INTER-UNIVERSAL TEICHMÜLLER THEORY IV: LOG-VOLUME COMPUTATIONS AND SET-THEORETIC FOUNDATIONS

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Abstract. The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers of the series, we introduced and studied the theory surrounding the **log-theta-lattice**, a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Consideration of various properties of the log-theta-lattice led naturally to the establishment, in the third paper of the series, of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"**. Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the present paper, estimates arising from these multiradial algorithms for splitting monoids of LGP-monoids are applied to verify various **diophantine results** which imply, for instance, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves. Finally, we examine — albeit from an extremely *naive/non-expert* point of view! — the *foundational/set-theoretic* issues surrounding the *vertical* and *horizontal arrows* of the log-theta-lattice by introducing and studying the basic properties of the notion of a **"species"**, which may be thought of as a sort of formalization, via set-theoretic formulas, of the intuitive notion of a *"type of mathematical object"*. These foundational issues are closely related to the central role played in the present series of papers by various results from **absolute anabelian geometry**, as well as to the idea of **gluing together** *distinct models of conventional scheme theory*, i.e., in a fashion that lies outside the framework of conventional scheme theory. Moreover, it is precisely these foundational issues surrounding the vertical and horizontal arrows of the log-theta-lattice that led naturally to the introduction of the term **"inter-universal"**.

Contents:

Introduction

The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers, [IUTchI], [IUTchII], and [IUTchIII], of the series, we introduced and studied the theory surrounding the **log-theta-lattice** [cf. the discussion of [IUTchIII], Introduction], a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper [IUTchI] of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$ [cf. [IUTchI], §I1]. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"** [cf. [IUTchIII], Theorem A]. Here, we recall that "multiradial algorithms" [cf. the discussion of the Introductions to [IUTchII], [IUTchIII]] are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the final portion of [IUTchIII], by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtained *estimates* for the *log-volume* of these LGP-monoids [cf. [IUTchIII], Theorem B]. In the present paper, these estimates will be applied to verify various **diophantine results**.

In §1 of the present paper, we start by discussing various *elementary estimates* for the *log-volume* of various *tensor products* of the modules obtained by applying the *p-adic logarithm* to the *local units* — i.e., in the terminology of [IUTchIII], *"tensor packets of log-shells"* [cf. the discussion of [IUTchIII], Introduction] — in terms of various well-known invariants, such as *differents*, associated to a *mixed-characteristic nonarchimedean local field* [cf. Propositions 1.1, 1.2, 1.3, 1.4]. We then discuss similar — but technically much simpler! — log-volume estimates in the case of *complex archimedean local fields* [cf. Proposition 1.5]. After reviewing a certain classical estimate concerning the *distribution of prime numbers* [cf. Proposition 1.6], as well as some elementary general nonsense concerning *weighted averages* [cf. Proposition 1.7] and well-known elementary facts concerning *elliptic curves* [cf. Proposition 1.8], we then proceed to *compute explicitly*, in more elementary language, the quantity that was estimated in [IUTchIII], Theorem B. These computations yield a *quite strong/explicit* **diophantine inequality** [cf. Theorem 1.10] concerning elliptic curves that are in **"sufficiently general position"**, so that one may apply the general theory developed in the first three papers of the series.

In §2 of the present paper, after reviewing another classical estimate concerning the *distribution of prime numbers* [cf. Proposition 2.1, (ii)], we then proceed to apply the theory of [GenEll] to **reduce** various diophantine results concerning an **arbitrary** *elliptic curve over a number field* to results of the type obtained in Theorem 1.10 concerning elliptic curves that are in **"sufficiently general position"** [cf. Corollary 2.2]. This reduction allows us to derive the following result [cf. Corollary 2.3], which constitutes the **main application** of the *"inter-universal Teichmüller theory"* developed in the present series of papers.

Theorem A. **(Diophantine Inequalities)** *Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\mathrm{def}}{=} X \backslash D$; d a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write ω_X for the canonical sheaf on X. Suppose that U_X is a* **hyperbolic curve***, i.e., that the degree of the line bundle $\omega_X(D)$ is* **positive***. Then, relative to the notation of [GenEll] [reviewed in the discussion preceding Corollary 2.2 of the present paper], one has an* **inequality of "bounded discrepancy classes"**

$$\mathrm{ht}_{\omega_X(D)} \ \lesssim \ (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \mathrm{ht}_{\omega_X(D)}$ is bounded below by a **constant** *on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), of the present paper].*

Thus, Theorem A asserts an *inequality* concerning the *canonical height* [i.e., "$\mathrm{ht}_{\omega_X(D)}$"], the *logarithmic different* [i.e., "log-diff_X"], and the *logarithmic conductor* [i.e., "log-cond_D"] of points of the curve U_X valued in number fields whose extension degree over $\mathbb{Q}$ is $\leq d$. In particular, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves all follow as special cases of Theorem A. We refer to [Vjt] for a detailed exposition of these conjectures.

Finally, in §3, we examine — albeit from an extremely *naive/non-expert* point of view! — certain **foundational issues** underlying the theory of the present series of papers. Typically in mathematical discussions [i.e., by mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] — such as, for instance, the theory developed in the present series of papers! — one defines various **"types of mathematical objects"** [i.e., such as groups, topological spaces, or schemes], together with a notion of *"morphisms"* between two particular examples of a specific type of mathematical object [i.e., morphisms between groups, between topological spaces, or between schemes]. Such objects and morphisms [typically] determine a *category*. On the other hand, if one restricts one's attention to such a category, then one must keep in mind the fact that the structure of the category — i.e., which consists *only of a collection of objects and morphisms satisfying certain properties!* — does not include any mention of the various sets and conditions satisfied by those sets that give rise to the "type of mathematical object" under consideration. For instance, the data consisting of the underlying set of a group, the group multiplication law on the group, and the properties satisfied by this group multiplication law *cannot be recovered* [at least in an *a priori* sense!] from the structure of the "*category* of groups". Put another way, although the notion of a "type of mathematical object" may give rise to a "category of such objects", the notion of a "type of mathematical object" is much *stronger* — in the sense that it involves much more *mathematical structure* — than the notion of a category. Indeed, a given "type of mathematical object" may have a *very complicated internal structure*, but may give rise to a category equivalent to a *one-morphism category* [i.e., a category with precisely one morphism]; in particular, in such cases, the structure of the associated category does not retain any information of interest concerning the internal structure of the "type of mathematical object" under consideration.

In Definition 3.1, (iii), we formalize this intuitive notion of a "type of mathematical object" by defining the notion of a **species** as, roughly speaking, a *collection of set-theoretic formulas* that gives rise to a category in any given *model of set theory* [cf. Definition 3.1, (iv)], but, unlike any *specific* category [e.g., of groups, etc.] is **not confined** to any **specific model of set theory**. In a similar vein, by working with *collections of set-theoretic formulas*, one may define a species-theoretic analogue of the notion of a *functor*, which we refer to as a **mutation** [cf. Definition 3.3, (i)]. Given a diagram of mutations, one may then define the notion of a "mutation that extracts, from the diagram, a certain portion of the types of mathematical objects that appear in the diagram that is *invariant* with respect to the mutations in the diagram"; we refer to such a mutation as a **core** [cf. Definition 3.3, (v)].

One fundamental example, in the context of the present series of papers, of a diagram of mutations is the usual set-up of **[absolute] anabelian geometry** [cf. Example 3.5 for more details]. That is to say, one begins with the *species* constituted by schemes satisfying certain conditions. One then considers the *mutation*

$$X \quad \rightsquigarrow \quad \Pi_X$$

that associates to such a scheme X its étale fundamental group Π_X [say, considered up to inner automorphisms]. Here, it is important to note that the codomain of this mutation is the *species* constituted by topological groups [say, considered up to inner automorphisms] that satisfy certain conditions which *do not include* any information concerning *how the group is related* [for instance, via some sort of étale fundamental group mutation] *to a scheme*. The notion of an **anabelian reconstruction algorithm** may then be formalized as a *mutation* that forms a *"mutation-quasi-inverse"* to the fundamental group mutation.

Another fundamental example, in the context of the present series of papers, of a diagram of mutations arises from the *Frobenius morphism* in positive characteristic scheme theory [cf. Example 3.6 for more details]. That is to say, one fixes a prime number p and considers the *species* constituted by reduced quasi-compact schemes of characteristic p and quasi-compact morphisms of schemes. One then considers the *mutation* that associates

$$S \quad \rightsquigarrow \quad S^{(p)}$$

to such a scheme S the scheme $S^{(p)}$ with the same topological space, but whose regular functions are given by the p-th powers of the regular functions on the original scheme. Thus, the domain and codomain of this mutation are given by the same species. One may also consider a *log scheme* version of this example, which, at the level of monoids, corresponds, in essence, to assigning

$$M \quad \rightsquigarrow \quad p \cdot M$$

to a torsion-free abelian monoid M the submonoid $p \cdot M \subseteq M$ determined by the image of multiplication by p. Returning to the case of schemes, one may then observe that the well-known constructions of the **perfection** and the **étale site**

$$S \quad \rightsquigarrow \quad S^{\mathrm{pf}}; \quad S \quad \rightsquigarrow \quad S_{\mathrm{\acute{e}t}}$$

associated to a reduced scheme S of characteristic p give rise to **cores** of the diagram obtained by considering iterates of the **"Frobenius mutation"** just discussed.

This last example of the *Frobenius mutation* and the associated core constituted by the *étale site* is of particular importance in the context of the present series of papers in that it forms the *"intuitive prototype"* that underlies the theory of the **vertical** and **horizontal** lines of the **log-theta-lattice** [cf. the discussion of Remark 3.6.1, (i)]. One notable aspect of this example is the [evident!] fact that the *domain* and *codomain* of the Frobenius mutation are given by the *same species*. That is to say, despite the fact that in the *construction* of the scheme $S^{(p)}$ [cf. the notation of the preceding paragraph] from the scheme S, the scheme $S^{(p)}$ is *"subordinate"* to the scheme S, the domain and codomain species of the resulting Frobenius mutation *coincide*, hence, in particular, are *on a par with one another*. This sort of situation served, for the author, as a sort of model for the **$\mathfrak{log}$-** and **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links** of the log-theta-lattice, which may be formulated as *mutations* between the *species* constituted by the notion of a $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater*. That is to say, although in the *construction* of either the $\mathfrak{log}$- or the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, the domain and codomain $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters are by no means on a "par" with one another, the domain and codomain $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters of the resulting $\mathfrak{log}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links are regarded as objects of the *same species*, hence, in particular, completely *on a par with one another*. This sort of **"relativization"** of **distinct models** *of conventional scheme theory over* $\mathbb{Z}$ via the notion of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. Fig. I.1 below; the discussion of "gluing together" such models of conventional scheme theory in [IUTchI], §I2] is one of the *most characteristic features* of the theory developed in the present series of papers and, in particular, lies [tautologically!] *outside the framework of conventional scheme theory over* $\mathbb{Z}$. That is to say, in the framework of conventional scheme theory over $\mathbb{Z}$, if one starts out with schemes over $\mathbb{Z}$ and constructs from them, say, by means of geometric objects such as the *theta function* on a Tate curve, some sort of Frobenioid that is isomorphic to a Frobenioid associated to $\mathbb{Z}$, then — unlike, for instance, the case of the *Frobenius morphism* in positive characteristic scheme theory —

> there is no way, within the framework of conventional scheme theory, to treat the newly constructed Frobenioid *"as if it is the Frobenioid associated to* $\mathbb{Z}$*, relative to some* **new** *version/model of conventional scheme theory"*.

...	non-scheme- / theoretic link	one model of conventional scheme theory over $\mathbb{Z}$	non-scheme- / theoretic link	another model of conventional scheme theory over $\mathbb{Z}$	non-scheme- / theoretic link	...

Fig. I.1: Relativized models of conventional scheme theory over $\mathbb{Z}$

If, moreover, one thinks of $\mathbb{Z}$ as being constructed, in the usual way, via *axiomatic set theory*, then one may interpret the *"absolute"* — i.e., *"tautologically* *unrelativizable"* — nature of conventional scheme theory over $\mathbb{Z}$ at a *purely set-theoretic level.* Indeed, from the point of view of the *"$\in$-structure"* of axiomatic set theory, there is *no way to treat* sets constructed at *distinct levels* of this $\in$-structure as being on a par with one another. On the other hand, if one focuses not on the level of the $\in$-structure to which a set belongs, but rather on *species*, then the notion of a species allows one to relate — i.e., to treat on a par with one another — objects belonging to the species that arise from sets constructed at distinct levels of the $\in$-structure. That is to say,

> the notion of a **species** allows one to **"simulate $\in$-loops"** *without violating the axiom of foundation of axiomatic set theory*

— cf. the discussion of Remark 3.3.1, (i).

As one constructs sets at *new levels* of the $\in$-structure of some model of axiomatic set theory — e.g., as one travels along *vertical* or *horizontal lines* of the *log-theta-lattice*! — one typically encounters *new schemes*, which give rise to *new Galois categories*, hence to *new Galois or étale fundamental groups*, which may only be constructed if one allows oneself to consider *new basepoints*, relative to *new universes.* In particular, one must continue to *extend the universe*, i.e., to *modify the model of set theory*, relative to which one works. Here, we recall in passing that such "extensions of universe" are possible on account of an **existence axiom** concerning **universes**, which is apparently attributed to the *"Grothendieck school"* and, moreover, cannot, apparently, be obtained as a consequence of the conventional ZFC axioms of axiomatic set theory [cf. the discussion at the beginning of §3 for more details]. On the other hand, ultimately in the present series of papers [cf. the discussion of [IUTchIII], Introduction], we wish to obtain **algorithms** for constructing various objects that arise in the context of the *new schemes/universes* discussed above — i.e., at *distant* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* of the log-theta-lattice — that *make sense* from the point of view of the *original schemes/universes* that occurred at the outset of the discussion. Again, the fundamental tool that makes this possible, i.e., that allows one to express constructions in the new universes in terms that makes sense in the original universe is precisely

> the **species-theoretic formulation** — i.e., the formulation via **set-theoretic formulas** that *do not depend on* **particular choices** *invoked in* **particular universes** — of the constructions of interest

— cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5, 3.6.2, 3.6.3. This is the point of view that gave rise to the term **"inter-universal"**. At a more concrete level, this "inter-universal" contact between constructions in distant models of conventional scheme theory in the log-theta-lattice is realized by considering [the *étale-like structures* given by] the various Galois or étale fundamental groups that occur as [the "type of mathematical object", i.e., *species* constituted by] **abstract topological groups** [cf. the discussion of Remark 3.6.3, (i); [IUTchI], §I3]. These abstract topological groups give rise to **vertical** or **horizontal cores** of the log-theta-lattice [cf. the discussion of [IUTchIII], Introduction; [IUTchIII], Theorem 1.5, (i), (ii)]. Moreover, once one obtains cores that are sufficiently *"nondegenerate"*, or *"rich in structure"*, so as to serve as *containers* for the *non-coric* portions of the various mutations [e.g., vertical and horizontal arrows of the log-theta-lattice] under consideration, then one may construct the desired algorithms, or **descriptions**, of these **non-coric portions** in terms of **coric containers**, up to certain *relatively mild* **indeterminacies** [i.e., which reflect the non-coric nature of these non-coric portions!] — cf. the illustration of this sort of situation given in Fig. I.2 below; Remark 3.3.1, (iii); Remark 3.6.1, (ii). In the context of the log-theta-lattice, this is precisely the sort of situation that was achieved in [IUTchIII], Theorem A [cf. the discussion of [IUTchIII], Introduction].

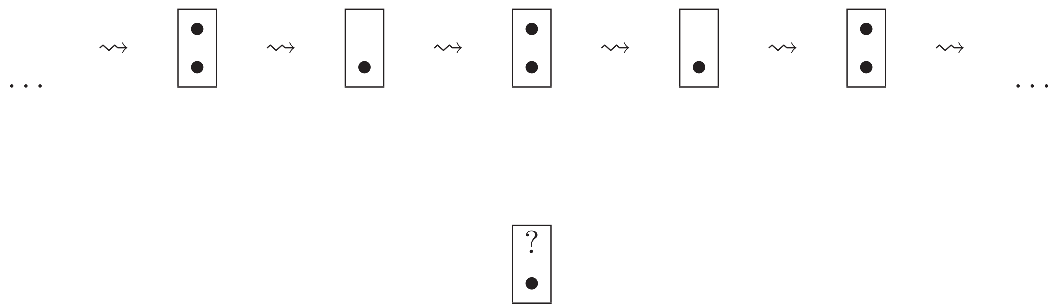

Fig. I.2: A coric container underlying a sequence of mutations

In the context of the above discussion of set-theoretic aspects of the theory developed in the present series of papers, it is of interest to note the following observation, relative to the analogy between the theory of the present series of papers and ***p*-adic Teichmüller theory** [cf. the discussion of [IUTchI], §I4]. If, instead of working *species-theoretically*, one attempts to document *all of the possible choices* that occur in various *newly introduced universes* that occur in a construction, then one finds that one is obliged to work with sets, such as sets obtained via **set-theoretic exponentiation**, of **very large cardinality**. Such sets of large cardinality are reminiscent of the **exponentially large** denominators that occur if one attempts to *p-adically formally integrate an arbitrary connection* as opposed to a **canonical crystalline connection** of the sort that occurs in the context of the **canonical liftings** of p-adic Teichmüller theory [cf. the discussion of Remark 3.6.2, (iii)]. In this context, it is of interest to recall the computations of [Finot], which assert, roughly speaking, that the canonical liftings of p-adic Teichmüller theory may, in certain cases, be characterized as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree**.

Finally, we observe that although, in the above discussion, we concentrated on the *similarities*, from an *"inter-universal"* point of view, between the *vertical* and *horizontal* arrows of the log-theta-lattice, there is one important *difference* between these vertical and horizontal arrows: namely,

· whereas the copies of the *full arithmetic fundamental group* — i.e., in particular, the copies of the **geometric fundamental group** — on either side of a **vertical** arrow are **identified** with one another,

· in the case of a **horizontal** arrow, only the **Galois groups** *of the local base fields* on either side of the arrow are identified with one another

— cf. the discussion of Remark 3.6.3, (ii). One way to understand the reason for this difference is as follows. In the case of the *vertical* arrows — i.e., the $\mathfrak{log}$-*links*, which, in essence, amount to the various *local p-adic logarithms* — in order to *construct* the $\mathfrak{log}$-link, it is necessary to make use, in an essential way, of the **local ring structures** at $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full arithmetic fundamental group*. By contrast, in order to construct the horizontal arrows — i.e., the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*links* — this local ring structure is *unnecessary*. On the other hand, in order to construct the horizontal arrows, it is necessary to work with structures that, up to isomorphism, are *common* to both the *domain* and the *codomain* of the arrow. Since the construction of the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link **depends**, in an essential way, on the *Gaussian monoids*, i.e., on the **labels** $\in \mathbb{F}_l^{\circledast}$ for the **theta values**, which are constructed from the *geometric fundamental group*, while the codomain only involves monoids arising from the local *q-parameters* "$\underline{\underline{q}}_{\underline{v}}$" [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], which are constructed in a fashion that is **independent** of these **labels**, in order to obtain an isomorphism between structures arising from the domain and codomain, it is necessary to restrict one's attention to the *Galois groups of the local base fields*, which are *free of any dependence on these labels*.

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Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: Log-volume Estimates

In the present §1, we perform various *elementary* **local computations** concerning nonarchimedean and archimedean local fields which allow us to obtain **more explicit versions** [cf. Theorem 1.10 below] of the **log-volume estimates** for **Θ-pilot objects** obtained in [IUTchIII], Corollary 3.12.

In the following, if $\lambda \in \mathbb{R}$, then we shall write

$$\lceil \lambda \rceil \text{ (respectively, } \lfloor \lambda \rfloor \text{)}$$

for the *smallest* (respectively, *largest*) $n \in \mathbb{Z}$ such that $n \geq \lambda$ (respectively, $n \leq \lambda$). Also, we shall write "$\log(-)$" for the *natural logarithm* of a positive real number.

Proposition 1.1. **(Multiple Tensor Products and Differents)** *Let p be a prime number, I a finite set of cardinality ≥ 2, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Write $\overline{R} \subseteq \overline{\mathbb{Q}}_p$ for the ring of integers of $\overline{\mathbb{Q}}_p$ and* $\mathrm{ord} : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ *for the natural p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that* $\mathrm{ord}(p) = 1$*; for $\lambda \in \mathbb{Q}$, we shall write p^λ for "some" [unspecified] element of $\overline{\mathbb{Q}}_p$ such that* $\mathrm{ord}(p^\lambda) = \lambda$. *For $i \in I$, let $k_i \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$; write $R_i \stackrel{\mathrm{def}}{=} \mathcal{O}_{k_i} = \overline{R} \bigcap k_i$ for the ring of integers of k_i and $\mathfrak{d}_i \in \mathbb{Q}_{\geq 0}$ for the order [i.e., "*$\mathrm{ord}(-)$*"] of any generator of the* **different ideal** *of R_i over $\mathbb{Z}_p$. Also, for any nonempty subset $E \subseteq I$, let us write*

$$R_E \stackrel{\mathrm{def}}{=} \bigotimes_{i \in E} R_i; \qquad \mathfrak{d}_E \stackrel{\mathrm{def}}{=} \sum_{i \in E} \mathfrak{d}_i$$

— where the tensor product is over $\mathbb{Z}_p$. Fix an element $* \in I$; write $I^* \stackrel{\mathrm{def}}{=} I \setminus \{*\}$. Then

$$p^{\mathfrak{d}_{I^*}} \cdot (R_I)^\sim \quad \subseteq \quad R_I \quad \subseteq \quad (R_I)^\sim$$

— where we write "$(-)^\sim$" for the **normalization** *of the [reduced] ring in parentheses in its ring of fractions, and we observe that it follows immediately from the definition of the "normalization" that the notation on the left-hand side of the first inclusion of the above display is well-defined for suitable "$p^{\mathfrak{d}_{I^*}}$" [such as products of elements $p^{\mathfrak{d}_i} \in R_i$, for $i \in I^*$] and independent of the choice of such suitable "$p^{\mathfrak{d}_{I^*}}$".*

Proof. Let us regard R_I as an R_**-algebra* in the evident fashion. It is immediate from the definitions that $R_I \subseteq (R_I)^\sim$. Now observe that

$$\overline{R} \otimes_{R_*} R_I \quad \subseteq \quad \overline{R} \otimes_{R_*} (R_I)^\sim \quad \subseteq \quad (\overline{R} \otimes_{R_*} R_I)^\sim$$

— where $(\overline{R} \otimes_{R_*} R_I)^\sim$ decomposes as a *direct sum* of finitely many copies of $\overline{R}$. In particular, one verifies immediately, in light of the fact the $\overline{R}$ is *faithfully flat* over R_*, that to complete the proof of Proposition 1.1, it suffices to verify that

$$p^{\mathfrak{d}_{I^*}} \cdot (\overline{R} \otimes_{R_*} R_I)^\sim \quad \subseteq \quad \overline{R} \otimes_{R_*} R_I$$

— where we observe that it follows immediately from the definition of the "normalization" that the notation on the left-hand side of the inclusion of the above display is well-defined and independent of the choice of "$p^{\mathfrak{d}_{I^*}}$". On the other hand, it follows immediately from *induction on the cardinality of* I that to verify this last inclusion, it suffices to verify the inclusion in the case where I is of *cardinality two*. But in this case, the desired inclusion follows immediately from the *definition of the different ideal*. This completes the proof of Proposition 1.1. ◯

Proposition 1.2. **(Differents and Logarithms)** *We continue to use the notation of Proposition 1.1. For $i \in I$, write e_i for the* **ramification index** *of k_i over $\mathbb{Q}_p$;*

$$a_i \stackrel{\text{def}}{=} \frac{1}{e_i} \cdot \lceil \frac{e_i}{p-2} \rceil \textit{ if } p > 2, \quad a_i \stackrel{\text{def}}{=} 2 \textit{ if } p = 2; \qquad b_i \stackrel{\text{def}}{=} \lfloor \frac{\log(p \cdot e_i/(p-1))}{\log(p)} \rfloor - \frac{1}{e_i}.$$

Thus,

$$\textit{if } p > 2 \textit{ and } e_i \leq p-2, \textit{ then } a_i = \frac{1}{e_i} = -b_i.$$

For any nonempty subset $E \subseteq I$, let us write

$$\log_p(R_E^\times) \stackrel{\text{def}}{=} \bigotimes_{i \in E} \log_p(R_i^\times); \qquad a_E \stackrel{\text{def}}{=} \sum_{i \in E} a_i; \qquad b_E \stackrel{\text{def}}{=} \sum_{i \in E} b_i$$

— where the tensor product is over $\mathbb{Z}_p$; we write "$\log_p(-)$" for the p-adic logarithm. For $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, we shall write $p^\lambda \cdot R_i$ for the fractional ideal of R_i generated by any element "p^λ" of k_i such that $\mathrm{ord}(p^\lambda) = \lambda$. *Let*

$$\phi : \log_p(R_I^\times) \otimes \mathbb{Q}_p \xrightarrow{\sim} \log_p(R_I^\times) \otimes \mathbb{Q}_p$$

be an **automorphism** *of the finite dimensional $\mathbb{Q}_p$-vector space $\log_p(R_I^\times) \otimes \mathbb{Q}_p$ that induces an automorphism of the submodule $\log_p(R_I^\times)$. Then:*

(i) We have:

$$p^{a_i} \cdot R_i \quad \subseteq \quad \log_p(R_i^\times) \quad \subseteq \quad p^{-b_i} \cdot R_i$$

— where the "$\subseteq$'s" are **equalities** *when $p > 2$ and $e_i \leq p-2$.*

(ii) We have:

$$\begin{aligned} \phi(p^\lambda \cdot (R_I)^\sim) \quad &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^\times) \\ &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim \end{aligned}$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above display is well-defined and independent of the various choices involved.] In particular, $\phi((R_I)^\sim) \subseteq p^{-\lceil \mathfrak{d}_I + a_I \rceil} \cdot \log_p(R_I^\times) \subseteq p^{-\lceil \mathfrak{d}_I + a_I \rceil - b_I} \cdot (R_I)^\sim$.

(iii) Suppose that $p > 2$, and that $e_i \leq p-2$ for all $i \in I$. Then we have:

$$\phi(p^\lambda \cdot (R_I)^\sim) \quad \subseteq \quad p^{\lambda - \mathfrak{d}_I - 1} \cdot (R_I)^\sim$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above display is well-defined and independent of the various choices involved.] In particular, $\phi((R_I)^\sim) \subseteq p^{-\mathfrak{d}_I - 1} \cdot (R_I)^\sim$.

(iv) If $p > 2$ and $e_i = 1$ for all $i \in I$, then $\phi((R_I)^\sim) \subseteq (R_I)^\sim$.

Proof. Since $a_i > \frac{1}{p-1}$, $\frac{p^{b_i + \frac{1}{e_i}}}{e_i} > \frac{1}{p-1}$ [cf. the definition of "$\lceil - \rceil$", "$\lfloor - \rfloor$"!], assertion (i) follows immediately from the well-known theory of the *p-adic logarithm* and *exponential* maps [cf., e.g., [Kobl], p. 81]. Next, we consider assertion (ii). Observe that it follows from the first displayed inclusion [*of R_I-modules!*] of Proposition 1.1 that

$$p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \quad \subseteq \quad \bigotimes_{i \in I} p^{a_i} \cdot R_i \quad \left(\subseteq \quad R_I \quad = \quad \bigotimes_{i \in I} R_i \right)$$

and hence that

$$\begin{aligned} p^\lambda \cdot (R_I)^\sim \quad & \subseteq \quad p^{\lambda - \mathfrak{d}_I - a_I} \cdot p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \\ & \subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \\ & \subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^\times) \quad \subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim \end{aligned}$$

— where, in the passage to the third and fourth inclusions following "$p^\lambda \cdot (R_I)^\sim$", we apply assertion (i). [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above two displays is well-defined and independent of the various choices involved.] Thus, assertion (ii) follows immediately from the fact that ϕ induces an automorphism of the submodule $\log_p(R_I^\times)$. Assertion (iii) follows from assertion (ii), together with the fact that if $p > 2$ and $e_i \leq p - 2$ for all $i \in I$, then we have $a_I = -b_I$, which implies that $\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I \geq \lambda - \mathfrak{d}_I - a_I - 1 - b_I \geq \lambda - \mathfrak{d}_I - 1$. Assertion (iv) follows from assertion (ii), together with the fact that if $p > 2$ and $e_i = 1$ for all $i \in I$, then we have $\mathfrak{d}_I = 0$, $a_I = -b_I \in \mathbb{Z}$. This completes the proof of Proposition 1.2. ○

Proposition 1.3. **(Estimates of Differents)** *We continue to use the notation of Proposition 1.2. Suppose that $k_0 \subseteq k_i$ is a subfield that contains $\mathbb{Q}_p$. Write $R_0 \stackrel{\text{def}}{=} \mathcal{O}_{k_0}$ for the ring of integers of k_0, $\mathfrak{d}_0$ for the order [i.e., "$\text{ord}(-)$"] of any generator of the different ideal of R_0 over $\mathbb{Z}_p$, e_0 for the ramification index of k_0 over $\mathbb{Q}_p$, $e_{i/0} \stackrel{\text{def}}{=} e_i/e_0$ $(\in \mathbb{Z})$, $[k_i : k_0]$ for the degree of the extension k_i/k_0, n_i for the unique nonnegative integer such that $[k_i : k_0]/p^{n_i}$ is an integer prime to p. Then:*

(i) We have:

$$\mathfrak{d}_i \geq \mathfrak{d}_0 + (e_{i/0} - 1)/(e_{i/0} \cdot e_0) = \mathfrak{d}_0 + (e_{i/0} - 1)/e_i$$

— *where the "$\geq$" is an* **equality** *if* k_i *is* **tamely ramified** *over* k_0.

(ii) Suppose that k_i *is a finite Galois extension of a subfield* $k_1 \subseteq k_i$ *such that* $k_0 \subseteq k_1$*, and* k_1 *is tamely ramified over* k_0*. Then we have:* $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$.

Proof. By replacing k_0 by an *unramified* extension of k_0 contained in k_i, we may assume without loss of generality in the following discussion that k_i is a *totally ramified* extension of k_0. First, we consider assertion (i). Let π_0 be a uniformizer of R_0. Then there exists an isomorphism of R_0-algebras $R_0[x]/(f(x)) \xrightarrow{\sim} R_i$, where $f(x) \in R_0[x]$ is a monic polynomial which is $\equiv x^{e_{i/0}} \pmod{\pi_0}$, that maps $x \mapsto \pi_i$ for some uniformizer π_i of R_i. Thus, the different $\mathfrak{d}_i$ may be computed as follows:

$$\begin{aligned}\mathfrak{d}_i - \mathfrak{d}_0 \;&=\; \mathrm{ord}(f'(\pi_i)) \;\geq\; \min(\mathrm{ord}(\pi_0), \mathrm{ord}(e_{i/0} \cdot \pi_i^{e_{i/0}-1})) \\ &\geq\; \min\left(\frac{1}{e_0}, \mathrm{ord}(\pi_i^{e_{i/0}-1})\right) \;=\; \min\left(\frac{1}{e_0}, \frac{e_{i/0}-1}{e_{i/0} \cdot e_0}\right) \;=\; \frac{e_{i/0}-1}{e_i}\end{aligned}$$

— where, for $\lambda, \mu \in \mathbb{R}$ such that $\lambda \geq \mu$, we define $\min(\lambda, \mu) \stackrel{\mathrm{def}}{=} \mu$. When k_i is *tamely ramified* over k_0, one verifies immediately that the inequalities of the above display are, in fact, equalities. This completes the proof of assertion (i).

Next, we consider assertion (ii). We apply *induction* on n_i. Since assertion (ii) follows immediately from assertion (i) when $n_i = 0$, we may assume that $n_i \geq 1$, and that assertion (ii) has been verified for smaller "n_i". By replacing k_1 by some tamely ramified extension of k_1 contained in k_i, we may assume without loss of generality that $\mathrm{Gal}(k_i/k_1)$ is a p-group. Since p-groups are solvable, and k_i is a *totally ramified* extension of k_0, it follows that there exists a subextension $k_1 \subseteq k_* \subseteq k_i$ such that k_i/k_* and k_*/k_1 are Galois extensions of degree p and p^{n_i-1}, respectively. Write $R_* \stackrel{\mathrm{def}}{=} \mathcal{O}_{k_*}$ for the ring of integers of k_*, $\mathfrak{d}_*$ for the order [i.e., "$\mathrm{ord}(-)$"] of any generator of the different ideal of R_* over $\mathbb{Z}_p$, and e_* for the ramification index of k_* over $\mathbb{Q}_p$. Thus, by the induction hypothesis, it follows that $\mathfrak{d}_* \leq \mathfrak{d}_0 + n_i - 1 + 1/e_0$. To verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$, it suffices to verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon$ for any positive real number ϵ. Thus, let us *fix* a positive real number ϵ. Then by possibly enlarging k_i and k_1, we may also assume without loss of generality that the tamely ramified extension k_1 of k_0 contains a *primitive* p*-th root of unity*, and, moreover, that the ramification index e_1 of k_1 over $\mathbb{Q}_p$ satisfies the inequality $e_1 \geq p/\epsilon$ [so $e_* \geq e_1 \geq p/\epsilon$]. Thus, k_i is a *Kummer extension* of k_*. In particular, there exists an *inclusion* of R_*-algebras $R_*[x]/(f(x)) \hookrightarrow R_i$, where $f(x) \in R_*[x]$ is a monic polynomial which is of the form $f(x) = x^p - \varpi_*$ for some element ϖ_* of R_* satisfying the estimates $0 \leq \mathrm{ord}(\varpi_*) \leq \frac{p-1}{e_*}$, that maps $x \mapsto \varpi_i$ for some element ϖ_i of R_i satisfying the estimates $0 \leq \mathrm{ord}(\varpi_i) \leq \frac{p-1}{p \cdot e_*}$. Now we compute:

$$\begin{aligned}\mathfrak{d}_i \;&\leq\; \mathrm{ord}(f'(\varpi_i)) + \mathfrak{d}_* \;\leq\; \mathrm{ord}(p \cdot \varpi_i^{p-1}) + \mathfrak{d}_0 + n_i - 1 + 1/e_0 \\ &=\; (p-1) \cdot \mathrm{ord}(\varpi_i) + \mathfrak{d}_0 + n_i + 1/e_0 \;\leq\; \frac{(p-1)^2}{p \cdot e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \\ &\leq\; \frac{p}{e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \;\leq\; \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon\end{aligned}$$

— thus completing the proof of assertion (ii). $\bigcirc$

Remark 1.3.1. Similar estimates to those discussed in Proposition 1.3 may be found in [Ih], Lemma A.

Proposition 1.4. (Nonarchimedean Normalized Log-volume Estimates) *We continue to use the notation of Proposition 1.2. Also, for $i \in I$, write $R_i^{\boldsymbol{\mu}} \subseteq R_i^{\times}$ for the torsion subgroup of $R_i^{\times}$, $R_i^{\times\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} R_i^{\times}/R_i^{\boldsymbol{\mu}}$, p^{f_i} for the cardinality of the residue field of k_i, and p^{m_i} for the order of the p-primary component of $R_i^{\boldsymbol{\mu}}$. Thus, the order of $R_i^{\boldsymbol{\mu}}$ is equal to $p^{m_i} \cdot (p^{f_i} - 1)$. Then:*

(i) The **log-volumes** *constructed in [AbsTopIII], Proposition 5.7, (i), on the various finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ may be suitably* **normalized** *[i.e., by dividing by the degree of the finite extension] so as to yield a notion of log-volume*

$$\mu^{\log}(-)$$

defined on compact open subsets of finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, valued in $\mathbb{R}$, and normalized so that $\mu^{\log}(R_i) = 0$, $\mu^{\log}(p \cdot R_i) = -\log(p)$, for each $i \in I$. Moreover, by applying the fact that tensor products of finitely many finite extensions of $\mathbb{Q}_p$ over $\mathbb{Z}_p$ decompose, naturally, as direct sums of finitely many finite extensions of $\mathbb{Q}_p$, we obtain a notion of log-volume — which, by abuse of notation, we shall also denote by "$\mu^{\log}(-)$" — defined on **compact open subsets of such tensor products***, valued in $\mathbb{R}$, and normalized so that $\mu^{\log}((R_E)^{\sim}) = 0$, $\mu^{\log}(p \cdot (R_E)^{\sim}) = -\log(p)$, for any nonempty set $E \subseteq I$.*

(ii) We have:

$$\mu^{\log}(\log_p(R_i^{\times})) = -\Big(\frac{1}{e_i} + \frac{m_i}{e_i f_i}\Big) \cdot \log(p)$$

[cf. [AbsTopIII], Proposition 5.8, (iii)].

(iii) Let $I^* \subseteq I$ be a subset such that for each $i \in I \setminus I^*$, it holds that $p - 2 \geq e_i$ (≥ 1). Then for any $\lambda \in \frac{1}{e_{i^\dagger}} \cdot \mathbb{Z}$, $i^\dagger \in I$, we have inclusions $\phi(p^{\lambda} \cdot (R_I)^{\sim}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim}$ and inequalities

$$\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times})) \ \leq \ \Big(- \lambda + \mathfrak{d}_I + 1 + 4 \cdot |I^*|/p \Big) \cdot \log(p);$$

$$\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim}) \ \leq \ \Big(- \lambda + \mathfrak{d}_I + 1 \Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}$$

— where we write "$|(-)|$" for the cardinality of the set "$(-)$". Moreover, $\mathfrak{d}_I + a_I \geq |I|$ if $p > 2$; $\mathfrak{d}_I + a_I \geq 2 \cdot |I|$ if $p = 2$.

(iv) If $p > 2$ and $e_i = 1$ for all $i \in I$, then $\phi((R_I)^{\sim}) \subseteq (R_I)^{\sim}$, and $\mu^{\log}((R_I)^{\sim}) = 0$.

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). We begin by *observing* that every compact open subset of $R_i^{\times\boldsymbol{\mu}}$ may be covered by a finite collection of compact open subsets of $R_i^{\times\boldsymbol{\mu}}$ that arise as images of compact open subsets of $R_i^\times$ that map *injectively* to $R_i^{\times\boldsymbol{\mu}}$. In particular, by applying this *observation*, we conclude that the log-volume on $R_i^\times$ determines, in a natural way, a log-volume on the quotient $R_i^\times \twoheadrightarrow R_i^{\times\boldsymbol{\mu}}$. Moreover, in light of the compatibility of the log-volume with "$\log_p(-)$" [cf. [AbsTopIII], Proposition 5.7, (i), (c)], it follows immediately that $\mu^{\log}(\log_p(R_i^\times)) = \mu^{\log}(R_i^{\times\boldsymbol{\mu}})$. Thus, it suffices to compute $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = e_i \cdot f_i \cdot \mu^{\log}(R_i^\times) - \log(p^{m_i} \cdot (p^{f_i} - 1))$. On the other hand, it follows immediately from the basic properties of the log-volume [cf. [AbsTopIII], Proposition 5.7, (i), (a)] that $e_i \cdot f_i \cdot \mu^{\log}(R_i^\times) = \log(1 - p^{-f_i})$, so $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = -(f_i + m_i) \cdot \log(p)$, as desired. This completes the proof of assertion (ii).

The inclusions of assertion (iii) follow immediately from Proposition 1.2, (ii). When $p = 2$, the fact that $\mathfrak{d}_I + a_I \geq 2 \cdot |I|$ follows immediately from the definition of "$\mathfrak{d}_i$" and "a_i" in Propositions 1.1, 1.2. When $p > 2$, it follows immediately from the definition of "a_i" in Proposition 1.2 that $a_i \geq 1/e_i$, for all $i \in I$; thus, since $\mathfrak{d}_i \geq (e_i - 1)/e_i$ for all $i \in I$ [cf. Proposition 1.3, (i)], we conclude that $\mathfrak{d}_i + a_i \geq 1$ for all $i \in I$, and hence that $\mathfrak{d}_I + a_I \geq |I|$, as asserted in the statement of assertion (iii). Next, let us observe that $\frac{1}{p-2} \leq \frac{4}{p}$ for $p \geq 3$; $\frac{p}{p-1} \leq 2$ for $p \geq 2$; $\frac{2}{p} \leq \frac{1}{\log(p)}$ for $p \geq 2$. Thus, it follows immediately from the definition of a_i, b_i in Proposition 1.2 that $a_i - \frac{1}{e_i} \leq \frac{4}{p} \leq \frac{2}{\log(p)}$, $(b_i + \frac{1}{e_i}) \cdot \log(p) \leq \log(2e_i) \leq 1 + \log(e_i)$ for $i \in I$; $a_i = \frac{1}{e_i} = -b_i$ for $i \in I \setminus I^*$. On the other hand, by assertion (i), we have $\mu^{\log}(R_I) \leq \mu^{\log}((R_I)^\sim) = 0$; by assertion (ii), we have $\mu^{\log}(\log_p(R_i^\times)) \leq -\frac{1}{e_i} \cdot \log(p)$. Now we compute:

$$\begin{aligned}
\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^\times)) \ &\leq \ \Big(-\lambda + \mathfrak{d}_I + a_I + 1\Big) \cdot \log(p) \ + \ \mu^{\log}(\log_p(R_I^\times)) \\
&= \ \Big(-\lambda + \mathfrak{d}_I + a_I + 1\Big) \cdot \log(p) \\
&\qquad + \ \Big\{\sum_{i \in I} \mu^{\log}(\log_p(R_i^\times))\Big\} \ + \ \mu^{\log}(R_I) \\
&\leq \ \Big\{-\lambda + \mathfrak{d}_I + 1 + \sum_{i \in I} (a_i - \frac{1}{e_i})\Big\} \cdot \log(p) \\
&\leq \ \Big(-\lambda + \mathfrak{d}_I + 1 + 4 \cdot |I^*|/p\Big) \cdot \log(p); \\
\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim) \ &\leq \ \Big(-\lambda + \mathfrak{d}_I + a_I + b_I + 1\Big) \cdot \log(p) \\
&\leq \ \Big(-\lambda + \mathfrak{d}_I + 1\Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}
\end{aligned}$$

— thus completing the proof of assertion (iii). Assertion (iv) follows immediately from assertion (i) and Proposition 1.2, (iv). ◯

Proposition 1.5. (Archimedean Metric Estimates) *In the following, we shall regard the* **complex archimedean field** $\mathbb{C}$ *as being equipped with its standard* **Hermitian metric***, i.e., the metric determined by the complex norm. Let us refer to as the* **primitive automorphisms** *of* $\mathbb{C}$ *the group of automorphisms [of order 8] of the underlying metrized real vector space of* $\mathbb{C}$ *generated by the operations of* **complex conjugation** *and* **multiplication** *by* ± 1 *or* $\pm\sqrt{-1}$.

(i) **(Direct Sum vs. Tensor Product Metrics)** *The metric on $\mathbb{C}$ determines a tensor product metric on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, as well as a direct sum metric on $\mathbb{C} \oplus \mathbb{C}$. Then, relative to these metrics, any* **isomorphism of topological rings** *[i.e., arising from the Chinese remainder theorem]*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ \stackrel{\sim}{\to}\ \mathbb{C} \oplus \mathbb{C}$$

is **compatible** *with these* **metrics***, up to a factor of* 2*, i.e., the metric on the right-hand side corresponds to* 2 *times the metric on the left-hand side. [Thus, lengths differ by a factor of* $\sqrt{2}$*.]*

(ii) **(Direct Sum vs. Tensor Product Automorphisms)** *Relative to the notation of (i), the* **direct sum decomposition** $\mathbb{C} \oplus \mathbb{C}$*, together with its Hermitian metric, is* **preserved***, relative to the displayed isomorphism of (i), by the automorphisms of* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ *induced by the various* **primitive automorphisms** *of the two copies of "*$\mathbb{C}$*" that appear in the tensor product* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$*.*

(iii) **(Direct Sums and Tensor Products of Multiple Copies)** *Let I, V be nonempty finite sets, whose cardinalities we denote by $|I|$, $|V|$, respectively. Write*

$$M \ \stackrel{\mathrm{def}}{=}\ \bigoplus_{v \in V} \mathbb{C}_v$$

for the direct sum of copies $\mathbb{C}_v \stackrel{\mathrm{def}}{=} \mathbb{C}$ of $\mathbb{C}$ labeled by $v \in V$, which we regard as equipped with the direct sum metric, and

$$M_I \ \stackrel{\mathrm{def}}{=}\ \bigotimes_{i \in I} M_i$$

for the tensor product over $\mathbb{R}$ of copies $M_i \stackrel{\mathrm{def}}{=} M$ of M labeled by $i \in I$, which we regard as equipped with the **tensor product metric** *[cf. the constructions of [IUTchIII], Proposition 3.2, (ii)]. Then the topological ring structure on each $\mathbb{C}_v$ determines a* **topological ring structure** *on M_I with respect to which M_I admits a unique* **direct sum decomposition** *as a direct sum of*

$$2^{|I|-1} \cdot |V|^{|I|}$$

copies of $\mathbb{C}$ [cf. [IUTchIII], Proposition 3.1, (i)]. The **direct sum metric** *on M_I — i.e., the metric determined by the natural metrics on these copies of $\mathbb{C}$ — is equal to*

$$2^{|I|-1}$$

times the original tensor product metric on M_I. Write

$$B_I \quad \subseteq \quad M_I$$

for the **"integral structure"** *[cf. the constructions of [IUTchIII], Proposition 3.1, (ii)] given by the direct product of the* **unit balls** *of the copies of $\mathbb{C}$ that occur in the direct sum decomposition of M_I. Then the tensor product metric on M_I, the direct sum decomposition of M_I, the direct sum metric on M_I, and the integral* *structure* $B_I \subseteq M_I$ *are* **preserved** *by the automorphisms of* M_I *induced by the various* **primitive automorphisms** *of the direct summands "*$\mathbb{C}_v$*" that appear in the factors "*M_i*" of the tensor product* M_I*.*

(iv) **(Tensor Product of Vectors of a Given Length)** *Suppose that we are in the situation of (iii). Fix* $\lambda \in \mathbb{R}_{>0}$*. Then*

$$M_I \quad \ni \quad \bigotimes_{i \in I} m_i \quad \in \quad \lambda^{|I|} \cdot B_I$$

for any collection of elements $\{m_i \in M_i\}_{i \in I}$ *such that the component of* m_i *in each direct summand "*$\mathbb{C}_v$*" of* M_i *is of* **length** λ*.*

Proof. Assertions (i) and (ii) are discussed in [IUTchIII], Remark 3.9.1, (ii), and may be verified by means of routine and elementary arguments. Assertion (iii) follows immediately from assertions (i) and (ii). Assertion (iv) follows immediately from the various definitions involved. $\bigcirc$

Proposition 1.6. **(The Prime Number Theorem)** *If* n *is a positive integer, then let us write* p_n *for the* n*-th smallest prime number. [Thus,* $p_1 = 2$*,* $p_2 = 3$*, and so on.] Then there exists an integer* n_0 *such that it holds that*

$$n \;\leq\; \tfrac{4p_n}{3 \cdot \log(p_n)}$$

for all $n \geq n_0$*. In particular, there exists a positive real number* η_{prm} *such that*

$$\sum_{p \leq \eta} 1 \quad \leq \quad \tfrac{4\eta}{3 \cdot \log(\eta)}$$

— where the sum ranges over the prime numbers $p \leq \eta$ *— for all positive real* $\eta \geq \eta_{\mathrm{prm}}$*.*

Proof. Relative to our notation, the *Prime Number Theorem* [cf., e.g., [DmMn], §3.10] implies that

$$\lim_{n \to \infty} \frac{n \cdot \log(p_n)}{p_n} \;=\; 1$$

— i.e., in particular, that for some positive integer n_0, it holds that

$$\frac{\log(p_n)}{p_n} \;\leq\; \frac{4}{3} \cdot \frac{1}{n}$$

for all $n \geq n_0$. The final portion of Proposition 1.6 follows formally. $\bigcirc$

Proposition 1.7. **(Weighted Averages)** *Let* E *be a nonempty finite set,* n *a positive integer. For* $e \in E$*, let* $\lambda_e \in \mathbb{R}_{>0}$*,* $\beta_e \in \mathbb{R}$*. Then, for any* $i = 1, \ldots, n$*, we have:*

$$\frac{\sum\limits_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}}}{\sum\limits_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} \quad = \quad \frac{\sum\limits_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}}{\sum\limits_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} \quad = \quad n \cdot \beta_{\mathrm{avg}}$$

— *where we write* $\beta_{\text{avg}} \stackrel{\text{def}}{=} \beta_E/\lambda_E$, $\beta_E \stackrel{\text{def}}{=} \sum_{e\in E} \beta_e \cdot \lambda_e$, $\lambda_E \stackrel{\text{def}}{=} \sum_{e\in E} \lambda_e$,

$$\beta_{\vec{e}} \stackrel{\text{def}}{=} \sum_{j=1}^{n} \beta_{e_j}; \qquad \lambda_{\Pi\vec{e}} \stackrel{\text{def}}{=} \prod_{j=1}^{n} \lambda_{e_j}$$

for any n*-tuple* $\vec{e} = (e_1, \ldots, e_n) \in E^n$ *of elements of* E.

Proof. We begin by observing that

$$\lambda_E^n = \sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\,; \qquad \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e}\in E^n} \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}$$

for any $i = 1, \ldots, n$. Thus, summing over i, we obtain that

$$n \cdot \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e}\in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi\vec{e}} = \sum_{\vec{e}\in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}$$

and hence that

$$\begin{aligned} n \cdot \beta_{\text{avg}} = n \cdot \beta_E \cdot \lambda_E^{n-1}/\lambda_E^n &= \Big(\sum_{\vec{e}\in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi\vec{e}}\Big) \cdot \Big(\sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\Big)^{-1} \\ &= \Big(\sum_{\vec{e}\in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}\Big) \cdot \Big(\sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\Big)^{-1} \end{aligned}$$

as desired. ○

Remark 1.7.1. In Theorem 1.10 below, we shall apply Proposition 1.7 to compute various **packet-normalized log-volumes** of the sort discussed in [IUTchIII], Proposition 3.9, (i) — i.e., log-volumes normalized by means of the **normalized weights** discussed in [IUTchIII], Remark 3.1.1, (ii). Here, we recall that the normalized weights discussed in [IUTchIII], Remark 3.1.1, (ii), were computed relative to the *non-normalized log-volumes* of [AbsTopIII], Proposition 5.8, (iii), (vi) [cf. the discussion of [IUTchIII], Remark 3.1.1, (ii); [IUTchI], Example 3.5, (iii)]. By contrast, in the discussion of the present §1, our computations are performed relative to *normalized log-volumes* as discussed in Proposition 1.4, (i). In particular, it follows that the *weights* $[K_{\underline{v}} : (F_{\text{mod}})_v]^{-1}$, where $\underline{\mathbb{V}} \ni \underline{v} \mid v \in \mathbb{V}_{\text{mod}}$, of the discussion of [IUTchIII], Remark 3.1.1, (ii), must be *replaced* — i.e., when one works with *normalized log-volumes* as in Proposition 1.4, (i) — by the **weights**

$$[K_{\underline{v}} : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot [K_{\underline{v}} : (F_{\text{mod}})_v]^{-1} = [(F_{\text{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}]$$

— where $\mathbb{V}_{\text{mod}} \ni v \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. This means that the *normalized weights* of the final display of [IUTchIII], Remark 3.1.1, (ii), must be replaced, when one works with *normalized log-volumes* as in Proposition 1.4, (i), by the **normalized weights**

$$\frac{\Big(\prod_{\alpha\in A} [(F_{\text{mod}})_{v_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}]\Big)}{\sum_{\{w_\alpha\}_{\alpha\in A}} \Big(\prod_{\alpha\in A} [(F_{\text{mod}})_{w_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}]\Big)}$$

— where the sum is over all collections $\{w_\alpha\}_{\alpha \in A}$ of [not necessarily distinct!] elements $w_\alpha \in \mathbb{V}_{\mathrm{mod}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$. Thus, in summary, when one works with *normalized log-volumes* as in Proposition 1.4, (i), the appropriate *normalized weights* are given by the expressions

$$\frac{\lambda_{\Pi \vec{e}^\dagger}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}}$$

[where $\vec{e}^\dagger \in E^n$] that appear in Proposition 1.7. Here, one takes "E" to be the set of elements of $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ lying over a fixed $v_{\mathbb{Q}}$; one takes "n" to be the cardinality of A, so that one can write $A = \{\alpha_1, \ldots, \alpha_n\}$ [where the α_i are *distinct*]; if $e \in E$ corresponds to $\underline{v} \in \underline{\mathbb{V}}$, $v \in \mathbb{V}_{\mathrm{mod}}$, then one takes

$$\text{“}\lambda_e\text{”} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \in \mathbb{R}_{>0}$$

and "β_e" to be a normalized log-volume of some compact open subset of $K_{\underline{v}}$.

Before proceeding, we review some *well-known elementary facts* concerning *elliptic curves*. In the following, we shall write $\mathcal{M}_{\mathrm{ell}}$ for the *moduli stack of elliptic curves* over $\mathbb{Z}$ and

$$\mathcal{M}_{\mathrm{ell}} \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$$

for the *natural compactification* of $\mathcal{M}_{\mathrm{ell}}$, i.e., the moduli stack of one-dimensional semi-abelian schemes over $\mathbb{Z}$. Also, if R is a $\mathbb{Z}$-algebra, then we shall write $(\mathcal{M}_{\mathrm{ell}})_R \stackrel{\mathrm{def}}{=} \mathcal{M}_{\mathrm{ell}} \times_{\mathbb{Z}} R$, $(\overline{\mathcal{M}}_{\mathrm{ell}})_R \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{\mathrm{ell}} \times_{\mathbb{Z}} R$.

Proposition 1.8. **(Torsion Points of Elliptic Curves)** *Let k be a perfect field, $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.*

(i) **("Serre's Criterion")** *Let $l \geq 3$ be a* **prime number** *that is* **invertible** *in k; suppose that $\overline{k} = k$. Let A be an* **abelian variety** *over k, equipped with a polarization λ. Write $A[l] \subseteq A(k)$ for the group of l-torsion points of $A(k)$. Then the natural map*

$$\phi : \quad \mathrm{Aut}_k(A, \lambda) \quad \to \quad \mathrm{Aut}(A[l])$$

from the group of automorphisms of the polarized abelian variety (A, λ) over k to the group of automorphisms of the abelian group $A[l]$ is **injective***.*

(ii) Let $E_{\overline{k}}$ be an **elliptic curve** *over $\overline{k}$ with origin $\epsilon_E \in E(\overline{k})$. For n a positive integer, write $E_{\overline{k}}[n] \subseteq E_{\overline{k}}(\overline{k})$ for the module of n-torsion points of $E_{\overline{k}}(\overline{k})$ and*

$$\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \quad \subseteq \quad \mathrm{Aut}_k(E_{\overline{k}})$$

for the respective groups of ϵ_E-preserving automorphisms of the $\overline{k}$-scheme $E_{\overline{k}}$ and the k-scheme $E_{\overline{k}}$. Then we have a **natural exact sequence**

$$1 \longrightarrow \mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \longrightarrow \mathrm{Aut}_k(E_{\overline{k}}) \longrightarrow G_k$$

— where the image $G_E \subseteq G_k$ *of the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *is* **open** *— and a* **natural representation**

$$\rho_n : \mathrm{Aut}_k(E_{\overline{k}}) \to \mathrm{Aut}(E_{\overline{k}}[n])$$

on the n*-torsion points of* $E_{\overline{k}}$*. The finite extension* k_E *of* k *determined by* G_E *is the* **minimal field of definition** *of* $E_{\overline{k}}$*, i.e., the field generated over* k *by the* **j-invariant** *of* $E_{\overline{k}}$*. Finally, if* $H \subseteq G_k$ *is any closed subgroup, which corresponds to an extension* k_H *of* k*, then the datum of a* **model** *of* $E_{\overline{k}}$ *over* k_H *[i.e., descent data for* $E_{\overline{k}}$ *from* $\overline{k}$ *to* k_H*] is equivalent to the datum of a* **section** *of the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *over* H*. In particular, the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *admits a section over* G_E*.*

(iii) In the situation of (ii), suppose further that $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) = \{\pm 1\}$*. Then the representation* ρ_2 **factors** *through* G_E *and hence defines a* **natural representation** $G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$*.*

(iv) In the situation of (ii), suppose further that $l \geq 3$ *is a* **prime number** *that is* **invertible** *in* k*, and that* $E_{\overline{k}}$ *descends to elliptic curves* E'_k *and* E''_k *over* k*, all of whose* l*-torsion points are* **rational** *over* k*. Then* E'_k *is* **isomorphic** *to* E''_k *over* k*.*

(v) In the situation of (ii), suppose further that k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$*, that* $l \geq 3$ *is a* **prime number** *that is* **invertible** *in* $\mathcal{O}_k$*, and that* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k*, all of whose* l*-torsion points are* **rational** *over* k*. Then* E_k *has* **semi-stable reduction** *over* $\mathcal{O}_k$ *[i.e., extends to a semi-abelian scheme over* $\mathcal{O}_k$*].*

(vi) In the situation of (iii), suppose further that 2 *is* **invertible** *in* k*, that* $G_E = G_k$*, and that the representation* $G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$ *is* **trivial***. Then* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k *which is defined by means of the* **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below]. If, moreover,* k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$ *such that* 2 *is* **invertible** *in* $\mathcal{O}_k$*, then* E_k *has* **semi-stable reduction** *over* $\mathcal{O}_{k'}$ *[i.e., extends to a semi-abelian scheme over* $\mathcal{O}_{k'}$*] for some finite extension* $k' \subseteq \overline{k}$ *of* k *such that* $[k' : k] \leq 2$*; if* E_k *has* **good reduction** *over* $\mathcal{O}_{k'}$ *[i.e., extends to an abelian scheme over* $\mathcal{O}_{k'}$*], then one may in fact take* k' *to be* k*.*

(vii) In the situation of (ii), suppose further that k *is a* **complete discrete valuation field** *with ring of integers* $\mathcal{O}_k$*, that* $E_{\overline{k}}$ *descends to an elliptic curve* E_k *over* k*, and that* n *is* **invertible** *in* $\mathcal{O}_k$*. If* E_k *has* **good reduction** *over* $\mathcal{O}_k$ *[i.e., extends to an abelian scheme over* $\mathcal{O}_k$*], then the action of* G_k *on* $E_{\overline{k}}[n]$ *is* **unramified***. If* E_k *has* **bad multiplicative reduction** *over* $\mathcal{O}_k$ *[i.e., extends to a non-proper semi-abelian scheme over* $\mathcal{O}_k$*], then the kernel of the action of* G_k *on* $E_{\overline{k}}[n]$ *determines a* **tamely ramified** *extension of* k *whose ramification index over* k *divides* n*.*

Proof. First, we consider assertion (i). Suppose that ϕ is *not injective*. Since $\mathrm{Aut}_k(A, \lambda)$ is well-known to be *finite* [cf., e.g., [Milne], Proposition 17.5, (a)], we thus conclude that there exists an $\alpha \in \mathrm{Ker}(\phi)$ of order $n \neq 1$. We may assume without loss of generality that n is *prime*. Now we follow the argument of [Milne], Proposition 17.5, (b). Since α acts trivially on $A[l]$, it follows immediately that the endomorphism of A given by $\alpha - \mathrm{id}_A$ [where id_A denotes the identity automorphism of A] may be written in the form $l \cdot \beta$, for β an endomorphism of A over k. Write $T_l(A)$ for the *l-adic Tate module* of A. Since $\alpha^n = \mathrm{id}_A$, it follows that the eigenvalues of the action of α on $T_l(A)$ are *n-th roots of unity*. On the other hand, the eigenvalues of the action of β on $T_l(A)$ are *algebraic integers* [cf. [Milne], Theorem 12.5]. We thus conclude that each eigenvalue ζ of the action of α on $T_l(A)$ is an n-th root of unity which, as an algebraic integer, is $\equiv 1 \pmod{l}$ [where $l \geq 3$], hence $= 1$. Since $\alpha^n = \mathrm{id}_A$, it follows that α acts on $T_l(A)$ as a *semi-simple matrix* which is also *unipotent*, hence equal to the *identity matrix*. But this implies that $\alpha = \mathrm{id}_A$ [cf. [Milne], Theorem 12.5]. This contradiction completes the proof of assertion (i).

Next, we consider assertion (ii). Since $E_{\overline{k}}$ is *proper* over $\overline{k}$, it follows [by considering the space of global sections of the structure sheaf of $E_{\overline{k}}$] that any automorphism of the scheme $E_{\overline{k}}$ lies over an automorphism of $\overline{k}$. This implies the existence of a *natural exact sequence* and *natural representation* as in the statement of assertion (ii). The relationship between k_E and the *j-invariant* of $E_{\overline{k}}$ follows immediately from the well-known theory of the j-invariant of an elliptic curve [cf., e.g., [Silv], Chapter III, Proposition 1.4, (b), (c)]. The final portion of assertion (ii) concerning *models* of $E_{\overline{k}}$ follows immediately from the definitions. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the fact that $\{\pm 1\}$ acts *trivially* on $E_{\overline{k}}[2]$.

Next, we consider assertion (iv). First, let us observe that it follows immediately from the final portion of assertion (ii) that a model E^*_k of $E_{\overline{k}}$ over k all of whose l-torsion points are *rational* over k corresponds to a closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ that lies in the kernel of ρ_l and, moreover, maps isomorphically to G_k. On the other hand, it follows from assertion (i) that the restriction of ρ_l to $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *injective*. Thus, the closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *uniquely determined* by the condition that it lie in the kernel of ρ_l and, moreover, map isomorphically to G_k. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that, by considering *l-level structures*, we obtain a *finite covering* of $S \to (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ which is *étale* over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ and *tamely ramified* over the divisor at infinity. Then it follows from assertion (i) that the algebraic stack S is in fact a *scheme*, which is, moreover, *proper* over $\mathbb{Z}[\frac{1}{l}]$. Thus, it follows from the valuative criterion for properness that any k-valued point of S determined by E_k — where we observe that such a point necessarily exists, in light of our assumption that the l-torsion points of E_k are *rational* over k — extends to an $\mathcal{O}_k$-valued point of S, hence also of $\overline{\mathcal{M}}_{\mathrm{ell}}$, as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). Since $G_E = G_k$, it follows from assertion (ii) that $E_{\overline{k}}$ descends to an elliptic curve E_k over k. Our assumption that the representation $G_k = G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$ of assertion (iii) is *trivial* implies that the 2-torsion points of E_k are *rational* over k. Thus, by considering suitable *global sections* of tensor powers of the line bundle on E_k determined by the origin *on which the automorphism "-1" of E_k acts via multiplication by* ± 1 [cf., e.g., [Harts], Chapter IV, the proof of Proposition 4.6], one concludes immediately that a suitable [possibly trivial] *twist* E'_k of E_k over k [i.e., such that E'_k and E_k are isomorphic over some quadratic extension k' of k] may be defined by means of the *Legendre form* of the Weierstrass equation. Now suppose that k is a *complete discrete valuation field* with ring of integers $\mathcal{O}_k$ such that 2 is *invertible* in $\mathcal{O}_k$, and that E_k is defined by means of the *Legendre form* of the Weierstrass equation. Then the fact that E_k has *semi-stable reduction* over $\mathcal{O}_{k'}$ for some finite extension $k' \subseteq \overline{k}$ of k such that $[k' : k] \leq 2$ follows from the explicit computations of the proof of [Silv], Chapter VII, Proposition 5.4, (c). These explicit computations also imply that if E_k has *good reduction* over $\mathcal{O}_{k'}$, then one may in fact take k' to be k. This completes the proof of assertion (vi).

Assertion (vii) follows immediately from [NerMod], §7.4, Theorem 5, in the case of *good reduction* and from [NerMod], §7.4, Theorem 6, in the case of *bad multiplicative reduction.* ◯

We are now ready to apply the elementary computations discussed above to give *more explicit log-volume estimates for* Θ*-pilot objects*. We begin by recalling some notation and terminology from [GenEll], §1.

Definition 1.9. Let F be a *number field* [i.e., a finite extension of the rational number field $\mathbb{Q}$], whose *set of valuations* we denote by $\mathbb{V}(F)$. Thus, $\mathbb{V}(F)$ decomposes as a disjoint union $\mathbb{V}(F) = \mathbb{V}(F)^{\text{non}} \bigcup \mathbb{V}(F)^{\text{arc}}$ of *nonarchimedean* and *archimedean* valuations. If $v \in \mathbb{V}(F)$, then we shall write F_v for the *completion* of F at v; if $v \in \mathbb{V}(F)^{\text{non}}$, then we shall write e_v for the *ramification index* of F_v over $\mathbb{Q}_{p_v}$, f_v for the *residue field degree* of F_v over $\mathbb{Q}_{p_v}$, and q_v for the cardinality of the *residue field* of F_v.

(i) An *[*$\mathbb{R}$*-]arithmetic divisor* $\mathfrak{a}$ on F is defined to be a finite formal sum

$$\sum_{v \in \mathbb{V}(F)} c_v \cdot v$$

— where $c_v \in \mathbb{R}$, for all $v \in \mathbb{V}(F)$. Here, we shall refer to the set

$$\text{Supp}(\mathfrak{a})$$

of $v \in \mathbb{V}(F)$ such that $c_v \neq 0$ as the *support of* $\mathfrak{a}$; if all of the c_v are ≥ 0, then we shall say that the arithmetic divisor is *effective*. Thus, the [$\mathbb{R}$-]arithmetic divisors on F naturally form a group $\text{ADiv}_{\mathbb{R}}(F)$. The assignment

$$\mathbb{V}(F)^{\text{non}} \ni v \mapsto \log(q_v); \quad \mathbb{V}(F)^{\text{arc}} \ni v \mapsto 1$$

determines a homomorphism

$$\deg_F : \text{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$$

which we shall refer to as the *degree* map. If $\mathfrak{a} \in \text{ADiv}_{\mathbb{R}}(F)$, then we shall refer to

$$\underline{\deg}(\mathfrak{a}) \quad \stackrel{\text{def}}{=} \quad \frac{1}{[F : \mathbb{Q}]} \cdot \deg_F(\mathfrak{a})$$

as the *normalized degree* of $\mathfrak{a}$. Thus, for any finite extension K of F, we have

$$\underline{\deg}(\mathfrak{a}|_K) = \underline{\deg}(\mathfrak{a})$$

— where we write $\underline{\deg}(\mathfrak{a}|_K)$ for the normalized degree of the pull-back $\mathfrak{a}|_K \in \mathrm{ADiv}_{\mathbb{R}}(K)$ [defined in the evident fashion] of $\mathfrak{a}$ to K.

(ii) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$, $E \subseteq \mathbb{V}(F)$ a nonempty set of elements lying over $v_{\mathbb{Q}}$. If $\mathfrak{a} = \sum_{v \in \mathbb{V}(F)} c_v \cdot v \in \mathrm{ADiv}_{\mathbb{R}}(F)$, then we shall write

$$\mathfrak{a}_E \stackrel{\mathrm{def}}{=} \sum_{v \in E} c_v \cdot v \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F); \quad \underline{\deg}_E(\mathfrak{a}) \stackrel{\mathrm{def}}{=} \frac{\deg(\mathfrak{a}_E)}{\sum_{v \in E} [F_v : \mathbb{Q}_{v_{\mathbb{Q}}}]}$$

for the portion of $\mathfrak{a}$ *supported in* E and the *"normalized E-degree"* of $\mathfrak{a}$, respectively. Thus, for any finite extension K of F, we have

$$\underline{\deg}_{E|_K}(\mathfrak{a}|_K) = \underline{\deg}_E(\mathfrak{a})$$

— where we write $E|_K \subseteq \mathbb{V}(K)$ for the set of valuations lying over valuations $\in E$.

Theorem 1.10. **(Log-volume Estimates for Θ-Pilot Objects)** *Fix a collection of* **initial Θ-data** *as in [IUTchI], Definition 3.1. Suppose that we are in the situation of [IUTchIII], Corollary 3.12, and that the elliptic curve E_F has* **good reduction** *at every valuation $\in \mathbb{V}(F)^{\mathrm{good}} \bigcap \mathbb{V}(F)^{\mathrm{non}}$ that does* **not divide** *$2l$. In the notation of [IUTchI], Definition 3.1, let us write $d_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} [F_{\mathrm{mod}} : \mathbb{Q}]$, $(1 \leq)\ e_{\mathrm{mod}}\ (\leq d_{\mathrm{mod}})$ for the* **maximal ramification index** *of F_{mod} [i.e., of valuations $\in \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}$] over $\mathbb{Q}$, $d^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}}$, $e^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}}\ (\leq d^*_{\mathrm{mod}})$, and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \stackrel{\mathrm{def}}{=} F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from F_{mod} by adjoining the fields of definition of the 2-torsion points of any model of $E_F \times_F \overline{F}$ over F_{mod} [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3 \cdot 5)$-torsion points of E_F are defined over F, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2 \cdot 3 \cdot 5]\ ) \quad \stackrel{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3 \cdot 5]\ )$$

— i.e., that F is obtained from F_{tpd} by adjoining $\sqrt{-1}$, together with the fields of definition of the $(3 \cdot 5)$-torsion points of a model $E_{F_{\mathrm{tpd}}}$ of the elliptic curve $E_F \times_F \overline{F}$ over F_{tpd} determined by the **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below; Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ over F, and from [IUTchI], Definition 3.1, (c), that $l \neq 5$.] If $F_{\mathrm{mod}} \subseteq F_{\Box} \subseteq K$ is any intermediate extension which is Galois over F_{mod}, then we shall write*

$$\mathfrak{d}^{F_{\Box}}_{\mathrm{ADiv}} \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F_{\Box})$$

for the effective arithmetic divisor determined by the **different ideal** *of* $F_\Box$ *over* $\mathbb{Q}$*,*

$$\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{R}}(F_\Box)$$

for the effective arithmetic divisor determined by the **q-parameters** *of the elliptic curve* E_F *at the elements of* $\mathbb{V}(F_\Box)^{\mathrm{bad}} \stackrel{\mathrm{def}}{=} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F_\Box)$ $(\neq \emptyset)$ *[cf. [GenEll], Remark 3.3.1],*

$$\mathfrak{f}^{F_\Box}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{R}}(F_\Box)$$

for the effective arithmetic divisor whose support coincides with $\mathrm{Supp}(\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}})$*, but all of whose coefficients are equal to* 1 *— i.e., the* **conductor** *— and*

$$\log(\mathfrak{d}^{F_\Box}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_v}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{d}^{F_\Box}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_{v_{\mathbb{Q}}}}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{d}^{F_\Box}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{d}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_v}(\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{q}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_{v_{\mathbb{Q}}}}(\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{q}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_\Box}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_v}(\mathfrak{f}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{f}^{F_\Box}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\Box)_{v_{\mathbb{Q}}}}(\mathfrak{f}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_\Box}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{f}^{F_\Box}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

— where $v \in \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} = \mathbb{V}(\mathbb{Q})$, $\mathbb{V}(F_\Box)_v \stackrel{\mathrm{def}}{=} \mathbb{V}(F_\Box) \times_{\mathbb{V}_{\mathrm{mod}}} \{v\}$, $\mathbb{V}(F_\Box)_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_\Box) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$*. Here, we observe that the various "*$\log(\mathfrak{q}_{(-)})$*'s" are independent of the choice of* $F_\Box$*, and that the quantity "*$|\log(\underline{\underline{q}})| \in \mathbb{R}_{>0}$*" defined in [IUTchIII], Corollary 3.12, is equal to* $\frac{1}{2l} \cdot \log(\mathfrak{q}) \in \mathbb{R}$ *[cf. the definition of "*$\underline{\underline{q}}_{v}$*" in [IUTchI], Example 3.2, (iv)]. Then one may take the constant "*$C_\Theta \in \mathbb{R}$*" of [IUTchIII], Corollary 3.12, to be*

$$\frac{l+1}{4 \cdot |\log(\underline{\underline{q}})|} \cdot \Big\{ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 10 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q}) \Big\} - 1$$

and hence, by applying the inequality "$C_\Theta \geq -1$*" of [IUTchIII], Corollary 3.12, conclude that*

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ &\leq \ (1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 20 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \\ &\leq \ (1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F})) + 20 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \end{aligned}$$

— where η_{prm} *is the positive real number of Proposition 1.6.*

Proof. For ease of reference, we divide our discussion into *steps*, as follows.

(i) We begin by recalling the following *elementary identities* for $n \in \mathbb{N}_{\geq 1}$:

(E1) $\frac{1}{n} \sum_{m=1}^{n} m \ = \ \frac{1}{2}(n+1);$

(E2) $\frac{1}{n} \sum_{m=1}^{n} m^2 \ = \ \frac{1}{6}(2n+1)(n+1).$

Also, we recall the following *elementary facts*:

(E3) For p a prime number, the cardinality $|GL_2(\mathbb{F}_p)|$ of $GL_2(\mathbb{F}_p)$ is given by $|GL_2(\mathbb{F}_p)| = p(p+1)(p-1)^2$.
(E4) For $p = 2, 3, 5$, the expression of (E3) may be computed as follows: $2(2+1)(2-1)^2 = 2\cdot 3$; $3(3+1)(3-1)^2 = 3\cdot 2^4$; $5(5+1)(5-1)^2 = 5\cdot 2^5\cdot 3$.
(E5) The degree of the extension $F_{\mathrm{mod}}(\sqrt{-1}\ )/F_{\mathrm{mod}}$ is ≤ 2.
(E6) We have: $0 \leq \log(2) \leq 1$, $\quad 1 \leq \log(3) \leq \log(\pi) \leq \log(5) \leq 2$.

(ii) Next, let us observe that the *inequality*

$$\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \ \leq\ \log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F})$$

follows immediately from Proposition 1.3, (i), and the various definitions involved. On the other hand, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F}) \ &\leq\ \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + \log(2^{11}\cdot 3^3\cdot 5^2)\\ &\leq\ \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 21\end{aligned}$$

follows by applying Proposition 1.3, (i), at the primes that do *not* divide $2\cdot 3\cdot 5$ [where we recall that the extension F/F_{tpd} is *tamely ramified* over such primes — cf. Proposition 1.8, (vi), (vii)] and applying Proposition 1.3, (ii), together with (E3), (E4), (E5), (E6), and the fact that we have a natural outer inclusion $\mathrm{Gal}(F/F_{\mathrm{tpd}}) \hookrightarrow GL_2(\mathbb{F}_3)\times GL_2(\mathbb{F}_5)\times \mathbb{Z}/2\mathbb{Z}$, at the primes that *divide* $2\cdot 3\cdot 5$. In a similar vein, since the extension K/F is *tamely ramified* at the primes that do *not* divide l, and we have a natural outer inclusion $\mathrm{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^{K}) \ \leq\ \log(\mathfrak{d}^{K}) + \log(\mathfrak{f}^{K}) \ &\leq\ \log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F}) + 2\cdot\log(l)\\ &\leq\ \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 2\cdot\log(l) + 21\end{aligned}$$

follows immediately from Proposition 1.3, (i), (ii). Finally, for later reference, we observe that

$$(1+\tfrac{4}{l})\cdot\log(\mathfrak{d}^{K}) \ \leq\ (1+\tfrac{4}{l})\cdot(\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2\cdot\log(l) + 46$$

— where we apply the estimates $\frac{\log(l)}{l} \leq \frac{1}{2}$ and $1+\frac{4}{l} \leq 2$, both of which may be regarded as consequences of the fact that $l \geq 5$ [cf. also (E6)].

(iii) If $F_{\mathrm{tpd}} \subseteq F_{\Box} \subseteq K$ is any intermediate extension which is *Galois* over F_{mod}, then we shall write

$$\mathbb{V}(F_{\Box})^{\mathrm{dst}} \ \subseteq\ \mathbb{V}(F_{\Box})^{\mathrm{non}}$$

for the set of **"distinguished"** nonarchimedean valuations $v \in \mathbb{V}(F_{\Box})^{\mathrm{non}}$, i.e., v that extend to a valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that *ramifies* over $\mathbb{Q}$. Now observe that it follows immediately from Proposition 1.8, (vi), (vii), together with our *assumption* on $\mathbb{V}(F)^{\mathrm{good}} \bigcap \mathbb{V}(F)^{\mathrm{non}}$, that

(D0) if $v \in \mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{non}}$ does *not divide* $2\cdot 3\cdot 5\cdot l$ and, moreover, is *not contained* in $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$, then the extension K/F_{tpd} is *unramified* over v.

Next, let us recall the well-known fact that the *determinant* of the Galois representation determined by the torsion points of an elliptic curve over a field of characteristic zero is the abelian Galois representation determined by the *cyclotomic character*. In particular, it follows [cf. the various definitions involved] that K contains a *primitive* $4 \cdot 3 \cdot 5 \cdot l$*-th root of unity*, hence is ramified over $\mathbb{Q}$ at any valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that divides $2 \cdot 3 \cdot 5 \cdot l$. Thus, one verifies immediately [i.e., by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v \in \mathbb{V}(F_{\square})^{\mathrm{non}}$ are *equivalent*:

(D1) $v \in \mathbb{V}(F_{\square})^{\mathrm{dst}}$.
(D2) The valuation v either divides $2 \cdot 3 \cdot 5 \cdot l$ or lies in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\square}}_{\mathrm{ADiv}})$.
(D3) The image of v in $\mathbb{V}(F_{\mathrm{tpd}})$ lies in $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$.

Let us write

$$\mathbb{V}^{\mathrm{dst}}_{\mathrm{mod}} \subseteq \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}; \quad \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}} \subseteq \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$$

for the respective *images* of $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$ in $\mathbb{V}_{\mathrm{mod}}$, $\mathbb{V}_{\mathbb{Q}}$ and, for $F_* \in \{F_{\mathrm{mod}}, \mathbb{Q}\}$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

$$\mathfrak{s}^{F_*}_{\mathrm{ADiv}} \stackrel{\mathrm{def}}{=} \sum_{v \in \mathbb{V}(F_*)^{\mathrm{dst}}} e_v \cdot v \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F_*)$$

$$\log(\mathfrak{s}^{F_*}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(F_*)_{v_{\mathbb{Q}}}}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{F_*}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

$$\mathfrak{s}^{\leq}_{\mathrm{ADiv}} \stackrel{\mathrm{def}}{=} \sum_{w_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})^{\mathrm{dst}}} \frac{\iota_{w_{\mathbb{Q}}}}{\log(p_{w_{\mathbb{Q}}})} \cdot w_{\mathbb{Q}} \ \in \ \mathrm{ADiv}_{\mathbb{R}}(\mathbb{Q})$$

$$\log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(\mathbb{Q})_{v_{\mathbb{Q}}}}(\mathfrak{s}^{\leq}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{\leq}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}(\mathfrak{s}^{\leq}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

— where we write $\mathbb{V}(F_*)_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_*) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$; we set $\iota_{w_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 1$ if $p_{w_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} \cdot l$, $\iota_{w_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 0$ if $p_{w_{\mathbb{Q}}} > e^*_{\mathrm{mod}} \cdot l$. Then one verifies immediately [again, by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ are *equivalent*:

(D4) $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$.
(D5) The valuation $v_{\mathbb{Q}}$ *ramifies* in K.
(D6) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$.
(D7) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F}_{\mathrm{ADiv}} + \mathfrak{d}^{F}_{\mathrm{ADiv}})$.

Here, we observe in passing that, for $v \in \mathbb{V}(F_{\square})$,

(R1) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l^4)$ if v *divides* l,
(R2) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l)$ if v *divides* $2 \cdot 3 \cdot 5$ or lies in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}})$ [hence does *not divide* l],
(R3) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}})$ if v does *not divide* $2 \cdot 3 \cdot 5 \cdot l$ and, moreover, is *not contained* in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}})$,

and hence that

(R4) if $e_v \geq p_v - 1 > p_v - 2$, then $p_v \ \leq \ 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l \ = \ e^*_{\mathrm{mod}} \cdot l$, and $\log(e_v) \ \leq \ -3 + 4 \cdot \log(e^*_{\mathrm{mod}} \cdot l)$

— cf. (E3), (E4), (E5), (E6); (D0); Proposition 1.8, (v), (vii); [IUTchI], Definition 3.1, (c). Next, for later reference, we observe that the *inequality*

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}})$$

holds for *any* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$; in particular, when $p_{v_{\mathbb{Q}}} = l \ (\geq 5)$, it holds that

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{1}{2}$$

— cf. (E6). On the other hand, it follows immediately from Proposition 1.3, (i), by considering the *various possibilities* for elements $\in \mathrm{Supp}(\mathfrak{s}^{F_{\mathrm{mod}}}_{\mathrm{ADiv}})$, that

$$\log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ 2 \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— and hence that

$$\tfrac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \ \leq \ \tfrac{2}{p_{v_{\mathbb{Q}}}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— for any $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $p_{v_{\mathbb{Q}}} \notin \{2, 3, 5, l\}$. In a similar vein, we conclude that

$$\begin{aligned} \log(\mathfrak{s}^{\mathbb{Q}}) \ &\leq \ 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + \log(2 \cdot 3 \cdot 5 \cdot l) \\ &\leq \ 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 5 + \log(l) \end{aligned}$$

and hence that

$$\tfrac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \ \leq \ \tfrac{8 \cdot d_{\mathrm{mod}}}{l} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 6$$

— cf. (E6); the fact that $l \geq 5$. Combining this last inequality with the inequality of the final display of Step (ii) yields the *inequality*

$$(1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K) + \tfrac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \ \leq \ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 52$$

— where we apply the estimate $d_{\mathrm{mod}} \geq 1$.

(iv) In order to estimate the constant "C_Θ" of [IUTchIII], Corollary 3.12, we must, according to the various definitions given in the statement of [IUTchIII], Corollary 3.12, compute an *upper bound* for the

> *procession-normalized mono-analytic log-volume* of the *holomorphic hull* of the *union* of the *possible images* of a Θ-*pilot object*, relative to the relevant *Kummer isomorphisms* [cf. [IUTchIII], Theorem 3.11, (ii)], in the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), which we regard as *subject* to the *indeterminacies* (Ind1), (Ind2), (Ind3) described in [IUTchIII], Theorem 3.11, (i), (ii).

Thus, we proceed to estimate this log-volume at *each* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Once one fixes $v_{\mathbb{Q}}$, this amounts to estimating the component of this log-volume in

$$\text{“}\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})\text{”}$$

[cf. the notation of [IUTchIII], Theorem 3.11, (i), (a)], for *each* $j \in \{1, \ldots, l^*\}$, which we shall also regard as an element of $\mathbb{F}_l^*$, and then computing the *average*, over $j \in \{1, \ldots, l^*\}$, of these estimates. Here, we recall [cf. [IUTchI], Proposition 6.9, (i); [IUTchIII], Proposition 3.4, (ii)] that $\mathbb{S}^{\pm}_{j+1} = \{0, 1, \ldots, j\}$. Also, we recall from [IUTchIII], Proposition 3.2, that “$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$” is, by definition, a *tensor product* of $j+1$ copies, indexed by the elements of $\mathbb{S}^{\pm}_{j+1}$, of the *direct sum* of the $\mathbb{Q}$-spans of the *log-shells* associated to each of the elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ [cf., especially, the second and third displays of [IUTchIII], Proposition 3.2]. In particular, for *each collection*

$$\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$$

of [not necessarily distinct!] elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$, we must estimate the component of the log-volume in question corresponding to the *tensor product* of the $\mathbb{Q}$-spans of the *log-shells* associated to this collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ and then compute the *weighted average* [cf. the discussion of Remark 1.7.1], over possible collections $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$, of these estimates.

(v) Let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iii), to the present situation, by taking

- “I” to be $\mathbb{S}^{\pm}_{j+1}$;
- “$I^* \subseteq I$” to be the set of $i \in I$ such that $e_{\underline{v}_i} > p_{v_{\mathbb{Q}}} - 2$;
- “k_i” to be $K_{\underline{v}_i}$ [so “R_i” will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$];
- “$i^\dagger$” to be $j \in \mathbb{S}^{\pm}_{j+1}$;
- “λ” to be 0 if $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}$;
- “λ” to be “$\mathrm{ord}(-)$” of the element $\underline{\underline{q}}_{\underline{v}_j}^{j^2}$ [cf. the definition of “$\underline{\underline{q}}_{\underline{v}}$” in [IUTchI], Example 3.2, (iv)] if $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Thus, the *inclusion* “$\phi(p^{\lambda} \cdot (R_I)^{\sim}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times})$” of Proposition 1.4, (iii), implies that the result of multiplying “$p^{\lfloor \lambda \rfloor - |I|} \cdot 2^{-|I|} \cdot \log_p(R_I^{\times})$” by a suitable *nonpositive* [cf. the inequalities concerning “$\mathfrak{d}_I + a_I$” that constitute the final portion of Proposition 1.4, (iii)] integer power of $p_{v_{\mathbb{Q}}}$ *contains* the “union of possible images of a Θ-pilot object” discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism “ϕ” [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the above-mentioned integer power of $p_{v_{\mathbb{Q}}}$ is *nonpositive*, which implies that the module obtained by multiplying by this power of $p_{v_{\mathbb{Q}}}$ *contains* “$p^{\lfloor \lambda \rfloor - |I|} \cdot 2^{-|I|} \cdot \log_p(R_I^{\times})$”. Thus, an upper bound on the component of the log-volume of the *holomorphic hull* under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side of the inclusion "$p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^\times) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim}$" of Proposition 1.4, (iii). Such an *upper bound*

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1\Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}\text{”}$$

is given in the second displayed inequality of Proposition 1.4, (iii). Here, we note that if $e_{\underline{v}_i} \leq p_{v_\mathbb{Q}} - 2$ for all $i \in I$, then this *upper bound* assumes the form

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1\Big) \cdot \log(p)\text{”}.$$

On the other hand, by (R4), if $e_{\underline{v}_i} > p_{v_\mathbb{Q}} - 2$ for some $i \in I$, then it follows that $p_{v_\mathbb{Q}} \leq e^*_{\mathrm{mod}} \cdot l$, and $\log(e_{\underline{v}_i}) \leq -3 + 4 \cdot \log(e^*_{\mathrm{mod}} \cdot l)$, so the *upper bound* in question may be taken to be

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1\Big) \cdot \log(p) \ + \ 4(j+1) \cdot l^*_{\mathrm{mod}}\text{”}$$

— where we write $l^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \log(e^*_{\mathrm{mod}} \cdot l)$. Also, we note that, unlike the other terms that appear in these *upper bounds*, "λ" is *asymmetric* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^\pm_{j+1}$. Since we would like to compute *weighted averages* [cf. the discussion of Remark 1.7.1], we thus observe that, after *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^\pm_{j+1}$, this *upper bound* may be written in the form

$$\text{“}\beta_{\vec{e}}\text{”}$$

[cf. the notation of Proposition 1.7] if, in the situation of Proposition 1.7, one takes

- "E" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_\mathbb{Q}}$;
- "n" to be $j+1$, so an element "$\vec{e} \in E^n$" corresponds precisely to a collection $\{v_i\}_{i \in \mathbb{S}^\pm_{j+1}}$;
- "λ_e", for an element $e \in E$ corresponding to $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be $[(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_\mathbb{Q}}] \in \mathbb{R}_{>0}$;
- "β_e", for an element $e \in E$ corresponding to $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be

$$\log(\mathfrak{d}^K_v) - \tfrac{j^2}{2l(j+1)} \cdot \log(\mathfrak{q}_v) + \tfrac{1}{j+1} \cdot \log(p_{v_\mathbb{Q}}) + 4 \cdot \iota_{v_\mathbb{Q}} \cdot l^*_{\mathrm{mod}}$$

— where we recall that $\iota_{v_\mathbb{Q}} \stackrel{\mathrm{def}}{=} 1$ if $p_{v_\mathbb{Q}} \leq e^*_{\mathrm{mod}} \cdot l$, $\iota_{v_\mathbb{Q}} \stackrel{\mathrm{def}}{=} 0$ if $p_{v_\mathbb{Q}} > e^*_{\mathrm{mod}} \cdot l$.

Here, we note that it follows immediately from the first equality of the first display of Proposition 1.7 that, after passing to *weighted averages*, the operation of *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^\pm_{j+1}$ does *not affect* the computation of the *upper bound* under consideration. Thus, by applying Proposition 1.7, we obtain that the resulting **"weighted average upper bound"** is given by

$$(j+1) \cdot \log(\mathfrak{d}^K_{v_\mathbb{Q}}) - \tfrac{j^2}{2l} \cdot \log(\mathfrak{q}_{v_\mathbb{Q}}) + \log(\mathfrak{s}^\mathbb{Q}_{v_\mathbb{Q}}) + 4(j+1) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_\mathbb{Q}})$$

— where we recall the notational conventions introduced in Step (iii). Thus, it remains to compute the *average over* $j \in \mathbb{F}_l^{\circledast}$. By *averaging* over $j \in \{1, \dots, l^{\circledast} = \frac{l-1}{2}\}$ and applying (E1), (E2), we obtain the **"procession-normalized upper bound"**

$$\begin{aligned}
&\tfrac{(l^{\circledast}+3)}{2} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{(2l^{\circledast}+1)(l^{\circledast}+1)}{12l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + 2(l^{\circledast}+3) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}}) \\
&\quad = \tfrac{l+5}{4} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{l+1}{24} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + (l+5) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}}) \\
&\quad \leq \tfrac{l+1}{4} \cdot \left\{ (1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{1}{6} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \tfrac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + \tfrac{20}{3} \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}}) \right\}
\end{aligned}$$

— where, in the passage to the final displayed inequality, we apply the estimates $\frac{1}{l+1} \leq \frac{1}{l}$ and $\frac{4(l+5)}{l+1} \leq \frac{20}{3}$, both of which may be regarded as consequences of the fact that $l \geq 5$.

(vi) Next, let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}} \setminus \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iv), to the present situation, by taking

- "I" to be $\mathbb{S}^{\pm}_{j+1}$;
- "k_i" to be $K_{\underline{v}_i}$ [so "R_i" will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$].

Here, we note that our assumption that $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}} \setminus \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$ implies that the hypotheses of Proposition 1.4, (iv), are satisfied. Thus, the *inclusion* "$\phi((R_I)^{\sim}) \subseteq (R_I)^{\sim}$" of Proposition 1.4, (iv), implies that the tensor product of log-shells under consideration *contains* the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism "ϕ" [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the "container of possible images" is *precisely equal* to the tensor product of log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side "$(R_I)^{\sim}$" of the above inclusion. Such an *upper bound*

"0"

is given in the final equality of Proposition 1.4, (iv). One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is clearly equal to 0.

(vii) Next, let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.5, (iii), (iv), to the present situation, by taking

- "I" to be $\mathbb{S}^{\pm}_{j+1}$ [so $|I| = j+1$];
- "V" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$;

· "$\mathbb{C}_v$" to be $K_{\underline{v}}$, where we write $\underline{v} \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ for the element determined by $v \in V$.

Then it follows from Proposition 1.5, (iii), (iv), that

$$\pi^{j+1} \cdot B_I$$

serves as a *container* for the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the fact that $B_I \subseteq M_I$ is *preserved* by *arbitrary automorphisms* of the type discussed in Proposition 1.5, (iii), while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that, by Proposition 1.5, (iv), together with our choice of the factor π^{j+1}, this "container of possible images" contains the elements of M_I obtained by forming the tensor product of elements of the log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of this container. Such an *upper bound*

$$(j+1) \cdot \log(\pi)$$

follows immediately from the fact that [in order to ensure compatibility with *arithmetic degrees* of arithmetic line bundles — cf. [IUTchIII], Proposition 3.9, (iii) — one is obliged to adopt normalizations which imply that] *the log-volume of B_I is equal to* 0. One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is given by

$$\tfrac{l+5}{4} \cdot \log(\pi) \;\leq\; \tfrac{l+1}{4} \cdot 4$$

— cf. (E1), (E6); the fact that $l \geq 5$.

(viii) Now we return to the discussion of Step (iv). In order to compute the desired *upper bound* for "C_Θ", it suffices to **sum over** $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ the various local **"procession-normalized upper bounds"** obtained in Steps (v), (vi), (vii) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. By applying the *inequality* of the final display of Step (iii), we thus obtain the following *upper bound* for "$C_\Theta \cdot |\log(\underline{\underline{q}})|$", i.e., the product of "C_Θ" and $\frac{1}{2l} \cdot \log(\mathfrak{q})$:

$$\tfrac{l+1}{4} \cdot \Big\{ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 56 - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q})$$

$$+ \tfrac{20}{3} \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \Big\} - \tfrac{1}{2l} \cdot \log(\mathfrak{q})$$

— where we apply the estimate $\frac{l+1}{4} \cdot \frac{1}{6} \cdot \frac{12}{l^2} \geq \frac{1}{2l}$ [cf. the fact that $l \geq 1$].

Now let us recall the constant "η_{prm}" of Proposition 1.6. By applying Proposition 1.6, we compute:

$$l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \log(e^*_{\mathrm{mod}} \cdot l) \cdot \sum_{p \,\leq\, e^*_{\mathrm{mod}} \cdot l} 1 \;\leq\; \tfrac{4}{3} \cdot \log(e^*_{\mathrm{mod}} \cdot l) \cdot \tfrac{e^*_{\mathrm{mod}} \cdot l}{\log(e^*_{\mathrm{mod}} \cdot l)}$$

$$= \tfrac{4}{3} \cdot e^*_{\mathrm{mod}} \cdot l$$

— where the sum ranges over the primes $p \leq e^*_{\mathrm{mod}} \cdot l$ — if $e^*_{\mathrm{mod}} \cdot l \geq \eta_{\mathrm{prm}}$;

$$l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \log(e^*_{\mathrm{mod}} \cdot l) \cdot \sum_{p \,\leq\, e^*_{\mathrm{mod}} \cdot l} 1 \;\leq\; \tfrac{4}{3} \cdot \log(\eta_{\mathrm{prm}}) \cdot \tfrac{\eta_{\mathrm{prm}}}{\log(\eta_{\mathrm{prm}})} \;=\; \tfrac{4}{3} \cdot \eta_{\mathrm{prm}}$$

— where the sum ranges over the primes $p \leq e^*_{\mathrm{mod}} \cdot l$ — if $e^*_{\mathrm{mod}} \cdot l < \eta_{\mathrm{prm}}$. Thus, we conclude that

$$l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \tfrac{4}{3} \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}})$$

[i.e., regardless of the size of $e^*_{\mathrm{mod}} \cdot l$]. Also, let us observe that

$$\tfrac{1}{3} \cdot \tfrac{4}{3} \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \;\geq\; \tfrac{1}{3} \cdot \tfrac{4}{3} \cdot e^*_{\mathrm{mod}} \cdot l \;\geq\; 2 \cdot 2 \cdot 2^{12} \cdot 3 \cdot 5 \cdot l \;\geq\; 2 \cdot \log(l) + 56$$

— where we apply the estimates $e_{\mathrm{mod}} \geq 1$, $2^{12} \cdot 3 \cdot 5 \geq 56$, $l \geq 5 \geq 1$, $l \geq \log(l)$ [cf. the fact that $l \geq 5$]. Thus, substituting back into our *original upper bound* for "$C_\Theta \cdot |\log(\underline{\underline{q}})|$", we obtain the following *upper bound* for "C_Θ":

$$\tfrac{l+1}{4 \cdot |\log(\underline{\underline{q}})|} \cdot \Big\{ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 10 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q}) \Big\} - 1$$

— where we apply the estimate $\frac{20+1}{3} \cdot \frac{4}{3} = \frac{7 \cdot 4}{3} \leq 10$ — i.e., as asserted in the statement of Theorem 1.10. The final portion of Theorem 1.10 follows immediately from [IUTchIII], Corollary 3.12, by applying the inequality of the first display of Step (ii), together with the estimates

$$(1 - \tfrac{12}{l^2})^{-1} \;\leq\; 2; \quad (1 - \tfrac{12}{l^2})^{-1} \cdot (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \;\leq\; 1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}$$

[cf. the fact that $l \geq 7$, $d_{\mathrm{mod}} \geq 1$]. ○

Remark 1.10.1. One of the main original motivations for the development of the theory discussed in the present series of papers was to create a *framework*, or *geometry*, within which a suitable analogue of the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII] could be realized in such a way that the *obstructions to diophantine applications* that arose in the scheme-theoretic formulation of [HASurI], [HASurII] [cf. the discussion of [HASurI], §1.5.1; [HASurII], Remark 3.7] could be *avoided*. From this point of view, it is of interest to observe that the computation of the *"leading term"* of the inequality of the final display of the statement of Theorem 1.10 — i.e., of the term

$$\tfrac{(l^{*}+3)}{2} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{(2l^{*}+1)(l^{*}+1)}{12l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}})$$

that occurs in the final display of Step (v) of the proof of Theorem 1.10 — via the identities (E1), (E2) is *essentially identical* to the computation of the leading term that occurs in the proof of [HASurI], Theorem A [cf. the discussion following the statement of Theorem A in [HASurI], §1.1]. That is to say, in some sense,

> the **computations** performed in the proof of Theorem 1.10 were already essentially known to the author around the year 2000; the problem then was to construct an appropriate **framework**, or **geometry**, in which these computations could be performed!

This sort of situation may be compared to the computations underlying the **Weil Conjectures** priori to the construction of a "Weil cohomology" in which those computations could be performed, or, alternatively, to various computations of invariants in topology or differential geometry that were motivated by computations in **physics**, again prior to the construction of a suitable mathematical framework in which those computations could be performed.

Remark 1.10.2. The computation performed in the proof of Theorem 1.10 may be thought of as the computation of a sort of **derivative** in the $\mathbb{F}_l^{\circledast}$**-direction**, which, relative to the analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich], corresponds to the *derivative* of the *canonical Frobenius lifting* — cf. the discussion of [IUTchIII], Remark 3.12.4, (iii). In this context, it is useful to recall the *arithmetic Kodaira-Spencer morphism* that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], §3]. In particular, in [HASurII], Corollary 3.6, it is shown that, when suitably formulated, a *"certain portion"* of this arithmetic Kodaira-Spencer morphism *coincides* with the usual *geometric Kodaira-Spencer morphism.* From the point of view of the *action* of $GL_2(\mathbb{F}_l)$ on the l-torsion points involved, this "certain portion" consists of the *unipotent matrices*

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$. By contrast, the $\mathbb{F}_l^{\circledast}$*-symmetries* that occur in the present series of papers correspond to the *toral matrices*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$ — cf. the discussion of [IUTchI], Example 4.3, (i). As we shall see in §2 below, in the present series of papers, we shall ultimately take l to be *"large"*. When l is "sufficiently large", $GL_2(\mathbb{F}_l)$ may be thought of as a *"good approximation"* for $GL_2(\mathbb{Z})$ or $GL_2(\mathbb{R})$ — cf. the discussion of [IUTchI], Remark 6.12.3, (i), (iii). In the case of $GL_2(\mathbb{R})$, "toral subgroups" may be thought of as corresponding to the *isotropy subgroups* [isomorphic to $\mathbb{S}^1$] of points that arise from the action of $GL_2(\mathbb{R})$ on the *upper half-plane*, i.e., subgroups which may be thought of as a sort of *geometric, group-theoretic representation* of **tangent vectors** at a point.

Remark 1.10.3. The *"terms involving l"* that occur in the inequality of the final display of Theorem 1.10 may be thought of as an *inevitable consequence* of the *fundamental role* played in the theory of the present series of papers by the *l-torsion points* of the elliptic curve under consideration. Here, we note that it is of crucial importance to work over the *field of rationality* of the l-torsion points [i.e., "K" as opposed to "F"] *not only* when considering the global portions of the various ΘNF-

and $\Theta^{\pm \mathrm{ell}}$*-Hodge theaters* involved, but also when considering the local portions — i.e., the *prime-strips* — of these ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge theaters. That is to say, these local portions are necessary, for instance, in order to *glue together* the ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge theaters that appear so as to form a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [cf. the discussion of [IUTchI], Remark 6.12.2]. In particular, to allow, within these local portions, any sort of *"Galois indeterminacy"* with respect to the l-torsion points — even, for instance, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, which, at first glance, might appear *irrelevant* to the theory of *Hodge-Arakelov-theoretic evaluation at* l*-torsion points* developed in [IUTchII] — would have the effect of *invalidating* the various *delicate manipulations involving* l*-torsion points* discussed in [IUTchI], §4, §6 [cf., e.g., [IUTchI], Propositions 4.7, 6.5].

Remark 1.10.4. The various *fluctuations in log-volume* — i.e., whose computation is the subject of Theorem 1.10! — that arise from the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), may be thought of as a sort of **"inter-universal analytic torsion"**. Indeed,

> in general, **"analytic torsion"** may be understood as a sort of *measure* — in **"metrized"** [e.g., log-volume!] terms — of the degree of **deviation** of the *"holomorphic functions"* [such as sections of a line bundle] on a variety — i.e., which depend, in an essential way, on the **holomorphic moduli** of the variety! — from the *"real analytic functions"* — i.e., which are **invariant** with respect to **deformations of the holomorphic moduli** of the variety.

For instance:

(a) In *"classical" Arakelov theory*, analytic torsion typically arises as [the logarithm of] a sort of **normalized determinant** of the **Laplacian** acting on some space of real analytic [or L^2-] sections of a line bundle on a complex variety equipped with a real analytic Kähler metric [cf., e.g., [Arak], Chapters V, VI]. Here, we recall that in this sort of situation, the space of *holomorphic* sections of the line bundle is given by the kernel of the Laplacian; the definition of the *Laplacian* depends, in an essential way, on the *Kähler metric*, hence, in particular, on the *holomorphic moduli* of the variety under consideration [cf., e.g., the case of the Poincaré metric on a hyperbolic Riemann surface!].

(b) In the *scheme-theoretic Hodge-Arakelov theory* discussed in [HASurI], [HASurII], the main theorem consists of a sort of *comparison isomorphism* [cf. [HASurI], Theorem A] between a certain subspace of the space of *global sections* of the pullback of an ample line bundle on an elliptic curve to the *universal vectorial extension of the elliptic curve* and the space of *set-theoretic functions on the torsion points* of the elliptic curve. That is to say, the former space of sections contains, in a natural way, the space of *holomorphic* sections of the ample line bundle on the elliptic curve, while the latter space of functions may be thought of as a sort of "discrete approximation" of the space of *real analytic* functions on the elliptic curve [cf. the discussion of [HASurI], §1.3.2, §1.3.4]. In this context, the **"Gaussian poles"** [cf. the discussion of [HASurI], §1.1] arise as a measure of the *discrepancy of integral structures* between these two spaces in a neighborhood of the divisor at infinity of the moduli stack of elliptic curves, hence may be thought of as a sort of *"analytic torsion at the divisor at infinity"* [cf. the discussion of [HASurI], §1.2].

(c) In the case of the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), the fluctuations of log-volume computed in Theorem 1.10 arise precisely as a result of the execution of a *comparison of an* **"alien" arithmetic holomorphic structure** to this **multiradial representation**, which is compatible with the *permutation symmetries of the étale-picture*, i.e., which is "invariant with respect to deformations of the arithmetic holomorphic moduli of the number field under consideration" in the sense that it makes sense *simultaneously with respect to distinct arithmetic holomorphic structures* [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)]. Here, it is of interest to observe that the object of this comparison consists of the *values of the theta function*, i.e., in essence, a "holomorphic section of an ample line bundle". In particular, the resulting fluctuations of log-volume may be thought as a sort of *"analytic torsion"*. By analogy to the terminology *"Gaussian poles"* discussed in (b) above, it is natural to think of the terms involving the *different* $\mathfrak{d}^K_{(-)}$ that appear in the computation underlying Theorem 1.10 [cf., e.g., the final display of Step (v) of the proof of Theorem 1.10] as **"differential poles"** [cf. the discussion of Remarks 1.10.1, 1.10.2]. Finally, in the context of the *normalized determinants* that appear in (a), it is interesting to note the role played by the **prime number theorem** — i.e., in essence, the **Riemann zeta function** [cf. Proposition 1.6 and its proof] — in the computation of **"inter-universal analytic torsion"** given in the proof of Theorem 1.10.

Remark 1.10.5. The above remarks focused on the *conceptual* aspects of the theory surrounding Theorem 1.10. Before proceeding, however, we pause to discuss briefly certain aspects of Theorem 1.10 that are of interest from a **computational** point of view, i.e., in the spirit of conventional *analytic number theory.*

(i) First, we begin by observing that, unlike the inequalities that appear in the various results [cf. Corollaries 2.2, (ii); 2.3] obtained in §2 below, the inequalities obtained in Theorem 1.10 involve only **essentially explicit constants** and, moreover, *do not require one to* **exclude** *some non-explicit* **finite set** of "isomorphism classes of exceptional elliptic curves". From this point of view,

> the inequalities obtained in Theorem 1.10 are suited to application to **computations** concerning various **explicit diophantine equations**, such as, for instance, the equations that appear in "Fermat's Last Theorem".

Such explicit computations in the case of specific diophantine equations are, however, beyond the scope of the present paper.

(ii) One topic of interest in the context of computational aspects of Theorem 1.10 is the **asymptotic behavior** of the bound that appears in, say, the first inequality of the final display of Theorem 1.10. Let us assume, for simplicity, that $F_{\mathrm{tpd}} = \mathbb{Q}$ [so $d_{\mathrm{mod}} = 1$]. Also, to simplify the notation, let us write $\delta \stackrel{\mathrm{def}}{=} \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) = \log(\mathfrak{f}^{F_{\mathrm{tpd}}})$. Then the bound under consideration assumes the form

$$\delta + * \cdot \frac{\delta}{l} + * \cdot l + *$$

— where, in the present discussion, the "$*$'s" are to be understood as denoting *fixed positive real numbers*. Thus, the *leading term* [cf. the discussion of Remark 1.10.1] is equal to δ. The remaining terms give rise to the *"ϵ terms"* [and bounded discrepancy] of the inequalities of Corollaries 2.2, (ii); 2.3, obtained in §2 below. Thus, if one ignores "bounded discrepancies", it is of interest to consider the behavior of the *"ϵ terms"*

$$* \cdot \frac{\delta}{l} \ + \ * \cdot l$$

as one allows the *initial* Θ-*data* under consideration to *vary* [i.e., subject to the condition "$F_{\mathrm{tpd}} = \mathbb{Q}$"]. In this context, one **fundamental observation** is the following: although l is subject to various other conditions, no matter how "skillfully" one chooses l, the resulting "ϵ terms" are always

$$\geq \ * \cdot \delta^{1/2}$$

— an estimate that may be obtained by thinking of l as $\approx \delta^{\alpha}$, for some real number α, and comparing δ^{α} and $\delta^{1-\alpha}$. This estimate is of particular interest in the context of various explicit examples constructed by Masser and others [cf. [Mss]; the discussion of [vFr], §2] in which explicit "*abc* sums" are constructed for which the quantity on the left-hand side of the inequality of Theorem 1.10 under consideration *exceeds* the order of δ +

$$* \cdot \frac{\delta^{1/2}}{\log(\delta)}$$

— cf. [vFr], Equation (6). In particular, the asymptotic estimates given by Theorem 1.10 are **consistent** with the known asymptotic behavior of these explicit *abc* sums. Indeed, the exponent "$\frac{1}{2}$" that appears in the *fundamental observation* discussed above *coincides precisely* with the "expectation" expressed by van Frankenhuijsen in the final portion of the discussion of [vFr], §2! In the present paper, although we are unable to in fact achieve bounds on the "ϵ terms" of the order $* \cdot \delta^{1/2}$, we do succeed in obtaining bounds on the "ϵ terms" of the order

$$* \cdot \delta^{1/2} \cdot \log(\delta)$$

— albeit under the assumption that the *abc* sums under consideration are **compactly bounded** away from **infinity** at the prime 2, as well as at the archimedean prime [cf. Corollary 2.2, (ii); Remark 2.2.1 below for more details].

(iii) In the context of the discussion of (ii), it is of interest to observe that the "$* \cdot l$" portion of the "ϵ terms" that appear arises from the estimates given in Step (viii) of the proof of Theorem 1.10 for the quantity "$\log(\mathfrak{s}^{\leq})$". From the point of view of the discussion of [vFr], §3, this quantity corresponds essentially to a "certain portion" of the quantity "$\omega(abc)$" associated to an *abc* sum. That is to say, whereas "$\omega(abc)$" denotes the *total number of prime factors* that occur in the product *abc*, the quantity "$\log(\mathfrak{s}^{\leq})$" corresponds, roughly speaking, to the number of these prime factors that are $\leq e^{*}_{\mathrm{mod}} \cdot l$. The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to "$\omega(abc)$" is of interest from the point of view of the discussion of [vFr], §3, partly since it is [*not precisely identical* to, but nonetheless] **reminiscent** of the various *refinements of the ABC Conjecture* proposed by Baker [i.e., which are the main topic of the discussion of

[vFr], §3]. The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to "$\omega(abc)$" is also of interest from the point of view of the explicit *abc* sums discussed in (ii) that give rise to asymptotic behavior $\geq * \cdot \frac{\delta^{1/2}}{\log(\delta)}$. That is to say, according to the discussion of [vFr], §3, Remark 1, this sort of *abc* sum tends to give rise to a

> **relatively large** value for $\omega(abc)$ — i.e., a state of affairs that is consistent with the *crucial role* played by the "ϵ term" related to $\omega(abc)$ in the computation of the lower bound "$\geq \ * \cdot \delta^{1/2}$" that appears in the *fundamental observation* of (ii).

By contrast, the *abc* sums of the form "$2^n = p + qr$" [where p, q, and r are prime numbers] considered in [vFr], §3, Remark 1, give rise to a

> **relatively small** value for $\omega(abc)$ [indeed, $\omega(abc) \leq 4$] — i.e., a situation that suggests *relatively small/essentially negligible* "ϵ terms" in the bound of Theorem 1.10 under consideration.

Such essentially negligible "ϵ terms" are, however, *consistent* with the fact [cf. [vFr], §3, Remark 1] that, for such *abc* sums, the left-hand side of the inequality of Theorem 1.10 under consideration is roughly $\approx \frac{1}{2}\cdot$ *the leading term* of the bound on the right-hand side, hence, in particular, is *amply bounded by the leading term* on the right-hand side, *without any "help" from the "ϵ terms"*.

Remark 1.10.6.

(i) In the context of the discussion of Remark 1.10.5, it is important to remember that

> the bound on "$\frac{1}{6} \cdot \log(\mathfrak{q})$" given in Theorem 1.10 only concerns the *q-parameters* at the nonarchimedean valuations contained in $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, all of which are necessarily of **odd residue characteristic**

— cf. [IUTchI], Definition 3.1, (b). This observation is of relevance to the examples of *abc* sums constructed in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], since it does not appear, at first glance, that there is any way to effectively control the *contributions at the prime* 2 in these examples, that is to say, in the notation of the Proposition of [Mss], to control the power of 2 that divides the integer "c" of the Proposition of [Mss], or, alternatively, in the notation of the proof of this Proposition on [Mss], p. 22, to control the power of 2 that divides the difference "$x_i - x_{i-1}$". On the other hand, it was pointed out to the author by *A. Venkatesh* that in fact it is not difficult to modify the construction of these examples of *abc* sums given in [Mss] so as to obtain *similar asymptotic estimates* to those obtained in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], *even without taking into account the contributions at the prime* 2.

(ii) In the context of the discussion of (i), it is of interest to recall **why** nonarchimedean primes of **even** *residue characteristic* where the elliptic curve under consideration has bad multiplicative reduction are *excluded* from $\mathbb{V}^{\text{bad}}_{\text{mod}}$ in the theory of the present series of papers. In a word, the reason that the theory encounters difficulties at primes over 2 is that it depends, in a quite essential way, on the theory of the **étale theta function** developed in [EtTh], which *fails* at primes over 2 [cf. the assumption that "*p* is *odd*" in [EtTh], Theorem 1.10, (iii); [EtTh], Definition 2.5; [EtTh], Corollary 2.18]. From the point of view of the theory of [IUTchI], [IUTchII], and [IUTchIII] [cf., especially, the theory of [IUTchII], §1, §2: [IUTchII], Corollary 1.12; [IUTchII], Corollary 2.4, (ii), (iii); [IUTchII], Corollary 2.6], one of the *key consequences* of the theory of [EtTh] is the **simultaneous multiradiality** of the algorithms that give rise to

(1) **constant multiple rigidity** and
(2) **cyclotomic rigidity**.

At a more concrete level, (1) is obtained by **evaluating** the usual series for the theta function [cf. [EtTh], Proposition 1.4] at the 2-torsion point in the "irreducible component labeled zero". One computes easily that the resulting "special value" is a **unit** for *odd* p, but is equal to a [nonzero] **non-unit** when $p = 2$. In particular, since (1) is established by *dividing* the series of [EtTh], Proposition 1.4 [i.e., the usual series for the theta function], by this special value, it follows that

(a) the *"integral structure"* on the theta function determined by this special value

coincides with

(b) the *"integral structure"* on the theta function determined by the natural integral structure on the pole at the origin

for **odd** p [cf. [EtTh], Theorem 1.10, (iii)], but **not** when $p = 2$. That is to say, when $p = 2$, a *nontrivial denominator* arises. Here, we recall that it is crucial to evaluate at 2-torsion points, i.e., as opposed to, say, more general points in the irreducible component labeled zero for reasons discussed in [IUTchII], Remark 2.5.1, (ii) [cf. also the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv)]. This nontrivial denominator is fundamentally incompatible with the *multiradiality* of the algorithms of (1), (2) in that it is incompatible with the *fundamental splitting*, or *"decoupling"*, into *"purely radial"* [i.e., roughly speaking, "value group"] and *"purely coric"* [i.e., roughly speaking, "unit"] components discussed in [IUTchII], Remarks 1.11.4, (i); 1.12.2, (vi) [cf. also the discussion of [IUTchII], Remark 1.11.5]. That is to say, on the one hand,

> the **multiradiality of (1)** may only be established if the possible values at the evaluation points in the irreducible component labeled zero are known, a priori, to be **units**, i.e., if one works relative to the **integral structure (a)**

— cf. the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv). On the other hand, if one tries to work

> **simultaneously** with the **integral structure (b)**, hence with the **non-trivial denominator** discussed above, then the **multiradiality of (2)** is **violated**.

Here, we recall that the *integral structure (b)*, which is referred to as the *"canonical integral structure"* in [EtTh], Proposition 1.4, (iii); [EtTh], Theorem 1.10, (iii), is in some sense the "integral structure of common sense".

(iii) It is not entirely clear to the author at the time of writing to what extent the *integral structure (b)* is *necessary* in order to carry out the theory developed in the present series of papers. Indeed, [EtTh], as well as the present series of papers, was written in a way that [unlike the discussion of (ii)!] "takes for granted" the fact that the two integral structures (a), (b) discussed above *coincide* for *odd* p, i.e., in a way which *identifies* these two integral structures and hence does not *specify*, at various key points in the discussion, whether one is in fact working with integral structure (a) or with integral structure (b). On the other hand, if it is indeed the case that *not only* the integral structure (a), *but also* the integral structure (b) plays an *essential role* in the present series of papers, then it follows [cf. the discussion of (ii)!] that the theory of the present series of papers is **fundamentally incompatible** with the inclusion in $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ of nonarchimedean primes of **even** *residue characteristic* where the elliptic curve under consideration has bad multiplicative reduction.

(iv) In the context of the discussion of (ii), (iii), it is perhaps useful to recall that the classical theory of theta functions also tends to [depending on your point of view!] *"break down"* or *"assume a completely different form"* at the prime 2. For instance, this phenomenon can be seen throughout Mumford's theory of *algebraic theta functions*, which may be thought of as a sort of predecessor to the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII], which, in turn, may be thought of as a sort of predecessor to the theory of the present series of papers. In a similar vein, it is of interest to recall that the prime 2 is also *excluded* in the *p-adic Teichmüller theory* of [pOrd], [pTeich]. This is done in order to avoid the complications that occur in the theory of the Lie algebra sl_2 over fields of characteristic 2.

Remark 1.10.7.

(i) Since $e^*_{\mathrm{mod}} \leq d^*_{\mathrm{mod}}$, one may *replace* "$e^*_{\mathrm{mod}}$" by "$d^*_{\mathrm{mod}}$" in the final two displays of the statement of Theorem 1.10.

(ii) By contrast, at least if one adheres to the framework of the theory of the present series of papers,

> it is **not** possible to **replace** "d_{mod}" by "e_{mod}" in the final two displays of the statement of Theorem 1.10.

The fundamental reason for this is that, in the construction of the **multiradial representation** of [IUTchIII], Theorem 3.11, (i), it is necessary to consider tensor products of copies, labeled by $j \in \mathbb{F}_l^{\circledast}$, of F_{mod} over $\mathbb{Q}$ [cf. [IUTchIII], Proposition

3.3!]. That is to say, it is *fundamentally impossible* [i.e., relative to the framework of the theory of the present series of papers] to *identify* the F_{mod}-linear structures for distinct labels j, since the various tensor packets that appear in the multiradial representation must be constructed in such a way as to depend only on the **additive structure** [i.e., not the module structure over some sort of *ring* such as F_{mod}!] of the **[mono-analytic!] log-shells** involved. Working with tensor powers of copies of F_{mod} over $\mathbb{Q}$ means that there is *no way to avoid*, when one localizes at a prime number p, working with tensor products between localizations of F_{mod} at *distinct primes* of F_{mod} that divide p. Moreover, whenever *even one* of these primes of F_{mod} is lies under a prime of K that *ramifies* over $\mathbb{Q}$ [cf. condition (D5) of Step (iii) of the proof of Theorem 1.10], the computation of Step (v) of the proof of Theorem 1.10 necessarily gives rise to a "$\log(p)$" term — i.e., that appears in "$\log(\mathfrak{s}^{\mathbb{Q}})$" — that arises from *"rounding up" non-integral powers of* p [i.e., as in the inclusions of Proposition 1.4, (iii)], since only *integral powers* of p make sense in the *multiradial representation*. That is to say, whereas integral powers of p only require the use of the *additive structure* of the **[mono-analytic!] log-shells** involved, *non-integral powers* only make sense if one is equipped with the module structure over some sort of *ring* such as F_{mod}!

Section 2: Diophantine Inequalities

In the present §2, we combine Theorem 1.10 with the theory of [GenEll] to give a proof of the **ABC Conjecture**, or, equivalently, **Vojta's Conjecture for hyperbolic curves** [cf. Corollary 2.3 below].

We begin by reviewing some well-known estimates.

Proposition 2.1. (Well-known Estimates)

(i) **(Linearization of Logarithms)** *We have* $\log(x) \leq x$ *for all* $(\mathbb{R} \ni)\ x \geq 1$.

(ii) **(The Prime Number Theorem)** *There exists a real number* $\xi_{\mathrm{prm}} \geq 5$ *such that*

$$\tfrac{2}{3} \cdot x \quad \leq \quad \theta(x) \stackrel{\mathrm{def}}{=} \sum_{p \leq x} \log(p) \quad \leq \quad \tfrac{4}{3} \cdot x$$

— where the sum ranges over the prime numbers p *such that* $p \leq x$ *— for all* $(\mathbb{R} \ni)\ x \geq \xi_{\mathrm{prm}}$. *In particular, if* $\mathcal{A}$ *is a finite set of prime numbers, and we write*

$$\theta_{\mathcal{A}} \stackrel{\mathrm{def}}{=} \sum_{p \in \mathcal{A}} \log(p)$$

[where we take the sum to be 0 *if* $\mathcal{A} = \emptyset$*], then there exists a prime number* $p \notin \mathcal{A}$ *such that* $p \leq 2(\theta_{\mathcal{A}} + \xi_{\mathrm{prm}})$.

Proof. Assertion (i) is well-known and entirely elementary. Assertion (ii) is a well-known consequence of the *Prime Number Theorem* [cf., e.g., [Edw], p. 76; [GenEll], Lemma 4.1; [GenEll], Remark 4.1.1]. ○

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. In the following discussion, we shall apply the *notation* and *terminology* of [GenEll]. Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\mathrm{def}}{=} X \backslash D$; d a positive integer. Write ω_X for the canonical sheaf on X. Suppose that U_X is a *hyperbolic curve*, i.e., that the degree of the line bundle $\omega_X(D)$ is *positive*. Then we recall the following notation:

- $U_X(\overline{\mathbb{Q}})^{\leq d} \subseteq U_X(\overline{\mathbb{Q}})$ denotes the subset of $\overline{\mathbb{Q}}$-rational points *defined over a finite extension field* of $\mathbb{Q}$ of *degree* $\leq d$ [cf. [GenEll], Example 1.3, (i)].

- log-diff_X denotes the *[normalized] log-different* function on $U_X(\overline{\mathbb{Q}})$ [cf. [GenEll], Definition 1.5, (iii)].

- log-cond_D denotes the *[normalized] log-conductor* function on $U_X(\overline{\mathbb{Q}})$ [cf. [GenEll], Definition 1.5, (iv)].

- $\mathrm{ht}_{\omega_X(D)}$ denotes the *[normalized] height* function on $U_X(\overline{\mathbb{Q}})$ associated to $\omega_X(D)$, which is well-defined up to a *"bounded discrepancy"* [cf. [GenEll], Proposition 1.4, (iii)].

In order to apply the theory of the present series of papers, it is neceesary to *construct suitable initial* Θ*-data*, as follows.

Corollary 2.2. **(Construction of Suitable Initial Θ-Data)** *Suppose that* $X = \mathbb{P}^1_{\mathbb{Q}}$ *is the* **projective line** *over* $\mathbb{Q}$*, and that* $D \subseteq X$ *is the divisor consisting of the* **three points** *"0", "1", and "∞". We shall regard X as the* **"λ-line"** *— i.e., we shall regard the standard coordinate on* $X = \mathbb{P}^1_{\mathbb{Q}}$ *as the "λ" in the* **Legendre form** *"$y^2 = x(x-1)(x-\lambda)$" of the Weierstrass equation defining an elliptic curve — and hence as being equipped with a natural* **classifying morphism** $U_X \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ *[cf. the discussion preceding Proposition 1.8]. Let*

$$\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$$

be a **compactly bounded subset** *[i.e., regarded as a subset of $X(\overline{\mathbb{Q}})$ — cf. Remark 2.3.1, (vi), below; [GenEll], Example 1.3, (ii)] whose* **support** *contains the nonarchimedean prime "2". Suppose further that $\mathcal{K}_V$ satisfies the following condition:*

($*_{j\text{-inv}}$) *If* $v \in \mathbb{V}(\mathbb{Q})$ *denotes the nonarchimedean prime "2", then the image of the subset* $\mathcal{K}_v \subseteq U_X(\overline{\mathbb{Q}}_v)$ *associated to* $\mathcal{K}_V$ *[cf. the notational conventions of [GenEll], Example 1.3, (ii)] via the* ***j*-invariant** $U_X \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ *is a* **bounded** *subset of* $\mathbb{A}^1_{\mathbb{Q}}(\overline{\mathbb{Q}}_v) = \overline{\mathbb{Q}}_v$*, i.e., is contained in a subset of the form* $2^{N_{j\text{-inv}}} \cdot \mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$*, where* $N_{j\text{-inv}} \in \mathbb{Z}$*, and* $\mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$ *denotes the ring of integers.*

Then:

(i) Write "$\log(\mathfrak{q}^{\forall}_{(-)})$" (respectively, "$\log(\mathfrak{q}^{\dagger 2}_{(-)})$") for the $\mathbb{R}$-valued function on $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$, hence also on $U_X(\overline{\mathbb{Q}})$, obtained by forming the normalized degree "$\underline{\deg}(-)$" of the effective arithmetic divisor determined by the ***q*-parameters** *of an elliptic curve over a number field at* **arbitrary** *nonarchimedean primes (respectively, at the nonarchimedean primes that do* **not divide** *2) [cf. the invariant "$\log(\mathfrak{q})$" associated, in the statement of Theorem 1.10, to the elliptic curve E_F]. Also, we shall write* ht_∞ *for the* **[normalized] height** *function on $U_X(\overline{\mathbb{Q}})$ — a function which is well-defined up to a* **"bounded discrepancy"** *[cf. the discussion preceding [GenEll], Proposition 3.4] — determined by the pull-back to X of the divisor at infinity of the natural compactification $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Q}}$ of $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$. Then we have an* **equality of "bounded discrepancy classes"** *[cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]*

$$\tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}_{(-)}) \;\approx\; \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}_{(-)}) \;\approx\; \tfrac{1}{6} \cdot \mathrm{ht}_\infty \;\approx\; \mathrm{ht}_{\omega_X(D)}$$

of functions on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$.

(ii) There exist

- *a positive real number H_{unif} which is* **independent** *of $\mathcal{K}_V$ and*
- *positive real numbers $C_{\mathcal{K}}$ and $H_{\mathcal{K}}$ which* **depend only** *on the choice of the* **compactly bounded subset** $\mathcal{K}_V$

such that the following property is satisfied: Let d be a positive integer, ϵ_d a positive real number ≤ 1. Set $\delta \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d$. Then there exists a **finite** *subset $\mathfrak{Exc}_d \subseteq U_X(\overline{\mathbb{Q}})^{\leq d}$ which depends only on $\mathcal{K}_V$, d, and ϵ_d, contains all points corresponding to elliptic curves that admit automorphisms of order > 2, and satisfies the following property:*

The function "$\log(\mathfrak{q}^{\forall}_{(-)})$" of (i) is

$$\leq \ H_{\mathrm{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$$

on $\mathfrak{Exc}_d$. Let E_F be an **elliptic curve** *over a number field $F \subseteq \overline{\mathbb{Q}}$ that determines a $\overline{\mathbb{Q}}$-valued point of $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ which lifts [not necessarily uniquely!] to a point $x_E \in U_X(F) \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ such that*

$$x_E \in \mathcal{K}_V, \quad x_E \notin \mathfrak{Exc}_d.$$

Write F_{mod} for the **minimal field of definition** *of the corresponding point $\in \mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$ and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \ \stackrel{\mathrm{def}}{=} \ F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from F_{mod} by adjoining the fields of definition of the 2-torsion points of any model of $E_F \times_F \overline{\mathbb{Q}}$ over F_{mod} [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3 \cdot 5)$-torsion points of E_F are defined over F, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2 \cdot 3 \cdot 5]\ ) \quad \stackrel{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3 \cdot 5]\ )$$

— i.e., that F is obtained from F_{tpd} by adjoining $\sqrt{-1}$, together with the fields of definition of the $(3 \cdot 5)$-torsion points of a model $E_{F_{\mathrm{tpd}}}$ of the elliptic curve $E_F \times_F \overline{\mathbb{Q}}$ over F_{tpd} determined by the **Legendre form** *of the Weierstrass equation discussed above [cf. Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ over F, so $x_E \in U_X(F_{\mathrm{tpd}}) \subseteq U_X(F)$; it follows from Proposition 1.8, (v), that E_F has* **stable reduction** *at every element of $\mathbb{V}(F)^{\mathrm{non}}$.] Write $\log(\mathfrak{q}^{\forall})$ (respectively, $\log(\mathfrak{q}^{\dagger 2})$) for the result of applying the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" (respectively, "$\log(\mathfrak{q}^{\dagger 2}_{(-)})$") of (i) to x_E. Then E_F and F_{mod} arise as the "E_F" and "F_{mod}" for a collection of* **initial Θ-data** *as in Theorem 1.10 that, in the notation of Theorem 1.10, satisfies the following conditions:*

(C1) $(\log(\mathfrak{q}^{\forall}))^{1/2} \ \leq \ l \ \leq \ 10\delta \cdot (\log(\mathfrak{q}^{\forall}))^{1/2} \cdot \log(2\delta \cdot \log(\mathfrak{q}^{\forall}))$;

(C2) we have inequalities

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ \leq \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}) \ &\leq \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) \\ &\leq \ (1 + \epsilon_E) \cdot (\text{log-diff}_X(x_E) + \text{log-cond}_D(x_E)) + C_{\mathcal{K}} \end{aligned}$$

— where we write

$$\epsilon_E \ \stackrel{\mathrm{def}}{=} \ (60\delta)^2 \cdot \frac{\log(2\delta \cdot (\log(\mathfrak{q}^{\forall})))}{(\log(\mathfrak{q}^{\forall}))^{1/2}}$$

[i.e., so ϵ_E **depends** *on the integer d, as well as on the elliptic curve E_F!], and we observe, relative to the notation of Theorem 1.10, that [it follows tautologically from the definitions that] we have an equality* $\text{log-diff}_X(x_E) = \log(\mathfrak{d}^{F_{\text{tpd}}})$*, as well as inequalities*

$$\log(\mathfrak{f}^{F_{\text{tpd}}}) \leq \text{log-cond}_D(x_E) \leq \log(\mathfrak{f}^{F_{\text{tpd}}}) + \log(2l).$$

(iii) The positive real number H_{unif} of (ii) [which is **independent** *of $\mathcal{K}_V$!] may be chosen in such a way that the following property is satisfied: Let d be a positive integer, ϵ_d and ϵ positive real numbers ≤ 1. Then there exists a* **finite** *subset $\mathfrak{Exc}_{\epsilon,d} \subseteq U_X(\overline{\mathbb{Q}})^{\leq d}$ which depends only on $\mathcal{K}_V$, ϵ, d, and ϵ_d such that the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" of (i) is*

$$\leq \; H_{\text{unif}} \cdot \epsilon^{-3} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d}$$

on $\mathfrak{Exc}_{\epsilon,d}$, and, moreover, in the notation of (ii), the invariant ϵ_E associated to an elliptic curve E_F as in (ii) [i.e., that satisfies certain conditions which **depend** *on $\mathcal{K}_V$ and d] satisfies the inequality $\epsilon_E \leq \epsilon$ whenever the point $x_E \in U_X(F)$ satisfies the condition $x_E \notin \mathfrak{Exc}_{\epsilon,d}$.*

Proof. First, we consider assertion (i). We begin by observing that, in light of the *condition* $(*_{j\text{-inv}})$ that was imposed on $\mathcal{K}_V$, it follows immediately from the various definitions involved that

$$\log(\mathfrak{q}^{\dagger 2}_{(-)}) \;\approx\; \log(\mathfrak{q}^{\forall}_{(-)})$$

— where we observe that the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" may be *identified* with the function "$\underline{\deg}_\infty$" of the discussion preceding [GenEll], Proposition 3.4 — on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. In a similar vein, since the *support* of $\mathcal{K}_V$ contains the unique archimedean prime of $\mathbb{Q}$, it follows immediately from the various definitions involved [cf. also Remark 2.3.1, (vi), below] that

$$\log(\mathfrak{q}^{\forall}_{(-)}) \;\approx\; \text{ht}_\infty$$

on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ [cf. the argument of the final paragraph of the proof of [GenEll], Lemma 3.7]. Thus, we conclude that $\log(\mathfrak{q}^{\dagger 2}_{(-)}) \;\approx\; \log(\mathfrak{q}^{\forall}_{(-)}) \;\approx\; \text{ht}_\infty$ on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. Finally, since [as is well-known] the pull-back to X of the divisor at infinity of the natural compactification $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ of $(\mathcal{M}_{\text{ell}})_{\mathbb{Q}}$ is of *degree* 6, while the line bundle $\omega_X(D)$ is of *degree* 1, the equality of BD-classes $\frac{1}{6} \cdot \text{ht}_\infty \;\approx\; \text{ht}_{\omega_X(D)}$ on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ follows immediately from [GenEll], Proposition 1.4, (i), (iii). This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us recall that if the once-punctured elliptic curve associated to E_F *fails to admit an F-core*, then there are only *four possibilities* for the j-invariant of E_F [cf. [CanLift], Proposition 2.7]. Thus, if we take the set $\mathfrak{Exc}_d$ to be the [*finite*!] collection of points corresponding to these four j-invariants, then we may assume that the once-punctured elliptic curve associated to E_F *admits an F-core*, hence, in particular, does not have any automorphisms of order > 2 over $\overline{\mathbb{Q}}$. In the discussion to follow, it will be necessary to *enlarge* the finite set $\mathfrak{Exc}_d$ several times, always in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d [i.e., but not on x_E!] and in such a way that the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" of (i) is $\leq H_{\mathrm{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$ [i.e., but not on d or ϵ_d!].

Next, let us write

$$h \stackrel{\mathrm{def}}{=} \log(\mathfrak{q}^{\forall}) \;=\; \frac{1}{[F:\mathbb{Q}]} \cdot \sum_{v \in \mathbb{V}(F)^{\mathrm{non}}} h_v \cdot f_v \cdot \log(p_v)$$

— that is to say, $h_v = 0$ for those v at which E_F has *good reduction*; $h_v \in \mathbb{N}_{\geq 1}$ is the *local height* of E_F [cf. [GenEll], Definition 3.3] for those v at which E_F has *bad multiplicative reduction*. Now it follows [by assertion (i); [GenEll], Proposition 1.4, (iv)] that the inequality $h^{1/2} < \xi_{\mathrm{prm}} + \eta_{\mathrm{prm}}$ [cf. the notation of Propositions 1.6; 2.1, (ii)] implies that there is only a *finite number of possibilities* for the j-invariant of E_F. Thus, by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathrm{unif}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$], we may assume without loss of generality that the inequality

$$h^{1/2} \;\geq\; \xi_{\mathrm{prm}} + \eta_{\mathrm{prm}} \;\geq\; 5$$

holds. Thus, since $[F:\mathbb{Q}] \leq \delta$ [cf. the properties (E3), (E4), (E5) in the proof of Theorem 1.10], it follows that

$$\begin{aligned}
\delta \cdot h^{1/2} \;\geq\; [F:\mathbb{Q}] \cdot h^{1/2} \;&=\; \sum_{v} h^{-1/2} \cdot h_v \cdot f_v \cdot \log(p_v) \;\geq\; \sum_{v} h^{-1/2} \cdot h_v \cdot \log(p_v) \\
&\geq \sum_{h_v \,\geq\, h^{1/2}} h^{-1/2} \cdot h_v \cdot \log(p_v) \;\geq\; \sum_{h_v \,\geq\, h^{1/2}} \log(p_v)
\end{aligned}$$

and

$$\begin{aligned}
2\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \;&\geq\; 2 \cdot [F:\mathbb{Q}] \cdot h^{1/2} \cdot \log(2 \cdot [F:\mathbb{Q}] \cdot h) \\
&\geq \sum_{h_v \neq 0} 2 \cdot h^{-1/2} \cdot \log(2 \cdot h_v \cdot f_v \cdot \log(p_v)) \cdot h_v \cdot f_v \cdot \log(p_v) \\
&\geq \sum_{h_v \neq 0} h^{-1/2} \cdot \log(h_v) \cdot h_v \;\geq\; \sum_{h_v \,\geq\, h^{1/2}} h^{-1/2} \cdot \log(h_v) \cdot h_v \\
&\geq \sum_{h_v \,\geq\, h^{1/2}} \log(h_v)
\end{aligned}$$

— where the sums are all over $v \in \mathbb{V}(F)^{\mathrm{non}}$ [possibly subject to various conditions, as indicated], and we apply the elementary estimate $2 \cdot \log(p_v) \geq 2 \cdot \log(2) = \log(4) \geq 1$ [cf. the property (E6) in the proof of Theorem 1.10].

Thus, in summary, we conclude from the estimates made above that if we take

$$\mathcal{A}$$

to be the [*finite*!] set of prime numbers p such that p either

(S1) is $\leq h^{1/2}$,
(S2) *divides a nonzero* h_v for some $v \in \mathbb{V}(F)^{\mathrm{non}}$, or
(S3) is *equal to* p_v for some $v \in \mathbb{V}(F)^{\mathrm{non}}$ for which $h_v \geq h^{1/2}$,

then it follows from Proposition 2.1, (ii), together with our assumption that $h^{1/2} \geq \xi_{\mathrm{prm}}$, that, in the notation of Proposition 2.1, (ii),

$$\begin{aligned} \theta_{\mathcal{A}} &\leq 2 \cdot h^{1/2} + \delta \cdot h^{1/2} + 2\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \\ &\leq 4\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \\ &\leq -\xi_{\mathrm{prm}} + 5\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \end{aligned}$$

— where we apply the estimates $\delta \geq 2$ and $\log(2\delta \cdot h) \geq \log(4) \geq 1$ [cf. the property (E6) in the proof of Theorem 1.10]. In particular, it follows from Proposition 2.1, (i), (ii), together with our assumption that $h^{1/2} \geq 5 \geq 1$, that there exists a *prime number* l such that

(P1) $(5 \leq)$ $h^{1/2} \leq l \leq 10\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h)$ $(\leq 20 \cdot \delta^2 \cdot h^2)$ [cf. the condition (C1) in the statement of Corollary 2.2];
(P2) l does *not divide* any nonzero h_v for $v \in \mathbb{V}(F)^{\mathrm{non}}$;
(P3) if $l = p_v$ for some $v \in \mathbb{V}(F)^{\mathrm{non}}$, then $h_v < h^{1/2}$.

Next, let us *observe* that, again by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that, in the terminology of [GenEll], Lemma 3.5,

(P4) E_F does *not* admit an *l-cyclic* subgroup scheme.

Indeed, the existence of an l-cyclic subgroup scheme of E_F would imply that

$$\tfrac{l-2}{24} \cdot \log(\mathfrak{q}^{\forall}) \leq 2 \cdot \log(l) + T_{\mathcal{K}}$$

— where we apply assertion (i), the displayed inequality of [GenEll], Lemma 3.5, and the final inequality of the display of [GenEll], Proposition 3.4; we take the "ϵ" of [GenEll], Lemma 3.5, to be 1; we write $T_{\mathcal{K}}$ for the positive real number [which depends only on the choice of the *compactly bounded subset* $\mathcal{K}_V$] that results from the various *"bounded discrepancies"* implicit in these inequalities. Since $l \geq 5$ [cf. (P1)], it follows that $1 \leq 2 \cdot \log(l) \leq 48 \cdot \frac{l-2}{24}$ [cf. the property (E6) in the proof of Theorem 1.10], and hence that the inequality of the preceding display implies that $\log(\mathfrak{q}^{\forall})$ is *bounded*. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a *finite number of possibilities* for the j-invariant of E_F. This completes the proof of the above *observation*.

Next, let us note that it follows immediately from (P1), together with Proposition 2.1, (i), that

$$\begin{aligned} h^{1/2} \cdot \log(l) &\leq h^{1/2} \cdot \log(20 \cdot \delta^2 \cdot h^2) \leq 2 \cdot h^{1/2} \cdot \log(5\delta \cdot h) \\ &\leq 8 \cdot h^{1/2} \cdot \log(2 \cdot \delta^{1/4} \cdot h^{1/4}) \leq 8 \cdot h^{1/2} \cdot 2 \cdot \delta^{1/4} \cdot h^{1/4} \\ &= 16 \cdot \delta^{1/4} \cdot h^{3/4} \end{aligned}$$

— where we apply the estimates $20 \leq 5^2$ and $5 \leq 2^4$. In particular, we *observe* that, again by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathrm{unif}} \cdot d + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that

(P5) if we write $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ for the set of nonarchimedean valuations $\in \mathbb{V}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$ that do *not divide* $2l$ and at which E_F has *bad multiplicative reduction*, then $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \neq \emptyset$.

Indeed, if $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} = \emptyset$, then it follows, in light of the definition of h, from (P3), assertion (i), and the computation performed above, that

$$h \;\approx\; \log(\mathfrak{q}^{\dagger 2}) \;\leq\; h^{1/2} \cdot \log(l) \;\leq\; 16 \cdot \delta^{1/4} \cdot h^{3/4}$$

— an inequality which implies that $h^{1/4}$, hence h itself, is *bounded*. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a *finite number of possibilities* for the j-invariant of E_F. This completes the proof of the above *observation*. This property (P5) implies that

(P6) the image of the outer homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{F}_l)$ determined by the l*-torsion points* of E_F *contains the subgroup* $SL_2(\mathbb{F}_l) \subseteq GL_2(\mathbb{F}_l)$.

Indeed, since, by (P5), E_F has *bad multiplicative reduction* at some valuation $\in \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \neq \emptyset$, (P6) follows formally from (P2), (P4), and [GenEll], Lemma 3.1, (iii) [cf. the proof of the final portion of [GenEll], Theorem 3.8].

Now it follows *formally* from (P1), (P2), (P5), and (P6) that, if one takes "$\overline{F}$" to be $\overline{\mathbb{Q}}$, "F" to be the number field F of the above discussion, "X_F" to be the once-punctured elliptic curve associated to E_F, "l" to be the prime number l of the above discussion, and "$\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$" to be the set $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ of (P5), then there exist data "$\underline{C}_K$", "$\underline{\mathbb{V}}$", and "$\underline{\epsilon}$" such that *all of the conditions of [IUTchI], Definition 3.1, (a), (b), (c), (d), (e), (f), are satisfied*, and, moreover, that

(P7) *the resulting* **initial Θ-data**

$$(\overline{F}/F,\; X_F,\; l,\; \underline{C}_K,\; \underline{\mathbb{V}},\; \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\; \underline{\epsilon})$$

satisfies the various conditions in the statement of Theorem 1.10.

Here, we note in passing that the crucial *existence* of data "$\underline{\mathbb{V}}$" and "$\underline{\epsilon}$" satisfying the requisite conditions follows, in essence, as a consequence of the fact [i.e., (P6)] that the Galois action on l-torsion points contains the *full special linear group* $SL_2(\mathbb{F}_l)$.

In light of (P7), we may apply Theorem 1.10 [cf. also Remark 1.10.7, (i)] to conclude that

$$\begin{aligned}
\tfrac{1}{6} \cdot \log(\mathfrak{q}) \;&\leq\; (1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \;+\; 20 \cdot (d^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \\
&\leq\; (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\
&\qquad\qquad\qquad\qquad +\; 200 \cdot \delta^2 \cdot h^{1/2} \cdot \log(2\delta \cdot h) \;+\; 20\eta_{\mathrm{prm}}
\end{aligned}$$

— where we apply (P1), as well as the estimates $20 \cdot d_{\mathrm{mod}} \leq d^*_{\mathrm{mod}} \leq \delta$.

Next, let us *observe* that it follows from (P3), together with the computation of the discussion preceding (P5), that

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}) - \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ &\leq\ \tfrac{1}{6} \cdot h^{1/2} \cdot \log(l) \ \leq\ \tfrac{1}{3} \cdot h^{1/2} \cdot \log(5\delta \cdot h) \\ &\leq\ h^{1/2} \cdot \log(2\delta \cdot h) \end{aligned}$$

— where we apply the estimates $1 \leq h$ and $5 \leq 2^3$. Thus, since, by assertion (i), the difference $\frac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) - \frac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2})$ is *bounded* by some positive real number $B_{\mathcal{K}}$ [which depends only on the choice of the *compactly bounded subset* $\mathcal{K}_V$], we conclude that

$$\begin{aligned} \tfrac{1}{6} \cdot h \ = \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) \ &\leq\ (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + \ (15\delta)^2 \cdot h^{1/2} \cdot \log(2\delta \cdot h) \ + \ \tfrac{1}{2} \cdot C_{\mathcal{K}} \\ &\leq\ (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + \ \tfrac{1}{6} \cdot h \cdot \tfrac{2}{5} \cdot (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \ + \ \tfrac{1}{2} \cdot C_{\mathcal{K}} \end{aligned}$$

— where we write $C_{\mathcal{K}} \stackrel{\mathrm{def}}{=} 40\eta_{\mathrm{prm}} + 2B_{\mathcal{K}}$, and we apply the estimate $6 \cdot 5 \leq 2 \cdot 4^2$.

Now let us set

$$\epsilon_E \ \stackrel{\mathrm{def}}{=}\ (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \ \ (\geq\ 5 \cdot \delta \cdot h^{-1/2});$$
$$\epsilon^*_d \ \stackrel{\mathrm{def}}{=}\ \tfrac{1}{16} \cdot \epsilon_d \ \ (< \tfrac{1}{2} \ \leq\ 1)$$

— where we apply the estimates $h \geq 1$, $\log(2\delta \cdot h) \ \geq\ \log(2\delta) \ \geq\ \log(4) \ \geq\ 1$ [cf. the property (E6) in the proof of Theorem 1.10], and $\epsilon_d \leq 1$. Note that the inequality

$$\begin{aligned} 1 \ < \epsilon_E \ &=\ (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \\ &=\ (\epsilon^*_d)^{-1} \cdot (60\delta)^2 \cdot h^{-1/2} \cdot \log(2^{\epsilon^*_d} \cdot \delta^{\epsilon^*_d} \cdot h^{\epsilon^*_d}) \\ &\leq\ (\epsilon^*_d)^{-1} \cdot (60\delta)^{2+\epsilon^*_d} \cdot h^{-(1/2-\epsilon^*_d)} \\ &\leq\ \left\{ (\epsilon^*_d)^{-3} \cdot (60\delta)^{4+\epsilon_d} \cdot h^{-1} \right\}^{(1/2-\epsilon^*_d)} \end{aligned}$$

— where we apply Proposition 2.1, (i), together with the estimates

$$\frac{1}{\frac{1}{2} - \epsilon^*_d} \ = \ \frac{16}{8 - \epsilon_d} \ \leq\ 3; \qquad \frac{2 + \epsilon^*_d}{\frac{1}{2} - \epsilon^*_d} \ = \ \frac{32 + \epsilon_d}{8 - \epsilon_d} \ \leq\ 4 + \epsilon_d \ \leq\ 5$$

[both of which are consequences of the fact that $0 < \epsilon_d \leq 1 \leq 3$], as well as the estimates $0 < \epsilon^*_d \leq 1$, $60\delta \geq 2\delta \geq 1$, and $h \geq 1$ — implies a *bound* on h, hence, [by assertion (i); [GenEll], Proposition 1.4, (iv)] that there is only a *finite number of possibilities* for the j-invariant of E_F. Thus, by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathrm{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that $\epsilon_E \leq 1$.

Thus, in summary, we obtain inequalities

$$\begin{aligned} \tfrac{1}{6} \cdot h \ &\leq\ (1 - \tfrac{2}{5} \cdot \epsilon_E)^{-1}(1 + \tfrac{1}{5} \cdot \epsilon_E) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + (1 - \tfrac{2}{5} \cdot \epsilon_E)^{-1} \cdot \tfrac{1}{2} \cdot C_{\mathcal{K}} \\ &\leq\ (1 + \epsilon_E) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + C_{\mathcal{K}} \end{aligned}$$

by applying the estimates

$$\frac{1 + \frac{1}{5} \cdot \epsilon_E}{1 - \frac{2}{5} \cdot \epsilon_E} \quad \leq \quad 1 + \epsilon_E; \qquad 1 - \tfrac{2}{5} \cdot \epsilon_E \quad \geq \quad \tfrac{1}{2}$$

— both of which are consequences of the fact that $0 < \epsilon_E \leq 1$. Thus, in light of (P1), together with the observation that it follows immediately from the definitions [cf. also Proposition 1.8, (vi)] that we have an equality $\text{log-diff}_X(x_E) = \log(\mathfrak{d}^{F_{\mathrm{tpd}}})$, as well as inequalities $\log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \leq \text{log-cond}_D(x_E) \leq \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + \log(2l)$, we conclude that both of the conditions (C1), (C2) in the statement of assertion (ii) hold for $C_{\mathcal{K}}$ as defined above. This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately by applying the argument applied above in the proof of assertion (ii) in the case of the inequality "$1 < \epsilon_E$" to the inequality "$\epsilon < \epsilon_E$". ○

Remark 2.2.1.

(i) Before proceeding, we pause to examine the **asymptotic behavior** of the **bound** obtained in Corollary 2.2, (ii), in the spirit of the discussion of Remark 1.10.5, (ii). For simplicity, we assume that $F_{\mathrm{tpd}} = \mathbb{Q}$ [so $d_{\mathrm{mod}} = 1$]; we write $h \stackrel{\mathrm{def}}{=} \log(\mathfrak{q}^{\forall})$ [cf. the proof of Corollary 2.2, (ii)] and $\delta \stackrel{\mathrm{def}}{=} \text{log-diff}_X(x_E) + \text{log-cond}_D(x_E) = \text{log-cond}_D(x_E)$ [i.e., notation that is closely related to the notation of Remark 1.10.5, (ii), but *differs* substantially from the notation of Corollary 2.2, (ii)]. Thus, it follows immediately from the definitions that $1 < \log(3) \leq \delta$ and $1 < \log(3) \leq h$. In particular, the bound under consideration may be written in the form

$$\tfrac{1}{6} \cdot h \ \leq\ \delta \ +\ * \cdot \delta^{1/2} \cdot \log(\delta)$$

— where "$*$" is to be understood as denoting a *fixed positive real number*; we observe that the *ratio* h/δ is always a positive real number which is *bounded below* by the definition of h and δ and *bounded above* precisely as a consequence of the bound under consideration. In this context, it is of interest to observe that the form of the "ϵ term" $\delta^{1/2} \cdot \log(\delta)$ is **strongly reminiscent** of well-known intepretations of the **Riemann hypothesis** in terms of the asymptotic behavior of the function defined by considering the number of prime numbers less than a given natural number. Indeed, from the point of view of **weights** [cf. also the discussion of Remark 2.2.2 below], it is natural to regard the [logarithmic] height of a line bundle as an object that has the *same weight* as a **single Tate twist**, or, from a more classical point of view, "$2\pi i$" raised to the *power* 1. On the other hand, again from the point of view of weights, the variable "s" of the *Riemann zeta function* $\zeta(s)$ may be thought of as corresponding precisely to the *number of Tate twists* under consideration, so a *single Tate twist* corresponds to "$s = 1$". Thus, from this point of view, "$s = \frac{1}{2}$", i.e., the *critical line* that appears in the *Riemann hypothesis*, corresponds precisely to the *square roots* of the [logarithmic] heights under consideration, i.e., to $h^{1/2}$, $\delta^{1/2}$. Moreover, from the point of view of the computations that underlie Theorem

1.10 and Corollary 2.2, (ii) [cf., especially, the proof of Corollary 2.2, (ii); Steps (v), (viii) of the proof of Theorem 1.10; the contribution of "b_i" in the *second displayed inequality* of Proposition 1.4, (iii)], this $\delta^{1/2}$ arises as a result of a sort of **"balance"**, or **"duality"** — i.e., that occurs as one *increases* the size of the auxiliary prime l [cf. the discussion of Remark 1.10.5, (ii)] — between the **archimedean decrease** in the "ϵ term" $\frac{\delta}{l}$ and the **nonarchimedean increase** in the "ϵ term" l [i.e., that arises from a certain estimate, in the proof of Proposition 1.2, (i), (ii), of the *radius of convergence* of the *p-adic logarithm*]. That is to say, such a **global arithmetic duality** is reminiscent of the **functional equation** of the Riemann zeta function [cf. the discussion of (iii) below].

(ii) In [vFr], §2, it is conjectured that, in the notation of the discussion of (i),

$$\limsup \ \frac{\log\left(\frac{1}{6}\cdot h-\delta\right)}{\log(h)} \ = \ \frac{1}{2}$$

and observed that the "$\frac{1}{2}$" that appears here is **strongly reminiscent** of the "$\frac{1}{2}$" that appears in the **Riemann hypothesis**. In the situation of Corollary 2.2, (ii), bounds are only obtained on *abc* sums that belong to the **compactly bounded subset** $\mathcal{K}_V$ under consideration; such bounds, i.e., as discussed in (i), thus imply that this lim sup is $\leq \frac{1}{2}$. On the other hand, it is shown in [vFr], §2 [cf. also the references quoted in [vFr]], that, if one allows *arbitrary abc* sums [i.e., which are not necessarily assumed to be contained in a single compactly bounded subset $\mathcal{K}_V$], then this lim sup is $\geq \frac{1}{2}$. It is not clear to the author at the time of writing whether or not such estimates [i.e., to the effect that the lim sup under consideration is $\geq \frac{1}{2}$] hold even if one imposes the restriction that the *abc* sums under consideration be contained in a single compactly bounded subset $\mathcal{K}_V$.

(iii) In the well-known classical theory of the **Riemann zeta function**, the Riemann zeta function is closely related to the **theta function**, i.e., by means of the **Mellin transform**. In light of the *central role* played by theta functions in the theory of the present series of papers, it is tempting to hope, especially in the context of the observations of (i), (ii), that perhaps some extension of the theory of the present series of papers — i.e., some sort of **"inter-universal Mellin transform"** — may be obtained that allows one to relate the theory of the present series of papers to the Riemann zeta function.

(iv) In the context of the discussion of (iii), it is of interest to recall that, relative to the analogy between *number fields* and *one-dimensional function fields over finite fields*, the theory of the present series of papers may be thought of as being analogous to the theory surrounding the **derivative** of a lifting of the **Frobenius morphism** [cf. the discussion of [IUTchI], §I4; [IUTchIII], Remark 3.12.4]. On the other hand, the analogue of the **Riemann hypothesis** for one-dimensional function fields over finite fields may be proven by considering the elementary geometry of the [graph of the] **Frobenius morphism**. This state of affairs suggests that perhaps some sort of **"integral"** of the theory of the present series of papers could shed light on the Riemann hypothesis in the case of number fields.

(v) One way to summarize the point of view discussed in (i), (ii), and (iii) is as follows: The *asymptotic behavior* discussed in (i) suggests that perhaps one should expect that the *inequality* constituted by well-known intepretations of the **Riemann hypothesis** in terms of the asymptotic behavior of the function defined by considering the number of prime numbers less than a given natural number may be obtained as some sort of *"restriction"*

$$\textbf{(ABC inequality)}|_{\textbf{canonical number}}$$

of some sort of *"ABC inequality"* [i.e., some sort of bound of the sort obtained in Corollary 2.2, (ii)] to some sort of *"canonical number"* [i.e., where the term "number" is to be understood as referring to an *abc* sum]. Here, the descriptive "canonical" is to be understood as expressing the idea that one is not so much interested in considering a *fixed explicit "number/abc sum"*, but rather some sort of *suitable abstraction* of the sort of sequence of numbers/*abc* sums that gives rise to the lim sup value of "$\frac{1}{2}$" discussed in (ii). Of course, it is by no means clear precisely how such an "abstraction" should be formulated, but the idea is that it should represent

some sort of **average** over **all possible addition operations**

in the number field [in this case, $\mathbb{Q}$] under consideration or [perhaps equivalently]

> some sort of **"arithmetic measure or distribution"** constituted by such a collection of **all possible addition operations** that somehow amounts to a sort of arithmetic analogue of the measure that gives rise to the classical **Mellin transform**

[i.e., that appears in the discussion of (iii)].

Remark 2.2.2. In the context of the discussion of **weights** in Remark 2.2.1, (i), it is of interest to recall the significance of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}$$

in the theory of the present series of papers [cf. [IUTchII], Introduction; [IUTchII], Remark 1.12.5, as well as Remark 1.10.1 of the present paper]. Indeed, typically discussions of the *Riemann zeta function* $\zeta(s)$, or more general *L-functions*, in the context of conventional arithmetic geometry are concerned principally with the behavior of such functions at *integral values* [i.e., $\in \mathbb{Z}$] of the variable s. Such integral values of the variable s correspond to *integral Tate twists*, i.e., at a more concrete level, to *integral powers* of the quantity $2\pi i$. If one neglects nonzero factors $\in \mathbb{Q}(i)$, then such integral powers may be regarded as *integral powers* of π [or 2π]. At the level of *classical integrals*, the notion of a single Tate twist may be thought of as corresponding to the integral

$$\int_{\mathbb{S}^1} d\theta \;=\; 2\pi$$

over the unit circle $\mathbb{S}^1$; at the level of *schemes*, the notion of a single Tate twist may be thought of as corresponding to the scheme $\mathbb{G}_{\mathrm{m}}$. On the other hand, whereas the conventional theory of Tate twists in arithmetic geometry only involves *integral powers* of a single Tate twist, i.e., corresponding, in essence, to integral powers of π, the Gaussian integral may be thought of as a sort of *fundamental integral representation* of the notion of a **"Tate semi-twist"**. From this point of view, *scheme-theoretic Hodge-Arakelov theory* may be thought of as a sort of *fundamental scheme-theoretic represention* of the notion of a **"Tate semi-twist"** [cf. the discussion of [IUTchII], Remark 1.12.5]. Thus, in summary,

(a) the **Gaussian integral**,
(b) **scheme-theoretic Hodge-Arakelov theory**,
(c) the **inter-universal Teichmüller theory** developed in the present series of papers, and
(d) the **Riemann hypothesis**,

may all be thought of as **"phenomena of weight $\frac{1}{2}$"**, i.e., at a concrete level, phenomena that revolve around *arithmetic versions of "$\sqrt{\pi}$"*. Moreover, we observe that in the first three of these four examples, the essential nature of the notion of "weight $\frac{1}{2}$" may be thought of as being reflected in some sort of **exponential** of a **quadratic form**. This state of affairs is strongly reminiscent of

(1) the **Griffiths semi-transversality** of the **crystalline theta object** that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], Theorem 2.8; [IUTchII], Remark 1.12.5, (i)], which corresponds essentially [cf. the discussion of the proof of [HASurII], Theorem 2.10] to the *quadratic form* that appears in the exponents of the well-known series expansion of the *theta function*;

(2) the *quadratic* nature of the **commutator** of the **theta group**, which is applied, in [EtTh] [cf. the discussion of [IUTchIII], Remark 2.1.1], to derive the various *rigidity properties* which are interpreted, in [IUTchII], §1, as *multiradiality* properties — an interpretation that is *strongly reminiscent*, if one interprets "multiradiality" in terms of *"connections"* and *"parallel transport"* [cf. [IUTchII], Remark 1.7.1], of the quadratic form discussed in (1);

(3) the essentially *quadratic* nature of the **"ϵ term"** $* \cdot \frac{\delta}{l} + * \cdot l$ [which, we recall, occurs at the level of addition of *heights*, i.e., *log-volumes*!] in the discussion of Remark 1.10.5, (ii).

Remark 2.2.3. The discussion of Remark 2.2.1 centers around the content of Corollary 2.2, (ii), in the case of *elliptic curves defined over* $\mathbb{Q}$. On the other hand, if, in the context of Corollary 2.2, (ii), (iii), one considers the case where d is an *arbitrary positive integer* [i.e., which is *not necessarily bounded*, as in the situation of Corollary 2.3 below!], then the inequalities obtained in (C2) of Corollary 2.2, (ii), may be regarded, by applying Corollary 2.2, (iii), as a sort of **"weak version"** of the so-called **"uniform ABC Conjecture"**. That is to say, these inequalities constitute only a "weak version" in the sense that they are restricted to rational points that lie in the **compactly bounded subset** $\mathcal{K}_V$, and, moreover, the bounds given for the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" [i.e., in essence, the *"height"*] on $\mathfrak{Exc}_d$ and $\mathfrak{Exc}_{\epsilon,d}$ **depend** on the positive integer d [cf. also Remark 2.3.2, (i), below].

Remark 2.2.4. Before proceeding, it is perhaps of interest to consider the ideas discussed in Remarks 2.2.1, 2.2.3 above in the context of the analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich] [cf. also [InpTch]].

(i) The analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich] [cf. also [InpTch]] is discussed in detail in [IUTchIII], Remark 1.4.1, (iii); [IUTchIII], Remark 3.12.4. In a word, this discussion concerns similarities between the *log-theta-lattice* considered in the present series of papers and the *canonical Frobenius lifting* on the ordinary locus of a *canonical curve* of the sort that appears in the theory of [pOrd]. Such a canonical curve is associated, in the theory of [pOrd], to a hyperbolic curve equipped with a nilpotent ordinary indigenous bundle over a **perfect field** of positive characteristic p. On the other hand, the theory of [pOrd] also addresses the *universal case*, i.e., of the tautological hyperbolic curve equipped with a nilpotent ordinary indigenous bundle over the *moduli stack* of such data in positive characteristic. In particular, one constructs, in the theory of [pOrd], a *canonical Frobenius lifting* over a canonical p-adic lifting of this moduli stack. This moduli stack is smooth of dimension $3g - 3 + r$ [i.e., in the case of hyperbolic curves of type (g, r)] over $\mathbb{F}_p$, hence, in particular, is **far from perfect** [i.e., as an algebraic stack in positive characteristic]. Thus, in some sense,

> the *gap* between the theory of the present series of papers, on the one hand, and the notion discussed in Remark 2.2.1, (v), of a **"canonical number/arithmetic measure/distribution"**, on the other, may be understood, in the context of the analogy with p-adic Teichmüller theory, as corresponding to the *gap* between the theory of [pOrd] specialized to the case of *"canonical curves"*, i.e., over **perfect** *base fields*, and the full, non-specialized version of the theory of [pOrd], i.e., which concerns *canonical Frobenius liftings* over the **non-perfect** *moduli stack* of hyperbolic curves equipped with a nilpotent ordinary indigenous bundle.

That is to say, in a word, one has a *correspondence*

$$\textbf{"canonical number"} \quad \longleftrightarrow \quad \textbf{modular Frobenius liftings}.$$

(ii) In general, the *gap* between *perfect* and *non-perfect* schemes in positive characteristic is reflected precisely in the extent to which the *Frobenius morphism* on the scheme under consideration *fails to be an isomorphism*. Put another way, the "phenomenon" of non-perfect schemes in positive characteristic may be thought of as a reflection of the *distortion* arising from the *Frobenius morphism* in positive characteristic. In the context of the theory of the present series of papers [cf. [IUTchIII], Remark 1.4.1, (iii)], the Frobenius morphism in positive characteristic corresponds to the **$\mathfrak{log}$-link**. Moreover, in the context of the inequalities obtained in Theorem 1.10, the term "$* \cdot l$" [cf. the discussion of Remark 1.10.5, (ii)] arises, in the computations that underlie the proof of Theorem 1.10, precisely by applying the *prime number theorem* [i.e., Proposition 1.6] to sum up the *log-volumes* of the *log-shells* [cf. Propositions 1.2, (ii); 1.4, (iii)] at various nonarchimedean primes of the number field. In this context, we make the following *observations*:

· These log-volumes of log-shells may be thought of as numerical measures of the **distortions** *of the integral structure* [i.e., relative to the "arithmetic holomorphic" integral structures determined by the various local rings of integers "$\mathcal{O}$"] that arise from the **$\mathfrak{log}$-link**.

· Estimates arising from the *prime number theorem* are closely related to the aspects of the **Riemann zeta function** that are discussed in Remark 2.2.1.

· The *prime number* l is, ultimately, in the computations of Corollary 2.2, (ii) [cf., especially, condition "(C1)"], taken to be roughly of the order of the square root of the *height* of the elliptic curve under consideration. That is to say, since the height of an elliptic curve *"roughly controls"* [i.e., up to finitely many possibilities] the *moduli* of the elliptic curve, the prime number l may be thought of as a sort of **rough numerical representation** of the **moduli** of the elliptic curve under consideration.

Thus, in summary, these observations strongly support the point of view that

> the *computations that underlie the proof of Theorem 1.10* may be thought of as constituting one *convincing piece of evidence* for the point of view discussed in (i) above.

(iii) In the context of the discussion of (i), (ii), it is of interest to recall that the *modular Frobenius liftings* of [pOrd] are *not defined* over the *algebraic* moduli stack of hyperbolic curves over $\mathbb{Z}_p$, but *rather* over the *p-adic formal algebraic stack* [which is *formally étale* over the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$] constituted by the canonical lifting to $\mathbb{Z}_p$ of the moduli stack of hyperbolic curves equipped with a nilpotent ordinary indigenous bundle. That is to say,

> the *gap* between this [*"p-adically analytic"*] p-adic formal algebraic stack parametrizing **"ordinary"** data and the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$ is highly reminiscent, in the context of Corollary 2.2, (ii) [cf. also Remark 2.2.3], of the *gap* between the [*"arithmetically analytic"*] **compactly bounded subsets** "$\mathcal{K}_V$" [i.e., consisting of elliptic curves that satisfy the condition of being in *"sufficiently general position"* — a condition that may be thought of as a sort of *"global arithmetic version of ordinariness"*] and the entire set of algebraic points "$\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$".

Ultimately, this gap between "$\mathcal{K}_V$" and "$\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$" will be bridged, in Corollary 2.3 below, by applying [GenEll], Theorem 2.1, which may be thought of as a sort of **arithmetic analytic continuation** by means of [noncritical] **Belyi maps** [cf. the discussion of Belyi maps in the Introduction to [GenEll]]. This state of affairs is reminiscent of the *"arithmetic analytic continuation via Belyi maps"* that occurs in the theory of [AbsTopIII] [i.e., in essence, the theory of *Belyi cuspidalizations*] that is applied in [IUTchI], §5 [cf. [IUTchI], Remark 5.1.4]. Finally, in this context, we recall that the *open immersion* "$\widehat{\kappa}$" that appears in the discussion towards the end of [InpTch], §2.6 — i.e., which embeds a sort of *perfection* of the p-adic formal algebraic stack discussed above into an essentially *algebraic* stack given by a certain pro-finite covering of the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$ determined by considering *representations of the geometric fundamental group into* $PGL_2(\mathbb{Z}_p)$ — may be thought of as a sort of *p-adic analytic continuation* to this corresponding algebraic moduli stack of the essentially "p-adically analytic" theory of modular Frobenius liftings developed in [pOrd].

(iv) Finally, in the context of the discussion of (i), (ii), (iii), we observe that the issue discussed in Remark 2.2.1 of considering the *asymptotic behavior* of the theory of the present series of papers when $l \to \infty$ may be thought of as the problem of understanding how the theory of the present series of papers behaves

> as one passes from the **discrete approximation** of the elliptic curve under consideration constituted by the *l-torsion points* of the elliptic curve to the **"full continuous theory"**

[cf. the discussion of [IUTchI], Remark 6.12.3, (i) ; [HASurI], §1.3.4]. This point of view is of interest in light of the theory of **Bernoulli numbers**, i.e., which, on the one hand, is, as is well-known, closely related to the **values** [at positive even integers] of the **Riemann zeta function** [cf. the discussion of Remark 2.2.1], and, on the other hand, is closely related to the passage from the

$$\textbf{discrete difference operator } f(x) \mapsto f(x+1) - f(x)$$

— for, say, real-valued real analytic functions $f(-)$ on the real line — to the

$$\textbf{continuous derivative operator } f(x) \mapsto \tfrac{d}{dx} f(x)$$

— where we recall that the operator $f(x) \mapsto f(x+1)$ may be thought of as the operator "$e^{\frac{d}{dx}}$" obtained by *exponentiating* this continuous derivative operator.

We are now ready to state and prove the *main theorem* of the present §2, which may also be regarded as the *main application* of the theory developed in the present series of papers.

Corollary 2.3. **(Diophantine Inequalities)** *Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\mathrm{def}}{=} X \backslash D$; d a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write ω_X for the canonical sheaf on X. Suppose that U_X is a* **hyperbolic curve**, *i.e., that the degree of the line bundle $\omega_X(D)$ is* **positive**. *Then, relative to the notation reviewed above, one has an* **inequality of "bounded discrepancy classes"**

$$\mathrm{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \mathrm{ht}_{\omega_X(D)}$ is bounded below by a **constant** *on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below].*

Proof. One verifies immediately that the content of the statement of Corollary 2.3 coincides precisely with the content of [GenEll], Theorem 2.1, (i). Thus, it follows from the *equivalence* of [GenEll], Theorem 2.1, that, in order to complete the proof of Corollary 2.3, it suffices to verify that [GenEll], Theorem 2.1, (ii), holds. That is to say, we may assume without loss of generality that:

· $X = \mathbb{P}^1_{\mathbb{Q}}$ is the *projective line* over $\mathbb{Q}$;
· $D \subseteq X$ is the divisor consisting of the *three points* "0", "1", and "∞";
· $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ is a *compactly bounded subset* [cf. Remark 2.3.1, (vi), below] whose *support* contains the nonarchimedean prime "2";
· $\mathcal{K}_V$ satisfies the condition "$(*_{j\text{-inv}})$" of Corollary 2.2.

[Here, we note, with regard to the condition "$(*_{j\text{-inv}})$" of Corollary 2.2, that this condition only concerns the behavior of $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ *as d varies*; that is to say, this condition is *entirely vacuous* in situations, i.e., such as the situation considered in [GenEll], Theorem 2.1, (ii), in which one is only concerned with $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ *for a fixed d*.] Then it suffices to show that the *inequality of BD-classes of functions* [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]

$$\mathrm{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

holds on $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$. But such an inequality follows immediately, in light of the [relevant] *equality of BD-classes* of Corollary 2.2, (i), from Corollary 2.2, (ii) [cf. condition (C2)], (iii) [where we note that it follows immediately from the various definitions involved that $d_{\mathrm{mod}} \leq d$]. This completes the proof of Corollary 2.3. ◯

Remark 2.3.1. We take this opportunity to correct some *unfortunate misprints* in [GenEll].

(i) The notation "$\mathrm{ord}_v(-) : F_v \to \mathbb{Z}$" in the final sentence of the first paragraph following [GenEll], Definition 1.1, should read "$\mathrm{ord}_v(-) : F_v^\times \to \mathbb{Z}$".

(ii) In [GenEll], Definition 1.2, (ii), the *non-resp'd* and *first resp'd* items in the display should be *reversed*! That is to say, the notation "$\alpha \lesssim_{\mathcal{F}} \beta$" corresponds to "$\alpha(x) - \beta(x) \leq C$"; the notation "$\alpha \gtrsim_{\mathcal{F}} \beta$" corresponds to "$\beta(x) - \alpha(x) \leq C$".

(iii) The first portion of the first sentence of the statement of [GenEll], Corollary 4.4, should read: "Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$; ...".

(iv) The "$\text{log-diff}_{\overline{\mathcal{M}}_{\mathrm{ell}}}([E_L]))$" in the second inequality of the final display of the statement of [GenEll], Corollary 4.4, should read "$\text{log-diff}_{\overline{\mathcal{M}}_{\mathrm{ell}}}([E_L])$".

(v) The *equality*

$$\mathrm{ht}_E \approx (\deg(E)/\deg(\omega_X)) \cdot \mathrm{ht}_{\omega_X}$$

implicit in the final "$\approx$" of the final display of the proof of [GenEll], Theorem 2.1, should be replaced by an *inequality*

$$\mathrm{ht}_E \lesssim 2 \cdot (\deg(E)/\deg(\omega_X)) \cdot \mathrm{ht}_{\omega_X}$$

[which follows immediately from [GenEll], Proposition 1.4, (ii)], and the expression "$\deg(E)/\deg(\omega_X)$" in the inequality imposed on the *choice* of ϵ' should be replaced by the expression "$2 \cdot (\deg(E)/\deg(\omega_X))$".

(vi) Suppose that we are in the situation of [GenEll], Example 1.3, (ii). Let $U \subseteq X$ be an open subscheme. Then a *"compactly bounded subset"*

$$\mathcal{K}_V \ \subseteq \ U(\overline{\mathbb{Q}}) \ \ (\subseteq X(\overline{\mathbb{Q}}))$$

of $U(\overline{\mathbb{Q}})$ is to be understood as a subset which forms a compactly bounded subset of $X(\overline{\mathbb{Q}})$ [i.e., in the sense discussed in [GenEll], Example 1.3, (ii)] and, moreover, satisfies the property that for each $v \in V^{\mathrm{arc}} \stackrel{\mathrm{def}}{=} V \bigcap \mathbb{V}(\mathbb{Q})^{\mathrm{arc}}$ (respectively, $v \in V^{\mathrm{non}} \stackrel{\mathrm{def}}{=} V \bigcap \mathbb{V}(\mathbb{Q})^{\mathrm{non}}$), the compact domain $\mathcal{K}_v \subseteq X^{\mathrm{arc}}$ (respectively, $\mathcal{K}_v \subseteq X(\overline{\mathbb{Q}}_v)$) is, in fact, *contained in*

$$U(\mathbb{C}) \subseteq X(\mathbb{C}) = X^{\mathrm{arc}} \text{ (respectively, } U(\overline{\mathbb{Q}}_v) \subseteq X(\overline{\mathbb{Q}}_v)).$$

In particular, this convention should be applied to the use of the term "compactly bounded subset" in the statements of [GenEll], Theorem 2.1; [GenEll], Lemma 3.7; [GenEll], Theorem 3.8; [GenEll], Corollary 4.4, as well as in the present paper [cf. the statement of Corollary 2.2; the proof of Corollary 2.3]. Although this convention was not discussed explicitly in [GenEll], Example 1.3, (ii), it is, in effect, discussed *explicitly* in the discussion of "compactly bounded subsets" at the beginning of the Introduction to [GenEll]. Moreover, this convention is *implicit* in the arguments involving compactly bounded subsets in the proof of [GenEll], Theorem 2.1.

(vii) In the discussion following the second display of [GenEll], Example 1.3, (ii), the phrase "(respectively, $X(\mathbb{Q}_v)$)" should read "(respectively, $X(\overline{\mathbb{Q}}_v)$)".

(viii) The first display of the paragraph immediately following [GenEll], Remark 3.3.1, should read as follows:

$$|\alpha|^2 \ \stackrel{\mathrm{def}}{=} \ \left| \int_{E_v} \alpha \wedge \overline{\alpha} \ \right|$$

[i.e., the integral should be replaced by the absolute value of the integral].

Remark 2.3.2.

(i) The reader will note that, by arguing with a "bit more care", it is not difficult to give **stronger** versions of the various **estimates** that occur in Theorem 1.10; Corollaries 2.2, 2.3 and their proofs. Such stronger estimates are, however, beyond the scope of the present series of papers, so we shall not pursue this topic further in the present paper.

(ii) On the other hand, we observe that the constant "1" in the inequality of the display of Corollary 2.3 **cannot be improved** — cf. the examples constructed in [Mss]; the discussion of Remark 1.10.5, (ii), (iii). This observation is closely related to discussions of how the theory of the present series of papers *breaks down* if one attempts to replace the **first power** of the **étale theta function** by its ***N*-th power** for some integer $N \geq 2$ [cf. the discussion in the final portion of Step (xi) of the proof of [IUTchIII], Corollary 3.12; the discussion of [IUTchIII], Remark 3.12.1, (ii)]. Such an "N-th power operation" may also be thought of as corresponding to the operation of replacing each Tate curve that occurs at an element $\in \underline{\mathbb{V}}^{\mathrm{bad}}$ by the Tate curve whose q-parameter is given by the *N-th power* of the q-parameter of the original Tate curve. This sort of operation on Tate curves may, in turn, be thought of as an **isogeny** of the sort that occurs in [GenEll], Lemma 3.5. On the other hand, the content of the proof of [GenEll], Lemma 3.5, consists essentially of a computation to the effect that even if one attempts to consider such "N-th power isogenies" at *certain* elements $\in \underline{\mathbb{V}}^{\mathrm{bad}}$, the *global height* of the elliptic curve over a number field that arises from such an isogeny will typically remain, up to a relatively small discrepancy, *unchanged*. In this context, we recall that this sort of **invariance**, up to a relatively small discrepancy, of the **global height** under **isogeny** is one of the essential observations that underlies the theory of [Falt] — a state of affairs that is also of interest in light of the observations of Remark 2.3.3 below.

Remark 2.3.3. Corollary 2.3 may be thought of as an **effective** version of the **Mordell Conjecture**. From this point of view, it is perhaps of interest to compare the **"essential ingredients"** that are applied in the proof of Corollary 2.3 [i.e., in effect, that are applied in the present series of papers!] with the "essential ingredients" applied in [Falt]. The following discussion benefited substantially from numerous e-mail and skype exchanges with *Ivan Fesenko* during the summer of 2015.

(i) Although the author does not wish to make any pretensions to completeness in any rigorous sense, perhaps a rough, informal list of "essential ingredients" in the case of [Falt] may be given as follows:

(a) results in elementary algebraic number theory related to the *"geometry of numbers"*, such as the theory of *heights* and the *Hermite-Minkowski theorem*;
(b) the *global class field theory* of *number fields*;
(c) the *p-adic* theory of *Hodge-Tate decompositions*;
(d) the *p-adic* theory of *finite flat group schemes*;
(e) generalities in algebraic geometry concerning *isogenies* and *Tate modules* of *abelian varieties*;
(f) generalities in algebraic geometry concerning *polarizations* of *abelian varieties*;
(g) the *logarithmic geometry* of toroidal compactifications of the moduli stack of abelian varieties.

With regard to the global class field theory of (b), we observe that there are *numerous different approaches* to *"dissecting"* the proofs of the main results of global class field theory into more primitive components. To some extent, these different approaches correspond to different points of view arising from subsequent research on topics related to global class field theory. Here, we wish to consider the approach taken in [Lang1], Chapters VIII, IX, X, XI, which is attibuted [cf. the Introduction to [Lang1], Part Two] to Weber. It is of interest, in the context of the discussion of (vii) below, that this is apparently the *oldest approach* to proving certain portions of global class field theory. It is also of interest that this approach motivates the approach to global class field theory via consideration of *density of primes in arithmetic progressions* and *splitting laws*. This aspect of this approach of [Lang1] is closely related to various issues that appear in [Falt] [cf. [Lang1], Chapter VIII,

§5]. Moreover, as we shall see in the following discussion, this approach of [Lang1] to global class field theory is well-suited to discussions of *comparisons* between the theory of [Falt] and the *inter-universal Teichmüller theory* developed in the present series of papers. At a technical level, the dissection of the global class field theory of (b), as developed in [Lang1], into more primitive components may be summarized as follows:

(b-1) the *local class field theory* of *p-adic local fields* [cf. [Lang1], Chapter IX, §3; [Lang1], Chapter XI, §4];
(b-2) the theory of *global density of primes* [cf. the discussion surrounding the *Universal Norm Index Inequality* in [Lang1], Chapter VIII, §3];
(b-3) results in elementary algebraic number theory related to the *"geometry of numbers"* that give rise to the *Unit Theorem* [cf. [Lang1], Chapter V, §1; [Lang1], Chapter IX, §4];
(b-4) the *global reciprocity law*, i.e., in effect, the *existence of a conductor for the Artin symbol* [cf. [Lang1], Chapter X, §2];
(b-5) *Kummer theory* [cf. [Lang1], Chapter XI, §1].

Here, we recall that (b-1), (b-2), and (b-3) are applied in [Lang1], Chapter IX, §5, to verify the *Universal Norm Index Equality* for cyclic extensions. This Universal Norm Index Equality is then applied in [Lang1], Chapter X, §1, and combined with the theory of *cyclotomic extensions* in [Lang1], Chapter X, §2, to verify (b-4). Finally, (b-4) is combined with (b-5) in [Lang1], Chapter XI, §2, to complete the proof of the *Existence Theorem* for class fields.

(ii) From the point of view of the theory of the present series of papers, (a), together with (b-3), is reminiscent of the elementary algebraic number theory characterization of **nonzero global integers** as **roots of unity**, which plays an important role in the theory of the present series of papers [cf. [IUTchIII], the proof of Proposition 3.10]. Moreover, (a) is also reminiscent of the **arithmetic degrees** of line bundles that appear, for instance, in the form of **global realified Frobenioids**, throughout the theory of the present series of papers. Next, we observe that (b-1) is reminiscent of the ***p*-adic absolute anabelian geometry** of [AbsTopIII] [cf., e.g., [AbsTopIII], Corollary 1.10, (i)]. On the other hand, (b-2) is reminiscent of repeated applications of the **Prime Number Theorem** in the present paper [cf. Propositions 1.6; 2.1, (ii)]; this comparison between (b-2) and the Prime Number Theorem will be discussed in more detail in (iv) below. Next, we observe [cf. the discussion of the latter portion of [IUTchIII], Remark 3.12.1, (iii)] that (b-4) is reminiscent of the application of the elementary fact "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$" in the multiradial algorithms for **cyclotomic rigidity isomorphisms** in the **number field** case [cf. [IUTchI], Example 5.1, (v), as well as the discussion of [IUTchIII], Remarks 2.3.2, 2.3.3], that is to say, not only in the sense that

> both are closely related to the various **cyclotomes** that appear in global class field theory theory or inter-universal Teichmüller theory,

but also in the sense that

> both may be regarded as analogues of the *usual* **product formula** [i.e., which appears at the level of *Frobenius-like* monoids isomorphic to the multiplicative group of nonzero elements of a number field] at the level of

certain [*étale-like*!] **profinite Galois groups** related to global number fields.

On the other hand, (b-5) is reminiscent of the central role played through inter-universal Teichmüller theory by constructions modeled on classical **Kummer theory**. In fact, these comparisons involving (b-4) and (b-5) are closely related to one another and will be discussed in more detail in (v), (vi), and (vii) below. Next, we recall that **Hodge-Tate decompositions** as in (c) play a central role in the proofs of the main results of [pGC], which, in turn, underlie the theory of [AbsTopIII]. The ramification computations concerning finite flat group schemes as in (d) are reminiscent of various p-adic ramification computations concerning **log-shells** in [AbsTopIII], as well as in Propositions 1.1, 1.2, 1.3, 1.4 of the present paper. Whereas [Falt] revolves around the *abelian/linear* theory of abelian varieties [cf. (e)], the theory of the present series of papers depends, in an essential way, on various intricate manipulations involving *finite étale coverings* of *hyperbolic curves*, such as the use of **Belyi maps** in [GenEll], as well as in the **Belyi cuspidalizations** applied in [AbsTopIII]. The theory of polarizations of abelian varieties applied in [Falt] [cf. (f)] is reminiscent of the essential role played by *commutators* of **theta groups** in the theory of [EtTh], which, in turn, plays a central role in the theory of the present series of papers. Finally, the *logarithmic geometry* of (g) is reminiscent of the **combinatorial anabelian geometry** of [SemiAnbd], which is applied, in [IUTchI], §2, to the logarithmic geometry of coverings of stable curves.

(iii) One way to summarize the discussion of (ii) is as follows:

> *many aspects of the theory of [Falt] may be regarded as* **"distant abelian ancestors"** *of certain aspects of the* **"anabelian-based theory"** *of inter-universal Teichmüller theory.*

Alternatively, one may observe that the overwhelmingly *scheme-theoretic* nature of the theory applied in [Falt] lies in stark contrast to the highly *non-scheme-theoretic* nature of the *absolute anabelian geometry* and theory of *monoids/Frobenioids* applied in the present series of papers: that is to say,

> *many aspects of the theory of [Falt] may be regarded as* **"distant arithmetically holomorphic ancestors"** *of certain aspects of the* **multiradial** *and* **mono-analytic** *[i.e.,* **"arithmetically real analytic"***] theory developed in inter-universal Teichmüller theory.*

One way to understand this **fundamental difference** between the theory of [Falt] and inter-universal Teichmüller theory is by considering the naive goal of constructing some sort of **"Frobenius morphism"** on a **number field** [cf. the discussion of [FrdI], §I3], i.e., which has the effect of **multiplying arithmetic degrees** by a positive factor > 1: whereas the theory of [Falt] [cf., e.g., the argument of the proof of [GenEll], Lemma 3.5, as discussed in Remark 2.3.2, (ii)] may regarded as a reflection of the point of view that,

> *so long as one respects the* **arithmetic holomorphic structure** *of scheme theory, such a "Frobenius morphism" on a number field* **cannot exist**,

the essential content of inter-universal Teichmüller theory may be summarized in a word as the assertion that,

> *if one dismantles this arithmetic holomorphic structure in a suitably canonical fashion and allows oneself to work with* **multiradial/mono-analytic** *[i.e.,* **"arithmetically quasi-conformal"***] structures, then one* **can indeed construct***, in a very* **canonical** *fashion, such a "Frobenius morphism" on a number field.*

(iv) In the context of the comparison discussed in (ii) concerning (b-2), it is of interest to note that the *fundamental difference* discussed in (iii) between the theory of [Falt] and inter-universal Teichmüller theory is, in some sense, reflected in the *difference* between the theory of *global density of primes* [i.e., (b-2)] and the *Prime Number Theorem*. That is to say, the **coherence** of the sorts of collections of primes that appear in the theory of global density of primes may be thought of as a sort of representation, in the context of *analytic number theory*, of the **arithmetic holomorphic structure** of conventional scheme theory. By contrast, in the context of the Prime Number Theorem, primes of a number field appear, so to speak, *one by one*, i.e., in a fashion that is only possible if one **deactivates**, in the context of analytic number theory, the *coherence* that underlies the **aggregrations of primes** that appear in the theory of global density of primes. That is to say, this approach to treating primes "one by one" may be thought of as corresponding to the *dismantling of arithmetic holomorphic structures* that occurs in the context of the **multiradial/mono-analytic** structures that appear in inter-universal Teichmüller theory. Here, it is also of interest to note that the way in which one "deactivates aggregations of primes" in the context of the Prime Number Theorem may be thought of [cf. the discussion of [IUTchIII], Remark 3.12.2, (i), (c)] as a sort of **dismantling** of the ring structure of a number field into its underlying **additive** [i.e., counting primes *"one by one"*!] and **multiplicative** structures [i.e., the very notion of a *prime*!].

(v) The *fundamental difference* discussed in (iii) between the theory of [Falt] and inter-universal Teichmüller theory may also be seen in the context of the comparison discussed in (ii) concerning (b-4). Indeed, the **global reciprocity law** of (b-4), which plays a *central role* in global class field theory, depends, in an essential way, on *nontrivial relationships* between **local units** [such as the unit determined by a prime number l at a nonarchimedean prime of a number field of residue characteristic $\neq l$] at one prime of a number field and elements of **local value groups** [such as the element determined by l at a nonarchimedean prime of a number field of residue characteristic l] at another prime of the number field. Such nontrivial relationships are

> **fundamentally incompatible** with the **splittings/decouplings** of *local units* and *local value groups* that play a *central role* in the **dismantling** of **arithmetic holomorphic structures** that occurs in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3, (i); [IUTchIII], Remark 3.12.2, (i), (a)].

This incompatibility [i.e., with *nontrivial relationships* between *local units* and *local value groups* at nonarchimedean primes with distinct residue characteristics] may also be seen quite *explicitly* in the structure of the various types of *prime-strips* that appear in inter-universal Teichmüller theory [cf. [IUTchI], Fig. I1.2]. That is to say, such nontrivial relationships, which form the content of the global reciprocity law of global class field theory, may be thought of as a sort of **global Galois-theoretic representation** of the **constraints** that constitute the **arithmetic holomorphic structure** of conventional scheme theory.

(vi) Another fundamental aspect of the comparison discussed in (ii) concerning (b-4) may be seen in the fact that whereas the global reciprocity law of global class field theory concerns the **global reciprocity map**, the cyclotomic rigidity algorithms of inter-universal Teichmüller theory to which (b-4) was compared appear in the context of **Kummer-theoretic isomorphisms**. That is to say, although both the global reciprocity map and Kummer-theoretic isomorphisms involve correspondences between multiplicative monoids associated to number fields and multiplicative monoids that arise from global Galois groups, one *fundamental difference* between these two types of correspondence lies in the fact that whereas

> **Kummer-theoretic isomorphisms** satisfy very **strong covariant** [with respect to *functions*] **functoriality** properties,

the *reciprocity maps* that appear in various versions of class field theory *tend not to satisfy such strong functoriality properties*. This presence or absence of strong functoriality properties is, to a substantial extent, a reflection of the fact that whereas *Kummer theory* may be performed in a *very straightforward, tautological, "general nonsense"* fashion in a *wide variety of situations*,

> **class field theory** may only be conducted in **very special arithmetic situations**.

This presence of strong functoriality properties [i.e., in the case of Kummer theory] is the essential reason for the *central role* played by *Kummer theory* [cf. (b-5)] in *inter-universal Teichmüller theory*, as well as in many situations that arise in *anabelian geometry* in general [cf., e.g., the theory of [Cusp]]. Indeed, the very **tautological/ubiquitous/strongly functorial nature of Kummer theory** makes it well-suited to the sort of **dismantling** of ring structures that occurs in inter-universal Teichmüller theory, as well as to the various **evaluation** operations of **functions** at **special points** that play a *central role*, in the context of **Galois evaluation**, in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3]. By contrast, although there exist various higher-dimensional versions of class field theory involving higher algebraic K-groups, these versions of class field theory are *fundamentally incompatible* with the crucial *evaluation of function* operations of the sort that occur in inter-universal Teichmüller theory. Indeed, more generally, except for very *exceptional classical cases* involving *exponential functions* in the case of $\mathbb{Q}$ or *modular and elliptic functions* in the case of imaginary quadratic fields,

> **class field theory** tends to be very **ill-suited** to situations that involve the **evaluation of special functions at special points**.

Moreover, even if one restricts one's attention, for instance, to functoriality with respect to passing to a finite extension field, the functoriality of the reciprocity maps that occur in class field theory are [unlike *Kummer-theoretic isomorphisms*!] **contravariant** [with respect to functions] and can only be made *covariant* if one applies some sort of nontrivial **duality** result to reverse the direction of the maps —

a state of affairs that makes class field theory very difficult to apply not only in inter-universal Teichmüller theory, but also in many situations that arise in anabelian geometry. On the other hand, in the context of inter-universal Teichmüller theory, the *price*, so to speak, that one pays for the very *convenient, "general nonsense"* nature of Kummer theory lies in

> the **highly nontrivial nature** — which may be seen, for instance, in the establishment of various **multiradiality** properties — of the **cyclotomic rigidity algorithms** that appear in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3].

Here, we recall that such cyclotomic rigidity algorithms — which *never appear* in discussions of conventional arithmetic geometry in which the *arithmetic holomorphic structure* is *held fixed* — play a *central role* in inter-universal Teichmüller theory precisely because of the *indeterminacies* that arise as a consequence of the *dismantling* of the arithmetic holomorphic structure. Finally, in this context, it is of interest to recall that, although *local class field theory* is, in a certain limited sense, applied in inter-universal Teichmüller theory, i.e., in order to obtain *cyclotomic rigidity algorithms* for *MLF-Galois pairs* [cf. [IUTchII], Proposition 1.3, (ii)], it is only *"of limited use"* in the sense that the resulting cyclotomic rigidity algorithms are **uniradial** [i.e., fail to be *multiradial* — cf. [IUTchIII], Figs. 2.1, 3.7, and the surrounding discussions].

(vii) The *fundamental incompatibility* — i.e., except in very *exceptional classical cases* involving *exponential functions* in the case of $\mathbb{Q}$ or *modular and elliptic functions* in the case of imaginary quadratic fields — discussed in (vi) of *class field theory* with situations that involve the *evaluation of special functions at special points* is highly reminiscent of the *original point of view* of class field theory in the early twentieth century [cf. **Kronecker's Jugendtraum**, **Hilbert's twelfth problem**], i.e., to the effect that further development of class field theory should proceed precisely by *extending* the theory involving *evaluation of special functions at special points* that exists in these "exceptional classical cases" to the case of *arbitrary number fields.* This state of affairs is, in turn, highly reminiscent of the fact that the approach taken in the above discussion to "dissecting global class field theory" is the **oldest/original approach** to global class field theory, as well as of the fact that this original approach is the most well-suited to discussions of *comparisons* between the theory of [Falt] and inter-universal Teichmüller theory. This state of affairs is also highly reminiscent of the discussion in [Pano], §3, §4, of the *numerous analogies* between inter-universal Teichmüller theory and the classical [i.e., dating back to the nineteenth century!] theory surrounding **Jacobi's identity for the theta function** on the upper half-plane and **Gaussian distributions/integrals**. Finally, this collection of observations, taken as a whole, may be summarized as follows:

> *Many of the ideas that appear in* **inter-universal Teichmüller theory** *bear a* **much closer resemblance** *to the mathematics of the* **late nineteenth** *and* **early twentieth centuries** *— i.e., to the mathematics of* **Gauss***,* **Jacobi***,* **Kummer***,* **Kronecker***,* **Weber***,* **Frobenius***,* **Hilbert***, and* **Teichmüller** *— than to the mathematics of the mid- to late twentieth century. This close resemblance suggests strongly that, relative to*

> *the mathematics of the late nineteenth and early twentieth centuries, the course of development of a substantial portion of the mathematics of the mid- to late twentieth century should* **not** *be regarded as* **"unique"** *or* **"inevitable"***, but rather as being merely* **one possible choice** *among many viable and fruitful alternatives that existed* **a priori***.*

Here, we note that although the use, in inter-universal Teichmüller theory, of **Belyi** maps, as well as of the ***p*-adic anabelian geometry** of the 1990's [i.e., [pGC]], may at first glance look like an incidence of "exceptions" to the "rule" constituted by this point of view, these "exceptions" may be thought of as "proving the rule" in the sense that they are *far from typical* of the mathematics of the late twentieth century.

Remark 2.3.4. Various aspects of the theory of the present series of papers are *substantially reminiscent* of the theory surrounding **Bogomolov's proof** of the **geometric** version of the **Szpiro Conjecture**, as discussed in [ABKP], [Zh]. Put another way, these aspects of the theory of the present series of papers may be thought of as **arithmetic analogues** of the geometric theory surrounding Bogomolov's proof. Alternatively, Bogomolov's proof may be thought of as a sort of **useful elementary guide**, or **blueprint** [perhaps even a sort of **Rosetta stone**!], for understanding substantial portions of the theory of the present series of papers. The author would like to express his gratitude to *Ivan Fesenko* for bringing to his attention, via numerous discussions in person, e-mails, and skype conversations between December 2014 and January 2015, the possibility of the existence of such fascinating connections between Bogomolov's proof and the theory of the present series of papers. We discuss these analogies in more detail in [BogIUT].

Remark 2.3.5. In [Par], a proof is given of the **Mordell Conjecture** *for function fields over the* **complex numbers**. Like the proof of Bogomolov discussed in Remark 2.3.4, Parshin's proof involves **metric estimates** of **"displacements"** that arise from actions of elements of the [usual topological] **fundamental group** of the complex hyperbolic curve that serves as the *base scheme of the given family of curves*. In particular, we observe that one may pose the following *question*:

> Is it possible to apply *some portion of the ideas of the* **inter-universal Teichmüller theory** developed in the present series of papers to obtain a *proof of the* **Mordell Conjecture** *over number fields* **without** making use of **Belyi maps** as in the proof of Corollary 2.3 [i.e., the proof of [GenEll], Theorem 2.1]?

This question was posed to the author by *Felipe Voloch* in an e-mail in September 2015. The answer to this question is, as far as the author can see at the time of writing, *"no"*. On the other hand, this question is interesting in the context of the discussion of Remarks 2.3.3 and 2.3.4 in that it serves to highlight various interesting aspects of inter-universal Teichmüller theory, as we explain in the following discussion.

(i) First, we recall [cf., e.g., [Lang2], Chapter I, §1, §2, for more details] that the starting point of the theory of the *Kobayashi distance* on a [Kobayashi] hyperbolic complex manifold is the well-known *Schwarz lemma* of elementary complex analysis and its consequences for the geometry of holomorphic maps from the open unit disc D in the complex plane to an arbitrary complex manifold. In the following discussion, we shall refer to this geometry as the *Schwarz-theoretic geometry* of D. Perhaps the most *fundamental difference* between the proofs of Parshin and Bogomolov lies in the fact that

(PB1) Whereas Parshin's proof revolves around *estimates of displacements* arising from actions of elements of the fundamental group on a certain *two-dimensional complete [Kobayashi] hyperbolic complex manifold* by means of the **holomorphic** geometry of the **Kobayashi distance**, i.e., in effect, the **Schwarz-theoretic geometry** of D, Bogomolov's proof [cf. the review of Bogomolov's proof given in [BogIUT]] revolves around *estimates of displacements* arising from actions of elements of the fundamental group on a *one-dimensional real analytic manifold* [i.e., a universal covering of a copy of the unit circle $\mathbb{S}^1$] by means of the **real analytic symplectic geometry** of the upper half-plane.

Here, it is already interesting to note that this fundamental gap, in the case of results over *complex function fields*, between the *holomorphic* geometry applied in Parshin's proof of the *Mordell Conjecture* and the *real analytic symplectic* geometry applied in Bogomolov's proof of the *Szpiro Conjecture* is highly reminiscent of the fundamental gap discussed in Remark 2.3.3, (iii), in the case of results over *number fields*, between the **arithmetically holomorphic** nature of the proof of the *Mordell Conjecture* given in [Falt] and the **"arithmetically quasi-conformal"** nature of the proof of the *Szpiro Conjecture* [cf. Corollary 2.3] via inter-universal Teichmüller theory given in the present series of papers. That is to say,

> Parshin's proof is best understood **not** as a *"weaker, or simplified, version of Bogomolov's proof obtained by extracting certain portions of Bogomolov's proof"*, but rather as a proof that reflects a *fundamentally qualitatively different geometry* — i.e., **holomorphic**, as opposed to **real analytic** — from Bogomolov's proof.

This point of view already suggests rather strongly, relative to the analogy between Bogomolov's proof and inter-universal Teichmüller theory [cf. [BogIUT]] that it is *unnatural/unrealistic* to expect to obtain a new proof of the Mordell Conjecture over number fields by applying *some portion of the ideas of the inter-universal Teichmüller theory.*

(ii) At a more technical level, the fundamental difference (PB1) discussed in (i) may be seen in the fact that

(PB2) whereas Parshin's proof involves **numerous holomorphic maps** from the open unit disc D into one- and two-dimensional complex manifolds [i.e., in essence, the universal coverings of the *base space* and *total space* of the family of curves under consideration], Bogomolov's proof revolves around the real analytic symplectic geometry of a **fixed copy** of the open unit disc D [or, equivalently, the upper half-plane], i.e., in Bogomolov's proof, *one never considers holomorphic maps from D to itself which are not biholomorphic.*

The essentially arbitrary nature of these numerous holomorphic maps that appear in Parshin's proof is reflected in the fact that

(PB3) Parshin's proof is well-suited to proving a **rough qualitative** [i.e., **"finiteness"**] result for families of curves of **arbitrary genus** ≥ 2, whereas Bogomolov's proof is well-suited to proving a much finer **explicit inequality**, but only in the case of families of **elliptic curves**.

Another technical aspect of the proofs of Parshin and Bogomolov that is closely related to both (PB2) and (PB3) is the fact that

(PB4) whereas the estimation apparatus of Bogomolov's proof depends in an essential way on special properties of particular types of elements — such as **unipotent** elements or **commutators** — of the fundamental group under consideration, the estimation apparatus of Parshin's proof is uniform for **arbitrary [“sufficiently small”] elements** of the fundamental group under consideration.

(iii) Although, as discussed in (ii), it is difficult to see how Parshin's proof could be *"embedded"* into [i.e., obtained as a *"suitable portion of"*] Bogomolov's proof, the **Schwarz-theoretic geometry** of D admits a *"natural embedding"* into [i.e., admits a natural analogy to a suitable portion of] inter-universal Teichmüller theory, namely, in the form of the theory of *categories of localizations* of the sort that appear in [GeoAnbd], §2; [AbsTopI], §4; [AbsTopII], §3. This theory of categories of localizations culminates in the theory of **Belyi cuspidalizations**, which is discussed in [AbsTopII], §3, and applied to obtained the **mono-anabelian reconstruction algorithms** of [AbsTopIII], §1. Moreover, the analogy between such categories of localizations and the classical *Schwarz-theoretic geometry* of D [or, equivalently, the upper half-plane] is discussed in the Introduction to [GeoAnbd], as well as in [IUTchI], Remark 5.1.4. This theory of categories of localizations may be summarized roughly as follows:

In the context of **absolute anabelian geometry** over number fields and their nonarchimedean localizations, **Belyi maps** play the role of the **Schwarz-theoretic geometry** of the open unit disc D, i.e., the role of realizing a sort of **arithmetic** version of **analytic continuation**.

This point of view is also interesting from the point of view of the discussion of Remark 2.2.4, (iii), i.e., to the effect that [noncritical] **Belyi maps** play the role of realizing a sort of **arithmetic** version of **analytic continuation** in the proof of [GenEll], Theorem 2.1. That is to say, from the point of view of the *question* posed at the beginning of the present Remark 2.3.5:

Even if, in the context of inter-universal Teichmüller theory, one attempts to search for an analogue of Parshin's proof in the form of a "suitable portion" of the inter-universal Teichmüller theory developed in [IUTchI], [IUTchII], [IUTchIII] [i.e., even if one *avoids* consideration of the application of [noncritical] Belyi maps in the proof of Corollary 2.3 via [GenEll], Theorem 2.1], one is ultimately led — i.e., from the point of view of considering arithmetic analogues of the classical complex theory of **analytic continuation** and the **Schwarz-theoretic geometry** of the open unit disc D — to the Belyi maps that appear in the **Belyi cuspidalizations** of [AbsTopII], §3; [AbsTopIII], §1.

Put another way, it appears that any *search* in the realm of inter-universal Teichmüller theory either for *some* proof of the Mordell Conjecture [over number fields] or for *some* analogue of Parshin's proof [of the Mordell Conjecture over complex function fields] appears to lead inevitably to *some* application of **Belyi maps** to realize *some* sort of arithmetic analogue of the classical complex theory of **analytic continuation** and the **Schwarz-theoretic geometry** of the open unit disc D.

Section 3: Inter-universal Formalism: the Language of Species

In the present §3, we develop — albeit from an extremely *naive/non-expert* point of view, relative to the theory of foundations! — the language of **species**. Roughly speaking, a "species" is a **"type of mathematical object"**, such as a "group", a "ring", a "scheme", etc. In some sense, this language may be thought of as an *explicit description* of certain tasks typically executed at an *implicit, intuitive level* by mathematicians [i.e., mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] via a sort of "mental arithmetic" in the course of interpreting various mathematical arguments. In the context of the theory developed in the present series of papers, however, it is useful to describe these intuitive operations explicitly.

In the following discussion, we shall work with various **models** — consisting of *"sets"* and a *relation* "$\in$" — of the standard *ZFC axioms* of axiomatic set theory [i.e., the nine axioms of *Zermelo-Fraenkel*, together with the *axiom of choice* — cf., e.g., [Drk], Chapter 1, §3]. We shall refer to such models as **ZFC-models**. Recall that a *(Grothendieck) universe* V is a set satisfying the following axioms [cf. [McLn], p. 194]:

(i) V is *transitive*, i.e., if $y \in x$, $x \in V$, then $y \in V$.
(ii) The set of *natural numbers* $\mathbb{N} \in V$.
(iii) If $x \in V$, then the *power set of* x also belongs to V.
(iv) If $x \in V$, then the *union of all members of* x also belongs to V.
(v) If $x \in V$, $y \subseteq V$, and $f : x \to y$ is a *surjection*, then $y \in V$.

We shall say that a set E is a V*-set* if $E \in V$.

The various ZFC-models that we work with may be thought of as [but are *not restricted* to be!] the ZFC-models determined by various *universes* that are sets relative to some *ambient ZFC-model* which, in addition to the standard axioms of ZFC set theory, satisfies the following **existence axiom** [attributed to the "Grothendieck school" — cf. the discussion of [McLn], p. 193]:

($\dagger^{\mathrm{G}}$) *Given any set x, there exists a universe V such that $x \in V$.*

We shall refer to a ZFC-model that also satisfies this additional axiom of the Grothendieck school as a *ZFCG-model.* This existence axiom ($\dagger^{\mathrm{G}}$) implies, in particular, that:

> *Given a set I and a collection of universes V_i, where $i \in I$, indexed by I [i.e., a 'function' $I \ni i \mapsto V_i$], there exists a [larger] universe V such that $V_i \in V$, for $i \in I$.*

Indeed, since the graph of the function $I \ni i \mapsto V_i$ is a *set*, it follows that $\{V_i\}_{i \in I}$ is a *set.* Thus, it follows from the *existence axiom* ($\dagger^{\mathrm{G}}$) that there exists a universe V such that $\{V_i\}_{i \in I} \in V$. Hence, by condition (i), we conclude that $V_i \in V$, for all $i \in I$, as desired. Note that this means, in particular, that there exist *infinite ascending chains of universes*

$$V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$$

— where n ranges over the natural numbers. On the other hand, by the *axiom of foundation*, there do not exist *infinite descending chains of universes*

$$V_0 \ni V_1 \ni V_2 \ni V_3 \ni \ldots \ni V_n \ni \ldots$$

— where n ranges over the natural numbers.

Although we shall not discuss in detail here the quite difficult issue of *whether or not there actually exist ZFCG-models*, we remark in passing that it may be possible to justify the stance of ignoring such issues in the context of the present series of papers — at least from the point of view of establishing the validity of various *"final results"* that may be formulated in ZFC-models — by invoking the work of *Feferman* [cf. [Ffmn]]. Precise statements concerning such issues, however, lie beyond the scope of the present paper [as well as of the level of expertise of the author!].

In the following discussion, we use the phrase *"set-theoretic formula"* as it is conventionally used in discussions of *axiomatic set theory* [cf., e.g., [Drk], Chapter 1, §2], with the following *proviso*: In the following discussion, it should be understood that every set-theoretic formula that appears is *"absolute"* in the sense that its validity for a collection of sets contained in some universe V relative to the model of set theory determined by V is *equivalent*, for any universe W such that $V \in W$, to its validity for the same collection of sets relative to the model of set theory determined by W [cf., e.g., [Drk], Chapter 3, Definition 4.2].

Definition 3.1.

(i) A 0-*species* $\mathfrak{S}_0$ is a collection of conditions given by a *set-theoretic formula*

$$\Phi_0(\mathfrak{E})$$

involving an ordered collection $\mathfrak{E} = (\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0})$ of sets $\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0}$ [which we think of as "indeterminates"], for some integer $n_0 \geq 1$; in this situation, we shall refer to $\mathfrak{E}$ as a *collection of species-data* for $\mathfrak{S}_0$. If $\mathfrak{S}_0$ is a 0-species given by a set-theoretic formula $\Phi_0(\mathfrak{E})$, then a 0-*specimen* of $\mathfrak{S}_0$ is a *specific* ordered collection of n_0 sets $E = (E_1, \ldots, E_{n_0})$ in some *specific* ZFC-model that satisfies $\Phi_0(E)$. If E is a 0-specimen of a 0-species $\mathfrak{S}_0$, then we shall write $E \in \mathfrak{S}_0$. If, moreover, it holds, in any ZFC-model, that the 0-specimens of $\mathfrak{S}_0$ form a *set*, then we shall refer to $\mathfrak{S}_0$ as 0-*small*.

(ii) Let $\mathfrak{S}_0$ be a 0-species. Then a 1-*species* $\mathfrak{S}_1$ *acting on* $\mathfrak{S}_0$ is a collection of *set-theoretic formulas* Φ_1, $\Phi_{1\circ 1}$ satisfying the following conditions:

(a) Φ_1 is a set-theoretic formula

$$\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$$

involving two collections of species-data $\mathfrak{E}$, $\mathfrak{E}'$ for $\mathfrak{S}_0$ [i.e., the conditions $\Phi_0(\mathfrak{E})$, $\Phi_0(\mathfrak{E}')$ hold] and an ordered collection $\mathfrak{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1})$ of ["indeterminate"] sets $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1}$, for some integer $n_1 \geq 1$; in this situation, we shall refer to $(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$ as a *collection of species-data* for $\mathfrak{S}_1$ and write

$\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$. If, in some ZFC-model, $E, E' \in \mathfrak{S}_0$, and F is a *specific* ordered collection of n_1 sets that satisfies the condition $\Phi_1(E, E', F)$, then we shall refer to the data (E, E', F) as a 1-*specimen* of $\mathfrak{S}_1$ and write $(E, E', F) \in \mathfrak{S}_1$; alternatively, we shall denote a 1-specimen (E, E', F) via the notation $F : E \to E'$ and refer to E (respectively, E') as the *domain* (respectively, *codomain*) of $F : E \to E'$.

(b) $\Phi_{1\circ 1}$ is a set-theoretic formula

$$\Phi_{1\circ 1}(\mathfrak{E}, \mathfrak{E}', \mathfrak{E}'', \mathfrak{F}, \mathfrak{F}', \mathfrak{F}'')$$

involving three collections of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$, $\mathfrak{F}' : \mathfrak{E}' \to \mathfrak{E}''$, $\mathfrak{F}'' : \mathfrak{E} \to \mathfrak{E}''$ for $\mathfrak{S}_1$ [i.e., the conditions $\Phi_0(\mathfrak{E})$; $\Phi_0(\mathfrak{E}')$; $\Phi_0(\mathfrak{E}'')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$; $\Phi_1(\mathfrak{E}', \mathfrak{E}'', \mathfrak{F}')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}'', \mathfrak{F}'')$ hold]; in this situation, we shall refer to $\mathfrak{F}''$ as a *composite of* $\mathfrak{F}$ *with* $\mathfrak{F}'$ and write $\mathfrak{F}'' = \mathfrak{F}' \circ \mathfrak{F}$ [which is, *a priori*, an abuse of notation, since there may exist *many* composites of $\mathfrak{F}$ with $\mathfrak{F}'$ — cf. (c) below]; we shall use similar terminology and notation for 1-specimens in specific ZFC-models.

(c) Given a pair of 1-specimens $F : E \to E'$, $F' : E' \to E''$ of $\mathfrak{S}_1$ in some ZFC-model, there *exists a unique composite* $F'' : E \to E''$ of F with F' in the given ZFC-model.

(d) Composition of 1-specimens $F : E \to E'$, $F' : E' \to E''$, $F'' : E'' \to E'''$ of $\mathfrak{S}_1$ in a ZFC-model is *associative*.

(e) For any 0-specimen E of $\mathfrak{S}_0$ in a ZFC-model, there exists a [necessarily unique] 1-specimen $F : E \to E$ of $\mathfrak{S}_1$ [in the given ZFC-model] — which we shall refer to as the *identity* 1-*specimen* id_E of E — such that for any 1-specimens $F' : E' \to E$, $F'' : E \to E''$ of $\mathfrak{S}_1$ [in the given ZFC-model] we have $F \circ F' = F'$, $F'' \circ F = F''$.

If, moreover, it holds, in any ZFC-model, that for any two 0-specimens E, E' of $\mathfrak{S}_0$, the 1-specimens $F : E \to E'$ of $\mathfrak{S}_1$ [i.e., the 1-specimens of $\mathfrak{S}_1$ with domain E and codomain E'] form a *set*, then we shall refer to $\mathfrak{S}_1$ as 1-*small*.

(iii) A *species* $\mathfrak{S}$ is defined to be a pair consisting of a 0-species $\mathfrak{S}_0$ and a 1-species $\mathfrak{S}_1$ acting on $\mathfrak{S}_0$. Fix a species $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$. Let $i \in \{0, 1\}$. Then we shall refer to an i-specimen of $\mathfrak{S}_i$ as an *i-specimen of* $\mathfrak{S}$. We shall refer to a 0-specimen (respectively, 1-specimen) of $\mathfrak{S}$ as a *species-object* (respectively, a *species-morphism*) of $\mathfrak{S}$. We shall say that $\mathfrak{S}$ is *i-small* if $\mathfrak{S}_i$ is i-small. We shall refer to a species-morphism $F : E \to E'$ as a *species-isomorphism* if there exists a species-morphism $F' : E' \to E$ such that the composites $F \circ F'$, $F' \circ F$ are *identity* species-morphisms; in this situation, we shall say that E, E' are *species-isomorphic*. [Thus, one verifies immediately that *composites* of *species-isomorphisms* are species-isomorphisms.] We shall refer to a species-isomorphism whose domain and codomain are equal as a *species-automorphism*. We shall refer to as *model-free* [cf. Remark 3.1.1 below] an i-specimen of $\mathfrak{S}$ equipped with a description via a *set-theoretic formula* that is "*independent* of the ZFC-model in which it is given" in the sense that for any pair of universes V_1, V_2 of some ZFC-model such that $V_1 \in V_2$, the set-theoretic formula determines the *same* i-specimen of $\mathfrak{S}$, whether interpreted relative to the ZFC-model determined by V_1 or the ZFC-model determined by V_2.

(iv) We shall refer to as the *category determined by* $\mathfrak{S}$ in a ZFC-model the *category* whose objects are the *species-objects* of $\mathfrak{S}$ in the given ZFC-model and whose arrows are the *species-morphisms* of $\mathfrak{S}$ in the given ZFC-model. [One verifies immediately that this description does indeed determine a category.]

Remark 3.1.1. We observe that any of the familiar descriptions of $\mathbb{N}$ [cf., e.g., [Drk], Chapter 2, Definitions 2.3, 2.9], $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$, or $\mathbb{R}$, for instance, yield *species* [all of whose species-morphisms are identity species-morphisms] each of which has a *unique* species-object in any given ZFC-model. Such species are *not to be confused* with such species as the species of "monoids isomorphic to $\mathbb{N}$ and monoid isomorphisms", which admits *many species-objects* [all of which are species-isomorphic] in any ZFC-model. On the other hand, the set-theoretic formula used, for instance, to define the former "species $\mathbb{N}$" may be applied to define a *"model-free species-object* $\mathbb{N}$*"* of the latter "species of monoids isomorphic to $\mathbb{N}$".

Remark 3.1.2.

(i) It is important to remember when working with species that

> the **essence** *of a species* lies *not in the specific sets* that occur as species-objects or species-morphisms of the species in various ZFC-models, but rather in the **collection of rules**, i.e., *set-theoretic formulas*, that govern the construction of such sets in an *unspecified, "indeterminate" ZFC-model.*

Put another way, the emphasis in the theory of species lies in the *programs* — i.e., **"software"** — that yield the desired output data, *not on the output data itself.* From this point of view, one way to describe the various set-theoretic formulas that constitute a species is as a *"deterministic algorithm"* [a term suggested to the author by Minhyong Kim] for constructing the sets to be considered.

(ii) One interesting point of view that arose in discussions between the author and F. Kato is the following. The relationship between the classical approach to discussing mathematics relative to a *fixed model of set theory* — an approach in which *specific sets* play a central role — and the *"species-theoretic"* approach considered here — in which the **rules**, given by set-theoretic formulas for constructing the sets of interest [i.e., not specific sets themselves!], play a central role — may be regarded as *analogous* to the relationship between *classical approaches to algebraic varieties* — in which specific sets of solutions of polynomial equations in an algebraically closed field play a central role — and *scheme theory* — in which the functor determined by a scheme, i.e., the polynomial equations, or "rules", that determine solutions, as opposed to specific sets of solutions themselves, play a central role. That is to say, in summary:

$$[\text{fixed model of set theory approach} : \text{species-theoretic approach}] \quad \longleftrightarrow \quad [\text{varieties} : \text{schemes}]$$

A similar analogy — i.e., of the form

$$[\text{fixed model of set theory approach} : \text{species-theoretic approach}] \quad \longleftrightarrow \quad [\text{groups of specific matrices} : \text{abstract groups}]$$

— may be made to the notion of an "abstract group", as opposed to a "group of specific matrices". That is to say, just as a "group of specific matrices may be thought of as a *specific representation* of an "abstract group", the category of objects determined by a species in a specific ZFC-model may be thought of as a *specific representation* of an "abstract species".

(iii) If, in the context of the discussion of (i), (ii), one tries to form a sort of *quotient*, in which "programs" that yield the same sets as "output data" are *identified*, then one must contend with the resulting *indeterminacy*, i.e., working with programs is only well-defined up to internal modifications of the programs in question that does not affect the final output. This leads to somewhat *intractable problems* concerning the internal structure of such programs — a topic that lies well beyond the scope of the present work.

Remark 3.1.3.

(i) Typically, in the discussion to follow, we shall not write out explicitly the various set-theoretic formulas involved in the definition of a species. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* of the mathematical objects involved. When applying such conventional descriptions, however, it is important to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(ii) The fact that the data involved in a species is given by abstract *set-theoretic formulas* imparts a certain **canonicality** to the mathematical notion constituted by the species, a canonicality that is **not shared**, for instance, by mathematical objects whose construction depends on an **invocation of the axiom of choice** in some particular ZFC-model [cf. the discussion of (i) above]. Moreover, by furnishing a stock of such "canonical notions", the theory of species allows one, in effect, to *compute the* **extent of deviation** of various *"non-canonical objects"* [i.e., whose construction depends upon the invocation of the axiom of choice!] from a sort of *"canonical norm"*.

Remark 3.1.4. Note that because the data involved in a species is given by abstract *set-theoretic formulas*, the mathematical notion constituted by the species is **immune** to, i.e., unaffected by, **extensions of the universe** — i.e., such as the ascending chain $V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$ that appears in the discussion preceding Definition 3.1 — in which one works. This is the sense in which we apply the term **"inter-universal"**. That is to say, *"inter-universal geometry"* allows one to relate the "geometries" that occur in distinct universes.

Remark 3.1.5. Similar remarks to the remarks made in Remarks 3.1.2, 3.1.3, and 3.1.4 concerning the significance of working with *set-theoretic formulas* may be made with regard to the notions of *mutations*, *morphisms of mutations*, *mutation-histories*, *observables*, and *cores* to be introduced in Definition 3.3 below.

One fundamental example of a species is the following.

Example 3.2. Categories. The notions of a [small] category and an isomorphism class of [covariant] functors between two given [small] categories yield an example of a *species*. That is to say, at a set-theoretic level, one may think of a [small] *category* as, for instance, a set of arrows, together with a set of composition relations, that satisfies certain properties; one may think of a [covariant] *functor* between [small] categories as the set given by the graph of the map on arrows determined by the functor [which satisfies certain properties]; one may think of an *isomorphism class of functors* as a collection of such graphs, i.e., the graphs determined by the functors in the isomorphism class, which satisfies certain properties. Then one has *"dictionaries"*

$$\text{0-species} \quad \longleftrightarrow \quad \text{the notion of a category}$$

$$\text{1-species} \quad \longleftrightarrow \quad \text{the notion of an isomorphism class of functors}$$

at the level of *notions* and

$$\text{a 0-specimen} \quad \longleftrightarrow \quad \text{a particular [small] category}$$

$$\text{a 1-specimen} \quad \longleftrightarrow \quad \text{a particular isomorphism class of functors}$$

at the level of *specific mathematical objects* in a specific ZFC-model. Moreover, one verifies easily that species-isomorphisms between 0-species correspond to isomorphism classes of equivalences of categories in the usual sense.

Remark 3.2.1. Note that in the case of Example 3.2, one could also define a notion of *"2-species"*, *"2-specimens"*, etc., via the notion of an "isomorphism of functors", and then take the 1-species under consideration to be the notion of a functor [i.e., not an isomorphism class of functors]. Indeed, more generally, one could define a notion of "n-species" for arbitrary integers $n \geq 1$. Since, however, this approach would only serve to add an *unnecessary level of complexity* to the theory, we choose here to take the approach of working with "functors considered up to isomorphism".

Definition 3.3. Let $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$ be *species*.

(i) A *mutation* $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ is defined to be a collection of *set-theoretic formulas* Ψ_0, Ψ_1 satisfying the following properties:

(a) Ψ_0 is a set-theoretic formula

$$\Psi_0(\mathfrak{E}, \underline{\mathfrak{E}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{E}}$ for $\underline{\mathfrak{S}}_0$; in this situation, we shall write $\mathfrak{M}(\mathfrak{E})$ for $\underline{\mathfrak{E}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{E} \in \underline{\mathfrak{S}}_0$ such that $\Psi_0(E, \underline{E})$ holds; in this situation, we shall write $\mathfrak{M}(E)$ for $\underline{E}$.

(b) Ψ_1 is a set-theoretic formula

$$\Psi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$ for $\mathfrak{S}_1$ and a collection of species-data $\underline{\mathfrak{F}} : \underline{\mathfrak{E}} \to \underline{\mathfrak{E}}'$ for $\underline{\mathfrak{S}}_1$, where $\underline{\mathfrak{E}} = \mathfrak{M}(\mathfrak{E})$, $\underline{\mathfrak{E}}' = \mathfrak{M}(\mathfrak{E}')$; in this situation, we shall write $\mathfrak{M}(\mathfrak{F})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $(F : E \to E') \in \mathfrak{S}_1$, then we require that there *exist a unique* $(\underline{F} : \underline{E} \to \underline{E}') \in \underline{\mathfrak{S}}_1$ such that $\Psi_0(E, E', F, \underline{F})$ holds; in this situation, we shall write $\mathfrak{M}(F)$ for $\underline{F}$. Finally, we require that the assignment $F \mapsto \mathfrak{M}(F)$ be compatible with *composites* and map *identity* species-morphisms of $\mathfrak{S}$ to identity species-morphisms of $\underline{\mathfrak{S}}$. In particular, if one fixes a ZFC-model, then $\mathfrak{M}$ determines a *functor* from the category determined by $\mathfrak{S}$ in the given ZFC-model to the category determined by $\underline{\mathfrak{S}}$ in the given ZFC-model.

There are evident notions of *"composition of mutations"* and *"identity mutations"*.

(ii) Let $\mathfrak{M}, \mathfrak{M}' : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be *mutations*. Then a *morphism of mutations* $\mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}'$ is defined to be a *set-theoretic formula* Ξ satisfying the following properties:

(a) Ξ is a set-theoretic formula

$$\Xi(\mathfrak{E}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{F}} : \mathfrak{M}(\mathfrak{E}) \to \mathfrak{M}'(\mathfrak{E})$ for $\mathfrak{S}_1$; in this situation, we shall write $\mathfrak{Z}(\mathfrak{E})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{F} \in \underline{\mathfrak{S}}_1$ such that $\Xi(E, \underline{F})$ holds; in this situation, we shall write $\mathfrak{Z}(E)$ for $\underline{F}$.

(b) Suppose, in some ZFC-model, that $F : E_1 \to E_2$ is a species-morphism of $\mathfrak{S}$. Then one has an equality of composite species-morphisms $\mathfrak{M}'(F) \circ \mathfrak{Z}(E_1) = \mathfrak{Z}(E_2) \circ \mathfrak{M}(F) : \mathfrak{M}(E_1) \to \mathfrak{M}'(E_2)$. In particular, if one fixes a ZFC-model, then a morphism of mutations $\mathfrak{M} \to \mathfrak{M}'$ determines a *natural transformation* between the functors determined by $\mathfrak{M}$, $\mathfrak{M}'$ in the ZFC-model — cf. (i).

There are evident notions of *"composition of morphisms of mutations"* and *"identity morphisms of mutations"*. If it holds that for every species-object E of $\mathfrak{S}$, $\mathfrak{Z}(E)$ is a *species-isomorphism*, then we shall refer to $\mathfrak{Z}$ as an *isomorphism of mutations*. In particular, one verifies immediately that $\mathfrak{Z}$ is an isomorphism of mutations if and only if there exists a morphism of mutations $\mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}$ such that the composite morphisms of mutations $\mathfrak{Z}' \circ \mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}$, $\mathfrak{Z} \circ \mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}'$ are the respective *identity* morphisms of the mutations $\mathfrak{M}$, $\mathfrak{M}'$.

(iii) Let $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be a *mutation*. Then we shall say that $\mathfrak{M}$ is a *mutation-equivalence* if there exists a mutation $\mathfrak{M}' : \underline{\mathfrak{S}} \rightsquigarrow \mathfrak{S}$, together with isomorphisms of mutations between the composites $\mathfrak{M} \circ \mathfrak{M}'$, $\mathfrak{M}' \circ \mathfrak{M}$ and the respective identity mutations. In this situation, we shall say that $\mathfrak{M}$, $\mathfrak{M}'$ are *mutation-quasi-inverses* to one another. Finally, we observe that, if we suppose further that $\mathfrak{S}$, $\underline{\mathfrak{S}}$ are 1-*small*, then for any two given species-objects in the domain species of a mutation-equivalence, the *mutation-equivalence* induces a *bijection* between the set of *species-morphisms* (respectively, *species-isomorphisms*) between the two given species-objects [of the domain species] and the set of species-morphisms (respectively, species-isomorphisms) between the two species-objects [of the codomain species] obtained by applying the mutation-equivalence to the two given species-objects.

(iv) Let $\vec{\Gamma}$ be an *oriented graph*, i.e., a graph Γ, which we shall refer to as the *underlying graph* of $\vec{\Gamma}$, equipped with the additional data of a total ordering, for each edge e of Γ, on the set [of cardinality 2] of *branches* of e [cf., e.g., [AbsTopIII], §0]. Then we define a *mutation-history* $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ *[indexed by* $\vec{\Gamma}$*]* to be a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a *species* $\mathfrak{S}^v$;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *mutation* $\mathfrak{M}^e : \mathfrak{S}^{v_1} \rightsquigarrow \mathfrak{S}^{v_2}$.

In this situation, we shall refer to the vertices, edges, and branches of $\vec{\Gamma}$ as vertices, edges, and branches of $\mathfrak{H}$. Thus, the notion of a "mutation-history" may be thought of as a *species-theoretic* version of the notion of a "diagram of categories" given in [AbsTopIII], Definition 3.5, (i).

(v) Let $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ be a *mutation-history*; $\underline{\mathfrak{S}}$ a *species*. For simplicity, we assume that the underlying graph of $\vec{\Gamma}$ is *simply connected*. Then we shall refer to as a(n) *[*$\underline{\mathfrak{S}}$*-valued] covariant* (respectively, *contravariant*) *observable* $\mathfrak{V}$ of the mutation-history $\mathfrak{H}$ a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a mutation $\mathfrak{V}^v : \mathfrak{S}^v \to \underline{\mathfrak{S}}$, which we shall refer to as the *observation mutation* at v;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *morphism of mutations* $\mathfrak{V}^e : \mathfrak{V}^{v_1} \to \mathfrak{V}^{v_2} \circ \mathfrak{M}^e$ (respectively, $\mathfrak{V}^e : \mathfrak{V}^{v_2} \circ \mathfrak{M}^e \to \mathfrak{V}^{v_1}$).

If $\mathfrak{V}$ is a covariant observable such that all of the morphisms of mutations "$\mathfrak{V}^e$" are *isomorphisms of mutations*, then we shall refer to the covariant observable $\mathfrak{V}$ as a *core*. Thus, one may think of a core $\mathfrak{C}$ of a mutation-history as lying *"under"* the entire mutation-history in a *"uniform fashion"*. Also, we shall refer to the "property [of an observable] of being a core" as the *"coricity"* of the observable. Finally, we note that the notions of an "observable" and a "core" given here may be thought of as simplified, *species-theoretic* versions of the notions of "observable" and "core" given in [AbsTopIII], Definition 3.5, (iii).

Remark 3.3.1.

(i) One well-known consequence of the *axiom of foundation* of axiomatic set theory is the assertion that **"$\in$-loops"**

$$a \in b \in c \in \ldots \in a$$

can *never occur* in the set theory in which one works. On the other hand, there are many situations in mathematics in which one wishes to somehow **"identify"** mathematical objects that arise at *higher levels* of the $\in$-structure of the set theory under consideration with mathematical objects that arise at *lower levels* of this $\in$-structure. In some sense, the notions of a *"set"* and of a *"bijection of sets"* allow one to achieve such "identifications". That is to say, the mathematical objects at both higher and lower levels of the $\in$-structure constitute examples of the *same mathematical notion of a "set"*, so that one may consider "bijections of sets" between those sets without violating the axiom of foundation. In some sense, the notion of a **species** may be thought of as a natural *extension* of this observation. That is to say,

> the notion of a "species" allows one to consider, for instance, *species-isomorphisms* between species-objects that occur at *different levels* of the $\in$-structure of the set theory under consideration — i.e., roughly speaking, to **"simulate $\in$-loops"** — *without violating the axiom of foundation.*

Moreover, typically the sorts of species-objects at different levels of the $\in$-structure that one wishes to somehow have "identified" with one another occur as the result of executing the *mutations* that arise in some sort of **mutation-history**

$$\dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}} \rightsquigarrow \underline{\underline{\mathfrak{S}}} \rightsquigarrow \dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \dots$$

[where $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$; $\underline{\underline{\mathfrak{S}}} = (\underline{\underline{\mathfrak{S}}}_0, \underline{\underline{\mathfrak{S}}}_1)$ are *species*] — e.g., the *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that arise from applying various mutations to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left.*

(ii) In the context of constructing *"loops"* in a mutation-history as in the final display of (i), we observe that

> the **simpler** the structure of the **species** involved, the **easier** it is to construct **"loops"**.

It is for this reason that species such as the species determined by the notion of a **category** [cf. Example 3.2] are easier to work with, from the point of view of constructing "loops", than more complicated species such as the species determined by the notion of a *scheme*. This is one of the *principal motivations* for the *"geometry of categories"* — of which *"absolute anabelian geometry"* is the special case that arises when the categories involved are Galois categories — i.e., for the theory of *representing scheme-theoretic geometries via categories* [cf., e.g., the Introductions of [MnLg], [SemiAnbd], [Cusp], [FrdI]]. At a more concrete level, the utility of working with categories to reconstruct objects that occurred at earlier stages of some sort of "series of constructions" [cf. the mutation-history of the final display of (i)!] may be seen in the *"reconstruction of the underlying scheme"* in various situations throughout [MnLg] by applying the *natural equivalence of categories* of the final display of [MnLg], Definition 1.1, (iv), from a certain category constructed from a log scheme, as well as in the theory of *"slim exponentiation"* discussed in the Appendix to [FrdI].

(iii) Again in the context of mutation-histories such as the one given in the final display of (i), although one may, on certain occasions, wish to apply various mutations that *fundamentally alter* the structure of the mathematical objects involved and hence give rise to *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that are related in a *highly nontrivial fashion* to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left*, it is also of interest to consider

> *"portions"* of the various mathematical objects that occur that are left **unaltered** by the various mutations that one applies.

This is precisely the reason for the introduction of the notion of a *core* of a mutation-history. One important consequence of the construction of various cores associated to a mutation-history is that often

> one may apply various cores associated to a mutation-history to **describe**, by means of **non-coric observables**, the portions of the various mathematical objects that occur which *are* **altered** by the various mutations that one applies *in terms of* the **unaltered** portions, i.e., **cores**.

Indeed, this point of view plays a *central role* in the theory of the present series of papers — cf. the discussion of Remark 3.6.1, (ii), below.

Remark 3.3.2. One somewhat *naive* point of view that constituted one of the original motivations for the author in the development of theory of the present series of papers is the following. In the classical theory of schemes, when considering **local systems** on a scheme, there is no reason to restrict oneself to considering local systems valued in, say, modules over a finite ring. If, moreover, there is no reason to make such a restriction, then one is naturally led to consider, for instance, local systems of *schemes* [cf., e.g., the theory of the "Galois mantle" in [pTeich]], or, indeed, local systems of more general **collections of mathematical objects**. One may then ask what happens if one tries to consider local systems on the schemes that occur as fibers of a local system of schemes. [More concretely, if X is, for instance, a connected scheme, then one may consider local systems $\mathcal{X}$ over X whose *fibers* are isomorphic to X; then one may repeat this process, by considering such local systems *over each fiber* of the local system $\mathcal{X}$ on X, etc.] In this way, one is eventually led to the consideration of **"systems of nested local systems"** — i.e., a local system over a local system over a local system, etc. It is precisely this point of view that underlies the notion of *"successive iteration of a given mutation-history"*, relative to the terminology formulated in the present §3. If, moreover, one thinks of such "successive iterates of a given mutation-history" as being a sort of abstraction of the naive idea of a "system of nested local systems", then the notion of a **core** may be thought of as a sort of mathematical object that is *invariant* with respect to the application of the operations that gave rise to the "system of nested local systems".

Example 3.4. Topological Spaces and Fundamental Groups.

(i) One verifies easily that the notions of a *topological space* and a *continuous map* between topological spaces determine an example of a *species* $\mathfrak{S}^{\mathrm{top}}$. In a similar vein, the notions of a *universal covering* $\widetilde{X} \to X$ of a pathwise connected topological space X and a *continuous map* between such universal coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ [i.e., a pair of compatible continuous maps $\widetilde{X} \to \widetilde{Y}$, $X \to Y$], considered up to composition with a *deck transformation* of the universal covering $\widetilde{Y} \to Y$, determine an example of a *species* $\mathfrak{S}^{\text{u-top}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structures of $\mathfrak{S}^{\mathrm{top}}$, $\mathfrak{S}^{\text{u-top}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\text{u-top}}$

are *collections* of continuous maps [between two given universal coverings], any two of which differ from one another by composition with a deck transformation.

(ii) One verifies easily that the notions of a *group* and an *outer homomorphism* between groups [i.e., a homomorphism considered up to composition with an inner automorphism of the codomain group] determine an example of a *species* $\mathfrak{S}^{\mathrm{gp}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structure of $\mathfrak{S}^{\mathrm{gp}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\mathrm{gp}}$ are *collections* of homomorphisms [between two given groups], any two of which differ from one another by composition with an inner automorphism.

(iii) Now one verifies easily that the assignment

$$(\widetilde{X} \to X) \quad \mapsto \quad \mathrm{Aut}(\widetilde{X}/X)$$

— where $(\widetilde{X} \to X)$ is a species-object of $\mathfrak{S}^{\text{u-top}}$, and $\mathrm{Aut}(\widetilde{X}/X)$ denotes the group of deck transformations of the universal covering $\widetilde{X} \to X$ — determines a *mutation* $\mathfrak{S}^{\text{u-top}} \rightsquigarrow \mathfrak{S}^{\mathrm{gp}}$. That is to say, the *"fundamental group"* may be thought of as a sort of mutation.

Example 3.5. Absolute Anabelian Geometry.

(i) Let $\mathcal{S}$ be a class of connected normal schemes that is closed under isomorphism [of schemes]. Suppose that there exists a set $E_{\mathcal{S}}$ of schemes describable by a *set-theoretic formula* with the property that every scheme of $\mathcal{S}$ is isomorphic to some scheme belonging to $E_{\mathcal{S}}$. Then just as in the case of universal coverings of topological spaces discussed in Example 3.4, (i), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{S}}$, that the *universal pro-finite étale coverings* $\widetilde{X} \to X$ of schemes X belonging to $\mathcal{S}$ and *isomorphisms* of such coverings considered up to composition with a *deck transformation* give rise to a *species* $\mathfrak{S}^{\mathcal{S}}$.

(ii) Let $\mathcal{G}$ be a class of topological groups that is closed under isomorphism [of topological groups]. Suppose that there exists a set $E_{\mathcal{G}}$ of topological groups describable by a *set-theoretic formula* with the property that every topological group of $\mathcal{G}$ is isomorphic to some topological group belonging to $E_{\mathcal{G}}$. Then just as in the case of abstract groups discussed in Example 3.4, (ii), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{G}}$, that *topological groups* belonging to $\mathcal{G}$ and [bi-continuous] *outer isomorphisms* between such topological groups give rise to a *species* $\mathfrak{S}^{\mathcal{G}}$.

(iii) Let $\mathcal{S}$ be as in (i). Then for an appropriate choice of $\mathcal{G}$, by associating to a universal pro-finite étale covering the resulting group of deck transformations, one obtains a *mutation*

$$\Pi : \quad \mathfrak{S}^{\mathcal{S}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{G}}$$

[cf. Example 3.4, (iii)]. Then one way to define the notion that the schemes belonging to the class $\mathcal{S}$ are **"[absolute] anabelian"** is to require the specification of a *mutation*

$$\mathbb{A} : \quad \mathfrak{S}^{\mathcal{G}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{S}}$$

which forms a *mutation-quasi-inverse* to Π. Here, we note that the existence of the bijections [i.e., *"fully faithfulness"*] discussed in Definition 3.3, (iii), is, in essence, the condition that is usually taken as the definition of "anabelian". By contrast, the *species-theoretic* approach of the present discussion may be thought of as an explicit mathematical formulation of the **algorithmic** *approach to [absolute] anabelian geometry* discussed in the Introduction to [AbsTopI].

(iv) The framework of [absolute] anabelian geometry [cf., e.g., the framework discussed above in (iii)] gives a good example of the importance of specifying *precisely* **what species** *one is working with* in a given "series of constructions" [cf., e.g., the mutation-history of the final display of Remark 3.3.1, (i)]. That is to say, there is a quite substantial difference between working with a

profinite group in its sole capacity as a profinite group

and working with the same profinite group — which may happen to arise as the *étale fundamental group* of some scheme! —

> *regarded as being equipped with various data that arise from the construction of the profinite group as the étale fundamental group of some scheme.*

It is precisely this sort of issue that constituted one of the original motivations for the author in the development of the theory of species presented here.

Example 3.6. The Étale Site and Frobenius.

(i) Let p be a *prime number*. If S is a *reduced* scheme over $\mathbb{F}_p$, then denote by $S^{(p)}$ the scheme with the same topological space as S, but whose structure sheaf is given by the *subsheaf*

$$\mathcal{O}_{S^{(p)}} \stackrel{\text{def}}{=} (\mathcal{O}_S)^p \subseteq \mathcal{O}_S$$

of p-th powers of sections of S. Thus, the natural inclusion $\mathcal{O}_{S^{(p)}} \hookrightarrow \mathcal{O}_S$ induces a morphism $\Phi_S : S \to S^{(p)}$. Moreover, "raising to the p-th power" determines a *natural isomorphism* $\alpha_S : S^{(p)} \xrightarrow{\sim} S$ such that the resulting composite $\alpha_S \circ \Phi_S : S \to S$ is the *Frobenius morphism* of S. Write

$$\mathfrak{S}^{p\text{-sch}}$$

for the *species* of reduced quasi-compact schemes over $\mathbb{F}_p$ and quasi-compact morphisms of schemes. Then consider the *[small] category* $S_{\text{ét}}$ — i.e., "the *small étale site* of S" — defined as follows:

> An *object* of $S_{\text{ét}}$ is a(n) [necessarily quasi-affine, by *Zariski's Main Theorem*!]
>
> > *étale morphism of finite presentation* $T \to S$ equipped with a *finite open cover* $\{U_i\}_{i\in I}$ of S, together with *factorizations* $T|_{U_i} \subseteq \mathbb{A}^{N_i}_{U_i} \to U_i$ for each $i \in I$
>
> — where I is a finite subset of the set of open subschemes of S; $\mathbb{A}^{N_i}_{U_i}$ denotes a standard copy of affine N_i-space over U_i, for some integer $N_i \geq 1$; the

"$\subseteq$" exhibits $T|_{U_i}$ as a finitely presented subscheme of $\mathbb{A}^{N_i}_{U_i}$; we observe that *any étale morphism of finite presentation $T \to S$ necessarily admits such auxiliary data parametrized by some index set I.* A *morphism* of $S_{\text{ét}}$ from an object $T_1 \to S$ to an object $T_2 \to S$ [each of which is equipped with auxiliary data] is a(n) [necessarily étale of finite presentation] S-morphism $T_1 \to T_2$.

In particular, one may construct an *assignment*

$$S \mapsto S_{\text{ét}}$$

that maps a species-object S of $\mathfrak{S}^{p\text{-sch}}$ to the [small] category $S_{\text{ét}}$ in such a way that the assignment $S \mapsto S_{\text{ét}}$ is *contravariantly functorial* with respect to species-morphisms $S_1 \to S_2$ of $\mathfrak{S}^{p\text{-sch}}$, and, moreover, may be described via *set-theoretic formulas.* Thus, such an assignment determines an *"étale site mutation"*

$$\mathfrak{M}^{\text{ét}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{\text{cat}}$$

— where we write $\mathfrak{S}^{\text{cat}}$ for the *species* of categories and isomorphism classes of contravariant functors [i.e., a slightly modified form of the species considered in Example 3.2]. Another natural assignment in the present context is the assignment

$$S \mapsto S^{\text{pf}}$$

which maps S to its *perfection* S^{pf}, i.e., the scheme determined by taking the inverse limit of the inverse system $\ldots \to S \to S \to S$ obtained by considering *iterates* of the *Frobenius morphism* of S. Thus, by considering the final copy of "S" in this inverse system, one obtains a natural morphism $\beta_S : S^{\text{pf}} \to S$. Finally, one obtains a *"perfection mutation"*

$$\mathfrak{M}^{\text{pf}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{\text{pf}}$.

(ii) Write

$$\mathfrak{F}^{p\text{-sch}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

for the **"Frobenius mutation"** obtained by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{(p)}$. Thus, one may formulate the well-known *"invariance of the étale site under Frobenius"* [cf., e.g., [FK], Chapter I, Proposition 3.16] as the statement that the **"étale site mutation"** $\mathfrak{M}^{\text{ét}}$ exhibits $\mathfrak{S}^{\text{cat}}$ as a **core** — i.e., an *"invariant piece"* — of the **"Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \ldots$$

determined by the *"Frobenius mutation"* $\mathfrak{F}^{p\text{-sch}}$. In this context, we observe that the **"perfection mutation"** $\mathfrak{M}^{\text{pf}}$ also yields a **core** — i.e., another *"invariant piece"* — of the Frobenius mutation-history. On the other hand, the natural morphism $\Phi_S : S \to S^{(p)}$ may be interpreted as a *covariant* **observable** of this mutation-history whose *observation mutations* are the identity mutations $\text{id}_{\mathfrak{S}^{p\text{-sch}}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$. Since Φ_S is not, in general, an isomorphism, it follows that this observable constitutes an example of an **non-coric** observable. Nevertheless, the natural morphism $\beta_S : S^{\mathrm{pf}} \to S$ may be interpreted as a *morphism of mutations* $\mathfrak{M}^{\mathrm{pf}} \to \mathrm{id}_{\mathfrak{S}^{p\text{-sch}}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{M}^{\mathrm{pf}}$.

(iii) One may also develop a version of (i), (ii) for *log schemes*; we leave the routine details to the interested reader. Here, we pause to mention that the theory of log schemes motivates the following **"combinatorial monoid-theoretic"** version of the *non-coric observable* on the *Frobenius mutation-history* of (ii). Write

$$\mathfrak{S}^{\mathrm{mon}}$$

for the species of *torsion-free abelian monoids* and *morphisms of monoids*. If M is a species-object of $\mathfrak{S}^{\mathrm{mon}}$, then write $M^{(p)} \stackrel{\mathrm{def}}{=} p \cdot M \subseteq M$. Then the assignment $M \mapsto M^{(p)}$ determines a **"monoid-Frobenius mutation"**

$$\mathfrak{F}^{\mathrm{mon}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

and hence a **"monoid-Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \ldots$$

which is equipped with a **non-coric** *contravariant* **observable** determined by the natural inclusion morphism $M^{(p)} \hookrightarrow M$ and the observation mutations given by the identity mutations $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$. On the other hand, the *p-perfection* M^{pf} of M, i.e., the inductive limit of the inductive system $M \hookrightarrow M \hookrightarrow M \hookrightarrow \ldots$ obtained by considering the inclusions given by *multiplying by* p, gives rise to a **"monoid-p-perfection mutation"**

$$\mathfrak{M}^{\text{pf-mon}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

— which may be interpreted as a **core** of the monoid-Frobenius mutation-history. Finally, the natural inclusion of monoids $M \hookrightarrow M^{\mathrm{pf}}$ may be interpreted as a *morphism of mutations* $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} \to \mathfrak{M}^{\text{pf-mon}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{M}^{\text{pf-mon}}$.

Remark 3.6.1.

(i) The various constructions of Example 3.6 may be thought of as providing, in the case of the phenomena of *"invariance of the étale site under Frobenius"* and *"invariance of the perfection under Frobenius"*, a *"species-theoretic intepretation"* — i.e., via consideration of

"coric" versus **"non-coric"** observables

— of the difference between **"étale-type"** and **"Frobenius-type"** structures [cf. the discussion of [FrdI], §I4]. This sort of approach via *"combinatorial patterns"* to expressing the difference between "étale-type" and "Frobenius-type" structures plays a *central role* in the theory of the present series of papers. Indeed, the mutation-histories and cores considered in Example 3.6, (ii), (iii), may be thought of as the **underlying motivating examples** for the theory of both

· the **vertical lines**, i.e., consisting of **$\mathfrak{log}$-links**, and

· the **horizontal lines**, i.e., consisting of **$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-links**,

of the **log-theta-lattice** [cf. [IUTchIII], Definitions 1.4, 3.8]. Finally, we recall that this approach to understanding the $\mathfrak{log}$-links may be seen in the introduction of the terminology of *"observables"* and *"cores"* in [AbsTopIII], Definition 3.5, (iii).

(ii) Example 3.6 also provides a good example of the *important theme* [cf. the discussion of Remark 3.3.1, (iii)] of

describing **non-coric** *data in terms of* **coric** *data*

— cf. the morphism $\beta_S : S^{\mathrm{pf}} \to S$ of Example 3.6, (ii); the natural inclusion $M \hookrightarrow M^{\mathrm{pf}}$ of Example 3.6, (iii). From the point of view of the *vertical* and *horizontal* lines of the **log-theta-lattice** [cf. the discussion of (i)], this theme may also be observed in the *vertically coric log-shells* that serve as a *common receptacle* for the various arrows of the **$\mathfrak{log}$-Kummer correspondences** of [IUTchIII], Theorem 3.11, (ii), as well as in the *multiradial representations* of [IUTchIII], Theorem 3.11, (i), which describe [certain aspects of] the **arithmetic holomorphic structure** on one vertical line of the log-theta-lattice in terms that may be understood relative to an **alien** *arithmetic holomorphic structure* on another vertical line — i.e., separated from the first vertical line by *horizontal arrows* — of the log-theta-lattice [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.2, (ii)].

Remark 3.6.2.

(i) In the context of the theme of *"coric descriptions of non-coric data"* discussed in Remark 3.6.1, (ii), it is of interest to observe the significance of the use of **set-theoretic formulas** [cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5] to realize such descriptions. That is to say, descriptions in terms of *arbitrary choices* that depend on a particular model of set theory [cf. Remark 3.1.3] do not allow one to **calculate** *in terms that make sense in one universe* the operations performed in an **alien universe**! This is precisely the sort of situation that one encounters when one considers the vertical and horizontal arrows of the log-theta-lattice [cf. (ii) below], where *distinct universes* arise from the *distinct* **scheme-theoretic basepoints** on either side of such an arrow that correspond to *distinct* **ring theories**, i.e., ring theories that *cannot be related* to one another by means of a *ring homomorphism* — cf. the discussion of Remark 3.6.3 below. Indeed,

it was precisely the need to understand this sort of situation that led the author to develop the **"inter-universal"** version of **Teichmüller theory** exposed in the present series of papers.

Finally, we observe that the *algorithmic approach* [i.e., as opposed to the *"fully faithfulness/Grothendieck Conjecture-style approach"* — cf. Example 3.5, (iii)] to reconstruction issues via *set-theoretic formulas* plays an essential role in this context. That is to say, although different algorithms, or *software*, may yield the *same output data*, it is only by working with **specific algorithms** that one may understand the *delicate inter-relations* that exist between various **components** of the structures that occur as one performs various operations [i.e., the mutations of a mutation-history]. In the case of the theory developed in the present series of papers, one central example of this phenomenon is the **cyclotomic rigidity isomorphisms** that underlie the theory of $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-link compatibility** discussed in [IUTchIII], Theorem 3.11, (iii), (c), (d) [cf. also [IUTchIII], Remarks 2.2.1, 2.3.2].

(ii) The **algorithmic** *approach to reconstruction* that is taken throughout the present series of papers, as well as, for instance, in [FrdI], [EtTh], and [AbsTopIII], was conceived by the author in the spirit of the **species-theoretic** *formulation* exposed in the present §3. Nevertheless, [cf. Remark 3.1.3, (i)] we shall not explicitly write out the various set-theoretic formulas involved in the various species, mutations, etc. that are *implicit* throughout the theory of the present series of papers. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* that are given of the mathematical objects involved. When applying such conventional descriptions, however, the reader is obliged to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(iii) The *sharp contrast* between

· the **canonicality** imparted by descriptions via *set-theoretic formulas* in the context of *extensions of the universe* in which one works

[cf. Remarks 3.1.3, 3.1.4] and

· the situation that arises if one allows, in one's descriptions, the various **arbitrary choices** arising from *invocations of the axiom of choice*

may be understood somewhat explicitly if one attempts to *"catalogue the various possibilities"* corresponding to various possible choices that may occur in one's description. That is to say, such a "cataloguing operation" typically obligates one to work with *"sets of very large cardinality"*, many of which must be constructed by means of **set-theoretic exponentiation** [i.e., such as the operation of passing from a set E to the power set "2^E" of all subsets of E]. Such a rapid outbreak of *"unwieldy large sets"* is reminiscent of the *rapid growth*, in the p-adic crystalline theory, of the p-adic valuations of the *denominators* that occur when one *formally integrates an* **arbitrary connection**, as opposed to a **"canonical connection"** of the sort that arises from a crystalline representation. In the p-adic theory, such "canonical connections" are typically related to **"canonical liftings"**, such as, for instance, those that occur in ***p*-adic Teichmüller theory** [cf. [pOrd], [pTeich]]. In this context, it is of interest to recall that the canonical liftings of p-adic Teichmüller theory may, under certain conditions, be thought of as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree** — cf. the computations of [Finot].

Remark 3.6.3.

(i) In the context of Remark 3.6.2, it is useful to recall the *fundamental reason* for the need to pursue **"inter-universality"** in the present series of papers [cf. the discussion of [IUTchIII], Remark 1.2.4; [IUTchIII], Remark 1.4.2], namely,

> since *étale fundamental groups* — i.e., in essence, **Galois groups** — are defined as certain *automorphism groups of fields/rings*, the definition of such a Galois group as a certain automorphism group of some ring structure is **fundamentally incompatible** with the **vertical** and **horizontal** arrows of the **log-theta-lattice** [i.e., which do *not* arise from ring homomorphisms]!

In this respect, "transformations" such as the vertical and horizontal arrows of the log-theta-lattice *differ*, quite fundamentally, from "transformations" that are *compatible with the ring structures* on the domain and codomain, i.e., **morphisms of rings/schemes**, which **tautologically** give rise to **functorial morphisms** between the respective étale fundamental groups. Put another way, in the notation of [IUTchI], Definition 3.1, (e), (f) [which will be applied throughout the remainder of the present Remark 3.6.3], for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$,

> the only *natural correspondence* that may be described by means of **set-theoretic formulas** between the isomorphs of the local base field Galois groups "$G_{\underline{v}}$" on either side of a vertical or horizontal arrow of the log-theta-lattice is the correspondence constituted by an **indeterminate isomorphism of topological groups**.

A similar statement may be made concerning the isomorphs of the *geometric fundamental group* $\Delta_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$ on either side of a *vertical* [but *not horizontal*! — cf. the discussion of (ii) below] arrow of the log-theta-lattice — that is to say,

> the only *natural correspondence* that may described by means of **set-theoretic formulas** between these isomorphs is the correspondence constituted by an **indeterminate isomorphism of topological groups** *equipped with some* **outer action** *by the respective isomorph of "$G_{\underline{v}}$"*

— cf. the discussion of [IUTchIII], Remark 1.2.4. Here, again we recall from the discussion of Remark 3.6.2, (i), (ii), that it is only by working with such correspondences that may be described by means of *set-theoretic formulas* that one may obtain descriptions that allow one to **calculate** the operations performed in *one universe* from the point of view of an **alien universe**.

(ii) One *fundamental difference* between the *vertical* and *horizontal* arrows of the log-theta-lattice is that whereas, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

(V1) one **identifies**, up to isomorphism, the isomorphs of the *full* arithmetic fundamental group "$\Pi_{\underline{v}}$" on either side of a **vertical** arrow,

(H1) one **distinguishes** the "$\Delta_{\underline{v}}$'s" on either side of a **horizontal** arrow, i.e., one *only identifies*, up to isomorphism, the local base field Galois groups "$G_{\underline{v}}$" on either side of a horizontal arrow.

— cf. the discussion of [IUTchIII], Remark 1.4.2. One way to understand the fundamental reason for this difference is as follows.

(V2) In order to construct the $\mathfrak{log}$**-link** — i.e., at a more concrete level, the power series that defines the $p_{\underline{v}}$-adic logarithm at $\underline{v}$ — it is necessary to avail oneself of the local **ring structures** at $\underline{v}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full* "$\Pi_{\underline{v}}$" [i.e., not from "$G_{\underline{v}}$ stripped of its structure as a *quotient* of $\Pi_{\underline{v}}$" — cf. the discussion of [IUTchIII], Remark 1.4.1, (i); [IUTchIII], Remark 2.1.1, (ii); [AbsTopIII], §I3].

(H2) In order to construct the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$**-links** — i.e., at a more concrete level, the correspondence

$$\underline{\underline{q}} \;\mapsto\; \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$$

[cf. [IUTchII], Remark 4.11.1] — it is necessary, in effect, to construct an *"isomorphism"* between a mathematical object [i.e., the *theta values* "$\underline{\underline{q}}^{j^2}$"] that **depends**, in an essential way, on regarding the various "j" as **distinct labels** [which are constructed from "$\Delta_{\underline{v}}$"!] and a mathematical object [i.e., "$\underline{\underline{q}}$"] that is **independent** *of these labels*; it is then a **tautology** that such an "isomorphism" may only be achieved if the labels — i.e., in essence, "$\Delta_{\underline{v}}$" — on either side of the "isomorphism" are kept **distinct** from one another.

Here, we observe in passing that the "apparently *horizontal* arrow-related" issue discussed in (H2) of **simultaneous realization** of **"label-dependent"** and **"label-free"** mathematical objects is reminiscent of the *vertical* arrow portion of the **bicoricity** theory of [IUTchIII], Theorem 1.5 — cf. the discussion of [IUTchIII], Remark 1.5.1, (i), (ii); Step (vii) of the proof of [IUTchIII], Corollary 3.12.

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