **number field** case — i.e., copies of [the union with $\{0\}$ of] the group "$F^{\times}_{\mathrm{mod}}$" — is [*unlike the case with the theta values* "$\underline{\underline{q}}_{\underline{v}}^{j^2}$"!] **stabilized** by the action of $\{\pm 1\}$

— cf. the discussion of Remark 3.11.4 below. Moreover, we observe in passing, in the context of the Galois evaluation operations in the number field case, that the copies of [the group] "$F^{\times}_{\mathrm{mod}}$" are constructed **globally** and in a fashion compatible with the $\mathbb{F}^{*}_{l}$**-symmetry** [cf. [IUTchII], Corollary 4.8, (i), (ii)], hence, in particular, in a fashion that does not require the establishment of *compatibility* properties [e.g., relating to the *"product formula"*] between constructions at distinct $\underline{v} \in \underline{\mathbb{V}}$.

$$\begin{array}{cccccccccc} & +1 & +1 & +1 & +1 & & +1 & +1 & +1 & +1 & \\ \dots & * & * & * & * & \dots & * & * & * & * & \dots \end{array}$$

Fig. 2.5: Orders [$\in \mathbb{Z}$] of zeroes/poles of the *theta function* at the cusps "$*$"

$$\begin{array}{cccccccccc} & 0 & 0 & +8 & -5 & & -6 & +3 & 0 & 0 & \\ \dots & * & * & * & * & \dots & * & * & * & * & \dots \end{array}$$

Fig. 2.6: Orders [$\in \mathbb{Z}$] of zeroes/poles of an *algebraic rational function* at the cusps "$*$"

(vii) In the context of the discussion of (vi), we observe that the *indeterminacy* issues discussed in (vi) may be thought of as a sort of **"multiple cusp version"** of the **"N-th power versus first power"** and **"linearity"** issues discussed in [IUTchII], Remark 3.6.4, (iii). Also, in this context, we recall from the discussion at the beginning of Remark 2.1.1 that the theory of **mono-theta-theoretic cyclotomic rigidity** satisfies the important property of being **compatible** with the **topology** of the **tempered fundamental group**. Such a compatibility contrasts sharply with the cyclotomic rigidity algorithms discussed in [IUTchI], Example 5.1, (v), which *depend* [cf. the discussion of (vi) above!], in an essential way, on the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} \ = \ \{1\}$$

— i.e., which is **fundamentally incompatible** *with the* **topology** *of the profinite groups involved* [as can be seen, for instance, by considering the fact that $\mathbb{N}_{\geq 1}$ forms a **dense** subset of $\widehat{\mathbb{Z}}$]. This close relationship between **cyclotomic rigidity** and [a sort of] **discrete rigidity** [i.e., the property of the above display] is reminiscent of the discussion given in [IUTchII], Remark 2.8.3, (ii), of such a relationship in the case of mono-theta environments.

(viii) In the context of the discussion of (vi), (vii), we observe that the *indeterminacy* issues discussed in (vi) also occur in the case of the cyclotomic rigidity algorithms discussed in [IUTchI], Definition 5.2, (vi), i.e., in the context of **mixed-characteristic local fields**. On the other hand, [cf. [IUTchII], Proposition 4.2, (i)] these algorithms in fact yield the *same cyclotomic rigidity isomorphism* as the cyclotomic rigidity isomorphisms that are applied in [AbsTopIII], Proposition 3.2, (iv) [i.e., the cyclotomic rigidity isomorphisms discussed in [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1]. Moreover, these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] are **manifestly compatible** with the **topology** of the profinite groups involved. From the point of view of the discussion of (vi),

this sort of *"de facto" compatibility with the topology of the profinite groups involved* may be thought of as a reflection of the fact that these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] amount, in essence, to applying the approach to cyclotomic rigidity by considering the *Kummer theory* of *algebraic rational functions* [i.e., the approach of (vi), or, alternatively, of [IUTchI], Example 5.1, (v)], in the case where the algebraic rational functions are taken to be the **uniformizers** — i.e., "rational functions" [any one of which is well-defined up to a unit] with precisely **one zero** of **order 1** and **no poles** [cf. the discussion of the theta function in (vi)!] — of the **mixed-characteristic local field** under consideration. Put another way, this sort of "de facto" compatibility may be regarded as a reflection of the fact that, unlike *number fields* [i.e., "NF's"] or *one-dimensional function fields* [i.e., "one-dim. FF's"], *mixed-characteristic local fields* [i.e., "MLF's"] are equipped with a **uniquely determined "canonical valuation"** — a situation that is reminiscent of the fact that the order [$\in \mathbb{Z}$] of the *zeroes/poles* of the *theta function* at *every cusp* is *equal to* 1 [i.e., the fact that "the set of *equivalences classes of cusps* relative to the equivalence relationship on cusps determined by considering the order [$\in \mathbb{Z}$] of the zeroes/poles of the theta function is of *cardinality one*"]. From the point of view of *"geometric containers"* discussed in (i) and (ii), this state of affairs may be summarized as follows:

> the **indeterminacy** issues that occur in the context of the discussion of **cyclotomic rigidity isomorphisms** in (vi) exhibit **similar qualitative** behavior in the
>
> **MLF/mono-theta** ($\longleftrightarrow$ **one valuation/cusp**)
>
> [i.e., where the expression "one cusp" is to be understood as referring to "one equivalence class of cusps", as discussed above] cases, as well as in the
>
> **NF/one-dim. FF** ($\longleftrightarrow$ **global** collection of **valuations/cusps**)
>
> cases.

Put another way, at least at the level of the *theory of valuations*,

> the theory of **theta** functions (respectively, **one-dimensional function fields**) serves as an accurate "**qualitative geometric model**" of the theory of **mixed-characteristic local fields** (respectively, **number fields**).

Finally, we observe that in this context, the crucial property "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$" that occurs in the discussion of the *number field/one-dimensional function field* cases is *highly reminiscent* of the *global nature of number fields* [i.e., such as $\mathbb{Q}$! — cf. the discussion of Remark 3.12.1, (iii), below].

(ix) The comparison given in (viii) of the special properties satisfied by the **theta function** with the corresponding properties of the **algebraic rational functions** that appear in the **number field** case is reminiscent of the analogy discussed in [IUTchI], Remark 6.12.3, (iii), with the **classical upper half-plane**. That is to say, the *eigenfunction* for the *additive symmetries* of the upper half-plane [i.e., which corresponds to the *theta* case]

$$q \stackrel{\text{def}}{=} e^{2\pi i z}$$

Aspect of the theory	*Theta case*	*Number field case*
multiradial geometric container	**theta functions** on **Tate curves**	**Belyi maps/ cuspidalizations**
radial arithmetic data via **Galois evaluation**	**theta values** "$\underline{\underline{q}}^{j^2}_{\underline{\underline{v}}}$"	copies of **number field** "F_{mod}" $\left(\supseteq F^{\times}_{\mathrm{mod}} \curvearrowleft \{\pm 1\}\right)$
Galois evaluation output data **dependence** on "Δ"	**simultaneously dependent** on **labels**, **holomorphic str.**	**indep.** of **labels**, **dependent** on **holomorphic str.**
cyclotomic rigidity isomorphism	**compatible** with $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, **tempered topology**	**incompatible** with $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, **profinite topology**
approach to eliminating **cyclo. rig. isom. indeterminacies**	order [$\in \mathbb{Z}$] of **zeroes/poles** of **theta function** at **every cusp = 1**	$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$, **non-invertibility** of nonconstant κ**-coric** rational functions
qualitative geometric model for **arithmetic**	**MLF/mono-theta** ($\longleftrightarrow$ **one valuation/cusp**) analogy	**NF/one-dim. FF** ($\longleftrightarrow$ **global** collection of **valuations/cusps**) analogy
analogy with **eigenfunctions** for **symmetries** of **upper half-plane**	**highly transcendental** function in z: $q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$	**algebraic rational** function of z: $w \stackrel{\mathrm{def}}{=} \frac{z-i}{z+i}$

Fig. 2.7: Comparison between the *theta* and *number field* cases

is **highly transcendental** in the coordinate z, whereas the *eigenfunction* for the *multiplicative symmetries* of the upper half-plane [i.e., which corresponds to the *number field* case]

$$w \stackrel{\text{def}}{=} \tfrac{z-i}{z+i}$$

is an **algebraic rational function** in the coordinate z.

(x) The various properties discussed above in the *theta* and *number field* cases are summarized in Fig. 2.7 above.

Remark 2.3.4. Before proceeding, it is perhaps of interest to review once more the essential content of [EtTh] in light of the various observations made in Remark 2.3.3.

(i) The starting point of the relationship between the theory of [EtTh] and the theory of the present series of papers lies [cf. the discussion of Remark 2.1.1, (i); [IUTchII], Remark 3.6.2, (ii)] in the various **non-ring/scheme-theoretic filters** [i.e., $\mathfrak{log}$-links and various types of Θ-links] between distinct ring/scheme theories that are constructed in the present series of papers. Such non-scheme-theoretic filters may only be constructed by making use of **Frobenius-like** structures. On the other hand, **étale-like structures** are important in light of their ability to relate structures on opposite sides of such non-scheme-theoretic filters. Then **Kummer theory** is applied to relate corresponding Frobenius-like and étale-like structures. Moreover, it is crucial that this Kummer theory be conducted in a **multiradial** fashion. This is achieved by means of certain **radial/coric decouplings**, by making use of **multiradial geometric containers**, as discussed in Remark 2.3.3, (i), (ii). That is to say, it is necessary to make use of such multiradial geometric containers and then to pass to theta values or number fields by means of **Galois evaluation**, since direct use of such theta values or number fields results in a Kummer theory that does **not** satisfy the desired multiradiality properties [cf. Remarks 2.2.1, 2.3.2].

(ii) The most naive approach to the Kummer theory of the *"functions"* that are to be used as *"multiradial geometric containers"* may be seen in the approach involving **algebraic rational functions** on the various algebraic curves under consideration, i.e., in the fashion of [IUTchI], Example 5.1, (v) [cf. also [IUTchI], Definition 5.2, (vi)]. On the other hand, in the context of the **local** theory at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, this approach suffers from the *fatal drawback* of being **incompatible** with the **profinite topology** of the profinite fundamental groups involved [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii); Figs. 2.5, 2.6]. Thus, in order to maintain compatibility with the profinite/tempered topology of the profinite/tempered fundamental groups involved, one is obliged to work with the **Kummer theory** of **theta functions**, **truncated modulo** N. On the other hand, the naive approach to this sort of [truncated modulo N] Kummer theory of theta functions suffers from the fatal drawback of being **incompatible** with **discrete rigidity** [cf. Remark 2.1.1, (v)]. This incompatibility with discrete rigidity arises from a lack of *"shifting automorphisms"* as in [EtTh], Proposition 2.14, (ii) [cf. also [EtTh], Remark 2.14.3], and is closely related to the **incompatibility** of this naive approach with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the discussion of [IUTchII], Remark 1.1.1,

(iv), (v)]. In order to surmount such incompatibilities, one is obliged to consider not the Kummer theory of theta functions in the naive sense, but rather, so to speak, the **Kummer theory** of [the **first Chern classes** of] the **line bundles** associated to theta functions [cf. the discussion of [IUTchII], Remark 1.1.1, (v)]. Thus, in summary:

[truncated] Kummer theory of **theta** [*not algebraic rational!*] **functions**
$\Longrightarrow$ **compatible** with **profinite/tempered topologies**;

[truncated] Kummer theory of [**first Chern classes** of] **line bundles**
[*not rational functions!*]
$\Longrightarrow$ **compatible** with **discrete rigidity,** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**.

(iii) To consider the "[truncated] Kummer theory of line bundles [associated to the theta function]" amounts, in effect, to considering the [partially truncated] *arithmetic fundamental group* of the $\mathbb{G}_m$-torsor determined by such a line bundle in a fashion that is **compatible** with the various **tempered Frobenioids** and **tempered fundamental groups** under consideration. Such a "[partially truncated] arithmetic fundamental group" corresponds precisely to the *"topological group"* portion of the data that constitutes a mono-theta or bi-theta environment [cf. [EtTh], Definition 2.13, (ii), (a); [EtTh], Definition 2.13, (iii), (a)]. In the context of the theory of theta functions, such "[partially truncated] arithmetic fundamental groups" are equipped with *two natural distinguished [classes of] sections*, namely, **theta sections** and **algebraic sections**. If one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *neither* with data corresponding to theta sections *nor* with data corresponding to algebraic sections, then the resulting mathematical object is necessarily subject to *indeterminacies* arising from *multiplication by constant units* [i.e., "$\mathcal{O}^\times$" of the base local field], hence, in particular, suffers from the drawback of being *incompatible* with *constant multiple rigidity* [cf. Remark 2.1.1, (iii)]. On the other hand, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *both* with data corresponding to theta sections *and* with data corresponding to algebraic sections, then the resulting mathematical object suffers from the *same lack of symmetries* as the [truncated] Kummer theory of theta functions [cf. the discussion of (ii)], hence, in particular, is *incompatible* with *discrete rigidity* [cf. Remark 2.1.1, (v)]. Finally, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *only* with data corresponding to *algebraic sections* [i.e., but not with data corresponding to theta sections!], then the resulting mathematical object is not equipped with sufficient data to apply the *crucial commutator property* of [EtTh], Proposition 2.12 [cf. also the discussion of [EtTh], Remark 2.19.2], hence, in particular, is *incompatible* with *cyclotomic rigidity* [cf. Remark 2.1.1, (iv)]. That is to say, it is only by thinking of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *only* with data corresponding to *theta sections* [i.e., but not with data corresponding to algebraic sections!] — i.e., in short, by working with **mono-theta environments** — that one may achieve a situation that is **compatible** with the **tempered topology** of the tempered fundamental groups involved, the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, and **all three** types of **rigidity** [cf. the initial portion of Remark 2.1.1; [IUTchII], Remark 3.6.4, (ii)]. Thus, in summary:

working **neither** *with theta sections* **nor** *with algebraic sections* $\Longrightarrow$

incompatible with **constant multiple rigidity**!

working with **bi-theta environments**, i.e.,
working simultaneously with **both** *theta sections* **and** *algebraic sections* $\Longrightarrow$
incompatible with **discrete rigidity,** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**!

working with *algebraic sections* **but not** *theta sections* $\Longrightarrow$
incompatible with **cyclotomic rigidity**!

working with **mono-theta environments**, i.e.,
working with *theta sections* **but not** *algebraic sections* $\Longrightarrow$
compatible with **tempered topology**, $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, **all three rigidities**!

(iv) Finally, we note that the approach of [EtTh] to the theory of theta functions differs substantially from *more conventional approaches* to the theory of theta functions such as

- the classical **function-theoretic** approach via **explicit series representations**, i.e., as given at the beginning of the Introduction to [IUTchII] [cf. also [EtTh], Proposition 1.4], and

- the **representation-theoretic** approach, i.e., by considering irreducible representations of **theta groups**.

Both of these more conventional approaches depend, in an essential way, on the **ring structures** — i.e., on both the **additive** and the **multiplicative** structures — of the various rings involved. [Here, we recall that explicit series are constructed precisely by adding and multiplying various functions on some space, whereas representations are, in effect, modules over suitable rings, hence, by definition, involve both additive and multiplicative structures.] In particular, although these more conventional approaches are well-suited to many situations in which one considers *"the"* theta function in some *fixed model of scheme/ring theory*, they are **ill-suited** to the situations treated in the present series of papers, i.e., where one must consider theta functions that appear in various **distinct ring/scheme theories**, which [cf. the discussion of (i)] may only be related to one another by means of suitable *Frobenius-like* and *étale-like* structures such as **tempered Frobenioids** and **tempered fundamental groups**. Here, we recall that these tempered Frobenioids correspond essentially to **multiplicative monoid** structures arising from the various rings of functions that appear, whereas tempered fundamental groups correspond to various **Galois actions**. That is to say, consideration of such multiplicative monoid structures and Galois actions is **compatible** with the **dismantling** of the additive and multiplicative structures of a ring, i.e., as considered in the present series of papers [cf. the discussion of Remark 3.12.2 below].

Definition 2.4.

(i) Let

$$^{\ddagger}\mathfrak{F}^{\vdash} = \{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}^{\vdash}$-*prime-strip*. Then recall from the discussion of [IUTchII], Definition 4.9, (ii), that at each $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the splittings of the split Frobenioid ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ determine *submonoids* "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$", as well as *quotient monoids* "$\mathcal{O}^{\perp}(-) \twoheadrightarrow \mathcal{O}^{\blacktriangleright}(-)$" [i.e., by forming the quotient of "$\mathcal{O}^{\perp}(-)$" by its torsion subgroup]. In a similar vein, for each $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{good}}$, the splitting of the split Frobenioid determined by [indeed, "constituted by", when $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ — cf. [IUTchI], Definition 5.2, (ii)] ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ determines a *submonoid "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$"* whose subgroup of units is *trivial* [cf. [IUTchII], Definition 4.9, (iv), when $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$]; in this case, we set $\mathcal{O}^{\blacktriangleright}(-) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\perp}(-)$. Write

$$ {}^{\ddagger}\mathfrak{F}^{\vdash\perp} \;=\; \{{}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad {}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright} \;=\; \{{}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

for the collections of data obtained by replacing the split Frobenioid portion of each ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ by the *Frobenioids* determined, respectively, by the subquotient monoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$", "$\mathcal{O}^{\blacktriangleright}(-)$" just defined.

(ii) We define [in the spirit of [IUTchII], Definition 4.9, (vii)] an $\mathcal{F}^{\vdash\perp}$-*prime-strip* to be a collection of data

$$ {}^{*}\mathfrak{F}^{\vdash\perp} = \{{}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ is a *Frobenioid* that is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ [cf. (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ consists of a Frobenioid and an object of $\mathbb{TM}^{\vdash}$ [cf. [IUTchI], Definition 5.2, (ii)] such that ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}$. In a similar vein, we define an $\mathcal{F}^{\vdash\blacktriangleright}$-*prime-strip* to be a collection of data

$$ {}^{*}\mathfrak{F}^{\vdash\blacktriangleright} = \{{}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ is a *Frobenioid* that is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ [cf. (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ consists of a Frobenioid and an object of $\mathbb{TM}^{\vdash}$ [cf. [IUTchI], Definition 5.2, (ii)] such that ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$. A *morphism of* $\mathcal{F}^{\vdash\perp}$- (respectively, $\mathcal{F}^{\vdash\blacktriangleright}$-) *prime-strips* is defined to be a collection of isomorphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips [cf. [IUTchI], Definition 5.2, (iii)].

(iii) We define [in the spirit of [IUTchII], Definition 4.9, (viii)] an $\mathcal{F}^{\Vdash\perp}$-*prime-strip* to be a collection of data

$$ {}^{*}\mathfrak{F}^{\Vdash\perp} \;=\; ({}^{*}\mathcal{C}^{\Vdash},\; \mathrm{Prime}({}^{*}\mathcal{C}^{\Vdash}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\; {}^{*}\mathfrak{F}^{\vdash\perp},\; \{{}^{*}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\Vdash}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^{\vdash}$-prime-strip is replaced by an $\mathcal{F}^{\vdash\perp}$-prime-strip. [We leave the routine details to the reader.] In a similar vein, we define an $\mathcal{F}^{\Vdash\blacktriangleright}$-*prime-strip* to be a collection of data

$$ {}^{*}\mathfrak{F}^{\Vdash\blacktriangleright} \;=\; ({}^{*}\mathcal{C}^{\Vdash},\; \mathrm{Prime}({}^{*}\mathcal{C}^{\Vdash}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\; {}^{*}\mathfrak{F}^{\vdash\blacktriangleright},\; \{{}^{*}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\Vdash}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^{\vdash}$-prime-strip is replaced by an $\mathcal{F}^{\vdash\blacktriangleright}$-prime-strip. [We leave the routine details to the reader.] A *morphism of* $\mathcal{F}^{\Vdash\perp}$- (respectively, $\mathcal{F}^{\Vdash\blacktriangleright}$-) *prime-strips* is defined to be an isomorphism between collections of data as discussed above.

Remark 2.4.1.

(i) Thus, by applying the constructions of Definition 2.4, (i), to the [underlying $\mathcal{F}^{\vdash}$-prime-strips associated to the] $\mathcal{F}^{\Vdash}$-prime-strips "$\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$" that appear in Corollary 2.3, one may regard the multiradiality of Corollary 2.3, (i), as implying a corresponding **multiradiality** assertion concerning the associated $\mathcal{F}^{\Vdash\perp}$-prime-strips "$\mathfrak{F}^{\Vdash\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$".

(ii) Suppose that we are in the situation discussed in (i). Then at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the submonoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\triangleright}(-)$" may be regarded, in a natural way [cf. Proposition 2.1, (ii); Theorem 2.2, (ii)], as *submonoids* of the monoids "${}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$" of Theorem 2.2, (ii), ($\mathrm{a}_{\underline{v}}$). Moreover, the resulting inclusion of monoids is **compatible** with the **multiradiality** discussed in (i) and the multiradiality of the data "${}^{\dagger}\mathfrak{R}^{\mathrm{bad}}$" of Corollary 2.3, ($\mathrm{c}_{\mathfrak{R}}$), that is implied by the multiradiality of Corollary 2.3, (i).

Remark 2.4.2.

(i) One verifies immediately that, just as one may associate to an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* a *pilot object* in the global realified Frobenioid portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [cf. [IUTchII], Definition 4.9, (viii)], one may associate to an $\mathcal{F}^{\Vdash\blacktriangleright}$-*prime-strip* a **pilot object** in the global realified Frobenioid portion of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip [i.e., in the notation of the final display of Definition 2.4, (iii), the global realified Frobenioid ${}^{*}\mathcal{C}^{\Vdash}$ of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip ${}^{*}\mathfrak{F}^{\Vdash\blacktriangleright}$].

(ii) For $\underline{v} \in \underline{\mathbb{V}}$ lying over $v \in \mathbb{V}_{\mathrm{mod}}$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$, write

- $r_{\underline{v}} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$,
- $r_{\underline{v}} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot \mathrm{ord}_{\underline{v}}(\underline{\underline{q}}_{\underline{v}}) \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

— where, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then $\mathrm{ord}_{\underline{v}} : K^{\times}_{\underline{v}} \to \mathbb{Q}$ denotes the natural $p_{\underline{v}}$-adic valuation normalized so that $\mathrm{ord}_{\underline{v}}(p_{\underline{v}}) = 1$, and $\underline{\underline{q}}_{\underline{v}}$ is as in [IUTchI], Example 3.2, (iv);

$$r^{\blacktriangleright}_{\underline{v}} \stackrel{\mathrm{def}}{=} -\frac{r_{\underline{v}}}{\sum\limits_{\underline{w}\in\underline{\mathbb{V}}^{\mathrm{bad}}} r_{\underline{w}}}$$

[cf. the constructions of [IUTchI], Example 3.5; [IUTchI], Remark 3.5.1; the discussion of *weights* in Remark 3.1.1, (ii), below].

(iii) In the notation of (ii), let M be any **ordered monoid** isomorphic [as an ordered monoid] to $\mathbb{R}$ [endowed with the usual additive and order structures]. Then M naturally determines a *collection of data*

$$(M,\ \{M_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho_{M_{\underline{v}}} : M_{\underline{v}} \xrightarrow{\sim} M\}_{\underline{v}\in\underline{\mathbb{V}}})$$

as follows: for each $\underline{v} \in \underline{\mathbb{V}}$, we take $M_{\underline{v}}$ to be a copy of M and $\rho_{M_{\underline{v}}} : M_{\underline{v}} \xrightarrow{\sim} M$ to be the isomorphism of monoids [that reverses the ordering!] given by *multiplying* by $r^{\blacktriangleright}_{\underline{v}} \in \mathbb{R}$.

(iv) In the notation of (ii), (iii), suppose, further, that we have been a **negative element**

$$\eta_M \in M$$

[i.e., an element < 0], which we shall refer to as a **pilot element**. Then, since, for $\underline{v} \in \underline{\mathbb{V}}$, $M_{\underline{v}}$ is defined to be a copy of M, η_M determines an element $\eta_{M_{\underline{v}}} \in M_{\underline{v}}$. Thus, the *pair* (M, η_M) naturally determines a *collection of data*

$$(M, \ \{M^{\blacktriangleright}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \ \{\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M\}_{\underline{v} \in \underline{\mathbb{V}}})$$

as follows: for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we take $M^{\blacktriangleright}_{\underline{v}} \subseteq M_{\underline{v}}$ to be the submonoid [isomorphic to $\mathbb{N}$] generated by $\eta_{M_{\underline{v}}}$ and $\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M$ to be the restriction of $\rho_{M_{\underline{v}}}$ to $M^{\blacktriangleright}_{\underline{v}}$; for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, we take $M^{\blacktriangleright}_{\underline{v}} \subseteq M_{\underline{v}}$ to be the submonoid [isomorphic to $\mathbb{R}_{\geq 0}$] given by the elements ≤ 0 and $\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M$ to be the restriction of $\rho_{M_{\underline{v}}}$ to $M^{\blacktriangleright}_{\underline{v}}$. In particular, it follows immediately from the construction of this data that

$$\rho_{M^{\blacktriangleright}_{\underline{v}}}(\eta_{M_{\underline{v}}}) \ = \ r^{\blacktriangleright}_{\underline{v}} \cdot \eta_M$$

for each $\underline{v} \in \underline{\mathbb{V}}$.

(v) Now we observe that the constructions of (iii) and (iv) allow one to give a sort of *"converse"* to the construction of the *pilot object* in (i). Indeed, consider the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip ${}^*\mathfrak{F}^{\Vdash\blacktriangleright}$ in the final display of Definition 2.4, (iii). Next, observe that the **"Picard group"** constructions "$\mathrm{Pic}_{\Phi}(-)$" and "$\mathrm{Pic}_{\mathcal{C}}(-)$" of [FrdI], Theorem 5.1, (i), applied to any object of the global realified Frobenioid ${}^*\mathcal{C}^{\Vdash}$ yield canonically isomorphic groups for *any object* of ${}^*\mathcal{C}^{\Vdash}$. In particular, it makes sense to speak of "$\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$". Moreover, it follows from [FrdI], Theorem 6.4, (i), (ii), that $\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ is equipped with a canonical structure of *ordered monoid*, with respect to which it is isomorphic to $\mathbb{R}$ [endowed with the usual additive and order structures]. Relative to this structure of ordered monoid, the *pilot object* discussed in (i) [cf. also the discussion of [IUTchII], Definition 4.9, (viii)] determines a **negative element** $\eta_{{}^*\mathcal{C}^{\Vdash}} \in \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$. Thus, one verifies immediately, by recalling the various definitions involved, that the *collection of data* "$(M, \{M_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \{\rho_{M_{\underline{v}}} : M_{\underline{v}} \xrightarrow{\sim} M\}_{\underline{v} \in \underline{\mathbb{V}}})$" constructed in (iii) from "M" is already sufficient to reconstruct, i.e., by taking $M \stackrel{\mathrm{def}}{=} \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$, the *collection of data*

$$({}^*\mathcal{C}^{\Vdash}, \ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}})$$

[cf. the notation of the final display of Definition 2.4, (iii)], while the *collection of data* "$(M, \{M^{\blacktriangleright}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \{\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M\}_{\underline{v} \in \underline{\mathbb{V}}})$" constructed in (iv) from the pair "(M, η_M)" is sufficient to reconstruct, i.e., by taking $M \stackrel{\mathrm{def}}{=} \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ and $\eta_M \stackrel{\mathrm{def}}{=} \eta_{{}^*\mathcal{C}^{\Vdash}}$, the *collection of data*

$$({}^*\mathcal{C}^{\Vdash}, \ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ \{\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \ \{{}^*\rho_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$$

where, for $\underline{v} \in \underline{\mathbb{V}}$, we write $\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}$ for the [constant!] *divisor monoid* [i.e., in effect, a single monoid isomorphic to $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$] determined by the Frobenioid structure [cf. [FrdI], Corollary 4.11, (iii); [FrdII], Theorem 1.2, (i)] on ${}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ [cf. the notation of the final display of Definition 2.4, (i)].

(vi) One immediate consequence of the discussion of (v) is the following:

If one starts from $M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$, then the resulting *collection of data*

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}})$$

yields a **common container**, namely, the Frobenioid ${}^*\mathcal{C}^{\Vdash}$ [regarded as an object reconstructed from $M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$!], in which **distinct choices** of the [*negative*!] **pilot element** $\in M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ — hence also the data

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{{}^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[which may be thought of as a sort of *"further rigidification"* on ${}^*\mathcal{C}^{\Vdash}$] reconstructed from such distinct choices of pilot element — may be **compared** with one another.

By contrast, if one attempts to compare the constructions of (v) applied to **positive** and **negative** "$\eta_M \in M$" [i.e., which amounts to *reversing the order structure* on M!], then already the corresponding Frobenioids "${}^*\mathcal{C}^{\Vdash}$" [i.e., attached to the same *group* "$\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$", but with *reversed order structures*!] involve *pre-steps* [i.e., in effect, the category-theoretic version of the notion of an *inclusion of line bundles* — cf. [FrdI], Definition 1.2, (iii)] going in *opposite directions.* That is to say, such Frobenioids may only be compared with one another if they are embedded in some sort of larger **ambient category** in which *the pre-steps are rendered invertible*; but this already implies that all objects arising from such Frobenioids become *isomorphic* in the ambient category. That is to say, working in such a larger ambient category already renders any sort of argument that requires one to *distinguish distinct elements* of $\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ — i.e., *distinct arithmetic degrees/heights* of arithmetic line bundles — *meaningless* [cf. the discussion of *positivity* in Remark 2.1.1, (v)].

Section 3: Multiradial Logarithmic Gaussian Procession Monoids

In the present §3, we apply the theory developed thus far in the present series of papers to give [cf. Theorem 3.11 below] **multiradial algorithms** for a slightly modified version of the **Gaussian monoids** discussed in [IUTchII], §4. This modification revolves around the combinatorics of *processions*, as developed in [IUTchI], §4, §5, §6, and is necessary in order to establish the desired multiradiality. At a more concrete level, these combinatorics require one to apply the theory of **tensor packets** [cf. Propositions 3.1, 3.2, 3.3, 3.4, 3.7, 3.9, below]. Finally, we observe in Corollary 3.12 that these multiradial algorithms give rise to certain **estimates** concerning the **log-volumes** of the **logarithmic Gaussian procession monoids** that occur. This observation forms the starting point of the theory to be developed in [IUTchIV].

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. Also, we shall write

$$\mathbb{V}_{\mathbb{Q}} \stackrel{\text{def}}{=} \mathbb{V}(\mathbb{Q})$$

[cf. [IUTchI], §0] and apply the notation of Definition 1.1 of the present paper. We begin by discussing the theory of **tensor packets**, which may be thought of as a sort of *amalgamation* of the theory of *log-shells* developed in §1 with the theory of *processions* developed in [IUTchI], §4, §5, §6.

Proposition 3.1. **(Local Holomorphic Tensor Packets)** *Let*

$$\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A} \;=\; \left\{\{{}^{\alpha}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right\}_{\alpha\in A}$$

be an ***n*-capsule***, with index set A, of $\mathcal{F}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], §0; [IUTchI], Definition 5.2, (i)]. Then [cf. the notation of Definition 1.1, (iii)] for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$, by considering* **invariants** *with respect to the natural action of various open subgroups of the topological group ${}^{\alpha}\Pi_{\underline{v}}$, one may regard $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ as an* **inductive limit** *of* **topological modules***, each of which is of finite dimension over $\mathbb{Q}_{v_{\mathbb{Q}}}$; we shall refer to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \;\mapsto\; \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \;\stackrel{\text{def}}{=}\; \bigoplus_{\underline{\mathbb{V}}\,\ni\,\underline{v}\,\mid\,v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

as the **[1-]tensor packet** *associated to the $\mathcal{F}$-prime-strip ${}^{\alpha}\mathfrak{F}$ and to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \;\mapsto\; \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \;\stackrel{\text{def}}{=}\; \bigotimes_{\alpha\in A} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \;=\; \bigoplus_{\{\underline{v}_{\alpha}\}_{\alpha\in A}} \left\{\bigotimes_{\alpha\in A} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}_{\alpha}})\right\}$$

— where the tensor products are to be understood as tensor products of ind-topological modules [i.e., as discussed above], and the direct sum is over all collections $\{\underline{v}_{\alpha}\}_{\alpha\in A}$ of [not necessarily distinct!] elements $\underline{v}_{\alpha} \in \underline{\mathbb{V}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$ —

as the **[n-]tensor packet** *associated to the collection of $\mathcal{F}$-prime-strips $\{^{\alpha}\mathfrak{F}\}_{\alpha \in A}$. Then:*

(i) **(Ring Structures)** *The* **ind-topological field structures** *on the various $\underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}})$ [cf. Definition 1.1, (i), (ii), (iii)], for $\alpha \in A$, determine an* **ind-topological ring structure** *on $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ with respect to which $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ may be regarded as an inductive limit of* **direct sums of ind-topological fields***. Such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ and, moreover, are compatible, for $\alpha \in A$, with the natural action of the topological group $^{\alpha}\Pi_{\underline{v}}$ [where $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$] on the direct summand with subscript $\underline{v}$ of the factor labeled α.*

(ii) **(Integral Structures)** *Fix elements $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $\underline{v} \mid v_{\mathbb{Q}}$. Relative to the tensor product in the above definition of $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, write*

$$\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}}) \otimes \Big\{ \bigotimes_{\beta \in A \setminus \{\alpha\}} \underline{\mathfrak{log}}(^{\beta}\mathcal{F}_{v_{\mathbb{Q}}}) \Big\} \subseteq \underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

for the ind-topological submodule determined by the tensor product of the factors labeled by $\beta \in A \setminus \{\alpha\}$ with the tensor product of the direct summand with subscript $\underline{v}$ of the factor labeled α. Then $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$ forms a direct summand of the ind-topological ring $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$; $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$ may be regarded as an inductive limit of **direct sums of ind-topological fields***; such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$. Moreover, by forming the tensor product with "1's" in the factors labeled by $\beta \in A \setminus \{\alpha\}$, one obtains a* **natural injective homomorphism** *of ind-topological rings*

$$\underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}}) \rightarrow \underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$$

that, for suitable choices [which are, in fact, cofinal] of objects appearing in the inductive limit descriptions given above for the domain and codomain, induces an **isomorphism** *of such an object in the domain onto each of the direct summand ind-topological fields of the object in the codomain. In particular, the integral structure*

$$\overline{\Psi}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})} \stackrel{\mathrm{def}}{=} \Psi_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})} \bigcup \{0\} \subseteq \underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}})$$

[cf. the notation of Definition 1.1, (i), (ii)] determines **integral structures** *on each of the direct summand ind-topological fields that appear in the inductive limit descriptions of $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$, $\underline{\mathfrak{log}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$.*

Proof. The various assertions of Proposition 3.1 follow immediately from the definitions and the references quoted in the statements of these assertions [cf. also Remark 3.1.1, (i), below]. ○

Remark 3.1.1.

(i) Let $\underline{v} \in \underline{\mathbb{V}}$. In the notation of [IUTchI], Definition 3.1, write $k \stackrel{\mathrm{def}}{=} K_{\underline{v}}$; let $\overline{k}$ be an algebraic closure of k. Then, roughly speaking, in the notation of Proposition 3.1,

$$\underline{\mathfrak{log}}(^{\alpha}\mathcal{F}_{\underline{v}}) \stackrel{\sim}{\rightarrow} \overline{k}; \quad \overline{\Psi}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})} \stackrel{\sim}{\rightarrow} \mathcal{O}_{\overline{k}};$$

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \quad \xrightarrow{\sim} \quad \bigotimes \ \overline{k} \quad \xrightarrow{\sim} \quad \varinjlim \ \bigoplus \ \overline{k} \quad \supseteq \quad \varinjlim \ \bigoplus \ \mathcal{O}_{\overline{k}}$$

— i.e., one verifies immediately that each ind-topological field $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ is isomorphic to $\overline{k}$; each $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ is a topological tensor product [say, over $\mathbb{Q}$] of copies of $\overline{k}$, hence may be described as an inductive limit of direct sums of copies of $\overline{k}$; each $\overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})}$ is a copy of the set [i.e., a ring, when $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$] of integers $\mathcal{O}_{\overline{k}} \subseteq \overline{k}$. In particular, the "integral structures" discussed in the final portion of Proposition 3.1, (ii), correspond to copies of $\mathcal{O}_{\overline{k}}$ contained in copies of $\overline{k}$.

(ii) Ultimately, for $\underline{v} \in \underline{\mathbb{V}}$, we shall be interested [cf. Proposition 3.9, (i), (ii), below] in considering **log-volumes** on the portion of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ corresponding to $K_{\underline{v}}$. On the other hand, let us recall that we do not wish to consider *all* of the valuations in $\mathbb{V}(K)$. That is to say, we wish to restrict ourselves to considering the subset $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$, equipped with the *natural bijection* $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\text{mod}}$ [cf. [IUTchI], Definition 3.1, (e)], which we wish to think of as a sort of *"local analytic section"* [cf. the discussion of [IUTchI], Remark 4.3.1, (i)] of the natural morphism $\text{Spec}(K) \to \text{Spec}(F)$ [or, perhaps more precisely, $\text{Spec}(K) \to \text{Spec}(F_{\text{mod}})$]. In particular, it will be necessary to consider these log-volumes on the portion of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ corresponding to $K_{\underline{v}}$ relative to the **weight** $[K_{\underline{v}} : (F_{\text{mod}})_v]^{-1}$, where we write $v \in \mathbb{V}_{\text{mod}}$ for the element determined [via the natural bijection just discussed] by $\underline{v}$ [cf. the discussion of [IUTchI], Example 3.5, (i), (ii), (iii), where similar factors appear]. When, moreover, we consider direct sums over all $\underline{v} \in \underline{\mathbb{V}}$ lying over a given $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ as in the case of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, it will be convenient to use the **normalized weight**

$$\frac{1}{[K_{\underline{v}} : (F_{\text{mod}})_v] \cdot \Big(\sum_{\mathbb{V}_{\text{mod}} \ni w | v_{\mathbb{Q}}} [(F_{\text{mod}})_w : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big)}$$

— i.e., *normalized* so that multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes by addition or subtraction [that is to say, depending on whether $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{arc}}$ or $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}}$] of the quantity $\log(p_{v_{\mathbb{Q}}}) \in \mathbb{R}$. In a similar vein, when we consider log-volumes on the portion of $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ corresponding to the tensor product of various $K_{\underline{v}_{\alpha}}$, where $\underline{\mathbb{V}} \ni \underline{v}_{\alpha} \mid v_{\mathbb{Q}}$, it will be necessary to consider these log-volumes relative to the **weight**

$$\frac{1}{\prod_{\alpha \in A} [K_{\underline{v}_{\alpha}} : (F_{\text{mod}})_{v_{\alpha}}]}$$

— where we write $v_{\alpha} \in \mathbb{V}_{\text{mod}}$ for the element determined by $\underline{v}_{\alpha}$. When, moreover, we consider direct sums over all possible choices for the data $\{\underline{v}_{\alpha}\}_{\alpha \in A}$, it will be convenient to use the **normalized weight**

$$\frac{1}{\Big(\prod_{\alpha \in A} [K_{\underline{v}_{\alpha}} : (F_{\text{mod}})_{v_{\alpha}}] \Big) \cdot \Big\{ \sum_{\{w_{\alpha}\}_{\alpha \in A}} \Big(\prod_{\alpha \in A} [(F_{\text{mod}})_{w_{\alpha}} : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big) \Big\}}$$

— where the sum is over all collections $\{w_{\alpha}\}_{\alpha \in A}$ of [not necessarily distinct!] elements $w_{\alpha} \in \mathbb{V}_{\text{mod}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$. Again, these normalized weights are *normalized* so that multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes by addition

or subtraction [that is to say, depending on whether $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ or $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$] of the quantity $\log(p_{v_{\mathbb{Q}}}) \in \mathbb{R}$.

(iii) In the discussion to follow, we shall, for simplicity, use the term *"measure space"* to refer to a *locally compact Hausdorff topological space* whose topology admits a *countable basis*, and which is equipped with a *complete Borel measure* in the sense of [Royden], Chapter 11, §1; [Royden], Chapter 14, §1. In particular, one may speak of the *product measure space* [cf. [Royden], Chapter 12, §4] of any finite nonempty collection of measure spaces. Then observe that *care must be exercised* when considering the various *weighted sums of log-volumes* discussed in (ii), since, unlike, for instance, the *log-volumes* discussed in [item (a) of] [AbsTopIII], Proposition 5.7, (i), (ii),

> such **weighted sums of log-volumes** do **not**, in general, arise as some positive real multiple of the [natural] logarithm of a **"volume"** or **"measure"** in the usual sense of measure theory.

In particular, when considering *direct sums* of the sort that appear in the second or third displays of the statement of Proposition 3.1, although it is *clear from the definitions* how to compute a weighted sum of log-volumes of the sort discussed in (ii) in the case of a *region* that arises as a *direct product* of, say, compact subsets of positive measure in each of the direct summands [i.e., since the volume/measure of such a compact subset may be computed as the *infimum* of the volume/measure of the compact open subsets that contain it], it is *not immediately clear* from the definitions how to compute such a weighted sum of log-volumes in the case of *more general regions*. In the following, for ease of reference, let us refer to such a

> *region* that arises as a *direct product* of compact subsets of positive measure in each of the direct summands as a **direct product region**

and to a

> *region* that arises as a *direct product* of relatively compact subsets in each of the direct summands as a **direct product pre-region**.

Then we observe in the remainder of the present Remark 3.1.1 that although, in the present series of papers,

> the regions that will actually be **of interest** in the development of the theory are, in fact, **direct product [pre-]regions**, in which case the computation of weighted sums of log-volumes is completely straightforward [cf. also the discussion of Remark 3.9.7, (ii), (iii), below],

in fact,

> **weighted sums of log-volumes** of the sort discussed in (ii) may be computed for, say, **arbitrary Borel sets** by applying the elementary construction discussed in (iv) below.

Here, in the context of the situation discussed in the final portion of (ii), we note that this construction in (iv) below is applied relative to the following given data:

· the finite set "V" is taken to be the direct product

$$\prod_{\alpha \in A} \underline{\mathbb{V}}_{v_{\mathbb{Q}}} \quad (\xrightarrow{\sim} \prod_{\alpha \in A} (\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}})$$

[where the subscript "$v_{\mathbb{Q}}$" denotes the fiber over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$];

· for "$v \in V$", the cardinality "N_v" is taken to be the product that appears in the discussion of (ii)

$$\prod_{\alpha \in A} [K_{\underline{v}_\alpha} : (F_{\mathrm{mod}})_{v_\alpha}]$$

[where we think of "$v \in V$" as a collection $\{\underline{v}_\alpha\}_{\alpha \in A}$ of elements of $\underline{\mathbb{V}}_{v_{\mathbb{Q}}}$ that lies over a collection $\{v_\alpha\}_{\alpha \in A}$ of elements of $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$], while "$M_v$" is taken to be the [*radial*, if $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}{}^{\mathrm{arc}}$] portion of the direct summand in the third display of the statement of Proposition 3.1 indexed by $v \in V$ that corresponds to the tensor product of the $\{K_{\underline{v}_\alpha}\}_{\alpha \in A}$.

[By the *"radial"* portion of a topological tensor product of a finite collection of complex archimedean fields, we mean the *direct product of the copies of* $\mathbb{R}_{>0}$ that arise by forming the *quotients by the units* [i.e., copies of $\mathbb{S}^1$] of each of the complex archimedean fields that appears in the *direct sum of fields* [cf. (i)] that arises from such a topological tensor product.] Then one verifies immediately that, in the case of *"direct product regions"* [as discussed above], the result of multiplying the [natural] logarithm of the *"$\mathcal{E}$-weighted measure $\mu_{\mathcal{E}}(-)$"* of (iv) by a suitable normalization factor [i.e., a suitable positive real number] yields the *weighted sums of log-volumes* discussed in (ii).

(iv) Let V a *nonempty finite set*; $\mathcal{E} \stackrel{\mathrm{def}}{=} \{E_v\}_{v \in V}$ a *collection of nonempty finite sets*; $\mathcal{M} \stackrel{\mathrm{def}}{=} \{(M_v, \mu_v)\}_{v \in V}$ a collection of nonempty *measure spaces* [cf. the discussion of (iii) above]. For $v \in V$, write

$$E \stackrel{\mathrm{def}}{=} \prod_{v' \in V} E_{v'}; \quad E_{\neq v} \stackrel{\mathrm{def}}{=} \prod_{V \ni v' \neq v} E_{v'};$$

$$E \times V \twoheadrightarrow W \stackrel{\mathrm{def}}{=} \coprod_{v' \in V} E_{\neq v'} \times \{v'\} \twoheadrightarrow V$$

— where the first arrow "$\twoheadrightarrow$" is defined by the condition that, for $v' \in V$, it restricts to the natural projection $E \times \{v'\} \twoheadrightarrow E_{\neq v'} \times \{v'\}$ on $E \times \{v'\}$; the second arrow "$\twoheadrightarrow$" is defined by the condition that, for $v' \in V$, it restricts to the natural projection $E_{\neq v'} \times \{v'\} \twoheadrightarrow \{v'\}$ on $E_{\neq v'} \times \{v'\}$. If $W \ni w \mapsto v \in V$ via the natural surjection $W \twoheadrightarrow V$ just discussed, then write $(M_w, \mu_w) \stackrel{\mathrm{def}}{=} (M_v, \mu_v)$. If Z is a subset of W or V, then we shall write

$$M_Z \stackrel{\mathrm{def}}{=} \prod_{z \in Z} M_z; \quad M_{E \times V} \stackrel{\mathrm{def}}{=} \prod_{(e,v) \in E \times V} M_v = \prod_{e \in E} M_V;$$

$$(M_{E \times V} \supseteq) \; M_{E * V} \stackrel{\mathrm{def}}{=} \Big\{ \{m_{e,v}\}_{(e,v) \in E \times V} \mid m_{e',v} = m_{e'',v}, \\ \forall (e', e'') \in E \times_{E_{\neq v}} E \subseteq E \times E \Big\} \stackrel{\sim}{\to} M_W$$

— where the bijection $M_{E*V} \stackrel{\sim}{\to} M_W$ is the map induced by the various natural projections $E \twoheadrightarrow E_{\neq v}$ that constitute the natural projection $E \times V \twoheadrightarrow W$; this bijection $M_{E*V} \stackrel{\sim}{\to} M_W$ is easily verified to be a *homeomorphism*. Thus, M_W,

M_V, and $M_{E\times V}$ are equipped with natural *product measure space* structures; the bijection $M_{E*V} \xrightarrow{\sim} M_W$, together with the measure space structure on M_W, induces a *measure space* structure on M_{E*V}. In particular, if $S \subseteq M_V$ is any *Borel set*, then the product

$$\prod_{e\in E} S \ \subseteq \ M_{E\times V}$$

is a Borel set of $M_{E\times V}$; the intersection of this product with M_{E*V}

$$S_E \ \stackrel{\mathrm{def}}{=} \ \Big\{\prod_{e\in E} S\Big\} \bigcap M_{E*V} \ \subseteq \ M_{E*V}$$

is a Borel set of M_{E*V} ($\xrightarrow{\sim} M_W$). Thus, in summary, for any *Borel set* $S \subseteq M_V$, one may speak of the *"$\mathcal{E}$-weighted measure"*

$$\mu_{\mathcal{E}}(S) \in \mathbb{R}_{\geq 0} \bigcup \{+\infty\}$$

of S, i.e., the *measure*, relative to the *measure space* structure of M_{E*V} ($\xrightarrow{\sim} M_W$), of S_E. Since, moreover, one verifies immediately that the above construction is *functorial* with respect to isomorphisms of the given data $(V, \mathcal{E}, \mathcal{M})$, it follows immediately that, in fact, $\mu_{\mathcal{E}}(-)$ is *completely determined* by the *cardinalities* $\mathcal{N} \stackrel{\mathrm{def}}{=} \{N_v\}_{v\in V}$ of the finite sets $\mathcal{E} = \{E_v\}_{v\in V}$, i.e., by the data $(V, \mathcal{N}, \mathcal{M})$. Finally, we observe that when $S \subseteq M_V$ is a **"direct product region"** [cf. the discussion of (iii)], i.e., a set of the form $\prod_{v\in V} S_v$, where $S_v \subseteq M_v$ is a compact subset of positive measure, then a straightforward computation reveals that

$$\tfrac{1}{N_{\mathcal{E}}} \cdot \mu^{\log}_{\mathcal{E}}(S) \ = \ \sum_{v\in V} \tfrac{1}{N_v} \cdot \mu^{\log}_v(S_v)$$

— where we write $N_{\mathcal{E}} = \prod_{v\in V} N_v$, and each superscript "log" denotes the natural logarithm of the corresponding quantity without a superscript.

Remark 3.1.2. The constructions involving **local holomorphic tensor packets** given in Proposition 3.1 may be applied to the capsules that appear in the various **$\mathcal{F}$-prime-strip processions** obtained by considering the evident $\mathcal{F}$-prime-strip analogues [cf. [IUTchI], Remark 5.6.1; [IUTchI], Remark 6.12.1] of the **holomorphic processions** discussed in [IUTchI], Proposition 4.11, (i); [IUTchI], Proposition 6.9, (i).

Proposition 3.2. **(Local Mono-analytic Tensor Packets)** *Let*

$$\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A} \ = \ \Big\{\{{}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\Big\}_{\alpha\in A}$$

be an ***n*-capsule**, *with index set A, of $\mathcal{D}^{\vdash}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], §0; [IUTchI], Definition 4.1, (iii)]. Then [cf. the notation of Proposition 1.2, (vi), (vii)] we shall refer to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \ \mapsto \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \bigoplus_{\underline{\mathbb{V}} \,\ni\, \underline{v} \,\mid\, v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

as the **[1-]tensor packet** *associated to the* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\alpha}\mathfrak{D}^{\vdash}$ *and to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \ \mapsto \ \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \bigotimes_{\alpha \in A} \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

— where the tensor product is to be understood as a tensor product of ind-topological modules — as the **[n-]tensor packet** *associated to the collection of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A}$*. For* $\alpha \in A$*,* $\underline{v} \in \underline{\mathbb{V}}$*,* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ *such that* $\underline{v} \mid v_{\mathbb{Q}}$*, we shall write*

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \ \subseteq \ \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

for the ind-topological submodule determined by the tensor product of the factors labeled by $\beta \in A \setminus \{\alpha\}$ *with the tensor product of the direct summand with subscript* $\underline{v}$ *of the factor labeled* α *[cf. Proposition 3.1, (ii)]. If the capsule of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A}$ *arises from a capsule of* $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$**-prime-strips**

$$\{{}^{\alpha}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}\}_{\alpha \in A} \ = \ \left\{ \{{}^{\alpha}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \right\}_{\alpha \in A}$$

[relative to the given initial Θ*-data — cf. [IUTchI], §0; [IUTchII], Definition 4.9, (vii)], then we shall use similar notation to the notation just introduced concerning* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A}$ *to denote objects associated to* $\{{}^{\alpha}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}\}_{\alpha \in A}$*, i.e., by* **replacing** *"*$\mathcal{D}^{\vdash}$*" in the above notational conventions by "*$\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$*" [cf. also the notation of Proposition 1.2, (vi), (vii)]. Then:*

(i) **(Mono-analytic/Holomorphic Compatibility)** *Suppose that the capsule of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A}$ *arises from the capsule of* $\mathcal{F}$*-prime-strips* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in A}$ *of Proposition 3.1; write* $\{{}^{\alpha}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}\}_{\alpha \in A}$ *for the capsule of* $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$*-prime-strips associated to* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in A}$*. Then the poly-isomorphisms "*$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})$*" of Proposition 1.2, (vi), (vii), induce* **natural poly-isomorphisms of ind-topological modules**

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{A}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

between the various **"mono-analytic" tensor packets** *of the present Proposition 3.2 and the* **"holomorphic" tensor packets** *of Proposition 3.1.*

(ii) **(Integral Structures)** *If* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$*, then the* **mono-analytic log-shells** *"*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*" of Proposition 1.2, (vi), determine [i.e., by forming suitable direct sums and tensor products] topological submodules*

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \subseteq \ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \subseteq \ \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \ \subseteq \ \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

— which may be regarded as **integral structures** *on the* $\mathbb{Q}$*-spans of these submodules. If* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*, then by regarding the* **mono-analytic log-shell** *"*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*" of Proposition 1.2, (vii), as the "closed unit ball" of a Hermitian metric on "*$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$*", and considering the induced direct sum Hermitian metric on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,*

together with the induced tensor product Hermitian metric on $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, one obtains Hermitian metrics on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,* $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$*, whose associated* **closed unit balls**

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

may be regarded as **integral structures** *on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,* $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$*, respectively. For arbitrary* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$*, we shall denote by "*$\mathcal{I}^{\mathbb{Q}}((-))$*" the* $\mathbb{Q}$**-span** *of "*$\mathcal{I}((-))$*"; also, we shall apply this notation involving "*$\mathcal{I}((-))$*", "*$\mathcal{I}^{\mathbb{Q}}((-))$*" with "*$\mathcal{D}^{\vdash}$*"* **replaced** *by "*$\mathcal{F}$*" or "*$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*" for the various objects obtained from the "*$\mathcal{D}^{\vdash}$*-versions" discussed above by applying the* **natural poly-isomorphisms** *of (i).*

Proof. The various assertions of Proposition 3.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.2.1. The issue of **estimating the discrepancy** between the **holomorphic** *integral structures* of Proposition 3.1, (ii), and the **mono-analytic** *integral structures* of Proposition 3.2, (ii), will form one of the main topics to be discussed in [IUTchIV] — cf. also Remark 3.9.1 below.

Remark 3.2.2. The constructions involving **local mono-analytic tensor packets** given in Proposition 3.2 may be applied to the capsules that appear in the various $\mathcal{D}^{\vdash}$**-prime-strip processions** — i.e., **mono-analytic processions** — discussed in [IUTchI], Proposition 4.11, (ii); [IUTchI], Proposition 6.9, (ii).

Proposition 3.3. (Global Tensor Packets) *Let*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

be a $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theater** *[relative to the given initial* Θ*-data] — cf. [IUTchI], Definition 6.13, (i). Thus,* $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *determines* Θ*NF- and* $\Theta^{\pm \mathrm{ell}}$*-Hodge theaters* $^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$*,* $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}}$ *as in [IUTchII], Corollary 4.8. Let* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$ *be an n-capsule of* $\mathcal{F}$*-prime-strips as in Proposition 3.1. Suppose, further, that A is a* **subset** *of the index set J that appears in the* Θ*NF-Hodge theater* $^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$*, and that, for each* $\alpha \in A$*, we are given a* $\mathfrak{log}$**-link**

$$^{\alpha}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathfrak{F}_{\alpha}$$

— i.e., a poly-isomorphism of $\mathcal{F}$*-prime-strips* $\mathfrak{log}({}^{\alpha}\mathfrak{F}) \overset{\sim}{\to} {}^{\dagger}\mathfrak{F}_{\alpha}$ *[cf. Definition 1.1, (iii)]. Next, recall the field* $^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}$ *discussed in [IUTchII], Corollary 4.8, (i); thus, one also has, for* $j \in J$*, a labeled version* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j$ *of this field [cf. [IUTchII], Corollary 4.8, (ii)]. We shall refer to*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A \overset{\mathrm{def}}{=} \bigotimes_{\alpha\in A} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$$

— *where the tensor product is to be understood as a tensor product of modules — as the* **global [n-]tensor packet** *associated to the subset* $A \subseteq J$ *and the* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$.

(i) **(Ring Structures)** *The* **field structure** *on the various* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$*, for* $\alpha \in A$*, determine a* **ring structure** *on* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A$ *with respect to which* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A$ *decomposes, uniquely, as a* **direct sum of number fields**. *Moreover, the various* **localization functors** *"*$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \rightarrow {}^{\dagger}\mathfrak{F}_j$*" considered in [IUTchII], Corollary 4.8, (iii), determine, by composing with the given* **$\mathfrak{log}$-links**, *a* **natural injective localization ring homomorphism**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A \quad \rightarrow \quad \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

to the product of the local holomorphic tensor packets considered in Proposition 3.1.

(ii) **(Integral Structures)** *Fix an element* $\alpha \in A$. *Then by forming the tensor product with "1's" in the factors labeled by* $\beta \in A \setminus \{\alpha\}$*, one obtains a* **natural ring homomorphism**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \rightarrow ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A$$

that induces an **isomorphism** *of the domain onto a subfield of each of the direct summand number fields of the codomain. For each* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$*, this homomorphism is* **compatible**, *in the evident sense, relative to the* **localization** *homomorphism of (i), with the natural homomorphism of ind-topological rings considered in Proposition 3.1, (ii). Moreover, for each* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$*, the composite of the above displayed homomorphism with the component at* $v_{\mathbb{Q}}$ *of the localization homomorphism of (i) maps the ring of integers of the number field* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *into the submodule constituted by the* **integral structure** *on* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ *considered in Proposition 3.1, (ii); for each* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*, the composite of the above displayed homomorphism with the component at* $v_{\mathbb{Q}}$ *of the localization homomorphism of (i) maps the set of archimedean integers [i.e., elements of absolute value* ≤ 1 *at all archimedean primes] of the number field* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *into the direct product of subsets constituted by the* **integral structures** *considered in Proposition 3.1, (ii), on the various direct summand ind-topological fields of* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$.

Proof. The various assertions of Proposition 3.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.3.1. One may perform analogous constructions to the constructions of Proposition 3.3 for the fields "$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$" of [IUTchII], Corollary 4.7, (ii) [cf. also the localization functors of [IUTchII], Corollary 4.7, (iii)], constructed from the associated $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$. These constructions are *compatible* with the corresponding constructions of Proposition 3.3, in the evident sense, relative to the various *labeled Kummer-theoretic isomorphisms* of [IUTchII], Corollary 4.8, (ii). We leave the routine details to the reader.

Remark 3.3.2.

(i) One may consider the **image** of the **localization** homomorphism of Proposition 3.3, (i), in the case of the various **local holomorphic tensor packets** arising from **processions**, as discussed in Remark 3.1.2. Indeed, at the level of the *labels* involved, this is immediate in the case of the *"$\mathbb{T}_l^{\circledast}$-processions"* of [IUTchI], Proposition 4.11, (i). On the other hand, in the case of the *"$|\mathbb{F}_l|$-processions"* of [IUTchI], Proposition 6.9, (i), this may be achieved by applying the *identifying isomorphisms* between the *zero label* $0 \in |\mathbb{F}_l|$ and the *diagonal label* $\langle \mathbb{F}_l^{\circledast} \rangle$ *associated to* $\mathbb{F}_l^{\circledast}$ discussed in [the final display of] [IUTchII], Corollary 4.6, (iii) [cf. also [IUTchII], Corollary 4.8, (ii)].

(ii) In a similar vein, one may *compose* the *"$\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater version"* discussed in Remark 3.3.1 of the **localization** homomorphism of Proposition 3.3, (i), with the product over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ of the inverses of the upper right-hand displayed isomorphisms at $v_{\mathbb{Q}}$ of Proposition 3.2, (i), and then consider the **image** of this composite morphism in the case of the various **local mono-analytic tensor packets** arising from **processions**, as discussed in Remark 3.2.2. Just as in the holomorphic case discussed in (i), in the case of the *"$|\mathbb{F}_l|$-processions"* of [IUTchI], Proposition 6.9, (ii), this obliges one to apply the *identifying isomorphisms* between the *zero label* $0 \in |\mathbb{F}_l|$ and the *diagonal label* $\langle \mathbb{F}_l^{\circledast} \rangle$ *associated to* $\mathbb{F}_l^{\circledast}$ discussed in [the final display of] [IUTchII], Corollary 4.5, (iii).

(iii) The various *images of global tensor packets* discussed in (i) and (ii) above may be *identified* — i.e., in light of the *injectivity* of the homomorphisms applied to construct these images — with the *global tensor packets* themselves. These **local holomorphic/local mono-analytic global tensor packet images** will play a *central role* in the development of the theory of the present §3 [cf., e.g., Proposition 3.7, below].

Remark 3.3.3. The **$\mathfrak{log}$-shifted** nature of the **localization** homomorphism of Proposition 3.3, (i), will play a crucial role in the development of the theory of present §3 — cf. the discussion of [IUTchII], Remark 4.8.2, (i), (iii).

$$\begin{array}{ccccccccccc}
 & & \underline{\underline{q}}^{1} \curvearrowright & & & & \underline{\underline{q}}^{j^2} \curvearrowright & & & & \underline{\underline{q}}^{(l^*)^2} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots /^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots \dots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}$$

Fig. 3.1: Splitting monoids of LGP-monoids acting on tensor packets

Proposition 3.4. **(Local Packet-theoretic Frobenioids)**

(i) **(Single Packet Monoids)** *In the situation of Proposition 3.1, fix elements $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $\underline{v} \mid v_{\mathbb{Q}}$. Then the operation of forming the* **image** *via the natural homomorphism $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \rightarrow \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ [cf. Proposition 3.1, (ii)]*

of the monoid $\Psi_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *[cf. the notation of Definition 1.1, (i), (ii)], together with its submonoid of units* $\Psi^{\times}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *and realification* $\Psi^{\mathbb{R}}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$*, determines monoids*

$$\Psi_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}, \quad \Psi^{\times}_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}, \quad \Psi^{\mathbb{R}}_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}$$

— which are equipped with $G_{\underline{v}}(^{\alpha}\Pi_{\underline{v}})$**-actions** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and, in the case of the first displayed monoid, with a pair consisting of an* Aut**-holomorphic orbispace** *and a* **Kummer structure** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$*. We shall think of these monoids as [possibly realified]* **subquotients** *of*

$$\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$$

that **act** *[multiplicatively] on suitable [possibly realified] subquotients of* $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$*. In particular, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$*, the first displayed monoid, together with its* $^{\alpha}\Pi_{\underline{v}}$*-action, determine a* **Frobenioid** *equipped with a natural isomorphism to* $\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})$*; when* $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$*, the first displayed monoid, together with its* Aut*-holomorphic orbispace and Kummer structure, determine a collection of data equipped with a natural isomorphism to* $\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})$*.*

(ii) **(Local Logarithmic Gaussian Procession Monoids)** *Let*

$$^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$$

be a $\mathfrak{log}$**-link** *of* $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *as in Proposition 1.3, (i) [cf. also the situation of Proposition 3.3]. Consider the* $\mathcal{F}$**-prime-strip processions** *that arise as the* $\mathcal{F}$*-prime-strip analogues [cf. Remark 3.1.2; [IUTchI], Remark 6.12.1] of the* **holomorphic processions** *discussed in [IUTchI], Proposition 6.9, (i), when the functor of [IUTchI], Proposition 6.9, (i), is applied to the* $\Theta^{\pm}$**-bridges** *associated to* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$*,* $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$*; we shall refer to such processions as "†-" or "‡-" processions. Here, we recall that for* $j \in \{1, \ldots, l^{*}\}$*, the index set of the* $(j+1)$*-capsule that appears in such a procession is denoted* $\mathbb{S}^{\pm}_{j+1}$*. Then by applying the various constructions of* **"single packet monoids"** *given in (i) in the case of the various capsules of* $\mathcal{F}$*-prime-strips that appear in a holomorphic ‡-procession — i.e., more precisely, in the case of the* **label** $j \in \{1, \ldots, l^{*}\}$ *[which we shall occasionally identify with its image in* $\mathbb{F}^{*}_{l} \subseteq |\mathbb{F}_{l}|$*] that appears in the* $(j+1)$**-capsule** *of the ‡-procession — to the pull-backs, via the* **poly-isomorphisms** *that appear in the definition [cf. Definition 1.1, (iii)] of the given* $\mathfrak{log}$**-link***, of the [collections of] monoids equipped with actions by* **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*]* $\Psi_{\mathcal{F}_{\text{gau}}}(^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$*,* $_{\infty}\Psi_{\mathcal{F}_{\text{gau}}}(^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ *of [IUTchII], Corollary 4.6, (iv), for* $\underline{v} \in \underline{\mathbb{V}}$*, one obtains a* **functorial algorithm** *in the* $\mathfrak{log}$**-link of** $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ *for constructing [collections of] monoids equipped with actions by* **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*]*

$$\underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi_{\mathcal{F}_{\text{LGP}}}(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}; \qquad \underline{\mathbb{V}} \ni \underline{v} \mapsto {}_{\infty}\Psi_{\mathcal{F}_{\text{LGP}}}(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}$$

— which we refer to as **"[local] LGP-monoids"***, or "logarithmic Gaussian procession monoids" [cf. Fig. 3.1 above]. Here, we note that the notation "*$(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})$*"*

constitutes a slight abuse of notation. Also, we note that this functorial algorithm requires one to apply the **compatibility** *of the given* $\mathfrak{log}$*-link with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** *involved [cf. Remark 1.3.2]. For* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*, the component labeled* $j \in \{1, \ldots, l^*\}$ *of the submonoid of* **Galois invariants** *[cf. (i)] of the* **entire** *LGP-monoid* $\Psi_{\mathcal{F}_{\text{LGP}}}({}^\dagger\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}$ *is a subset of*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^\pm_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

[i.e., where the notation "$;\ddagger$*" denotes the result of applying the discussion of (i) to the case of* $\mathcal{F}$*-prime-strips labeled "*$\ddagger$*"; cf. also the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^\pm_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$ *[cf. the constructions of [IUTchII], Corollary 3.6, (ii)]. For any* $\underline{v} \in \underline{\mathbb{V}}$*, the component labeled* $j \in \{1, \ldots, l^*\}$ *of the submodule of* **Galois invariants** *[cf. (i) when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$*; this Galois action is trivial when* $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$*] of the* **unit portion** $\Psi_{\mathcal{F}_{\text{LGP}}}({}^\dagger\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})^\times_{\underline{v}}$ *of such an LGP-monoid is a subset of*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^\pm_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

[cf. the discussion of (i); the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^\pm_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$ *[cf. the constructions of [IUTchII], Corollary 3.6, (ii); [IUTchII], Proposition 4.2, (iv); [IUTchII], Proposition 4.4, (iv)].*

Proof. The various assertions of Proposition 3.4 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Proposition 3.5. (Kummer Theory and Upper Semi-compatibility for Vertically Coric Local LGP-Monoids) *Let* $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}\}_{n,m\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from a* **Gaussian log-theta-lattice** *[cf. Definition 1.4]. For each* $n \in \mathbb{Z}$*, write*

$$ {}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}$$

for the $\mathcal{D}$*-*$\Theta^{\pm\text{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$*, where* $m \in \mathbb{Z}$*, via the* **vertical coricity** *of Theorem 1.5, (i).*

(i) **(Vertically Coric Local LGP-Monoids and Associated Kummer Theory)** *Write*

$$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_\succ)_t$$

for the $\mathcal{F}$*-prime-strip associated [cf. [IUTchII], Remark 4.5.1, (i)] to the labeled collection of monoids "*$\Psi_{\text{cns}}({}^{n,\circ}\mathfrak{D}_\succ)_t$*" of [IUTchII], Corollary 4.5, (iii) [i.e., where we take "*$\dagger$*" to be "*$n,\circ$*"]. Recall the constructions of Proposition 3.4, (ii), involving* $\mathcal{F}$*-prime-strip processions. Then by applying these constructions to the* $\mathcal{F}$*-prime-strips "*$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_\succ)_t$*" and the various full* $\mathfrak{log}$*-links associated [cf. the discussion of Proposition 1.2, (ix)] to these* $\mathcal{F}$*-prime-strips — which we consider in a fashion* **compatible** *with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** *involved [cf. Remark 1.3.2; Proposition*

3.4, (ii)] — we obtain a **functorial algorithm** *in the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing [collections of] monoids*

$$\underline{\mathbb{V}} \ni \underline{v} \;\mapsto\; \Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}; \qquad \underline{\mathbb{V}} \ni \underline{v} \;\mapsto\; {}_{\infty}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

equipped with actions by **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*] — which we refer to as* **"vertically coric [local] LGP-monoids"**. *For each* $n, m \in \mathbb{Z}$*, this functorial algorithm is* **compatible** *[in the evident sense] with the functorial algorithm of Proposition 3.4, (ii) — i.e., where we take "†" to be "*n, m*" and "‡" to be "*$n, m-1$*" — relative to the* **Kummer isomorphisms** *of labeled data*

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

of [IUTchII], Corollary 4.6, (iii), and the evident identification, for $m' = m, m-1$*, of* ${}^{n,m'}\mathfrak{F}_t$ *[i.e., the* $\mathcal{F}$*-prime-strip that appears in the associated* $\Theta^{\pm}$*-bridge] with the* $\mathcal{F}$*-prime-strip associated to* $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$*. In particular, for each* $n, m \in \mathbb{Z}$*, we obtain* **Kummer isomorphisms** *of [collections of] monoids*

$$\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \;\stackrel{\sim}{\to}\; \Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

$${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \;\stackrel{\sim}{\to}\; {}_{\infty}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

equipped with actions by **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], for* $\underline{v} \in \underline{\mathbb{V}}$.

(ii) **(Upper Semi-compatibility)** *The Kummer isomorphisms of the final two displays of (i) are* **"upper semi-compatible"** *— cf. the discussion of "upper semi-commutativity" in Remark 1.2.2, (iii) — with the various* $\mathfrak{log}$**-links of** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[where* $m \in \mathbb{Z}$*] of the Gaussian log-theta-lattice under consideration in the following sense. Let* $j \in \{0, 1, \ldots, l^*\}$*. Then:*

(a) **(Nonarchimedean Primes)** *For* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$*, the topological module*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow "j" to be 0], in the **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] —* **contains** *the images of the submodules of* **Galois invariants** *[where we recall the Galois actions that appear in the data of [IUTchII], Corollary 4.6, (i), (iii)] of the* **groups of units** $(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$*, for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$ *and* $|t| \in \{0, \ldots, j\}$*, via* **both**

(1) the tensor product, over such $|t|$*, of the [relevant]* **Kummer isomorphisms** *of (i),* **and**

(2) the tensor product, over such $|t|$*, of the pre-composite of these Kummer isomorphisms with the* m'*-th* **iterates** *[cf. Remark*

1.1.1] of the **$\mathfrak{log}$-links***, for* $m' \geq 1$*, of the* n*-th column of the Gaussian log-theta-lattice under consideration [cf. the discussion of Remark 1.2.2, (i), (iii)].*

(b) **(Archimedean Primes)** *For* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{arc}}$*, the closed unit ball*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow "j*" to be* 0*], in the* **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] —* **contains** *the image, via the tensor product, over* $|t| \in \{0, \dots, j\}$*, of the [relevant]* **Kummer isomorphisms** *of (i), of* **both**

(1) the **groups of units** $(\Psi_{\text{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$*, for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$*,* **and**

(2) the **closed balls of radius** π *inside* $(\Psi_{\text{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\text{gp}}_{\underline{v}}$ *[cf. the notational conventions of Definition 1.1], for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$*.*

Here, we recall from the discussion of Remark 1.2.2, (ii), (iii), that, if we regard each $\mathfrak{log}$*-link as a correspondence that only concerns the* **units** *that appear in its domain [cf. Remark 1.1.1], then a closed ball as in (2) contains, for each* $m' \geq 1$*, a subset that* **surjects***, via the* m'*-th* **iterate** *of the* **$\mathfrak{log}$-link** *of the* n*-th column of the Gaussian log-theta-lattice under consideration, onto the subset of the group of units* $(\Psi_{\text{cns}}({}^{n,m-m'}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$ *on which this iterate is defined.*

(c) **(Bad Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*; suppose that* $j \neq 0$*. Recall that the various monoids "*$\Psi_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}}$*", "*${}_{\infty}\Psi_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}}$*" constructed in Proposition 3.4, (ii), as well as the monoids "*$\Psi_{\text{LGP}}(-)_{\underline{v}}$*", "*${}_{\infty}\Psi_{\text{LGP}}(-)_{\underline{v}}$*" constructed in (i) above, are equipped with natural* **splittings up to torsion***. Write*

$$\Psi^{\perp}_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}} \subseteq \Psi_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}}; \quad {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}} \subseteq {}_{\infty}\Psi_{\mathcal{F}_{\text{LGP}}}(-)_{\underline{v}}$$

$$\Psi^{\perp}_{\text{LGP}}(-)_{\underline{v}} \subseteq \Psi_{\text{LGP}}(-)_{\underline{v}}; \quad {}_{\infty}\Psi^{\perp}_{\text{LGP}}(-)_{\underline{v}} \subseteq {}_{\infty}\Psi_{\text{LGP}}(-)_{\underline{v}}$$

for the submonoids corresponding to these splittings [cf. the submonoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$*" discussed in Definition 2.4, (i), in the case of "*$\Psi^{\perp}$*"; the notational conventions of Theorem 2.2, (ii), in the case of "*${}_{\infty}\Psi^{\perp}$*"]. [Thus, the subgroup of units of "*$\Psi^{\perp}$*" consists of the* $2l$*-torsion subgroup of "*Ψ*", while the subgroup of units of "*${}_{\infty}\Psi^{\perp}$*" contains the entire torsion subgroup of "*${}_{\infty}\Psi$*".] Then, as* m **ranges** *over the elements of* $\mathbb{Z}$*, the actions, via the [relevant]* **Kummer isomorphisms** *of (i), of the various monoids* $\Psi^{\perp}_{\mathcal{F}_{\text{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}$ $(\subseteq {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\text{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}})$ *on the ind-topological modules*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}})$$

[where $j = 1, \dots, l^{*}$*] — i.e., that arise from applying the constructions of Proposition 3.4, (ii), in the* **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] — are* **mutually**

compatible, *relative to the* $\mathfrak{log}$**-links** *of the* n*-th column of the Gaussian log-theta-lattice under consideration, in the sense that the only portions of these actions that are possibly related to one another via these* $\mathfrak{log}$*-links are the* **indeterminacies** *with respect to* **multiplication by roots of unity** *in the domains of the* $\mathfrak{log}$*-links, that is to say, indeterminacies at* m *that correspond, via the* $\mathfrak{log}$*-link, to* **"addition by zero"** *— i.e., to* **no indeterminacy**! *— at* $m+1$.

Now let us think of the submodules of **Galois invariants** *[cf. the discussion of Proposition 3.4, (ii)] of the various* **groups of units**, *for* $\underline{v} \in \underline{\mathbb{V}}$,

$$(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}, \quad \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})^{\times}_{\underline{v}}$$

and the **splitting monoids**, *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

as acting on various portions of the modules, for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

not *via a* **single Kummer isomorphism** *as in (i) — which* **fails to be compatible** *with the* $\mathfrak{log}$**-links** *of the Gaussian log-theta-lattice! — but rather via the* **totality** *of the various pre-composites of Kummer isomorphisms with* **iterates** *[cf. Remark 1.1.1] of the* $\mathfrak{log}$**-links** *of the Gaussian log-theta-lattice — i.e., precisely as was described in detail in (a), (b), (c) above [cf. also the discussion of Remark 3.11.4 below]. Thus, one obtains a sort of* **"**$\mathfrak{log}$**-Kummer correspondence"** *between the* **totality**, *as* m *ranges over the elements of* $\mathbb{Z}$, *of the various groups of units and splitting monoids just discussed [i.e., which are labeled by "*n, m*"] and their* **actions** *[as just described] on the "*$\mathcal{I}^{\mathbb{Q}}$*" labeled by "*$n, \circ$*" which is* **invariant** *with respect to the* **translation symmetries** *[cf. Proposition 1.3, (iv)] of the* n*-th column of the Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].*

Proof. The various assertions of Proposition 3.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Example 3.6. Concrete Representations of Global Frobenioids. Before proceeding, we pause to take a closer look at the Frobenioid "${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$" of [IUTchI], Example 5.1, (iii), i.e., more concretely speaking, the Frobenioid of *arithmetic line bundles on the stack* "S_{mod}" of [IUTchI], Remark 3.1.5. Let us write

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}$$

for the Frobenioid "${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$" of [IUTchI], Example 5.1, (iii), in the case where the data denoted by the label "†" arises [in the evident sense] from data as discussed in [IUTchI], Definition 3.1. In the following discussion, we shall use the notation of [IUTchI], Definition 3.1.

(i) **(Rational Function Torsor Version)** For each $\underline{v} \in \underline{\mathbb{V}}$, the valuation on $K_{\underline{v}}$ determined by $\underline{v}$ determines a *group homomorphism* $\beta_{\underline{v}} : F^{\times}_{\mathrm{mod}} \to K^{\times}_{\underline{v}}/\mathcal{O}^{\times}_{K_{\underline{v}}}$ [cf. Remark 3.6.1 below]. Then let us define a *category* $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ as follows. An *object* $\mathcal{T} = (T, \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ consists of a collection of data

(a) an $F^{\times}_{\mathrm{mod}}$-*torsor* T;

(b) for each $\underline{v} \in \underline{\mathbb{V}}$, a *trivialization* $t_{\underline{v}}$ of the torsor $T_{\underline{v}}$ obtained from T by executing the "change of structure group" operation determined by the homomorphism $\beta_{\underline{v}}$

subject to the condition that there exists an element $t \in T$ such that $t_{\underline{v}}$ *coincides* with the trivialization of $T_{\underline{v}}$ determined by t *for all but finitely many* $\underline{v}$. An *elementary morphism* $\mathcal{T}_1 = (T_1, \{t_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \to \mathcal{T}_2 = (T_2, \{t_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ between objects of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is defined to be an isomorphism $T_1 \xrightarrow{\sim} T_2$ of $F^{\times}_{\mathrm{mod}}$-torsors which is *integral* at each $\underline{v} \in \underline{\mathbb{V}}$, i.e., maps the trivialization $t_{1,\underline{v}}$ to an element of the $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$-*orbit* of $t_{2,\underline{v}}$. There is an evident notion of *composition of elementary morphisms*, as well as an evident notion of *tensor powers* $\mathcal{T}^{\otimes n}$, for $n \in \mathbb{Z}$, of an object $\mathcal{T}$ of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$. A *morphism* $\mathcal{T}_1 = (T_1, \{t_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \to \mathcal{T}_2 = (T_2, \{t_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ between objects of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is defined to consist of a positive integer n and an elementary morphism $(\mathcal{T}_1)^{\otimes n} \to \mathcal{T}_2$. There is an evident notion of *composition of morphisms.* Thus, $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ forms a *category.* In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ admits a natural *Frobenioid* structure [cf. [FrdI], Definition 1.3], for which the *base category* is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the *linear morphisms*, and the positive integer "n" that appears in the definition of a morphism of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is the *Frobenius degree* of the morphism. Moreover, by associating to an arithmetic line bundle on S_{mod} the $F^{\times}_{\mathrm{mod}}$-torsor determined by restricting the line bundle to the generic point of S_{mod} and the local trivializations at $\underline{v} \in \underline{\mathbb{V}}$ determined by the various local integral structures, one verifies immediately that there exists a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}$$

that induces the *identity morphism* $F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10].

(ii) **(Local Fractional Ideal Version)** Let us define a *category* $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ as follows. An *object*

$$\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ consists of a collection of *"fractional ideals"* $J_{\underline{v}} \subseteq K_{\underline{v}}$ for each $\underline{v} \in \underline{\mathbb{V}}$ — i.e., a finitely generated nonzero $\mathcal{O}_{K_{\underline{v}}}$-submodule of $K_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; a positive real multiple of $\mathcal{O}_{K_{\underline{v}}} \stackrel{\mathrm{def}}{=} \{\lambda \in K_{\underline{v}} \mid |\lambda| \leq 1\} \subseteq K_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ — such that $J_{\underline{v}} = \mathcal{O}_{K_{\underline{v}}}$ *for all but finitely many* $\underline{v}$. If $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, then for any element $f \in F^{\times}_{\mathrm{mod}}$, one obtains an object $f \cdot \mathcal{J} = \{f \cdot J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ by multiplying each of the fractional ideals $J_{\underline{v}}$ by f. Moreover, if $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, then for any $n \in \mathbb{Z}$, there is an evident notion of the n-th tensor power $\mathcal{J}^{\otimes n}$ of $\mathcal{J}$. An *elementary morphism* $\mathcal{J}_1 = \{J_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathcal{J}_2 = \{J_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ between objects of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is

defined to be an element $f \in F^{\times}_{\text{mod}}$ that is *integral* with respect to $\mathcal{J}_1$ and $\mathcal{J}_2$ in the sense that $f \cdot J_{1,\underline{v}} \subseteq J_{2,\underline{v}}$ for each $\underline{v} \in \underline{\mathbb{V}}$. There is an evident notion of *composition of elementary morphisms*. A *morphism* $\mathcal{J}_1 = \{J_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathcal{J}_2 = \{J_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ between objects of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is defined to consist of a positive integer n and an elementary morphism $(\mathcal{J}_1)^{\otimes n} \to \mathcal{J}_2$. There is an evident notion of *composition of morphisms*. Thus, $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ forms a *category*. In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ admits a natural *Frobenioid* structure [cf. [FrdI], Definition 1.3], for which the *base category* is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the *linear morphisms*, and the positive integer "n" that appears in the definition of a morphism of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is the *Frobenius degree* of the morphism. Moreover, by associating to an object $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ the arithmetic line bundle on S_{mod} obtained from the trivial arithmetic line bundle on S_{mod} by modifying the integral structure of the trivial line bundle at $\underline{v} \in \underline{\mathbb{V}}$ in the fashion prescribed by $J_{\underline{v}}$, one verifies immediately that there exists a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}} \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\text{mod}}$$

that induces the *identity morphism* $F^{\times}_{\text{mod}} \to F^{\times}_{\text{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10].

(iii) By composing the isomorphisms of Frobenioids of (i) and (ii), one thus obtains a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}} \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\text{MOD}}$$

that induces the *identity morphism* $F^{\times}_{\text{mod}} \to F^{\times}_{\text{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10]. One verifies immediately that although the above isomorphism of Frobenioids is not necessarily determined by the condition that it induce the identity morphism on $F^{\times}_{\text{mod}}$, the induced isomorphism between the respective *perfections* [hence also on *realifications*] of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, $\mathcal{F}^{\circledast}_{\text{MOD}}$ is *completely determined* by this condition.

Remark 3.6.1. Note that, as far as the various constructions of Example 3.6, (i), are concerned, the various homomorphisms $\beta_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$, may be thought of, alternatively, as a collection of

subquotients of the perfection $(F^{\times}_{\text{mod}})^{\text{pf}}$ of $F^{\times}_{\text{mod}}$

— each of which is equipped with a *submonoid of "nonnegative elements"* — that are *completely determined* by the *ring structure* of the field F_{mod} [i.e., equipped with its structure as the *field of moduli* of X_F].

Remark 3.6.2.

(i) In the theory to be developed below, we shall be interested in relating certain Frobenioids — which will, in fact, be isomorphic to the *realification* of $\mathcal{F}^{\circledast}_{\text{mod}}$ — that lie on opposite sides of [a certain enhanced version of] the $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-link** to one another. In particular, at the level of objects of the Frobenioids involved, it only

makes sense to work with **isomorphism classes** of objects that are preserved by the isomorphisms of Frobenioids that appear. Here, we note that the isomorphism classes of the sort of Frobenioids that appear in this context are determined by the **divisor** and **rational function monoids** of the [model] Frobenioid in question [cf. the constructions given in [FrdI], Theorem 5.2, (i), (ii)]. In this context, we observe that the rational function monoid $F^{\times}_{\mathrm{mod}}$ of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ satisfies the following *fundamental property*:

[the union with $\{0\}$ of] $F^{\times}_{\mathrm{mod}}$ admits a natural **additive structure**.

In this context, we note that this property is *not* satisfied by

(a) the rational function monoids of the *perfection* or *realification* of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$

(b) subgroups $\Gamma \subseteq F^{\times}_{\mathrm{mod}}$ — such as, for instance, the trivial subgroup $\{1\}$ or the subgroup of *S-units*, for $S \subseteq \mathbb{V}_{\mathrm{mod}}$ a nonempty finite subset — that do not arise as the multiplicative group of some subfield of F_{mod} [cf. [AbsTopIII], Remark 5.10.2, (iv)].

The significance of this fundamental property is that it allows one to represent the objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ **additively**, i.e., as *modules* — cf. the point of view of Example 3.6, (ii). At a more concrete level, if, in the notation of (b), one considers the result of *"adding"* two elements of a Γ-*torsor* [cf. the point of view of Example 3.6, (i)!], then the resulting "sum" can only be rendered meaningful, relative to the given Γ-torsor, if Γ is *additively closed*. The **additive representation** of objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ will be of *crucial importance* in the theory of the present series of papers since it will allow us to relate objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ on opposite sides of [a certain enhanced version of] the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link to one another — which, a priori, are only related to one another at the level of *realifications* in a **multiplicative** fashion — by means of **["additive"] mono-analytic log-shells** [cf. the discussion of [IUTchII], Remark 4.7.2].

(ii) One way to understand the content of the discussion of (i) is as follows: whereas

the construction of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ depends on the **additive** structure of $F^{\times}_{\mathrm{mod}}$

in an essential way,

the construction of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is strictly **multiplicative** in nature.

Indeed, the construction of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ given in Example 3.6, (i), is essentially the same as the construction of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ given in [FrdI], Example 6.3 [i.e., in effect, in [FrdI], Theorem 5.2, (i)]. From this point of view, it is natural to **identify** $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ with $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ via the natural isomorphism of Frobenioids of Example 3.6, (i). We shall often do this in the theory to be developed below.

Proposition 3.7. (Global Packet-theoretic Frobenioids)

(i) **(Single Packet Rational Function Torsor Version)** *In the notation of Proposition 3.3: For each $\alpha \in A$, there is an* **algorithm** *for constructing, as discussed in Example 3.6, (i) [cf. also Remark 3.6.1], from the [number] field given by the* **image**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

of the composite

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \ \to \ ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A} \ \to \ \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$$

of the homomorphisms of Proposition 3.3, (i), (ii), a **Frobenioid** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, *together with a* **natural isomorphism of Frobenioids**

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha} \ \xrightarrow{\sim} \ ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

[cf. the notation of [IUTchII], Corollary 4.8, (ii)] that induces the **tautological** *isomorphism* $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *on the associated rational function monoids [cf. Example 3.6, (i)]. We shall often use this isomorphism of Frobenioids to* **identify** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *with* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *[cf. Remark 3.6.2, (ii)]. Write* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$ *for the* **realification** *of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$.

(ii) **(Single Packet Local Fractional Ideal Version)** *In the notation of Propositions 3.3, 3.4: For each* $\alpha \in A$, *there is an* **algorithm** *for constructing, as discussed in Example 3.6, (ii), from the [number] field* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha} \stackrel{\mathrm{def}}{=} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *[cf. (i)] and the* **Galois invariants** *of the local monoids*

$$\Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} \ \subseteq \ \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

for $\underline{v} \in \underline{\mathbb{V}}$ *of Proposition 3.4, (i) — i.e., so the corresponding local "fractional ideal* $J_{\underline{v}}$*" of Example 3.6, (ii), is a* **subset** *[indeed a submodule when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] of* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *whose* $\mathbb{Q}$*-span is equal to* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *[cf. the notational conventions of Proposition 3.2, (ii)] — a* **Frobenioid** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$, *together with* **natural isomorphisms of Frobenioids**

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \ \xrightarrow{\sim} \ ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}; \quad ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \ \xrightarrow{\sim} \ ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

that induce the **tautological** *isomorphisms* $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_{\alpha}$, $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *on the associated rational function monoids [cf. the natural isomorphism of Frobenioids of (i); Example 3.6, (ii), (iii)]. Write* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_{\alpha}$ *for the* **realification** *of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$.

(iii) **(Global Realified** LGP**-Frobenioids)** *In the notation of Proposition 3.4: By applying the composites of the isomorphisms of Frobenioids "*${}^{\dagger}\mathcal{C}^{\Vdash}_{j} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{j}$*" of [IUTchII], Corollary 4.8, (iii), with the realifications "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$*" of the isomorphisms of Frobenioids of (i) above to the global realified Frobenioid portion* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}$ *of the* $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *of [IUTchII], Corollary 4.10, (ii) [cf. Remarks 1.5.3, (iii); 3.3.2, (i)], one obtains a* **functorial algorithm** *in the* **$\mathfrak{log}$-link of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *of Proposition 3.4, (ii), for constructing a Frobenioid*

$$\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$$

— which we refer to as a **"global realified LGP-Frobenioid"**. *Here, we note that the notation "*$({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$*" constitutes a slight abuse of notation. In particular, the global realified Frobenioid* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, *together with*

the collection of data $\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *constructed in Proposition 3.4, (ii), give rise, in a natural fashion, to an* $\mathcal{F}^{\Vdash}$**-prime-strip**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \;=\; ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}},\ \{{}^{\dagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

— cf. the construction of the $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *in [IUTchII], Corollary 4.10, (ii) — together with a* **natural isomorphism**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips [i.e., that arises* **tautologically** *from the construction of* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$*!].*

(iv) **(Global Realified** $\mathfrak{lgp}$**-Frobenioids)** *In the situation of (iii) above, write* $\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}} \stackrel{\mathrm{def}}{=} {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}$*. Then by replacing, in the construction of (iii), the isomorphisms "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$*" by the natural isomorphisms "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_{\alpha}$*" [cf. (ii)], one obtains a* **functorial algorithm** *in the* $\mathfrak{log}$**-link of** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *of Proposition 3.4, (ii), for constructing a Frobenioid*

$$\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$$

— which we refer to as a **"global realified** $\mathfrak{lgp}$**-Frobenioid"** *— as well as an* $\mathcal{F}^{\Vdash}$**-prime-strip**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} \;=\; ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}},\ \{{}^{\dagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

— where we write ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *— together with* **tautological isomorphisms**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips [cf. (iii)].*

(v) **(Realified Product Embeddings and Non-realified Global Frobenioids)** *The constructions of* $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *given in (iii) and (iv) above give rise to a commutative diagram of categories*

$$\begin{array}{ccc}
\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) & \hookrightarrow & \prod_{j\in\mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{j} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) & \hookrightarrow & \prod_{j\in\mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_{j}
\end{array}$$

— where the horizontal arrows are **embeddings** *that arise tautologically from the constructions of (iii) and (iv) [cf. [IUTchII], Remark 4.8.1, (i)]; the vertical arrows are* **isomorphisms***; the left-hand vertical arrow arises from the second isomorphism that appears in the final display of (iv); the right-hand vertical arrow is the product of the* **realifications** *of copies of the inverse of the second isomorphism that appears in the final display of (ii). In particular, by applying the definition of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{j}$ *—*

i.e., in terms of **local fractional ideals** *[cf. (ii)] — together with the products of* **realification functors**

$$\prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \;\to\; \prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j$$

[cf. [FrdI], Proposition 5.3], one obtains an **algorithm** *for constructing, in a fashion compatible [in the evident sense] with the* **local isomorphisms** $\{{}^{\dagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, $\{{}^{\dagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *of (iii) and (iv), objects of the [global!] categories* $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$, $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ *from the local fractional ideals generated by elements of the* **monoids** *[cf. (iv); Proposition 3.4, (ii)]*

$$\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Proof. The various assertions of Proposition 3.7 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Definition 3.8.

(i) In the situation of Proposition 3.7, (iv), (v), write $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}(-)_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the notation of Proposition 3.5, (ii), (c)]. Then we shall refer to the object of

$$\prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j \quad \text{or} \quad \prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$$

— as well as its *realification*, regarded as an object of ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} = \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ or ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} = \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ [cf. Proposition 3.7, (iii), (iv), (v)] — determined by any collection, indexed by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to torsion of the monoids $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}$ as a **Θ-pilot object** [cf. also Remark 3.8.1 below]. We shall refer to the object of the [global realified] Frobenioid

$${}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle}$$

of [IUTchII], Corollary 4.10, (i), determined by any collection, indexed by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to torsion of the splitting monoid associated to the split Frobenioid ${}^{\dagger}\mathcal{F}^{\vdash}_{\triangle,\underline{v}}$ [i.e., the data indexed by $\underline{v}$ of the $\mathcal{F}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle}$ of [IUTchII], Corollary 4.10, (i)] — that is to say, at a more concrete level, determined by the "$\underline{\underline{q}}_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the notation of [IUTchI], Example 3.2, (iv)] — as a ***q*-pilot object** [cf. also Remark 3.8.1 below].

(ii) Let

$${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

be a $\mathfrak{log}$-*link of* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* and

$$^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [all relative to the given initial Θ-data]. Recall the $\mathcal{F}^{\Vdash}$-prime-strip

$$^{*}\mathfrak{F}^{\Vdash}_{\triangle}$$

constructed from $^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ in [IUTchII], Corollary 4.10, (i). Following the notational conventions of [IUTchII], Corollary 4.10, (iii), let us write $^{*}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\triangle}$ (respectively, $^{\dagger}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\mathrm{LGP}}$; $^{\dagger}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\mathfrak{lgp}}$) for the $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip associated to the $\mathcal{F}^{\Vdash}$-prime-strip $^{*}\mathfrak{F}^{\Vdash}_{\triangle}$ (respectively, $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$; $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$) [cf. Proposition 3.7, (iii), (iv); [IUTchII], Definition 4.9, (viii); the functorial algorithm described in [IUTchII], Definition 4.9, (vi)]. Then — in the style of [IUTchII], Corollary 4.10, (iii) — we shall refer to the full poly-isomorphism of $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strips $^{\dagger}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{*}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\triangle}$ as the **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} \quad {}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to $^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the $\mathfrak{log}$-link $^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, and to the full poly-isomorphism of $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strips $^{\dagger}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\mathfrak{lgp}} \stackrel{\sim}{\to} {}^{*}\mathfrak{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}_{\triangle}$ as the **$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} \quad {}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to $^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the $\mathfrak{log}$-link $^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$.

(iii) Let $\{^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}\}_{n,m \in \mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers. Then we shall refer to the first (respectively, second) diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & & \vdots & & & \\
 & & \uparrow \mathfrak{log} & & & \uparrow \mathfrak{log} & & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & \dots & \\
 & & \uparrow \mathfrak{log} & & & \uparrow \mathfrak{log} & & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & \dots & \\
 & & \uparrow \mathfrak{log} & & & \uparrow \mathfrak{log} & & & \\
 & & \vdots & & & \vdots & & &
\end{array}
$$

— where the *vertical* arrows are the *full* $\mathfrak{log}$*-links*, and the *horizontal* arrow of the first (respectively, second) diagram from ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ is the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$- (respectively, $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-) link from ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the full $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [cf. (ii)] — as the [LGP**-Gaussian**] (respectively, [$\mathfrak{lgp}$**-Gaussian**]) **log-theta-lattice**. Thus, [cf. Definition 1.4] either of these diagrams may be represented symbolically by an *oriented graph*

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

— where the "$\bullet$'s" correspond to the "${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$".

Remark 3.8.1. The LGP-Gaussian and $\mathfrak{lgp}$-Gaussian log-theta-lattices are, of course, closely related, but, in the theory to be developed below, we shall mainly be interested in the LGP**-Gaussian log-theta-lattice** [for reasons to be explained in Remark 3.10.1, (ii), below]. On the other hand, our computation of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link will involve the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, as well as related **Θ-pilot objects**, in an essential way. Here, we note, for future reference, that both the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$- and the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link map **$\Theta$-pilot objects** to **$q$-pilot objects**. Also, we observe that this terminology of *"Θ-pilot/q-pilot objects"* is *consistent* with the notion of a **"pilot object"** associated to a $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip, as defined in [IUTchII], Definition 4.9, (viii).

Remark 3.8.2. One verifies immediately that the *main results* obtained so far concerning Gaussian log-theta-lattices — namely, Theorem 1.5, Proposition 2.1,

Corollary 2.3 [cf. also Remark 2.3.2], and Proposition 3.5 — generalize immediately [indeed, "formally"] to the case of LGP- or $\mathfrak{lgp}$-**Gaussian log-theta-lattices**. Indeed, the substantive content of these results concerns portions of the log-theta-lattices involved that are *substantively unaffected* by the transition from "Gaussian" to "LGP- or $\mathfrak{lgp}$-Gaussian".

Remark 3.8.3. In the definition of the various **horizontal arrows** of the **log-theta-lattices** discussed in Definition 3.8, (iii), it may appear to the reader, at first glance, that, instead of working with $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips, it might in fact be sufficient to replace the unit [i.e., $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip] portions of these prime-strips by the associated **log-shells** [cf. Proposition 1.2, (vi), (vii)], on which, at nonarchimedean $\underline{v} \in \underline{\mathbb{V}}$, the associated local Galois groups act *trivially.* In fact, however, this is *not* the case. That is to say, the *nontrivial Galois action* on the local unit portions of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips involved is necessary in order to consider the **Kummer theory** [cf. Proposition 3.5, (i), (ii), as well as Proposition 3.10, (i), (iii); Theorem 3.11, (iii), (c), (d), below] of the various local and global objects for which the log-shells serve as **"multiradial containers"** [cf. the discussion of Remark 1.5.2]. Here, we recall that this Kummer theory plays a crucial role in the theory of the present series of papers in relating corresponding *Frobenius-like* and *étale-like* objects [cf. the discussion of Remark 1.5.4, (i)].

Proposition 3.9. (Log-volume for Packets and Processions)

(i) **(Local Holomorphic Packets)** *In the situation of Proposition 3.2, (i), (ii): Suppose that* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$, $\alpha \in A$. *Then the* $p_{v_{\mathbb{Q}}}$*-adic log-volume on each of the direct summand* $p_{v_{\mathbb{Q}}}$*-adic fields of* $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, *and* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *— cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of* **normalized weights** *in Remark 3.1.1, (ii), (iii), (iv) — determines [cf. [AbsTopIII], Proposition 5.7, (i)]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where we write "$\mathfrak{M}(-)$*" for the set of* **nonempty compact open subsets** *of "*$(-)$*" — such that the log-volume of each of the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii) — i.e., the elements*

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A,\alpha}\mathcal{F}_{\underline{v}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

of "$\mathfrak{M}(-)$*" given by the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand* $p_{v_{\mathbb{Q}}}$*-adic fields — is equal to* **zero**. *Here, we assume that these log-volumes are normalized so that multiplication of an element of "*$\mathfrak{M}(-)$*" by* $p_{\underline{v}}$ *corresponds to adding the quantity* $-\log(p_{\underline{v}}) \in \mathbb{R}$*; we shall refer to this normalization as the* **packet-normalization**. *Suppose that* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$, $\alpha \in A$. *Then the sum of the radial log-volumes on each of the direct summand complex archimedean fields of* $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, *and* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *— cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of* **normalized**

weights *in Remark 3.1.1, (ii), (iii), (iv) — determines [cf. [AbsTopIII], Proposition 5.7, (ii)]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where we write "$\mathfrak{M}(-)$" for the set of **compact closures of nonempty open subsets** *of "$(-)$" — such that the log-volume of each of the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii) — i.e., the elements*

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A,\alpha}\mathcal{F}_{\underline{v}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

of "$\mathfrak{M}(-)$" given by the products of the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand complex archimedean fields — is equal to **zero**. *Here, we assume that these log-volumes are normalized so that multiplication of an element of "$\mathfrak{M}(-)$" by $e = 2.71828\ldots$ corresponds to adding the quantity $1 = \log(e) \in \mathbb{R}$; we shall refer to this normalization as the* **packet-normalization**. *In both the nonarchimedean and archimedean cases, "$\mu^{\log}_{A,v_{\mathbb{Q}}}$" is* **invariant** *with respect to* **permutations** *of A. Finally, when working with collections of capsules in a procession, as in Proposition 3.4, (ii), we obtain, in both the nonarchimedean and archimedean cases, log-volumes on the products of the "$\mathfrak{M}(-)$" associated to the various capsules under consideration, which we normalize by taking the* **average**, *over the various capsules under consideration; we shall refer to this normalization as the* **procession-normalization** *[cf. Remark 3.9.3 below].*

(ii) **(Mono-analytic Compatibility)** *In the situation of Proposition 3.2, (i), (ii): Suppose that $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Then by applying the $p_{v_{\mathbb{Q}}}$-adic log-volume, when $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$, or the radial log-volume, when $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$, on the* **mono-analytic log-shells** *"$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$" of Proposition 1.2, (vi), (vii), (viii), and adjusting appropriately [cf. Remark 3.9.1 below for more details] to account for the* **discrepancy** *between the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii), and the* **"mono-analytic" integral structures** *of Proposition 3.2, (ii), one obtains [by a slight abuse of notation]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where "$\mathfrak{M}(-)$" is as in (i) above — which are **compatible** *with the log-volumes obtained in (i), relative to the* **natural poly-isomorphisms** *of Proposition 3.2, (i). In particular, these log-volumes may be constructed via a* **functorial algorithm** *from the $\mathcal{D}^{\vdash}$-prime-strips under consideration. If one considers the* **mono-analyticization** *[cf. [IUTchI], Proposition 6.9, (ii)] of a holomorphic procession as in Proposition 3.4, (ii), then taking the average, as in (i) above, of the* **packet-normalized** *log-volumes of the above display gives rise to* **procession-normalized** *log-volumes, which are compatible, relative to the natural poly-isomorphisms of Proposition 3.2, (i), with the procession-normalized log-volumes of (i). Finally, by* **replacing** *"$\mathcal{D}^{\vdash}$" by "$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$" [cf. also the discussion of Proposition 1.2, (vi),*

(vii), (viii)], one obtains a similar theory of log-volumes for the various objects associated to the mono-analytic log-shells "$\mathcal{I}_{\dagger \mathcal{F}^{\vdash}_{\underline{v}} \times \mu}$", which is **compatible** *with the theory obtained for "$\mathcal{D}^{\vdash}$" relative to the various* **natural poly-isomorphisms** *of Proposition 3.2, (i).*

(iii) **(Global Compatibility)** *In the situation of Proposition 3.7, (i), (ii): Write*

$$\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) = \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

and

$$\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \subseteq \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

for the subset of elements whose components, indexed by $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, have **zero log-volume** *[cf. (i)] for all but finitely many $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Then, by adding the log-volumes of (i) [all but finitely many of which are zero!] at the various $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, one obtains a* **global log-volume**

$$\mu^{\log}_{A,\mathbb{V}_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

which is **invariant** *with respect to* **multiplication** *by elements of*

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} = ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$$

as well as with respect to **permutations** *of A, and, moreover, satisfies the following property concerning [the elements of "$\mathfrak{M}(-)$" determined by] objects "$\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$" of $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$ [cf. Example 3.6, (ii); Proposition 3.7, (ii)]: the* **global log-volume** *$\mu^{\log}_{A,\mathbb{V}_{\mathbb{Q}}}(\mathcal{J})$ is equal to the* **degree** *of the* **arithmetic line bundle** *determined by $\mathcal{J}$ [cf. the discussion of Example 3.6, (ii); the natural isomorphism $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \stackrel{\sim}{\rightarrow} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ of Proposition 3.7, (ii)], relative to a* **suitable normalization**.

(iv) **($\mathfrak{log}$-link Compatibility)** *Let $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m \in \mathbb{Z}}$ be a* **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from an* LGP-**Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. Then [cf. also the discussion of Remark 3.9.4 below]:*

(a) *For $n, m \in \mathbb{Z}$, the* **log-volumes** *constructed in (i), (ii), (iii) above determine log-volumes on the various "$\mathcal{I}^{\mathbb{Q}}((-))$" that appear in the construction of the* **local/global** LGP-**/$\mathfrak{lgp}$-monoids/Frobenioids** *that appear in the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$ constructed in Proposition 3.7, (iii), (iv), relative to the* **$\mathfrak{log}$-link** *${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.*

(b) *At the level of the $\mathbb{Q}$-spans of* **log-shells** *"$\mathcal{I}^{\mathbb{Q}}((-))$" that arise from the various* **$\mathcal{F}$-prime-strips** *involved, the log-volumes of (a) indexed by (n, m) are* **compatible** *— in the sense discussed in Propositions 1.2, (iii); 1.3, (iii) — with the corresponding log-volumes indexed by $(n, m-1)$, relative to the $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.*

Proof. The various assertions of Proposition 3.9 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.9.1. In the spirit of the *explicit descriptions* of Remark 3.1.1, (i) [cf. also Remark 1.2.2, (i), (ii)], we make the following observations.

(i) Suppose that $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$. Write $\{\underline{v}_1, \ldots, \underline{v}_{n_{v_{\mathbb{Q}}}}\}$ for the [distinct!] elements of $\underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$. For each $i = 1, \ldots, n_{v_{\mathbb{Q}}}$, set $k_i \stackrel{\mathrm{def}}{=} K_{\underline{v}_i}$; write $\mathcal{O}_{k_i} \subseteq k_i$ for the ring of integers of k_i,

$$\mathcal{I}_i \stackrel{\mathrm{def}}{=} (p_{v_{\mathbb{Q}}}^*)^{-1} \cdot \log_{k_i}(\mathcal{O}_{k_i}^{\times}) \ \subseteq \ k_i$$

— where $p_{v_{\mathbb{Q}}}^* = p_{\underline{v}}$ if $p_{v_{\mathbb{Q}}}$ is *odd*, $p_{v_{\mathbb{Q}}}^* = p_{v_{\mathbb{Q}}}^2$ if $p_{v_{\mathbb{Q}}}$ is *even* — cf. Remark 1.2.2, (i). Then, roughly speaking, in the notation of Proposition 3.9, (i), the **mono-analytic integral structures** of Proposition 3.2, (ii), in

$$\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \bigoplus_{i=1}^{n_{v_{\mathbb{Q}}}} k_i; \quad \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \bigotimes_{\alpha \in A} \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

are given by

$$\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \bigoplus_{i=1}^{n_{v_{\mathbb{Q}}}} \mathcal{I}_i; \quad \mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \bigotimes_{\alpha \in A} \mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

while the **local holomorphic integral structures**

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \ \subseteq \ \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \ \subseteq \ \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

of Proposition 3.9, (i), in the ind-topological rings $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ — both of which are *direct sums* of finite extensions of $\mathbb{Q}_{p_{v_{\mathbb{Q}}}}$ — are given by the *subrings of integers* in $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Thus, by applying the formula of the final display of [AbsTopIII], Proposition 5.8, (iii), for the log-volume of $\mathcal{I}_i$, [one verifies easily that] one may compute the *log-volumes*

$$\mu_{\alpha, v_{\mathbb{Q}}}^{\log}(\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})), \quad \mu_{A, v_{\mathbb{Q}}}^{\log}(\mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

entirely in terms of the given **initial Θ-data**. We leave the routine details to the reader.

(ii) Suppose that $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$. Write $\{\underline{v}_1, \ldots, \underline{v}_{n_{v_{\mathbb{Q}}}}\}$ for the [distinct!] elements of $\underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$. For each $i = 1, \ldots, n_{v_{\mathbb{Q}}}$, set $k_i \stackrel{\mathrm{def}}{=} K_{\underline{v}_i}$; write $\mathcal{O}_{k_i} \stackrel{\mathrm{def}}{=} \{\lambda \in k_i \mid |\lambda| \leq 1\} \subseteq k_i$ for the "set of integers" of k_i,

$$\mathcal{I}_i \stackrel{\mathrm{def}}{=} \pi \cdot \mathcal{O}_{k_i} \ \subseteq \ k_i$$

— cf. Remark 1.2.2, (ii). Then, roughly speaking, in the notation of Proposition 3.9, (i), the **discrepancy** between the **mono-analytic integral structures** of

Proposition 3.2, (ii), determined by the $\mathcal{I}({}^{\dagger}\mathcal{F}_{\underline{v}_i}) \xrightarrow{\sim} \mathcal{I}_i \subseteq k_i$ and the **local holomorphic integral structures**

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \bigoplus_{i=1}^{n_{v_{\mathbb{Q}}}} k_i$$

$$\mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \bigotimes_{\alpha \in A} \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

of Proposition 3.9, (i), in the topological rings $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ — both of which are direct sums of complex archimedean fields — determined by taking the product [relative to this direct sum decomposition] of the respective *"subsets of integers"* may be computed entirely in terms of the given **initial Θ-data**, by applying the following two [easily verified] observations:

(a) Equip $\mathbb{C}$ with its standard Hermitian metric, i.e., the metric determined by the complex norm. This metric on $\mathbb{C}$ determines a tensor product metric on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, as well as a direct sum metric on $\mathbb{C} \oplus \mathbb{C}$. Then, relative to these metrics, any *isomorphism of topological rings* [i.e., arising from the Chinese remainder theorem]

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \oplus \mathbb{C}$$

is **compatible** with these **metrics**, up to a factor of 2, i.e., the metric on the right-hand side corresponds to 2 times the metric on the left-hand side.

(b) Relative to the notation of (a), the **direct sum decomposition** $\mathbb{C} \oplus \mathbb{C}$, together with its Hermitian metric, is **preserved**, relative to the displayed isomorphism of (a), by the operation of conjugation on either of the two copies of "$\mathbb{C}$" that appear in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, as well as by the operations of multiplying by ± 1 or $\pm\sqrt{-1}$ via either of the two copies of "$\mathbb{C}$" that appear in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

We leave the routine details to the reader.

(iii) The computation of the discrepancy between local holomorphic and mono-analytic integral structures will be discussed in more detail in [IUTchIV], §1.

Remark 3.9.2. In the situation of Proposition 3.9, (iii), one may construct **["mono-analytic"] algorithms** for recovering the **subquotient** of the *perfection* of $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} = ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ associated to $\underline{w} \in \underline{\mathbb{V}}$ [cf. Remark 3.6.1], together with the *submonoid of "nonnegative elements"* of such a subquotient, by considering the effect of multiplication by elements of $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} = ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ on the *log-volumes* defined on the various $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$ [cf. Proposition 3.9, (ii)].

Remark 3.9.3. With regard to the **procession-normalizations** discussed in Proposition 3.9, (i), (ii), the reader might wonder the following: Is it possible to work with

more general **weighted averages**, i.e., as opposed to just *averages*, in the usual sense, over the capsules that appear in the procession?

The answer to this question is *"no"*. Indeed, in the situation of Proposition 3.4, (ii), for $j \in \{1, \ldots, l^*\}$, the *packet-normalized* log-volume corresponding to the capsule with index set $\mathbb{S}^{\pm}_{j+1}$ may be thought of as a log-volume that arises from *"any one of the log-shells whose label* $\in \{0, 1, \ldots, j\}$*"*. In particular, if $j', j_1, j_2 \in \{1, \ldots, l^*\}$, and $j' \leq j_1, j_2$, then log-volumes corresponding to the **same log-shell** labeled j' might give rise to packet-normalized log-volumes corresponding to *either* of [the capsules with index sets] $\mathbb{S}^{\pm}_{j_1+1}$, $\mathbb{S}^{\pm}_{j_2+1}$. That is to say, in order for the resulting notion of a *procession-normalized log-volume* to be **compatible** with the appearance of the component labeled j' in various distinct capsules of the procession — i.e., compatible with the various **inclusion morphisms** of the procession! — one has no choice but to assign the *same weights* to [the capsules with index sets] $\mathbb{S}^{\pm}_{j_1+1}$, $\mathbb{S}^{\pm}_{j_2+1}$.

Remark 3.9.4. The **$\mathfrak{log}$-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), may be *formulated somewhat more explicitly* by applying various elementary observations, as follows.

(i) Let (M, μ_M) be a *measure space* [i.e., in the sense of the discussion of Remark 3.1.1, (iii)]. We shall say that a subset $S \subseteq M$ is *pre-ample* if S is a relatively compact Borel set, that a pre-ample subset $S \subseteq M$ is *ample* if $\mu_M(S) > 0$, and that (M, μ_M) is *ample* if there exists an ample subset of M. In the following, for the sake of simplicity, we assume that (M, μ_M) is *ample*. Also, to simplify the notation, we shall often denote the dependence of objects constructed from the *pair* (M, μ_M) by means of the notation "(M)" [i.e., as opposed to "(M, μ_M)"]. Write

$$\mathrm{Sub}(M)$$

for the *set of pre-ample subsets* of M and

$$\mathrm{Fn}(M)$$

for the *set of Borel measurable functions* $f : M \to \mathbb{R}_{\geq 0}$ such that the image $f(M) \subseteq \mathbb{R}_{\geq 0}$ of f is a *finite set*, and, moreover, $M \supseteq f^{-1}(\mathbb{R}_{>0}) \in \mathrm{Sub}(M)$. Observe that $\mathrm{Fn}(M)$ is equipped with a *natural monoid structure* [induced by the natural monoid structure on $\mathbb{R}_{\geq 0}$], as well as a *natural action* by $\mathbb{R}_{>0}$ [induced by the natural action by multiplication of $\mathbb{R}_{>0}$ on $\mathbb{R}_{\geq 0}$]. By assigning to an element $S \in \mathrm{Sub}(M)$ the *characteristic function* $\chi_S : M \to \mathbb{R}_{\geq 0}$ [i.e., which is $= 1$ on S and $= 0$ on $M \setminus S$], we shall regard $\mathrm{Sub}(M)$ as a *subset* of $\mathrm{Fn}(M)$. Note that *integration over* M, relative to the *measure* μ_M, determines an $\mathbb{R}_{>0}$*-equivariant surjection*

$$\int_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}_{\geq 0}$$

whose restriction to $\mathrm{Sub}(M)$ maps $\mathrm{Sub}(M) \ni S \mapsto \mu_M(S) \in \mathbb{R}_{\geq 0}$. In particular, if we write $\mathrm{Fn}\mathbb{R}\mathrm{ss}_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}\mathrm{ss}(M)$ for the natural map to the *quotient set* of $\mathrm{Fn}(M)$ [i.e., the set of equivalence classes of elements of $\mathrm{Fn}(M)$] determined by $\int_M$

[so $\mathbb{R}\mathrm{ss}(M)$ also admits a *natural monoid structure*, as well as a *natural action* by $\mathbb{R}_{>0}$], then we obtain a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids*

$$\int_M^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$$

such that $\int_M = \int_M^{\mathbb{R}\mathrm{ss}} \circ \mathrm{Fn}\mathbb{R}\mathrm{ss}_M$. Here,

we wish to think of **integration** $\int_M$ — and hence of the **quotient**

$$\mathrm{Fn}\mathbb{R}\mathrm{ss}_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}\mathrm{ss}(M)$$

— as a sort of **"realified semi-simplification"** of (M, μ_M)

[i.e., roughly in the spirit of the *Grothendieck group* associated to an additive category], that is to say, a *quotient* in the category of *commutative monoids with* $\mathbb{R}_{>0}$*-action*, whose restriction to $\mathrm{Sub}(M) \subseteq \mathrm{Fn}(M)$

- · *identifies* $S_1, S_2 \in \mathrm{Sub}(M)$ such that $\mu_M(S_1) = \mu_M(S_2)$ [such as, for instance, *additive translates* of an element $S \in \mathrm{Sub}(M)$ relative to an *additive structure* on M with respect to which μ_M is *invariant*];
- · *maps* $S_1 \cup S_2 \in \mathrm{Sub}(M)$ to the *sum* [relative to the *monoid structure* on the *quotient*] of the images of $S_1, S_2 \in \mathrm{Sub}(M)$ whenever $S_1, S_2 \in \mathrm{Sub}(M)$ are *disjoint* [i.e., as subsets of M].

We shall refer to a subset $E \subseteq \mathrm{Fn}(M)$ as *ample* if $(\mathbb{R}_{\geq 0} \supseteq)\ \mathbb{R}_{>0} \cap \int_M(E) \neq \emptyset$. Thus, if, for instance, $S \in \mathrm{Sub}(M)$ is *ample* and *compact*, then the pair $(S, \mu_M|_S)$ obtained by restricting μ_M to S is an *ample measure space* that determines *compatible natural inclusions* $\mathrm{Sub}(S) \hookrightarrow \mathrm{Sub}(M)$, $\mathrm{Fn}(S) \hookrightarrow \mathrm{Fn}(M)$ [the latter of which is defined by *extension by zero*] — which we shall use to regard $\mathrm{Sub}(S)$, $\mathrm{Fn}(S)$ as subsets of $\mathrm{Sub}(M)$, $\mathrm{Fn}(M)$, respectively — such that the subsets $\mathrm{Sub}(S), \mathrm{Fn}(S) \subseteq \mathrm{Fn}(M)$ are *ample*. If $E \subseteq \mathrm{Fn}(M)$ is *ample*, then the image of E in $\mathbb{R}\mathrm{ss}(M)$ determines a *natural subset* $\mathbb{R}\mathrm{ss}(E) \subseteq \mathbb{R}\mathrm{ss}(M)$, whose $\mathbb{R}_{\geq 0}$-orbit $\mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(E)$ is equal to $\mathbb{R}\mathrm{ss}(M)$. In particular, if $S \in \mathrm{Sub}(M)$ is *ample* and *compact*, then we obtain *natural* $\mathbb{R}_{>0}$*-equivariant isomorphisms of monoids*

$$\mathbb{R}\mathrm{ss}(S) \xrightarrow{\sim} \mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(\mathrm{Fn}(S)) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(M)$$

— where, the notation "$\mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(\mathrm{Fn}(S))$" is intended relative to the interpretation of $\mathrm{Fn}(S)$ as a subset of $\mathrm{Fn}(M)$ — such that the *composite isomorphism* $\mathbb{R}\mathrm{ss}(S) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(M)$ is *compatible* with the isomorphisms $\int_S^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(S) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$, $\int_M^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$. Finally, we observe that if (M_1, μ_{M_1}) and (M_2, μ_{M_2}) are *ample measure spaces*, then the *product measure space* $(M_1 \times M_2, \mu_{M_1 \times M_2})$ is also an ample measure space; moreover, there is a *natural map*

$$\mathrm{Sub}(M_1) \times \mathrm{Sub}(M_2) \rightarrow \mathrm{Sub}(M_1 \times M_2)$$

that maps $(S_1, S_2) \mapsto S_1 \times S_2$ and induces a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids*

$$\mathbb{R}\mathrm{ss}(M_1) \otimes \mathbb{R}\mathrm{ss}(M_2) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(M_1 \times M_2)$$

that is *compatible* with the isomorphisms $\int_{M_1}^{\mathbb{R}\mathrm{ss}} \otimes \int_{M_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M_1) \otimes \mathbb{R}\mathrm{ss}(M_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$, $\int_{M_1 \times M_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M_1 \times M_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$. [Here, we observe that there is a natural notion of *"tensor product of monoids isomorphic to* $\mathbb{R}_{\geq 0}$*"* since such a monoid may be thought of, by passing to the groupification of such a monoid, as a *one-dimensional* $\mathbb{R}$*-vector space equipped with a subset [which forms a* $\mathbb{R}_{>0}$*-torsor] of "positive elements"*.]

(ii) One *very rough* approach to understanding the $\mathfrak{log}$*-link compatibility* of *log-volumes* is the following. Suppose that instead of knowing this property, one only knows that

> each application of the $\mathfrak{log}$-link has the effect of **dilating volumes** by a factor $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$.

[Here, relative to the notation of (i), we observe that this sort of situation in which *volumes* are *dilated* in a nontrivial fashion may be seen in the following example:

> Suppose that $M \stackrel{\mathrm{def}}{=} \mathbb{Q}_p$, for some prime number p, equipped with the [additive] Haar measure $\mu_{\mathbb{Q}_p}$ normalized so that $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ has measure 1, so $(\mathbb{Q}_p, \mu_{\mathbb{Q}_p})$ is an *ample measure space* in the sense of (i). Then *multiplication by* p induces a bijection $\alpha_p : \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p$. Moreover, α_p induces *compatible bijections* $\mathrm{Sub}(\alpha_p) : \mathrm{Sub}(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Sub}(\mathbb{Q}_p)$, $\mathrm{Fn}(\alpha_p) : \mathrm{Fn}(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Fn}(\mathbb{Q}_p)$, $\mathbb{R}\mathrm{ss}(\alpha_p) : \mathbb{R}\mathrm{ss}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(\mathbb{Q}_p)$. On the other hand, [*unlike* the situation discussed in (i) concerning the "composite isomorphism $\mathbb{R}\mathrm{ss}(S) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(M)$"!] in the present context, $\mathbb{R}\mathrm{ss}(\alpha_p)$ is *not compatible* with the isomorphisms $\int_{\mathbb{Q}_p}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ in the domain and codomain of $\mathbb{R}\mathrm{ss}(\alpha_p)$, i.e., it is only compatible *up to a factor* p^{-1} $(\neq 1)$!]

Then in order to *compute log-volumes* in a fashion that is *consistent* with the various arrows [i.e., both Kummer isomorphisms and $\mathfrak{log}$-links!] of the "systems" constituted by the **$\mathfrak{log}$-Kummer correspondences** discussed in Proposition 3.5, (ii), it would be necessary to regard the various "log-volumes" computed as only giving rise to well-defined elements [not $\in \mathbb{R}$, but rather]

$$\in \quad \mathbb{R}/\mathbb{Z} \cdot \log(\lambda) \quad (\cong \ \mathbb{S}^1)$$

— a situation which is *not acceptable*, relative to the goal of obtaining *log-volume estimates* [i.e., as in Corollary 3.12 below] for the various objects for which log-shells serve as *"multiradial containers"* [cf. the discussion of Remark 1.5.2; the content of Theorem 3.11 below].

(iii) In the following discussion, we use the notation of Remark 1.2.2, (i). Thus, we regard k as being equipped with the [additive] Haar measure μ_k normalized so that $\mu_k(\mathcal{O}_k) = 1$ [cf. [AbsTopIII], Proposition 5.7, (i)]. Then (k, μ_k) is an *ample measure space* in the sense of (i); $\mathcal{O}_k^\times \subseteq k$ is an *ample subset*; for any *compact ample subset* $S \subseteq \mathcal{O}_k^\times$ on which $\log_k : \mathcal{O}_k^\times \to k$ is *injective*, we have $\mu_k(S) = \mu_k(\log_k(S))$ [cf. [AbsTopIII], Proposition 5.7, (i), (c)]. In particular, by applying the *formalism* of **realified semi-simplifications** introduced in (i), we conclude that the diagram

$$\begin{array}{ccccccc} k & \supseteq & \mathcal{O}_k^\times & \stackrel{\log_k}{\longrightarrow} & \log_k(\mathcal{O}_k^\times) & \subseteq & k \\ & & \cup & & \cup & & \\ & & S & \xrightarrow{\sim} & \log_k(S) & & \end{array}$$

induces a *commutative diagram*

$$\begin{array}{ccccccccccc}
\mathbb{R}\mathrm{ss}(k) & \overset{\sim}{\leftarrow} & \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) & \overset{\sim}{\leftarrow} & \mathbb{R}\mathrm{ss}(S) & \overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(\log_k(S)) & \overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times)) & \overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(k) \\
\big\downarrow \int_k^{\mathbb{R}\mathrm{ss}} & & \big\downarrow \int_{\mathcal{O}_k^\times}^{\mathbb{R}\mathrm{ss}} & & \big\downarrow \int_S^{\mathbb{R}\mathrm{ss}} & & \big\downarrow \int_{\log_k(S)}^{\mathbb{R}\mathrm{ss}} & & \big\downarrow \int_{\log_k(\mathcal{O}_k^\times)}^{\mathbb{R}\mathrm{ss}} & & \big\downarrow \int_k^{\mathbb{R}\mathrm{ss}} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}$$

— where the vertical arrows are $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*, and the *composite [*$\mathbb{R}_{>0}$*-equivariant] isomorphism [of monoids]* $\mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times))$ is easily verified to be *independent* of the choice of the *compact ample subset* $S \subseteq \mathcal{O}_k^\times$. [Also, we observe that it is easily verified that there *exist* compact ample subsets $S \subseteq \mathcal{O}_k^\times$ for which the induced map $S \twoheadrightarrow \log_k(S)$ is *injective*.] One may then compose this diagram with the bijection

$$\log : \mathbb{R}_{\geq 0} \overset{\sim}{\rightarrow} \mathbb{R} \ \cup \ \{-\infty\}$$

determined by the *natural logarithm* and then multiply by a *suitable normalization factor* $\in \mathbb{R}_{>0}$ to conclude that

> *the diagram*
> $$k \ \supseteq \ \mathcal{O}_k^\times \ \overset{\log_k}{\longrightarrow} \ \log_k(\mathcal{O}_k^\times) \ \subseteq \ k$$
> *induces* $\mathbb{R}_{>0}$**-equivariant isomorphisms of monoids** *on the respective* **realified semi-simplifications** *"*$\mathbb{R}\mathrm{ss}(-)$*", all of which are* **compatible** *with the* **log-volume** *maps on each of the "*$\mathbb{R}\mathrm{ss}(-)$*'s", i.e., which restrict to the "usual log-volume maps" on the respective "*$\mathrm{Sub}(-)$*'s", relative to the natural maps "*$\mathrm{Sub}(-) \rightarrow \mathbb{R}\mathrm{ss}(-)$*".*

This is one way to formulate the **log-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Finally, we observe that this log-*link compatibility* with *log-volumes* is itself *compatible* with passing to **finite extensions** of k [or, more generally, $\mathbb{Q}_{p_{\underline{v}}}$], as follows. Let $k_1 \subseteq k_2$ be finite field extensions of $\mathbb{Q}_{p_{\underline{v}}}$. We shall use analogous notation for objects associated to k_1 and k_2 to the notation that was used above for objects associated to k. Then observe that since $\mathcal{O}_{k_2}$ is a *finite free* $\mathcal{O}_{k_1}$*-module of rank* $[k_2 : k_1]$, it follows that the [additive] *compact topological group* $\mathcal{O}_{k_2}$ is isomorphic to the product of $[k_2 : k_1]$ copies of the [additive] compact topological group $\mathcal{O}_{k_1}$. In particular, since the Haar measure of a compact topological group is *invariant* with respect to arbitrary automorphisms of the topological group, we thus conclude [cf. the discussion of *product measure spaces* in (i)] that the *inclusion of topological fields* $k_1 \hookrightarrow k_2$ induces *natural* $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*

$$\mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \ \overset{\sim}{\leftarrow} \ \mathbb{R}\mathrm{ss}(\mathcal{O}_{k_1})^{\otimes[k_2:k_1]} \ \overset{\sim}{\rightarrow} \ \mathbb{R}\mathrm{ss}(\mathcal{O}_{k_2}) \ \overset{\sim}{\rightarrow} \ \mathbb{R}\mathrm{ss}(k_2)$$

such that the *composite isomorphism* $\mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(k_2)$ is *compatible* with the $\mathbb{R}_{>0}$-equivariant isomorphisms of monoids

$$\Big(\int_{k_1}^{\mathbb{R}\mathrm{ss}}\Big)^{\otimes[k_2:k_1]} : \mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0}$$

and $\int_{k_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(k_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$.

(iv) In the notation of (iii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; write $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} \underline{\underline{q}}_{\underline{v}}$. Thus, we have a *submonoid*

$$\mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}} \subseteq k$$

of the underlying multiplicative monoid of k. Then the various arrows of the **$\mathfrak{log}$-Kummer correspondence** discussed in Proposition 3.5, (ii), may be thought of, from the point of view of a *vertically coric étale holomorphic* copy of "k" [i.e., a copy *labeled "$n, \circ$"*, as in Proposition 3.5, (i)], as corresponding to the *operations*

$$\begin{aligned} k \ \rightsquigarrow\ & \mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}} \ (\subseteq k) \\ \rightsquigarrow\ & \mathcal{O}_k^{\times} \ (\subseteq k) \\ \rightsquigarrow\ & \log_k(\mathcal{O}_k^{\times}) \ (\subseteq k) \ \rightsquigarrow\ k \end{aligned}$$

— i.e., of passing first from k to the multiplicative submonoid $\mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}}$, then to the multiplicative submonoid $\mathcal{O}_k^{\times}$, then applying $\log_k$ to obtain an additive submonoid of k, and finally passing from this submonoid back to k itself. Then the **log-volume compatibility** discussed in (iii) may be understood, in the context of the **$\mathfrak{log}$-Kummer correspondence**, as the statement that

> the *operations* of the above display induce $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*
>
> $$\mathbb{R}\mathrm{ss}(k) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(\mathcal{O}_k^{\times}) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^{\times})) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(k)$$
>
> that are *compatible* with the respective [normalized] *log-volume maps* to $\mathbb{R} \cup \{-\infty\}$ [cf. the discussion of (iii)], in such a way as to *avoid any interference*, up to multiplication by roots of unity, with the submonoid $\underline{\underline{q}}^{\mathbb{N}} \subseteq k$, which induces, by applying the [normalized] *log-volume* to the image of $\mathcal{O}_k \subseteq k$ via multiplication by elements of this submonoid, an embedding
>
> $$\mathbb{N} \hookrightarrow \mathbb{R}\mathrm{ss}(k) \xrightarrow{\sim} \mathbb{R} \cup \{-\infty\}$$
>
> that maps $\mathbb{N} \ni 1 \mapsto -\log(\underline{\underline{q}}) \in \mathbb{R}$

[where we write $\log(\underline{\underline{q}}) \stackrel{\mathrm{def}}{=} \mathrm{ord}_{\underline{v}}(\underline{\underline{q}}_{\underline{v}}) \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ — cf. the notation of Remark 2.4.2, (ii)]. A similar interpretation of *log-volume compatibility* in the context of the $\mathfrak{log}$-*Kummer correspondence* may be given in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ by simply omitting the portion of the above discussion concerning "$\underline{\underline{q}}$".

(v) In the notation of Remark 3.9.1, (i), we observe that the discussion of (iii), (iv), may be extended to *topological tensor products* of the form

$$k_{i_A} \stackrel{\mathrm{def}}{=} \bigotimes_{\alpha \in A} k_{i_\alpha}$$

— where $i_\alpha \in \{1, \ldots, n_{v_{\mathbb{Q}}}\}$, for each $\alpha \in A$, and we regard k_{i_A} as being equipped with the [additive] Haar measure normalized [cf. Proposition 3.9, (i)] so that the

ring of integers $\mathcal{O}_{k_{i_A}} \subseteq k_{i_A}$ [i.e., the integral structure discussed in Proposition 3.1, (ii)] has Haar measure $= 1$. Indeed, each of the *direct summand fields* of k_{i_A} [cf. Proposition 3.1, (i)] may be taken to be a [finite extension of a] "k" as in (iii), (iv). In particular, the measure space k_{i_A} may be regarded as a *product measure space* of [finite extensions of] "k" as in (iii), (iv), so one may extend (iii), (iv) to k_{i_A} by applying (iii), (iv) to *each factor* of this product measure space [cf. the discussion of *product measure spaces* in (i)]. [We leave the routine details to the reader.] On the other hand, in this context, it is also of interest to observe that it follows immediately from the discussion of *compatibility* with *finite extensions* in (iii), together with the discussion of *product measure spaces* in (i), that, for each $\alpha \in A$, the natural structure of k_{i_A} as a k_{i_α}-*algebra* determines *natural* $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*

$$\mathbb{R}\text{ss}(\mathbb{Q}_{p_{\underline{v}}})^{\otimes d_A} \ \stackrel{\sim}{\to}\ \mathbb{R}\text{ss}(k_{i_\alpha})^{\otimes d_\alpha} \ \stackrel{\sim}{\to}\ \mathbb{R}\text{ss}(k_{i_A})$$

— where we write $d_\alpha \stackrel{\text{def}}{=} \prod_{\beta \in A \setminus \{\alpha\}} [k_{i_\beta} : \mathbb{Q}_{p_{\underline{v}}}]$, $d_A \stackrel{\text{def}}{=} d_\alpha \cdot [k_{i_\alpha} : \mathbb{Q}_{p_{\underline{v}}}]$. In particular,

> the $\mathfrak{log}$**-link compatibility** of **log-volumes** [as discussed above] for the *realified semi-simplification*
>
> $$\mathbb{R}\text{ss}(k_{i_A})$$
>
> of the *topological tensor product* k_{i_A} may be understood, for any $\alpha \in A$, as the [functorially induced!] d_α*-th tensor power* of the $\mathfrak{log}$-*link compatibility* of *log-volumes* for the *realified semi-simplification*
>
> $$\mathbb{R}\text{ss}(k_{i_\alpha})$$
>
> of k_{i_α} or, alternatively/equivalently, as the [functorially induced!] d_A*-th tensor power* of the $\mathfrak{log}$-*link compatibility* of *log-volumes* for the *realified semi-simplification*
>
> $$\mathbb{R}\text{ss}(\mathbb{Q}_{p_{\underline{v}}})$$
>
> of $\mathbb{Q}_{p_{\underline{v}}}$

— where we note that the latter "alternative/equivalent" approach has the virtue of being *independent* of the choice of $\alpha \in A$.

(vi) In the following discussion, we use the notation of Remark 1.2.2, (ii). We regard the *complex archimedean field* k as being equipped with the *standard Euclidean metric* [cf. the discussion of *"metrics"* in Remark 1.2.1, (ii)], with respect to which $\mathcal{O}_k^\times \subseteq k$ has length 2π. This metric on k thus determines *measures* $\mu_{|k|}$ on $|k| \stackrel{\text{def}}{=} k/\mathcal{O}_k^\times$ and $\mu_{\mathcal{O}_k^\times}$ on $\mathcal{O}_k^\times \subseteq k$ [cf. the situation discussed in [AbsTopIII], Proposition 5.7, (ii)] such that $(|k|, \mu_{|k|})$ and $(\mathcal{O}_k^\times, \mu_{\mathcal{O}_k^\times})$ are *ample measure spaces* in the sense of (i). Moreover, by

(a) thinking of $\mathcal{O}_k^\times$ as a *union* of *closed arcs* [i.e., whose *interiors* are *disjoint*] of measure $\mu_{\mathcal{O}_k^\times}(-) < \epsilon$, for some positive real number ϵ,

(b) considering *additive translates* of such *closed arcs* that map one of the endpoints of the arc to $0 \in k$,

(c) *projecting such additive translates* via the natural surjection $k \twoheadrightarrow |k|$, and
(d) passing to the *limit* $\epsilon \to 0$,

one verifies immediately that we obtain, by applying the *formalism* of **realified semi-simplifications** introduced in (i), a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids* $\rho_k : \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(|k|)$, together with a *commutative diagram*

$$\begin{array}{ccccccc}
\mathbb{R}\mathrm{ss}(|k|) & \xleftarrow{\sim} & \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) & \xleftarrow{\sim} & \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times)) & \xrightarrow{\sim} & \mathbb{R}\mathrm{ss}(|k|) \\
\downarrow \int^{\mathbb{R}\mathrm{ss}}_{|k|} & & \downarrow \int^{\mathbb{R}\mathrm{ss}}_{\mathcal{O}_k^\times} & & \downarrow \int^{\mathbb{R}\mathrm{ss}}_{\log_k(\mathcal{O}_k^\times)} & & \downarrow \int^{\mathbb{R}\mathrm{ss}}_{|k|} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}$$

— where the vertical arrows are $\mathbb{R}_{>0}$*-equivariant isomorphisms of monoids*; the first "$\xleftarrow{\sim}$" is ρ_k; we regard $\log_k(\mathcal{O}_k^\times) \stackrel{\mathrm{def}}{=} \exp_k^{-1}(\mathcal{O}_k^\times)$ as being equipped with the *measure* $\mu_{\log_k(\mathcal{O}_k^\times)}$ [such that $(\log_k(\mathcal{O}_k^\times), \mu_{\log_k(\mathcal{O}_k^\times)})$ is an *ample measure space*] obtained by pulling back $\mu_{|k|}$ via the *homeomorphism* $\log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k|$ induced by restricting the natural surjection $k \twoheadrightarrow |k|$ to $\log_k(\mathcal{O}_k^\times) \subseteq k$; the second "$\xleftarrow{\sim}$" is the natural $\mathbb{R}_{>0}$-equivariant isomorphism of monoids naturally induced [i.e., by considering *ample* $S \subseteq \log_k(\mathcal{O}_k^\times)$ that map *bijectively* to $\exp_k(S) \subseteq \mathcal{O}_k^\times$ — cf. [AbsTopIII], Proposition 5.7, (ii), (c)] by the *universal covering map* $\exp_k|_{\log_k(\mathcal{O}_k^\times)} : \log_k(\mathcal{O}_k^\times) \to \mathcal{O}_k^\times$; the "$\xrightarrow{\sim}$" is the natural $\mathbb{R}_{>0}$-equivariant isomorphism of monoids induced by the *homeomorphism* $\log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k|$. One may then compose this diagram with the bijection

$$\log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R} \cup \{-\infty\}$$

determined by the *natural logarithm* and then multiply by a *suitable normalization factor* $\in \mathbb{R}_{>0}$ to conclude that

> *the diagram*
>
> $$|k| \twoheadleftarrow k \supseteq \mathcal{O}_k^\times \xleftarrow{\exp_k} \log_k(\mathcal{O}_k^\times) \subseteq k \twoheadrightarrow |k|$$
>
> *induces* $\mathbb{R}_{>0}$**-equivariant isomorphisms of monoids** *on the respective* **realified semi-simplifications** *"*$\mathbb{R}\mathrm{ss}(-)$*" of* $|k|$, $\mathcal{O}_k^\times$, $\log_k(\mathcal{O}_k^\times)$, *and* $|k|$*; each of these isomorphisms is* **compatible** *with the* **log-volume** *map on "*$\mathbb{R}\mathrm{ss}(-)$*", i.e., which restricts to the "usual radial/angular log-volume map" on "*$\mathrm{Sub}(-)$*" [that is to say, the map uniquely determined by the radial/angular log-volume map of [AbsTopIII], Proposition 5.7, (ii), (a)] relative to the natural map "*$\mathrm{Sub}(-) \to \mathbb{R}\mathrm{ss}(-)$*".*

This is one way to formulate the **log-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. One verifies immediately that one also has analogues for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of (iv), (v). [We leave the routine details to the reader.]

Remark 3.9.5. In situations that involve consideration of *various sorts of regions* [cf. the discussion of Remarks 3.1.1, (iii), (iv); 3.9.4] to which the *log-volume* may be applied, it is often of use to consider the notion of the *holomorphic hull* of a region.

(i) Suppose that we are in the situation of Proposition 3.9, (i). Let

$$^{\alpha}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \quad (\text{respectively, } {}^{A}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad {}^{A,\alpha}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{A,\alpha}\mathcal{F}_{\underline{v}}))$$

be a subset that contains a relatively compact subset whose *log-volume* [cf. the discussion of Remark 3.1.1, (iii), (iv), as well as Remark 3.9.7, (ii), below] is *finite* [i.e., $> -\infty$]. If ${}^{\alpha}\mathcal{U}$ (respectively, ${}^{A}\mathcal{U}$; ${}^{A,\alpha}\mathcal{U}$) is *relatively compact*, then we define the **holomorphic hull** of ${}^{\alpha}\mathcal{U}$ (respectively, ${}^{A}\mathcal{U}$; ${}^{A,\alpha}\mathcal{U}$) to be the smallest subset of the form

$$^{\alpha}\mathcal{H} \stackrel{\text{def}}{=} \lambda \cdot \mathcal{O}_{^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \quad (\text{respectively, } {}^{A}\mathcal{H} \stackrel{\text{def}}{=} \lambda \cdot \mathcal{O}_{^{A}\mathcal{F}_{v_{\mathbb{Q}}}}; \quad {}^{A,\alpha}\mathcal{H} \stackrel{\text{def}}{=} \lambda \cdot \mathcal{O}_{^{A,\alpha}\mathcal{F}_{\underline{v}}})$$

— where, relative to the direct sum decomposition of $\mathcal{I}^{\mathbb{Q}}((-))$ as a direct sum of fields [cf. the discussion of Proposition 3.9, (i)], $\lambda \in \mathcal{I}^{\mathbb{Q}}((-))$ is an element such that each *component* of λ [i.e., relative to this direct sum decomposition] is *nonzero* — that *contains* ${}^{\alpha}\mathcal{U}$ (respectively, ${}^{A}\mathcal{U}$; ${}^{A,\alpha}\mathcal{U}$). If ${}^{\alpha}\mathcal{U}$ (respectively, ${}^{A}\mathcal{U}$; ${}^{A,\alpha}\mathcal{U}$) is *not relatively compact*, then we define the **holomorphic hull** of ${}^{\alpha}\mathcal{U}$ (respectively, ${}^{A}\mathcal{U}$; ${}^{A,\alpha}\mathcal{U}$) to be $\mathcal{I}^{\mathbb{Q}}(^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$ (respectively, $\mathcal{I}^{\mathbb{Q}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$; $\mathcal{I}^{\mathbb{Q}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$). One verifies immediately that the holomorphic hull is well-defined [under the conditions stated].

(ii) In the remainder of the discussion of the present Remark 3.9.5, for the sake of simplicity, we shall refer to "holomorphic hulls" as *"hulls"*. Write

$$\mathbb{P} \stackrel{\text{def}}{=} \{P \subseteq \mathcal{I}^{\mathbb{Q}}((-)) \mid P \text{ is a } \textit{direct product region} \text{ [cf. Remark 3.1.1, (iii)]}\};$$
$$\mathbb{H} \stackrel{\text{def}}{=} \{H \subsetneq \mathcal{I}^{\mathbb{Q}}((-)) \mid H \text{ is a } \textit{hull} \text{ [cf. (i)]}\}$$

— where the argument "$(-)$" is "${}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}$", "${}^{A}\mathcal{F}_{v_{\mathbb{Q}}}$", or "${}^{A,\alpha}\mathcal{F}_{\underline{v}}$" [cf. (i)], and we observe that $\mathbb{H} \subseteq \mathbb{P}$. Then it is essentially a *tautology* that the operation of *forming the* **hull** discussed in (i)

$$^{\square}\mathcal{U} \mapsto {}^{\square}\mathcal{H}$$

— where "$\square$" is "α", "A", or "A, α" — determines a map

$$\phi : \mathbb{P} \to \mathbb{H}$$

that may be **characterized uniquely** by the following properties

(P1) $\phi(H) = H$, for any $H \in \mathbb{H}$;
(P2) $P \subseteq \phi(P)$, for any $P \in \mathbb{P}$;
(P3) $\phi(P_1) \subseteq \phi(P_2)$, for any $P_1, P_2 \in \mathbb{P}$ such that $P_1 \subseteq P_2$.

Indeed, since, as is easily verified, any intersection of elements of $\mathbb{H}$ which is of *finite log-volume* necessarily determines an element of $\mathbb{H}$, it follows formally from (P1), (P2), (P3) that

$$\phi(P) = \bigcap_{\mathbb{H} \ni H \supseteq P} H$$

for any $P \in \mathbb{P}$. Put another way, this map ϕ may be thought of as a sort of **adjoint**, or **push forward** in the opposite direction, of the inclusion $\mathbb{H} \subseteq \mathbb{P}$. Alternatively, ϕ may be thought of as a sort of **canonical splitting** of the inclusion $\mathbb{H} \subseteq \mathbb{P}$, or, in the spirit of the discussion of Remark 3.9.4, as a sort of **integration** operation. The *compatibility* [cf. (P2), (P3)] of ϕ with the *pre-order structure* on $\mathbb{P}$ determined

by inclusion of direct product regions will play an *important role* in the context of various *log-volume estimates* of regions.

(iii) Now we consider the various *log-volumes* $\mu^{\log}_{(-)}$ [where the argument "$(-)$" is "$\alpha, v_{\mathbb{Q}}$", "$A, v_{\mathbb{Q}}$", or "$A, \alpha, \underline{v}$" — cf. (ii)] of Proposition 3.9, (i) [cf. also Remark 3.1.1, (iii)], in the context of the discussion of (ii). In the following, for the sake of simplicity, we shall denote "$\mu^{\log}_{(-)}$" by $\mu^{\log}$. For $P \in \mathbb{P}$, write

$$\Phi(P) \stackrel{\text{def}}{=} \{H \in \mathbb{H} \mid \phi(P) \supseteq H, \ (\mu^{\log}(\phi(P)) \geq) \ \mu^{\log}(H) \geq \mu^{\log}(P)\} \ \subseteq \ \mathbb{H};$$

$$\Xi(P) \stackrel{\text{def}}{=} \{H \in \mathbb{H} \mid \phi(P) \supseteq H, \ (\mu^{\log}(\phi(P)) \geq) \ \mu^{\log}(H) = \mu^{\log}(P)\} \ \subseteq \ \Phi(P);$$

$$H_{\Phi(P)} \stackrel{\text{def}}{=} \bigcup_{H \in \Phi(P)} H \ \subseteq \ \phi(P); \quad H_{\Xi(P)} \stackrel{\text{def}}{=} \bigcup_{H \in \Xi(P)} H \ \subseteq \ H_{\Phi(P)} \ \subseteq \ \phi(P).$$

Thus, one may think of elements $\in \Phi(P)$ or $\in \Xi(P)$ as

"log-volume approximations" of P by means of *hulls* $\in \mathbb{H}$.

If one thinks of distinct elements $\in \Phi(P)$ or $\in \Xi(P)$ — i.e., of the issue of constructing a **"log-volume hull-approximant"** of P — as a sort of *indeterminacy* [i.e., in the assignment to P of a *specific element* $\in \mathbb{H}$!], then

> this **indeterminacy** is **compact**, i.e., in the sense that all possible choices of an element $\in \Phi(P)$ or $\in \Xi(P)$ are contained in the *compact set* $\phi(P) \in \mathbb{H}$.

Indeed, developing the theory in such a way that

all the **indeterminacies** that occur in the theory are **compact**

is in some sense one important theme in the present series of papers. Note that this *compactness* would *not be valid* if, in the definition of $\Phi(-)$ or $\Xi(-)$, one *omits* the condition "$H \subseteq \phi(P)$".

(iv) In the context of (iii), we observe that

$$\phi(P) \in \Phi(P), \text{ so } \phi(P) = H_{\Phi(P)},$$

but the issue of whether or not $\Xi(P) = \emptyset$ is *not so immediate*. Indeed:

(Ξ1) If *either* of the following conditions is satisfied, then it is easily verified that $\Xi(P) \neq \emptyset$:

- (Ξ^{non}1) if we write K^{cl} for the *Galois closure* of K over $\mathbb{Q}$, then the *residue field extension degree* of each valuation $\in \mathbb{V}(K^{\text{cl}})$ that divides $v_{\mathbb{Q}} \in \mathbb{V}^{\text{non}}_{\mathbb{Q}}$ is $= 1$, and, moreover, $\mu^{\log}(P) = \mu^{\log}(Q)$, for some $Q \in \mathbb{P}$ which is a $\mathbb{Z}_{p_{v_{\mathbb{Q}}}}$*-submodule* of $\mathcal{I}^{\mathbb{Q}}((-))$;
- (Ξ^{arc}1) $v_{\mathbb{Q}} \in \mathbb{V}^{\text{arc}}_{\mathbb{Q}}$.

(Ξ2) If one allows the $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ in the present discussion to *vary*, and one considers *global situations* [i.e., which necessarily involve the *unique valuation* $\in \mathbb{V}^{\text{arc}}_{\mathbb{Q}}$!] as in Proposition 3.9, (iii), then it is easily verified that the *global analogue* of "$\Xi(P)$" is *nonempty*.

On the other hand, *in general, it is not so clear whether or not* $\Xi(P) \neq \emptyset$. In this context, it is also of interest to observe that if $P \in \mathbb{H}$, then

$$\{\phi(P)\} = \Phi(P) = \Xi(P), \text{ so } P = \phi(P) = H_{\Phi(P)} = H_{\Xi(P)},$$

but in general, even in the situation of (Ξ1), the *inclusion* $P \subseteq \phi(P) = H_{\Phi(P)}$, as well as the induced *inequality of log-volumes* $\mu^{\log}(P) \leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))$, is *strict*. Indeed, for instance,

(Ξ3) in the situation of (Ξ1), if $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ [cf. the notational conventions of Remark 3.1.1, (iii)] and A are of *cardinality* ≥ 2, then one verifies easily that there exist $P \in \mathbb{P}$ for which $\mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)})$ ($\leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))$).

This sort of phenomenon may be seen in the following *example*:

Let p be a prime number. Write $I \stackrel{\mathrm{def}}{=} \mathbb{Q}_p \times \mathbb{Q}_p$;

$$H_0 \stackrel{\mathrm{def}}{=} \mathbb{Z}_p \times \mathbb{Z}_p \ \subseteq \ I; \quad H_1 \stackrel{\mathrm{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times (p \cdot \mathbb{Z}_p) \ \subseteq \ I;$$
$$P \stackrel{\mathrm{def}}{=} H_0 \ \cup \ (\{p^{-1}\} \times \mathbb{Z}_p) \ \subseteq \ I; \quad H_P \stackrel{\mathrm{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times \mathbb{Z}_p \ \subseteq \ I$$

— where we think of I as being equipped with the *Haar measure* μ_I normalized so that $\mu_I(H_0) = 1$. Thus, p corresponds to "$p_{v_{\mathbb{Q}}}$" such that "$(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$" is of *cardinality* 2; I corresponds to "$\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$"; P corresponds to "P"; H_P corresponds to "$\phi(P)$"; H_0 and H_1 correspond to elements of "$\Xi(P)$", so $H_0 \cup H_1$ corresponds to a subset of "$H_{\Xi(P)}$", hence also a subset of "$H_{\Phi(P)} = \phi(P)$". Then

$$\mu_I(P) \ = \ \mu_I(H_0) \ = \ 1 < \ 2 - p^{-1} \ = \ \mu_I(H_0 \cup H_1)$$

— i.e., the *inequality of [log-]volumes* in question is *strict*. In fact, by considering various *translates* of H_0, H_1 by *automorphisms* of the $\mathbb{Z}_p$-module H_P, one verifies immediately that H_P corresponds not only to "$\phi(P) = H_{\Phi(P)}$", but also to "$H_{\Xi(P)}$". That is to say, this is a situation in which one has "$H_{\Xi(P)} = H_{\Phi(P)} = \phi(P)$", hence also "$\mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)}) = \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))$".

(v) Let E be a set, $S \subseteq E$ a *proper subset* of E of *cardinality* ≥ 2 [so $S \neq \emptyset \neq E \setminus S$]. Write

$$(E \twoheadrightarrow) \ E \,\bar{\wedge}\, S \ \stackrel{\mathrm{def}}{=} \ (E \setminus S) \ \coprod \ \{S\}$$

[i.e., "*E upper S*"] for the *set-theoretic quotient* of E by S, i.e., the quotient of E obtained by identifying the elements of S and leaving $E \setminus S$ unaffected. Write $\bar{\wedge}_S \stackrel{\mathrm{def}}{=} \{S\} \subseteq E \,\bar{\wedge}\, S$. Then observe that

any set-theoretic map

$$(E \supseteq) \ S_1 \ \to \ S_2 \ (\subseteq E)$$

between *nonempty subsets* $S_1, S_2 \subseteq S$ ($\subseteq E$) induces, upon passing to the *quotient* $E \twoheadrightarrow E \,\bar{\wedge}\, S$, the *identity map*

$$(E \,\bar{\wedge}\, S \supseteq) \ \bar{\wedge}_S \ \to \ \bar{\wedge}_S \ (\subseteq E \,\bar{\wedge}\, S)$$

between the *images* [i.e., both of which are equal to $\barwedge_S$!] of S_1, S_2 in $E \barwedge S$, hence *lies over* the *identity map* $E \barwedge S \to E \barwedge S$ on $E \barwedge S$.

Moreover, this map may be *"extended"* to the case where S_i [for $i \in \{1, 2\}$] is *empty* if this S_i is treated as a *"formal intersection"* [cf. our hypothesis that the *cardinality* of S is ≥ 2] — i.e., a *"category-theoretic formal fiber product, or inverse system, over E"* — of some collection of *nonempty subsets* of S. That is to say, such an inverse system induces, upon passing to the *quotient* $E \twoheadrightarrow E \barwedge S$, a system that consists of identity maps between copies of $\barwedge_S$. In particular,

> if one thinks in terms of such *formal inverse systems*, then *"formal empty sets"* $\subseteq S$ $(\subseteq E)$ also map to $\barwedge_S \subseteq E \barwedge S$.

Finally, we observe that the above discussion may be thought of as an

> **abstract set-theoretic formalization** of the notions of **upper semi-commutativity/semi-compatibility**, as discussed in Remark 1.2.2, (iii); Remark 1.5.4, (iii); Proposition 3.5, (ii)

— i.e., where [cf. the notational conventions of Propositions 3.2, (ii); 3.5, (ii)] one takes the $S \subseteq E$ of the present discussion to be "$\mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$", and we observe that, in the context of upper semi-commutativity/semi-compatibility, the *empty set* always arises as an intersection between a nonempty set and the *domain of definition* [cf. the discussion of Remark 1.1.1] of the *"set-theoretic logarithm map"* under consideration.

(vi) Let us return to the discussion of (ii), (iii), (iv). Let $P \in \mathbb{P}$. Then let us observe that

> the *abstract set-theoretic "$\barwedge$-formalism"* of (v) — i.e., where one takes "$S \subseteq E$" to be $\phi(P) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ — yields a *convenient tool* for **identifying** P with its various **log-volume hull-approximants** $\in \Phi(P)$ or $\in \Xi(P)$ [all of which are nonempty subsets of $\phi(P) \in \mathbb{H}$ — cf. the discussion of (iii)], i.e., of passing to a **quotient** in which the **indeterminacy** discussed in (iii) is **eliminated**.

Moreover, one verifies easily that this *identification* is achieved in such a way that *images* of *distinct* $H_1, H_2 \in \mathbb{H}$ map to the *same subset* of $\mathcal{I}^{\mathbb{Q}}((-)) \barwedge \phi(P)$ if and only if $H_1, H_2 \subseteq \phi(P)$. That is to say, the *equivalence relation* on $\mathbb{H}$ induced by the quotient map $\mathcal{I}^{\mathbb{Q}}((-)) \twoheadrightarrow \mathcal{I}^{\mathbb{Q}}((-)) \barwedge \phi(P)$ is the *"expected equivalence relation"*

$$\mathbb{H} \ni H_1 \sim H_2 \in \mathbb{H} \qquad \iff \qquad H_1, H_2 \subseteq \phi(P)$$

on $\mathbb{H}$. Finally, we observe that the discussion of the present (vi) may be applied not only to *single elements* $P \in \mathbb{P}$, but also to *bounded families of elements* $P_B \stackrel{\text{def}}{=} \{P_\beta\}_{\beta \in B}$ indexed by some index set B [i.e., collections of elements $P_\beta \in \mathbb{P}$ such that $\cup_{\beta \in B} P_\beta \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ is *relatively compact*], by taking "$S \subseteq E$" in the discussion of (v) to be

$$\phi(P_B) \stackrel{\text{def}}{=} \bigcap_{\mathbb{H} \ni H \supseteq P_\beta, \ \forall \beta \in B} H \quad \subseteq \quad \mathcal{I}^{\mathbb{Q}}((-))$$

[cf. the representation of $\phi(P)$ as an intersection in (ii)].

(vii) The operation of forming the **hull** will play a *crucial role* in the context of Corollary 3.12 below, for the following reason:

> the *output of "possible images"* [cf. the statement of Corollary 3.12] that arises from applying the *multiradial algorithms* of Theorem 3.11 below **cannot** be **directly compared** [i.e., at least in any *a priori* sense] to the *objects* in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Definition 3.8, (i), (ii); [IUTchII], Definition 4.9, (viii)] determined by the **arithmetic line bundle** that gives rise to the ***q*-pilot object**.

The *obstructions* to performing such a *comparison* may be *eliminated* in the following way [cf., especially, the display of (Ob5)]:

(Ob1) **$\mathcal{O}^{\times}$-Indeterminacies acting on tensor packets of log-shells**: The various *"possible images"* that occur as the *output* of the *multiradial algorithms* under consideration are *regions* — i.e., in essence, elements $\in \mathbb{P}$ — contained in *tensor packets* of *log-shells* $\mathcal{I}_k$ [where, for simplicity, we apply the notational conventions of Remark 1.2.2, (i), at *nonarchimedean* valuations]. By contrast, the *arithmetic line bundle* that gives rise to the *q-pilot object* arises, locally, as an *ideal*, i.e., an $\mathcal{O}_k$*-submodule*, contained in the $\mathcal{O}_k$*-module* $\mathcal{O}_k$, which, to avoid confusion, we denote by $\mathcal{O}_k^{\mathrm{mdl}}$. Here, we observe that

> *unlike* the *ring* $\mathcal{O}_k$, the $\mathcal{O}_k$*-module* $\mathcal{O}_k^{\mathrm{mdl}}$ does **not** admit a **canonical generator** [i.e., a canonical element corresponding to the element $1 \in \mathcal{O}_k$]; by contrast, $\mathcal{I}_k \subseteq k$ *can only be defined* by using the **ring structure** of $\mathcal{O}_k$ and is **not**, in general, **stabilized** by the natural action [via multiplication] by $\mathcal{O}_k^{\times}$.

That is to say, $\mathcal{O}_k^{\mathrm{mdl}}$ only admits a *"canonical generator"* up to an *indeterminacy* given by *multiplication by* $\mathcal{O}_k^{\times}$, i.e., an indeterminacy that does *not stabilize* $\mathcal{I}_k$.

(Ob2) **From arbitrary regions to arithmetic vector bundles, i.e., hulls**: Thus, by passing from an *arbitrary given region* $\in \mathbb{P}$ to the associated **hull** $\phi(P) \in \mathbb{H}$, we obtain a region $\phi(P) \in \mathbb{H}$ that is *stabilized* by the natural action of $\mathcal{O}_k^{\times}$ [cf. (Ob1)] and, moreover, [unlike an arbitrary element $\in \mathbb{P}$!] may be regarded as defining the *local* portion of a *global* **arithmetic vector bundle** relative to the ring structure labeled by some $\alpha \in A$ [i.e., which is typically taken, when $0 \in A \subseteq |\mathbb{F}_l|$, to be the *zero label* $0 \in |\mathbb{F}_l|$].

(Ob3) **From arithmetic vector bundles to arithmetic line bundles via "$\det_{\otimes M}(-)$"**: Moreover, by forming the **determinant** of the *arithmetic vector bundle* constituted by a *hull* $\in \mathbb{H}$, one obtains an **arithmetic line bundle**, i.e., which does indeed yield *objects* in the *local* and [by allowing $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $\underline{v} \in \underline{\mathbb{V}}$ to *vary*] *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, hence may be **compared**, in a meaningful way, to the objects determined by the *arithmetic line bundle* that gives rise to the ***q*-pilot object**. Here, we observe that:

(Ob3-1) **Weighted tensor powers/determinant**: In fact, when forming such a *"determinant"*, it is necessary to perform the following

operations:

(Ob3-1-1) In order to obtain a *"determinant"* that is *consistent* with the computation of the **log-volume** by means of certain **weighted sums** [cf. the discussion of Remark 3.1.1], it is necessary to work with **suitable positive tensor powers** [i.e., corresponding to the *weights* — cf. the various products of "N_v's" in the discussion of the final portion of Remark 3.1.1, (iv)] of the determinant line bundles corresponding to the various *direct summands* [as in the second and third displays of Proposition 3.1] of the tensor packet of log-shells "$\mathcal{I}^{\mathbb{Q}}((-))$".

(Ob3-1-2) In order to obtain a *"determinant"* that is *consistent* with the **normalization** of the **log-volume** given by "$\mathcal{O}_{(-)}$" [cf. Proposition 3.9, (i)], it is necessary to *tensor* the "determinant" of (Ob3-1-1) with the *inverse* of the "determinant" [in the sense of (Ob3-1-1)] of the *structure sheaf* [i.e., "$\mathcal{O}_{(-)}$"], which may be thought of as a sort of adjustment to take into account the *ramification* that occurs in the various local fields involved.

[We leave the routine details to the reader.]

(Ob3-2) **Positive tensor powers of the determinant**: In the context of (Ob3-1), we observe that there is *no particular reason* to require that the various *exponents* [i.e., which correspond to *weights* — cf. the various products of "N_v's" in the discussion of the final portion of Remark 3.1.1, (iv)] of these "suitable positive tensor powers" are necessarily *relatively prime*. In particular, the resulting *"determinant"* might in fact be more accurately described as a *"determinant raised to some positive tensor power"*. In the following, we shall denote this operation of forming the *"determinant raised to some positive tensor power"* by means of the notation

$$\text{“}\det_{\otimes M}(-)\text{”}$$

— where M denotes the [uniquely determined] positive integer [cf. the positive integer "$N_{\mathcal{E}}$" that appears in the final portion of the discussion of Remark 3.1.1, (iv)] such that this operation "$\det_{\otimes M}(-)$" maps [the result of tensoring the "$\mathcal{O}_{(-)}$" of Proposition 3.9, (i), with] an *arithmetic line bundle* to the ***M*-th tensor power** of the *arithmetic line bundle*. Thus, for instance, by taking M to be *sufficiently large* [in the "multiplicative sense", i.e., "sufficiently divisible"], we may, for the sake of simplicity, assume [cf. the "stack-theoretic twists" at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, arising from the structure of the stack-theoretic quotient discussed in [IUTchI], Remark 3.1.5] that the *localization* at each $\underline{v} \in \underline{\mathbb{V}}$ of any *arithmetic line bundle* that appears as the *output* of the operation $\det_{\otimes M}(-)$ is always *trivial*.

(Ob3-3) **Determinants and log-volumes**: Finally, we observe in passing that since [cf. the situation discussed in Proposition 3.9, (iii)]

the *arithmetic degree* of such an *arithmetic line bundle* may be interpreted, by working with *suitable normalization factors*, as the *log-volume* of the *original arithmetic vector bundle* [i.e., to which the operation $\det_{\otimes M}(-)$ was applied — cf. (Ob3-1), (Ob3-2); the discussion of Remark 3.1.1], *this intermediate step of applying* $\det_{\otimes M}(-)$ *may be omitted* in discussions in which one is only interested in *computing log-volumes*.

(Ob4) **Positive tensor powers of arithmetic line bundles**: From the point of view of the *original goal* [cf. the discussion at the beginning of the present (vii)] of obtaining objects that may be *compared* to the objects in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link determined by the *arithmetic line bundle* that gives rise to the *q-pilot object*, we thus conclude from (Ob3) that

> applying the operation $\det_{\otimes M}(-)$ yields objects that may indeed be **compared** to the objects in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link determined by the **arithmetic line bundle** that gives rise to the $\boldsymbol{M}$**-th tensor power** of the $\boldsymbol{q}$**-pilot object**.

Since, however, the *internal structure* of these *local* and *global Frobenioids* [as well of as the *localization functors* that relate local to global Frobenioids] remains *unaffected*, the **latter "slightly modified goal"** [i.e., of *comparison* with M*-th tensor powers* of objects that arise from the q-pilot object, as opposed to the *"original goal"* of comparison with objects that arise from the original q-pilot object] *does* **not** *result in any substantive problems* such as, for instance, an *indeterminacy* arising from *confusion* between a given arithmetic line bundle and its M-th tensor power [i.e., an indeterminacy analogous to the indeterminacy involving "$\mathbb{I}^{\mathrm{ord}}$" discussed in Remark 2.3.3, (vi)]. One way to understand this situation is as follows:

(Ob4-1) **From non-tensor-power to tensor-power Frobenioids via naive Frobenius functors**: One may think of the *local* and *global Frobenioids* [as well as of the *localization functors* that relate local to global Frobenioids] that appear in the *"slightly modified goal"* as **"$\boldsymbol{M}$-th tensor power versions"** of the *local* and *global Frobenioids* that appear in the *"original goal"*. That is to say, one may think of these *"tensor-power Frobenioids"* as *copies* of the *"non-tensor-power Frobenioids"* obtained by applying the **naive Frobenius functor of degree $\boldsymbol{M}$** of [FrdI], Proposition 2.1, (i). In particular, we conclude [i.e., from [FrdI], Proposition 2.1, (i)] that the *non-tensor-power Frobenioids completely determine the tensor-power Frobenioids*.

(Ob4-2) **From tensor-power to non-tensor-power Frobenioids via tensor power roots**: Alternatively, one may think of the *non-tensor-power Frobenioids* [i.e., that appear in the *"original goal"*] as being obtained from the *tensor-power Frobenioids* [i.e., that appear in the *"slightly modified goal"*] by **"extracting $\boldsymbol{M}$-th power roots"**. Since the *rational function monoids* [cf. [FrdI],

Theorem 5.2, (ii)] that give rise to the various *local Frobenioids* under consideration [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] are *not divisible* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the *tensor-power Frobenioids only determine the non-tensor-power Frobenioids up to certain twists*. Of course, these *twists* may be eliminated [cf. (Ob4-1)!] simply by applying the *naive Frobenius functor of degree M*.

(Ob4-3) **Tensor-power-twist indeterminacies**: In particular, if one thinks of the *output* of the crucial operation $\det_{\otimes M}(-)$ [cf. (Ob3)] as lying in the *tensor-power Frobenioids*, then one may always **"reconstruct"** the *non-tensor-power Frobenioids* from the *tensor-power Frobenioids* simply by considering *new copies of the tensor-power Frobenioids* which are related to the *given copies of tensor-power Frobenioids* by applying the *naive Frobenius functor of degree M* whose *domain* is the *new copies*, and whose *codomain* is the *given copies*. On the other hand, these *reconstructed non-tensor-power Frobenioids*, though completely determined up to isomorphism, are only related to one another, when regarded over the *given copies of tensor-power Frobenioids*, up to **certain twists** — i.e., up to a **"tensor-power-twist indeterminacy"** — as discussed in (Ob4-2). Since, however, we shall ultimately [e.g., in the context of Corollary 3.12] only be interested in *estimates of log-volumes*, such *tensor-power-twist indeterminacies* will *not* have any substantive effect on our computations [i.e., of log-volumes — cf. the discussion of (Ob3-3)].

(Ob5) **Independence of the "indeterminacy of possibilities"**: The issue of selecting a **specific element** in some collection of *"possible regions"* $\in \mathbb{P}$ that appears in the *output* of the *multiradial algorithm* is an issue that is **internal** to the algorithm. In particular, in order to **compare**, in a meaningful way, the output of the algorithm to some object — i.e., such as the *arithmetic line bundle* that gives rise to the *q-pilot object* — that is essentially **external** to the algorithm, it is necessary to work with objects that are **independent** of the **choice** of such a specific element/possibility. This may be achieved by

working with the **hull** [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4)]

$$\phi(P_B)$$

associated to the *[bounded] collection of possible regions* $P_B \stackrel{\mathrm{def}}{=} \{P_\beta\}_{\beta \in B}$ [cf. the discussion in the final portion of (vi)] that appears as the *output* of the *multiradial algorithms* under consideration and applying the **abstract set-theoretic** $\overline{\curlywedge}$**-formalism** of (v) [cf. also (vi)].

Here, we observe that this $\overline{\curlywedge}$-formalism of (v) may be applied *not only* to $\phi(P_B)$ *but also* [cf. the discussion of (Ob3)] to $\det_{\otimes M}(\phi(P_B))$ and $\mu^{\log}(\phi(P_B))$ [and in a *compatible* fashion].

(Ob6) **Hull-approximants for the log-volume of a given region**: Since one is ultimately interested in *estimating log-volumes* [cf. the discussion of

(iii), (iv)], it is tempting to consider *simply replacing a given region* $P \in \mathbb{P}$ by $\mu^{\log}(P)$. On the other hand, in order to obtain objects *comparable* with the *q-pilot object* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], one is obliged to work with *hulls* $\in \mathbb{H}$ [cf. also the discussion of (Ob7) below]. This state of affairs suggests working with, for instance, $\Xi(P) \subseteq \mathbb{H}$, i.e., with **hull-approximants** for $\mu^{\log}(P)$ [cf. the discussion of (iii), (vi)]. In this context, it is useful to recall [cf. the discussion of (iv)] that, in general, it is not so clear whether or not $\Xi(P) = \emptyset$. This already makes it more natural to consider $\Phi(P) \subseteq \mathbb{H}$ [cf. the discussion of (iii)], i.e., as opposed to $\Xi(P) \subseteq \mathbb{H}$. On the other hand, the issue of **independence** of the **choice** of a *specific possibility internal to the algorithm* under consideration [cf. (Ob5)] already means that one must consider $\mu^{\log}(H_{\Phi(P)})$ or $\mu^{\log}(H_{\Xi(P)})$, as opposed to $\mu^{\log}(P)$, which, in general, *may be* $>$ $\mu^{\log}(P)$ *and indeed* $= \mu^{\log}(\phi(P))$ [cf. the discussion of (iv)].

(Ob7) **Compatibility with $\mathfrak{log}$-Kummer correspondences**: In (Ob6), the discussion of the issue of *simply replacing a given region* $P \in \mathbb{P}$ by $\mu^{\log}(P)$ — i.e., put another away, of passing to the **quotient** [cf. the discussion of Remark 3.9.4, as well as of (viii) below] given by taking the **log-volume** — was subject to the **constraint** that one must construct, i.e., by working with *hulls* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], objects that may be *meaningfully compared* to *objects* in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. This *constraint* prompts the following question:

> Why is it that one cannot simply adopt **log-volumes** as the **"ultimate stage for comparison"** — that is to say, *without* passing through hulls or objects in the local and global Frobenioids referred to above?

At a more technical level, this question may be reformulated as follows:

> Why is it that one cannot **eliminate** the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip portion* [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$,
>
> the **local Galois groups** "$G_{\underline{v}}$" and **units** "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$"
>
> [cf. the notation of the discussion surrounding [IUTchI], Fig. I1.2; here and in the following discussion, we regard "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" as being equipped with the *auxiliary structure*, i.e., a *collection of submodules* [cf. [IUTchII], Definition 4.9, (i)] or *system of compatible surjections* [cf. [IUTchII], Definition 4.9, (v)], with which it is equipped in the definition of an $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip] — from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link?

[Closely related issues are discussed in (ix), (x) below.] The essential reason for this may be understood as follows:

(Ob7-1) **Local Galois groups**: The local Galois groups "$G_{\underline{v}}$" [for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] satisfy the important property of being **invariant**, up to isomorphism, with respect to the transformations constituted

by the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$- and $\mathfrak{log}$-links — cf. the **vertical** and **horizontal coricity** properties discussed in Theorem 1.5, (i), (ii), as well as the discussion of [IUTchII], Remark 3.6.2, (ii). These coricity properties play a *fundamental role* in the theory of the present paper, i.e., by allowing one to relate, via these coricity properties, objects on either side of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$- and $\mathfrak{log}$-links which do *not* satisfy such invariance properties. In particular, the theory of the present series of papers *cannot* function properly if the local Galois groups are eliminated from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(Ob7-2) **Units**: Thus, it remains to consider what happens if one eliminates the *[Frobenius-like!] units* [but *not* the *local Galois groups* — cf. (Ob7-1)!] from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. This amounts to replacing the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link by the *associated* $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strips* [cf. Definition 2.4, (iii)]. Of course, since one still has the *local Galois groups*, one can consider the *étale-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}(G_{\underline{v}})$" [i.e., "$\mathcal{O}^{\times\boldsymbol{\mu}}(G)$", in the case where one takes "G" to be $G_{\underline{v}}$] of [IUTchII], Example 1.8, (iv). On the other hand, these *étale-like units* **differ fundamentally** from their *Frobenius-like counterparts* in the following respect:

- The **Frobenius-like units** "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link are [*tautologically*!] related **only** to the Frobenius-like units at the **same vertical coordinate** [i.e., in a vertical column of the log-theta-lattice] as the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration, i.e., **not** to the Frobenius-like units at other vertical coordinates in this vertical column. In particular, these Frobenius-like units arise from the **same underlying multiplicative** structure [i.e., of the **ring structure** determined, on various Frobenius-like multiplicative monoids, by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater to which they belong] as the local and global [*Frobenius-like*!] **value group** portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration. Put another way, the **splittings** of unit group and value group portions that appear in the *intrinsic structure* of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips under consideration [cf. [IUTchII], Definition 4.9, (vi), (viii)] are **consistent** with the *underlying multiplicative* structure of the *ring structure* determined [on various Frobenius-like multiplicative monoids] by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater under consideration.

- By contrast, the **étale-like counterparts** of these Frobenius-like units are *constrained* by their *vertical coricity* [cf. Theorem 1.5, (i)] to be related, via the relevant $\mathfrak{log}$-*Kummer correspondences*, **simultaneously** to the corresponding Frobenius-like units at **every verti-**

> **cal coordinate** in a vertical column of the log-theta-lattice as the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration. In particular, the relationship between these étale-like units and the corresponding Frobenius-like units at various vertical coordinates in the vertical column under consideration is **subject** to the action of arbitrary iterates of the **$\mathfrak{log}$-link**, hence to a **complicated confusion** between the *unit group* and *value group* portions at various vertical coordinates of this vertical column. This *complicated confusion* is **inconsistent** with the *intrinsic structure* of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips under consideration [cf. Definition 2.4, (iii)], that is to say, with treating the *local and global value group portions* of these $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips as objects that are *not subject* to any *constraints* in their relationship to the *étale-like units*, i.e., to the *local Galois group portions* of these $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips. Put another way, if one regards the *étale-like units* as the **sole access route**, from the point of view of the Frobenius-like units in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration, to the Frobenius-like units in the *domain* of this $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, then one obtains a situation in which the data in the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips [i.e., *"non-mutually constrained local/global value group portions and local Galois groups*] is **"over-constrained/over-determined"**.

Thus, in summary, one **cannot eliminate** the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip portion* [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" — from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. One *important consequence* of the fact that the local Galois group and unit portions are indeed included in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is the [*"proper functioning"*, as described in the present paper, of the] theory of **$\mathfrak{log}$-Kummer correspondences** and **log-shells** — which serve as **"multiradial containers"** [cf. Remarks 1.5.2, 2.3.3, 2.3.4, 3.8.3] — both of which play a *central role* in the present paper.

(Ob8) **Vertical shifts in the output data**: One *important consequence* of the theory of $\mathfrak{log}$-*Kummer correspondences* lies in the fact that it allows one to **transport/relate** [i.e., by applying the theory of $\mathfrak{log}$-*Kummer correspondences*!] the *output* of the *multiradial algorithms* under consideration to **different vertical coordinates** within a vertical column of the log-theta-lattice. In fact,

> this **output** — even if one works with *hulls* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)]! — yields, *a priori*, objects in local and global Frobenioids that **differ**, strictly speaking, from the corresponding [*multiplicative*!] local and global Frobenioids that

appear in the **input data** of the algorithm [i.e., the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link — cf. Definition 3.8, (i), (ii)] by a **"vertical shift"** in the log-theta-lattice, i.e., more concretely, by an application of the **$\mathfrak{log}$-link** [that is to say, which produces *additive* log-shells from the *multiplicative* "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" in the *input data*].

In particular, it is precisely by applying the theory of $\mathfrak{log}$*-Kummer correspondences* that we will ultimately be able to obtain *objects* [i.e., objects in local and global Frobenioids] arising from the *output* of the *multiradial algorithms* under consideration that may indeed be **meaningfully compared** with objects in the local and global Frobenioids that appear in the *input data* of the algorithm [cf. Step (xi-d) of the proof of Corollary 3.12 below]. On the other hand, in this context, it is important to note that since such *comparable objects* are obtained by applying the $\mathfrak{log}$-Kummer correspondence, the local and global Frobenioids to which these *comparable objects* belong are necessarily **subject** to the indeterminacies of the relevant $\mathfrak{log}$-Kummer correspondence, i.e., in more concrete terms, to *arbitrary iterates* of the **$\mathfrak{log}$-link** [cf. the discussion of the final portion of Remark 3.12.2, (v)].

(Ob9) **Hulls in the context of the $\mathfrak{log}$-link and log-volumes**: In the context of the discussion of the final portion of (Ob8), we observe that the operation of *passing to* **realified semi-simplications** [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] in situations where one considers the $\mathfrak{log}$*-link compatibility* of the *log-volume*, is a **quotient operation** on both the *domain* and the *codomain* of the $\mathfrak{log}$-link that induces a **natural bijection** between **log-volumes** of **hulls** in the domain and codomain of the $\mathfrak{log}$-link. That is to say, the fact that this *quotient operation* [i.e., of passing to *realified semi-simplifications*] induces such a *natural bijection* is *not affected* — i.e., unlike the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)! — by the fact that the operation of *passing to realified semi-simplications* [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] involves, at various intermediate steps, the use of various **regions** which are **not hulls**. The *fundamental qualitative difference* between the present situation, on the one hand, and the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) [i.e., which required the formation of *hulls*!], on the other, may be understood as follows:

(Ob9-1) **Formal indeterminacies acting on comparable objects**: Once the passage to **comparable** objects via $\det_{\otimes M}(-)$ of a suitable **hull** has been achieved [cf. the discussion of (Ob5)], the various **formal, or stack-theoretic/diagram-theoretic, indeterminacies** that arose from this passage to comparable objects — i.e.,

- the *tensor-power-twist indeterminacies* of (Ob4-3),
- the application of the $\overline{\wedge}$*-formalism* in (Ob5), and
- the *indeterminacy* with respect to application of *arbitrary iterates* of the $\mathfrak{log}$*-link* of (Ob8)

— have **no effect** on the **comparability** of the objects obtained

in (Ob5). That is to say, these indeterminacies function solely as *compatibility conditions* that must be satisfied [e.g., by applying the theory of *realified semi-simplications*, as developed in Remark 3.9.4, (iii), (iv), (v), (vi)] when passing to *"coarse/set-theoretic invariants"* such as the *log-volume*.

(Ob9-2) **Non-explicit relationships between comparable and non-comparable objects**: By contrast, the situation discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) was one in which — until the *"final conclusion"* of this discussion in (Ob5) — *comparable objects had not yet been obtained*. Put another way, prior to this *"final conclusion"*, the *precise relationship* between the *non-comparable objects* that occurred as the *a priori output* of the *multiradial algorithms* under consideration, on the one hand, and *comparable objects*, on the other, had **not** yet been **explicitly computed**.

Closely related issues are discussed in (ix) below.

(viii) In the context of (vi), (vii), it is of interest to observe that, just as in the case of the operations of

(sQ1) **Kummer-detachment**, i.e., passing from *Frobenius-like* [that is to say, strictly speaking, Frobenius-like structures that contain certain *étale-like structures*] to ["purely"] *étale-like* structures [cf. Remark 1.5.4, (i), as well as the *vertical arrows* of the *commutative diagram* of Remark 3.10.2 below], and

(sQ2) **Galois evaluation** [cf. [IUTchII], Remark 1.12.4, as well as the *horizontal arrows* of the *commutative diagram* of Remark 3.10.2 below],

the operations of

(sQ3) passing from more general regions to **positive tensor powers** of **determinants** of **hulls** and then applying the **abstract set-theoretic $\overline{\wedge}$-formalism** of (v) [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7)],

(sQ4) adjusting the **vertical shifts** [i.e., in the vertical column of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times \boldsymbol{\mu}}_{\text{LGP}}$-link under consideration] in the output of the multiradial algorithm by applying the **$\mathfrak{log}$-Kummer correspondence** [cf. the discussion of (Ob8)], as well as of

(sQ5) passing to **log-volumes** [cf. (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], via the formalism of **realified semi-simplifications** discussed in Remark 3.9.4,

may be regarded as **intricate (sub)quotient** — or [cf. the discussion of (ii)] **push forward/splitting/integration** — operations. Indeed, from this point of view, the content of the *entire theory of the present series of papers* may be regarded as the development of

a suitable collection of **(sub)quotient operation algorithms** for constructing a

relatively simple, concrete (sub)quotient

of the complicated apparatus constituted by the full **log-theta-lattice**.

The *goal* of this construction of *(sub)quotient operation algorithms* — i.e., of the *entire theory of the present series of papers* — may then be understood as

the **computation** of the **projection**, via the resulting *relatively simple, concrete (sub)quotient*, of

the **"Θ-intertwining"** [i.e., the structure on an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the Θ-*pilot object* appearing in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* of Definition 3.8, (ii)]

onto structures arising from the *vertical column* in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link*, that is to say, where

the **"*q*-intertwining"** [i.e., the structure on an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the *q-pilot object* appearing in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* of Definition 3.8, (ii)]

is *in force* [cf. the discussion of Remark 3.12.2, (ii), below].

This *computation*, when suitably interpreted, amounts, *essentially tautologically*, to the **inequality** of Corollary 3.12 below. Here, we observe that each of these (sub)quotient operations (sQ1), (sQ2), (sQ3), (sQ4), (sQ5) may be understood as an operation whose purpose is to **simplify** the quite complicated apparatus constituted by the full *log-theta-lattice* by allowing the introduction of various **indeterminacies**. Put another way, the **nontriviality** of these various (sub)quotient operations lies

in the **very delicate balance** between **minimizing** the **indeterminacies** that *arise from passing to a quotient*, while at the same time ensuring **compatibility** with the structures that exist *prior to formation of the quotient*.

Indeed:

· In the case of (sQ1), i.e., the case of *Kummer-detachment indeterminacies*, this delicate balance is discussed in detail in Remarks 1.5.4, 2.1.1, 2.2.1, 2.2.2, 2.3.3, as well as Remark 3.10.1, (ii), (iii), below.

· In the case of (sQ2), i.e., the case of *Galois evaluation*, the delicate issue of *compatibility* with *Kummer theory* is discussed in [IUTchII], Remark 1.12.4.

· In the case of (sQ3), i.e., the case of passing to *hulls*, various delicate issues — such as, for instance, the delicate issues of *tensor-power-twist indeterminacies* [cf. (Ob4-3)], the $\overline{\wedge}$-*formalism* [cf. (Ob5)], and *compatibility* with $\mathfrak{log}$-*Kummer correspondences* [cf. (Ob7)] — are discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7) [cf. also (ix), (x) below].

· In the case of (sQ4), the adjustment of *vertical shifts* via $\mathfrak{log}$-*Kummer correspondences* results in an indeterminacy with respect to application

of *arbitrary iterates* of the $\mathfrak{log}$**-link**, i.e., in the vertical column of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration [cf. (Ob8)].

· In the case of (sQ5), i.e., the case of passing to *log-volumes*, various subtleties surrounding the *compatibility* of the log-volume with the $\mathfrak{log}$-*link* are discussed in detail in Remark 3.9.4, as well as in (vii) of the present Remark 3.9.5 [cf., especially, the discussion of (Ob9)].

Finally, in this context, we observe that, in light of the **rigidity** of **étale-like** structures [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)], i.e., at a more concrete level, of objects constructed via **anabelian algorithms**, the construction of *suitable subquotients* of the *étale-like* portion of the *log-theta-lattice* — that is to say, as in the case of (sQ2), (sQ3), (sQ4), (sQ5) — is a particularly **nontrivial** issue.

(ix) In the context of the discussion of (vii) [cf., especially, (Ob7), (Ob8), (Ob9)], (viii), it is important to observe that there is a *fundamental qualitative difference* between (sQ3), (sQ4), on the one hand, and (sQ5), on the other:

(cQ3) **Compatibility of (sQ3) with $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip data**: The fact that [in the notation of (Ob1), (Ob2)] *hulls* $\in \mathbb{H}$ are *stabilized* by multiplication by elements of $\mathcal{O}_k$ implies that, by taking [a suitable positive tensor power of] the determinant [cf. (Ob3)], they determine *objects* [i.e., the "$\det_{\otimes M}(\phi(P_B))$" of (Ob5)] in the *local* and [by allowing $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $\underline{v} \in \underline{\mathbb{V}}$ to *vary*] *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In particular, by considering suitable *pull-back morphisms* in these *local Frobenioids* [i.e., which correspond to base-change morphisms in conventional scheme theory — cf. [FrdI], Definition 1.3, (i)], one obtains *objects* equipped with natural faithful actions by the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" [cf. the notation of (Ob7)], i.e., the data that corresponds to the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Moreover, as discussed in (vi), the *quotient induced on* $\mathbb{H}$ by the *set-theoretic* $\overline{\wedge}$-*formalism* of (v) [cf. the display of (Ob5)] may be understood as corresponding to the consideration of the *"$\overline{\wedge}$-category"* consisting of

($\overline{\wedge}^{\mathrm{lc}}_1$) *objects* in the *local Frobenioid* under consideration equipped with a *"structure poly-morphism"* to the *original object arising from a hull*, i.e., the "$\det_{\otimes M}(\phi(P_B))$" of (Ob5), given by the $\mathrm{Aut}(\det_{\otimes M}(\phi(P_B)))$-orbit of a *linear morphism* in the *local Frobenioid* [cf. [FrdI], Definition 1.2, (i)] to $\det_{\otimes M}(\phi(P_B))$ and

($\overline{\wedge}^{\mathrm{lc}}_2$) *morphisms* between such objects that are compatible with the *structure poly-morphism*.

[Alternatively, one could consider a slightly modified version of this "$\overline{\wedge}$-category" by restricting the objects to be objects that arise from *hull-approximants for the log-volume*, i.e., as in the discussion of (Ob6).] By considering suitable *pull-back morphisms* in this $\overline{\wedge}$-*category*, we again obtain *objects* equipped with *mutually compatible* [i.e., relative to varying the object within the $\overline{\wedge}$-category] natural faithful actions by the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" [cf. the notation of (Ob7)], i.e., the data

that corresponds to the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Next, we observe that one may consider *categories of "local-global $\overline{\wedge}$-collections of objects"*, i.e., categories whose *objects* are collections consisting of

($\overline{\wedge}_1^{\text{lc-gl}}$) a *"local"* object in the $\overline{\wedge}$*-category* at each $\underline{v} \in \underline{\mathbb{V}}$,

($\overline{\wedge}_2^{\text{lc-gl}}$) a *"global"* object in the *global realified Frobenioid* of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration, and

($\overline{\wedge}_3^{\text{lc-gl}}$) *localization isomorphisms* between the image of the *local* object at each $\underline{v} \in \underline{\mathbb{V}}$ in the realification of the local Frobenioid at $\underline{v}$ and the localization of the *global object* at the element $\in \mathrm{Prime}(-)$ of the global realified Frobenioid corresponding to $\underline{v}$

[and whose *morphisms* are compatible collections of morphisms between the respective portions of the data $\overline{\wedge}_1^{\text{lc-gl}}$ and $\overline{\wedge}_2^{\text{lc-gl}}$] — cf. the discussion of the [closely related] *functors* in the final displays of [FrdII], Example 5.6, (iii), (iv). In particular, just as the *tensor-power-twist indeterminacies* of (sQ3) [cf. (Ob4-3)] and the *indeterminacy* with respect to application of *arbitrary iterates* of the $\mathfrak{log}$*-link* of (sQ4) [cf. (Ob8)] may be understood as **"formal, or stack-theoretic/diagram-theoretic, quotients"** [i.e., as opposed to *"coarse/set-theoretic quotients"* given by *set-theoretic invariants* such as the *log-volume* — cf. the discussion of (Ob9-1)], the *pair* consisting of

($\overline{\wedge}_1^{\text{fQ}}$) such a *category of "local-global $\overline{\wedge}$-collections"* [cf. $\overline{\wedge}_1^{\text{lc-gl}}$, $\overline{\wedge}_2^{\text{lc-gl}}$, $\overline{\wedge}_3^{\text{lc-gl}}$] and

($\overline{\wedge}_2^{\text{fQ}}$) the analogous *category of "local-global collections"*, i.e., where the "$\overline{\wedge}$-category" at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. $\overline{\wedge}_1^{\text{lc-gl}}$] is replaced by the original *local Frobenioid* [portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration] at each $\underline{v} \in \underline{\mathbb{V}}$,

may also be regarded as the *"formal, or stack-theoretic, quotient"* corresponding to the operation of considering "$\overline{\wedge}_2^{\text{fQ}}$ *modulo* $\overline{\wedge}_1^{\text{fQ}}$".

(cQ4) **Compatibility of (sQ4) with $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip data**: Since the *adjustment of vertical shifts* in (sQ4) is obtained precisely by applying the $\mathfrak{log}$*-Kummer correspondence*, this adjustment operation is *tautologically compatible* [cf. the *vertical coricity* of Theorem 1.5, (i)] with *suitable isomorphisms* between the *local Galois groups* "$G_{\underline{v}}$" and the *étale-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}(G_{\underline{v}})$" [cf. the notation of (Ob7-1), (Ob7-2)] that appear. Alternatively, this adjustment operation is *tautologically compatible* with *suitable isomorphisms* between the *local Galois groups* "$G_{\underline{v}}$" and the *Frobenius-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{F_{\underline{v}}}$" [cf. the notation of (Ob7)], so long as one allows for an **indeterminacy** with respect to application of *arbitrary iterates* of the **$\mathfrak{log}$-link** [cf. the discussion of (Ob8)].

(iQ5) **Incompatibility of (sQ5) with $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip data**: By contrast, unlike the situation with (sQ3), (sQ4), passing to *log-volumes* [i.e., (sQ5)] amounts precisely to *forgetting* the *local Galois groups* and *Frobenius-like units*, i.e., the data that corresponds to the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion

of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the discussion of (Ob7)].

Here, we recall that (sQ3), (sQ4), (sQ5) all occur *within the vertical column* of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration. In particular, the various *local Galois groups* "$G_{\underline{v}}$" are all equipped with *rigidifications* as *quotients* of [isomorphs of] "$\Pi_{\underline{v}}$" [cf. the notation of the discussion surrounding [IUTchI], Fig. I1.2]. Put another way [cf. also the discussion of (Ob9)]:

- One may think of the *compatibility* properties (cQ3), (cQ4) as a sort of **arithmetic holomorphicity** [relative to the vertical column under consideration] or, alternatively, as a sort of *compatibility* with the **$\mathfrak{log}$-Kummer correspondence** of this vertical column. This point of view is reminiscent of the use of the descriptive *"holomorphic"* in the term *"holomorphic hull"*.
- Conversely, one may think of the *incompatibility* property (iQ5) as corresponding to the operation of *forgetting* this arithmetic holomorphic structure or, alternatively, as a sort of *incompatibility* with the $\mathfrak{log}$-*Kummer correspondence* of this vertical column.

From the point of view of the *computation* of the *projection* of the Θ-*intertwining* onto the *q-intertwining* discussed in (viii), this *fundamental qualitative difference* — i.e., (cQ3), (cQ4) versus (iQ5) — has a *very substantive consequence*:

It is precisely by *passing through (sQ3), (sQ4)* — i.e., *before applying (sQ5)!* [cf. also the discussion of (Ob7)] — that the **chain** of *poly-isomorphisms of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [i.e., including the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips!] that

- begins with the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the ***q*-pilot object** in the **codomain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link,
- passes through the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** to the **domain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link,
- passes through the various *poly-isomorphisms of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [cf. the diagram of Remark 3.10.2 below; the discussion of *"IPL"* in Remark 3.11.1, (iii), below] induced by **(sQ1), (sQ2)**, and
- finally, passes through **(sQ3), (sQ4)**, which satisfy the *compatibility* property with the $\mathfrak{log}$-*Kummer correspondence* discussed above [i.e., (cQ3), (cQ4)]

forms a **closed loop**, i.e., up to the introduction of the **"formal quotient indeterminacies"** discussed in (cQ3), (cQ4) [cf. also the discussion of (Ob9-1)].

In this context, we observe that a *non-closed loop* would yield a situation from which *no nontrivial conclusions may be drawn*, for essentially the *same reason* [that no nontrivial conclusions may be drawn] as in the case of the *"distinct labels approach"* of Remark 3.11.1, (vii), below [cf. also Remark 3.12.2, (ii), (c^{itw}), (c^{toy}), below]. That it to say, it is only by constructing such a *closed loop* that one can *complete* the *computation* of the *projection* [that is to say, as discussed in (viii)] of the Θ-*intertwining* onto the *q-intertwining*, i.e.,

> *complete the computation of the* Θ**-intertwining** *structure, up to suitable indeterminacies, on a* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** *that is* **constrained** *to be subject to the* $\boldsymbol{q}$**-intertwining**.

Here, we recall from the discussion of (Ob7) [cf. also (x) below for a discussion of a related topic] that the *construction* of this sort of mathematical structure — i.e.,

> a $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** that is **simultaneously equipped** with **two intertwinings**, namely, the the Θ*-intertwining*, up to indeterminacies, and the q*-intertwining*

— *cannot* be achieved if one omits various subportions of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip. It is this *computation/construction* that will allow us, in Corollary 3.12 below, to conclude *nontrivial*, albeit *essentially tautological*, consequences from the theory of the present series of papers, such as the *inequality* of Corollary 3.12 [cf. Substeps (xi-d), (xi-e), (xi-f), (xi-g) of the proof of Corollary 3.12; Fig. 3.8 below]. Put another way, if one attempts to *skip either (sQ3) or (sQ4)* and *pass directly* from (sQ2) to (sQ4) or (sQ5) [or from (sQ3) to (sQ5)], then the *resulting chain* of poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips *no longer forms a closed loop*, and one can *no longer* conclude any *nontrivial* consequences from the theory of the present series of papers.

(x) In the context of the discussion of (vii), (viii), (ix), it is of interest to observe that it is **not possible** [at least in any immediate sense!] to

> work with regions $\in \mathbb{P}$ that do not necessarily belong to $\mathbb{H}$ — and hence *avoid the operation of passing to the hull*! — by replacing the *local and global Frobenioids* [i.e., categories of local and global *arithmetic line bundles*] that appear in the definition of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [cf. [IUTchII], Definition 4.9, (viii)] by *"more general* **categories of regions** $\in \mathbb{P}$*"*.

Indeed, any sort of category of regions $\in \mathbb{P}$ necessarily requires consideration of the **multi-dimensional underlying space** of $\mathcal{I}^{\mathbb{Q}}((-))$ [cf. (ii)], i.e., in essence, an **additive module** of **rank** > 1. Put another way, the only natural way to relate various *"lines"* [i.e., rank 1 submodules] within this space to one another is by invoking the *additive structure* of this module. On the other hand, since the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is **not compatible** with the **additive structures** in its domain and codomain, it is of *crucial importance* that the categories that are *glued together* via the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link be **purely multiplicative** in nature, i.e., independent, at least in an *a priori* sense, of the *additive structures* in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In particular, one must, in effect, work with **arithmetic line bundles** [which — unlike *arithmetic vector bundles of rank* > 1! — may indeed be defined in a way that only uses the *multiplicative* structures of the rings involved] — cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4). Of course, instead of working [as we in fact do in the present series of papers] with arithmetic line bundles over F_{mod}, up to certain "stack-theoretic twists" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. [IUTchI], Remark 3.1.5], where we work with local arithmetic line bundles over $K_{\underline{v}}$ [which are necessary in order to accommodate the use of various powers of $\underline{\underline{q}}_{\underline{v}}$! — cf. [IUTchI], Example 3.2, (iv)], one could instead consider working with *arithmetic line bundles over* $\mathbb{Q}$. Relative to the arithmetic line bundles over F_{mod} or $K_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, that in fact appear in

the present series of papers, working with arithmetic line bundles over $\mathbb{Q}$ amounts, in effect, to applying some sort of

> **norm**, or **determinant** operation, from F_{mod} down to $\mathbb{Q}$ or, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, from $K_{\underline{v}}$ down to $\mathbb{Q}_{p_{\underline{v}}}$

[followed by tensor product with a certain fixed arithmetic line bundle on $\mathbb{Q}$ or $\mathbb{Q}_{p_{\underline{v}}}$, in order to take into account the *ramification* of F_{mod} over $\mathbb{Q}$ or $K_{\underline{v}}$ over $\mathbb{Q}_{p_{\underline{v}}}$ — cf. the discussion of (Ob3-1-2)]. On the other hand, if we write $G_{p_{\underline{v}}} \subseteq \mathrm{Gal}(\overline{F}/\mathbb{Q})$ for the unique decomposition group of $p_{\underline{v}}$ that contains $G_{\underline{v}}$, then one verifies immediately that the fact that $G_{p_{\underline{v}}}$ does **not** admit a **splitting**

$$\text{“}G_{p_{\underline{v}}} \stackrel{\sim}{\to} G_{\underline{v}} \times (G_{p_{\underline{v}}}/G_{\underline{v}})\text{”}$$

implies that this sort of *norm operation* from $K_{\underline{v}}$ down to $\mathbb{Q}_{p_{\underline{v}}}$ **cannot** be extended, in any meaningful sense, to any sort of **Galois-equivariant** [i.e., $G_{\underline{v}}$-equivariant] operation on algebraic closures of $K_{\underline{v}}$ and $\mathbb{Q}_{p_{\underline{v}}}$. Since the *faithful action* of $G_{\underline{v}}$ on the *unit portion* of the local Frobenioids in an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip plays a *central role* [cf. the discussion of (Ob7)] in the theory of **$\mathfrak{log}$-Kummer correspondences** and **log-shells** [which play a *central role* in the present paper!], the *incompatibility* of any sort of *norm operation* with the *local Galois group* $G_{\underline{v}}$ makes such a norm operation *fundamentally unsuited* for defining the *gluings* that constitute the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

Remark 3.9.6. In the context of Proposition 3.9, (iii), (iv) [cf. also Remark 3.9.4], we make the following observation. The $\mathfrak{log}$*-link compatibility* of Proposition 3.9, (iv) [cf. also Proposition 1.2, (iii); Proposition 1.3, (iii); Remark 3.9.4] amounts to a *coincidence of log-volumes* — *not* of *arbitrary* regions that appear in the domain and codomain of the $\mathfrak{log}$-link, but rather — of *certain types of "sufficiently small" regions* that appear in the domain and codomain of the $\mathfrak{log}$-link. On the other hand, the *"product formula"* — i.e., at a more concrete level, the *"ratios of conversion"* [cf. [IUTchI], Remark 3.5.1, (ii)] between *log-volumes* at distinct $\underline{v} \in \underline{\mathbb{V}}$ — may be formulated [without loss of generality!] in terms of such "sufficiently small" regions. Thus, in summary, we conclude that

> the **$\mathfrak{log}$-link compatibility** of Proposition 3.9, (iv), implies a **compatibility** of **"product formulas"**, i.e., of **"ratios of conversion"** between *log-volumes* at distinct $\underline{v} \in \underline{\mathbb{V}}$, in the domain and codomain of the $\mathfrak{log}$-link.

In particular, in the context of Proposition 3.9, (iii), we conclude that Proposition 3.9, (iv), implies a **compatibility** between **global arithmetic degrees** in the domain and codomain of the $\mathfrak{log}$-link.

Remark 3.9.7. When computing *log-volumes of various regions* of the sort considered in Proposition 3.9, it is useful to keep the following *elementary observations* in mind:

(i) In the context of Proposition 3.9, (iii), the defining condition "**zero log-volume** for all but finitely many $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$" for

$$\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \subseteq \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

that is imposed on the various components indexed by $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ of the direct product of the above display may be satisfied by considering elements of this direct product such that for *all but finitely many* of the elements $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}}$ for which $p_{w_{\mathbb{Q}}}$ is **unramified** in K, the component at $w_{\mathbb{Q}}$ is given by $\mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Indeed, for each such $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}}$, the subset

$$\mathcal{O}_{(-)} = \mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$$

[cf. the notation of Proposition 3.2, (ii); Proposition 3.9, (i); the final sentence of [AbsTopIII], Definition 5.4, (iii)] has *zero log-volume*. Finally, in the context of Proposition 3.9, (ii), we observe that, for each such $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}}$, the subset $\mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ is a *mono-analytic invariant*, which, moreover, [cf. Remark 3.9.5, (i)] is equal to its own *holomorphic hull*.

(ii) In the context of Proposition 3.9, (i), (ii), we observe that one may consider the **log-volume** of **more general**, say, **relatively compact subsets** $E \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ [cf. the discussion of Remark 3.1.1, (iii)] than the sets which belong to $\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$, i.e.,

> simply by *defining* the *log-volume of* E to be the **infimum** of the log-volumes of the sets $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$.

This definition means that one must allow for the possibility that the *log-volume of* E is $-\infty$. Alternatively [and *essentially equivalently*!], one can treat such E by thinking of such an E as corresponding to the

inverse system of $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$.

Here, when E is a **direct product pre-region**, it is natural to consider instead the inverse system of **direct product regions** $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$ [cf. the discussion of Remark 3.1.1, (iii)]. This *approach via inverse systems* of regions each of which has *finite log-volume* has the advantage that it allows one to always work with **finite log-volumes**.

(iii) In a similar vein, in the context of Proposition 3.9, (iii), we observe that one may consider the **log-volume** of **more general collections of relatively compact subsets** [cf. the discussion of Remark 3.1.1, (iii)] than the collections of sets of the sort considered in the discussion of (i) above. Indeed, if

$$\{E_{v_{\mathbb{Q}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})\}_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}$$

is a collection of subsets such that, for *some* collection of sets $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ of the sort considered in the discussion of (i), it holds that $E_{v_{\mathbb{Q}}} \subseteq E^*_{v_{\mathbb{Q}}}$, for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, then one may

> simply *define* the *log-volume of* $\{E_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ to be the **infimum** of the log-volumes of the collections of sets $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ of the sort considered in the discussion of (i) above such that $E_{v_{\mathbb{Q}}} \subseteq E^*_{v_{\mathbb{Q}}}$, for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$

[in which case one must allow for the possibility that the *log-volume of* E is $-\infty$]; alternatively [and *essentially equivalently*!], one may think of such a collection $\{E_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ as corresponding to the

inverse system of collections $\{E^*_{v_\mathbb{Q}}\}_{v_\mathbb{Q}}$ of the sort considered in the discussion of (i) above such that $E_{v_\mathbb{Q}} \subseteq E^*_{v_\mathbb{Q}}$, for each $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}$

[an approach that has the advantage that it allows one to always work with **finite log-volumes**]. Here, in the case where each $E_{v_\mathbb{Q}}$, for $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}$, is a **direct product pre-region**, it is natural to consider instead inverse systems $\{E^*_{v_\mathbb{Q}}\}_{v_\mathbb{Q}}$ as above such that each $E^*_{v_\mathbb{Q}}$, for $v_\mathbb{Q} \in \mathbb{V}_\mathbb{Q}$, is a **direct product region** [cf. the discussion of Remark 3.1.1, (iii)].

Proposition 3.10. (Global Kummer Theory and Non-interference with Local Integers) *Let* $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from an* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii); Proposition 3.5; Remark 3.8.2]. For each* $n \in \mathbb{Z}$*, write*

$$ {}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} $$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, where* $m \in \mathbb{Z}$*, via the* **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Vertically Coric Global** LGP**-,** $\mathfrak{lgp}$**-Frobenioids and Associated Kummer Theory)** *Recall the constructions of various global Frobenioids in Proposition 3.7, (i), (ii), (iii), (iv), in the context of* $\mathcal{F}$*-prime-strip processions. Then by applying these constructions to the* $\mathcal{F}$*-prime-strips "*$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$*" [cf. the notation of Proposition 3.5, (i)] and the various full* $\mathfrak{log}$*-links associated [cf. the discussion of Proposition 1.2, (ix)] to these* $\mathcal{F}$*-prime-strips — which we consider in a fashion* **compatible** *with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** *involved [cf. Remark 1.3.2; Proposition 3.4, (ii)] — we obtain* **functorial algorithms** *in the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing [number]* **fields***,* **monoids***, and* **Frobenioids** *equipped with natural isomorphisms*

$$ \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha = \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha $$
$$ \supseteq \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha = \mathbb{M}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha $$

$$ \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \supseteq \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha $$

$$ \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \overset{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \overset{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha $$

[cf. the number fields, monoids, and Frobenioids "$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \supseteq \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$*", "*$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$*" of [IUTchII], Corollary 4.7, (ii)] for* $\alpha \in A$*, where* A *is a subset of* J *[cf. Proposition 3.3], as well as* $\mathcal{F}^{\Vdash}$**-prime-strips** *equipped with natural isomorphisms*

$$ \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{gau}} \overset{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}} \overset{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}} $$

— [all of] which we shall refer to as being **"vertically coric"***. For each* $n, m \in \mathbb{Z}$*, these functorial algorithms are* **compatible** *[in the evident sense] with the ["non-vertically coric"] functorial algorithms of Proposition 3.7, (i), (ii), (iii), (iv) —*

i.e., where [in Proposition 3.7, (iii), (iv)] we take "†" to be "n, m" and "‡" to be "n, m − 1" — relative to the **Kummer isomorphisms** *of labeled data*

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

$$({}^{n,m'}\mathbb{M}^{\circledast}_{\mathrm{mod}})_j \stackrel{\sim}{\to} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j; \quad ({}^{n,m'}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \stackrel{\sim}{\to} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j$$

[cf. [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (ii)] and the evident identification, for $m' = m, m-1$, of ${}^{n,m'}\mathfrak{F}_t$ [i.e., the $\mathcal{F}$-prime-strip that appears in the associated $\Theta^{\pm}$-bridge] with the $\mathcal{F}$-prime-strip associated to $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$ [cf. Proposition 3.5, (i)]. In particular, for each $n, m \in \mathbb{Z}$, we obtain **"Kummer isomorphisms"** *of fields, monoids, Frobenioids, and $\mathcal{F}^{\Vdash}$-prime-strips*

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_\alpha \stackrel{\sim}{\to} \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_\alpha \stackrel{\sim}{\to} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_\alpha \stackrel{\sim}{\to} \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha \stackrel{\sim}{\to} \mathbb{M}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_\alpha \stackrel{\sim}{\to} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{mod}})_\alpha \stackrel{\sim}{\to} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{mod}})_\alpha \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad {}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \stackrel{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{gau}}$$

$${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \stackrel{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}; \quad {}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} \stackrel{\sim}{\to} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

that are **compatible** *with the various equalities, natural inclusions, and natural isomorphisms discussed above.*

(ii) **(Non-interference with Local Integers)** *In the notation of Propositions 3.2, (ii); 3.4, (i); 3.7, (i), (ii); 3.9, (iii), we have*

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha \bigcap \prod_{\underline{v}\in\underline{\mathbb{V}}} \Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} = ({}^{\dagger}\mathbb{M}^{\circledast\boldsymbol{\mu}}_{\mathrm{MOD}})_\alpha$$

$$\left(\subseteq \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) = \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) = \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \right)$$

— where we write $({}^{\dagger}\mathbb{M}^{\circledast\boldsymbol{\mu}}_{\mathrm{MOD}})_\alpha \subseteq ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha$ for the [finite] subgroup of torsion elements, i.e., **roots of unity***; for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, we identify the product $\prod_{\underline{\mathbb{V}}\ni\underline{v}|v_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ with $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Now let us think of the various* **groups**

$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_j$$

[of nonzero elements of a number field] as acting on various portions of the modules

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\mathbb{V}_{\mathbb{Q}}})$$

[i.e., where the subscript "$\mathbb{V}_{\mathbb{Q}}$" denote the direct product over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ — cf. the notation of Proposition 3.5, (ii)] **not** *via a* **single Kummer isomorphism** *as in (i), but rather via the* **totality** *of the various pre-composites of Kummer isomorphisms with* **iterates** *[cf. Remark 1.1.1] of the* **$\mathfrak{log}$-links** *of the* LGP*-Gaussian*

log-theta-lattice — where we observe that these actions are **mutually compatible** *up to [harmless!]* **"identity indeterminacies"** *at an adjacent "m", precisely as a consequence of the equality of the first display of the present (ii) [cf. the discussion of Remark 1.2.3, (ii); the discussion of Definition 1.1, (ii), concerning quotients by "$\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$; the discussion of Definition 1.1, (iv), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] — cf. also the discussion of Remark 3.11.4 below. Thus, one obtains a sort of* **"$\mathfrak{log}$-Kummer correspondence"** *between the* **totality***, as m ranges over the elements of $\mathbb{Z}$, of the various groups [of nonzero elements of a number field] just discussed [i.e., which are labeled by "n, m"] and their actions [as just described] on the "$\mathcal{I}^{\mathbb{Q}}$" labeled by "$n, \circ$" which is* **invariant** *with respect to the* **translation symmetries** *[cf. Proposition 1.3, (iv)] of the n-th column of the* LGP*-Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].*

(iii) **(Frobenioid-theoretic $\mathfrak{log}$-Kummer Correspondences)** *The relevant Kummer isomorphisms of (i) induce, via the "$\mathfrak{log}$-Kummer correspondence" of (ii) [cf. also Proposition 3.7, (i); Remarks 3.6.1, 3.9.2],* **isomorphisms of Frobenioids**

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha \ \stackrel{\sim}{\rightarrow}\ \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_\alpha \ \stackrel{\sim}{\rightarrow}\ \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

that are **mutually compatible***, as m varies over the elements of $\mathbb{Z}$, with the $\mathfrak{log}$-links of the* LGP*-Gaussian log-theta-lattice. Moreover, these compatible isomorphisms of Frobenioids, together with the relevant Kummer isomorphisms of (i), induce, via the* **global** *"$\mathfrak{log}$-Kummer correspondence" of (ii) and the* **splitting monoid** *portion of the "$\mathfrak{log}$-Kummer correspondence" of Proposition 3.5, (ii),* **isomorphisms of associated $\mathcal{F}^{\Vdash\perp}$-prime-strips** *[cf. Definition 2.4, (iii)]*

$${}^{n,m}\mathfrak{F}^{\Vdash\perp}_{\mathrm{LGP}} \ \stackrel{\sim}{\rightarrow}\ \mathfrak{F}^{\Vdash\perp}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$$

that are **mutually compatible***, as m varies over the elements of $\mathbb{Z}$, with the $\mathfrak{log}$-links of the* LGP*-Gaussian log-theta-lattice.*

Proof. The various assertions of Proposition 3.10 follow immediately from the definitions and the references quoted in the statements of these assertions. Here, we observe that the computation of the *intersection* of the first display of (ii) is an immediate consequence of the well-known fact that the set of nonzero elements of a number field that are *integral* at all of the places of the number field consists of the set of *roots of unity* contained in the number field [cf. the discussion of Remark 1.2.3, (ii); [Lang], p. 144, the proof of Theorem 5]. ○

Remark 3.10.1.

(i) Note that the $\mathfrak{log}$-*Kummer correspondence* of Proposition 3.10, (ii), induces isomorphisms of Frobenioids as in the first display of Proposition 3.10, (iii), precisely because the construction of "$({}^\dagger\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha$" only involves the group "$({}^\dagger\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha$", together with the collection of *subquotients* of its perfection indexed by $\underline{\mathbb{V}}$ [cf. Proposition 3.7, (i); Remarks 3.6.1, 3.9.2]. By contrast, the construction of "$({}^\dagger\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha$" also involves the *local monoids* "$\Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$" in an essential way [cf.

Proposition 3.7, (ii)]. These local monoids are subject to a somewhat more complicated "$\mathfrak{log}$-Kummer correspondence" [cf. Proposition 3.5, (ii)] that revolves around *"upper semi-compatibility"*, i.e., in a word, *one-sided inclusions*, as opposed to precise equalities. The imprecise nature of such one-sided inclusions is *incompatible* with the construction of "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha$". In particular, one cannot construct $\mathfrak{log}$-*link-compatible* isomorphisms of Frobenioids for "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha$" as in the first display of Proposition 3.10, (iii).

(ii) The **precise compatibility** of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" with the $\mathfrak{log}$-links of the LGP-Gaussian log-theta-lattice [cf. the discussion of (i); the first "mutual compatibility" of Proposition 3.10, (iii)] makes it more suited [i.e., by comparison to "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$"] to the task of computing the **Kummer-detachment indeterminacies** [cf. Remark 1.5.4, (i), (iii)] that arise when one attempts to pass from the *Frobenius-like structures* constituted by the global portion of the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*links* of the LGP-Gaussian log-theta-lattice to corresponding *étale-like structures*. That is to say, the mutual compatibility of the isomorphisms

$$^{n,m}\mathfrak{F}^{\Vdash\perp}_{\mathrm{LGP}} \overset{\sim}{\to} \mathfrak{F}^{\Vdash\perp}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$$

of the second display of Proposition 3.10, (iii), asserts, in effect, that such *Kummer-detachment indeterminacies do not arise.* This is precisely the reason why we wish to work with the LGP-, as opposed to the $\mathfrak{lgp}$-, Gaussian log-theta lattice [cf. Remark 3.8.1]. On the other hand, the essentially **multiplicative** nature of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" [cf. Remark 3.6.2, (ii)] makes it *ill-suited* to the task of computing the **étale-transport indeterminacies** [cf. Remark 1.5.4, (i), (ii)] that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(iii) By contrast, whereas the **additive nature** of the local modules [i.e., local fractional ideals] that occur in the construction of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" renders "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" ill-suited to the computation of Kummer-detachment indeterminacies [cf. the discussion of (i), (ii)], the *close relationship* [cf. Proposition 3.9, (i), (ii), (iii)] of these local modules to the **mono-analytic log-shells** that are **coric** with respect to the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Theorem 1.5, (iv); Remark 3.8.2] renders "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" *well-suited* to the computation of the **étale-transport indeterminacies** that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. That is to say, although various *distortions* of these local modules arise as a result of both [the Kummer-detachment indeterminacies constituted by] the local **"upper semi-compatibility"** of Proposition 3.5, (ii), and [the étale-transport indeterminacies constituted by] the **discrepancy** *between* **local holomorphic** *and* **mono-analytic** *integral structures* [cf. Remark 3.9.1, (i), (ii)], one may nevertheless compute — i.e., if one takes into account the various distortions that occur, **"estimate"** — the **global arithmetic degrees** of objects of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" by computing **log-volumes** [cf. Proposition 3.9, (iii)], which are **bi-coric**, i.e., coric with respect to both the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links [cf. Proposition 3.9, (ii)] and the $\mathfrak{log}$-links [cf. Proposition 3.9, (iv)] of the LGP-Gaussian log-theta-lattice. This **computability** is precisely the topic of Corollary 3.12 below. On the other hand, the issue of obtaining *concrete estimates* will be treated in [IUTchIV].

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$/<u>LGP-*structures*</u>	$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$/<u>$\mathfrak{lgp}$-*structures*</u>
biased toward **multiplicative** structures	biased toward **additive** structures
easily related to **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link	easily related to **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, i.e., **mono-analytic log-shells**
admits **precise $\mathfrak{log}$-Kummer correspondence**	only admits **"upper semi-compatible" $\mathfrak{log}$-Kummer correspondence**
rigid, but not suited to explicit computation	subject to **substantial distortion**, but suited to **explicit estimates**

Fig. 3.2: $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$/LGP-structures versus $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$/$\mathfrak{lgp}$-structures

(iv) The various properties of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" and "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" discussed in (i), (ii), (iii) above are summarized in Fig. 3.2 above. In this context, it is of interest to observe that the natural isomorphisms of Frobenioids

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha} \ \stackrel{\sim}{\to}\ \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha}$$

as well as the resulting isomorphisms of $\mathcal{F}^{\Vdash}$-prime-strips

$$\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}} \ \stackrel{\sim}{\to}\ \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

of Proposition 3.10, (i), play the *highly nontrivial* role of relating [cf. the discussion of [IUTchII], Remark 4.8.2, (i)] the "*multiplicatively biased* $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" to the "*additively biased* $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" by means of the **global ring structure** of the number field $\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha} = \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha}$. A similar statement holds concerning the tautological isomorphism of $\mathcal{F}^{\Vdash}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$ of Proposition 3.7, (iv).

Remark 3.10.2. In the context of the various *Kummer isomorphisms* discussed in the final display of Proposition 3.10, (i), it is useful to recall that the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$, ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$ that appear in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-*links* in Definition 3.8, (ii), were constructed from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ [associated to the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$] of [IUTchII], Corollary 4.10, (ii), in a

fashion that we review as follows. First, we remark that, in the present discussion, it is convenient for us to think of ourselves as working with objects arising from the LGP-*Gaussian log-theta-lattice* of Definition 3.8, (iii) [so "†" will be replaced by "(n, m)" or "$(n, \circ)$"]. Now recall, from the theory developed so far in the present series of papers, that we have a *commutative diagram of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips*

$$\begin{array}{ccccccc}
\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{env}} & \stackrel{\sim}{\rightarrow} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{gau}} & \stackrel{\sim}{\rightarrow} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{LGP}} & \stackrel{\sim}{\rightarrow} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathfrak{lgp}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
{}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} & \stackrel{\sim}{\rightarrow} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} & \stackrel{\sim}{\rightarrow} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} & \stackrel{\sim}{\rightarrow} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}
\end{array}$$

— where

· for simplicity, we use the abbreviated version "${}^{n,\circ}$" of the notation "${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$" of Proposition 3.10, (i);

· the *first vertical arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *Kummer isomorphism* [whose codomain includes an argument "${}^{\dagger}\mathfrak{D}_{>}$", which we denote here by "${}^{n,\circ}$"] of the final display of Proposition 2.1, (ii) [cf. also Proposition 2.1, (iii), (iv), (v)];

· the *second, third, and fourth vertical arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *Kummer isomorphisms* of the final display of Proposition 3.10, (i);

· the *first lower horizontal arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *evaluation isomorphism* of the final display of [IUTchII], Corollary 4.10, (ii);

· the *second and third lower horizontal arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *tautological isomorphisms* of the final displays of Proposition 3.7, (iii), (iv);

· the *first upper horizontal arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *étale-like evaluation isomorphism* implicit in the construction [via [IUTchII], Corollary 4.6, (iv), (v)] of the evaluation isomorphism of the final display of [IUTchII], Corollary 4.10, (ii);

· the *second and third upper horizontal arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *natural isomorphisms* of the second display of Proposition 3.10, (i).

That is to say, in summary,

> the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$ that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-*links* of Definition 3.8, (iii), were *constructed from the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ and *related to this* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ via the lower horizontal arrows of the above commutative diagram; moreover, each of these *lower horizontal arrows* may be constructed by *conjugating the corresponding upper horizontal arrow* by the relevant *Kummer isomorphisms*, i.e., by the *vertical arrows* in the diagram.

We are now ready to discuss the *main theorem* of the present series of papers.

Theorem 3.11. (Multiradial Algorithms via LGP-Monoids/Frobenioids) *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Let

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$$

be a **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from an* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. For each* $n \in \mathbb{Z}$, *write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, *where* $m \in \mathbb{Z}$, *via the* **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Multiradial Representation)** *Consider the* **procession of** $\mathcal{D}^{\vdash}$**-prime-strips** $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$

$$\{{}^{n,\circ}\mathfrak{D}^{\vdash}_0\} \hookrightarrow \{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_1\} \hookrightarrow \ \dots\ \hookrightarrow \{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_1,\ \dots,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_{l^*}\}$$

obtained by applying the natural functor of [IUTchI], Proposition 6.9, (ii), to [the $\mathcal{D}$-$\Theta^{\pm}$*-bridge associated to]* ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. *Consider also the following data:*

(a) for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $j \in |\mathbb{F}_l|$, *the* **topological modules** *and* **mono-analytic integral structures**

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \subseteq\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \quad \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \ \subseteq\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

— where the notation "$;n,\circ$" denotes the result of applying the construction in question to the case of $\mathcal{D}^{\vdash}$*-prime-strips labeled "$n,\circ$" — of Proposition 3.2, (ii) [cf. also the notational conventions of Proposition 3.4, (ii)], which we regard as equipped with the* **procession-normalized mono-analytic log-volumes** *of Proposition 3.9, (ii);*

(b) for $\underline{\mathbb{V}}^{\mathrm{bad}} \ni \underline{v}$, *the* **splitting monoid**

$$\Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

of Proposition 3.5, (ii), (c) [cf. also the notation of Proposition 3.5, (i)], which we regard — via the natural poly-isomorphisms

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \ \stackrel{\sim}{\to}\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \ \stackrel{\sim}{\to}\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}})$$

for $j \in \mathbb{F}_l^{\circledast}$ *[cf. Proposition 3.2, (i), (ii)] — as a* **subset** *of*

$$\prod_{j \in \mathbb{F}_l^{\circledast}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

equipped with a(n) [multiplicative] **action** *on* $\prod_{j \in \mathbb{F}_l^{\circledast}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$*;*

(c) *for* $j \in \mathbb{F}_l^{\circledast}$, *the* **number field**

$$\begin{aligned} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \; &= \; \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \\ &\subseteq \; \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) \; \stackrel{\mathrm{def}}{=} \; \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \end{aligned}$$

[cf. the natural poly-isomorphisms discussed in (b); Proposition 3.9, (iii); Proposition 3.10, (i)], together with **natural isomorphisms** *between the associated* **global non-realified/realified Frobenioids**

$$\mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \; \xrightarrow{\sim} \; \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j$$

$$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \; \xrightarrow{\sim} \; \mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j$$

[cf. Proposition 3.10, (i)], whose associated **"global degrees"** *may be computed by means of the* **log-volumes** *of (a) [cf. Proposition 3.9, (iii)].*

Write

$${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$$

for the **collection of data** *(a), (b), (c) regarded up to* **indeterminacies** *of the following two types:*

(Ind1) *the indeterminacies induced by the* **automorphisms** *of the* **procession of** $\mathcal{D}^{\vdash}$**-prime-strips** $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*;*

(Ind2) *for each* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *(respectively,* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*), the indeterminacies induced by the action of* **independent** *copies of* Ism *[cf. Proposition 1.2, (vi)] (respectively, copies of each of the automorphisms of order* 2 *whose orbit constitutes the poly-automorphism discussed in Proposition 1.2, (vii)) on each of the* **direct summands** *of the* $j+1$ **factors** *appearing in the tensor product used to define* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ *[cf. (a) above; Proposition 3.2, (ii)] — where we recall that the cardinality of the collection of direct summands is equal to the cardinality of the set of* $\underline{v} \in \underline{\mathbb{V}}$ *that lie over* $v_{\mathbb{Q}}$.

Then ${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$ *may be constructed via an* **algorithm** *in the procession of* $\mathcal{D}^{\vdash}$*-prime-strips* $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$ *that is* **functorial** *with respect to isomorphisms of processions of* $\mathcal{D}^{\vdash}$*-prime-strips. For* $n, n' \in \mathbb{Z}$*, the* **permutation symmetries** *of the* **étale-picture** *discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce* **compatible poly-isomorphisms**

$$\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T) \; \xrightarrow{\sim} \; \mathrm{Prc}({}^{n',\circ}\mathfrak{D}^{\vdash}_T); \quad {}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}} \; \xrightarrow{\sim} \; {}^{n',\circ}\mathfrak{R}^{\mathrm{LGP}}$$

which are, moreover, compatible with the poly-isomorphisms

$$^{n,\circ}\mathfrak{D}_0^{\vdash} \ \stackrel{\sim}{\to} \ ^{n',\circ}\mathfrak{D}_0^{\vdash}$$

induced by the **bi-coricity** *poly-isomorphisms of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].*

(ii) **($\mathfrak{log}$-Kummer Correspondence)** *For $n, m \in \mathbb{Z}$, the* **Kummer isomorphisms** *of labeled data*

$$\Psi_{\text{cns}}(^{n,m}\mathfrak{F}_{\succ})_t \ \stackrel{\sim}{\to} \ \Psi_{\text{cns}}(^{n,\circ}\mathfrak{D}_{\succ})_t$$

$$\{\pi_1^{\kappa\text{-sol}}(^{n,m}\mathcal{D}^{\circledast}) \ \curvearrowright \ ^{n,m}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \ \stackrel{\sim}{\to} \ \{\pi_1^{\kappa\text{-sol}}(^{n,\circ}\mathcal{D}^{\circledast}) \ \curvearrowright \ \mathbb{M}^{\circledast}_{\infty\kappa}(^{n,\circ}\mathcal{D}^{\circledcirc})\}_j$$

$$(^{n,m}\overline{\mathbb{M}}^{\circledast}_{\text{mod}})_j \stackrel{\sim}{\to} \overline{\mathbb{M}}^{\circledast}_{\text{mod}}(^{n,\circ}\mathcal{D}^{\circledcirc})_j$$

— *where $t \in \text{LabCusp}^{\pm}(^{n,\circ}\mathfrak{D}_{\succ})$ — of [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (i), (ii) [cf. also Propositions 3.5, (i); 3.10, (i), of the present paper] induce* **isomorphisms** *between the* **vertically coric** *data (a), (b), (c) of (i) [which we regard, in the present (ii), as data which has* **not yet** *been subjected to the indeterminacies* (Ind1), (Ind2) *discussed in (i)] and the corresponding data arising from each $\Theta^{\pm\text{ell}}$NF-Hodge theater $^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$, i.e.:*

(a) *for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$, $j \in |\mathbb{F}_l|$, isomorphisms with* **local mono-analytic tensor packets** *and their $\mathbb{Q}$-spans*

$$\mathcal{I}(^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \mathcal{I}(^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \mathcal{I}(^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

$$\mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \ \stackrel{\sim}{\to} \ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

$$\mathcal{I}(^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}_{\underline{v}}) \ \stackrel{\sim}{\to} \ \mathcal{I}(^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \ \stackrel{\sim}{\to} \ \mathcal{I}(^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

$$\mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}_{\underline{v}}) \ \stackrel{\sim}{\to} \ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \ \stackrel{\sim}{\to} \ \mathcal{I}^{\mathbb{Q}}(^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

[cf. Propositions 3.2, (i), (ii); 3.4, (ii); 3.5, (i)], all of which are **compatible** *with the respective* **log-volumes** *[cf. Proposition 3.9, (ii)];*

(b) *for $\underline{\mathbb{V}}^{\text{bad}} \ni \underline{v}$, isomorphisms of* **splitting monoids**

$$\Psi^{\perp}_{\mathcal{F}_{\text{LGP}}}(^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}} \ \stackrel{\sim}{\to} \ \Psi^{\perp}_{\text{LGP}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}$$

[cf. Proposition 3.5, (i); Proposition 3.5, (ii), (c)];

(c) *for $j \in \mathbb{F}_l^{*}$, isomorphisms of* **number fields** *and* **global non-realified/ realified Frobenioids**

$$(^{n,m}\overline{\mathbb{M}}^{\circledast}_{\text{MOD}})_j \ \stackrel{\sim}{\to} \ \overline{\mathbb{M}}^{\circledast}_{\text{MOD}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j; \quad (^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_j \ \stackrel{\sim}{\to} \ \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j$$

$$(^{n,m}\mathcal{F}^{\circledast}_{\text{MOD}})_j \ \stackrel{\sim}{\to} \ \mathcal{F}^{\circledast}_{\text{MOD}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j; \quad (^{n,m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \ \stackrel{\sim}{\to} \ \mathcal{F}^{\circledast}_{\mathfrak{mod}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j$$

$$(^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\text{MOD}})_j \ \stackrel{\sim}{\to} \ \mathcal{F}^{\circledast\mathbb{R}}_{\text{MOD}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j; \quad (^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j \ \stackrel{\sim}{\to} \ \mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}(^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j$$

which are compatible with the respective natural isomorphisms between "MOD"- and "$\mathfrak{mod}$"-subscripted versions [cf. Proposition 3.10, (i)]; here, the isomorphisms of the third line of the display induce isomorphisms of the **global realified Frobenioid** *portions*

$$^{n,m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}); \quad {}^{n,m}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$$

of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, and $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. Propositions 3.7, (iii), (iv), (v); 3.10, (i)].

Moreover, as one varies $m \in \mathbb{Z}$, the various isomorphisms of (b) and of the first line in the first display of (c) are **mutually compatible** *with one another, relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the domains of these isomorphisms that are possibly related to one another via the $\mathfrak{log}$-links consist of* **roots of unity** *in the domains of the $\mathfrak{log}$-links [multiplication by which corresponds, via the $\mathfrak{log}$-link, to an* **"addition by zero"** *indeterminacy, i.e., to* **no indeterminacy**! *] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii). This mutual compatibility of the isomorphisms of the first line in the first display of (c) implies a corresponding* **mutual compatibility** *between the isomorphisms of the second and third lines in the first display of (c) that* **involve the subscript "MOD"** *[but* **not** *between the isomorphisms that involve the subscript "$\mathfrak{mod}$"! — cf. Proposition 3.10, (iii); Remark 3.10.1]. On the other hand, the isomorphisms of (a) are subject to a certain* **"indeterminacy"** *as follows:*

(Ind3) *as one varies $m \in \mathbb{Z}$, the isomorphisms of (a) are* **"upper semi-compatible"**, *relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in a sense that involves certain* **natural inclusions "$\subseteq$"** *at $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ and certain* **natural surjections "$\twoheadrightarrow$"** *at $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$ — cf. Proposition 3.5, (ii), (a), (b), for more details.*

Finally, as one varies $m \in \mathbb{Z}$, the isomorphisms of (a) are [precisely!] **compatible**, *relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, with the respective* **log-volumes** *[cf. Proposition 3.9, (iv)].*

(iii) **($\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-Link Compatibility)** *The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various* **horizontal arrows** *— i.e., $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-links — of the* LGP*-Gaussian log-theta-lattice under consideration as follows:*

(a) *The first Kummer isomorphism of the first display of (ii) induces — by applying the $\mathbb{F}_l^{\rtimes \pm}$-***symmetry** *of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ — a* **Kummer isomorphism** ${}^{n,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle})$ *[cf. Theorem 1.5, (iii)]. Relative to this Kummer isomorphism, the full poly-isomorphism of $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strips*

$$\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n+1,\circ}\mathfrak{D}^{\vdash}_{\triangle})$$

is **compatible** *with the full poly-isomorphism of $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strips*

$$^{n,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration [cf. Theorem 1.5, (iii)].*

(b) The $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$, $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})$ [cf. Proposition 2.1, (ii)] that appear **implicitly** *in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)] admit* **natural isomorphisms** *of associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips ${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}$, $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})$ [cf. Proposition 2.1, (vi)]. Relative to these natural isomorphisms and to the Kummer isomorphism discussed in (a) above, the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n+1,\circ}\mathfrak{D}_{>})$$

is **compatible** *with the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration [cf. Corollary 2.3, (iii)].*

(c) Recall the data "${}^{n,\circ}\mathfrak{R}$" [cf. Corollary 2.3, (ii)] associated to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ — data which appears **implicitly** *in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)]. This data that arises from ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is related to corresponding data that arises from the projective system of mono-theta environments associated to the tempered Frobenioids of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ via the* **Kummer isomorphisms** *and* **poly-isomorphisms of projective systems of mono-theta environments** *discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii)] and Theorem 1.5, (iii) [cf. also (a), (b) above], (v). The* **algorithmic construction** *of these Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments, as well as of the poly-isomorphism*

$${}^{n,\circ}\mathfrak{R} \xrightarrow{\sim} {}^{n+1,\circ}\mathfrak{R}$$

induced by any **permutation symmetry** *of the* **étale-picture** *[cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2] ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is* **compatible** *with the* **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration, e.g., with the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by these horizontal arrows [cf. Corollary 2.3, (iv)], in the sense that these constructions are **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP*-Gaussian log-theta-lattice. Finally, the* **algorithmic construction** *of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various* **evaluation** *maps implicit in the portion of the* **$\mathfrak{log}$-Kummer correspondence** *discussed in (ii), (b), are* **compatible** *with the* **horizontal arrows** *of the* LGP*-Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP*-Gaussian log-theta-lattice.*

(d) *The* **algorithmic construction** *of the Kummer isomorphisms of the first display of (ii) [cf. also (a), (b) above; the* **gluing** *discussed in [IUTchII], Corollary 4.6, (iv); the Kummer compatibilities discussed in [IUTchII], Corollary 4.8, (iii); the relationship to the notation of [IUTchI], Definition 5.2, (vi), (viii), referred to in [IUTchII], Propositions 4.2, (i), and 4.4, (i)], as well as of the poly-isomorphisms between the data*

$$\left[\begin{array}{l}\{\pi_1^{\kappa\text{-sol}}({}^{n,\circ}\mathcal{D}^{\circledast}) \ \curvearrowright \ \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n,\circ}\mathcal{D}^{\circledcirc})\}_j \\ \quad \to \quad \mathbb{M}_{\infty\kappa v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j}) \ \subseteq \ \mathbb{M}_{\infty\kappa\times v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j})\end{array}\right]_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\overset{\sim}{\to} \quad \left[\begin{array}{l}\{\pi_1^{\kappa\text{-sol}}({}^{n+1,\circ}\mathcal{D}^{\circledast}) \ \curvearrowright \ \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n+1,\circ}\mathcal{D}^{\circledcirc})\}_j \\ \quad \to \quad \mathbb{M}_{\infty\kappa v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j}) \ \subseteq \ \mathbb{M}_{\infty\kappa\times v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j})\end{array}\right]_{\underline{v}\in\underline{\mathbb{V}}}$$

[i.e., of the second line of the first display of [IUTchII], Corollary 4.7, (iii)] induced by any **permutation symmetry** *of the* **étale-picture** *[cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2]* ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *are* **compatible** *[cf. the discussion of Remark 2.3.2] with the full poly-isomorphism of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \overset{\sim}{\to} \ {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP*-Gaussian log-theta-lattice under consideration, in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP*-Gaussian log-theta-lattice. Finally, the* **algorithmic construction** *of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various* **evaluation** *maps implicit in the portion of the* **$\mathfrak{log}$-Kummer correspondence** *discussed in (ii), (c), are* **compatible** *with the* **horizontal arrows** *of the* LGP*-Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions*

are **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP-*Gaussian log-theta-lattice.*

Proof. The various assertions of Theorem 3.11 follow immediately from the definitions and the references quoted in the statements of these assertions — cf. also the various related observations of Remarks 3.11.1, 3.11.2, 3.11.3, 3.11.4 below. ◯

Remark 3.11.1.

(i) One way to summarize the content of Theorem 3.11 is as follows:

Theorem 3.11 gives an **algorithm** for describing, up to certain relatively **mild indeterminacies**, the **LGP-monoids** [cf. Fig. 3.1] — i.e., in essence, the **theta values**

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*}$$

— which are constructed relative to the **scheme/ring structure**, i.e., **"arithmetic holomorphic structure"**, associated to *one* vertical line [i.e., "$(n, \circ)$" for some *fixed* $n \in \mathbb{Z}$] in the LGP-Gaussian log-theta-lattice under consideration, in terms of the *a priori* **alien** arithmetic holomorphic structure of *another* vertical line [i.e., "$(n+1, \circ)$"] in the LGP-Gaussian log-theta-lattice under consideration [cf., especially, the final portion of Theorem 3.11, (i), concerning **functoriality** and *compatibility* with the **permutation symmetries** of the **étale-picture**].

This point of view is consistent with the point of view of the discussion of Remark 1.5.4; [IUTchII], Remark 3.8.3, (iii).

(ii) Although the various versions of the Θ-*link* are defined [cf. Definition 3.8, (ii)] as *gluings* of

the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* whose associated pilot object [cf. [IUTchII], Definition 4.9, (viii)] is some sort of Θ-*pilot object* in the *domain* of the Θ-link

to

the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* whose associated pilot object is some sort of *q-pilot object* in the *codomain* of the Θ-link,

in fact it is not difficult to see that the theory developed in the present series of papers remains **essentially unaffected**

even if one **replaces** this q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip in the *codomain of the* Θ-*link* by **some other** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**

such as, for instance, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip whose associated pilot object is the q^λ-*pilot object* [i.e., the λ-th power of the q-pilot object, for some positive integer $\lambda > 1$] — cf. the discussion of Remark 3.12.1, (ii), below. One way to formulate this observation is as follows: The Θ-*link compatibility* described in Theorem 3.11, (iii), may be interpreted as an assertion to the effect that the **functorial construction**

algorithm for the Θ-*pilot object* up to certain *mild indeterminacies* [i.e., (Ind1), (Ind2), (Ind3)] that is given in Theorem 3.11 may be regarded as

> an algorithm whose **input data** is an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** [i.e., the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip that appears in the *codomain of the* Θ-*link*], and whose **functoriality** is with respect to **arbitrary isomorphisms** of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* that appear as input data of the algorithm.

From the point of view of the **gluing** given by the Θ-*link*, this *functoriality* in the *input data* given by an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* may be interpreted in the following way:

> this **functoriality** allows one to regard the **functorial construction algorithm** for the Θ-*pilot object* up to certain *mild indeterminacies* that is given in Theorem 3.11 as an algorithm with respect to which the *codomain* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* of the Θ-link [together with the other $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters in the *same vertical line* of the log-theta-lattice as this *codomain* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater*] — i.e., in effect, the $\boldsymbol{q}$**-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, equipped with the *rigidification* determined by the **arithmetic holomorphic structure** constituted by this *vertical line* of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters — is **"coric"**, i.e., "remains **invariant**" / "may be regarded as being held **fixed**" throughout the execution of the various operations of the algorithm.

This interpretation will play a *crucial role* in the application of Theorem 3.11 to Corollary 3.12 below.

(iii) On the other hand, the **étale-picture permutation symmetries** discussed in the final portion of Theorem 3.11, (i) [cf. also the references to these symmetries in Theorem 3.11, (iii), (c), (d)], may be interpreted as follows: The **output data** of the **functorial construction algorithm** of Theorem 3.11 consists of a representation of the data of Theorem 3.11, (i), (b), (c) [cf. also Theorem 3.11, (iii), (c), (d)], up to certain *mild indeterminacies* on the *mono-analytic étale-like log-shells* of Theorem 3.11, (i), (a), that satisfies the following *properties*:

· **(Input prime-strip link (IPL))** This output data is constructed in such a way that it is **linked/related**, via full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** induced by operations in the algorithm, to the **input data prime-strip**, i.e., the **"coric"/"fixed"** $\boldsymbol{q}$**-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, equipped with its *rigidifying* **arithmetic holomorphic structure** [cf. the discussion of (ii)]. In particular, we note that *each of these "intermediate"* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* that appears in the construction may *itself* be taken to be **both**

> the **input data** of the **functorial algorithm** of Theorem 3.11 [cf. the discussion of (ii)]

and

> [by applying the full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that link/relate it to the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip] the **input data** for the **Kummer theory** surrounding the $\boldsymbol{q}$**-pilot object** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** in its rigidifying $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. the discussion of (ii)].

At a more *explicit level*, the linking isomorphisms of "intermediate" $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips are given by *composing*

- the *inverses* of the *first two lower horizontal arrows* of the commutative diagram of Remark 3.10.2, followed by
- the *first vertical arrow* of this diagram — corresponding to the **Kummer theory** portion of Theorem 3.11, (iii), (c), (d) — followed by
- the *three upper horizontal arrows* of the diagram — corresponding to the **evaluation map** portion of Theorem 3.11, (iii), (c), (d).

Here, we observe that the *final evaluation map* portion of this composite involves a construction of the Θ-*pilot object* up to certain indeterminacies [i.e., (Ind1), (Ind2), (Ind3)], which, by applying the discussion of Remark 2.4.2, (v), (vi), may be *interpreted* — *provided* that certain *sign conditions* [cf. the discussion of Remark 2.4.2, (iv), (vi)] are satisfied, and one takes into account the considerations discussed in Remarks 3.9.6 [concerning the *product formula*], 3.9.7 [concerning *inverse systems of direct product regions*] — as a construction of the *global realified Frobenioid* portion of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip, together with *various possibilities* [corresponding to the indeterminacies] for the *"further rigidification"* determined by the pilot object.

- **(Simultaneous holomorphic expressibility (SHE))** The construction of this output data, as well as the output data itself, is expressed in terms that are **simultaneously** valid/executable/well-defined relative to **both**

 the **arithmetic holomorphic structure** that gives rise to the **Θ-pilot object** in the *domain* of the Θ-link — i.e., in more technical language, in terms of/as a **function** of structures in the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the domain of the Θ-link —

 and

 the **arithmetic holomorphic structure** that gives rise to the **input data prime-strip** [i.e., such as the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip, as discussed in (ii)] in the *codomain* of the Θ-link — i.e., in more technical language, in terms of/as a **function** of structures in the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the codomain of the Θ-link.

 In passing, we observe that this property "SHE" may be understood, in a slightly more concrete way, as corresponding to the fact that the **chain** of (sub)quotients considered in Remark 3.9.5, (viii), (ix), forms a **closed loop**.

These two *fundamental properties* of the output data of the algorithm of Theorem 3.11 will play a *central role* in the application of Theorem 3.11 to Corollary 3.12 below. In the context of these two fundamental properties, it is interesting to observe that, relative to the analogy between *multiradiality* and *crystals/connections* [cf. [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii), (iii)],

the distinction between *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* and various *specific realizations of such* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [e.g., arising from the structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater]

may be understood as corresponding to

the distinction between *reduced characteristic* p *schemes* [where p is a prime number] and *thickenings of such schemes over* $\mathbb{Z}_p$

in the context of p-adic crystals.

(iv) The **SHE** property discussed in (iii) may be thought of as a sort of *"parallel transport"* mechanism for the Θ-pilot object [cf. the analogy between *multiradiality* and *connections*, as discussed in [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii)], up to certain mild indeterminacies, from the [arithmetic holomorphic structure represented by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the] *domain* of the Θ-link to the [arithmetic holomorphic structure represented by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the] *codomain* of the Θ-link. On the other hand, in this context, it is important to observe that:

· **(Algorithmic parallel transport (APT))** This parallel transport mechanism does **not** consist of a simple instance of transport of *some set-theoretic region* [such as the region in the tensor packet of log-shells determined by the Θ*-pilot object* in the *domain* of the Θ-link] *via some set-theoretic function*. Rather, it consists of a **construction algorithm** that is **simultaneously valid/executable/well-defined** with respect to the arithmetic holomorphic structures in the domain and codomain of the Θ-link [cf. the discussion of (iii)].

[In this context, it is important to remember that although this *construction algorithm* may yield, as *output*, various *"possible regions"*, such possible regions **cannot** necessarily be **directly compared** with various structures in the *codomain* of the Θ-link. That is to say, such *comparisons* typically require the application of further techniques, as discussed in Remark 3.9.5, (vii).] In particular, if one takes the point of view — *as will be done in Corollary 3.12 below!* — that one is only interested in considering the **qualitative logical aspects/consequences** of the *construction algorithm* of Theorem 3.11, then:

· **(Hidden internal structures (HIS))** One may [and, indeed, it is often useful to] regard this *construction algorithm* of Theorem 3.11 as a construction algorithm for producing **"some sort of output data"** satisfying various properties [cf. (iii)] associated to **"some sort of input data"** [cf. (ii)] and **forget** that this construction algorithm of Theorem 3.11 has *anything to do with* **theta functions** *[e.g., the theory of [EtTh]]* *or* **theta values** *[i.e., the* $\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$*]*!

That is to say, theta functions/theta values may be regarded as **HIS** of the construction algorithm of Theorem 3.11 — somewhat like the internal structure of the *CPU or operating system of a computer*! — i.e., internal structures whose technical details are [of course, of *crucial importance* from the point of view of the *actual functioning of the construction algorithm*, but nonetheless] *irrelevant* or *uninteresting* from the point of view of the *"end user"*, who is only interested in applying the

construction algorithm to certain *input data* to obtain certain *output data*. [Here, we observe in passing that, relative to this analogy with the internal structure of the *CPU or operating system of a computer*, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that occur in the Θ-link may be thought of as a sort **connecting cable**, i.e., of the sort that is used to link distinct computers via the **internet**. That is to say, despite the fact that such a connecting cable may have a *very simple internal structure* by comparison to the computers that it connects, the connection that it furnishes has *highly nontrivial consequences* [e.g., as in the case of the *internet*!] — cf. the discussion in (iii) of the **input prime-strip link (IPL)** and the analogy with **crystals/connections**.] On the other hand, we observe that, unlike Corollary 3.12 below, which only concerns *qualitative logical aspects/consequences* of the construction algorithm of Theorem 3.11, the **explicit computation** to be performed in [IUTchIV], §1, of the **log-volumes** that occur in the statement of Corollary 3.12 makes *essential use* of the way in which **theta values** occur in the construction algorithm of Theorem 3.11.

(v) Thus, in summary, the above discussion yields a slightly different, and in some sense more detailed, way [by comparison to (i)] to summarize the content of the construction algorithm of Theorem 3.11 [cf. also the discussion of Remark 3.12.2, (ii), below]: The *functorial construction algorithm* of Theorem 3.11 is an algorithm whose

· **input data** consists solely of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, regarded up to **isomorphism** [cf. (ii)], and whose

· **output data** consists of certain data that is **linked/related**, via full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** induced by operations in the algorithm, to the **input data prime-strip**, and, moreover, whose construction algorithm may be expressed in terms that are **simultaneously** valid/executable/well-defined relative to **both** the *arithmetic holomorphic structure* that gives rise to the Θ*-pilot object* in the *domain* of the Θ-link **and** the *arithmetic holomorphic structure* that gives rise to the **input data prime-strip** [i.e., such as the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip].

This construction algorithm of Theorem 3.11 makes *crucial use* of certain **HIS** such as *theta functions* and *theta values*, but these HIS may be **ignored**, if one is only interested in the qualitative logical aspects/consequences of the input and output data of the algorithm.

(vi) In the context of the **input prime-strip link (IPL)** and **simultaneous holomorphic expressibility (SHE)** properties discussed in (iii), it is perhaps of interest to consider what happens in the case of the very *simple, naive example* discussed in Remark 2.2.2, (i). That is to say, suppose that one considers the **"naive version"** of the **Θ-link** given by a correspondence of the form

$$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$$

— where $\lambda > 1$ is a *positive integer* — relative to a **single** arithmetic holomorphic structure, i.e., in effect, *ring structure* "R". [Here, we remark that, unlike the situation considered in the discussion of (ii), where "$\underline{\underline{q}}^{\lambda}$" appears in the *codomain* of some modified version of the Θ-link, the "$\underline{\underline{q}}^{\lambda}$" in the present discussion appears in the *domain* of some modified version of the Θ-link.] Then the *very definition* of

this *naive version* of the Θ-link yields an *explicit construction algorithm* for "$\underline{\underline{q}}^\lambda$", namely, as the λ-th power of "$\underline{\underline{q}}$". That is to say, this [essentially tautological!] explicit construction algorithm for "$\underline{\underline{q}}^\lambda$" satisfies the **SHE** property considered in (iii) in the sense that

> the *tautological construction algorithm* given by taking "the λ-th power of $\underline{\underline{q}}$" may be regarded as *simultaneously executable* relative to *both* the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to "$\underline{\underline{q}}$" *and* the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to "$\underline{\underline{q}}^\lambda$".

On the other hand, we observe that this sort of [essentially tautological!] SHE property is achieved as the cost of **sacrificing** the establishment of the analogue of the **IPL** property of (iii), in the sense that

> if one restricts oneself to considering "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^\lambda$" inside the **fixed container** constituted by the given **arithmetic holomorphic structure** [i.e., in effect, ring structure "R"] that gives rise to "$\underline{\underline{q}}$", then the tautological construction algorithm considered above does **not induce** any sort of **identification** between "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^\lambda$".

(vii) We maintain the notation of (vi). One may then approach the issue of establishing the analogue of the **IPL** property of (iii) by introducing a **formal symbol** "$*$" [corresponding to the abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the Θ-link] and then considering one of the following *two approaches*:

- **(Distinct labels)** It is essentially a tautology that in order to consider **both** of the assignments $* \mapsto \underline{\underline{q}}$ and $* \mapsto \underline{\underline{q}}^\lambda$ **simultaneously** [i.e., in order to establish the analogue of the IPL property of (iii)!], it is necessary to introduce **distinct labels** "†" and "‡" for the arithmetic holomorphic structures [i.e., in effect, ring structures] that give rise to "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^\lambda$", respectively. That is to say, it is a tautology that one may consider the assignments

$$* \mapsto {}^{\ddagger}\underline{\underline{q}}^\lambda, \quad * \mapsto {}^{\dagger}\underline{\underline{q}}$$

simultaneously and **without** introducing any **inconsistencies**. On the other hand, this approach via the introduction of *tautologically distinct labels* — which may be summarized via the diagram

$$\begin{array}{ccccc} * & \mapsto & {}^{\ddagger}\underline{\underline{q}}^\lambda \in & & {}^{\ddagger}R \\ & & & & \vdots \\ \| & & & & ?? \\ & & & & \vdots \\ * & \mapsto & {}^{\dagger}\underline{\underline{q}} \in & & {}^{\dagger}R \end{array}$$

IPL: holds
SHE: ??

— has the *drawback* that it is by no means clear, at least in any *a priori* sense, how to establish the analogue of the **SHE** property of (iii), since it

is by no means clear, at least in any *a priori* sense, *how to "compute" the relationship between the "†" and "‡" arithmetic holomorphic structures [i.e., in effect, ring structures].*

· **(Forced identification of arithmetic holomorphic structures)** Of course, one may then attempt to remedy the *drawback* that appeared in the *distinct labels approach* by simply **arbitrarily identifying** the "†" and "‡" arithmetic holomorphic structures [i.e., in effect, ring structures], that is to say, by simply **deleting/forgetting** the distinct labels "†" and "‡". This approach — which may be summarized via the diagram

$$\begin{array}{ccccc} * & \mapsto & \underline{\underline{q}}^{\lambda} \in & R \\ \vdots & & & \\ ?? & & & \parallel \\ \vdots & & & \\ * & \mapsto & \underline{\underline{q}} \in & R \end{array}$$

IPL: ??
SHE: holds

— allows one to apply the [tautological!] *construction algorithm* discussed in (vi). On the other hand, this approach has the *drawback* that, in order to consider the assignments

$$* \mapsto \underline{\underline{q}}^{\lambda}, \quad * \mapsto \underline{\underline{q}}$$

simultaneously and **consistently** [i.e., in order to establish the analogue of the IPL property of (iii)!], one is led [at least in the absence of more sophisticated machinery!] to regard "$\underline{\underline{q}}$" as being **only well-defined up to possible confusion with "$\underline{\underline{q}}^{\lambda^n}$"**, for some indeterminate $n \in \mathbb{Z}$. That is to say, in summary, this approach gives rise to a sort of **"uninteresting/trivial multiradial representation** of "$\underline{\underline{q}}^{\lambda}$" via

$$\text{"}\{\underline{\underline{q}}^{\lambda^n}\}_{n\in\mathbb{Z}}\text{"}$$

— which [despite being uninteresting/trivial!] does indeed satisfy the *formal analogues* of the *IPL* and *SHE* properties of (iii).

(viii) We conclude our discussion of the *simple, naive examples* discussed in (vi) and (vii) by considering the *relationship* between these simple, naive examples and the *theory of the present series of papers*. We begin by observing that the *"trivial multiradial representation* $\{\underline{\underline{q}}^{\lambda^n}\}_{n\in\mathbb{Z}}$*"* discussed in (vii) is, on the one hand, of interest, in the context of the *IPL* and *SHE* properties of (iii), in that it constitutes a **useful elementary "toy model"** for considering the **qualitative logical aspects** of these fundamental properties satisfied by the *multiradial construction algorithm* of Theorem 3.11. On the other hand, this "trivial multiradial representation" is **useless** from the point of view of applications such as the log-volume estimates given in Corollary 3.12 below [cf. the discussion of the final portion of (iv)]

for the following reasons: This **"trivial multiradial representation $\{\underline{\underline{q}}^{\lambda^n}\}_{n\in\mathbb{Z}}$"** is obtained by

· allowing for **indeterminacies** in the **value group** portion [i.e., "$\underline{\underline{q}}^{\mathbb{Z}}$"] of the data under consideration,

· while the **unit group** portion [i.e., the "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" associated to the local fields that appear] of the data under consideration is held **rigid** [i.e., not subject to indeterminacies];

· only working with the **multiplicative structure** constituted by the **value group** portion of the rings involved, and

· **ignoring** issues related to the **additive structure** of the rings involved, especially, issues related to the **intertwining** between the *additive* and *multiplicative* structures of these rings [cf. the discussion of Remark 3.12.2, (ii), below].

By contrast, the *log-volume estimates* of Corollary 3.12 below rely, in an essential way, on the fact that in the **multiradial construction algorithm** of Theorem 3.11:

· the **value group** portions of the data under consideration [i.e., the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips associated to the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the Θ-link] are held **rigid** [i.e., are not subject to indeterminacies],

· while the **unit group** portions of the data under consideration [i.e., the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips associated to the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the Θ-link] are subject to the **indeterminacies** (Ind1), (Ind2), (Ind3);

· the multiradial construction algorithm makes use, via the **$\mathfrak{log}$-Kummer correspondence**, of the structure of the **intertwining** between the **additive** and **multiplicative** structures of the rings involved [cf. the discussion of Remark 3.12.2, (ii), (iii), (iv), (v), below].

Finally, we observe that the technique of assigning **distinct labels** that appears in the *distinct labels approach* discussed in (vii) is *formalized* in the theory of the present series of papers by means of the notion of **Frobenius-like structures**, i.e., at a more concrete level, mathematical objects that, at least *a priori*, only make sense within the $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater labeled "*(n,m)*"* [where $n,m\in\mathbb{Z}$] of the *log-theta-lattice*. The problem of relating objects arising from $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters with *distinct labels "*(n,m)*"* is then resolved in the present series of papers — **not** by means of **"forced identification"** [i.e., in the style of the discussion of (vii)] of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters with distinct labels, but rather — by considering the **permutation symmetries** [i.e., of the sort discussed in the final portion of Theorem 3.11, (i)] satisfied by **étale-like structures**. Here, it is perhaps useful to recall that the *fundamental model* for such permutation symmetries is, in the notation of [IUTchII], Example 1.8, (i),

$$\Pi \quad\longrightarrow\quad G \quad\longleftarrow\quad \Pi$$

— where the arrows "$\longrightarrow$" and "$\longleftarrow$" denote the poly-morphism given by composing the natural surjection $\Pi \twoheadrightarrow \Pi/\Delta$ with the full poly-isomorphism $\Pi/\Delta \overset{\sim}{\to} G$,

and we observe that the diagram of this display admits a *permutation symmetry* that switches these two arrows "$\longrightarrow$" and "$\longleftarrow$".

Remark 3.11.2.

(i) In Theorem 3.11, (i), we do not apply the *formalism* or *language* developed in [IUTchII], §1, for discussing multiradiality. Nevertheless, the approach taken in Theorem 3.11, (i) — i.e., by regarding the collection of data (a), (b), (c) up to the indeterminacies given by (Ind1), (Ind2) — to constructing **"multiradial representations"** amounts, in essence, to a special case of the **tautological** approach to constructing multiradial environments discussed in [IUTchII], Example 1.9, (ii). That is to say, this tautological approach is applied to the **vertically coric** constructions of Proposition 3.5, (i); 3.10, (i), which, *a priori*, are *uniradial* in the sense that they depend, in an essential way, on the **arithmetic holomorphic structure** constituted by a *particular* vertical line — i.e., "$(n, \circ)$" for some *fixed* $n \in \mathbb{Z}$ — in the LGP-Gaussian log-theta-lattice under consideration.

(ii) One important underlying aspect of the *tautological approach to multiradiality* discussed in (i) is the treatment of the various **labels** that occur in the **multiplicative** and **additive combinatorial Teichmüller theory** associated to the $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theater ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}$ under consideration [cf. the theory of [IUTchI], §4, §6]. The various transitions between types of labels is illustrated in Fig. 3.3 below. Here, we recall that:

(a) the passage from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** to **labels** $\in \mathbb{F}_l$ forms the content of the associated $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-*Hodge theater* [cf. [IUTchI], Remark 6.6.1];

(b) the passage from **labels** $\in \mathbb{F}_l$ to **labels** $\in |\mathbb{F}_l|$ forms the content of the *functorial algorithm* of [IUTchI], Proposition 6.7;

(c) the passage from **labels** $\in |\mathbb{F}_l|$ to $\pm$**-processions** forms the content of [IUTchI], Proposition 6.9, (ii);

(d) the passage from the $\mathbb{F}_l^{\circledast}$**-symmetry** to **labels** $\in \mathbb{F}_l^{\circledast}$ forms the content of the associated $\mathcal{D}$-Θ*NF-Hodge theater* [cf. [IUTchI], Remark 4.7.2, (i)];

(e) the passage from **labels** $\in \mathbb{F}_l^{\circledast}$ to $\circledast$**-processions** forms the content of [IUTchI], Proposition 4.11, (ii);

(f) the compatibility between $\circledast$**-processions** and $\pm$**-processions**, relative to the natural inclusion of **labels** $\mathbb{F}_l^{\circledast} \hookrightarrow |\mathbb{F}_l|$, forms the content of [IUTchI], Proposition 6.9, (iii).

Here, we observe in passing that, in order to perform these various *transitions*, it is absolutely necessary to work with *all of the labels in* $\mathbb{F}_l$ *or* $|\mathbb{F}_l|$, i.e., one does not have the option of *"arbitrarily omitting certain of the labels"* [cf. the discussion of [IUTchII], Remark 2.6.3; [IUTchII], Remark 3.5.2]. Also, in this context, it is important to note that there is a fundamental difference between the **labels** $\in \mathbb{F}_l, |\mathbb{F}_l|, \mathbb{F}_l^{\circledast}$ — which are essentially **arithmetic holomorphic** in the sense that they depend, in an essential way, on the various local and global *arithmetic fundamental groups* involved — and the **index sets of the mono-analytic** $\pm$**-processions**

that appear in the multiradial representation of Theorem 3.11, (i). Indeed, these index sets are just *"naked sets"* which are determined, up to isomorphism, by their *cardinality*. In particular,

> *the construction of these index sets is* **independent** *of the various* **arithmetic holomorphic structures** *involved.*

Indeed, it is precisely this property of these index sets that renders them suitable for use in the construction of the *multiradial representations* of Theorem 3.11, (i). As discussed in [IUTchI], Proposition 6.9, (i), for $j \in \{0, \ldots, l^{\divideontimes}\}$, there are precisely $j+1$ *possibilities* for the "element labeled j" in the index set of cardinality $j+1$; this leads to a total of $(l^{\divideontimes}+1)! = l^{\pm}!$ *possibilities* for the "label identification" of elements of index sets of capsules appearing in the mono-analytic $\pm$-processions of Theorem 3.11, (i). Finally, in this context, it is of interest to recall that the *"rougher approach to symmetrization"* that arises when one works with *mono-analytic processions* is *["downward"] compatible* with the *finer* arithmetically holomorphic approach to symmetrization that arises from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. [IUTchII], Remark 3.5.3; [IUTchII], Remark 4.5.2, (ii); [IUTchII], Remark 4.5.3, (ii)].

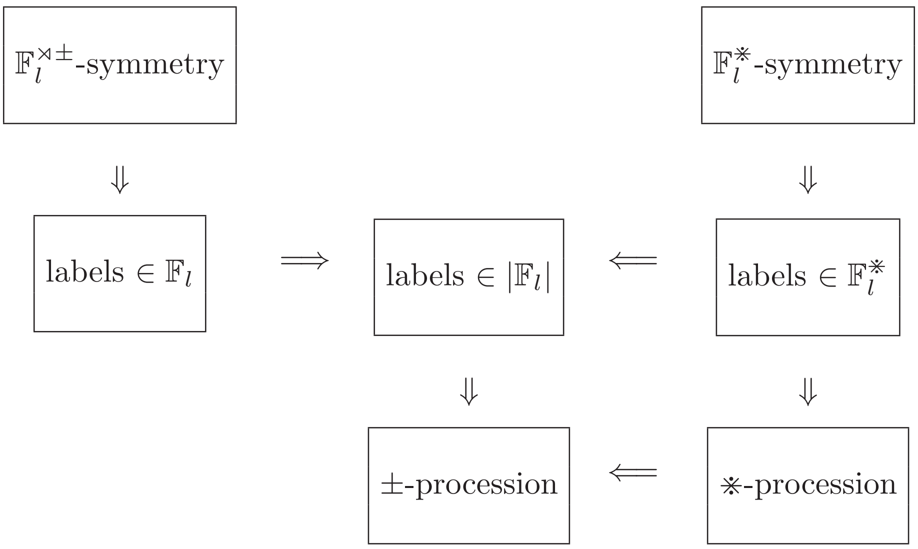

Fig. 3.3: Transitions from symmetries to labels to processions in a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

(iii) Observe that the **"Kummer isomorphism of global realified Frobenioids"** that appears in the theory of [IUTchII], §4 — i.e., more precisely, the various versions of the isomorphism of Frobenioids "${}^{\ddagger}\mathcal{C}^{\Vdash} \xrightarrow{\sim} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$" discussed in [IUTchII], Corollary 4.6, (ii), (v) — is constructed by considering isomorphisms between **local value groups** obtained by forming the **quotient** of the multiplicative groups associated to the various local fields that appear by the subgroups of **local units** [cf. [IUTchII], Propositions 4.2, (ii); 4.4, (ii)]. In particular, such "Kummer isomorphisms" *fail* to give rise to a **"$\mathfrak{log}$-Kummer correspondence"**, i.e., they fail to satisfy **mutual compatibility** properties of the sort discussed in the final portion of Theorem 3.11, (ii). Indeed, as discussed in Remark 1.2.3, (i) [cf. also [IUTchII], Remark 1.12.2, (iv)], at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the operation of forming a *multiplicative* quotient by *local units* corresponds, on the opposite side of the $\mathfrak{log}$-link, to

forming an *additive* quotient by the submodule obtained as the *$p_{\underline{v}}$-adic logarithm of these local units*. This is precisely why, in the context of Theorem 3.11, (ii), we wish to work with the global non-realified/realified Frobenioids "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$", "$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$" that arise from copies of "F_{mod}" which satisfy a "$\mathfrak{log}$-Kummer correspondence", as described in the final portion of Theorem 3.11, (ii) [cf. the discussion of Remark 3.10.1]. On the other hand, the *pathologies/indeterminacies* that arise from working with global arithmetic line bundles by means of various *local data* at $\underline{v} \in \underline{\mathbb{V}}$ in the context of the $\mathfrak{log}$-*link* are *formalized* via the theory of the global Frobenioids "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$", together with the **"upper semi-compatibility"** of local units discussed in the final portion of Theorem 3.11, (ii) [cf. also the discussion of Remark 3.10.1].

(iv) In the context of the discussion of *global realified Frobenioids* given in (iii), we observe that, in the case of the global realified Frobenioids [constructed by means of "$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$"!] that appear in the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$ [cf. Theorem 3.11, (ii), (c)], the various **localization functors** that appear [i.e., the various "${}^{\ddagger}\rho_{\underline{v}}$" of [IUTchI], Definition 5.2, (iv); cf. also the isomorphisms of the second display of [IUTchII], Corollary 4.6, (v)] may be **reconstructed**, in the spirit of the discussion of Remark 3.9.2, "by considering the *effect of multiplication* by elements of the *[non-realified] global monoids* under consideration on the *log-volumes* of the various local mono-analytic tensor packets that appear". [We leave the routine details to the reader.] This reconstructibility, together with the *mutual incompatibilities* observed in (iii) above that arise when one attempts to work *simultaneously* with *log-shells* and with the *splitting monoids* of the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, are the primary reasons for our omission of explicit mention of the splitting monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ [which in fact appear as part of the data "${}^{n,\circ}\mathfrak{R}$" considered in the discussion of Theorem 3.11, (iii), (c)] from the statement of Theorem 3.11 [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b); Theorem 3.11, (iii), (c), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$].

Remark 3.11.3. Before proceeding, we pause to discuss the relationship between the **$\mathfrak{log}$-Kummer correspondence** of Theorem 3.11, (ii), and the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link compatibility** of Theorem 3.11, (iii).

(i) First, we recall [cf. Remarks 1.4.1, (i); 3.8.2] that the various squares that appear in the *[*LGP-*Gaussian] log-theta-lattice* are *far from being [*1-*]commutative*! On the other hand, the **bi-coricity** of *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips* and *mono-analytic log-shells* discussed in Theorem 1.5, (iii), (iv), may be intepreted as the statement that

> the various squares that appear in the *[*LGP-*Gaussian] log-theta-lattice* **are** in fact **[1-]commutative** with respect to [the portion of the data associated to each "•" in the log-theta-lattice that is constituted by] these bi-coric *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips* and *mono-analytic log-shells*.

(ii) Next, let us observe that in order to relate both the *unit* and *value group* portions of the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link corresponding to *adjacent vertical lines* — i.e., $(n-1,*)$ and $(n,*)$ — of the [LGP-Gaussian] log-theta-lattice to one another,

> it is necessary to relate these **unit** and **value group** portions to one another by means of a **single $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link**, i.e., from $(n-1,m)$ to (n,m).

That is to say, from the point of view of constructing the various LGP-*monoids* that appear in the multiradial representation of Theorem 3.11, (i), one is tempted to work with correspondences between *value groups* on adjacent vertical lines that lie in a **vertically once-shifted** position — i.e., say, at $(n-1, m)$ and (n, m) — relative to the correspondence between *unit groups* on adjacent vertical lines, i.e., say, at $(n-1, m-1)$ and $(n, m-1)$. On the other hand, such an approach *fails*, at least from an *a priori* point of view, precisely on account of the *noncommutativity* discussed in (i). Finally, we observe that in order to relate both unit and value groups by means of a *single* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link*,

> it is necessary to avail oneself of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link compatibility* properties discussed in Theorem 3.11, (iii) — i.e., of the theory of §2 and [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) — so as to **insulate** the **cyclotomes** that appear in the **Kummer theory** surrounding the **étale theta function** and **κ-coric functions** from the $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$-**indeterminacies** that act on the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips involved as a result of the application of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link

— cf. the discussion of Remarks 2.2.1, 2.3.2.

(iii) As discussed in (ii) above, a "vertically once-shifted" approach to relating units on adjacent vertical lines *fails* on account of the *noncommutativity* discussed in (i). Thus, one natural approach to treating the units in a "vertically once-shifted" fashion — which, we recall, is necessary in order to relate the LGP-*monoids* on adjacent vertical lines to one another! — is to apply the **bi-coricity** *of mono-analytic log-shells* discussed in (i). On the other hand, to take this approach means that one must work in a *framework* that allows one to *relate* [cf. the discussion of Remark 1.5.4, (i)] the "Frobenius-like" structure constituted by the *Frobenioid-theoretic* units [i.e., which occur in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] to corresponding étale-like structures **simultaneously** via both

(a) the usual **Kummer isomorphisms** — i.e., so as to be compatible with the application of the compatibility properties of Theorem 3.11, (iii), as discussed in (ii) — and

(b) the **composite** of the usual Kummer isomorphisms with [a single iterate of] the **$\mathfrak{log}$-link** — i.e., so as to be compatible with the bi-coric treatment of mono-analytic log-shells [as well as the closely related construction of LGP-monoids] proposed above.

Such a framework may only be realized if one relates Frobenius-like structures to étale-like structures in a fashion that is **invariant** with respect to pre-composition with various iterates of the **$\mathfrak{log}$-link** [cf. the final portions of Propositions 3.5, (ii); 3.10, (ii)]. This is precisely what is achieved by the $\mathfrak{log}$-*Kummer correspondences* of the final portion of Theorem 3.11, (ii).

(iv) The discussion of (i), (ii), (iii) above may be summarized as follows: The **$\mathfrak{log}$-Kummer correspondences** of the final portion of Theorem 3.11, (ii), allow one to

(a) relate both the **unit** and the **value group** portions of the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link corresponding to *adjacent vertical lines* of the [LGP-Gaussian] log-theta-lattice to one another, in a fashion that

(b) **insulates** the **cyclotomes/Kummer theory** surrounding the **étale theta function** and **κ-coric functions** involved from the $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** that act on the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips involved as a result of the application of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Theorem 3.11, (iii)], and, moreover,

(c) is **compatible** with the **bi-coricity** of the **mono-analytic log-shells** [cf. Theorem 1.5, (iv)], hence also with the operation of relating the LGP**-monoids** that appear in the multiradial representation of Theorem 3.11, (i), corresponding to *adjacent vertical lines* of the [LGP-Gaussian] log-theta-lattice to one another.

These observations will play a *key role* in the proof of Corollary 3.12 below.

Remark 3.11.4. In the context of the *compatibility* discussed in the final portion of Theorem 3.11, (iii), (c), (d), we make the following observations.

(i) First of all, we observe that consideration of the $\mathfrak{log}$**-Kummer correspondence** in the context of the **compatibility** discussed in the final portion of Theorem 3.11, (iii), (c), (d), amounts precisely to **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s" [cf. the notation of the final display of Proposition 1.3, (iv)], i.e., to **identifying** data associated to these Frobenius-like "$\bullet$'s" with the corresponding data associated to the **étale-like** "$\circ$". In particular, [cf. the discussion of Theorem 3.11, (ii), preceding the statement of (Ind3)] *multiplication* of the data considered in Theorem 3.11, (ii), (b), (c), by *roots of unity* must be *"identified"* with the *identity automorphism*. Put another way, this data of Theorem 3.11, (ii), (b), (c), may only be considered **up to multiplication by roots of unity**. Thus, for instance, it only makes sense to consider **orbits** *of this data relative to multiplication by roots of unity* [i.e., as opposed to *specific elements* within such orbits]. This does not cause any problems in the case of the **theta values** considered in Theorem 3.11, (ii), (b), precisely because the theory developed so far was *formulated* precisely in such a way as to be **invariant** with respect to such **indeterminacies** [i.e., multiplication of the theta values by $2l$-th roots of unity — cf. the *left-hand portion* of Fig. 3.4 below]. In the case of the **number fields** [i.e., copies of F_{mod}] considered in Theorem 3.11, (ii), (c), the resulting indeterminacies do not cause any problems precisely because, in the theory of the present series of papers, ultimately one is only interested in the **global Frobenioids** [i.e., copies of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" and "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" and their realifications] associated to these number fields by means of constructions that only involve

- **local data**, together with
- the **entire set** — i.e., which, unlike *specific elements* of this set, is **stabilized** by multiplication by roots of unity of the number field [cf. the *left-hand portion* of Fig. 3.5 below] — constituted by the **number field** under consideration

[cf. the constructions of Example 3.6, (i), (ii); the discussion of Remark 3.9.2]. In this context, we recall from the discussion of Remark 2.3.3, (vi), that the operation of **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s" also gives rise to various **indeterminacies** in the **cyclotomic rigidity isomorphisms** applied in the $\mathfrak{log}$-Kummer correspondence. On the other hand, in the case of the **theta values** considered in Theorem 3.11, (ii), (b), we recall from this discussion of

Remark 2.3.3, (vi), that such indeterminacies are in fact **trivial** [cf. the *right-hand portion* of Fig. 3.4 below]. In the case of the **number fields** [i.e., copies of F_{mod}] considered in Theorem 3.11, (ii), (c), we recall from this discussion of Remark 2.3.3, (vi), that such cyclotomic rigidity isomorphism indeterminacies amount to a possible indeterminacy of **multiplication by** ± 1 on copies of the multiplicative group $F^{\times}_{\mathrm{mod}}$ [cf. the *right-hand portion* of Fig. 3.5 below], i.e., indeterminacies which do not cause any problems, again, precisely as a consequence of the fact that such indeterminacies **stabilize** the **entire set** [i.e., as opposed to *specific elements* of this set] constituted by the **number field** under consideration. Finally, in this context, we observe [cf. the discussion at the beginning of Remark 2.3.3, (viii)] that, in the case of the various **local data** at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ that appears in Theorem 3.11, (ii), (a), and gives rise to the **holomorphic log-shells** that serve as **containers** for the data considered in Theorem 3.11, (ii), (b), (c), the corresponding **cyclotomic rigidity isomorphism indeterminacies** are in fact **trivial**. Indeed, this *triviality* may be understood as a consequence of the fact the following observation: Unlike the case with the cyclotomic rigidity isomorphisms that are applied in the context of the **geometric containers** [cf. the discussion of Remark 2.3.3, (i)] that appear in the case of the data of Theorem 3.11, (ii), (b), (c), i.e., which give rise to **"vicious circles"/"loops"** consisting of identification morphisms that differ from the usual *natural identification* by multiplication by elements of the *submonoid* $\mathbb{I}^{\mathrm{ord}} \subseteq \pm\mathbb{N}_{\geq 1}$ [cf. the discussion of Remark 2.3.3, (vi)],

> the cyclotomic rigidity isomorphisms that are applied in the context of this **local data** — even when subject to the *various identifications* arising from **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s"! — only give rise to **natural isomorphisms** between **"geometric" cyclotomes** arising from the *geometric fundamental group* and **"arithmetic" cyclotomes** arising from copies of the *absolute Galois group* of the base [local] field [cf. [AbsTopIII], Corollary 1.10, (c); [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1].

That is to say, **no "vicious circles"/"loops"** arise since there is *never any confusion* between such "geometric" and "arithmetic" cyclotomes. [A similar phenomenon may be observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ with regard to the *Kummer structures* considered in [IUTchI], Example 3.4, (i).] Thus, in summary,

> the various **indeterminacies** that, *a priori*, might arise in the context of the portions of the **$\mathfrak{log}$-Kummer correspondence** that appear in the final portion of Theorem 3.11, (iii), (c), (d), are in fact **"invisible"**, i.e., they have **no substantive effect** on the objects under consideration

[cf. also the discussion of (ii) below]. This is precisely the sense in which the **"compatibility"** stated in the final portion of Theorem 3.11, (iii), (c), (d), is to be understood.

(ii) In the context of the discussion of (i), we make the following *observation*:

> the discussion in (i) of **indeterminacies** that, *a priori*, might arise in the context of the portions of the **$\mathfrak{log}$-Kummer correspondence** that appear in the final portion of Theorem 3.11, (iii), (c), (d), is **complete**, i.e., there are *no further possible indeterminacies* that might appear.

Indeed, this *observation* is a consequence of the "general nonsense" observation [cf., e.g., the discussion of [FrdII], Definition 2.1, (ii)] that, in general, **"Kummer isomorphisms"** are **completely determined** by the following data:

(a) **isomorphisms** between the respective **cyclotomes** under consideration;
(b) the **Galois action** on roots of elements of the monoid under consideration.

That is to say, the **compatibility** of all of the various constructions that appear with the actions of the relevant **Galois groups** [or arithmetic fundamental groups] is **tautological**, so there is no possibility that further indeterminacies might arise with respect to the data of (b). On the other hand, the effect of the indeterminacies that might arise with respect to the data of (a) was precisely the content of the latter portion of the discussion of (i) [i.e., of the discussion of Remark 2.3.3, (vi), (viii)].

(iii) In the context of the discussion of (i), we observe that the **"invisible indeterminacies"** discussed in (i) in the case of the data considered in Theorem 3.11, (ii), (b), (c), may be thought of as a sort of *analogue* for this data of the **indeterminacy** (Ind3) [cf. the discussion of the final portion of Theorem 3.11, (ii)] to which the data of Theorem 3.11, (ii), (a), is subject. By contrast, the **multiradiality** and **radial/coric decoupling** discussed in Remarks 2.3.2, 2.3.3 [cf. also Theorem 3.11, (iii), (c), (d)] may be understood as asserting precisely that the **indeterminacies** (Ind1), (Ind2) discussed in Theorem 3.11, (i), which act, essentially, on the data of Theorem 3.11, (ii), (a), have **no effect** on the **geometric containers** [cf. the discussion of Remark 2.3.3, (i)] that underlie [i.e., prior to execution of the relevant **evaluation** operations] the data considered in Theorem 3.11, (ii), (b), (c).

$$\boldsymbol{\mu}_{2l} \quad \curvearrowright \quad \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*} \quad \curvearrowleft \quad \{1\} \quad (\subseteq \pm\mathbb{N}_{\geq 1})$$

Fig. 3.4: Invisible indeterminacies acting on *theta values*

$$\boldsymbol{\mu}(F^{\times}_{\mathrm{mod}}) \quad \curvearrowright \quad F^{\times}_{\mathrm{mod}} \quad \curvearrowleft \quad \{\pm 1\} \quad (\subseteq \pm\mathbb{N}_{\geq 1})$$

Fig. 3.5: Invisible indeterminacies acting on *copies of* $F^{\times}_{\mathrm{mod}}$

The following result may be thought of as a relatively *concrete consequence* of the somewhat abstract content of Theorem 3.11.

Corollary 3.12. (Log-volume Estimates for Θ-Pilot Objects) *Suppose that we are in the situation of Theorem 3.11. Write*

$$- |\log(\underline{\underline{\Theta}})| \in \mathbb{R} \bigcup \{+\infty\}$$

for the **procession-normalized mono-analytic log-volume** *[i.e., where the average is taken over* $j \in \mathbb{F}_l^{*}$ *— cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a)] of the* **holomorphic hull** *[cf. Remark 3.9.5, (i)] of the* **union** *of the* **possible images** *of a* **Θ-pilot object** *[cf. Definition 3.8, (i)],*

relative to the relevant **Kummer isomorphisms** *[cf. Theorem 3.11, (ii)], in the* **multiradial representation** *of Theorem 3.11, (i), which we regard as* **subject** *to the* **indeterminacies** (Ind1), (Ind2), (Ind3) *described in Theorem 3.11, (i), (ii). Write*

$$-\ |\log(\underline{\underline{q}})|\ \in\ \mathbb{R}$$

for the **procession-normalized mono-analytic log-volume** *of the image of a* **q-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorem 3.11, (ii)], in the* **multiradial representation** *of Theorem 3.11, (i), which we do* **not** *regard as subject to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in Theorem 3.11, (i), (ii). Here, we recall the definition of the symbol "$\triangle$" as the result of identifying the* **labels**

$$\text{“}0\text{” and “}\langle \mathbb{F}_l^{\circledast} \rangle\text{”}$$

[cf. [IUTchII], Corollary 4.10, (i)]. In particular, $|\log(\underline{\underline{q}})| > 0$ *is easily computed in terms of the various* **q-parameters** *of the elliptic curve* E_F *[cf. [IUTchI], Definition 3.1, (b)] at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ $(\neq \emptyset)$. *Then it holds that* $-\ |\log(\underline{\underline{\Theta}})| \in \mathbb{R}$, *and*

$$-\ |\log(\underline{\underline{\Theta}})|\ \geq\ -\ |\log(\underline{\underline{q}})|$$

— i.e., $C_\Theta \geq -1$ *for any real number* $C_\Theta \in \mathbb{R}$ *such that* $-\ |\log(\underline{\underline{\Theta}})|\ \leq\ C_\Theta \cdot |\log(\underline{\underline{q}})|$.

Proof. We begin by observing that, since $|\log(\underline{\underline{q}})| > 0$, we may *assume without loss of generality* in the remainder of the proof that

$$-\ |\log(\underline{\underline{\Theta}})|\ <\ 0$$

whenever $-\ |\log(\underline{\underline{\Theta}})| \in \mathbb{R}$ [i.e., since an inequality $-\ |\log(\underline{\underline{\Theta}})|\ \geq\ 0$ would imply that $-\ |\log(\underline{\underline{\Theta}})|\ \geq\ 0\ >\ -|\log(\underline{\underline{q}})|$]. Now suppose that we are in the situation of Theorem 3.11. For $n \in \mathbb{Z}$, write

$$^{n,\circ}\mathcal{U}\ \ \stackrel{\mathrm{def}}{=}\ \ \left\{ ^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}} \right\}_{j \in |\mathbb{F}_l|, v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}\ \ \subseteq\ \ ^{n,\circ}\mathcal{U}^{\mathbb{Q}}\ \ \stackrel{\mathrm{def}}{=}\ \ \left\{ ^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} \right\}_{j \in |\mathbb{F}_l|, v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}$$

[where we observe that the "$\subseteq$" constitutes a slight abuse of notation] for the collection of subsets $^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}} \subseteq {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1}; {}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ [cf. Theorem 3.11, (i), (a)] given by the various **unions**, for $j \in |\mathbb{F}_l|$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, of the **possible images** of a **Θ-pilot object** [cf. Definition 3.8, (i)], relative to the relevant **Kummer isomorphisms** [cf. Theorem 3.11, (ii)], in the **multiradial representation** of Theorem 3.11, (i), which we regard as **subject** to the **indeterminacies** (Ind1), (Ind2), (Ind3) described in Theorem 3.11, (i), (ii);

$$^{n,\circ}\overline{\mathcal{U}}\ \ =\ \ \left\{ ^{n,\circ}\overline{\mathcal{U}}_{j,v_{\mathbb{Q}}} \right\}_{j \in |\mathbb{F}_l|, v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}\ \ \subseteq\ \ ^{n,\circ}\mathcal{U}^{\mathbb{Q}}\ \ =\ \ \left\{ ^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} \right\}_{j \in |\mathbb{F}_l|, v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}$$

[where we observe that the "$\subseteq$" constitutes a slight abuse of notation] for the collection of subsets $^{n,\circ}\overline{\mathcal{U}}_{j,v_{\mathbb{Q}}} \subseteq {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} = \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1}; {}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ [cf. Theorem 3.11, (i), (a)] given by the various **holomorphic hulls** [cf. Remark 3.9.5, (i)] of the subsets

${}^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, relative to the arithmetic holomorphic structure labeled "$n, \circ$". Here, we observe that one concludes easily from the [easily verified] *compactness* of the ${}^{1,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}}$ [where $j \in |\mathbb{F}_l|$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$], together with the definition of the *log-volume*, that the quantity $-\,|\log(\underline{\underline{\Theta}})|$ is *finite*, hence *negative* [by our *assumption* at the beginning of the present proof!]. In particular, we observe [cf. Remark 2.4.2, (iv), (v), (vi); Remark 3.9.6; Remark 3.9.7; the discussion of "IPL" in Remark 3.11.1, (iii)] that

> we may restrict our attention to **possible images** of a **Θ-pilot object** that correspond to data [i.e., collections of regions] that may be interpreted as an $\mathcal{F}^{\Vdash\blacktriangleright}$**-prime-strip**.

Now we proceed to review precisely what is achieved by the various portions of Theorem 3.11 and, indeed, by the theory developed thus far in the present series of papers. This review leads naturally to an interpretation of the theory that gives rise to the *inequality* asserted in the statement of Corollary 3.12. For ease of reference, we divide our discussion into *steps*, as follows.

(i) In the following discussion, we concentrate on a *single arrow* — i.e., a *single* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-link*

$$ {}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}}{\longrightarrow} {}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} $$

— of the [LGP-Gaussian] log-theta-lattice under consideration. This arrow consists of the *full poly-isomorphism* of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{1,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle} $$

[cf. Definition 3.8, (ii)]. This poly-isomorphism may be thought of as consisting of a **"unit portion"** constituted by the associated [full] poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{1,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} $$

and a **"value group portion"** constituted by the associated [full] poly-isomorphism of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{1,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\triangle} $$

[cf. Definition 2.4, (iii)]. This value group portion of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link maps Θ*-pilot objects* of ${}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to *q-pilot objects* of ${}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Remark 3.8.1].

(ii) Whereas the units of the Frobenioids that appear in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip ${}^{0,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{LGP}}$ are subject to $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$-indeterminacies [i.e., "(Ind1), (Ind2)" — cf. Theorem 3.11, (iii), (a), (b)], the **cyclotomes** that appear in the Kummer theory surrounding the **étale theta function** and κ**-coric functions**, i.e., which give rise to the "value group portion" ${}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\mathrm{LGP}}$, are **insulated** from these $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** — cf. Theorem 3.11, (iii), (c), (d); the discussion of Remark 3.11.3, (iv); Fig. 3.6 below. Here, we recall that in the case of the étale theta function, this follows from the theory of §2, i.e., in essence, from the **cyclotomic rigidity of mono-theta environments**, as discussed in [EtTh]. On the other hand, in the case of κ-coric functions, this follows from the algorithms discussed in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii).

	Θ-related objects	*NF-related objects*
require *mono-analytic containers*, **Kummer theory incompatible** with (Ind1), (Ind2)	local LGP-monoids [cf. Proposition 3.4, (ii)]	copies of F_{mod} [cf. Proposition 3.7, (i)]
independent of *mono-analytic containers*, **Kummer theory compatible** with (Ind1), (Ind2) [cf. Remark 2.3.3]	étale theta function, mono-theta environments [cf. Corollary 2.3]	global $_{\infty}\kappa$-coric, local $_{\infty}\kappa$-, $_{\infty}\kappa\times$-coric structures [cf. Remark 2.3.2]

Fig. 3.6: Relationship of theta- and number field-related objects to mono-analytic containers

(iii) In the following discussion, it will be of crucial importance to relate **simultaneously** both the **unit** and the **value group** portions of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link(s) involved on the 0-*column* [i.e., the vertical line indexed by 0] of the log-theta-lattice under consideration to the corresponding unit and value group portions on the 1-*column* [i.e., the vertical line indexed by 1] of the log-theta-lattice under consideration. On the other hand, if one attempts to relate the *unit* portions via one $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [say, from $(0, m)$ to $(1, m)$] and the *value group* portions via another $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [say, from $(0, m')$ to $(1, m')$, for $m' \neq m$], then the **non-commutativity** of the log-theta-lattice renders it practically impossible to obtain conclusions that require one to relate both the unit and the value group portions *simultaneously* [cf. the discussion of Remark 3.11.3, (i), (ii)]. This is precisely why we concentrate on a **single** $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** [cf. (i)].

(iv) The issue discussed in (iii) is relevant in the context of the present discussion for the following reason. Ultimately, we wish to apply the **bi-coricity** of the **units** [cf. Theorem 1.5, (iii), (iv)] in order to compute the 0-*column* Θ-*pilot object* in terms of the arithmetic holomorphic structure of the 1-*column*. In order to do this, one must work with *units* that are **vertically once-shifted** [i.e., lie at $(n, m-1)$] relative to the value group structures involved [i.e., which lie at (n, m)] — cf. the discussion of Remark 3.11.3, (ii). The solution to the problem of simultaneously accommodating these apparently *contradictory requirements* — i.e., "vertical shift" vs. "impossibility of vertical shift" [cf. (iii)] — is given precisely by working, on the 0-column, with structures that are **invariant** with respect to

vertical shifts [i.e., "$(0, m) \mapsto (0, m+1)$"] of the log-theta-lattice [cf. the discussion surrounding Remark 1.2.2, (iii), (a)] such as **vertically coric structures** [i.e., indexed by "$(n, \circ)$"] that are related to the *"Frobenius-like"* structures which are *not* vertically coric by means of the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii). Here, we note that this "solution" may be implemented only at the cost of admitting the *"indeterminacy"* constituted by the **upper semi-compatibility** of (Ind3).

(v) Thus, we begin our computation of the 0-*column* Θ-*pilot object* in terms of the arithmetic holomorphic structure of the 1-*column* by relating the units on the 0- and 1-columns by means of the **unit portion**

$$^{0,0}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{1,0}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}$$

of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link from $(0, 0)$ to $(1, 0)$ [cf. (i)] and then applying the **bi-coricity** of the **units** of Theorem 1.5, (iii), (iv). In particular, the **mono-analytic log-shell** interpretation of this bi-coricity given in Theorem 1.5, (iv), will be applied to regard these mono-analytic log-shells as **"multiradial mono-analytic containers"** [cf. the discussion of Remark 1.5.2, (i), (ii), (iii)] for the various [local and global] value group structures that constitute the Θ-pilot object on the 0-column — cf. Fig. 3.6 above. [Here, we observe that the parallel treatment of *"theta-related"* and *"number field-related"* objects is reminiscent of the discussion of [IUTchII], Remark 4.11.2, (iv).] That is to say, we will relate the various Frobenioid-theoretic [i.e., "Frobenius-like" — cf. Remark 1.5.4, (i)]

- *local units* at $\underline{v} \in \underline{\mathbb{V}}$,
- *splitting monoids* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and
- *global Frobenioids*

indexed by $(0, m)$, for $m \in \mathbb{Z}$, to the *vertically coric* [i.e., indexed by "$(0, \circ)$"] versions of these bi-coric mono-analytic containers by means of the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (a), (b), (c) — i.e., by *varying* the *"Kummer input index"* $(0, m)$ along the 0-column.

(vi) In the context of (v), it is useful to recall that the $\mathfrak{log}$-Kummer correspondences of Theorem 3.11, (ii), (b), (c), are obtained precisely as a consequence of the **splittings**, up to roots of unity, of the relevant monoids into **unit** and **value group** portions constructed by applying the **Galois evaluation** operations discussed in Remarks 2.2.2, (iii) [in the case of Theorem 3.11, (ii), (b)], and 2.3.2 [in the case of Theorem 3.11, (ii), (c)]. Moreover, we recall that the Kummer theory surrounding the local LGP-monoids of Proposition 3.4, (ii), depends, in an essential way, on the theory of [IUTchII], §3 [cf., especially, [IUTchII], Corollaries 3.5, 3.6], which, in turn, depends, in an essential way, on the Kummer theory surrounding **mono-theta environments** established in [EtTh]. Thus, for instance, we recall that the **discrete rigidity** established in [EtTh] is applied so as to avoid working, in the tempered Frobenioids that occur, with **"$\widehat{\mathbb{Z}}$-divisors/line bundles"** [i.e., "$\widehat{\mathbb{Z}}$-completions" of $\mathbb{Z}$-modules of divisors/line bundles], which are *fundamentally incompatible* with conventional notions of divisors/line bundles, hence, in particular, with *mono-theta-theoretic cyclotomic rigidity* [cf. Remark 2.1.1, (v)]. Also, we recall that **"isomorphism class compatibility"** — i.e., in the terminology of

[EtTh], *"compatibility with the* **topology** *of the* **tempered fundamental group***"* [cf. the discussion at the beginning of Remark 2.1.1] — allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the **ring-theoretic basepoints** that occur on either side of the **$\mathfrak{log}$-link** [cf. Remarks 2.1.1, (ii), and 2.3.3, (vii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the $\mathfrak{log}$-*Kummer correspondence for the splitting monoids of local LGP-monoids*, whose construction depends, in an essential way [cf. the theory of [IUTchII], §3, especially, [IUTchII], Corollaries 3.5, 3.6], on the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**. That is to say,

> it is precisely by establishing this conjugate synchronization arising from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry relative to *these basepoints* that occur on either side of the $\mathfrak{log}$-link that one is able to conclude the crucial **compatibility** *of this conjugate synchronization with the* $\mathfrak{log}$-*link* [cf. Remark 1.3.2].

A similar observation may be made concerning the *MLF-Galois pair* approach to the *cyclotomic rigidity isomorphism* that is applied at $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$ [cf. [IUTchII], Corollary 1.11, (a); [IUTchII], Remark 1.11.1, (i), (a); [IUTchII], Proposition 4.2, (i); [AbsTopIII], Proposition 3.2, (iv), as well as Remark 2.3.3, (viii), of the present paper], which amounts, in essence, to

> computations involving the Galois cohomology groups of *various subquotients — such as torsion subgroups [i.e., roots of unity] and associated value groups — of the [multiplicative] module of nonzero elements* of an algebraic closure of the mixed characteristic local field involved

[cf. the proof of [AbsAnab], Proposition 1.2.1, (vii)] — i.e., algorithms that are *manifestly compatible with the* **topology** *of the profinite groups involved* [cf. the discussion of Remark 2.3.3, (viii)], in the sense that they do not require one to pass to *Kummer towers* [cf. the discussion of [IUTchII], Remark 3.6.4, (i)], which are *fundamentally incompatible* with the *ring structure* of the fields involved. Here, we note in passing that the corresponding property for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ [cf. [IUTchII], Proposition 4.4, (i)] holds as a consequence of the interpretation discussed in [IUTchI], Remark 3.4.2, of **Kummer structures** in terms of **co-holomorphicizations**. On the other hand, the approaches to cyclotomic rigidity just discussed for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ and $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$ *differ quite fundamentally* from the approach to cyclotomic rigidity taken in the case of *[global] number fields* in the algorithms described in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii), which depend, in an essential way, on the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

— i.e., which is **fundamentally incompatible** *with the* **topology** *of the profinite groups involved* [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii)] in the sense that it clearly cannot be obtained as some sort of limit of corresponding properties of $(\mathbb{Z}/N\mathbb{Z})^{\times}$! Nevertheless, with regard to uni-/multi-radiality issues, this approach to cyclotomic rigidity in the case of the number fields resembles the theory of mono-theta-theoretic cyclotomic rigidity at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ in that it admits a *natural* **multiradial** *formulation* [cf. Theorem 3.11, (iii), (d); the discussion of Remarks 2.3.2, 3.11.3], in sharp contrast to the essentially **uniradial** nature of the approach to cyclotomic rigidity via MLF-Galois pairs at $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$ [cf. the discussion of [IUTchII], Remark 1.11.3]. These observations are summarized in Fig. 3.7 below.

Finally, we recall that [one verifies immediately that] the various approaches to cyclotomic rigidity just discussed are *mutually compatible* in the sense that they yield the same cyclotomic rigidity isomorphism in any setting in which more than one of these approaches may be applied.

Approach to cyclotomic rigidity	*Uni-/multi-radiality*	*Compatibility with $\mathbb{F}_l^{\rtimes\pm}$-symmetry, profinite/tempered topologies, ring structures, $\mathfrak{log}$-link*
mono-theta environments	**multiradial**	**compatible**
MLF-Galois pairs, via Brauer groups	**uniradial**	**compatible**
number fields, via $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$	**multiradial**	**incompatible**

Fig. 3.7: Three approaches to cyclotomic rigidity

(vii) In the context of the discussion in the final portion of (vi), it is of interest to recall that the constructions underlying the crucial **bi-coricity** theory of Theorem 1.5, (iii), (iv), depend, in an essential way, on the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, which allows one to relate the local monoids and Galois groups at distinct labels $\in |\mathbb{F}_l|$ to one another in a fashion that is *simultaneously compatible* both with

· the **vertically coric** structures and **Kummer theory** that give rise to the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii),

and with

· the property of **distinguishing** [i.e., not identifying] data indexed by **distinct labels** $\in |\mathbb{F}_l|$

— cf. the discussion of Remark 1.5.1, (i), (ii). Since, moreover, this crucial conjugate synchronization is *fundamentally incompatible* with the $\mathbb{F}_l^{\circledast}$**-symmetry**, it is necessary to work with these two symmetries *separately*, as was done in [IUTchI], §4, §5, §6 [cf. [IUTchII], Remark 4.7.6]. Here, it is useful to recall that the $\mathbb{F}_l^{\circledast}$-symmetry also plays a crucial role, in that it allows one to *"descend to F_{mod}" at the level of* **absolute Galois groups** [cf. [IUTchII], Remark 4.7.6]. On the other hand, *both* the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries share the property of being compatible with the **vertical coricity** and relevant **Kummer isomorphisms** of the 0-column — cf. the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii), (b) [in the case of the

$\mathbb{F}_l^{\rtimes\pm}$-symmetry], (c) [in the case of the $\mathbb{F}_l^{\circledast}$-symmetry]. Here, we recall that the vertically coric versions of both the $\mathbb{F}_l^{\rtimes\pm}$- and the $\mathbb{F}_l^{\circledast}$-symmetries depend, in an essential way, on the **arithmetic holomorphic structure** of the 0-column, hence give rise to **multiradial** structures via the **tautological** approach to constructing such structures discussed in Remark 3.11.2, (i), (ii).

(viii) In the context of (vii), it is useful to recall that in order to construct the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, it is necessary to make use of **global ±-synchronizations** of various local ±-indeterminacies. Since the local tempered fundamental groups at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ do not extend to a "global tempered fundamental group", these global ±-synchronizations give rise to **profinite conjugacy indeterminacies** in the vertically coric construction of the LGP-monoids [i.e., the theta values at torsion points] given in [IUTchII], §2, which are *resolved* by applying the theory of [IUTchI], §2 — cf. the discussion of [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii).

(ix) In the context of (vii), it is also useful to recall the important role played, in the theory of the present series of papers, by the various "copies of F_{mod}", i.e., more concretely, in the form of the various copies of the **global Frobenioids** "$\mathcal{F}^{\circledast}_{\text{MOD}}$", "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" and their realifications. That is to say, the **ring structure** of the global field F_{mod} allows one to bridge the gap — i.e., furnishes a **translation apparatus** — between the **multiplicative** structures constituted by the global realified Frobenioids related via the $\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$-link and the **additive** representations of these global Frobenioids that arise from the "mono-analytic containers" furnished by the *mono-analytic log-shells* [cf. (v)]. Here, the **precise compatibility** of the ingredients for "$\mathcal{F}^{\circledast}_{\text{MOD}}$" with the $\mathfrak{log}$-Kummer correspondence renders "$\mathcal{F}^{\circledast}_{\text{MOD}}$" better suited to describing the relation to the $\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$-link [cf. Remark 3.10.1, (ii)]. On the other hand, the local portion of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" — i.e., which is subject to **"upper semi-compatibility"** [cf. (Ind3)], hence only *"approximately compatible"* with the $\mathfrak{log}$-Kummer correspondence — renders it better suited to **explicit estimates** of global arithmetic degrees, by means of **log-volumes** [cf. Remark 3.10.1, (iii)].

(x) Thus, one may summarize the discussion thus far as follows. The theory of **"Kummer-detachment"** — cf. Remarks 1.5.4, (i); 2.1.1; 3.10.1, (ii), (iii) — furnished by Theorem 3.11, (ii), (iii), allows one to relate the **Frobenioid-theoretic** [i.e., "Frobenius-like"] structures that appear in the domain [i.e., at $(0,0)$] of the $\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$**-link** [cf. (i)] to the **multiradial representation** described in Theorem 3.11, (i), (a), (b), (c), but only at the cost of introducing the **indeterminacies**

(Ind1) — which may be thought of as arising from the requirement of *compatibility* with the **permutation symmetries** of the **étale-picture** [cf. Theorem 3.11, (i)];

(Ind2) — which may be thought of as arising from the requirement of *compatibility* with the $\text{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** that act on the domain/codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$**-link** [cf. (ii); Theorem 3.11, (i), (iii)], i.e., with the **horizontal arrows** of the log-theta-lattice;

(Ind3) — which may be thought of as arising from the requirement of *compatibility* with the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii), i.e., with the **vertical arrows** of the log-theta-lattice.

The various indeterminacies (Ind1), (Ind2), (Ind3) to which the multiradial representation is subject may be thought of as data that describes some sort of **"formal quotient"**, like the **"fine moduli spaces"** that appear in algebraic geometry. In this context, the **procession-normalized mono-analytic log-volumes** [i.e., where the average is taken over $j \in \mathbb{F}_l^{\circledast}$] of Theorem 3.11, (i), (a), (c), furnish a means of constructing a sort of associated **"coarse space"** or **"inductive limit"** [of the "inductive system" constituted by this "formal quotient"] — i.e., in the sense that [one verifies immediately — cf. Proposition 3.9, (ii) — that] the resulting log-volumes $\in \mathbb{R}$ are **invariant** with respect to the indeterminacies (Ind1), (Ind2), and have the effect of converting the indeterminacy (Ind3) into an **inequality** [from above]. Moreover, the **$\mathfrak{log}$-link compatibility** of the various log-volumes that appear [cf. Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] ensures that these log-volumes are *compatible* with [the portion of the "formal quotient"/"inductive system" constituted by] the various arrows [i.e., *Kummer isomorphisms* and $\mathfrak{log}$-*links*] of the **$\mathfrak{log}$-Kummer correspondence** of Theorem 3.11, (ii). Here, we note that the *averages over* $j \in \mathbb{F}_l^{\circledast}$ that appear in the definition of the procession-normalized volumes involved may be thought of as a consequence of the **$\mathbb{F}_l^{\circledast}$-symmetry** acting on the labels of the theta values that give rise to the LGP-monoids — cf. also the definition of the symbol "$\triangle$" in [IUTchII], Corollary 4.10, (i), via the *identification of the symbols* "0" and "$\langle \mathbb{F}_l^{\circledast} \rangle$"; the discussion of Remark 3.9.3. Also, in this context, it is of interest to observe that the various **tensor products** that appear in the various local mono-analytic tensor packets that arise in the multiradial representation of Theorem 3.11, (i), (a), have the effect of **identifying** the operation of "multiplication by elements of $\mathbb{Z}$" — and hence also the effect on **log-volumes** of such multiplication operations! — at *different labels* $\in \mathbb{F}_l^{\circledast}$.

(xi) For ease of reference, we divide this *step* into *substeps*, as follows.

(xi-a) Consider a ***q*-pilot object** at $(1,0)$, which we think of — relative to the relevant copy of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" — in terms of the **holomorphic log-shells** constructed at $(1,0)$ [cf. the discussion of Remark 3.12.2, (iv), (v), below]. Then the **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link** from $(0,0)$ to $(1,0)$ may be interpreted as a sort of **gluing isomorphism** that relates the **arithmetic holomorphic structure** — i.e., the *"conventional ring/scheme-theory"* — at $(1,0)$ to the arithmetic holomorphic structure at $(0,0)$ in such a way that the **Θ-pilot object** at $(0,0)$ [thought of as an object of the relevant global realified Frobenioid] corresponds to the ***q*-pilot object** at $(1,0)$ [cf. (i); the discussion of Remark 3.12.2, (ii), below].

(xi-b) On the other hand, the multiradial construction algorithm of Theorem 3.11, which was summarized in the discussion of (x), yields a construction of a *collection of possibilities of output data* contained in

$$({}^{0,\circ}\mathcal{U}^{\mathbb{Q}} \supseteq) \quad {}^{0,\circ}\mathcal{U} \quad \overset{\sim}{\to} \quad {}^{1,\circ}\mathcal{U} \quad (\subseteq {}^{1,\circ}\mathcal{U}^{\mathbb{Q}})$$

— where the isomorphism "$\overset{\sim}{\to}$" arises from the **permutation symmetries** discussed in the final portion of Theorem 3.11, (i) — that satisfies the **input prime-strip link (IPL)** and **simultaneous holomorphic expressibility (SHE)** properties discussed in Remark 3.11.1, (iii), (iv), (v) [cf. also the discussion of *"possible*

images" at the beginning of the present proof]. Here, with regard to (IPL), we observe that the $\mathcal{F}^{\Vdash\blacktriangleright}$-*prime-strip* portion of the **link/relationship** of this collection of possibilities of output data to the **input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip** [cf. Remark 3.11.1, (ii)] consists precisely of **(full poly-)isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips**, while the corresponding *link/relationship* for $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips is *somewhat more complicated*, as a result of the *indeterminacies* (Ind1), (Ind2), (Ind3). Also, in this context, we observe that, although the multiradial construction algorithm of Theorem 3.11 in fact involves the Θ-*pilot object* at $(0, 0)$, in the present discussion of Step (xi), we shall *only be concerned* with **qualitative logical aspects/consequences** of this construction algorithm, i.e., with the

· **input prime-strip link (IPL)**,
· **simultaneous holomorphic expressibility (SHE)**, and
· **algorithmic parallel transport (APT)**

properties discussed in Remark 3.11.1, (iii), (iv), (v). That is to say, we shall take the point of view of **"temporarily forgetting"** — cf. the discussion of **hidden internal structures (HIS)** in Remark 3.11.1, (iv) — the fact that the multiradial construction algorithm of Theorem 3.11 in fact involves Θ-*pilot objects*, *theta functions/values*, *mono-theta environments*. Alternatively, in the discussion to follow, we shall, roughly speaking, think of the multiradial construction algorithm of Theorem 3.11 as

> **"some"** algorithm that transforms a certain type of **input data** into a certain type of **output data** and, moreover, satisfies **certain properties (IPL)** and **(SHE)**.

(xi-c) Thus, the discussion of the (IPL) and (SHE) properties in (xi-b) may be summarized as follows:

> The *multiradial construction algorithm of Theorem* 3.11 yields a collection of possibilities of output data in ${}^{1,\circ}\mathcal{U}$ ($\subseteq {}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$) that are **linked/related** [cf. (IPL)], via **isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips**, to the representation [via the $\mathfrak{log}$-Kummer correspondence in the 1-column] of the ***q*-pilot object** at $(1, 0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$, and, moreover, whose construction may be expressed entirely relative to the **arithmetic holomorphic structure** in the 1-column [cf. (SHE)].

Here, we recall that, in more concrete language, this "*arithmetic holomorphic structure* in the 1-column" amounts, in essence, to the **ring structure** labeled "$1, \circ$". Moreover, by slightly enlarging the collection of possibilities of output data under consideration by working with the **holomorphic hull** ${}^{1,\circ}\overline{\mathcal{U}}$ ($\supseteq {}^{1,\circ}\mathcal{U}$), we obtain output data that is expressed — *not* in terms of regions contained in various *tensor products of local fields* labeled "$1, \circ$" [i.e., more concretely, various isomorphs of "$K_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}$], but rather — in terms of **localizations of arithmetic vector bundles** over certain *local rings* labeled "$1, \circ$" [i.e., more concretely, various isomorphs of "$\mathcal{O}_{K_{\underline{v}}}$", for $\underline{v} \in \underline{\mathbb{V}}$] — cf. the discussion of Remarks 3.9.5, (vii), (Ob1), (Ob2), (Ob5); 3.12.2, (v), below. Such an expression in terms of *"localizations of arithmetic vector bundles"* is necessary in order to render the output data in a form that is **comparable** to the representation of the ***q*-pilot object** [i.e., which arises from a certain *arithmetic line bundle*] at $(1, 0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$.

(xi-d) The discussion of (xi-c) thus yields the following conclusion:

> The *multiradial construction algorithm* of Theorem 3.11, followed by formation of the *holomorphic hull*, yields a collection of possibilities of output data in ${}^{1,\circ}\overline{\mathcal{U}}$ that are **linked/related** [cf. (IPL)], via **isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips**, to the representation [via the $\mathfrak{log}$-*Kummer correspondence* in the 1-*column*] of the ***q*-pilot object** at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$, and, moreover, whose construction may be expressed entirely relative to **localizations of arithmetic vector bundles** over **rings** that arise in the **arithmetic holomorphic structure** in the 1-column [cf. (SHE)].

Here, we observe that these "localizations of arithmetic vector bundles" are [unlike the *arithmetic line bundle* that gives rise to the *q-pilot object*] of *rank* > 1. Moreover, the q-pilot object is defined at the level of *realifications* of Frobenioids of [global] arithmetic line bundles. Thus, it is only by forming [a suitable positive tensor power of] the *determinant* of the localizations of arithmetic vector bundles mentioned in the above display [cf. Remark 3.9.5, (vii), (Ob3), (Ob4)] and then applying the [suitably *normalized*, with respect to $j \in |\mathbb{F}_l|$] **log-volume** to various regions — i.e., the *region* ${}^{1,\circ}\overline{\mathcal{U}}$ and the *region* that arises from the representation of the q-pilot object at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$ — in ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$ [cf. Remark 3.9.5, (vii), (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], that we are able to obtain **completely comparable objects** [cf. Remarks 3.9.5, (vii), (Ob5), (Ob6), (Ob7), (Ob8), (Ob9); 3.9.5, (viii), (ix)], namely,

$$\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \stackrel{\text{def}}{=} \{\lambda \in \mathbb{R} \mid \lambda \leq -|\log(\underline{\underline{\Theta}})|\} \subseteq \mathbb{R}; \quad -|\log(\underline{\underline{q}})| \in \mathbb{R}$$

— where we recall that, by definition, $-|\log(\underline{\underline{\Theta}})|$ is the [*negative* — cf. the discussion of *"possible images"* at the beginning of the present proof] log-volume of ${}^{1,\circ}\overline{\mathcal{U}}$, while $-|\log(\underline{\underline{q}})|$ is the log-volume of the region that arises from the representation of the q-pilot object at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$. In this context, it is useful to recall from Proposition 3.9, (iii) [cf. also the discussion of Remarks 3.9.2, 3.9.6], that **global arithmetic degrees** of objects of global realified Frobenioids may be interpreted as **log-volumes** [cf. also the discussion of Remarks 1.5.2, (iii); 3.10.1, (iv), as well as of Remark 3.12.2, (v), below]. Finally, in this context, we observe [cf. the first display of the present (xi-d)] that it is of *crucial importance* to apply the **$\mathfrak{log}$-Kummer correspondence** in the **1-column** [cf. the discussion of $\mathfrak{log}$-*Kummer correspondences* in Remark 3.9.5, (vii), (Ob7), (Ob8); Remark 3.9.5, (viii), (sQ4); Remark 3.9.5, (ix); the final portion of Remark 3.9.5, (x); the discussion of the final portion of Remark 3.12.2, (v), below], in order to *rectify the* **vertical shift/mismatch** [cf. the discussion of (iii), (iv) in the case of the 0-*column*] between the **unit portion** of ${}^{1,0}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$ and the **log-shells** arising from [the image via the relevant Kummer isomorphisms of] this unit portion, which give rise to the *tensor packets of log-shells* that constitute ${}^{1,\circ}\mathcal{U}$.

(xi-e) Next, let us recall that the relationship, i.e., that arises by applying the *log-volume to the pilot-object*, between the **pilot-object log-volume** $-|\log(\underline{\underline{q}})| \in \mathbb{R}$ and the **input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip** is precisely the relationship **prescribed/imposed** by the **arithmetic holomorphic structure** in the 1-column, i.e., via the representation of the input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip on ${}^{1,\circ}\mathcal{U}$ relative

to this 1-column arithmetic holomorphic structure. That is to say, *"expressibility relative to the arithmetic holomorphic structure in the* 1*-column"* [cf. (SHE)] amounts *precisely* to

> *"expressibility via operations that are valid/executable/well-defined even when* **subject** *to the* **condition** *that the* **pilot-object log-volume** *associated to the input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip [which is, of course, linked/ related, via isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips, to the possible output data $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips!] be equal to the* **fixed value** $-\ |\log(\underline{\underline{q}})| \in \mathbb{R}$*".*

In particular, the discussion of (xi-d) thus yields the following conclusion:

> The *multiradial construction algorithm* of Theorem 3.11, followed by formation of the *holomorphic hull* and application of the *log-volume*, yields a collection of **possible log-volumes of pilot-object output data**
>
> $$\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \ \subseteq \ \mathbb{R}$$
>
> that are **linked/related** [cf. (IPL)], via **isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips**, to the **pilot-object log-volume**
>
> $$-\ |\log(\underline{\underline{q}})| \in \mathbb{R}$$
>
> of the **input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip** [cf. (SHE)].

(xi-f) Thus, we conclude from (xi-e) that

> the construction of the subset $\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \ \subseteq \ \mathbb{R}$ of **possible pilot-object log-volumes** of output data is **subject** to the **condition** that this construction of output data possibilities constitutes, in particular, a construction [perhaps only up to some sort of *"approximation"*, as a result of various *indeterminacies*] of the **pilot-object log-volume** of the **input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip**, namely, $-\ |\log(\underline{\underline{q}})| \in \mathbb{R}$.

The inclusion $-\ |\log(\underline{\underline{q}})| \in \mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|}$, hence also the inequality

$$-\ |\log(\underline{\underline{q}})| \ \leq \ -\ |\log(\underline{\underline{\Theta}})| \ \in \ \mathbb{R}$$

— i.e., the conclusion that $C_\Theta \geq -1$ for any $C_\Theta \in \mathbb{R}$ such that $-\ |\log(\underline{\underline{\Theta}})| \ \leq C_\Theta \cdot |\log(\underline{\underline{q}})|$ — in the statement of Corollary 3.12, then follows formally.

(xi-g) Thus, in summary,

> *the* **multiradial construction algorithm** *of Theorem 3.11, followed by formation of the* **holomorphic hull** *and application of the* **log-volume**, *yields* **two tautologically equivalent ways to compute** *the log-volume of the q-pilot object at* $(1, 0)$ — cf. Fig. 3.8 below.

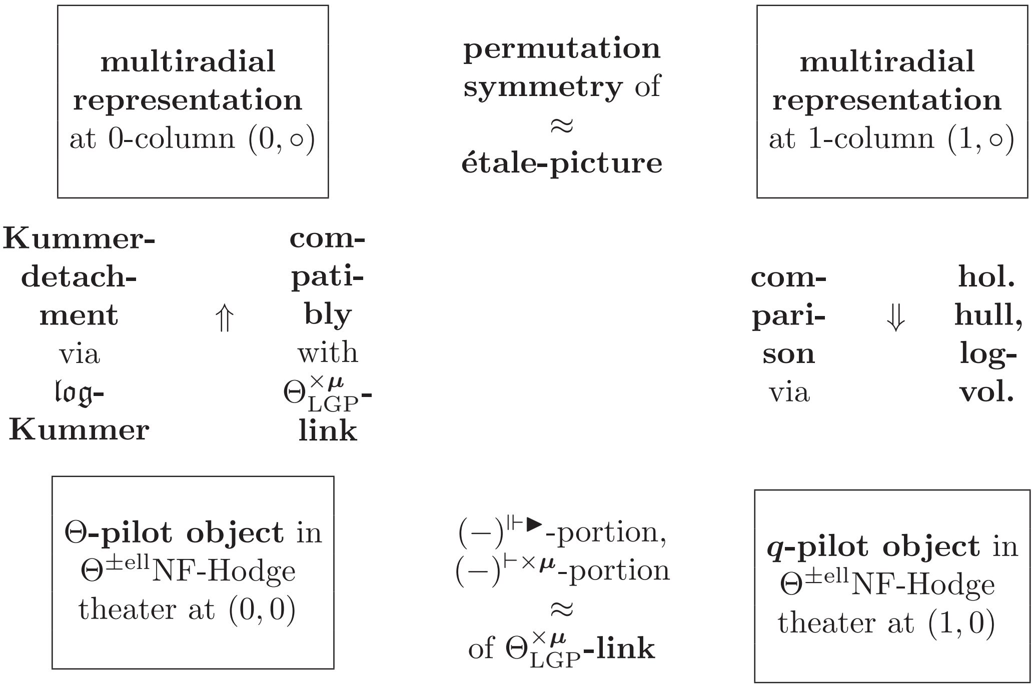

Fig. 3.8: Two tautologically equivalent ways to compute the log-volume of the q-pilot object at $(1, 0)$

(xi-h) In this context, it is useful to recall that the above argument depends, in an essential way [cf. the discussion of (ii), (vi)], on the theory of [EtTh], which *does not admit any evident generalization* to the case of N-th tensor powers of Θ-pilot objects, for $N \geq 2$. That is to say, the *log-volume* of such an N-th tensor power of a Θ-pilot object must always be computed as the result of *multiplying the log-volume of the original* Θ*-pilot object by* N — cf. Remark 2.1.1, (iv); [IUTchII], Remark 3.6.4, (iii), (iv). In particular, although the analogue of the above argument for such an N-th tensor power would lead to **sharper inequalities** than the inequalities obtained here, it is difficult to see how to obtain such sharper inequalities via a routine generalization of the above argument. In fact, as we shall see in [IUTchIV], these sharper inequalities are known to be **false** [cf. [IUTchIV], Remark 2.3.2, (ii)].

(xii) In the context of the argument of (xi), it is useful to observe the important role played by the **global** realified Frobenioids that appear in the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link. That is to say, since ultimately one is only concerned with the computation of *log-volumes*, it might appear, at first glance, that it is possible to dispense with the use of such global Frobenioids and instead work only with the various *local Frobenioids*, for $\underline{v} \in \underline{\mathbb{V}}$, that are directly related to the computation of log-volumes. On the other hand, observe that since the isomorphism of [local or global!] Frobenioids arising from the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link only preserves **isomorphism classes of objects** of these Frobenioids [cf. the discussion of Remark 3.6.2, (i)], to work only with local Frobenioids means that one must contend with the **indeterminacy** of not knowing whether, for instance, such a local Frobenioid object at some $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ corresponds to a *given open submodule* of the log-shell at $\underline{v}$ or to, say, the $p_{\underline{v}}^N$*-multiple of this submodule*, for $N \in \mathbb{Z}$. Put another way, one must contend with the indeterminacy arising from the fact that, unlike the case with the global Frobenioids "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$", "$\mathcal{F}^{\circledast \mathbb{R}}_{\mathrm{MOD}}$", objects of the various local Frobenioids that arise admit **endomorphisms**

which are **not automorphisms**. This indeterminacy has the effect of rendering *meaningless* any attempt to perform a precise log-volume computation as in (xi). ○

Remark 3.12.1.

(i) In [IUTchIV], we shall be concerned with obtaining *more explicit upper bounds* on $-|\log(\underline{\underline{\Theta}})|$, i.e., *estimates* "$C_\Theta$" as in the statement of Corollary 3.12.

(ii) It is not difficult to verify that, for $\lambda \in \mathbb{Q}_{>0}$, one may obtain a similar theory to the theory developed in the present series of papers [cf. the discussion of Remark 3.11.1, (ii)] for *"generalized* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-links"* of the form

$$\underline{\underline{q}}^{\lambda} \quad \mapsto \quad \underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}}$$

— i.e., so the theory developed in the present series of papers corresponds to the case of $\lambda = 1$. This sort of "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link" is roughly reminiscent of — but by no means equivalent to! — the sort of issues considered in the discussion of Remark 2.2.2, (i). Here, we observe that raising to the λ-th power on the "$\underline{\underline{q}}$ side" *differs quite fundamentally* from raising to the λ-th power on the "$\underline{\underline{q}}^{(1^2 \ldots (l^*)^2)}$ side", an issue that is discussed briefly [in the case of $\lambda = N$] in the final portion of Step (xi) of the proof of Corollary 3.12. That is to say, "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links" as in the above display *differ fundamentally* both from the situation of Remark 2.2.2, (i), and the situation discussed in the final portion of Step (xi) of the proof of Corollary 3.12 in that the theory of the **first power** of the **étale theta function** is left unchanged [i.e., relative to the theory developed in the present series of papers] — cf. the discussion of Remark 2.2.2, (i); Step (xi) of the proof of Corollary 3.12. At any rate, in the case of "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links" as in the above display, one may apply the same arguments as the arguments used to prove Corollary 3.12 to conclude the *inequality*

$$C_\Theta \geq -\lambda$$

— i.e., which is *sharper*, for $\lambda < 1$, than the inequality obtained in Corollary 3.12 in the case of $\lambda = 1$. In fact, however, such sharper inequalities will not be of interest to us, since, in [IUTchIV], our estimates for the upper bound C_Θ will be *sufficiently rough as to be unaffected* by adding a constant of absolute value ≤ 1.

(iii) In the context of the discussion of (ii) above, it is of interest to note that the **multiradial** theory of **mono-theta-theoretic cyclotomic rigidity**, and, in particular, the theory of the **first power** of the **étale theta function**, may be regarded as a theory that concerns a sort of **"canonical profinite volume"** on the elliptic curves under consideration associated to the **first power** of the ample line bundle corresponding to the étale theta function. This point of view is also of interest in the context of the discussion of various approaches to *cyclotomic rigidity* summarized in Fig. 3.7 [cf. also the discussion of Remark 2.3.3]. Indeed, the elementary fact "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$", which plays a key role in the **multiradial** algorithms for cyclotomic rigidity isomorphisms in the **number field** case

[cf. [IUTchI], Example 5.1, (v), as well as the discussion of Remarks 2.3.2, 2.3.3 of the present paper], may be regarded as an immediate consequence of an easy interpretation of the *product formula* in terms of the *geometry* of the *domain* in the archimedean completion of the number field $\mathbb{Q}$ determined by the inequality "≤ 1", i.e., a domain which may be thought of as a sort of concrete geometric representation of a **"canonical unit of volume"** of the number field $\mathbb{Q}$.

Remark 3.12.2.

(i) One of the main themes of the present series of papers is the issue of *dismantling the* **two underlying combinatorial dimensions** of a number field — cf. Remarks 1.2.2, (vi), of the present paper, as well as [IUTchI], Remarks 3.9.3, 6.12.3, 6.12.6; [IUTchII], Remarks 4.7.5, 4.7.6, 4.11.2, 4.11.3, 4.11.4. The principle examples of this topic may be summarized as follows:

(a) splittings of various monoids into **unit** and **value group** portions;

(b) separating the "$\mathbb{F}_l$" arising from the l-torsion points of the elliptic curve — which may be thought of as a sort of *"finite approximation"* of $\mathbb{Z}$! — into a [*multiplicative*] $\mathbb{F}_l^{\circledast}$**-symmetry** — which may also be thought of as corresponding to the *global arithmetic* portion of the arithmetic fundamental groups involved — and a(n) [*additive*] $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** — which may also be thought of as corresponding to the *geometric* portion of the arithmetic fundamental groups involved;

(c) separating the ring structures of the various **global number fields** that appear into their respective underlying **additive** structures — which may be related directly to the various *log-shells* that appear — and their respective underlying **multiplicative** structures — which may be related directly to the various *Frobenioids* that appear.

From the point of view of Theorem 3.11, example (a) may be seen in the **"non-interference"** properties that underlie the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (b), (c), as well as in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link compatibility** properties discussed in Theorem 3.11, (ii), (c), (d).

(ii) On the other hand, another important theme of the present §3 consists of the issue of **"reassembling"** *these two dismantled combinatorial dimensions* by means of the **multiradial mono-analytic containers** furnished by the **mono-analytic log-shells** — cf. Fig. 3.6 — i.e., of exhibiting the extent to which these two dismantled combinatorial dimensions **cannot be separated** from one another, at least in the case of the **Θ-pilot object**, by describing the **"structure of the intertwining"** between these two dimensions that existed prior to their separation. From this point of view, one may think of the **multiradial representations** discussed in Theorem 3.11, (i) [cf. also Theorem 3.11, (ii), (iii)], as the *final output* of this "reassembling procedure" for Θ-pilot objects. From the point of view of example (a) of the discussion of (i), this "reassembling procedure" allows one to **compute/estimate the value group portions** of various monoids of arithmetic interest in terms of the **unit group portions** of these monoids. It is precisely these estimates that give rise to the **inequality** obtained in Corollary 3.12. That is to say, from the point of view of *dismantling/reassembling the intertwining between*

value group and unit group portions, the argument of the proof of Corollary 3.12 may be summarized as follows:

(a^{itw}) When considered from the point of view of **log-volumes** of **Θ-pilot** and ***q*-pilot** objects, the correspondence of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [i.e., that sends Θ-pilot objects to q-pilot objects] may seem a bit **"mysterious"** or even, at first glance, *"self-contradictory"* to some readers.

(b^{itw}) On the other hand, this correspondence of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is made possible by the fact that one works with Θ-pilot or q-pilot objects in terms of **"sufficiently weakened data"** [namely, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link], i.e., data that is "sufficiently weak" that *one can no longer distinguish between Θ-pilot and q-pilot objects*.

(c^{itw}) Thus, if one thinks of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link as a **"single abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip"** that is *regarded/only known up to isomorphism*, then the issue of *which log-volume* such an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip corresponds to [cf. (a^{itw})] is precisely the issue of "**which intertwining** *between* **value group** *and* **unit group** *portions"* one considers, i.e., the issue of *"which arithmetic holomorphic structure"* [of the arithmetic holomorphic structures that appear in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] that one works in. Put another way, it is essentially a *tautological consequence* of the fact that these two arithmetic holomorphic structures in the *domain* and *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link are **distinguished** from one another that the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link yields a situation in which *both* the **Θ-intertwining** [i.e., the intertwining associated to the Θ-pilot object in the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] and the ***q*-intertwining** [i.e., the intertwining associated to the q-pilot object in the codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] are **simultaneously valid**, i.e.,

$$\Big(q\text{-intertwining holds}\Big) \ \wedge \ \Big(\Theta\text{-intertwining holds}\Big)$$

— cf. the discussion of the *"distinct labels approach"* in Remark 3.11.1, (vii).

(d^{itw}) On the other hand, from the point of view of the analogy between **multiradiality** and the classical theory of **parallel transport** via **connections** [cf. [IUTchII], Remark 1.7.1], the **multiradial representation** of Theorem 3.11 [cf. also the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)] asserts that, up to the *relatively mild "monodromy"* constituted by the **indeterminacies** (Ind1), (Ind2), (Ind3), one may **"parallel transport"** or **"confuse"** the Θ-pilot object in the **domain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, i.e., the Θ-pilot object represented relative to its *"native intertwining/arithmetic holomorphic structure"*, with the Θ-pilot object represented relative to the *"alien intertwining/arithmetic holomorphic structure"* in the **codomain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(e^{itw}) In particular, one may **fix** the arithmetic holomorphic structure of the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, i.e., the *"native intertwining/arithmetic holomorphic structure"* associated to the *q-pilot object* in the *codomain* of the

$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, and then, by *applying* ($\mathrm{d}^{\mathrm{itw}}$) and working up to the *indeterminacies* (Ind1), (Ind2), (Ind3) [cf. also the subtleties discussed in (iv), (v) below; Remark 3.9.5, (vii), (viii), (ix)], **construct** the *"native intertwining/arithmetic holomorphic structure"* associated to the **Θ-pilot object** in the **domain** of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link as a mathematical structure that is **intrinsically associated** to the underlying structure of — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the *"native intertwining/arithmetic holomorphic structure"* associated to the ***q*-pilot object** in the **codomain** of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)]. Indeed, this point of view is precisely the point of view that is taken in the proof of Corollary 3.12 [cf., especially, Step (xi)].

($\mathrm{f}^{\mathrm{itw}}$) One way of summarizing the situation described in ($\mathrm{e}^{\mathrm{itw}}$) is in terms of *logical relations* as follows. The **multiradial representation** of Theorem 3.11 [cf. also the discussion of Remark 3.11.1] may be thought of [cf. the *first* "$\Longrightarrow$" of the following display] as an algorithm for constructing, **up to suitable indeterminacies** [cf. the discussion of ($\mathrm{e}^{\mathrm{itw}}$)], the **"Θ-intertwining"** as a mathematical structure that is **intrinsically associated** to the underlying structure of — hence, in particular, **simultaneously with/without invalidating** [cf. the *logical relator "AND"*, i.e., "$\wedge$"] the conditions imposed by — the **"*q*-intertwining"**, while holding the **"single abstract $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip"** of the discussion of ($\mathrm{b}^{\mathrm{itw}}$), ($\mathrm{c}^{\mathrm{itw}}$) **fixed**, i.e., in symbols:

$$\Big(q\text{-itw.}\Big) \quad \Longrightarrow \quad \Big(q\text{-itw.}\Big) \wedge \Big(\Theta\text{-itw./indets.}\Big) \quad \Longrightarrow \quad \Big(\Theta\text{-itw./indets.}\Big)$$

— where the *second* "$\Longrightarrow$" of the above display is *purely formal*; "itw." and "/indets." are to be understood, respectively as abbreviations for "intertwining holds" and "up to suitable indeterminacies". Here, we observe that

> the "$\wedge$" of the above display may be regarded as the *"image"* of, hence, in particular, as a *consequence* of, the "$\wedge$" in the display of ($\mathrm{c}^{\mathrm{itw}}$), via the various **(sub)quotient operations** discussed in Remark 3.9.5, (viii), i.e., whose subtle *compatibility properties* allow one to conclude the "$\wedge$" of the above display from the "$\wedge$" in the display of ($\mathrm{c}^{\mathrm{itw}}$).

Thus, at the level of *logical relations*,

> the ***q*-intertwining**, hence also the *log-volume* of the *q-pilot object* in the codomain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, may be thought of as a **special case** of the **Θ-intertwining**, i.e., at a more concrete level, of the *log-volume* of the Θ-*pilot object* in the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, regarded up to *suitable indeterminacies*.

Corollary 3.12 then follows, *essentially formally*.

Alternatively, from the point of view of *"[very rough!] toy models"*, i.e., whose goal lies solely in representing certain *overall qualitative aspects* of a situation, one may think of the discussion of ($\mathrm{a}^{\mathrm{itw}}$) $\sim$ ($\mathrm{f}^{\mathrm{itw}}$) given above in the following terms:

(a^{toy}) Consider **two distinct copies** ${}^q\mathbb{R}$ and ${}^\Theta\mathbb{R}$ of the **topological field of real numbers** $\mathbb{R}$, equipped with labels "q" and "Θ", together with an *abstract symbol* "$*$" and *assignments*

$$\lambda_q : * \mapsto {}^q(-h) \in {}^q\mathbb{R}, \qquad \lambda_\Theta : * \mapsto {}^\Theta(-2h) \in {}^\Theta\mathbb{R},$$

— where, in the present discussion, we shall write "${}^q(-)$", "${}^\Theta(-)$" to denote the respective elements/subsets of ${}^q\mathbb{R}$, ${}^\Theta\mathbb{R}$ determined by an element/subset "$(-)$" of $\mathbb{R}$; $h \in \mathbb{R}_{>0}$ is a positive real number that we are interested in **bounding** from above. If one **forgets** the *distinct labels* "q" and "Θ", then these two assignments λ_q, λ_Θ are **mutually incompatible** and *cannot be considered simultaneously*, i.e., they *contradict* one another [in the sense that $\mathbb{R} \ni -h \neq -2h \in \mathbb{R}$].

(b^{toy}) One aspect of the situation of (a^{toy}) that renders the *simultaneous consideration* of the two assignments λ_q, λ_Θ *valid* — i.e., at the level of *logical relations*,

$$\Big(* \mapsto {}^q(-h) \in {}^q\mathbb{R}\Big) \wedge \Big(* \mapsto {}^\Theta(-2h) \in {}^\Theta\mathbb{R}\Big)$$

— is the use of the **abstract symbol** "$*$", i.e., which is, *a priori*, entirely **unrelated** to any copies of $\mathbb{R}$ [such as ${}^q\mathbb{R}$, ${}^\Theta\mathbb{R}$].

(c^{toy}) The other aspect of the situation of (a^{toy}) that renders the *simultaneous consideration* of the two assignments λ_q, λ_Θ *valid*— i.e., at the level of *logical relations*,

$$\Big(* \mapsto {}^q(-h) \in {}^q\mathbb{R}\Big) \wedge \Big(* \mapsto {}^\Theta(-2h) \in {}^\Theta\mathbb{R}\Big)$$

— is the use of the **distinct labels** "q", "Θ" for the copies of $\mathbb{R}$ that appear in the assignments λ_q, λ_Θ.

(d^{toy}) Now let us consider an *alternative approach to constructing* the assignment λ_Θ: We construct λ_Θ as the **"assignment with indeterminacies"**

$$\lambda_\Theta^{\mathrm{Ind}} : * \mapsto {}^\Theta\mathbb{R}_{\leq -2h+\epsilon} \subseteq {}^\Theta\mathbb{R}$$

— where $\mathbb{R}_{\leq -2h+\epsilon} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R} \mid x \leq -2h+\epsilon\} \subseteq \mathbb{R}$; $\epsilon \in \mathbb{R}_{>0}$ is some positive number.

(e^{toy}) Now suppose that one **verifies** that one may **construct** the *"assignment with indeterminacies"* $\lambda_\Theta^{\mathrm{Ind}}$ of (d^{toy}) as a mathematical structure that is **intrinsically associated** to the underlying structure of the assignment λ_q — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the assignment λ_q, even if one *forgets the labels* "q", "Θ" that were appended to copies of $\mathbb{R}$, i.e., even if one *identifies* ${}^q\mathbb{R}$, ${}^\Theta\mathbb{R}$, in the usual way, with $\mathbb{R}$ [cf. the properties (IPL), (SHE) of Remark 3.11.1, (iii)]. That is to say, we suppose that one can show that the assignments determined, respectively, by λ_q, $\lambda_\Theta^{\mathrm{Ind}}$, by identifying copies of $\mathbb{R}$, namely,

$$* \mapsto -h \in \mathbb{R}, \qquad * \mapsto \mathbb{R}_{\leq -2h+\epsilon} \subseteq \mathbb{R}$$

— where the latter assignment may be considered as the assignment that maps $*$ to *"some [undetermined] element* $\in \mathbb{R}_{\leq -2h+\epsilon}$*"* — are such that one may construct the latter assignment as a mathematical structure that is **intrinsically associated** to — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the former assignment. Here, we note that it is *not particularly relevant* that "$\mathbb{R}_{\leq -2h+\epsilon}$" arose as some sort of *"perturbation via indeterminacies of* $2h$*"* [cf. the property (HIS) of Remark 3.11.1, (iv)].

($\mathrm{f}^{\mathrm{toy}}$) The discussion of ($\mathrm{e}^{\mathrm{toy}}$) may be summarized at the level of *logical relations* [cf. the displays of ($\mathrm{b}^{\mathrm{toy}}$), ($\mathrm{c}^{\mathrm{toy}}$)] as follows:

$$\Big(* \mapsto -h\Big) \implies \Big(* \mapsto -h\Big) \wedge \Big(* \mapsto \mathbb{R}_{\leq -2h+\epsilon}\Big) \implies \Big(* \mapsto \mathbb{R}_{\leq -2h+\epsilon}\Big)$$

— that is to say, "$* \mapsto -h$" may be regarded as a **special case** of "$* \mapsto \mathbb{R}_{\leq -2h+\epsilon}$", which, in turn, may be regarded as a *"version with indeterminacies"* of "$* \mapsto -2h$". One then concludes formally that $-h \in \mathbb{R}_{\leq -2h+\epsilon}$ and hence that

$$-h \leq -2h + \epsilon, \quad \text{i.e.,} \quad h \leq \epsilon$$

— that is to say, the *desired upper bound on* h.

(iii) One fundamental aspect of the theory that renders possible the "reassembling procedure" discussed in (ii) [cf. the discussion of Step (iv) of the proof of Corollary 3.12] is the **"juggling of ⊞, ⊠"** [cf. the discussion of Remark 1.2.2, (vi)] effected by the **$\mathfrak{log}$-links**, i.e., the **vertical arrows** of the log-theta-lattice. This "juggling of ⊞, ⊠" may be thought of as a sort of combinatorial way of representing the **arithmetic holomorphic structure** associated to a **vertical line** of the log-theta-lattice. Indeed, at *archimedean primes*, this juggling amounts essentially to *multiplication by* $\pm i$, which is a well-known method [cf. the notion of an *"almost complex structure"*!] for representing holomorphic structures in the classical theory of differential manifolds. On the other hand, it is important to recall in this context that this "juggling of ⊞, ⊠" is precisely what gives rise to the **upper semi-compatibility** indeterminacy (Ind3) [cf. Proposition 3.5, (ii); Remark 3.10.1, (i)].

(iv) In the context of the discussion of (ii), (iii), it is of interest to **compare**, in the cases of the **0-** and **1-columns** of the log-theta-lattice, the way in which the theory of **$\mathfrak{log}$-Kummer correspondences** associated to a vertical column of the log-theta-lattice is applied in the proof of Corollary 3.12, especially in Steps (x) and (xi). We begin by observing that the *vertical column* [i.e., 0- or 1-column] under consideration may be depicted [*"horizontally"*!] in the fashion of the diagram of

the third display of Proposition 1.3, (iv)

$$\begin{array}{ccccccccc} & & & & \bullet_0 & & & & \\ & & & & \| & & & & \\ \dots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

— where the "$\bullet_0$" in the first line of the diagram denotes the portion with *vertical coordinate* 0 [i.e., the portion at $(0,0)$ or $(1,0)$] of the vertical column under consideration. As discussed in Step (iii) of the proof of Corollary 3.12, since the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is **fundamentally incompatible** with the **distinct arithmetic holomorphic structures** — i.e., **ring structures** — that exist in the 0- and 1-columns, one is obliged to work with the **Frobenius-like** versions of the *unit group* and *value group* portions of monoids arising from "$\bullet_0$" in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link precisely in order to avoid the need to contend, in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, with the issue of describing the **"structure of the intertwining"** [cf. the discussion of (ii)] between these unit group and value group portions determined by the *distinct arithmetic holomorphic structures* — i.e., *ring structures* — that exist in the 0- and 1-columns. On the other hand, one is *also* obliged to work with the **étale-like** "$\circ$" versions of various objects since it is precisely these *vertically coric* versions that allow one to **access**, i.e., by serving as **containers** [cf. the discussion of (ii)] for, the other "$\bullet$'s" in the vertical column under consideration. That is to say, although the various **Kummer isomorphisms** that relate various portions of the Frobenius-like "$\bullet_0$" to the corresponding portions of the étale-like "$\circ$" may at first give the impression that either "$\bullet_0$" or "$\circ$" is superfluous or unnecessary in the theory, in fact

> *both "$\bullet_0$" and "$\circ$"* play an **essential** and **by no means superfluous** role in the theory of the vertical columns of the log-theta-lattice.

This aspect of the theory is essentially the same in the case of both the 0- and the 1-columns. The **$\mathfrak{log}$-link compatibility** of the various **log-volumes** that appear [cf. the discussion of Step (x) of the proof of Corollary 3.12; Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] is another aspect of the theory that is essentially the same in the case of both the 0- and the 1-columns. Also, although the discussion of the **"non-interference"** properties that underlie the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (b), (c), was only given expicitly, in effect, in the case of the 0-column, i.e., concerning **Θ-pilot objects**, entirely similar "non-interference" properties hold for **$\boldsymbol{q}$-pilot objects**. [Indeed, this may be seen, for instance, by applying the *same arguments* as the arguments that were applied in the case of Θ-pilot objects, or, for instance, by *specializing* the non-interference properties obtained for Θ-pilot objects to the index "$j = 1$" as in the discussion of *"pivotal distributions"* in [IUTchI], Example 5.4, (vii).] These *similarities* between the 0- and 1-columns are summarized in the upper portion of Fig. 3.9 below.

(v) In the discussion of (iv), we highlighted various *similarities* between the 0- and 1-columns of the log-theta-lattice in the context of Steps (x), (xi) of the

proof of Corollary 3.12. By contrast, one *significant difference* between the theory of $\mathfrak{log}$-Kummer correspondences in the 0- and 1-columns is

> the **lack** of analogues for ***q*-pilot objects** of the crucial **multiradiality** properties summarized in Theorem 3.11, (iii), (c)

Aspect of the theory	*0-column/ Θ-pilot objects*	*1-column/ q-pilot objects*
essential role of *both* "$\bullet_0$" *and* "$\circ$"	similar	similar
$\mathfrak{log}$**-link compatibility** of **log-volumes**	similar	similar
"non-interference" properties of $\mathfrak{log}$**-Kummer correspondences**	similar	similar
multiradiality properties of Θ-/q-pilot objects	hold	do *not* hold
treatment of **log-shells/ unit group** portions	used as **mono-analytic containers** for **regions**	**tautological** documenting device for **logarithmic** relationship betw. ring structures
resulting **indeterminacies** acting on **log-shells**	(Ind1), (Ind2), (Ind3)	absorbed by applying **holomorphic hulls**, **log-volumes**

Fig. 3.9: Similarities and differences, in the context of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, between the 0- and 1-columns of the log-theta-lattice

— i.e., in effect, the lack of an analogue for the q-pilot objects of the theory of *rigidity properties* developed in [EtTh] [cf. the discussion of Remark 2.2.2, (i)]. Another *significant difference* between the theory of $\mathfrak{log}$-Kummer correspondences

in the 0- and 1-columns lies in the way in which the associated *vertically coric holomorphic* **log-shells** [cf. Proposition 1.2, (ix)] are treated in their relationship to the **unit group** portions of monoids that occur in the various "•'s" of the $\mathfrak{log}$-Kummer correspondence. That is to say, in the case of the **0-column**, these log-shells are used as **containers** [cf. the discussion of (ii)] for the various **regions** [i.e., subsets] arising from these unit group portions via various composites of arrows in the $\mathfrak{log}$-Kummer correspondence. This approach has the *advantage* of admitting an *interpretation* — i.e., in terms of subsets of **mono-analytic log-shells** — that makes sense even relative to the *distinct arithmetic holomorphic structures* that appear in the 1-column of the log-theta-lattice [cf. Remark 3.11.1]. On the other hand, it has the *drawback* that it gives rise to the **upper semi-compatibility** indeterminacy (Ind3) discussed in the final portion of Theorem 3.11, (ii). By contrast,

> in the case of the **1-column**, since the associated **arithmetic holomorphic structure** is held **fixed** and *regarded* [cf. the discussion of Step (xi) of the proof of Corollary 3.12] as the *standard* with respect to which constructions arising from the 0-column are to be *computed*, there is **no need** [i.e., in the case of the 1-column] to require that the constructions applied **admit mono-analytic interpretations**.

That is to say, in the case of the 1-column, the various unit group portions of monoids at the various "•'s" simply serve as a means of *documenting the* **"logarithmic"** *relationship* [cf. the definition of the $\mathfrak{log}$-link given in Definition 1.1, (i), (ii)!] between the *ring structures* in the domain and codomain of the $\mathfrak{log}$-link. These ring structures give rise to the local copies of sets of integral elements "$\mathcal{O}$" with respect to which the "$\mathfrak{mod}$" *versions* [cf. Example 3.6, (ii)] of *categories of arithmetic line bundles* are defined at the various "•'s". Since the objects of these categories of arithmetic line bundles are **not equipped** with **local trivializations** at the various $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of **isomorphism classes of objects** of Frobenioids in Remark 3.6.2, (i)],

> **regions** in log-shells may only be related to such categories of arithmetic line bundles at the expense of allowing for an **indeterminacy** with respect to **"$\mathcal{O}^{\times}$"-multiples** at each $\underline{v} \in \underline{\mathbb{V}}$.

It is precisely this indeterminacy that necessitates the introduction, in Step (xi) of the proof of Corollary 3.12, of **holomorphic hulls**, i.e., which have the effect of *absorbing* this indeterminacy [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Finally, in Step (xi) of the proof of Corollary 3.12,

> the **indeterminacy** in the *specification of a particular member* of the collection of ring structures just discussed — i.e., arising from the *choice of a particular composite* of arrows in the $\mathfrak{log}$-Kummer correspondence that is used to specify a **particular ring structure** among its various "logarithmic conjugates" — is **absorbed** by passing to **log-volumes**

— i.e., by applying the **$\mathfrak{log}$-link compatibility** [cf. (iv)] of the various log-volumes associated to these ring structures [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Thus, unlike the case of the 0-column, where the *mono-analytic* interpretation via *regions* of mono-analytic log-shells gives rise only to *upper bounds* on log-volumes, the approach just discussed in the case of the 1-column — i.e., which makes essential use of the **ring structures** that are available

as a consequence of the fact that the **arithmetic holomorphic structure** is held **fixed** — gives rise to **precise equalities** [i.e., not just inequalities!] concerning log-volumes. These *differences* between the 0- and 1-columns are summarized in the lower portion of Fig. 3.9.

Remark 3.12.3.

(i) Let S be a *hyperbolic Riemann surface of finite type* of genus g_S with r_S punctures. Write $\chi_S \stackrel{\mathrm{def}}{=} -(2g_S - 2 + r_S)$ for the *Euler characteristic* of S and $d\mu_S$ for the Kähler metric on S [i.e., the $(1,1)$-form] determined by the *Poincaré metric* on the upper half-plane. Recall the *analogy* discussed in [IUTchI], Remark 4.3.3, between the theory of **log-shells**, which plays a key role in the theory developed in the present series of papers, and the **classical metric geometry of hyperbolic Riemann surfaces**. Then, relative to this analogy, the **inequality** obtained in Corollary 3.12 may be regarded as corresponding to the inequality

$$\chi_S \;=\; -\int_S d\mu_S \;<\; 0$$

— i.e., in essence, a statement of the **hyperbolicity** of S — arising from the classical **Gauss-Bonnet formula**, together with the **positivity** of $d\mu_S$. Relative to the analogy between *real analytic Kähler metrics* and *ordinary Frobenius liftings* discussed in [pOrd], Introduction, §2 [cf. also the discussion of [pTeich], Introduction, §0], the *local* property constituted by this positivity of $d\mu_S$ may be thought of as corresponding to the [*local* property constituted by the] *Kodaira-Spencer isomorphism of an indigenous bundle* — i.e., which gives rise to the *ordinarity* of the corresponding Frobenius lifting on the ordinary locus — in the p-adic theory. As discussed in [AbsTopIII], §I5, these properties of indigenous bundles in the p-adic theory may be thought of as corresponding, in the theory of *log-shells*, to the *"maximal incompatibility"* between the various *Kummer isomorphisms* and the *corically constructed data* of the Frobenius-picture of Proposition 1.2, (x). On the other hand, it is just this "maximal incompatibility" that gives rise to the *"upper semi-commutativity"* discussed in Remark 1.2.2, (iii), i.e., [from the point of view of the theory of the present §3] the **upper semi-compatibility** indeterminacy (Ind3) of Theorem 3.11, (ii), that underlies the **inequality** of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12].

(ii) The *"metric aspect"* of Corollary 3.12 discussed in (i) is reminiscent of the analogy between the theory of the present series of papers and *classical complex Teichmüller theory* [cf. the discussion of [IUTchI], Remark 3.9.3] in the following sense:

> Just as *classical complex Teichmüller theory* is concerned with relating distinct holomorphic structures in a sufficiently **canonical** way as to **minimize** the resulting **conformality distortion**, the **canonical** nature of the algorithms discussed in Theorem 3.11 for relating **alien arithmetic holomorphic structures** [cf. Remark 3.11.1] gives rise to a relatively **strong estimate** of the **[log-]volume distortion** [cf. Corollary 3.12] resulting from such a deformation of the arithmetic holomorphic structure.

Remark 3.12.4. In light of the discussion of Remark 3.12.3, it is of interest to reconsider the analogy between the theory of the present series of papers and the *p-adic Teichmüller theory* of [*p*Ord], [*p*Teich], in the context of Theorem 3.11, Corollary 3.12.

(i) First, we observe that the **splitting monoids** at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b)] may be regarded as analogous to the **canonical coordinates** of p-adic Teichmüller theory [cf., e.g., [*p*Teich], Introduction, §0.9] that are constructed over the *ordinary locus* of a canonical curve. In particular, it is natural to regard the **bad primes** $\in \underline{\mathbb{V}}^{\text{bad}}$ as corresponding to the **ordinary** locus of a canonical curve and the **good primes** $\in \underline{\mathbb{V}}^{\text{good}}$ as corresponding to the **supersingular** locus of a canonical curve. This point of view is reminiscent of the discussion of [IUTchII], Remark 4.11.4, (iii).

(ii) On the other hand, the **bi-coric mono-analytic log-shells** — i.e., the various local "$\mathcal{O}^{\times \boldsymbol{\mu}}$" — that appear in the tensor packets of Theorem 3.11, (i), (a); Theorem 3.11, (ii), (a), may be thought of as corresponding to the **[multiplicative!] Teichmüller representatives** associated to the various Witt rings that appear in p-adic Teichmüller theory. Within a *fixed arithmetic holomorphic structure*, these *mono-analytic* log-shells arise from **"local holomorphic units"** — i.e., "$\mathcal{O}^{\times}$" — which are subject to the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**. These "local holomorphic units" may be thought of as corresponding to the **positive characteristic ring structures** on [the positive characteristic reductions of] Teichmüller representatives. Here, the **uniradial**, i.e., **"non-multiradial"**, nature of these "local holomorphic units" [cf. the discussion of [IUTchII], Remark 4.7.4, (ii); [IUTchII], Figs. 4.1, 4.2] may be regarded as corresponding to the *mixed characteristic nature of Witt rings*, i.e., the **incompatibility** of Teichmüller representatives with the **additive structure** of Witt rings.

(iii) The set $\mathbb{F}_l^{\circledast}$ of $l^{\circledast}$ **"theta value labels"**, which plays an important role in the theory of the present series of papers, may be thought of as corresponding to the **"factor of p"** that appears in the *"mod p/p^2 portion"*, i.e., the gap separating the *"mod p"* and *"mod p^2"* portions, of the rings of Witt vectors that occur in the p-adic theory. From this point of view, one may think of the *procession-normalized volumes* obtained by taking **averages over** $j \in \mathbb{F}_l^{\circledast}$ [cf. Corollary 3.12] as corresponding to the operation of **dividing by p** to relate the "mod p/p^2 portion" of the Witt vectors to the "mod p portion" of the Witt vectors [i.e., the characteristic p theory]. In this context, the **multiradial representation** of Theorem 3.11, (i), by means of *mono-analytic log-shells labeled by elements of* $\mathbb{F}_l^{\circledast}$ may be thought of as corresponding to the **derivative** of the **canonical Frobenius lifting** on a canonical curve in the p-adic theory [cf. the discussion of [AbsTopIII], §I5] in the sense that this multiradial representation may be regarded as a sort of **comparison** of the **canonical splitting monoids** discussed in (i) to the **"absolute constants"** [cf. the discussion of (ii)] constituted by the **bi-coric** *mono-analytic log-shells*. This *"absolute comparison"* is precisely what results in the **indeterminacies** (Ind1), (Ind2) of Theorem 3.11, (i).

(iv) In the context of the discussion of (iii), we note that the set of labels $\mathbb{F}_l^{\circledast}$ may, alternatively, be thought of as corresponding to the **infinitesimal moduli** of the positive characteristic curve under consideration in the p-adic theory [cf. the

discussion of [IUTchII], Remark 4.11.4, (iii), (d)]. That is to say, the *"deformation dimension"* constituted by the *horizontal dimension* of the log-theta-lattice in the theory of the present series of papers or by the deformations modulo various powers of p in the p-adic theory [cf. Remark 1.4.1, (iii); Fig. 1.3] is **highly canonical** in nature, hence may be thought of as being equipped with a natural isomorphism to the **"absolute moduli"** — i.e., so to speak, the *"moduli over $\mathbb{F}_1$"* — of the *given number field equipped with an elliptic curve*, in the theory of the present series of papers, or of the *given positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle*, in p-adic Teichmüller theory.

Inter-universal Teichmüller theory	*p-adic Teichmüller theory*
splitting monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$	**canonical coordinates** on the *ordinary locus*
bad primes $\in \underline{\mathbb{V}}^{\mathrm{bad}}$	**ordinary** locus of a can. curve
good primes $\in \underline{\mathbb{V}}^{\mathrm{good}}$	**supersing.** locus of a can. curve
mono-analytic log-shells "$\mathcal{O}^{\times\boldsymbol{\mu}}$"	**[multiplicative!] Teich. reps.**
uniradial "local hol. units $\mathcal{O}^{\times}$" subject to $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**	**pos. char. ring structures** on [pos. char. reductions of] Teich. reps.
set of **"theta value labels"** $\mathbb{F}_l^{*}$	**factor p** in mod p/p^2 portion of Witt vectors
multiradial rep. via $\mathbb{F}_l^{*}$-labeled **mono-analytic log-shells** [cf. (Ind1), (Ind2), (Ind3)]	**derivative** of the **canonical Frobenius lifting**
set of **"theta value labels"** $\mathbb{F}_l^{*}$	implicit **"absolute moduli/$\mathbb{F}_1$"**
inequality arising from **upper semi-compatibility** [cf. (Ind3)]	**inequality** arising from interference between **Frobenius conjugates**

Fig. 3.10: The analogy between inter-universal Teichmüller theory and p-adic Teichmüller theory

(v) Let A be the ring of Witt vectors of a perfect field k of positive characteristic p; X a *smooth, proper hyperbolic curve* over A of genus g_X which is **canonical** in the sense of p-adic Teichmüller theory; $\widehat{X}$ the p-adic formal scheme associated to X; $\widehat{U} \subseteq \widehat{X}$ the *ordinary locus* of $\widehat{X}$. Write ω_{X_k} for the canonical bundle of $X_k \stackrel{\text{def}}{=} X \times_A k$. Then when [cf. the discussion of (iii)] one computes the **derivative** of the **canonical Frobenius lifting** $\Phi : \widehat{U} \to \widehat{U}$ on $\widehat{U}$, one must contend with *"interference phenomena"* between the various *copies* of some positive characteristic algebraic geometry set-up — i.e., at a more concrete level, the various *Frobenius conjugates* "t^{p^n}" [where t is a local coordinate on X_k] associated to various $n \in \mathbb{N}_{\geq 1}$. In particular, this derivative only yields [upon dividing by p] an *inclusion* [i.e., not an isomorphism!] of line bundles

$$\omega_{X_k} \hookrightarrow \Phi^* \omega_{X_k}$$

— also known as the **"[square] Hasse invariant"** [cf. [pOrd], Chapter II, Proposition 2.6; the discussion of "generalities on ordinary Frobenius liftings" given in [pOrd], Chapter III, §1]. Thus, at the level of *global degrees of line bundles*, we obtain an *inequality* [i.e., not an equality!]

$$(1-p)(2g_X - 2) \leq 0$$

— which may be thought of as being, in essence, a statement of the **hyperbolicity** of X [cf. the inequality of the display of Remark 3.12.3, (i)]. Since the *"Frobenius conjugate dimension"* [i.e., the "n" that appears in "t^{p^n}"] in the p-adic theory corresponds to the *vertical dimension* of the log-theta-lattice in the theory of the present series of papers [cf. Remark 1.4.1, (iii); Fig. 1.3], we thus see that the inequality of the above display in the p-adic case arises from circumstances that are *entirely analogous* to the circumstances — i.e., the **upper semi-compatibility** indeterminacy (Ind3) of Theorem 3.11, (ii) — that underlie the **inequality** of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12; the discussion of Remark 3.12.3, (i)].

(vi) The analogies of the above discussion are summarized in Fig. 3.10 above.

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