The present paper forms the fourth and final paper in a series of papers concerning “inter-universal Teichmüller theory”. In the first three papers of the series, we introduced and studied the theory surrounding the log-theta-lattice, a highly non-commutative two-dimensional diagram of “miniature models of conventional scheme theory”, called $\Theta^{\pm,\mathrm{ell}}_{\mathrm{NF}}$-Hodge theaters, that were associated, in the first paper of the series, to certain data, called initial $\Theta$-data. This data includes an elliptic curve $E_F$ over a number field $F$, together with a prime number $l \geq 5$. Consideration of various properties of the log-theta-lattice led naturally to the establishment, in the third paper of the series, of multiradial algorithms for constructing “splitting monoids of LGP-monoids”. Here, we recall that “multiradial algorithms” are algorithms that make sense from the point of view of an “alien arithmetic holomorphic structure”, i.e., the ring/scheme structure of a $\Theta^{\pm,\mathrm{ell}}_{\mathrm{NF}}$-Hodge theater related to a given $\Theta^{\pm,\mathrm{ell}}_{\mathrm{NF}}$-Hodge theater by means of a non-ring/scheme-theoretic horizontal arrow of the log-theta-lattice. In the present paper, estimates arising from these multiradial algorithms for splitting monoids of LGP-monoids are applied to verify various diophantine results which imply, for instance, the so-called Vojta Conjecture for hyperbolic curves, the ABC Conjecture, and the Szpiro Conjecture for elliptic curves. Finally, we examine — albeit from an extremely naive/non-expert point of view! — the foundational/set-theoretic issues surrounding the vertical and horizontal arrows of the log-theta-lattice by introducing and studying the basic properties of the notion of a “species”, which may be thought of as a sort of formalization, via set-theoretic formulas, of the intuitive notion of a “type of mathematical object”. These foundational issues are closely related to the central role played in the present series of papers by various results from absolute anabelian geometry, as well as to the idea of gluing together distinct models of conventional scheme theory, i.e., in a fashion that lies outside the framework of conventional scheme theory. Moreover, it is precisely these foundational issues surrounding the vertical and horizontal arrows of the log-theta-lattice that led naturally to the introduction of the term “inter-universal”.

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§3. Inter-universal Formalism: the Language of Species
The present paper forms the fourth and final paper in a series of papers concerning “inter-universal Teichmüller theory”. In the first three papers, [IUTchI], [IUTchII], and [IUTchIII], of the series, we introduced and studied the theory surrounding the log-theta-lattice [cf. the discussion of [IUTchIII], Introduction], a highly non-commutative two-dimensional diagram of “miniature models of conventional scheme theory”, called $\Theta^{\pm}\text{ell}\text{NF-Hodge theaters}$, that were associated, in the first paper [IUTchI] of the series, to certain data, called initial $\Theta$-data. This data includes an elliptic curve $E_F$ over a number field $F$, together with a prime number $l \geq 5$ [cf. [IUTchI], §I]. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of multiradial algorithms for constructing “splitting monoids of LGP-monoids” [cf. [IUTchIII], Theorem A]. Here, we recall that “multiradial algorithms” [cf. the discussion of the Introductions to [IUTchII], [IUTchIII]] are algorithms that make sense from the point of view of an “alien arithmetic holomorphic structure”, i.e., the ring/scheme structure of a $\Theta^{\pm}\text{ell}\text{NF-Hodge theater}$ related to a given $\Theta^{\pm}\text{ell}\text{NF-Hodge theater}$ by means of a non-ring/scheme-theoretic horizontal arrow of the log-theta-lattice. In the final portion of [IUTchIII], by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtained estimates for the log-volume of these LGP-monoids [cf. [IUTchIII], Theorem B]. In the present paper, these estimates will be applied to verify various diophantine results.

In §1 of the present paper, we start by discussing various elementary estimates for the log-volume of various tensor products of the modules obtained by applying the $p$-adic logarithm to the local units — i.e., in the terminology of [IUTchIII], “tensor packets of log-shells” [cf. the discussion of [IUTchIII], Introduction] — in terms of various well-known invariants, such as differents, associated to a mixed-characteristic nonarchimedean local field [cf. Propositions 1.1, 1.2, 1.3, 1.4]. We then discuss similar — but technically much simpler! — log-volume estimates in the case of complex archimedean local fields [cf. Proposition 1.5]. After reviewing a certain classical estimate concerning the distribution of prime numbers [cf. Proposition 1.6], as well as some elementary general nonsense concerning weighted averages [cf. Proposition 1.7] and well-known elementary facts concerning elliptic curves [cf. Proposition 1.8], we then proceed to compute explicitly, in more elementary language, the quantity that was estimated in [IUTchIII], Theorem B. These computations yield a quite strong/explicit diophantine inequality [cf. Theorem 1.10] concerning elliptic curves that are in “sufficiently general position”, so that one may apply the general theory developed in the first three papers of the series.

In §2 of the present paper, after reviewing another classical estimate concerning the distribution of prime numbers [cf. Proposition 2.1, (ii)], we then proceed to apply the theory of [GenEll] to reduce various diophantine results concerning an arbitrary elliptic curve over a number field to results of the type obtained in Theorem 1.10 concerning elliptic curves that are in “sufficiently general position” [cf. Corollary 2.2]. This reduction allows us to derive the following result [cf. Corollary 2.3], which constitutes the main application of the “inter-universal Teichmüller theory” developed in the present series of papers.
Theorem A. (Diophantine Inequalities) Let $X$ be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \overset{\text{def}}{=} X \setminus D$; $d$ a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write $\omega_X$ for the canonical sheaf on $X$. Suppose that $U_X$ is a hyperbolic curve, i.e., that the degree of the line bundle $\omega_X(D)$ is positive. Then, relative to the notation of [GenEll] [reviewed in the discussion preceding Corollary 2.2 of the present paper], one has an inequality of “bounded discrepancy classes”

$$\text{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \text{ht}_{\omega_X(D)}$ is bounded below by a constant on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), of the present paper].

Thus, Theorem A asserts an inequality concerning the canonical height [i.e., “$\text{ht}_{\omega_X(D)}$”], the logarithmic different [i.e., “$\text{log-diff}_X$”], and the logarithmic conductor [i.e., “$\text{log-cond}_D$”] of points of the curve $U_X$ valued in number fields whose extension degree over $\mathbb{Q}$ is $\leq d$. In particular, the so-called Vojta Conjecture for hyperbolic curves, the ABC Conjecture, and the Szpiro Conjecture for elliptic curves all follow as special cases of Theorem A. We refer to [Vjt] for a detailed exposition of these conjectures.

Finally, in §3, we examine — albeit from an extremely naive/non-expert point of view! — certain foundational issues underlying the theory of the present series of papers. Typically in mathematical discussions [i.e., by mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] — such as, for instance, the theory developed in the present series of papers! — one defines various “types of mathematical objects” [i.e., such as groups, topological spaces, or schemes], together with a notion of “morphisms” between two particular examples of a specific type of mathematical object [i.e., morphisms between groups, between topological spaces, or between schemes]. Such objects and morphisms [typically] determine a category. On the other hand, if one restricts one’s attention to such a category, then one must keep in mind the fact that the structure of the category — i.e., which consists only of a collection of objects and morphisms satisfying certain properties! — does not include any mention of the various sets and conditions satisfied by those sets that give rise to the “type of mathematical object” under consideration. For instance, the data consisting of the underlying set of a group, the group multiplication law on the group, and the properties satisfied by this group multiplication law cannot be recovered [at least in an a priori sense!] from the structure of the “category of groups”. Put another way, although the notion of a “type of mathematical object” may give rise to a “category of such objects”, the notion of a “type of mathematical object” is much stronger — in the sense that it involves much more mathematical structure — than the notion of a category. Indeed, a given “type of mathematical object” may have a very complicated internal structure, but may give rise to a category equivalent to a one-morphism category [i.e., a category with precisely one morphism]; in particular, in such cases, the structure of the associated category does not retain any information of interest concerning the internal structure of the “type of mathematical object” under consideration.
In Definition 3.1, (iii), we formalize this intuitive notion of a “type of mathematical object” by defining the notion of a species as, roughly speaking, a collection of set-theoretic formulas that gives rise to a category in any given model of set theory [cf. Definition 3.1, (iv)], but, unlike any specific category [e.g., of groups, etc.] is not confined to any specific model of set theory. In a similar vein, by working with collections of set-theoretic formulas, one may define a species-theoretic analogue of the notion of a functor, which we refer to as a mutation [cf. Definition 3.3, (i)]. Given a diagram of mutations, one may then define the notion of a “mutation that extracts, from the diagram, a certain portion of the types of mathematical objects that appear in the diagram that is invariant with respect to the mutations in the diagram”; we refer to such a mutation as a core [cf. Definition 3.3, (v)].

One fundamental example, in the context of the present series of papers, of a diagram of mutations is the usual set-up of [absolute] anabelian geometry [cf. Example 3.5 for more details]. That is to say, one begins with the species constituted by schemes satisfying certain conditions. One then considers the mutation

\[ X \rightsquigarrow \Pi_X \]

that associates to such a scheme \( X \) its étale fundamental group \( \Pi_X \) [say, considered up to inner automorphisms]. Here, it is important to note that the codomain of this mutation is the species constituted by topological groups [say, considered up to inner automorphisms] that satisfy certain conditions which do not include any information concerning how the group is related [for instance, via some sort of étale fundamental group mutation] to a scheme. The notion of an anabelian reconstruction algorithm may then be formalized as a mutation that forms a “mutation quasi-inverse” to the fundamental group mutation.

Another fundamental example, in the context of the present series of papers, of a diagram of mutations arises from the Frobenius morphism in positive characteristic scheme theory [cf. Example 3.6 for more details]. That is to say, one fixes a prime number \( p \) and considers the species constituted by reduced schemes of characteristic \( p \). One then considers the mutation that associates

\[ S \rightsquigarrow S^{(p)} \]

to such a scheme \( S \) the scheme \( S^{(p)} \) with the same topological space, but whose regular functions are given by the \( p \)-th powers of the regular functions on the original scheme. Thus, the domain and codomain of this mutation are given by the same species. One may also consider a log scheme version of this example, which, at the level of monoids, corresponds, in essence, to assigning

\[ M \rightsquigarrow p \cdot M \]

to a torsion-free abelian monoid \( M \) the submonoid \( p \cdot M \subseteq M \) determined by the image of multiplication by \( p \). Returning to the case of schemes, one may then observe that the well-known constructions of the perfection and the étale site

\[ S \rightsquigarrow S^{\text{perf}}; \quad S \rightsquigarrow S_{\text{ét}} \]

associated to a reduced scheme \( S \) of characteristic \( p \) give rise to cores of the diagram obtained by considering iterates of the “Frobenius mutation” just discussed.
This last example of the Frobenius mutation and the associated core constituted by the étale site is of particular importance in the context of the present series of papers in that it forms the “intuitive prototype” that underlies the theory of the vertical and horizontal lines of the log-theta-lattice [cf. the discussion of Remark 3.6.1, (i)]. One notable aspect of this example is the [evident!] fact that the domain and codomain of the Frobenius mutation are given by the same species. That is to say, despite the fact that in the construction of the scheme $S^{(p)}$ [cf. the notation of the preceding paragraph] from the scheme $S$, the scheme $S^{(p)}$ is “subordinate” to the scheme $S$, the domain and codomain species of the resulting Frobenius mutation coincide, hence, in particular, are on a par with one another. This sort of situation served, for the author, as a sort of model for the log- and $\Theta^{\pm\ell}_{\text{LGP}}$-links of the log-theta-lattice, which may be formulated as mutations between the species constituted by the notion of a $\Theta^{\pm\ell}_{\text{NF-Hodge theater}}$. That is to say, although in the construction of either the log- or the $\Theta^{\pm\ell}_{\text{LGP}}$-link, the domain and codomain $\Theta^{\pm\ell}_{\text{NF-Hodge theaters}}$ are by no means on a “par” with one another, the domain and codomain $\Theta^{\pm\ell}_{\text{NF-Hodge theaters}}$ of the resulting log-/$\Theta^{\pm\ell}_{\text{LGP}}$-links are regarded as objects of the same species, hence, in particular, completely on a par with one another. This sort of “relativization” of distinct models of conventional scheme theory over $\mathbb{Z}$ via the notion of a $\Theta^{\pm\ell}_{\text{NF-Hodge theater}}$ [cf. Fig. I.1 below; the discussion of “gluing together” such models of conventional scheme theory in [IUTchI], §12] is one of the most characteristic features of the theory developed in the present series of papers and, in particular, lies [tautologically!] outside the framework of conventional scheme theory over $\mathbb{Z}$. That is to say, in the framework of conventional scheme theory over $\mathbb{Z}$, if one starts out with schemes over $\mathbb{Z}$ and constructs from them, say, by means of geometric objects such as the theta function on a Tate curve, some sort of Frobenioid that is isomorphic to a Frobenioid associated to $\mathbb{Z}$, then — unlike, for instance, the case of the Frobenius morphism in positive characteristic scheme theory — there is no way, within the framework of conventional scheme theory, to treat the newly constructed Frobenioid “as if it is the Frobenioid associated to $\mathbb{Z}$, relative to some new version/model of conventional scheme theory”.

![Fig. I.1: Relativized models of conventional scheme theory over \(\mathbb{Z}\)](image)

If, moreover, one thinks of $\mathbb{Z}$ as being constructed, in the usual way, via axiomatic set theory, then one may interpret the “absolute” — i.e., “tautologically
unrelativizable” — nature of conventional scheme theory over \( \mathbb{Z} \) at a purely set-theoretic level. Indeed, from the point of view of the “\( \in \)-structure” of axiomatic set theory, there is no way to treat sets constructed at distinct levels of this \( \in \)-structure as being on a par with one another. On the other hand, if one focuses not on the level of the \( \in \)-structure to which a set belongs, but rather on species, then the notion of a species allows one to relate — i.e., to treat on a par with one another — objects belonging to the species that arise from sets constructed at distinct levels of the \( \in \)-structure. That is to say,

the notion of a **species** allows one to **simulate \( \in \)-loops** without violating the axiom of foundation of axiomatic set theory

— cf. the discussion of Remark 3.3.1, (i).

As one constructs sets at higher and higher levels of the \( \in \)-structure of some model of axiomatic set theory — e.g., as one travels along vertical or horizontal lines of the log-theta-lattice! — one typically encounters new schemes, which give rise to new Galois categories, hence to new Galois or étale fundamental groups, which may only be constructed if one allows oneself to consider new basepoints, relative to new universes. In particular, one must continue to extend the universe, i.e., to modify the model of set theory, relative to which one works. Here, we recall in passing that such “extensions of universe” are possible on account of an **existence axiom** concerning universes, which is apparently attributed to the “Grothendieck school” and, moreover, cannot, apparently, be obtained as a consequence of the conventional ZFC axioms of axiomatic set theory [cf. the discussion at the beginning of §3 for more details]. On the other hand, ultimately in the present series of papers [cf. the discussion of [IUTchIII], Introduction], we wish to obtain **algorithms** for constructing various objects that arise in the context of the new schemes/universes discussed above — i.e., at distant \( \Theta^{\pm \text{ell}} \)-Hodge theaters of the log-theta-lattice — that make sense from the point of view of the original schemes/universes that occurred at the outset of the discussion. Again, the fundamental tool that makes this possible, i.e., that allows one to express constructions in the new universes in terms that makes sense in the original universe is precisely

the **species-theoretic formulation** — i.e., the formulation via set-theoretic formulas that do not depend on particular choices invoked in particular universes — of the constructions of interest

— cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5, 3.6.2, 3.6.3. This is the point of view that gave rise to the term “**inter-universal**”. At a more concrete level, this “inter-universal” contact between constructions in distant models of conventional scheme theory in the log-theta-lattice is realized by considering [the étale-like structures given by] the various Galois or étale fundamental groups that occur as [the “type of mathematical object”, i.e., species constituted by] **abstract topological groups** [cf. the discussion of Remark 3.6.3, (i); [IUTchI], §I3]. These abstract topological groups give rise to vertical or horizontal cores of the log-theta-lattice. Moreover, once one obtains cores that are sufficiently “non-degenerate”, or “rich in structure”, so as to serve as containers for the non-coric portions of the various mutations [e.g., vertical and horizontal arrows of the log-theta-lattice] under consideration, then one may construct the desired algorithms,
or descriptions, of these non-coric portions in terms of coric containers, up to certain relatively mild indeterminacies [i.e., which reflect the non-coric nature of these non-coric portions!] — cf. the illustration of this sort of situation given in Fig. 1.2 below; Remark 3.3.1, (iii); Remark 3.6.1, (ii). In the context of the log-theta-lattice, this is precisely the sort of situation that was achieved in [IUTchIII], Theorem A [cf. the discussion of [IUTchIII], Introduction].

\[ \vdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \vdots \]

Fig. 1.2: A coric container underlying a sequence of mutations

In the context of the above discussion of set-theoretic aspects of the theory developed in the present series of papers, it is of interest to note the following observation, relative to the analogy between the theory of the present series of papers and \textit{p}-adic Teichmüller theory [cf. the discussion of [IUTchI], §I4]. If, instead of working species-theoretically, one attempts to document all of the possible choices that occur in various newly introduced universes that occur in a construction, then one finds that one is obliged to work with sets, such as sets obtained via set-theoretic exponentiation, of very large cardinality. Such sets of large cardinality are reminiscent of the exponentially large denominators that occur if one attempts to \textit{p-}adically formally integrate an arbitrary connection as opposed to a canonical crystalline connection of the sort that occurs in the context of the canonical liftings of \textit{p}-adic Teichmüller theory [cf. the discussion of Remark 3.6.2, (iii)]. In this context, it is of interest to recall the computations of [Finot], which assert, roughly speaking, that the canonical liftings of \textit{p}-adic Teichmüller theory may, in certain cases, be characterized as liftings “of minimal complexity” in the sense that their Witt vector coordinates are given by polynomials of minimal degree.

Finally, we observe that although, in the above discussion, we concentrated on the similarities, from an “inter-universal” point of view, between the vertical and horizontal arrows of the log-theta-lattice, there is one important difference between these vertical and horizontal arrows: namely,

- whereas the copies of the full arithmetic fundamental group — i.e., in particular, the copies of the geometric fundamental group — on either side of a vertical arrow are identified with one another,

- in the case of a horizontal arrow, only the Galois groups of the local base fields on either side of the arrow are identified with one another

— cf. the discussion of Remark 3.6.3, (ii). One way to understand the reason for this difference is as follows. In the case of the vertical arrows — i.e., the log-links, which, in essence, amount to the various local \textit{p-}adic logarithms — in order
to *construct* the log-link, it is necessary to make use, in an essential way, of the **local ring structures** at \( v \in V \) [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full arithmetic fundamental group*. By contrast, in order to construct the horizontal arrows — i.e., the \( \Theta^{\times \mu}_{\text{LGP}} \)-links — this local ring structure is *unnecessary*. On the other hand, in order to construct the horizontal arrows, it is necessary to work with structures that, up to isomorphism, are common to both the domain and the codomain of the arrow. Since the construction of the domain of the \( \Theta^{\times \mu}_{\text{LGP}} \)-link *depends*, in an essential way, on the *Gaussian monoids*, i.e., on the labels \( \in \mathbb{F}_l^* \) for the *theta values*, which are constructed from the *geometric fundamental group*, while the codomain only involves monoids arising from the local \( q \)-parameters \( \frac{q_v}{q} \) [for \( v \in V^{\text{bad}} \)], which are constructed in a fashion that is *independent* of these labels, in order to obtain an isomorphism between structures arising from the domain and codomain, it is necessary to restrict one’s attention to the *Galois groups of the local base fields*, which are free of any dependence on these labels.

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**Notations and Conventions:**

We shall continue to use the “Notations and Conventions” of [IUTchI], §0.
Section 1: Log-volume Estimates

In the present §1, we perform various elementary local computations concerning nonarchimedean and archimedean local fields which allow us to obtain more explicit versions [cf. Theorem 1.10 below] of the log-volume estimates for \( \Theta \)-pilot objects obtained in [IUTchIII], Corollary 3.12.

In the following, if \( \lambda \in \mathbb{R} \), then we shall write \( \lceil \lambda \rceil \) (respectively, \( \lfloor \lambda \rfloor \)) for the smallest (respectively, largest) \( n \in \mathbb{Z} \) such that \( n \geq \lambda \) (respectively, \( n \leq \lambda \)). Also, we shall write “log(−)” for the natural logarithm of a positive real number.

Proposition 1.1. (Multiple Tensor Products and Differents) Let \( p \) be a prime number, \( I \) a finite set of cardinality \( \geq 2 \), \( \overline{\mathbb{Q}}_p \) an algebraic closure of \( \mathbb{Q}_p \). Write \( R \subseteq \overline{\mathbb{Q}}_p \) for the ring of integers of \( \mathbb{Q}_p \) and \( \text{ord} : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q} \) for the natural \( p \)-adic valuation on \( \mathbb{Q}_p \), normalized so that \( \text{ord}(p) = 1 \). For \( i \in I \), let \( k_i \subseteq \overline{\mathbb{Q}}_p \) be a finite extension of \( \mathbb{Q}_p \); write \( R_i \) def \( = \mathfrak{O}_{k_i} = R \cap k_i \) for the ring of integers of \( k_i \) and \( \mathfrak{d}_i \in \mathbb{Q}_{\geq 0} \) for the order \([\text{ord}(−)]\) of any generator of the different ideal of \( R_i \) over \( \mathbb{Z}_p \). Also, for any nonempty subset \( E \subseteq I \), let us write

\[
R_E \overset{\text{def}}{=} \bigotimes_{i \in E} R_i; \quad \mathfrak{d}_E \overset{\text{def}}{=} \sum_{i \in E} \mathfrak{d}_i
\]

— where the tensor product is over \( \mathbb{Z}_p \). Fix an element \( * \in I \); write \( I^* \overset{\text{def}}{=} I \setminus \{*\} \).

Then

\[
R_I \subseteq (R_I)^\sim; \quad \ p^{[\mathfrak{d}_I]} \cdot (R_I)^\sim \subseteq R_I
\]

— where we write “(−)^\sim” for the normalization of the [reduced] ring in parentheses in its ring of fractions.

Proof. Let us regard \( R_I \) as an \( R_* \)-algebra in the evident fashion. It is immediate from the definitions that \( R_I \subseteq (R_I)^\sim \). Now observe that

\[
\overline{R} \otimes_{R_*} R_I \subseteq \overline{R} \otimes_{R_*} (R_I)^\sim \subseteq (\overline{R} \otimes_{R_*} R_I)^\sim
\]

— where \((\overline{R} \otimes_{R_*} R_I)^\sim\) decomposes as a direct sum of finitely many copies of \( \overline{R} \). In particular, one verifies immediately, in light of the fact the \( \overline{R} \) is faithfully flat over \( R_* \), that to complete the proof of Proposition 1.1, it suffices to verify that

\[
P^{[\mathfrak{d}_I]} \cdot (\overline{R} \otimes_{R_*} R_I)^\sim \subseteq \overline{R} \otimes_{R_*} R_I
\]

— or, indeed, that

\[
P^{[\mathfrak{d}_I]} \cdot (\overline{R} \otimes_{R_*} R_I)^\sim \subseteq \overline{R} \otimes_{R_*} R_I
\]
— where, for \( \lambda \in \mathbb{Q} \), we write \( p^\lambda \) for any element of \( \overline{\mathbb{Q}_p} \) such that \( \text{ord}(p^\lambda) = \lambda \).

On other hand, it follows immediately from induction on the cardinality of \( I \) that to verify this last inclusion, it suffices to verify the inclusion in the case where \( I \) is of cardinality two. But in this case, the desired inclusion follows immediately from the definition of the different ideal. This completes the proof of Proposition 1.1. \( \Box \)

**Proposition 1.2. (Differents and Logarithms)** We continue to use the notation of Proposition 1.1. For \( i \in I \), write \( e_i \) for the ramification index of \( k_i \) over \( \mathbb{Q}_p \):

\[
a_i \overset{\text{def}}{=} \frac{1}{e_i} \cdot \left\lfloor \frac{e_i}{p-2} \right\rfloor \quad \text{if } p > 2, \quad a_i \overset{\text{def}}{=} 2 \quad \text{if } p = 2; \quad b_i \overset{\text{def}}{=} \left\lfloor \frac{\log(p \cdot e_i/(p-1))}{\log(p)} \right\rfloor - \frac{1}{e_i}.
\]

Thus,

\[
\text{if } p > 2 \text{ and } e_i \leq p - 2, \text{ then } a_i = \frac{1}{e_i} = -b_i.
\]

For any nonempty subset \( E \subseteq I \), let us write

\[
\log_p \left( R_E^\times \right) \overset{\text{def}}{=} \bigotimes_{i \in E} \log_p \left( R_i^\times \right); \quad a_E \overset{\text{def}}{=} \sum_{i \in E} a_i; \quad b_E \overset{\text{def}}{=} \sum_{i \in E} b_i
\]

— where the tensor product is over \( \mathbb{Z}_p \); we write \( \log_p(\mathbf{Z}) \) for the \( p \)-adic logarithm.

For \( \lambda \in \frac{1}{e_i} \cdot \mathbb{Z} \), we shall write \( p^\lambda \cdot R_i \) for the fractional ideal of \( R_i \) generated by any element “\( p^\lambda \)” of \( k_i \) such that \( \text{ord}(p^\lambda) = \lambda \). Let

\[
\phi : \log_p \left( R_I^\times \right) \otimes \mathbb{Q}_p \xrightarrow{\sim} \log_p \left( R_I^\times \right) \otimes \mathbb{Q}_p
\]

be an automorphism of the finite dimensional \( \mathbb{Q}_p \)-vector space \( \log_p \left( R_I^\times \right) \otimes \mathbb{Q}_p \) that induces an automorphism of the submodule \( \log_p \left( R_I^\times \right) \). Then:

(i) We have:

\[
p^{a_i} \cdot R_i \subseteq \log_p(R_i^\times) \subseteq p^{-b_i} \cdot R_i
\]

— where the “\( \subseteq \)”’s are equalities when \( p > 2 \) and \( e_i \leq p - 2 \).

(ii) We have:

\[
\phi(p^\lambda \cdot R_i \otimes R_i \cdot (R_i)^\sim) \subseteq p^{[\lambda] - [\varpi_i] - [\alpha_i]} \cdot \log_p(R_i^\times)
\]

\[
\subseteq p^{[\lambda] - [\varpi_i] - [\alpha_i] - [b_i]} \cdot (R_i)^\sim
\]

for any \( \lambda \in \frac{1}{e_i} \cdot \mathbb{Z}, i \in I \). In particular, \( \phi((R_I)^\sim) \subseteq p^{-[\varpi_i] - [\alpha_i]} \cdot \log_p(R_I^\times) \subseteq p^{-[\varpi_i] - [\alpha_i] - [b_i]} \cdot (R_I)^\sim \).

(iii) Suppose that \( p > 2 \), and that \( e_i \leq p - 2 \) for all \( i \in I \). Then we have:

\[
\phi(p^\lambda \cdot R_i \otimes R_i \cdot (R_i)^\sim) \subseteq p^{[\lambda] - [\varpi_i] - 1} \cdot (R_i)^\sim
\]

for any \( \lambda \in \frac{1}{e_i} \cdot \mathbb{Z}, i \in I \). In particular, \( \phi((R_I)^\sim) \subseteq p^{-[\varpi_i] - 1} \cdot (R_I)^\sim \).
(iv) If \( p > 2 \) and \( e_i = 1 \) for all \( i \in I \), then \( \phi((R_I)\sim) \subseteq (R_I)\sim \).

**Proof.** Since \( a_i > \frac{1}{p-1} \), \( p^\frac{b_i}{e_i} > \frac{1}{p-1} \) [cf. the definition of “\([-\)”, “[\(-)!”], assertion (i) follows immediately from the well-known theory of the \( p \)-adic logarithm and exponential maps [cf., e.g., [Kobl], p. 81]. Next, let us observe that to verify assertions (ii) and (iii), it suffices to consider the case where \( \lambda = 0 \). Now it follows from the second displayed inclusion of Proposition 1.1 that

\[
p^{[\sigma_I]} \cdot (R_I)\sim \subseteq R_I = \bigotimes_{i \in I} R_i
\]

and hence that

\[
p^{[\sigma_I] + [a_I]} \cdot (R_I)\sim \subseteq \bigotimes_{i \in I} p^{\sigma_i} \cdot R_i \subseteq \bigotimes_{i \in I} \log_p(R_i) = \log_p(R_I) \subseteq p^{-b_I} \cdot (R_I)\sim
\]

— where, in the context of the first and third inclusions, we recall that \( (R_I)\sim \) decomposes as a *direct sum* of rings of integers of finite extensions of \( \mathbb{Q}_p \); in the context of the second and third inclusions, we apply assertion (i). Thus, assertion (ii) follows immediately from the fact that \( \phi \) induces an automorphism of the submodule \( \log_p(R_I) \). When \( p > 2 \) and \( e_i \leq p - 2 \) for all \( i \in I \), we thus obtain that

\[
p^{[\sigma_I] + [a_I]} \cdot \phi((R_I)\sim) \subseteq \bigotimes_{i \in I} p^{\sigma_i} \cdot R_i \subseteq p^{[a_I]} \cdot (R_I)\sim
\]

— where the equality follows from assertion (i), and the final inclusion follows immediately from the fact that \( (R_I)\sim \) decomposes as a *direct sum* of rings of integers of finite extensions of \( \mathbb{Q}_p \). Thus, assertions (iii) and (iv) follow immediately from the fact that \( [a_I] - [a_I] \geq -1 \), together with the fact that \( a_i = 1, \vartheta_i = 0 \) whenever \( p > 2, e_i = 1 \). This completes the proof of Proposition 1.2. \( \Box \)

**Proposition 1.3.** *(Estimates of Differents)* We continue to use the notation of Proposition 1.2. Suppose that \( k_0 \subseteq k_i \) is a subfield that contains \( \mathbb{Q}_p \). Write \( R_0 \overset{\text{def}}{=} O_{k_0} \) for the ring of integers of \( k_0 \), \( \vartheta_0 \) for the order [i.e., “ord(–)”] of any generator of the different ideal of \( R_0 \) over \( \mathbb{Z}_p \), \( e_0 \) for the ramification index of \( k_0 \) over \( \mathbb{Q}_p \), \( e_{i/0} \overset{\text{def}}{=} e_i/e_0 \) (\( \in \mathbb{Z} \)), \( [k_i : k_0] \) for the degree of the extension \( k_i/k_0 \), \( n_i \) for the unique nonnegative integer such that \( [k_i : k_0]/p^{n_i} \) is an integer prime to \( p \). Then:

(i) We have:

\[
\vartheta_i \geq \vartheta_0 + (e_{i/0} - 1)/(e_{i/0} \cdot e_0) = \vartheta_0 + (e_{i/0} - 1)/e_i
\]

— where the “\( \geq \)” is an **equality** when \( k_i \) is tamely ramified over \( k_0 \).

(ii) Suppose that \( k_i \) is a finite Galois extension of a subfield \( k_1 \subseteq k_i \) such that \( k_0 \subseteq k_1 \), and \( k_1 \) is tamely ramified over \( k_0 \). Then we have: \( \vartheta_i \leq \vartheta_0 + n_i + 1/e_0 \).
Proof. First, we consider assertion (i). By replacing $k_0$ by an unramified extension of $k_0$ contained in $k_i$, we may assume without loss of generality that $k_i$ is a totally ramified extension of $k_0$. Let $\pi_0$ be a uniformizer of $R_0$. Then there exists an isomorphism of $R_0$-algebras $R_0[x]/(f(x)) \cong R_i$, where $f(x) \in R_0[x]$ is a monic polynomial which is $\equiv x^{e_i/0} \pmod{\pi_0}$, that maps $x \mapsto \pi_i$ for some uniformizer $\pi_i$ of $R_i$. Thus, the different $\mathcal{D}_i$ may be computed as follows:

$$\mathcal{D}_i - \mathcal{D}_0 = \text{ord}(f'(\pi_i)) \geq \min(\text{ord}(\pi_0), \text{ord}(e_{i/0} \cdot \pi_i)),$$

$$\geq \min\left(\frac{1}{e_0}, \text{ord}(\pi_i^{e_i/0 - 1})\right) = \min\left(\frac{1}{e_0}, \frac{e_{i/0} - 1}{e_{i/0} \cdot e_0}\right) = \frac{e_{i/0} - 1}{e_i}$$

— where, for $\lambda, \mu \in \mathbb{R}$ such that $\lambda \geq \mu$, we define $\min(\lambda, \mu) \overset{\text{def}}{=} \mu$. When $k_i$ is tamely ramified over $k_0$, one verifies immediately that the inequalities of the above display are, in fact, equalities. This completes the proof of assertion (i).

Next, we consider assertion (ii). We apply induction on $n_i$. Since assertion (ii) follows immediately from assertion (i) when $n_i = 0$, we may assume that $n_i \geq 1$, and that assertion (ii) has been verified for smaller “$n_i$”. By replacing $k_1$ by some tamely ramified extension of $k_1$ contained in $k_i$, we may assume without loss of generality that $\text{Gal}(k_i/k_1)$ is a $p$-group. Since $p$-groups are solvable, it follows that there exists a subextension $k_1 \subseteq k_r \subseteq k_i$ such that $k_i/k_r$ and $k_r/k_1$ are Galois extensions of degree $p$ and $p^{n_i-1}$, respectively. Write $R_r \overset{\text{def}}{=} \mathcal{O}_{k_r}$ for the ring of integers of $k_r$, $\mathcal{D}_r$ for the order [i.e., “ord(−)”] of any generator of the different ideal of $R_r$ over $\mathbb{Z}_p$, and $e_r$ for the ramification index of $k_r$ over $\mathbb{Q}_p$. Thus, by the induction hypothesis, it follows that $\mathcal{D}_r \leq \mathcal{D}_0 + n_i - 1 + 1/e_0$. To verify that $\mathcal{D}_i \leq \mathcal{D}_0 + n_i + 1/e_0$, it suffices to verify that $\mathcal{D}_i \leq \mathcal{D}_0 + n_i + 1/e_0 + \epsilon$ for any positive real number $\epsilon$. Thus, let us fix a positive real number $\epsilon$. Then by possibly enlarging $k_i$ and $k_1$, we may also assume without loss of generality that the tamely ramified extension $k_1$ of $k_0$ contains a primitive $p$-th root of unity, and, moreover, that the ramification index $e_1$ of $k_1$ over $\mathbb{Q}_p$ satisfies $e_1 \geq p/\epsilon$ [so $e_r \geq e_1 \geq p/\epsilon$]. Thus, $k_i$ is a Kummer extension of $k_r$. In particular, there exists an inclusion of $R_r$-algebras $R_r[x]/(f(x)) \hookrightarrow R_i$, where $f(x) \in R_r[x]$ is a monic polynomial which is of the form $f(x) = x^p - \varpi$, for some element $\varpi$ of $R_i$, satisfying $0 \leq \text{ord}(\varpi) \leq \frac{p-1}{e_r}$. That maps $x \mapsto \varpi$ for some element $\varpi$ of $R_i$, satisfying $0 \leq \text{ord}(\varpi) \leq \frac{p-1}{p \cdot e_r}$. Now we compute:

$$\mathcal{D}_i \leq \text{ord}(f'(\varpi_i)) + \mathcal{D}_r \leq \text{ord}(p \cdot \varpi_i^{p-1}) + \mathcal{D}_0 + n_i - 1 + 1/e_0 = (p - 1) \cdot \text{ord}(\varpi) + \mathcal{D}_0 + n_i + 1/e_0 \leq \frac{(p - 1)^2}{p \cdot e_r} + \mathcal{D}_0 + n_i + 1/e_0 \leq \frac{p}{e_r} + \mathcal{D}_0 + n_i + 1/e_0 \leq \mathcal{D}_0 + n_i + 1/e_0 + \epsilon$$

— thus completing the proof of assertion (ii). ∎

Remark 1.3.1. Similar estimates to those discussed in Proposition 1.3 may be found in [Ih], Lemma A.

Proposition 1.4. (Nonarchimedean Normalized Log-volume Estimates)
We continue to use the notation of Proposition 1.2. Also, for $i \in I$, write $R_i^{\mu} \subseteq R_i^\times$
for the torsion subgroup of $R_i^\times$, $R_i^{\times \mu} \overset{\text{def}}{=} R_i^\times/R_i^\mu$, $p^{f_i}$ for the cardinality of the residue field of $k_i$, and $p^{m_i}$ for the order of the $p$-primary component of $R_i^\mu$. Thus, the order of $R_i^\mu$ is equal to $p^{m_i} \cdot (p^{f_i} - 1)$. Then:

(i) The log-volumes constructed in [AbsTopIII], Proposition 5.7, (i), on the various finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ may be suitably normalized [i.e., by dividing by the degree of the finite extension] so as to yield a notion of log-volume

$$\mu^{\log}(-)$$

defined on compact open subsets of finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, valued in $\mathbb{R}$, and normalized so that $\mu^{\log}(R_i) = 0$, $\mu^{\log}(p \cdot R_i) = -\log(p)$, for each $i \in I$. Moreover, by applying the fact that tensor products of finitely many finite extensions of $\mathbb{Q}_p$ over $\mathbb{Z}_p$ decompose, naturally, as direct sums of finitely many finite extensions of $\mathbb{Q}_p$, we obtain a notion of log-volume — which, by abuse of notation, we shall also denote by “$\mu^{\log}(-)$” — defined on compact open subsets of finitely generated $\mathbb{Z}_p$-submodules of such tensor products, valued in $\mathbb{R}$, and normalized so that $\mu^{\log}((R_E)^\sim) = 0$, $\mu^{\log}(p \cdot (R_E)^\sim) = -\log(p)$, for any nonempty set $E \subseteq I$.

(ii) We have:

$$\mu^{\log}(\log_p(R_i^\times)) = -\left(\frac{1}{e_i} + \frac{m_i}{e_i f_i}\right) \cdot \log(p)$$

[cf. [AbsTopIII], Proposition 5.8, (iii)].

(iii) Let $I^* \subseteq I$ be a subset such that for each $i \in I \setminus I^*$, it holds that $p - 2 \geq e_i$ ($\geq 1$). Then for any $\lambda \in \frac{1}{e_i^{1+}} \cdot \mathbb{Z}$, $i^{1+} \in I$, we have inclusions $\phi(p^{\lambda} \cdot R_i \otimes R_i) (R_i)^\sim \subseteq p^{[\lambda]-[\delta i]-[a_i]} \cdot \log_p(R_i^\times) \subseteq p^{[\lambda]-[\delta i]-[a_i]-[\beta i]} \cdot (R_i)^\sim$ and inequalities

$$\mu^{\log}(p^{[\lambda]-[\delta i]-[a_i]} \cdot \log_p(R_i^\times)) \leq \left(-\lambda + \delta_i + 3 + 4 \cdot |I^*|/p\right) \cdot \log(p);$$

$$\mu^{\log}(p^{[\lambda]-[\delta i]-[a_i]-[\beta i]} \cdot (R_i)^\sim) \leq \left(-\lambda + \delta_i + 4\right) \cdot \log(p) + \sum_{i \in I^*} \left\{3 + \log(e_i)\right\}$$

— where we write “$|(-)|$” for the cardinality of the set “$(-)$”. Moreover, $[\delta_i] + [a_i] \geq |I|$ if $p > 2$; $[\delta_i] + [a_i] \geq 2 \cdot |I|$ if $p = 2$.

(iv) If $p > 2$ and $e_i = 1$ for all $i \in I$, then $\phi((R_i)^\sim) \subseteq (R_i)^\sim$, and $\mu^{\log}((R_i)^\sim) = 0$.

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). We begin by observing that every compact open subset of $R_i^{\times \mu}$ may be covered by a finite collection of compact open subsets of $R_i^{\times \mu}$ that arise as images of compact open subsets of $R_i^\times$ that map injectively to $R_i^{\times \mu}$. In particular, by applying this observation, we conclude that the log-volume on $R_i^\times$ determines, in a natural way, a log-volume on the quotient $R_i^\times \to R_i^{\times \mu}$. Moreover, in light of the compatibility of the log-volume with “$\log_p(-)$” [cf. [AbsTopIII], Proposition 5.7, (i), (c)], it follows immediately that $\mu^{\log}(\log_p(R_i^\times)) = \mu^{\log}(R_i^{\times \mu})$. Thus, it
suffices to compute \( e_i \cdot f_i \cdot \mu^{\log(R_i^x)} = e_i \cdot f_i \cdot \mu^{\log(R_i^x)} - \log(p^{m_i} \cdot (p^{f_i} - 1)) \). On the other hand, it follows immediately from the basic properties of the log-volume \([\text{cf. AbsTopIII, Proposition 5.7, (i), (a)}]\) that \( e_i \cdot f_i \cdot \mu^{\log(R_i^x)} = \log(1 - p^{-f_i}) \), so \( e_i \cdot f_i \cdot \mu^{\log(R_i^x)} = -(f_i + m_i) \cdot \log(p) \), as desired. This completes the proof of assertion (ii).

The inclusions of assertion (iii) follow immediately from Proposition 1.2, (ii). When \( p = 2 \), the fact that \( \lceil d_i \rceil + \lceil a_i \rceil \geq 2 \cdot |I| \) follows immediately from the definition of “\( a_i \)” in Proposition 1.2. When \( p > 2 \), it follows immediately from the definition of “\( a_i \)” in Proposition 1.2 that \( a_i \geq 1/e_i \), for all \( i \in I \); thus, since \( d_i \geq (e_i - 1)/e_i \) for all \( i \in I \) \([\text{cf. Proposition 1.3, (i)}]\), we conclude that \( d_i + a_i \geq 1 \) for all \( i \in I \), and hence that \( \lceil d_i \rceil + \lceil a_i \rceil \geq d_i + a_i \geq |I| \), as asserted in the statement of assertion (iii). Next, let us observe that \( \frac{1}{p-2} \leq \frac{1}{p} \) for \( p \geq 3 \); \( \frac{p}{p-1} \leq 2 \) for \( p \geq 2 \); \( \frac{2}{p} \leq \frac{1}{\log(p)} \) for \( p \geq 2 \). Thus, it follows immediately from the definition of \( a_i, b_i \) in Proposition 1.2 that \( a_i - \frac{1}{e_i} \leq \frac{2}{\log(p)} \cdot (b_i + \frac{1}{e_i}) \cdot \log(p) \leq \log(2e_i) \leq 1 + \log(e_i) \) for \( i \in I \); \( a_i = \frac{1}{e_i} = -b_i \) for \( i \in I \setminus I^* \). On the other hand, by assertion (i), we have \( \mu^{\log(R_i)} \leq \mu^{\log((R_i)^\sim)} = 0 \); by assertion (ii), we have \( \mu^{\log(\log_p(R_i^x))} \leq -\frac{1}{e_i} \cdot \log(p) \).

Now we compute:

\[
\mu^{\log\left(p^{\lceil d_i \rceil - \lceil a_i \rceil} \cdot \log_p(R_i^x)\right)} \leq \left(-\lambda + d_i + a_i + 3\right) \cdot \log(p) + \mu^{\log(\log_p(R_i^x))} \\
\leq \left(-\lambda + d_i + a_i + 3\right) \cdot \log(p) \\
+ \left\{ \sum_{i \in I} \mu^{\log(\log_p(R_i^x))} \right\} + \mu^{\log(R_i)} \\
\leq \left\{ -\lambda + d_i + 3 + \sum_{i \in I} \left(a_i - \frac{1}{e_i}\right) \right\} \cdot \log(p) \\
\leq \left(-\lambda + d_i + 3 + 4 \cdot |I^*/p| \right) \cdot \log(p);
\]

\[
\mu^{\log\left(p^{\lceil d_i \rceil - \lceil a_i \rceil - \lceil b_i \rceil} \cdot (R_i^\sim)\right)} \leq \left(-\lambda + d_i + a_i + b_i + 4\right) \cdot \log(p) \\
\leq \left(-\lambda + d_i + 4\right) \cdot \log(p) + \sum_{i \in I^*} \left\{ 3 + \log(e_i) \right\}
\]

— thus completing the proof of assertion (iii). Assertion (iv) follows immediately from assertion (i) and Proposition 1.2, (iv). \( \square \)

**Proposition 1.5.** (Archimedean Metric Estimates) In the following, we shall regard the complex archimedean field \( \mathbb{C} \) as being equipped with its standard Hermitian metric, i.e., the metric determined by the complex norm. Let us refer to as the primitive automorphisms of \( \mathbb{C} \) the group of automorphisms \([\text{of order 8}]\) of the underlying metrized real vector space of \( \mathbb{C} \) generated by the operations of complex conjugation and multiplication by \( \pm 1 \) or \( \pm \sqrt{-1} \).

(i) (Direct Sum vs. Tensor Product Metrics) The metric on \( \mathbb{C} \) determines a tensor product metric on \( \mathbb{C} \otimes \mathbb{C} \), as well as a direct sum metric on \( \mathbb{C} \oplus \mathbb{C} \).
Then, relative to these metrics, any isomorphism of topological rings [i.e., arising from the Chinese remainder theorem]

\[ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \oplus \mathbb{C} \]

is compatible with these metrics, up to a factor of 2, i.e., the metric on the right-hand side corresponds to 2 times the metric on the left-hand side. [Thus, lengths differ by a factor of \( \sqrt{2} \).]

(ii) (Direct Sum vs. Tensor Product Automorphisms) Relative to the notation of (i), the direct sum decomposition \( \mathbb{C} \oplus \mathbb{C} \), together with its Hermitian metric, is preserved, relative to the displayed isomorphism of (i), by the automorphisms of \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \) induced by the various primitive automorphisms of the two copies of “\( \mathbb{C} \)” that appear in the tensor product \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \).

(iii) (Direct Sums and Tensor Products of Multiple Copies) Let \( I, V \) be nonempty finite sets, whose cardinalities we denote by \( |I|, |V| \), respectively. Write

\[ M \overset{\text{def}}{=} \bigoplus_{v \in V} \mathbb{C}_v \]

for the direct sum of copies \( \mathbb{C}_v \overset{\text{def}}{=} \mathbb{C} \) of \( \mathbb{C} \) labeled by \( v \in V \), which we regard as equipped with the direct sum metric, and

\[ M_I \overset{\text{def}}{=} \bigotimes_{i \in I} M_i \]

for the tensor product over \( \mathbb{R} \) of copies \( M_i \overset{\text{def}}{=} M \) of \( M \) labeled by \( i \in I \), which we regard as equipped with the tensor product metric [cf. the constructions of [IUTchIII], Proposition 3.2, (ii)]. Then the topological ring structure on each \( \mathbb{C}_v \) determines a topological ring structure on \( M_I \) with respect to which \( M_I \) admits a unique direct sum decomposition as a direct sum of

\[ 2^{|I|} - 1 \cdot |V|^{|I|} \]

copies of \( \mathbb{C} \) [cf. [IUTchIII], Proposition 3.1, (i)]. The direct sum metric on \( M_I \) — i.e., the metric determined by the natural metrics on these copies of \( \mathbb{C} \) — is equal to

\[ 2^{|I| - 1} \]

times the original tensor product metric on \( M_I \). Write

\[ B_I \subseteq M_I \]

for the “integral structure” [cf. the constructions of [IUTchIII], Proposition 3.1, (ii)] given by the direct product of the unit balls of the copies of \( \mathbb{C} \) that occur in the direct sum decomposition of \( M_I \). Then the tensor product metric on \( M_I \), the direct sum decomposition of \( M_I \), the direct sum metric on \( M_I \), and the integral structure \( B_I \subseteq M_I \) are preserved by the automorphisms of \( M_I \) induced by the various primitive automorphisms of the direct summands “\( \mathbb{C}_v \)” that appear in the factors “\( M_i \)” of the tensor product \( M_I \).
(iv) (Tensor Product of Vectors of a Given Length) Suppose that we are in the situation of (iii). Fix \( \lambda \in \mathbb{R}_{>0} \). Then

\[
M_I \ni \bigotimes_{i \in I} m_i \in \lambda^{|I|} \cdot B_I
\]

for any collection of elements \( \{m_i \in M_i\}_{i \in I} \) such that the component of \( m_i \) in each direct summand “\( C_v \)” of \( M_i \) is of length \( \lambda \).

Proof. Assertions (i) and (ii) are discussed in [IUTchIII], Remark 3.9.1, (ii), and may be verified by means of routine and elementary arguments. Assertion (iii) follows immediately from assertions (i) and (ii). Assertion (iv) follows immediately from the various definitions involved. \( \square \)

Proposition 1.6. (The Prime Number Theorem) If \( n \) is a positive integer, then let us write \( p_n \) for the \( n \)-th smallest prime number. [Thus, \( p_1 = 2, p_2 = 3, \) and so on.] Then there exists an integer \( n_0 \) such that it holds that

\[
n \leq \frac{4p_n}{3\log(p_n)}
\]

for all \( n \geq n_0 \). In particular, there exists a positive real number \( \eta_{\text{prm}} \) such that

\[
\sum_{p \leq \eta} 1 \leq \frac{4\eta}{3\log(\eta)}
\]

— where the sum ranges over the prime numbers \( p \leq \eta \) — for all positive real \( \eta \geq \eta_{\text{prm}} \).

Proof. Relative to our notation, the Prime Number Theorem [cf., e.g., [DmMn], §3.10] implies that

\[
\lim_{n \to \infty} \frac{n \cdot \log(p_n)}{p_n} = 1
\]

— i.e., in particular, that for some positive integer \( n_0 \), it holds that

\[
\frac{\log(p_n)}{p_n} \leq \frac{4}{3} \cdot \frac{1}{n}
\]

for all \( n \geq n_0 \). The final portion of Proposition 1.6 follows formally. \( \square \)

Proposition 1.7. (Weighted Averages) Let \( E \) be a nonempty finite set, \( n \) a positive integer. For \( e \in E \), let \( \lambda_e \in \mathbb{R}_{>0}, \beta_e \in \mathbb{R} \). Then, for any \( i = 1, \ldots, n \), we have:

\[
\frac{\sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} = \frac{\sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} = n \cdot \beta_{\text{avg}}
\]
where we write

$$\beta_{\text{avg}} \overset{\text{def}}{=} \beta_E / \lambda_E, \quad \beta_E \overset{\text{def}}{=} \sum_{e \in E} \beta_e \cdot \lambda_e, \quad \lambda_E \overset{\text{def}}{=} \sum_{e \in E} \lambda_e,$$

$$\beta_{\vec{e}} \overset{\text{def}}{=} \sum_{j=1}^n \beta_{e_j}, \quad \lambda_{\Pi \vec{e}} \overset{\text{def}}{=} \prod_{j=1}^n \lambda_{e_j}$$

for any n-tuple $\vec{e} = (e_1, \ldots, e_n) \in E^n$ of elements of $E$.

**Proof.** We begin by observing that

$$\lambda_E^n = \sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}; \quad \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e} \in E^n} \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}$$

for any $i = 1, \ldots, n$. Thus, summing over $i$, we obtain that

$$n \cdot \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}} = \sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}$$

and hence that

$$n \cdot \beta_{\text{avg}} = n \cdot \beta_E \cdot \lambda_E^{n-1} / \lambda_E^n = \left( \sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}} \right) \cdot \left( \sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}} \right)^{-1}$$

$$= \left( \sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}} \right) \cdot \left( \sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}} \right)^{-1}$$

as desired. $\Box$

**Remark 1.7.1.** In Theorem 1.10 below, we shall apply Proposition 1.7 to compute various packet-normalized log-volumes of the sort discussed in [IUTchIII], Proposition 3.9, (i) — i.e., log-volumes normalized by means of the normalized weights discussed in [IUTchIII], Remark 3.1.1, (ii). Here, we recall that the normalized weights discussed in [IUTchIII], Remark 3.1.1, (ii), were computed relative to the non-normalized log-volumes of [AbsTopIII], Proposition 5.8, (iii), (vi) [cf. the discussion of [IUTchIII], Remark 3.1.1, (ii); [IUTchI], Example 3.5, (iii)]. By contrast, in the discussion of the present §1, our computations are performed relative to normalized log-volumes as discussed in Proposition 1.4, (i). In particular, it follows that the weights $[K_v : (F_{\text{mod}})_{v}]^{-1}$, where $\mathbb{V} \ni v \ni \mathbb{V}_{\text{mod}}$, of the discussion of [IUTchIII], Remark 3.1.1, (ii), must be replaced — i.e., when one works with normalized log-volumes as in Proposition 1.4, (i) — by the weights

$$[K_v : Q_{vQ}] \cdot [K_v : (F_{\text{mod}})_{v}]^{-1} = [(F_{\text{mod}})_{v} : Q_{vQ}]$$

— where $\mathbb{V}_{\text{mod}} \ni v \ni \mathbb{V}_{Q}$. This means that the normalized weights of the final display of [IUTchIII], Remark 3.1.1, (ii), must be replaced, when one works with normalized log-volumes as in Proposition 1.4, (i), by the normalized weights

$$\left( \prod_{\alpha \in A} [(F_{\text{mod}})_{v_{\alpha}} : Q_{v_{\alpha}}] \right) \sum_{\{w_{\alpha}\} \ni A} \left( \prod_{\alpha \in A} [(F_{\text{mod}})_{w_{\alpha}} : Q_{v_{\alpha}}] \right)$$
where the sum is over all collections \( \{ w_\alpha \}_{\alpha \in A} \) of [not necessarily distinct!] elements \( w_\alpha \in V_{\text{mod}} \) lying over \( v \in Q \) and indexed by \( \alpha \in A \). Thus, in summary, when one works with normalized log-volumes as in Proposition 1.4, (i), the appropriate normalized weights are given by the expressions

\[
\sum_{\bar{e} \in \bar{E}} \lambda_\bar{e} \prod_{\bar{e}^\dagger} \lambda_{\bar{e}^\dagger}
\]

[where \( \bar{e}^\dagger \in E^n \)] that appear in Proposition 1.7. Here, one takes “\( E \)” to be the set of elements of \( \mathcal{V} \) lying over a fixed \( v \in Q \); one takes “\( n \)” to be the cardinality of \( A \), so that one can write \( A = \{ \alpha_1, \ldots, \alpha_n \} \) [where the \( \alpha_i \) are distinct]; if \( e \in E \) corresponds to \( v \in \mathcal{V}, v \in V_{\text{mod}} \), then one takes

\[
\lambda_e \overset{\text{def}}{=} [(F_{\text{mod}})_v : Q_v] \in \mathbb{R}_{>0}
\]

and “\( \beta_e \)” to be a normalized log-volume of some compact open subset of \( K_\Sigma \).

Before proceeding, we review some well-known elementary facts concerning elliptic curves. In the following, we shall write \( \mathcal{M}_{\text{ell}} \) for the moduli stack of elliptic curves over \( \mathbb{Z} \) and

\[
\mathcal{M}_{\text{ell}} \subseteq \overline{\mathcal{M}}_{\text{ell}}
\]

for the natural compactification of \( \mathcal{M}_{\text{ell}} \), i.e., the moduli stack of one-dimensional semi-abelian schemes over \( \mathbb{Z} \). Also, if \( R \) is a \( \mathbb{Z} \)-algebra, then we shall write \( (\mathcal{M}_{\text{ell}})_R \overset{\text{def}}{=} \mathcal{M}_{\text{ell}} \times_{\mathbb{Z}} R \), \( (\overline{\mathcal{M}}_{\text{ell}})_R \overset{\text{def}}{=} \overline{\mathcal{M}}_{\text{ell}} \times_{\mathbb{Z}} R \).

**Proposition 1.8. (Torsion Points of Elliptic Curves)** Let \( k \) be a perfect field, \( \overline{k} \) an algebraic closure of \( k \). Write \( G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k) \).

(i) (“Serre’s Criterion”) Let \( l \geq 3 \) be a prime number that is invertible in \( k \); suppose that \( \overline{k} = k \). Let \( A \) be an abelian variety over \( k \), equipped with a polarization \( \lambda \). Write \( A[l] \subseteq A(k) \) for the group of \( l \)-torsion points of \( A(k) \). Then the natural map

\[
\phi : \text{Aut}_k(A, \lambda) \rightarrow \text{Aut}(A[l])
\]

from the group of automorphisms of the polarized abelian variety \( (A, \lambda) \) over \( k \) to the group of automorphisms of the abelian group \( A[l] \) is injective.

(ii) Let \( E_{\overline{k}} \) be an elliptic curve over \( \overline{k} \) with origin \( \epsilon_E \in E(\overline{k}) \). For \( n \) a positive integer, write \( E_{\overline{k}}[n] \subseteq E_{\overline{k}}(\overline{k}) \) for the module of \( n \)-torsion points of \( E_{\overline{k}}(\overline{k}) \) and

\[
\text{Aut}_{\overline{k}}(E_{\overline{k}}) \subseteq \text{Aut}_k(E_{\overline{k}})
\]

for the respective groups of \( \epsilon_E \)-preserving automorphisms of the \( \overline{k} \)-scheme \( E_{\overline{k}} \) and the \( k \)-scheme \( E_{\overline{k}} \). Then we have a natural exact sequence

\[
1 \rightarrow \text{Aut}_{\overline{k}}(E_{\overline{k}}) \rightarrow \text{Aut}_k(E_{\overline{k}}) \rightarrow G_k
\]
— where the image \( G_E \subseteq G_k \) of the homomorphism \( \text{Aut}_k(E_k^\alpha) \to G_k \) is open — and a natural representation
\[
\rho_n : \text{Aut}_k(E_k^\alpha) \to \text{Aut}(E_k^\alpha[n])
\]
on the \( n \)-torsion points of \( E_k^\alpha \). The finite extension \( k_E \) of \( k \) determined by \( G_E \) is the minimal field of definition of \( E_k^\alpha \), i.e., the field generated over \( k \) by the \( j \)-invariant of \( E_k^\alpha \). Finally, if \( H \subseteq G_k \) is any closed subgroup, which corresponds to an extension \( k_H \) of \( k \), then the datum of a model of \( E_k^\alpha \) over \( k_H \) [i.e., descent data for \( E_k^\alpha \) from \( \overline{k} \) to \( k_H \)] is equivalent to the datum of a section of the homomorphism \( \text{Aut}_k(E_k^\alpha) \to G_k \) over \( H \). In particular, the homomorphism \( \text{Aut}_k(E_k^\alpha) \to G_k \) admits a section over \( G_k \).

(iii) In the situation of (ii), suppose further that \( \text{Aut}_{\overline{k}}(E_{\overline{k}}^\alpha) = \{ \pm 1 \} \). Then the representation \( \rho_2 \) factors through \( G_E \) and hence defines a natural representation \( G_E \to \text{Aut}(E_{\overline{k}}^\alpha[2]) \).

(iv) In the situation of (ii), suppose further that \( l \geq 3 \) is a prime number that is invertible in \( k \), and that \( E_k^\alpha \) descends to elliptic curves \( E_k' \) and \( E_k'' \) over \( k \), all of whose \( l \)-torsion points are rational over \( k \). Then \( E_k' \) is isomorphic to \( E_k'' \) over \( k \).

(v) In the situation of (iii), suppose further that \( k \) is a complete discrete valuation field with ring of integers \( \mathcal{O}_k \), that \( l \geq 3 \) is a prime number that is invertible in \( \mathcal{O}_k \), and that \( E_k^\alpha \) descends to an elliptic curve \( E_k \) over \( k \), all of whose \( l \)-torsion points are rational over \( k \). Then \( E_k \) has semi-stable reduction over \( \mathcal{O}_k \) [i.e., extends to a semi-abelian scheme over \( \mathcal{O}_k \)].

(vi) In the situation of (iii), suppose further that \( 2 \) is invertible in \( k \), that \( G_E = G_k \), and that the representation \( G_E \to \text{Aut}(E_{\overline{k}}^\alpha[2]) \) is trivial. Then \( E_k^\alpha \) descends to an elliptic curve \( E_k \) over \( k \) which is defined by means of the Legendre form of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below]. If, moreover, \( k \) is a complete discrete valuation field with ring of integers \( \mathcal{O}_k \) such that \( 2 \) is invertible in \( \mathcal{O}_k \), then \( E_k \) has semi-stable reduction over \( \mathcal{O}_k' \) [i.e., extends to a semi-abelian scheme over \( \mathcal{O}_k' \)] for some finite extension \( k' \subseteq \overline{k} \) of \( k \) such that \( [k' : k] \leq 2 \); if \( E_k \) has good reduction over \( \mathcal{O}_k' \) [i.e., extends to an abelian scheme over \( \mathcal{O}_k' \)], then one may in fact take \( k' \) to be \( k \).

(vii) In the situation of (ii), suppose further that \( k \) is a complete discrete valuation field with ring of integers \( \mathcal{O}_k \), that \( E_k^\alpha \) descends to an elliptic curve \( E_k \) over \( k \), and that \( n \) is invertible in \( \mathcal{O}_k \). If \( E_k \) has good reduction over \( \mathcal{O}_k \) [i.e., extends to an abelian scheme over \( \mathcal{O}_k \)], then the action of \( G_k \) on \( E_k^\alpha[n] \) is unramified. If \( E_k \) has bad multiplicative reduction over \( \mathcal{O}_k \) [i.e., extends to a non-proper semi-abelian scheme over \( \mathcal{O}_k \)], then the action of \( G_k \) on \( E_k^\alpha[n] \) is tamely ramified.

Proof. First, we consider assertion (i). Suppose that \( \phi \) is not injective. Since \( \text{Aut}_k(A, \lambda) \) is well-known to be finite [cf., e.g., [Milne], Proposition 17.5, (a)], we thus conclude that there exists an \( \alpha \in \text{Ker}(\phi) \) of order \( n \neq 1 \). We may assume without loss of generality that \( n \) is prime. Now we follow the argument of [Milne],...
Proposition 17.5, (b). Since $\alpha$ acts trivially on $A[l]$, it follows immediately that the endomorphism of $A$ given by $\alpha - \text{id}_A$ [where $\text{id}_A$ denotes the identity automorphism of $A$] may be written in the form $l \cdot \beta$, for $\beta$ an endomorphism of $A$ over $k$. Write $T_l(A)$ for the $l$-adic Tate module of $A$. Since $\alpha^n = \text{id}_A$, it follows that the eigenvalues of the action of $\alpha$ on $T_l(A)$ are $n$-th roots of unity. On the other hand, the eigenvalues of the action of $\beta$ on $T_l(A)$ are algebraic integers [cf. [Milne], Theorem 12.5]. We thus conclude that each eigenvalue $\zeta$ of the action of $\alpha$ on $T_l(A)$ is an $n$-th root of unity which, as an algebraic integer, is $\equiv 1 \pmod{l}$ [where $l \geq 3$], hence $= 1$. Since $\alpha^n = \text{id}_A$, it follows that $\alpha$ acts on $T_l(A)$ as a semi-simple matrix which is also unipotent, hence equal to the identity matrix. But this implies that $\alpha = \text{id}_A$ [cf. [Milne], Theorem 12.5]. This contradiction completes the proof of assertion (i).

Next, we consider assertion (ii). Since $E_{\overline{k}}$ is proper over $\overline{k}$, it follows [by considering the space of global sections of the structure sheaf of $E_{\overline{k}}$] that any automorphism of the scheme $E_{\overline{k}}$ lies over an automorphism of $\overline{k}$. This implies the existence of a natural exact sequence and natural representation as in the statement of assertion (ii). The relationship between $k_E$ and the $j$-invariant of $E_{\overline{k}}$ follows immediately from the well-known theory of the $j$-invariant of an elliptic curve [cf., e.g., [Silv], Chapter III, Proposition 1.4, (b), (c)]. The final portion of assertion (ii) concerning models of $E_{\overline{k}}$ follows immediately from the definitions. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the fact that $\{\pm 1\}$ acts trivially on $E_{\overline{k}}[2]$.

Next, we consider assertion (iv). First, let us observe that it follows immediately from the final portion of assertion (ii) that a model $E'_k$ of $E_{\overline{k}}$ over $k$ all of whose $l$-torsion points are rational over $k$ corresponds to a closed subgroup $H^* \subseteq \text{Aut}_k(E_{\overline{k}})$ that lies in the kernel of $\rho_l$ and, moreover, maps isomorphically to $G_k$. On the other hand, it follows from assertion (i) that the restriction of $\rho_l$ to $\text{Aut}_{\overline{k}}(E_{\overline{k}}) \subseteq \text{Aut}_k(E_{\overline{k}})$ is injective. Thus, the closed subgroup $H^* \subseteq \text{Aut}_k(E_{\overline{k}})$ is uniquely determined by the condition that it lie in the kernel of $\rho_l$ and, moreover, map isomorphically to $G_k$. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that, by considering level structures, we obtain a finite covering of $S \to (\mathcal{M}_{\text{ell}})_{\mathbb{Z}[\frac{1}{4}]}$ which is étale over $(\mathcal{M}_{\text{ell}})_{\mathbb{Z}[\frac{1}{4}]}$ and tamely ramified over the divisor at infinity. Then it follows from assertion (i) that the algebraic stack $S$ is in fact a scheme, which is, moreover, proper over $\mathbb{Z}[\frac{1}{4}]$. Thus, it follows from the valuative criterion for properness that any $k$-valued point of $S$ determined by $E_k$ — where we observe that such a point necessarily exists, in light of our assumption that the $l$-torsion points of $E_k$ are rational over $k$ — extends to an $\mathcal{O}_k$-valued point of $S$, hence also of $\overline{\mathcal{M}}_{\text{ell}}$, as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). Since $G_E = G_k$, it follows from assertion (ii) that $E_{\overline{k}}$ descends to an elliptic curve $E_k$ over $k$. Our assumption that the representation $G_k = G_E \to \text{Aut}(E_{\overline{k}}[2])$ of assertion (iii) is trivial implies that the 2-torsion points of $E_k$ are rational over $k$. Thus, by considering suitable global sections of tensor powers of the line bundle on $E_k$ determined by the origin on which the automorphism “$-1$” of $E_k$ acts via multiplication by $\pm 1$ [cf., e.g., [Harts], Chapter IV, the proof of Proposition 4.6], one concludes immediately that a suitable [possibly trivial] twist $E'_k$ of $E_k$ over $k$ [i.e., such that $E'_k$ and $E_k$ are isomorphic over
some quadratic extension \( k' \) of \( k \) may be defined by means of the Legendre form of the Weierstrass equation. Now suppose that \( k \) is a complete discrete valuation field with ring of integers \( \mathcal{O}_k \) such that 2 is invertible in \( \mathcal{O}_k \), and that \( E_k \) is defined by means of the Legendre form of the Weierstrass equation. Then the fact that \( E_k \) has semi-stable reduction over \( \mathcal{O}_k' \) for some finite extension \( k' \subseteq k \) such that \([k' : k] \leq 2\) follows from the explicit computations of the proof of [Silv], Chapter VII, Proposition 5.4, (c). These explicit computations also imply that if \( E_k \) has good reduction over \( \mathcal{O}_k' \), then one may in fact take \( k' \) to be \( k \). This completes the proof of assertion (vi).

Assertion (vii) follows immediately from [NerMod], §7.4, Theorem 5, in the case of good reduction and from [NerMod], §7.4, Theorem 6, in the case of bad multiplicative reduction.

We are now ready to apply the elementary computations discussed above to give more explicit log-volume estimates for \( \Theta \)-pilot objects. We begin by recalling some notation and terminology from [GenEll], §1.

**Definition 1.9.** Let \( F \) be a number field [i.e., a finite extension of the rational number field \( \mathbb{Q} \)], whose set of valuations we denote by \( \mathcal{V}(F) \). Thus, \( \mathcal{V}(F) \) decomposes as a disjoint union \( \mathcal{V}(F) = \mathcal{V}(F)^{\text{non}} \cup \mathcal{V}(F)^{\text{arc}} \) of nonarchimedean and archimedean valuations. If \( v \in \mathcal{V}(F) \), then we shall write \( F_v \) for the completion of \( F \) at \( v \); if \( v \in \mathcal{V}^{\text{non}} \), then we shall write \( \text{ord}_v(-) : F_v^* \to \mathbb{Z} \) for the order defined by \( v \), \( e_v \) for the ramification index of \( F_v \) over \( \mathbb{Q}_{p_v} \), and \( q_v \) for the cardinality of the residue field of \( F_v \).

(i) An \([\mathbb{R}]-\)arithmetic divisor \( a \) on \( F \) is defined to be a finite formal sum

\[
\sum_{v \in \mathcal{V}(F)} c_v \cdot v
\]

— where \( c_v \in \mathbb{R} \), for all \( v \in \mathcal{V}(F) \). Here, we shall refer to the set

\[
\text{Supp}(a)
\]

of \( v \in \mathcal{V}(F) \) such that \( c_v \neq 0 \) as the support of \( a \); if all of the \( c_v \) are \( \geq 0 \), then we shall say that the arithmetic divisor is effective. Thus, the \([\mathbb{R}]-\)arithmetic divisors on \( F \) naturally form a group \( \text{ADiv}_\mathbb{R}(F) \). The assignment

\[
\mathcal{V}^{\text{non}} \ni v \mapsto \log(q_v); \quad \mathcal{V}^{\text{arc}} \ni v \mapsto 1
\]

determines a homomorphism

\[
\deg_F : \text{ADiv}_\mathbb{R}(F) \to \mathbb{R}
\]

which we shall refer to as the degree map. If \( a \in \text{ADiv}_\mathbb{R}(F) \), then we shall refer to

\[
\deg(a) \overset{\text{def}}{=} \frac{1}{[F : \mathbb{Q}]} \cdot \deg_F(a)
\]
as the \textit{normalized degree} of $a$. Thus, for any finite extension $K$ of $F$, we have

$$\deg(a|_K) = \deg(a)$$

— where we write $\deg(a|_K)$ for the normalized degree of the pull-back $a|_K \in \text{ADiv}_R(K)$ [defined in the evident fashion] of $a$ to $K$.

(ii) Let $v_Q \in \mathcal{V}_Q \overset{\text{def}}{=} \mathcal{V}(Q)$, $E \subseteq \mathcal{V}(F)$ a nonempty set of elements lying over $v_Q$. If $a = \sum_{v \in \mathcal{V}(F)} c_v \cdot v \in \text{ADiv}_R(F)$, then we shall write

$$a_E \overset{\text{def}}{=} \sum_{v \in E} c_v \cdot v \in \text{ADiv}_R(F); \quad \deg_E(a) \overset{\text{def}}{=} \sum_{v \in E} \deg(a_E)$$

for the portion of $a$ supported in $E$ and the \textit{“normalized $E$-degree”} of $a$, respectively. Thus, for any finite extension $K$ of $F$, we have

$$\deg_{E|_K}(a|_K) = \deg_E(a)$$

— where we write $E|_K \subseteq \mathcal{V}(K)$ for the set of valuations lying over valuations $\in E$.

\textbf{Theorem 1.10.} (Log-volume Estimates for $\Theta$-Pilot Objects) Fix a collection of initial $\Theta$-data as in [IUTchI], Definition 3.1. Suppose that we are in the situation of [IUTchIII], Corollary 3.12, and that the elliptic curve $E_F$ has \textit{good reduction} at every valuation $\in \mathcal{V}(F)^{\text{good}} \bigcap \mathcal{V}(F)^{\text{non}}$ that does not divide $2l$. In the notation of [IUTchI], Definition 3.1, let us write $d_{\text{mod}} \overset{\text{def}}{=} [F_{\text{mod}} : \mathbb{Q}]$, $d_{\text{mod}}^* \overset{\text{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\text{mod}}$, and

$$F_{\text{mod}} \subseteq F_{\text{tpd}} \overset{\text{def}}{=} F_{\text{mod}}[E_{F_{\text{mod}}[2]}] \subseteq F$$

for the \textit{“tripodal”} intermediate field obtained from $F_{\text{mod}}$ by adjoining the fields of definition of the 2-torsion points of any model of $E_F \times_F \overline{F}$ over $F_{\text{mod}}$ [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3\cdot5)$-torsion points of $E_F$ are defined over $F$, and that

$$F = F_{\text{mod}}(\sqrt{-1}, E_{F_{\text{mod}}[2 \cdot 3 \cdot 5]}) \overset{\text{def}}{=} F_{\text{tpd}}(\sqrt{-1}, E_{F_{\text{tpd}}[3 \cdot 5]})$$

— i.e., that $F$ is obtained from $F_{\text{tpd}}$ by adjoining $\sqrt{-1}$, together with the fields of definition of the $(3\cdot5)$-torsion points of a model $E_{F_{\text{tpd}}}$ of the elliptic curve $E_F \times_F \overline{F}$ over $F_{\text{tpd}}$ determined by the \textbf{Legendre form} of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below; Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that $E_F \cong E_{F_{\text{tpd}}} \times_{F_{\text{tpd}}} F$ over $F$, and from [IUTchI], Definition 3.1, (c), that $l \neq 5$.] If $F_{\text{mod}} \subseteq F_{\square} \subseteq K$ is any intermediate extension which is Galois over $F_{\text{mod}}$, then we shall write

$$d_{\text{ADiv}}^{F_{\square}} \in \text{ADiv}_R(F_{\square})$$
for the effective divisor determined by the \textbf{different ideal} of \( F \) over \( \mathbb{Q} \),

\[
q^F_{\text{ADiv}} \in \text{ADiv}_{\mathbb{R}}(F)
\]

for the effective arithmetic divisor determined by the \( q \)-\textbf{parameters} of the elliptic curve \( E_F \) at the elements of \( \mathbb{V}(F) \) \( \text{bad} = \mathbb{V}_{\text{mod}} \times \mathbb{V}_{\text{mod}} \mathbb{V}(F) \) \((\neq \emptyset) \) \cite{[GenEll]}, Corollary 3.12, to be

\[
\mathfrak{f}^F_{\text{ADiv}} \in \text{ADiv}_{\mathbb{R}}(F)
\]

for the effective arithmetic divisor whose support coincides with \( \text{Supp}(q^F_{\text{ADiv}}) \), but all of whose coefficients are equal to 1 — \( \text{i.e., the conductor} \) — and

\[
\log(\mathfrak{d}_v^F) = \deg_{\mathbb{V}(F)}(\mathfrak{d}_v^F)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{d}_v^F) = \deg_{\mathbb{V}(F)}(\mathfrak{d}_v^F)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}
\]

\[
\log(\mathfrak{q}_v) = \deg_{\mathbb{V}(F)}(\mathfrak{q}_v)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{q}_v) = \deg_{\mathbb{V}(F)}(\mathfrak{q}_v)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}
\]

\[
\log(\mathfrak{f}_v^F) = \deg_{\mathbb{V}(F)}(\mathfrak{f}_v^F)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{f}_v^F) = \deg_{\mathbb{V}(F)}(\mathfrak{f}_v^F)_{\text{ADiv}} \in \mathbb{R}_{\geq 0}
\]

where \( v \in \mathbb{V}_{\text{mod}} = \mathbb{V}(F)_{\text{mod}} \), \( v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} = \mathbb{V}(Q) \), \( \mathbb{V}(F)_{\text{v}} = \mathbb{V}(F) \times \mathbb{V}_{\text{mod}} \{v\} \), \( \mathbb{V}(F)_{\text{v}_q} = \mathbb{V}(F) \times \mathbb{V}_{\mathbb{Q}} \{v_{\mathbb{Q}}\} \). Here, we observe that the various “\( \log(\mathfrak{q}_\cdot) \)’s” are independent of the choice of \( F \), and that the quantity “\( |\log(q)| \in \mathbb{R}_{\geq 0} \)” defined in \cite{[IUTchIII]}, Corollary 3.12, is equal to \( \frac{1}{2} \cdot \log(q) \in \mathbb{R} \) \cite{[GenEll]} the definition of “\( q \)” in \cite{[IUTchI]}, Example 3.2, (iv)]. Then one may take the constant \( “\mathcal{C}_\Theta \in \mathbb{R}” \) of \cite{[IUTchIII]}, Corollary 3.12, to be

\[
\frac{l+1}{8 \cdot |\log(q)|} \cdot \left\{ (1 + \frac{36 \cdot d_{\text{mod}}}{l}) \cdot \left( \log(\mathfrak{d}_{\text{tpd}}) + \log(\mathfrak{f}_{\text{tpd}}) \right) + 10 \cdot (d_{\text{mod}} \cdot l + \eta_{\text{prm}}) \right\} - 1
\]

and hence, by applying the inequality “\( \mathcal{C}_\Theta \geq -1 \)” of \cite{[IUTchIII]}, Corollary 3.12, conclude that

\[
\frac{1}{6} \cdot \log(q) \leq (1 + \frac{80 \cdot d_{\text{mod}}}{l}) \cdot \left( \log(\mathfrak{d}_{\text{tpd}}) + \log(\mathfrak{f}_{\text{tpd}}) \right) + 20 \cdot (d_{\text{mod}} \cdot l + \eta_{\text{prm}})
\]

\[
\leq (1 + \frac{80 \cdot d_{\text{mod}}}{l}) \cdot \left( \log(\mathfrak{d}) + \log(\mathfrak{f}) \right) + 20 \cdot (d_{\text{mod}} \cdot l + \eta_{\text{prm}})
\]

where \( \eta_{\text{prm}} \) is the positive real number of Proposition 1.6.

\textit{Proof.} For ease of reference, we divide our discussion into \textit{steps}, as follows.

(i) We begin by recalling the following \textit{elementary identities} for \( n \in \mathbb{N}_{\geq 1} \):

\[
(E1) \quad \frac{1}{n} \sum_{m=1}^{n} m = \frac{1}{2} (n+1);
\]

\[
(E2) \quad \frac{1}{n} \sum_{m=1}^{n} m^2 = \frac{1}{6} (2n+1)(n+1).
\]
Also, we recall the following elementary facts:

(E3) For $p$ a prime number, the cardinality $|GL_2(\mathbb{F}_p)|$ of $GL_2(\mathbb{F}_p)$ is given by $|GL_2(\mathbb{F}_p)| = p(p + 1)(p - 1)^2$.

(E4) For $p = 2, 3, 5$, the expression of (E3) may be computed as follows: $2(2+1)(2-1)^2 = 2\cdot3$; $3(3+1)(3-1)^2 = 3\cdot2^4$; $5(5+1)(5-1)^2 = 5\cdot2^5\cdot3$.

(E5) The degree of the extension $F_{\text{mod}}(\sqrt{-1})/F_{\text{mod}}$ is $\leq 2$.

(E6) We have: $0 \leq \log(2) \leq 1$, $1 \leq \log(3) \leq \log(\pi) \leq \log(5) \leq 2$.

(ii) Next, let us observe that the inequality

$$\log(\mathfrak{o}^{F_{\text{tpd}}}) + \log(f^{F_{\text{tpd}}}) \leq \log(\mathfrak{o}^F) + \log(f^F)$$

follows immediately from Proposition 1.3, (i), and the various definitions involved. On the other hand, the inequality

$$\log(\mathfrak{o}^F) + \log(f^F) \leq \log(\mathfrak{o}^{F_{\text{tpd}}}) + \log(f^{F_{\text{tpd}}}) + \log(2^{11\cdot3^3\cdot5^2})$$

$$\leq \log(\mathfrak{o}^{F_{\text{tpd}}}) + \log(f^{F_{\text{tpd}}}) + 21$$

follows by applying Proposition 1.3, (i), at the primes that do not divide $2\cdot3\cdot5$ [where we recall that the extension $F/F_{\text{tpd}}$ is tamely ramified over such primes — cf. Proposition 1.8, (vi), (vii)] and applying Proposition 1.3, (ii), together with (E3), (E4), (E5), (E6), and the fact that we have a natural outer inclusion $\text{Gal}(F/F_{\text{tpd}}) \rightarrow GL_2(\mathbb{F}_3) \times GL_2(\mathbb{F}_5) \times \mathbb{Z}/2\mathbb{Z}$, at the primes that divide $2\cdot3\cdot5$. In a similar vein, since the extension $K/F$ is tamely ramified at the primes that do not divide $l$, and we have a natural outer inclusion $\text{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$, the inequality

$$\log(\mathfrak{o}^K) \leq \log(\mathfrak{o}^K) + \log(f^K) \leq \log(\mathfrak{o}^F) + \log(f^F) + 2\cdot\log(l)$$

$$\leq \log(\mathfrak{o}^{F_{\text{tpd}}}) + \log(f^{F_{\text{tpd}}}) + 2\cdot\log(l) + 46$$

— where we apply the estimates $\frac{\log(l)}{l} \leq \frac{1}{2}$ and $1 + \frac{4}{l} \leq 2$, both of which may be regarded as consequences of the fact that $l \geq 5$ [cf. also (E6)].

(iii) If $F_{\text{tpd}} \subseteq F\subseteq K$ is any intermediate extension which is Galois over $F_{\text{mod}}$, then we shall write

$$V(F\subseteq)^{\text{dist}} \subseteq V(F\subseteq)^{\text{non}}$$

for the set of “distinguished” nonarchimedean valuations $v \in V(F\subseteq)^{\text{non}}$, i.e., $v$ that extend to a valuation $\in V(K)^{\text{non}}$ that ramifies over $\mathbb{Q}$. Now observe that it follows immediately from Proposition 1.8, (vi), (vii), together with our assumption on $V(F)^{\text{good}} \cap V(F)^{\text{non}}$, that

(D0) if $v \in V(F_{\text{tpd}})^{\text{non}}$ does not divide $2\cdot3\cdot5\cdot l$ and, moreover, is not contained in $\text{Supp}(q^{F_{\text{tpd}}})$, then the extension $K/F_{\text{tpd}}$ is unramified over $v$. 


Next, let us recall the well-known fact that the determinant of the Galois representation determined by the torsion points of an elliptic curve over a field of characteristic zero is the abelian Galois representation determined by the cyclotomic character. In particular, it follows [cf. the various definitions involved] that $K$ contains a primitive $4 \cdot 3 \cdot 5 \cdot l$-th root of unity, hence is ramified over $\mathbb{Q}$ at any valuation $v \in \mathbb{V}(K)$ that divides $2 \cdot 3 \cdot 5 \cdot l$. Thus, one verifies immediately [i.e., by applying (D0); cf. also [IUTChI], Definition 3.1, (c)] that the following conditions on a valuation $v \in \mathbb{V}(F)$ are equivalent:

(D1) $v \in \mathbb{V}(F)^{\text{dst}}$.

(D2) The valuation $v$ either divides $2 \cdot 3 \cdot 5 \cdot l$ or lies in $\text{Supp}(q_{\text{ADiv}}^F + \mathfrak{d}_{\text{ADiv}}^F)$.

(D3) The image of $v$ in $\mathbb{V}(F_{\text{tpd}})$ lies in $\mathbb{V}(F_{\text{tpd}})^{\text{dst}}$.

Let us write

$$
\mathbb{V}^{\text{dst}}_{\text{mod}} \subseteq \mathbb{V}^{\text{non}}_{\text{mod}}, \quad \mathbb{V}^{\text{dst}}_Q \subseteq \mathbb{V}^{\text{non}}_Q
$$

for the respective images of $\mathbb{V}(F_{\text{tpd}})^{\text{dst}}$ in $\mathbb{V}_{\text{mod}}, \mathbb{V}_Q$ and, for $F_* \in \{F, F_{\text{mod}}, \mathbb{Q}\}$ and $v_Q \in \mathbb{V}_Q$,

$$
\mathfrak{s}_{\text{ADiv}}^F \overset{\text{def}}{=} \sum_{v \in \mathbb{V}(F_*)^{\text{dst}}} e_v \cdot v \in \text{ADiv}_R(F_*)
$$

$$
\log(\mathfrak{s}^F_{v_Q}) \overset{\text{def}}{=} \text{deg}_{\mathbb{V}(F_*)_{v_Q}}(\mathfrak{s}_{\text{ADiv}}^F) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^F) \overset{\text{def}}{=} \text{deg}(\mathfrak{s}_{\text{ADiv}}^F) \in \mathbb{R}_{\geq 0}
$$

$$
\mathfrak{s}^{\leq}_{\text{ADiv}} \overset{\text{def}}{=} \sum_{v_Q \in \mathbb{V}(Q)^{\text{dst}}} \frac{\tau_{v_Q} \cdot w_Q}{\log(p_{v_Q})} \in \text{ADiv}_R(\mathbb{Q})
$$

$$
\log(\mathfrak{s}^{\leq}_{v_Q}) \overset{\text{def}}{=} \text{deg}_{\mathbb{V}(Q)_{v_Q}}(\mathfrak{s}^{\leq}_{\text{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{\leq}) \overset{\text{def}}{=} \text{deg}(\mathfrak{s}^{\leq}_{\text{ADiv}}) \in \mathbb{R}_{\geq 0}
$$

where we write $\mathbb{V}(F_*)_{v_Q} \overset{\text{def}}{=} \mathbb{V}(F_*) \times_{\mathbb{V}(Q)} \{v_Q\}$; we set $\tau_{v_Q} \overset{\text{def}}{=} 1$ if $p_{v_Q} \leq d_{\text{mod}}^* \cdot l$, $\tau_{v_Q} \overset{\text{def}}{=} 0$ if $p_{v_Q} > d_{\text{mod}}^* \cdot l$. Then one verifies immediately [again, by applying (D0); cf. also [IUTChI], Definition 3.1, (c)] that the following conditions on a valuation $v_Q \in \mathbb{V}^{\text{non}}_Q$ are equivalent:

(D4) $v_Q \in \mathbb{V}^{\text{dst}}_Q$.

(D5) The valuation $v_Q$ ramifies in $K$.

(D6) Either $p_{v_Q} | 2 \cdot 3 \cdot 5 \cdot l$ or $v_Q$ lies in the image of $\text{Supp}(q_{\text{ADiv}}^F + \mathfrak{d}_{\text{ADiv}}^F)$.

(D7) Either $p_{v_Q} | 2 \cdot 3 \cdot 5 \cdot l$ or $v_Q$ lies in the image of $\text{Supp}(q_{\text{ADiv}}^F + \mathfrak{d}_{\text{ADiv}}^F)$.

Here, we observe in passing that, for $v \in \mathbb{V}(F)$,

(R1) $\log(e_v) \leq \log(2^{11} \cdot 3^{3} \cdot 5 \cdot d_{\text{mod}} \cdot l^4)$ if $v$ divides $l$,

(R2) $\log(e_v) \leq \log(2^{11} \cdot 3^{3} \cdot 5 \cdot d_{\text{mod}} \cdot l)$ if $v$ divides $2 \cdot 3 \cdot 5$ or lies in $\text{Supp}(q_{\text{ADiv}}^F)$ [hence does not divide $l$],

(R3) $\log(e_v) \leq \log(2^{11} \cdot 3^{3} \cdot 5 \cdot d_{\text{mod}})$ if $v$ is does not divide $2 \cdot 3 \cdot 5 \cdot l$ and, moreover, is not contained in $\text{Supp}(q_{\text{ADiv}}^F)$,

and hence that
(R4) If \( e_v \geq p_v - 1 > p_v - 2 \), then \( p_v \leq 2^{12} \cdot 3^3 \cdot 5 \cdot \text{mod} \cdot l = d^{*}_{\text{mod}} \cdot l \), and
\[
\log(e_v) \leq -3 + 4 \cdot \log(d^{*}_{\text{mod}} \cdot l)
\]

— cf. (E3), (E4), (E5), (E6); [IUTchII], Definition 3.1, (c). Next, for later reference, we observe that the inequality
\[
\frac{1}{p_{vQ}} \cdot \log(s^{F\text{mod}}_{vQ}) \leq \frac{1}{p_{vQ}} \cdot \log(p_{vQ})
\]
holds for any \( vQ \in V_Q \); in particular, when \( p_{vQ} = l (\geq 5) \), it holds that
\[
\frac{1}{p_{vQ}} \cdot \log(s^{F\text{mod}}_{vQ}) \leq \frac{1}{p_{vQ}} \cdot \log(p_{vQ}) \leq \frac{1}{2}
\]

— cf. (E6). On the other hand, it follows immediately from Proposition 1.3, (i), by considering the various possibilities for elements \( \in \text{Supp}(s^{F\text{mod}}_{ADV}) \), that
\[
\log(s^{F\text{mod}}_{vQ}) \leq 2 \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ}))
\]

— and hence that
\[
\frac{1}{p_{vQ}} \cdot \log(s^{F\text{mod}}_{vQ}) \leq \frac{2}{p_{vQ}} \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ}))
\]

— for any \( vQ \in V_Q \) such that \( p_{vQ} \not\in \{2, 3, 5, l\} \). In a similar vein, we conclude that
\[
\log(s^Q) \leq 2 \cdot d_{\text{mod}} \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ})) + \log(2 \cdot 3 \cdot 5 \cdot l)
\leq 2 \cdot d_{\text{mod}} \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ})) + 5 + \log(l)
\]

and hence that
\[
\frac{16}{l} \cdot \log(s^Q) \leq \frac{32 \cdot d_{\text{mod}}}{l} \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ})) + 24
\]

— cf. (E6); the fact that \( l \geq 5 \). Combining this last inequality with the inequality of the final display of Step (ii) yields the inequality
\[
(1 + \frac{4}{l}) \cdot \log(s^Q) + \frac{16}{l} \cdot \log(s^Q) \leq (1 + \frac{36 \cdot d_{\text{mod}}}{l}) \cdot (\log(s^{F_{1\text{mod}}}_{vQ}) + \log(f^{F_{1\text{mod}}}_{vQ})) + 2 \cdot \log(l) + 70
\]

— where we apply the estimate \( d_{\text{mod}} \geq 1 \).

(iv) In order to estimate the constant “\( C_{\Theta} \)” of [IUTchIII], Corollary 3.12, we must, according to the various definitions given in the statement of [IUTchIII], Corollary 3.12, compute an upper bound for the

procession-normalized mono-analytic log-volume of the holomorphic hull of the union of the possible images of a \( \Theta \)-pilot object, relative to the relevant Kummer isomorphisms [cf. [IUTchIII], Theorem 3.11, (ii)], in the multiradial representation of [IUTchIII], Theorem 3.11, (i), which we regard as subject to the indeterminacies (Ind1), (Ind2), (Ind3) described in [IUTchIII], Theorem 3.11, (i), (ii).
Thus, we proceed to estimate this log-volume at each \( v_Q \in V_Q \). Once one fixes \( v_Q \),
this amounts to estimating the component of this log-volume in
\[
\mathcal{L}^Q(S^+_{j+1}; m, D^\perp_{v_Q})^{-n}
\]
[cf. the notation of \([\text{IUTchIII}], \text{Theorem 3.11, (i), (a)}\), for each \( j \in \{1, \ldots, l^*\} \),
which we shall also regard as an element of \( \mathbb{F}_l^* \), and then computing the average,
over \( j \in \{1, \ldots, l^*\} \), of these estimates. Here, we recall [cf. \([\text{IUTchI}], \text{Proposition 6.9, (i); [IUTchIII], Proposition 3.4, (ii)}\)]
that \( S_{j+1}^+ = \{0, 1, \ldots, j\} \). Also, we recall from \([\text{IUTchIII}], \text{Proposition 3.2}\),
that \( \mathcal{L}^Q(S^+_{j+1}; m, D^\perp_{v_Q})^{-n} \) is, by definition, a tensor product
of \( j + 1 \) copies, indexed by the elements of \( S_{j+1}^+ \), of the direct sum of the \( Q \)-
spans of the log-shells associated to each of the elements of \( V(F_{\text{mod}})_{v_Q} \) [cf.,
especially, the second and third displays of \([\text{IUTchIII}], \text{Proposition 3.2}\)]. In particular, for each collection
\[
\{v_i\}_{i \in S_{j+1}^+}
\]
of [not necessarily distinct!] elements of \( V(F_{\text{mod}})_{v_Q} \), we must estimate the component
of the log-volume in question corresponding to the tensor product of the \( Q \)-spans of the log-shells associated to this collection \( \{v_i\}_{i \in S_{j+1}^+} \) and then compute
the weighted average [cf. the discussion of Remark 1.7.1], over possible collections
\( \{v_i\}_{i \in S_{j+1}^+} \), of these estimates.

(v) Let \( v_Q \in V_{\text{dist}}^Q \). Fix \( j \), \( \{v_i\}_{i \in S_{j+1}^+} \) as in Step (iv).
Write \( \tilde{v}_j \in \mathcal{V} \sim \mathcal{V}_{\text{mod}} = V(F_{\text{mod}}) \) for the element corresponding to \( v_i \).
We would like to apply Proposition 1.4, (iii), to the present situation, by taking
\begin{itemize}
  \item \( \Gamma \) to be \( S_{j+1}^+ \);
  \item \( \Gamma^* \subseteq \Gamma \) to be the set of \( i \in I \) such that \( e_{\tilde{v}_j} > p_{v_Q} - 2 \);
  \item \( k_i \) to be \( K_{\tilde{v}_j} \) [so \( \Gamma_i \) will be the ring of integers \( O_{K_{\tilde{v}_j}} \) of \( K_{\tilde{v}_j} \)];
  \item \( i^{\Gamma} \) to be \( j \in S_{j+1}^+ \);
  \item \( \lambda \) to be \( 0 \) if \( \tilde{v}_j \in V_{\text{good}}^Q \);
  \item \( \lambda \) to be \( \text{ord}(-) \) of the element \( q_{	ilde{v}_j}^2 \) [cf. the definition of \( q_{\tilde{v}_j}^2 \) in
[\text{IUTchI}], Example 3.2, (iv)] if \( \tilde{v}_j \in V_{\text{bad}}^Q \).
\end{itemize}
Thus, the inclusion \( \phi(p^{\lambda} \cdot R_{i^\Gamma} \otimes R_{i^\Gamma}(R_i)^{\sim-}) \subseteq p^{[\lambda]-[\delta_i]-[a_{\Gamma}]} \cdot \log_p(R_i)^{\sim} \) of Proposition
1.4, (iii), implies that the result of multiplying the tensor product of log-shells under consideration by a suitable nonpositive [cf. the inequalities concerning
\( [\delta_i] + [a_{\Gamma}] \) that constitute the final portion of Proposition 1.4, (iii)] integer power of \( p_{v_Q} \) contains the “union of possible images of a \( \Theta \)-pilot object” discussed in Step (iv).
That is to say, the indeterminacies (Ind1) and (Ind2) are taken into account by the arbitrary nature of the automorphism \( \phi \) [cf. Proposition 1.2], while the indeterminacy (Ind3) is taken into account by the fact that we are considering upper bounds [cf. the discussion of Step (x) of the proof of \([\text{IUTchIII}], \text{Corollary 3.12}\)],
together with the fact that the above-mentioned integer power of \( p_{v_Q} \) is nonpositive,
which implies that the module obtained by multiplying by this power of \( p_{v_Q} \) contains
the tensor product of log-shells under consideration. Thus, an upper bound
on the component of the log-volume of the holomorphic hull under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side of the inclusion \( p^{[\lambda - |\alpha| - |\beta|]} \cdot \log_{p}(R_{I}^{2}) \subseteq p^{[\lambda - |\alpha| - |\beta|]} \cdot (R_{I})^{\wedge} \) of Proposition 1.4, (iii). Such an upper bound

\[
\left( - \lambda + \vartheta I + 4 \right) \cdot \log(p) + \sum_{i \in I^*} \{ 3 + \log(e_{i}) \}
\]

is given in the second displayed inequality of Proposition 1.4, (iii). Here, we note that if \( e_{\varepsilon_{i}} \leq p_{v_{0}} - 2 \) for all \( i \in I \), then this upper bound assumes the form

\[
\left( - \lambda + \vartheta I + 4 \right) \cdot \log(p).
\]

On the other hand, by (R4), if \( e_{\varepsilon_{i}} > p_{v_{0}} - 2 \) for some \( i \in I \), then it follows that \( p_{v_{0}} \leq d_{\text{mod}}^{*} \cdot l \), and \( \log(e_{\varepsilon_{i}}) \leq -3 + 4 \cdot \log(d_{\text{mod}}^{*} \cdot l) \), so the upper bound in question may be taken to be

\[
\left( - \lambda + \vartheta I + 4 \right) \cdot \log(p) + 4(j + 1) \cdot l^{*}_{\text{mod}}
\]

— where we write \( l^{*}_{\text{mod}} \overset{\text{def}}{=} \log(d_{\text{mod}}^{*} \cdot l) \). Also, we note that, unlike the other terms that appear in these upper bounds, \( \lambda \) is asymmetric with respect to the choice of \( \iota \in I \) in \( S_{j+1}^{\pm} \). Since we would like to compute weighted averages [cf. the discussion of Remark 1.7.1], we thus observe that, after symmetrizing with respect to the choice of \( \iota \in I \) in \( S_{j+1}^{\pm} \), this upper bound may be written in the form

\[
\beta_{e}
\]

[cf. the notation of Proposition 1.7] if, in the situation of Proposition 1.7, one takes

\[
\cdot \quad \text{“E” to be } V(F_{\text{mod}}_{v_{0}});
\cdot \quad \text{“n” to be } j + 1, \text{ so an element “} \varepsilon \in E^{n} \text{” corresponds precisely to a collection } \{ v_{i} \}_{i \in S_{j+1}^{\pm}};
\cdot \quad \text{“} \lambda_{e} \text{” for an element } e \in E \text{ corresponding to } v \in V(F_{\text{mod}}) = V_{\text{mod}}, \text{ to be } [(F_{\text{mod}})_{v} : \mathbb{Q}_{v}] \in \mathbb{R}_{>0};
\cdot \quad \text{“} \beta_{e} \text{” for an element } e \in E \text{ corresponding to } v \in V, v \in V(F_{\text{mod}}) = V_{\text{mod}}, \text{ to be }
\]

\[
\log(\delta_{v}^{K}) - \frac{j^{2}}{2(l+1)} \cdot \log(q_{v}) + \frac{4}{j+1} \cdot \log(p_{v_{0}}) + 4 \cdot t_{v_{0}} \cdot l^{*}_{\text{mod}}
\]

— where we recall that \( t_{v_{0}} \overset{\text{def}}{=} 1 \) if \( p_{v_{0}} \leq d_{\text{mod}}^{*} \cdot l \), \( t_{v_{0}} \overset{\text{def}}{=} 0 \) if \( p_{v_{0}} > d_{\text{mod}}^{*} \cdot l \).

Here, we note that it follows immediately from the first equality of the first display of Proposition 1.7 that, after passing to weighted averages, the operation of symmetrizing with respect to the choice of \( \iota \in I \) in \( S_{j+1}^{\pm} \) does not affect the computation of the upper bound under consideration. Thus, by applying Proposition 1.7, we obtain that the resulting “weighted average upper bound” is given by

\[
(j + 1) \cdot \log(\delta_{v_{0}}^{K}) - \frac{j^{2}}{2l} \cdot \log(q_{v_{0}}) + 4 \cdot \log(s_{Q}^{v_{0}}) + 4(j + 1) \cdot l^{*}_{\text{mod}} \cdot \log(s_{v_{0}}^{\leq})
\]
— where we recall the notational conventions introduced in Step (iii). Thus, it remains to compute the average over \( j \in \mathbb{F}_l^* \). By averaging over \( j \in \{1, \ldots, l^* = \frac{l-1}{2} \} \) and applying (E1), (E2), we obtain the "procession-normalized upper bound"

\[
\frac{(l^* + 3)}{2} \cdot \log(\delta^K_{vq}) - \frac{(2l^* + 1)(l^* + 1)}{2l} \cdot \log(q_{vq}) + 4 \cdot \log(s^Q_{vq})^* + 2(l^* + 3) \cdot l^* \cdot \log(s^L_{vq}) \\
= \frac{l+5}{4} \cdot \log(\delta^K_{vq}) - \frac{l+1}{2} \cdot \log(q_{vq}) + 4 \cdot \log(s^Q_{vq})^* + (l + 5) \cdot l^* \cdot \log(s^L_{vq}) \\
\leq \frac{l+1}{4} \left\{ (1 + \frac{4}{7}) \cdot \log(\delta^K_{vq}) - \frac{1}{8} \cdot \log(q_{vq}) + \frac{16}{7} \cdot \log(s^Q_{vq})^* + \frac{20}{3} \cdot l^* \cdot \log(s^L_{vq}) \right\}
\]

— where, in the passage to the final displayed inequality, we apply the estimates \( \frac{1}{7} \leq \frac{1}{8} \) and \( \frac{4}{7} \leq \frac{20}{3} \), both of which may be regarded as consequences of the fact that \( l \geq 5 \).

(vi) Next, let \( v_Q \in \mathbb{V}^\text{non} \setminus \mathbb{V}^\text{dst}_Q \). Fix \( j, \{v_i\}_{i \in S_{j+1}^\pm} \) as in Step (iv). Write \( v_j \in \mathbb{V} \sim \mathbb{V}_\text{mod} = \mathbb{V}(F_{\text{mod}}) \) for the element corresponding to \( v_i \). We would like to apply Proposition 1.4, (iv), to the present situation, by taking

- "I" to be \( S_{j+1}^\pm \);
- "k_i" to be \( K_{v_i} \) [so "R_i" will be the ring of integers \( O_{K_{v_i}} \) of \( K_{v_i} \)].

Here, we note that our assumption that \( v_Q \in \mathbb{V}_Q^\text{non} \setminus \mathbb{V}_Q^\text{dst} \) implies that the hypotheses of Proposition 1.4, (iv), are satisfied. Thus, the inclusion "\( \phi((R_I)^{\sim}) \subseteq (R_I)^{\sim} \)" of Proposition 1.4, (iv), implies that the tensor product of log-shells under consideration contains the "union of possible images of a \( \Theta \)-pilot object" discussed in Step (iv). That is to say, the indeterminacies (Ind1) and (Ind2) are taken into account by the arbitrary nature of the automorphism "\( \phi \)" [cf. Proposition 1.2], while the indeterminacy (Ind3) is taken into account by the fact that we are considering upper bounds [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 5.12], together with the fact that the "container of possible images" is precisely equal to the tensor product of log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side "((R_I)^{\sim})" of the above inclusion. Such an upper bound "0"

is given in the final equality of Proposition 1.4, (iv). One may then compute a "weighted average upper bound" and then a "procession-normalized upper bound", as was done in Step (v). The resulting "procession-normalized upper bound" is clearly equal to 0.

(vii) Next, let \( v_Q \in \mathbb{V}_Q^\text{arc} \). Fix \( j, \{v_i\}_{i \in S_{j+1}^\pm} \) as in Step (iv). Write \( v_j \in \mathbb{V} \sim \mathbb{V}_\text{mod} = \mathbb{V}(F_{\text{mod}}) \) for the element corresponding to \( v_i \). We would like to apply Proposition 1.5, (iii), (iv), to the present situation, by taking

- "I" to be \( S_{j+1}^\pm \) [so \( |I| = j + 1 \)];
- "V" to be \( \mathbb{V}(F_{\text{mod}})_{v_i} \);
· "C_v" to be $K_v$, where we write $v \in V \stackrel{\sim}{\rightarrow} V_{\text{mod}}$ for the element determined by $v \in V$.

Then it follows from Proposition 1.5, (iii), (iv), that

$$\pi^{j+1} \cdot B_I$$

serves as a container for the “union of possible images of a $\Theta$-pilot object” discussed in Step (iv). That is to say, the indeterminacies (Ind1) and (Ind2) are taken into account by the fact that $B_I \subseteq M_I$ is preserved by arbitrary automorphisms of the type discussed in Proposition 1.5, (iii), while the indeterminacy (Ind3) is taken into account by the fact that we are considering upper bounds [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that, by Proposition 1.5, (iv), together with our choice of the factor $\pi^{j+1}$, this “container of possible images” contains the elements of $M_I$ obtained by forming the tensor product of elements of the log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of this container. Such an upper bound

$$(j + 1) \cdot \log(\pi)$$

follows immediately from the fact that [in order to ensure compatibility with arithmetic degrees of arithmetic line bundles — cf. [IUTchIII], Proposition 3.9, (iii) — one is obliged to adopt normalizations which imply that] the log-volume of $B_I$ is equal to 0. One may then compute a “weighted average upper bound” and then a “procession-normalized upper bound”, as was done in Step (v). The resulting “procession-normalized upper bound” is given by

$$\frac{l+1}{4} \cdot \log(\pi) \leq \frac{l+1}{4} \cdot 4$$

— cf. (E1), (E6); the fact that $l \geq 5$.

(viii) Now we return to the discussion of Step (iv). In order to compute the desired upper bound for “$C_\Theta$”, it suffices to sum over $v_Q \in V_Q$ the various local “procession-normalized upper bounds” obtained in Steps (v), (vi), (vii) for $v_Q \in V_Q$. By applying the inequality of the final display of Step (iii), we thus obtain the following upper bound for “$C_\Theta \cdot \log(q)$”, i.e., the product of “$C_\Theta$” and $\frac{1}{2l} \cdot \log(q)$:

$$\frac{l+1}{4} \cdot \left\{ (1 + \frac{36 \cdot d_{\text{mod}}}{l}) \cdot (\log(d^{F_{\text{ind}}}) + \log(f^{F_{\text{ind}}})) + 2 \cdot \log(l) + 74 - \frac{1}{6} \cdot (1 - \frac{12}{l^2}) \cdot \log(q) \\
+ \frac{20}{3} \cdot l^*_{\text{mod}} \cdot \log(s^{\leq}) \right\} - \frac{1}{2l} \cdot \log(q)$$

— where we apply the estimate $\frac{l+1}{4} \cdot \frac{1}{6} \cdot \frac{12}{l^2} \geq \frac{1}{2l}$ [cf. the fact that $l \geq 1$].

Now let us recall the constant “$\eta_{\text{prim}}$” of Proposition 1.6. By applying Proposition 1.6, we compute:

$$l^*_{\text{mod}} \cdot \log(s^{\leq}) = \log(d^*_{\text{mod}} \cdot l) \cdot \sum_{p \leq d^*_{\text{mod}}} 1 \leq \frac{4}{3} \cdot \log(d^*_{\text{mod}} \cdot l) \cdot \frac{d^*_{\text{mod}}}{\log(d^*_{\text{mod}} \cdot l)} = \frac{4}{3} \cdot d^*_{\text{mod}} \cdot l$$
— where the sum ranges over the primes \( p \leq d_{\text{mod}}^* \cdot l \) — if \( d_{\text{mod}}^* \cdot l \geq \eta_{\text{prm}}; \)

\[
l_{\text{mod}}^* \cdot \log(s^\leq) = \log(d_{\text{mod}}^* \cdot l) \cdot \sum_{p \leq d_{\text{mod}}^* \cdot l} 1 \leq \frac{4}{3} \cdot \log(\eta_{\text{prm}}) \cdot \frac{\eta_{\text{prm}}}{\log(\eta_{\text{prm}})} = \frac{4}{3} \cdot \eta_{\text{prm}}
\]

— where the sum ranges over the primes \( p \leq d_{\text{mod}}^* \cdot l \) — if \( d_{\text{mod}}^* \cdot l < \eta_{\text{prm}} \).

Thus, we conclude that

\[
l_{\text{mod}}^* \cdot \log(s^\leq) \leq \frac{4}{3} \cdot (d_{\text{mod}}^* \cdot l + \eta_{\text{prm}})
\]

[i.e., regardless of the size of \( d_{\text{mod}}^* \cdot l \)]. Also, let us observe that

\[
\frac{1}{3} \cdot \frac{4}{3} \cdot (d_{\text{mod}}^* \cdot l + \eta_{\text{prm}}) \geq \frac{1}{3} \cdot \frac{4}{3} \cdot d_{\text{mod}}^* \cdot l \geq 2 \cdot 2^{12} \cdot 3 \cdot 5 \cdot l \geq 2 \cdot \log(l) + 74
\]

— where we apply the estimates \( d_{\text{mod}} \geq 1, \quad 2^{12} \cdot 3 \cdot 5 \geq 74, \quad l \geq 5 \geq 1, \quad l \geq \log(l) \) [cf. the fact that \( l \geq 5 \)]. Thus, substituting back into our original upper bound for \( "C_{\Theta} \cdot |\log(q)|" \), we obtain the following upper bound for \( "C_{\Theta}" \):

\[
\frac{l+1}{4 \cdot |\log(q)|} \cdot \left\{ (1 + \frac{36 \cdot d_{\text{mod}}}{l}) \cdot (\log(d_{F,\text{mod}}^*) + \log(f_{F,\text{mod}}^*)) + 10 \cdot (d_{\text{mod}}^* \cdot l + \eta_{\text{prm}}) - \frac{1}{6} \cdot (1 - \frac{12}{l^2}) \cdot \log(q) \right\} - 1
\]

— where we apply the estimate \( \frac{20+1}{4} \cdot \frac{4}{3} = \frac{7 \cdot 4}{3} \leq 10 \) — i.e., as asserted in the statement of Theorem 1.10. The final portion of Theorem 1.10 follows immediately from [IUTchIII], Corollary 3.12, by applying the inequality of the first display of Step (ii), together with the estimates

\[
(1 - \frac{12}{l^2})^{-1} \leq 2; \quad (1 + \frac{12}{l^2})^{-1} \cdot (1 + \frac{36 \cdot d_{\text{mod}}}{l}) \leq 1 + \frac{80 \cdot d_{\text{mod}}}{l}
\]

[cf. the fact that \( l \geq 5, d_{\text{mod}} \geq 1 \)]. ⊓⊔

**Remark 1.10.1.** One of the main original motivations for the development of the theory discussed in the present series of papers was to create a framework, or *geometry*, within which a suitable analogue of the *scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII]* could be realized in such a way that the *obstructions to diophantine applications* that arose in the scheme-theoretic formulation of [HASurI], [HASurII] [cf. the discussion of [HASurI], §1.5.1; [HASurII], Remark 3.7] could be avoided. From this point of view, it is of interest to observe that the computation of the “leading term” of the inequality of the final display of the statement of Theorem 1.10 — i.e., of the term

\[
\frac{(l^*+3)}{2} \cdot \log(q_{vq}^K) - \frac{(2l^*+1)(l^*+1)}{12l} \cdot \log(q_{vq})
\]

that occurs in the final display of Step (v) of the proof of Theorem 1.10 — via the identities (E1), (E2) is essentially identical to the computation of the leading term that occurs in the proof of [HASurI], Theorem A [cf. the discussion following the statement of Theorem A in [HASurI], §1.1]. That is to say, in some sense,
the computations performed in the proof of Theorem 1.10 were already essentially known to the author around the year 2000; the problem then was to construct an appropriate framework, or geometry, in which these computations could be performed!

This sort of situation may be compared to the computations underlying the Weil Conjectures prior to the construction of a “Weil cohomology” in which those computations could be performed, or, alternatively, to various computations of invariants in topology or differential geometry that were motivated by computations in physics, again prior to the construction of a suitable mathematical framework in which those computations could be performed.

Remark 1.10.2. The computation performed in the proof of Theorem 1.10 may be thought of as the computation of a sort of derivative in the $\mathbb{F}_l^*$-direction, which, relative to the analogy between the theory of the present series of papers and the $p$-adic Teichmüller theory of $[p\text{Ord}], [p\text{Teich}]$, corresponds to the derivative of the canonical Frobenius lifting — cf. the discussion of [IUTchIII], Remark 3.12.4, (iii). In this context, it is useful to recall the arithmetic Kodaira-Spencer morphism that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], §3]. In particular, in [HASurII], Corollary 3.6, it is shown that, when suitably formulated, a “certain portion” of this arithmetic Kodaira-Spencer morphism coincides with the usual geometric Kodaira-Spencer morphism. From the point of view of the action of $GL_2(\mathbb{F}_l)$ on the $l$-torsion points involved, this “certain portion” consists of the unipotent matrices

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$. By contrast, the $\mathbb{F}_l^*$-symmetries that occur in the present series of papers correspond to the toral matrices

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$ — cf. the discussion of [IUTchI], Example 4.3, (i). As we shall see in §2 below, in the present series of papers, we shall ultimately take $l$ to be “large”. When $l$ is “sufficiently large”, $GL_2(\mathbb{F}_l)$ may be thought of as a “good approximation” for $GL_2(\mathbb{Z})$ or $GL_2(\mathbb{R})$ — cf. the discussion of [IUTchI], Remark 6.12.3, (i), (iii). In the case of $GL_2(\mathbb{R})$, “toral subgroups” may be thought of as corresponding to the isotropy subgroups [isomorphic to $\mathbb{S}^1$] of points that arise from the action of $GL_2(\mathbb{R})$ on the upper half-plane, i.e., subgroups which may be thought of as a sort of geometric, group-theoretic representation of tangent vectors at a point.

Remark 1.10.3. The “terms involving $l$” that occur in the inequality of the final display of Theorem 1.10 may be thought of as an inevitable consequence of the fundamental role played in the theory of the present series of papers by the $l$-torsion points of the elliptic curve under consideration. Here, we note that it is of crucial importance to work over the field of rationality of the $l$-torsion points [i.e., “$K$” as opposed to “$F$”] not only when considering the global portions of the various $\Theta NF$-
and $\Theta^{\pm\text{ell}}$-Hodge-theaters involved, but also when considering the local portions — i.e., the prime-strips — of these $\Theta^{\text{NF}}$- and $\Theta^{\pm\text{ell}}$-Hodge-theaters. That is to say, these local portions are necessary, for instance, in order to glue together the $\Theta^{\text{NF}}$- and $\Theta^{\pm\text{ell}}$-Hodge-theaters that appear so as to form a $\Theta^{\pm\text{ell}}$ NF-Hodge-theater [cf. the discussion of [IUTchI], Remark 6.12.2]. In particular, to allow, within these local portions, any sort of “Galois indeterminacy” with respect to the $l$-torsion points — even, for instance, at $v \in \mathbb{V}^{\text{good}} \cap \mathbb{V}^{\text{non}}$, which, at first glance, might appear irrelevant to the theory of Hodge-Arakelov-theoretic evaluation at $l$-torsion points developed in [IUTchII] — would have the effect of invalidating the various delicate manipulations involving $l$-torsion points discussed in [IUTchI], §4, §6 [cf., e.g., [IUTchI], Propositions 4.7, 6.5].

**Remark 1.10.4.** The various fluctuations in log-volume — i.e., whose computation is the subject of Theorem 1.10! — that arise from the multiradial representation of [IUTchIII], Theorem 3.11, (i), may be thought of as a sort of “inter-universal analytic torsion”. Indeed,

in general, “analytic torsion” may be understood as a sort of measure — in “metrized” [e.g., log-volume!] terms — of the degree of deviation of the “holomorphic functions” [such as sections of a line bundle] on a variety — i.e., which depend, in an essential way, on the holomorphic moduli of the variety! — from the “real analytic functions” — i.e., which are invariant with respect to deformations of the holomorphic moduli of the variety.

For instance:

(a) In “classical” Arakelov theory, analytic torsion typically arises as [the logarithm of] a sort of normalized determinant of the Laplacian acting on some space of real analytic [or $L^2$-] sections of a line bundle on a complex variety equipped with a real analytic Kähler metric [cf., e.g., [Arak], Chapters V, VI]. Here, we recall that in this sort of situation, the space of holomorphic sections of the line bundle is given by the kernel of the Laplacian; the definition of the Laplacian depends, in an essential way, on the Kähler metric, hence, in particular, on the holomorphic moduli of the variety under consideration [cf. e.g., the case of the Poincaré metric on a hyperbolic Riemann surface!].

(b) In the scheme-theoretic Hodge-Arakelov theory discussed in [HASurI], [HASurII], the main theorem consists of a sort of comparison isomorphism [cf. [HASurI], Theorem A] between a certain subspace of the space of global sections of the pull-back of an ample line bundle on an elliptic curve to the universal vectorial extension of the elliptic curve and the space of set-theoretic functions on the torsion points of the elliptic curve. That is to say, the former space of sections contains, in a natural way, the space of holomorphic sections of the ample line bundle on the elliptic curve, while the latter space of functions may be thought of as a sort of “discrete approximation” of the space of real analytic functions on the elliptic curve [cf. the discussion of [HASurI], §1.3.2, §1.3.4]. In this context, the “Gaussian poles” [cf. the discussion of [HASurI], §1.1] arise as a measure of the discrepancy of integral structures between these two spaces in a neighborhood of the divisor at infinity of
the moduli stack of elliptic curves, hence may be thought of as a sort of “analytic torsion at the divisor at infinity” [cf. the discussion of [HASurI], §1.2].

(c) In the case of the multiradial representation of [IUTchIII], Theorem 3.11, (i), the fluctuations of log-volume computed in Theorem 1.10 arise precisely as a result of the execution of a comparison of an “alien” arithmetic holomorphic structure to this multiradial representation, which is compatible with the permutation symmetries of the étale-picture, i.e., which is “invariant with respect to deformations of the arithmetic holomorphic moduli of the number field under consideration” in the sense that it makes sense simultaneously with respect to distinct arithmetic holomorphic structures [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)]. Here, it is of interest to observe that the object of this comparison consists of the values of the theta function, i.e., in essence, a “holomorphic section of an ample line bundle”. In particular, the resulting fluctuations of log-volume may be thought as a sort of “analytic torsion”. By analogy to the terminology “Gaussian poles” discussed in (b) above, it is natural to think of the terms involving the different \( \delta^K_\cdot \) that appear in the computation underlying Theorem 1.10 [cf., e.g., the final display of Step (v) of the proof of Theorem 1.10] as “differential poles” [cf. the discussion of Remarks 1.10.1, 1.10.2]. Finally, in the context of the normalized determinants that appear in (a), it is interesting to note the role played by the prime number theorem — i.e., in essence, the Riemann zeta function [cf. Proposition 1.6 and its proof] — in the computation of “inter-universal analytic torsion” given in the proof of Theorem 1.10.

Remark 1.10.5. The above remarks focused on the conceptual aspects of the theory surrounding Theorem 1.10. Before proceeding, however, we pause to discuss briefly certain aspects of Theorem 1.10 that are of interest from a computational point of view, i.e., in the spirit of conventional analytic number theory.

(i) First, we begin by observing that, unlike the inequalities that appear in the various results [cf. Corollaries 2.2, (ii); 2.3] obtained in §2 below, the inequalities obtained in Theorem 1.10 involve only essentially explicit constants and, moreover, do not require one to exclude some non-explicit finite set of “isomorphism classes of exceptional elliptic curves”. From this point of view, the inequalities obtained in Theorem 1.10 are suited to application to computations concerning various explicit diophantine equations, such as, for instance, the equations that appear in “Fermat’s Last Theorem”.

Such explicit computations in the case of specific diophantine equations are, however, beyond the scope of the present paper.

(ii) One topic of interest in the context of computational aspects of Theorem 1.10 is the asymptotic behavior of the bound that appears in, say, the first inequality of the final display of Theorem 1.10. Let us assume, for simplicity, that \( F_{tpd} = \mathbb{Q} \) [so \( d_{mod} = 1 \)]. Also, to simplify the notation, let us write \( \delta \overset{\text{def}}{=} \log(\delta^{F_{tpd}}) + \log(\delta^{F_{tpd}}) = \log(\delta^{F_{tpd}}) \). Then the bound under consideration assumes the form

\[
\delta + \ast \cdot \frac{d}{l} + \ast \cdot l + \ast
\]
— where, in the present discussion, the “∗’s” are to be understood as denoting fixed positive real numbers. Thus, the leading term [cf. the discussion of Remark 1.10.1] is equal to δ. The remaining terms give rise to the “ε terms” [and bounded discrepancy] of the inequalities of Corollaries 2.2, (ii); 2.3, obtained in §2 below. Thus, if one ignores “bounded discrepancies”, it is of interest to consider the behavior of the “ε terms”

\[ * \cdot \frac{\delta}{2} + * \cdot l \]

as one allows the initial Θ-data under consideration to vary [i.e., subject to the condition “\( F_{\text{tpd}} = \mathbb{Q} \)”). In this context, one fundamental observation is the following: although \( l \) is subject to various other conditions, no matter how “skillfully” one chooses \( l \), the resulting “ε terms” are always

\[ \geq * \cdot \delta^{1/2} \]

— an estimate that may be obtained by thinking of \( l \) as \( \approx \delta^\alpha \), for some real number \( \alpha \), and comparing \( \delta^\alpha \) and \( \delta^{1-\alpha} \). This estimate is of particular interest in the context of various explicit examples constructed by Masser and others [cf. [Mss]; the discussion of [vFr], §2] in which explicit “abc sums” are constructed for which the quantity on the left-hand side of the inequality of Theorem 1.10 under consideration exceeds the order of \( \delta + * \cdot \frac{\delta^{1/2}}{\log(\delta)} \)

— cf. [vFr], Equation (6). In particular, the asymptotic estimates given by Theorem 1.10 are consistent with the known asymptotic behavior of these explicit abc sums. Indeed, the exponent “1/2” that appears in the fundamental observation discussed above coincides precisely with the “expectation” expressed by van Frankenhuijsen in the final portion of the discussion of [vFr], §2! In the present paper, although we are unable to in fact achieve bounds on the “ε terms” of the order \( * \cdot \delta^{1/2} \), we do succeed in obtaining bounds on the “ε terms” of the order

\[ * \cdot \delta^{1/2} \cdot \log(\delta) \]

— albeit under the assumption that the abc sums under consideration are compactly bounded away from infinity at the prime 2, as well as at the archimedean prime [cf. Corollary 2.2, (ii); Remark 2.2.1 below for more details].

(iii) In the context of the discussion of (ii), it is of interest to observe that the “∗ · l” portion of the “ε terms” that appear arises from the estimates given in Step (viii) of the proof of Theorem 1.10 for the quantity “\( \log(s^{\leq}) \)”.

From the point of view of the discussion of [vFr], §3, this quantity corresponds essentially to a “certain portion” of the quantity “\( \omega(abc) \)” associated to an abc sum. That is to say, whereas “\( \omega(abc) \)” denotes the total number of prime factors that occur in the product abc, the quantity “\( \log(s^{\leq}) \)” corresponds, roughly speaking, to the number of these prime factors that are \( \leq d_{\text{mod}} \cdot l \). The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to “\( \omega(abc) \)” is of interest from the point of view of the discussion of [vFr], §3, partly since it is [not precisely identical to, but nonetheless] reminiscent of the various refinements of the ABC Conjecture proposed by Baker [i.e., which are the main topic of the discussion of
[vFr], §3. The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to “ω(abc)” is also of interest from the point of view of the explicit abc sums discussed in (ii) that give rise to asymptotic behavior \( \geq * \cdot \frac{\delta^{1/2}}{\log(\delta)} \). That is to say, according to the discussion of [vFr], §3, Remark 1, this sort of abc sum tends to give rise to a

**relatively large** value for \( \omega(abc) \) — i.e., a state of affairs that is consistent with the crucial role played by the “\( \epsilon \) term” related to \( \omega(abc) \) in the computation of the lower bound \( \geq * \cdot \delta^{1/2} \) that appears in the fundamental observation of (ii).

By contrast, the abc sums of the form “\( 2^n = p + qr \)” [where \( p, q, \) and \( r \) are prime numbers] considered in [vFr], §3, Remark 1, give rise to a

**relatively small** value for \( \omega(abc) \) [indeed, \( \omega(abc) \leq 4 \)] — i.e., a situation that suggests relatively small/essentially negligible “\( \epsilon \) terms” in the bound of Theorem 1.10 under consideration.

Such essentially negligible “\( \epsilon \) terms” are, however, consistent with the fact [cf. [vFr], §3, Remark 1] that, for such abc sums, the left-hand side of the inequality of Theorem 1.10 under consideration is roughly \( \approx \frac{1}{2} \cdot \text{the leading term} \) of the bound on the right-hand side, hence, in particular, is amply bounded by the leading term on the right-hand side, without any “help” from the “\( \epsilon \) terms”.

**Remark 1.10.6.**

(i) In the context of the discussion of Remark 1.10.5, it is important to remember that

the bound on \( \frac{1}{5} \cdot \log(q) \) given in Theorem 1.10 only concerns the \( q \)-parameters at the nonarchimedean valuations contained in \( \mathcal{V}_{\text{bad}} \), all of which are necessarily odd residue characteristic — cf. [IUTchI], Definition 3.1, (b). This observation is of relevance to the examples of abc sums constructed in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], since it does not appear, at first glance, that there is any way to effectively control the contributions at the prime 2 in these examples, that is to say, in the notation of the Proposition of [Mss], to control the power of 2 that divides the integer “\( \epsilon \)” of the Proposition of [Mss], or, alternatively, in the notation of the proof of this Proposition on [Mss], p. 22, to control the power of 2 that divides the difference \( x_i - x_{i-1} \). On the other hand, it was pointed out to the author by A. Venkatesh that in fact it is not difficult to modify the construction of these examples of abc sums given in [Mss] so as to obtain similar asymptotic estimates to those obtained in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], **even without taking into account the contributions at the prime 2**.

(ii) In the context of the discussion of (i), it is of interest to recall why nonarchimedean primes of even residue characteristic where the elliptic curve under
consideration has bad multiplicative reduction are excluded from $V_{\text{bad mod}}$ in the theory of the present series of papers. In a word, the reason that the theory encounters difficulties at primes over 2 is that it depends, in a quite essential way, on the theory of the \textit{étale theta function} developed in [EtTh], which fails at primes over 2 [cf. the assumption that “$p$ is odd” in [EtTh], Theorem 1.10, (iii); [EtTh], Definition 2.5; [EtTh], Corollary 2.18]. From the point of view of the theory of [IUTchI], [IUTchII], and [IUTchIII] [cf., especially, the theory of [IUTchII], §1, §2: [IUTchII], Corollary 1.12; [IUTchII], Corollary 2.4, (ii), (iii); [IUTchII], Corollary 2.6], one of the key consequences of the theory of [EtTh] is the \textit{simultaneous multiradiality} of the algorithms that give rise to

\begin{enumerate}
\item constant multiple rigidity
\item cyclotomic rigidity
\end{enumerate}

At a more concrete level, (1) is obtained by evaluating the usual series for the theta function [cf. [EtTh], Proposition 1.4] at the 2-torsion point in the “irreducible component labeled zero”. One computes easily that the resulting “special value” is a unit for odd $p$, but is equal to a [nonzero] non-unit when $p = 2$. In particular, since (1) is established by dividing the series of [EtTh], Proposition 1.4 [i.e., the usual series for the theta function], by this special value, it follows that

\begin{enumerate}
\item the “integral structure” on the theta function determined by this special value coincides with
\item the “integral structure” on the theta function determined by the natural integral structure on the pole at the origin
\end{enumerate}

for odd $p$ [cf. [EtTh], Theorem 1.10, (iii)], but not when $p = 2$. That is to say, when $p = 2$, a nontrivial denominator arises. Here, we recall that it is crucial to evaluate at 2-torsion points, i.e., as opposed to, say, more general points in the irreducible component labeled zero for reasons discussed in [IUTchII], Remark 2.5.1, (ii) [cf. also the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv)]. This nontrivial denominator is fundamentally incompatible with the \textit{multiradiality} of the algorithms of (1), (2) in that it is incompatible with the \textit{fundamental splitting}, or “decoupling”, into “purely radial” [i.e., roughly speaking, “value group”] and “purely coric” [i.e., roughly speaking, “unit”] components discussed in [IUTchII], Remarks 1.11.4, (i); 1.12.2, (vi) [cf. also the discussion of [IUTchII], Remark 1.11.5]. That is to say, on the one hand,

the \textit{multiradiality} of (1) may only be established if the possible values at the evaluation points in the irreducible component labeled zero are known, a priori, to be units, i.e., if one works relative to the integral structure (a)

— cf. the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv). On the other hand, if one tries to work simultaneously with

---
the integral structure (b), hence with the nontrivial denominator discussed above, then the multiradiality of (2) is violated.

Here, we recall that the integral structure (b), which is referred to as the “canonical integral structure” in [EtTh], Proposition 1.4, (iii); [EtTh], Theorem 1.10, (iii), is in some sense the “integral structure of common sense”.

(iii) It is not entirely clear to the author at the time of writing to what extent the integral structure (b) is necessary in order to carry out the theory developed in the present series of papers. Indeed, [EtTh], as well as the present series of papers, was written in a way that [unlike the discussion of (ii)!] “takes for granted” the fact that the two integral structures (a), (b) discussed above coincide for odd $p$, i.e., in a way which identifies these two integral structures and hence does not specify, at various key points in the discussion, whether one is in fact working with integral structure (a) or with integral structure (b). On the other hand, if it is indeed the case that not only the integral structure (a), but also the integral structure (b) plays an essential role in the present series of papers, then it follows [cf. the discussion of (ii)!] that the theory of the present series of papers is fundamentally incompatible with the inclusion in $V_{\text{bad}}^{\text{mod}}$ of nonarchimedean primes of even residue characteristic where the elliptic curve under consideration has bad multiplicative reduction.

(iv) In the context of the discussion of (ii), (iii), it is perhaps useful to recall that the classical theory of theta functions also tends to [depending on your point of view!] “break down” or “assume a completely different form” at the prime 2. For instance, this phenomenon can be seen throughout Mumford’s theory of algebraic theta functions, which may be thought of as a sort of predecessor to the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII], which, in turn, may be thought of as a sort of predecessor to the theory of the present series of papers. In a similar vein, it is of interest to recall that the prime 2 is also excluded in the $p$-adic Teichmüler theory of [pOrd], [pTeich]. This is done in order to avoid the complications that occur in the theory of the Lie algebra $sl_2$ over fields of characteristic 2.
Section 2: Diophantine Inequalities

In the present §2, we combine Theorem 1.10 with the theory of [GenEll] to give a proof of the **ABC Conjecture**, or, equivalently, **Vojta’s Conjecture for hyperbolic curves** [cf. Corollary 2.3 below].

We begin by reviewing some well-known estimates.

**Proposition 2.1. (Well-known Estimates)**

(i) **(Linearization of Logarithms)** We have \( \log(x) \leq x \) for all \( (\mathbb{R} \ni) x \geq 1 \).

(ii) **(The Prime Number Theorem)** There exists a real number \( \xi_{\text{prm}} \geq 5 \) such that
\[
\frac{2}{3} \cdot x \leq \theta(x) \overset{\text{def}}{=} \sum_{p \leq x} \log(p) \leq \frac{4}{3} \cdot x
\]
— where the sum ranges over the prime numbers \( p \) such that \( p \leq x \) — for all \( (\mathbb{R} \ni) x \geq \xi_{\text{prm}} \). In particular, if \( \mathcal{A} \) is a finite set of prime numbers, and we write
\[
\theta_{\mathcal{A}} \overset{\text{def}}{=} \sum_{p \in \mathcal{A}} \log(p)
\]
[where we take the sum to be 0 if \( \mathcal{A} = \emptyset \)], then there exists a prime number \( p \not\in \mathcal{A} \) such that \( p \leq 2(\theta_{\mathcal{A}} + \xi_{\text{prm}}) \).

**Proof.** Assertion (i) is well-known and entirely elementary. Assertion (ii) is a well-known consequence of the Prime Number Theorem [cf., e.g., [Edw], p. 76; [GenEll], Lemma 4.1; [GenEll], Remark 4.1.1].

Let \( \overline{\mathbb{Q}} \) be an algebraic closure of \( \mathbb{Q} \). In the following discussion, we shall apply the notation and terminology of [GenEll]. Let \( X \) be a smooth, proper, geometrically connected curve over a number field; \( D \subseteq X \) a reduced divisor; \( U_X \overset{\text{def}}{=} X \setminus D; \) \( d \) a positive integer. Write \( \omega_X \) for the canonical sheaf on \( X \). Suppose that \( U_X \) is a **hyperbolic curve**, i.e., that the degree of the line bundle \( \omega_X(D) \) is positive. Then we recall the following notation:

- \( U_X(\overline{\mathbb{Q}}) \overset{\leq d}{\subseteq} U_X(\overline{\mathbb{Q}}) \) denotes the subset of \( \overline{\mathbb{Q}} \)-rational points defined over a finite extension field of \( \mathbb{Q} \) of degree \( \leq d \) [cf. [GenEll], Example 1.3, (i)].
- \( \log \text{-diff}_X \) denotes the (normalized) log-different function on \( U_X(\overline{\mathbb{Q}}) \) [cf. [GenEll], Definition 1.5, (iii)].
- \( \log \text{-cond}_D \) denotes the (normalized) log-conductor function on \( U_X(\overline{\mathbb{Q}}) \) [cf. [GenEll], Definition 1.5, (iv)].
- \( \text{ht}_{\omega_X(D)} \) denotes the (normalized) height function on \( U_X(\overline{\mathbb{Q}}) \) associated to \( \omega_X(D) \), which is well-defined up to a “bounded discrepancy” [cf. [GenEll], Proposition 1.4, (iii)].
In order to apply the theory of the present series of papers, it is necessary to construct suitable initial Θ-data, as follows.

**Corollary 2.2. (Construction of Suitable Initial Θ-Data)** Suppose that $X = \mathbb{P}^1_\mathbb{Q}$ is the projective line over $\mathbb{Q}$, and that $D \subseteq X$ is the divisor consisting of the three points “0”, “1”, and “∞”. We shall regard $X$ as the “λ-line” — i.e., we shall regard the standard coordinate on $X = \mathbb{P}^1_\mathbb{Q}$ as the “λ” in the Legendre form $y^2 = x(x-1)(x-\lambda)$ of the Weierstrass equation defining an elliptic curve — and hence as being equipped with a natural classifying morphism $U_X \rightarrow (\mathcal{M}_{\text{ell}})_\mathbb{Q}$ [cf. the discussion preceding Proposition 1.8]. Let

$$K_V \subseteq U_X(\overline{\mathbb{Q}})$$

be a compactly bounded subset [i.e., regarded as a subset of $X(\overline{\mathbb{Q}})$ — cf. Remark 2.3.1, (vi), below; [GenEll], Example 1.3, (ii)] whose support contains the nonarchimedean prime “2”. Suppose further that $K_V$ satisfies the following condition:

\((*)_{j\text{-inv}}\) If $v \in V(\mathbb{Q})$ denotes the nonarchimedean prime “2”, then the image of the subset $K_v \subseteq U_X(\overline{\mathbb{Q}}_v)$ associated to $K_V$ [cf. the notational conventions of [GenEll], Example 1.3, (ii)] via the $j$-invariant $U_X \rightarrow (\mathcal{M}_{\text{ell}})_\mathbb{Q} \rightarrow \mathbb{A}^1_\mathbb{Q}$ is a bounded subset of $\mathbb{A}^1_\mathbb{Q}(\overline{\mathbb{Q}}_v) = \overline{\mathbb{Q}}_v$, i.e., is contained in a subset of the form $2^{N_{j\text{-inv}}} \cdot \mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$, where $N_{j\text{-inv}} \in \mathbb{Z}$, and $\mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$ denotes the ring of integers.

Then:

(i) Write “$\log(q_v^{1/2})$” (respectively, “$\log(q_v^{1/2})$”) for the $\mathbb{R}$-valued function on $\mathcal{M}_{\text{ell}}(\overline{\mathbb{Q}})$, hence also on $U_X(\overline{\mathbb{Q}})$, obtained by forming the normalized degree “$\deg(-)$” of the effective arithmetic divisor determined by the $q$-parameters of an elliptic curve over a number field at arbitrary nonarchimedean primes (respectively, at the nonarchimedean primes that do not divide 2) [cf. the invariant “$\log(q)$” associated, in the statement of Theorem 1.10, to the elliptic curve $E_F$]. Also, we shall write $\text{ht}_\infty$ for the (normalized) height function on $U_X(\overline{\mathbb{Q}})$ — a function which is well-defined up to a “bounded discrepancy” [cf. the discussion preceding [GenEll], Proposition 3.4] — determined by the pull-back to $X$ of the divisor at infinity of the natural compactification $(\mathcal{M}_{\text{ell}})_\mathbb{Q}$ of $(\mathcal{M}_{\text{ell}})_\mathbb{Q}$. Then we have an equality of “bounded discrepancy classes” [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]

$$\frac{1}{6} \cdot \log(q_v^{1/2}) \approx \frac{1}{6} \cdot \log(q_v^{1/2}) \approx \frac{1}{6} \cdot \text{ht}_\infty \approx \text{ht}_{\omega_X(D)}$$

of functions on $K_V \subseteq U_X(\overline{\mathbb{Q}})$.

(ii) There exist

- a positive real number $H_{\text{unif}}$ which is independent of $K_V$ and
- positive real numbers $C_K$ and $H_K$ which depend only on the choice of the compactly bounded subset $K_V$.
such that the following property is satisfied: Let \( d \) be a positive integer, \( \epsilon_d \) a positive real number \( \leq 1 \). Set \( d'^* \overset{\text{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d \). Then there exists a finite subset \( \mathcal{E}_{\mathfrak{c}} d \subseteq U_X(\overline{\mathbb{Q}}) \leq d \) which depends only on \( K_V, d, \) and \( \epsilon_d \), contains all points corresponding to elliptic curves that admit automorphisms of order \( > 2 \), and satisfies the following property:

The function \( \log(q^{\varphi}_\gamma(\cdot)) \) of (i) is

\[
\leq H_{\text{unif}} \cdot \epsilon_d^{-3} \cdot d^{1+\epsilon_d} + H_K
\]
on \( \mathcal{E}_{\mathfrak{c}} d \). Let \( E_F \) be an elliptic curve over a number field \( F \subset \mathbb{Q} \) that determines a \( \overline{\mathbb{Q}} \)-valued point of \( (\mathcal{M}_{\text{ell}})_{\mathbb{Q}} \) which lifts [not necessarily uniquely!] to a point \( x_E \in U_X(F) \cap U_X(\overline{\mathbb{Q}}) \leq d \) such that

\[
x_E \in K_V, \quad x_E \notin \mathcal{E}_{\mathfrak{c}} d.
\]

Write \( F_{\text{mod}} \) for the minimal field of definition of the corresponding point \( x \in \mathcal{M}_{\text{ell}}(\overline{\mathbb{Q}}) \) and

\[
F_{\text{mod}} \subseteq F_{\text{tpd}} \overset{\text{def}}{=} F_{\text{mod}}( E_{F_{\text{mod}}}[2] ) \subseteq F
\]

for the “tripodal” intermediate field obtained from \( F_{\text{mod}} \) by adjoining the fields of definition of the 2-torsion points of any model of \( E_F \times F \overline{\mathbb{Q}} \) over \( F_{\text{mod}} \) [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the \( (3\cdot 5) \)-torsion points of \( E_F \) are defined over \( F \), and that

\[
F = F_{\text{mod}}( \sqrt{-1}, E_{F_{\text{mod}}}[2 \cdot 3 \cdot 5] ) \overset{\text{def}}{=} F_{\text{tpd}}( \sqrt{-1}, E_{F_{\text{tpd}}}[3 \cdot 5] )
\]

— i.e., that \( F \) is obtained from \( F_{\text{tpd}} \) by adjoining \( \sqrt{-1} \), together with the fields of definition of the \( (3 \cdot 5) \)-torsion points of a model \( E_{F_{\text{tpd}}} \) of the elliptic curve \( E_F \times F \overline{\mathbb{Q}} \) over \( F_{\text{tpd}} \) determined by the Legendre form of the Weierstrass equation discussed above [cf. Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that \( E_F \cong E_{F_{\text{tpd}}} \times F_{\text{tpd}} \) over \( F \), so \( x_E \in U_X(F_{\text{tpd}}) \subseteq U_X(F) \); it follows from Proposition 1.8, (v), that \( E_F \) has stable reduction at every element of \( \mathbb{V}(F)^{\text{non}}. \) Write \( \log(q^{\varphi}) \) (respectively, \( \log(q^{\varphi^2}) \)) for the result of applying the function \( \log(q^{\varphi}_\gamma(\cdot)) \) (respectively, \( \log(q^{\varphi^2}_\gamma(\cdot)) \)) of (i) to \( x_E \). Then \( E_F \) and \( F_{\text{mod}} \) arise as the “\( E_F \)” and “\( F_{\text{mod}} \)” for a collection of initial \( \Theta \)-data as in Theorem 1.10 that, in the notation of Theorem 1.10, satisfies the following conditions:

(C1) \( (\log(q^{\varphi}))^{1/2} \leq l \leq 10d^* \cdot (\log(q^{\varphi}))^{1/2} \cdot \log(2d^* \cdot \log(q^{\varphi})) \);

(C2) we have inequalities

\[
\frac{1}{5} \cdot \log(q^{\varphi}) \leq \frac{1}{6} \cdot \log(q^{\varphi^2}) \leq \frac{1}{6} \cdot \log(q^{\varphi})
\]

\[
\leq (1 + \epsilon_E) \cdot (\log-diff_X(x_E) + \log-\text{cond}_D(x_E)) + C_K
\]

— where we write

\[
\epsilon_E \overset{\text{def}}{=} (60d^*)^2 \cdot \frac{\log(2d^* \cdot \log(q^{\varphi}))}{(\log(q^{\varphi}))^{1/2}}
\]
[i.e., so $\epsilon_E$ depends on the integer $d$, as well as on the elliptic curve $E_F$!], and we observe, relative to the notation of Theorem 1.10, that [it follows tautologically from the definitions that] we have an equality

$$\log\text{-}\text{diff}_X(x_E) = \log(d^{F_{tpd}}),$$

as well as inequalities

$$\log(d^{F_{tpd}}) \leq \log\text{-}\text{cond}_D(x_E) \leq \log(d^{F_{tpd}}) + \log(2l).$$

(iii) The positive real number $H_{\text{unit}}$ of (ii) [which is independent of $K_V$!] may be chosen in such a way that the following property is satisfied: Let $d$ be a positive integer, $\epsilon_d$ and $\epsilon$ positive real numbers $\leq 1$. Then there exists a finite subset $\text{Exc}_{\epsilon,d} \subseteq U_X(Q)$ which depends only on $K_V$, $\epsilon$, $d$, and $\epsilon_d$ such that the function “$\log(q^{\epsilon}_{(-)})$” of (i) is

$$\leq H_{\text{unit}} \cdot \epsilon^{-3} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d}$$

on $\text{Exc}_{\epsilon,d}$, and, moreover, in the notation of (ii), the invariant $\epsilon_E$ associated to an elliptic curve $E_F$ as in (ii) [i.e., that satisfies certain conditions which depend on $K_V$ and $d$] satisfies the inequality $\epsilon_E \leq \epsilon$ whenever the point $x_E \in U_X(F)$ satisfies the condition $x_E \notin \text{Exc}_{\epsilon,d}$.

Proof. First, we consider assertion (i). We begin by observing that, in light of the condition $(\ast_{j\text{-inv}})$ that was imposed on $K_V$, it follows immediately from the various definitions involved that

$$\log(q^{\epsilon}_{(-)}) \approx \log(q^{\epsilon}_{(-)})$$

where we observe that the function “$\log(q^{\epsilon}_{(-)})$” may be identified with the function “$\text{deg}_{\infty}$” of the discussion preceding [GenEll], Proposition 3.4 — on $K_V \subseteq U_X(Q)$. In a similar vein, since the support of $K_V$ contains the unique archimedean prime of $Q$, it follows immediately from the various definitions involved [cf. also Remark 2.3.1, (vi), below] that

$$\log(q^{\epsilon}_{(-)}) \approx \text{ht}_{\infty}$$

on $K_V \subseteq U_X(Q)$ [cf. the argument of the final paragraph of the proof of [GenEll], Lemma 3.7]. Thus, we conclude that $\log(q^{\epsilon}_{(-)}) \approx \log(q^{\epsilon}_{(-)}) \approx \text{ht}_{\infty}$ on $K_V \subseteq U_X(Q)$. Finally, since [as is well-known] the pull-back to $X$ of the divisor at infinity of the natural compactification $(\mathcal{M}_{\text{ell}})_Q$ of $(\mathcal{M}_{\text{ell}})_Q$ is of degree 6, while the line bundle $\omega_X(D)$ is of degree 1, the equality of BD-classes $\frac{1}{6} \cdot \text{ht}_{\infty} \approx \text{ht}_{\omega_X(D)}$ on $K_V \subseteq U_X(Q)$ follows immediately from [GenEll], Proposition 1.4, (i), (iii). This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us recall that if the once-punctured elliptic curve associated to $E_F$ fails to admit an $F$-core, then there are only four possibilities for the $j$-invariant of $E_F$ [cf. [CanLift], Proposition 2.7]. Thus, if we take the set $\text{Exc}_d$ to be the [finite!] collection of points corresponding to these four $j$-invariants, then we may assume that the once-punctured elliptic curve associated to $E_F$ admits an $F$-core, hence, in particular, does not have any automorphisms of order $> 2$ over $Q$. In the discussion to follow, it will be necessary to enlarge
the finite set $\mathfrak{E}_d$ several times, always in a fashion that depends only on $K_V$, $d$, and $\epsilon_d$ [i.e., but not on $x_E$!] and in such a way that the function “$\log(q_v)$” of (i) is $\leq H_{\text{unif}} \cdot \epsilon_d^{-3} \cdot d^4 + H_K$ on $\mathfrak{E}_d$ for some positive real number $H_{\text{unif}}$ that is independent of $K_V$ and some positive real number $H_K$ that depends only on $K_V$ [i.e., but not on $d$ or $\epsilon_d$].

Next, let us write

$$h \overset{\text{def}}{=} \log(q^V) = \prod_{v \in \mathbb{V}(F)^{\text{non}}} h_v \cdot f_v \cdot \log(p_v)$$

— that is to say, $h_v = 0$ for those $v$ at which $E_F$ has good reduction; $h_v \in \mathbb{N}_{\geq 1}$ is the local height of $E_F$ [cf. [GenEll], Definition 3.3] for those $v$ at which $E_F$ has bad multiplicative reduction. Now it follows [by assertion (i); [GenEll], Proposition 1.4, (iv)] that the inequality $h^{1/2} \leq \xi_{\text{prim}} + \eta_{\text{prim}}$ [cf. the notation of Propositions 1.6; 2.1, (ii)] implies that there is only a finite number of possibilities for the $j$-invariant of $E_F$. Thus, by possibly enlarging the finite set $\mathfrak{E}_d$ in a fashion that depends only on $K_V$, $d$, and $\epsilon_d$ and in such a way that $h \leq H_{\text{unif}}$ on $\mathfrak{E}_d$ for some positive real number $H_{\text{unif}}$ that is independent of $K_V$, we may assume without loss of generality that the inequality

$$h^{1/2} \geq \xi_{\text{prim}} + \eta_{\text{prim}} \geq 5$$

holds. Thus, since $[F : \mathbb{Q}] \leq d^*$ [cf. the properties (E3), (E4), (E5) in the proof of Theorem 1.10], it follows that

$$d^* \cdot h^{1/2} \geq [F : \mathbb{Q}] \cdot h^{1/2} = \sum_{v} h^{-1/2} \cdot h_v \cdot f_v \cdot \log(p_v) \geq \sum_{h_v \geq h^{1/2}} h^{-1/2} \cdot h_v \cdot \log(p_v)$$

and

$$2d^* \cdot h^{1/2} \cdot \log(2d^* \cdot h) \geq 2 \cdot [F : \mathbb{Q}] \cdot h^{1/2} \cdot \log(2 \cdot [F : \mathbb{Q}] \cdot h)$$

$$\geq \sum_{h_v \neq 0} 2 \cdot h^{-1/2} \cdot \log(2 \cdot h_v \cdot f_v \cdot \log(p_v)) \cdot h_v \cdot f_v \cdot \log(p_v)$$

$$\geq \sum_{h_v \neq 0} h^{-1/2} \cdot \log(h_v) \cdot h_v \geq \sum_{h_v \geq h^{1/2}} h^{-1/2} \cdot \log(h_v) \cdot h_v$$

$$\geq \sum_{h_v \geq h^{1/2}} \log(h_v)$$

— where the sums are all over $v \in \mathbb{V}(F)^{\text{non}}$ [possibly subject to various conditions, as indicated], and we apply the elementary estimate $2 \cdot \log(p_v) \geq 2 \cdot \log(2) = \log(4) \geq 1$ [cf. the property (E6) in the proof of Theorem 1.10].

Thus, in summary, we conclude from the estimates made above that if we take

$$A$$

to be the [finite!] set of prime numbers $p$ such that $p$ either
(S1) is \(\leq h^{1/2}\),
(S2) divides a nonzero \(h_v\) for some \(v \in \mathbb{V}(F)_{\text{non}}\), or
(S3) is equal to \(p_v\) for some \(v \in \mathbb{V}(F)_{\text{non}}\) for which \(h_v \geq h^{1/2}\),

then it follows from Proposition 2.1, (ii), together with our assumption that \(h^{1/2} \geq \xi_{\text{prim}}\), that, in the notation of Proposition 2.1, (ii),

\[
\theta_A \leq 2 \cdot h^{1/2} + d^* \cdot h^{1/2} + 2d^* \cdot h^{1/2} \cdot \log(2d^* \cdot h)
\]

\[
\leq 4d^* \cdot h^{1/2} \cdot \log(2d^* \cdot h)
\]

\[
\leq -\xi_{\text{prim}} + 5d^* \cdot h^{1/2} \cdot \log(2d^* \cdot h)
\]

— where we apply the estimates \(d^* \geq 2\) and \(\log(2d^* \cdot h) \geq \log(4) \geq 1\) [cf. the property (E6) in the proof of Theorem 1.10]. In particular, it follows from Proposition 2.1, (i), (ii), together with our assumption that \(h^{1/2} \geq 5 \geq 1\), that there exists a prime number \(l\) such that

(P1) \((5 \leq)\ h^{1/2} \leq l \leq 10d^* \cdot h^{1/2} \cdot \log(2d^* \cdot h) (\leq 20 \cdot (d^*)^2 \cdot h^2)\) [cf. the condition (C1) in the statement of Corollary 2.2];

(P2) \(l\) does not divide any nonzero \(h_v\) for \(v \in \mathbb{V}(F)_{\text{non}}\);

(P3) if \(l = p_v\) for some \(v \in \mathbb{V}(F)_{\text{non}}\), then \(h_v < h^{1/2}\).

Next, let us observe that, again by possibly enlarging the finite set \(\mathcal{E} \mathcal{F} \mathcal{C}_d\) [in a fashion that depends only on \(K_V\), \(d\), and \(\varepsilon_d\) and in such a way that \(h \leq H_K\) on \(\mathcal{E} \mathcal{F} \mathcal{C}_d\) for some positive real number \(H_K\) that depends only on \(K_V\)], we may assume without loss of generality that, in the terminology of [GenEll], Lemma 3.5,

(P4) \(E_F\) does not admit an \(l\)-cyclic subgroup scheme.

Indeed, the existence of an \(l\)-cyclic subgroup scheme of \(E_F\) would imply that

\[
\frac{\log(q^\gamma)}{24} \leq 2 \cdot \log(l) + T_K
\]

— where we apply assertion (i), the displayed inequality of [GenEll], Lemma 3.5, and the final inequality of the display of [GenEll], Proposition 3.4; we take the “\(\varepsilon\)” of [GenEll], Lemma 3.5, to be 1; we write \(T_K\) for the positive real number [which depends only on the choice of the compactly bounded subset \(K_V\)] that results from the various “bounded discrepancies” implicit in these inequalities. Since \(l \geq 5\) [cf. (P1)], it follows that \(1 \leq 2 \cdot \log(l) \leq 48 \cdot \frac{\log(q^\gamma)}{24}\) [cf. the property (E6) in the proof of Theorem 1.10], and hence that the inequality of the preceding display implies that \(\log(q^\gamma)\) is bounded. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a finite number of possibilities for the \(j\)-invariant of \(E_F\). This completes the proof of the above observation.

Next, let us observe that it follows immediately from (P1), together with Proposition 2.1, (i), that

\[
h^{1/2} \cdot \log(l) \leq h^{1/2} \cdot \log(20 \cdot (d^*)^2 \cdot h^2) \leq 2 \cdot h^{1/2} \cdot \log(5d^* \cdot h)
\]

\[
\leq 8 \cdot h^{1/2} \cdot \log(2 \cdot (d^*)^{1/4} \cdot h^{1/4}) \leq 8 \cdot h^{1/2} \cdot 2 \cdot (d^*)^{1/4} \cdot h^{1/4}
\]

\[
= 16 \cdot (d^*)^{1/4} \cdot h^{3/4}
\]
— where we apply the estimates $20 \leq 5^2$ and $5 \leq 2^4$. In particular, it follows that, again by possibly enlarging the finite set $\mathfrak{C}_{\mathfrak{r}_d}$ [in a fashion that depends only on $\mathcal{K}_V$, $d$, and $\epsilon_d$ and in such a way that $h \leq H_{\text{unif}} \cdot d + H_{K}$ on $\mathfrak{C}_{\mathfrak{r}_d}$ for some positive real number $H_{\text{unif}}$ that is independent of $\mathcal{K}_V$ and some positive real number $H_K$ that depends only on $\mathcal{K}_V$], we may assume without loss of generality that

(P5) if we write $\mathbb{V}_{\text{bad}} \mod$ for the set of nonarchimedean valuations $\in \mathbb{V}_{\text{mod}}$ that do not divide $2l$ and at which $E_F$ has bad multiplicative reduction, then $\mathbb{V}_{\text{bad}} \mod \neq \emptyset$.

Indeed, if $\mathbb{V}_{\text{bad}} \mod = \emptyset$, then it follows, in light of the definition of $h$, from (P3), assertion (i), and the computation performed above, that

$$h \approx \log(q^{1/2}) \leq h^{1/2} \cdot \log(l) \leq 16 \cdot (d^*)^{1/4} \cdot h^{3/4}$$

— an inequality which implies that $h^{1/4}$, hence $h$ itself, is bounded. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a finite number of possibilities for the $j$-invariant of $E_F$. This completes the proof of the above observation. This property (P5) implies that

(P6) the image of the outer homomorphism $\text{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{F}_l)$ determined by the $l$-torsion points of $E_F$ contains the subgroup $SL_2(\mathbb{F}_l) \subseteq GL_2(\mathbb{F}_l)$.

Indeed, since, by (P5), $E_F$ has bad multiplicative reduction at some valuation $\in \mathbb{V}_{\text{mod}} \neq \emptyset$, (P6) follows formally from (P2), (P4), and [GenEll], Lemma 3.1, (iii) [cf. the proof of the final portion of [GenEll], Theorem 3.8].

Now it follows formally from (P1), (P2), (P5), and (P6) that, if one takes $\overline{F}$ to be $\overline{\mathbb{Q}}$, “$F$” to be the number field $F$ of the above discussion, “$X_F$” to be the once-punctured elliptic curve associated to $E_F$, “$l$” to be the prime number $l$ of the above discussion, and “$\mathbb{V}_{\text{bad}} \mod$” to be the set $\mathbb{V}_{\text{mod}}$ of (P5), then there exist data “$\mathcal{C}_K$”, “$\mathcal{V}$”, and “$\xi$” such that all of the conditions of [IUTchI], Definition 3.1, (a), (b), (c), (d), (e), (f), are satisfied, and, moreover, that

(P7) the resulting initial $\Theta$-data

$$\{(\overline{F}/F, X_F, l, \mathcal{C}_K, \mathcal{V}, \mathbb{V}_{\text{mod}} \mod, \xi)\}$$

satisfies the various conditions in the statement of Theorem 1.10.

Here, we note in passing that the crucial existence of data “$\mathcal{V}$” and “$\xi$” satisfying the requisite conditions follows, in essence, as a consequence of the fact [i.e., (P6)] that the Galois action on $l$-torsion points contains the full special linear group $SL_2(\mathbb{F}_l)$.

In light of (P7), we may apply Theorem 1.10 to conclude that

$$\frac{1}{6} \cdot \log(q) \leq (1 + \frac{80 \cdot d_{\text{mod}} \cdot l}{d^*}) \cdot (\log(\xi^{F_{\text{mod}}}) + \log(f^{F_{\text{mod}}})) + 20 \cdot (d_{\text{mod}} \cdot l + \eta_{\text{prim}})$$

$$\leq (1 + d^* \cdot h^{-1/2}) \cdot (\log(\xi^{F_{\text{mod}}}) + \log(f^{F_{\text{mod}}})) + 200 \cdot (d^*)^2 \cdot h^{1/2} \cdot \log(2d^* \cdot h) + 20 \eta_{\text{prim}}$$

— where we apply (P1), as well as the estimates $80 \cdot d_{\text{mod}} \leq d^* \mod \leq d^*$. 
Next, let us observe that it follows from (P3), together with the computation of the discussion preceding (P5), that

\[
\frac{1}{6} \cdot \log(q^2) - \frac{1}{6} \cdot \log(q) \leq \frac{1}{6} \cdot h^{1/2} \cdot \log(l) \leq \frac{1}{3} \cdot h^{1/2} \cdot \log(5d^* \cdot h) \\
\leq h^{1/2} \cdot \log(2d^* \cdot h)
\]

— where we apply the estimates \(1 \leq h \leq 5 \leq 2^3\). Thus, since, by assertion (i), the difference \(\frac{1}{6} \cdot \log(q^2) - \frac{1}{6} \cdot \log(q)\) is bounded by some positive real number \(B_K\) [which depends only on the choice of the compactly bounded subset \(K_V\)], we conclude that

\[
\frac{1}{6} \cdot h = \frac{1}{6} \cdot \log(q^\vee) \leq (1 + d^* \cdot h^{-1/2}) \cdot (\log(o^{F_{np}}) + \log(f^{F_{np}})) \\
\hfill + (15d^*)^2 \cdot h^{1/2} \cdot \log(2d^* \cdot h) + \frac{1}{2} \cdot C_K \\
\hfill \leq (1 + d^* \cdot h^{-1/2}) \cdot (\log(o^{F_{np}}) + \log(f^{F_{np}})) \\
\hfill + \frac{1}{6} \cdot h \cdot \frac{2}{5} \cdot (60d^*)^2 \cdot h^{-1/2} \cdot \log(2d^* \cdot h) + \frac{1}{2} \cdot C_K
\]

— where we write \(C_K \overset{\text{def}}{=} 40\eta_{\text{prm}} + 2B_K\), and we apply the estimate \(6 \cdot 5 \leq 2 \cdot 4^2\).

Now let us set

\[
\epsilon_E \overset{\text{def}}{=} (60d^*)^2 \cdot h^{-1/2} \cdot \log(2d^* \cdot h) \quad (\geq 5 \cdot d^* \cdot h^{-1/2});
\]

\[
\epsilon_d^* \overset{\text{def}}{=} \frac{1}{16} \cdot \epsilon_d \quad (< \frac{1}{2} \leq 1)
\]

— where we apply the estimates \(h \geq 1, \log(2d^* \cdot h) \geq \log(2d^*) \geq \log(4) \geq 1\) [cf. the property (E6) in the proof of Theorem 1.10], and \(\epsilon_d \leq 1\). Note that the inequality

\[
1 < \epsilon_E = (60d^*)^2 \cdot h^{-1/2} \cdot \log(2d^* \cdot h) \\
= (\epsilon_d^*)^{-1} \cdot (60d^*)^2 \cdot h^{-1/2} \cdot \log(2^{\epsilon_d^*} \cdot (d^*)^{\epsilon_d^*} \cdot h^{\epsilon_d^*}) \\
\leq (\epsilon_d^*)^{-1} \cdot (60d^*)^{2+\epsilon_d^*} \cdot h^{-(1/2-\epsilon_d^*)} \\
\leq \left\{ (\epsilon_d^*)^{-3} \cdot (60d^*)^{4+\epsilon_d} \cdot h^{-1} \right\}^{(1/2-\epsilon_d^*)}
\]

— where we apply Proposition 2.1, (i), together with the estimates

\[
\frac{1}{2} - \epsilon_d^* = \frac{16}{8 - \epsilon_d} \leq 3; \quad \frac{2 + \epsilon_d}{\frac{1}{2} - \epsilon_d^*} = \frac{32 + \epsilon_d}{8 - \epsilon_d} \leq 4 + \epsilon_d \leq 5
\]

[both of which are consequences of the fact that \(0 < \epsilon_d \leq 1 \leq 3\)], as well as the estimates \(0 < \epsilon_d^* \leq 1, 60d^* \geq 2d^* \geq 1,\) and \(h \geq 1 —\) implies a bound on \(h\), hence, [by assertion (i)] [GenEll], Proposition 1.4, (iv)] that there is only a finite number of possibilities for the \(j\)-invariant of \(E_F\). Thus, by possibly enlarging the finite set \(\mathfrak{E}_{d,K}\) in a fashion that depends only on \(K_V, d,\) and \(\epsilon_d\) and in such a way that \(h \leq H_{\text{unif} \cdot \epsilon_d^{-3} \cdot d^{1+\epsilon_d} + H_K}\) on \(\mathfrak{E}_{d,K}\) for some positive real number \(H_{\text{unif}}\) that is independent of \(K_V\) and some positive real number \(H_K\) that depends only on \(K_V\), we may assume without loss of generality that \(\epsilon_E \leq 1\).
Thus, in summary, we obtain inequalities
\[
\frac{1}{6} \cdot h \leq \left(1 - \frac{2}{5} \cdot \epsilon_E\right)^{-1} (1 + \frac{1}{5} \cdot \epsilon_E) \cdot (\log(d_{F_{\text{tpd}}}) + \log(f^F_{\text{tpd}})) + (1 - \frac{2}{5} \cdot \epsilon_E)^{-1} \cdot \frac{1}{2} \cdot C_K
\]
\[
\leq (1 + \epsilon_E) \cdot (\log(d_{F_{\text{tpd}}}) + \log(f^F_{\text{tpd}})) + C_K
\]
by applying the estimates
\[
\frac{1 + \frac{1}{5} \cdot \epsilon_E}{1 - \frac{2}{5} \cdot \epsilon_E} \leq 1 + \epsilon_E; \quad 1 - \frac{2}{5} \cdot \epsilon_E \geq \frac{1}{2}
\]
— both of which are consequences of the fact that \(0 < \epsilon_E \leq 1\). Thus, in light of (P1), together with the observation that it follows immediately from the definitions [cf. also Proposition 1.8, (vi)] that we have an equality \(\log\text{-diff}_X(x_E) = \log(d_{F_{\text{tpd}}})\), as well as inequalities \(\log(f^F_{\text{tpd}}) \leq \log\text{-cond}_D(x_E) \leq \log(f^F_{\text{tpd}}) + \log(2l)\), we conclude that both of the conditions (C1), (C2) in the statement of assertion (ii) hold for \(C_K\) as defined above. This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately by applying the argument applied above in the proof of assertion (ii) in the case of the inequality \(1 < \epsilon_E\) to the inequality \(\epsilon < \epsilon_E\).

**Remark 2.2.1.**

(i) Before proceeding, we pause to examine the asymptotic behavior of the bound obtained in Corollary 2.2, (ii), in the spirit of the discussion of Remark 1.10.5, (ii). For simplicity, we assume that \(F_{\text{tpd}} = Q\) [so \(d_{\text{mod}} = 1\)]; we write \(h \overset{\text{def}}{=} \log(q^\psi)\) [cf. the proof of Corollary 2.2, (ii)] and \(\delta \overset{\text{def}}{=} \log\text{-diff}_X(x_E) + \log\text{-cond}_D(x_E) = \log\text{-cond}_D(x_E)\). Thus, it follows immediately from the definitions that \(1 < \log(3) \leq \delta\) and \(1 < \log(3) \leq h\). In particular, the bound under consideration may be written in the form
\[
\frac{1}{6} \cdot h \leq \delta + * \cdot \delta^{1/2} \cdot \log(\delta)
\]
— where \(*\) is to be understood as denoting a fixed positive real number; we observe that the ratio \(h/\delta\) is always a positive real number which is bounded below by the definition of \(h\) and \(\delta\) and bounded above precisely as a consequence of the bound under consideration. In this context, it is of interest to observe that the form of the \(\epsilon\) term \(\delta^{1/2} \cdot \log(\delta)\) is strongly reminiscent of well-known interpretations of the Riemann hypothesis in terms of the asymptotic behavior of the function defined by considering the number of prime numbers below a given number. Indeed, from the point of view of weights [cf. also the discussion of Remark 2.2.2 below], it is natural to regard the logarithmic height of a line bundle as an object that has the same weight as a single Tate twist, or, from a more classical point of view, \(2\pi i\) raised to the power 1. On the other hand, again from the point of view of weights, the variable \(s\) of the Riemann zeta function \(\zeta(s)\) may be thought of as corresponding precisely to the number of Tate twists under consideration, so a single Tate twist corresponds to \(s = 1\). Thus, from this point of view, \(s = \frac{1}{2}\), i.e., the critical line that appears in the Riemann hypothesis, corresponds precisely to the square roots of the logarithmic heights under consideration, i.e., to \(h^{1/2}, \delta^{1/2}\). Moreover, from the point of view of the computations that underlie Theorem
1.10 and Corollary 2.2, (ii) [cf., especially, the proof of Corollary 2.2, (ii); Steps (v), (viii) of the proof of Theorem 1.10; the contribution of “$b_i$” in the second displayed inequality of Proposition 1.4, (iii)], this $\delta^{1/2}$ arises as a result of a sort of “balance”, or “duality” — i.e., that occurs as one increases the size of the auxiliary prime $l$ [cf. the discussion of Remark 1.10.5, (ii)] — between the archimedean decrease in the “$\epsilon$ term” $\frac{1}{6}$ and the nonarchimedean increase in the “$\epsilon$ term” $l$ [i.e., that arises from a certain estimate, in the proof of Proposition 1.2, (i), (ii), of the radius of convergence of the $p$-adic logarithm]. That is to say, such a global arithmetic duality is reminiscent of the functional equation of the Riemann zeta function [cf. the discussion of (iii) below].

(ii) In [vFr], §2, it is conjectured that, in the notation of the discussion of (i),

$$\limsup \frac{\log\left(\frac{1}{6}h-\delta\right)}{\log(h)} = \frac{1}{2}$$

and observed that the “$\frac{1}{2}$” that appears here is strongly reminiscent of the “$\frac{1}{2}$” that appears in the Riemann hypothesis. In the situation of Corollary 2.2, (ii), bounds are only obtained on $abc$ sums that belong to the compactly bounded subset $K_V$ under consideration; such bounds, i.e., as discussed in (i), thus imply that this lim sup is $\leq \frac{1}{2}$. On the other hand, it is shown in [vFr], §2 [cf. also the references quoted in [vFr]], that, if one allows arbitrary $abc$ sums [i.e., which are not necessarily assumed to be contained in a single compactly bounded subset $K_V$], then this lim sup is $\geq \frac{1}{2}$. It is not clear to the author at the time of writing whether or not such estimates [i.e., to the effect that the lim sup under consideration is $\geq \frac{1}{2}$] hold even if one imposes the restriction that the $abc$ sums under consideration be contained in a single compactly bounded subset $K_V$.

(iii) In the well-known classical theory of the Riemann zeta function, the Riemann zeta function is closely related to the theta function, i.e., by means of the Mellin transform. In light of the central role played by theta functions in the theory of the present series of papers, it is tempting to hope, especially in the context of the observations of (i), (ii), that perhaps some extension of the theory of the present series of papers — i.e., some sort of “inter-universal Mellin transform” — may be obtained that allows one to relate the theory of the present series of papers to the Riemann zeta function.

(iv) In the context of the discussion of (iii), it is of interest to recall that, relative to the analogy between number fields and one-dimensional function fields over finite fields, the theory of the present series of papers may be thought of as being analogous to the theory surrounding the derivative of a lifting of the Frobenius morphism [cf. the discussion of [IUTchI], §I4; [IUTchIII], Remark 3.12.4]. On the other hand, the analogue of the Riemann hypothesis for one-dimensional function fields over finite fields may be proven by considering the elementary geometry of the [graph of the] Frobenius morphism. This state of affairs suggests that perhaps some sort of “integral” of the theory of the present series of papers could shed light on the Riemann hypothesis in the case of number fields.

(v) One way to summarize the point of view discussed in (i), (ii), and (iii) is as follows: The asymptotic behavior discussed in (i) suggests that perhaps one
should expect that the inequality constituted by well-known interpretations of the
Riemann hypothesis in terms of the asymptotic behavior of the function defined
by considering the number of prime numbers below a given number may be obtained
as some sort of “restriction”

\[(\text{ABC inequality})|_{\text{canonical number}}\]
of some sort of “ABC inequality” [i.e., some sort of bound of the sort obtained
in Corollary 2.2, (ii)] to some sort of “canonical number” [i.e., where the term
“number” is to be understood as referring to an abc sum]. Here, the descriptive
“canonical” is to be understood as expressing the idea that one is not so much
interested in considering a fixed explicit “number/abc sum”, but rather some sort
of suitable abstraction of the sort of sequence of numbers/abc sums that gives rise
to the limsup value of “\(\frac{1}{2}\)” discussed in (ii). Of course, it is by no means clear
precisely how such an “abstraction” should be formulated, but the idea is that it
should represent

some sort of average over all possible addition operations

in the number field [in this case, \(\mathbb{Q}\)] under consideration or [perhaps equivalently]

some sort of “arithmetic measure or distribution” constituted by
such a collection of all possible addition operations that somehow
amounts to a sort of arithmetic analogue of the measure that gives rise to
the classical Mellin transform

[i.e., that appears in the discussion of (iii)].

Remark 2.2.2. In the context of the discussion of weights in Remark 2.2.1, (i),
it is of interest to recall the significance of the Gaussian integral

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}
\]
in the theory of the present series of papers [cf. [IUTchII], Introduction; [IUTchII],
Remark 1.12.5, as well as Remark 1.10.1 of the present paper]. Indeed, typically
discussions of the Riemann zeta function \(\zeta(s)\), or more general \(L\)-functions, in the
context of conventional arithmetic geometry are concerned principally with the
behavior of such functions at integral values [i.e., \(\in \mathbb{Z}\)] of the variable \(s\). Such
integral values of the variable \(s\) correspond to integral Tate twists, i.e., at a more
concrete level, to integral powers of the quantity \(2\pi i\). If one neglects nonzero factors
\(\in \mathbb{Q}(i)\), then such integral powers may be regarded as integral powers of \(\pi\) [or \(2\pi\)].
At the level of classical integrals, the notion of a single Tate twist may be thought
of as corresponding to the integral

\[
\int_{\mathbb{S}^1} d\theta = 2\pi
\]

over the unit circle \(\mathbb{S}^1\); at the level of schemes, the notion of a single Tate twist may
be thought of as corresponding to the scheme \(\mathbb{G}_m\). On the other hand, whereas
the conventional theory of Tate twists in arithmetic geometry only involves integral powers of a single Tate twist, i.e., corresponding, in essence, to integral powers of $\pi$, the Gaussian integral may be thought of as a sort of fundamental integral representation of the notion of a “Tate semi-twist”. From this point of view, scheme-theoretic Hodge-Arakelov theory may be thought of as a sort of fundamental scheme-theoretic representation of the notion of a “Tate semi-twist” [cf. the discussion of [IUTchII], Remark 1.12.5]. Thus, in summary,

(a) the Gaussian integral,
(b) scheme-theoretic Hodge-Arakelov theory,
(c) the inter-universal Teichmüller theory developed in the present series of papers, and
(d) the Riemann hypothesis.

may all be thought of as “phenomena of weight $\frac{1}{2}$”, i.e., at a concrete level, phenomena that revolve around arithmetic versions of “$\sqrt{\pi}$”. Moreover, we observe that in the first three of these four examples, the essential nature of the notion of “weight $\frac{1}{2}$” may be thought of as being reflected in some sort of exponential of a quadratic form. This state of affairs is strongly reminiscent of

(1) the Griffiths semi-transversality of the crystalline theta object that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], Theorem 2.8; [IUTchII], Remark 1.12.5, (i)], which corresponds essentially [cf. the discussion of the proof of [HASurII], Theorem 2.10] to the quadratic form that appears in the exponents of the well-known series expansion of the theta function;

(2) the quadratic nature of the commutator of the theta group, which is applied, in [EtTh] [cf. the discussion of [IUTchIII], Remark 2.1.1], to derive the various rigidity properties which are interpreted, in [IUTchII], §1, as multiradiality properties — an interpretation that is strongly reminiscent, if one interprets “multiradiality” in terms of “connections” and “parallel transport” [cf. [IUTchII], Remark 1.7.1], of the quadratic form discussed in (1);

(3) the essentially quadratic nature of the “$\epsilon$ term” $\ast \cdot \frac{d}{\pi} + \ast \cdot l$ [which, we recall, occurs at the level of addition of heights, i.e., log-volumes!] in the discussion of Remark 1.10.5, (ii).

Remark 2.2.3. The discussion of Remark 2.2.1 centers around the content of Corollary 2.2, (ii), in the case of elliptic curves defined over $\mathbb{Q}$. On the other hand, if, in the context of Corollary 2.2, (ii), (iii), one considers the case where $d$ is an arbitrary positive integer [i.e., which is not necessarily bounded, as in the situation of Corollary 2.3 below!], then the inequalities obtained in (C2) of Corollary 2.2, (ii), may be regarded, by applying Corollary 2.2, (iii), as a sort of “weak version” of the so-called “uniform ABC Conjecture”. That is to say, these inequalities constitute only a “weak version” in the sense that they are restricted to rational points that lie in the compactly bounded subset $K_V$, and, moreover, the bounds
given for the function \( \log(q)^{-\epsilon} \) [i.e., in essence, the “height”] on \( \mathfrak{d} \) and \( \mathfrak{e}_{\epsilon,d} \) depend on the positive integer \( d \) [cf. also Remark 2.3.2, (i), below].

We are now ready to state and prove the main theorem of the present §2, which may also be regarded as the main application of the theory developed in the present series of papers.

**Corollary 2.3. (Diophantine Inequalities)** Let \( X \) be a smooth, proper, geometrically connected curve over a number field; \( D \subseteq X \) a reduced divisor; \( U_X \overset{\text{def}}{=} X \setminus D \); \( d \) a positive integer; \( \epsilon \in \mathbb{R}_{>0} \) a positive real number. Write \( \omega_X \) for the canonical sheaf on \( X \). Suppose that \( U_X \) is a hyperbolic curve, i.e., that the degree of the line bundle \( \omega_X(D) \) is positive. Then, relative to the notation reviewed above, one has an inequality of “bounded discrepancy classes”

\[
\text{ht}_{\omega_X(D)} \lesssim (1 + \epsilon)(\text{log-diff}_X + \text{log-cond}_D)
\]

of functions on \( U_X(\overline{Q}) \leq d \) — i.e., the function \((1 + \epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \text{ht}_{\omega_X(D)}\) is bounded below by a constant on \( U_X(\overline{Q}) \leq d \) [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below].

**Proof.** One verifies immediately that the content of the statement of Corollary 2.3 coincides precisely with the content of [GenEll], Theorem 2.1, (i). Thus, it follows from the equivalence of [GenEll], Theorem 2.1, that, in order to complete the proof of Corollary 2.3, it suffices to verify that [GenEll], Theorem 2.1, (ii), holds. That is to say, we may assume without loss of generality that:

- \( X = \mathbb{P}^1_Q \) is the projective line over \( Q \);
- \( D \subseteq X \) is the divisor consisting of the three points “0”, “1”, and “\( \infty \)”;
- \( \mathcal{K}_V \subseteq U_X(\overline{Q}) \) is a compactly bounded subset [cf. Remark 2.3.1, (vi), below] whose support contains the nonarchimedean prime “2”;
- \( \mathcal{K}_V \) satisfies the condition “\((*)\text{-inv}\)” of Corollary 2.2.

[Here, we note, with regard to the condition “\((*)\text{-inv}\)" of Corollary 2.2, that this condition only concerns the behavior of \( \mathcal{K}_V \bigcap U_X(\overline{Q}) \leq d \) as \( d \) varies; that is to say, this condition is entirely vacuous in situations, i.e., such as the situation considered in [GenEll], Theorem 2.1, (ii), in which one is only concerned with \( \mathcal{K}_V \bigcap U_X(\overline{Q}) \leq d \) for a fixed \( d \).] Then it suffices to show that the inequality of BD-classes of functions [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]

\[
\text{ht}_{\omega_X(D)} \lesssim (1 + \epsilon)(\text{log-diff}_X + \text{log-cond}_D)
\]

holds on \( \mathcal{K}_V \bigcap U_X(\overline{Q}) \leq d \). But such an inequality follows immediately, in light of the [relevant] equality of BD-classes of Corollary 2.2, (i), from Corollary 2.2, (ii) [cf. condition (C2)], (iii) [where we note that it follows immediately from the various definitions involved that \( d \mod \leq d \)]. This completes the proof of Corollary 2.3.

**Remark 2.3.1.** We take this opportunity to correct some unfortunate misprints in [GenEll].
(i) The notation “$\text{ord}_v(-) : F_v \to \mathbb{Z}$” in the final sentence of the first paragraph following [GenEll], Definition 1.1, should read “$\text{ord}_v(-) : F_v^\times \to \mathbb{Z}$”.

(ii) In [GenEll], Definition 1.2, (ii), the non-resp’d and first resp’d items in the display should be reversed! That is to say, the notation “$\alpha \lesssim F \beta$” corresponds to “$\alpha(x) - \beta(x) \leq C$”; the notation “$\alpha \gtrsim F \beta$” corresponds to “$\beta(x) - \alpha(x) \leq C$”.

(iii) The first portion of the first sentence of the statement of [GenEll], Corollary 4.4, should read: “Let $Q$ be an algebraic closure of $\mathbb{Q}$; ...”.

(iv) The “$\log\text{-diff}_{\overline{\mathcal{M}}_{ell}}([E_L])$” in the second inequality of the final display of the statement of [GenEll], Corollary 4.4, should read “$\log\text{-diff}_{\mathcal{M}_{ell}}([E_L])$”.

(v) The equality

$$\text{ht}_E \approx (\deg(E)/\deg(\omega_X)) \cdot \text{ht}_{\omega_X}$$

implicit in the final “$\approx$” of the final display of the proof of [GenEll], Theorem 2.1, should be replaced by an inequality

$$\text{ht}_E \lesssim 2 \cdot (\deg(E)/\deg(\omega_X)) \cdot \text{ht}_{\omega_X}$$

[which follows immediately from [GenEll], Proposition 1.4, (ii)], and the expression “$\deg(E)/\deg(\omega_X)$” in the inequality imposed on the choice of $\epsilon'$ should be replaced by the expression “$2 \cdot (\deg(E)/\deg(\omega_X))$”.

(vi) Suppose that we are in the situation of [GenEll], Example 1.3, (ii). Let $U \subseteq X$ be an open subscheme. Then a “compactly bounded subset”

$$\mathcal{K}_V \subseteq U(\overline{\mathbb{Q}}) \ (\subseteq X(\overline{\mathbb{Q}}))$$

of $U(\overline{\mathbb{Q}})$ is to be understood as a subset which forms a compactly bounded subset of $X(\overline{\mathbb{Q}})$ [i.e., in the sense discussed in [GenEll], Example 1.3, (ii)] and, moreover, satisfies the property that for each $v \in V^{\text{arc}} \overset{\text{df}}{=} V \cap V(\mathbb{Q})^{\text{arc}}$ (respectively, $v \in V^{\text{non}} \overset{\text{df}}{=} V \cap V(\mathbb{Q})^{\text{non}}$), the compact domain $\mathcal{K}_v \subseteq X^{\text{arc}}$ (respectively, $\mathcal{K}_v \subseteq X(\overline{\mathbb{Q}}_v)$) is, in fact, contained in

$$U(\mathbb{C}) \subseteq X(\mathbb{C}) = X^{\text{arc}} \ (\text{respectively, } U(\overline{\mathbb{Q}}_v) \subseteq X(\overline{\mathbb{Q}}_v)).$$

In particular, this convention should be applied to the use of the term “compactly bounded subset” in the statements of [GenEll], Theorem 2.1; [GenEll], Lemma 3.7; [GenEll], Theorem 3.8; [GenEll], Corollary 4.4, as well as in the present paper [cf. the statement of Corollary 2.2; the proof of Corollary 2.3]. Although this convention was not discussed explicitly in [GenEll], Example 1.3, (ii), it is, in effect, discussed explicitly in the discussion of “compactly bounded subsets” at the beginning of the Introduction to [GenEll]. Moreover, this convention is implicit in the arguments involving compactly bounded subsets in the proof of [GenEll], Theorem 2.1.

(vii) In the discussion following the second display of [GenEll], Example 1.3, (ii), the phrase “(respectively, $X(\mathbb{Q}_v)$)” should read “(respectively, $X(\overline{\mathbb{Q}}_v)$)”.
Remark 2.3.2.

(i) The reader will note that, by arguing with a “bit more care”, it is not difficult to give stronger versions of the various estimates that occur in Theorem 1.10; Corollaries 2.2, 2.3 and their proofs. Such stronger estimates are, however, beyond the scope of the present series of papers, so we shall not pursue this topic further in the present paper.

(ii) On the other hand, we recall that the constant “1” in the inequality of the display of Corollary 2.3 cannot be improved — cf. the examples constructed in [Mss]; the discussion of Remark 1.10.5, (ii), (iii).

Remark 2.3.3. Corollary 2.3 may be thought of as an effective version of the Mordell Conjecture. From this point of view, it is perhaps of interest to compare the “essential ingredients” that are applied in the proof of Corollary 2.3 [i.e., in effect, that are applied in the present series of papers!] with the “essential ingredients” applied in [Falt]. Although the author does not wish to make any pretensions to completeness in any rigorous sense, perhaps a rough, informal list of “essential ingredients” in the case of [Falt] may be given as follows:

(a) results in elementary algebraic number theory concerning the “geometry of numbers”, such as the theory of heights and the Hermite-Minkowski theorem;
(b) the global class field theory of number fields;
(c) the p-adic theory of Hodge-Tate decompositions;
(d) the p-adic theory of finite flat group schemes;
(e) generalities in algebraic geometry concerning isogenies and Tate modules of abelian varieties;
(f) generalities in algebraic geometry concerning polarizations of abelian varieties;
(g) the logarithmic geometry of toroidal compactifications of the moduli stack of abelian varieties.

Here, we note that the global class field theory of (b) may be thought of as a sort of concatenation of

(b-1) the local class field theory of p-adic local fields;
(b-2) results concerning the global density of primes such as the Tchebotarev density theorem.

From the point of view of the theory of the present series of papers, (a) is reminiscent of the arithmetic degrees of line bundles that appear, for instance, in the form of global realized Frobenoioids, throughout the theory of the present series of papers; (a) is also reminiscent of the elementary algebraic number theory characterization of nonzero global integers as roots of unity, which plays an important role in the theory of the present series of papers [cf. [IUTchIII], the proof of Proposition 3.10; the discussion of the latter portion of [IUTchIII], Remark 3.12.1, (iii)]. If one thinks of (b) as a concatenation of (b-1) and (b-2), then (b-1) is reminiscent of the p-adic absolute anabelian geometry of [AbsTopIII] [cf., e.g., [AbsTopIII], Corollary 1.10, (i)], while (b-2) is reminiscent of repeated applications of the Prime Number Theorem in the present paper [cf. Propositions 1.6; 2.1, (ii)]. Next, we recall that
Hodge-Tate decompositions as in (c) play a central role in the proofs of the main results of \([pGC]\), which, in turn, underlie the theory of \([Abs\text{TopIII}]\). The ramification computations concerning finite flat group schemes as in (d) are reminiscent of various \(p\)-adic ramification computations concerning log-shells in \([Abs\text{TopIII}]\), as well as in Propositions 1.1, 1.2, 1.3, 1.4 of the present paper. Whereas \([Falt]\) revolves around the abelian/linear theory of abelian varieties [cf. (e)], the theory of the present series of papers depends, in an essential way, on various intricate manipulations involving finite étale coverings of hyperbolic curves, such as the use of Belyi maps in \([GenEll]\), as well as in the Belyi cuspidalizations applied in \([Abs\text{TopIII}]\). The theory of polarizations of abelian varieties applied in \([Falt]\) [cf. (f)] is reminiscent of the essential role played by commutators of theta groups in the theory of \([Et\text{Th}]\), which, in turn, plays a central role in the theory of the present series of papers. Finally, the logarithmic geometry of (f) is reminiscent of the combinatorial anabelian geometry of \([SemiAnbd]\), which is applied, in \([IUTchI]\), §2, to the logarithmic geometry of coverings of stable curves. Thus, in summary,

many aspects of the theory of \([Falt]\) may be regarded as “distant abelian ancestors” of certain aspects of the “anabelian-based theory” of the present series of papers.

Alternatively, one may observe that the overwhelmingly scheme-theoretic nature of the theory applied in \([Falt]\) lies in stark contrast to the highly non-scheme-theoretic nature of the absolute anabelian geometry and theory of monoids/Frobenioids applied in the present series of papers: that is to say,

many aspects of the theory of \([Falt]\) may be regarded as “distant arithmetically holomorphic ancestors” of certain aspects of the multiradial and mono-analytic [i.e., “arithmetically real analytic”] theory developed in the present series of papers.
Section 3: Inter-universal Formalism: the Language of Species

In the present §3, we develop — albeit from an extremely naive/non-expert point of view, relative to the theory of foundations! — the language of species. Roughly speaking, a “species” is a “type of mathematical object”, such as a “group”, a “ring”, a “scheme”, etc. In some sense, this language may be thought of as an explicit description of certain tasks typically executed at an implicit, intuitive level by mathematicians [i.e., mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] via a sort of “mental arithmetic” in the course of interpreting various mathematical arguments. In the context of the theory developed in the present series of papers, however, it is useful to describe these intuitive operations explicitly.

In the following discussion, we shall work with various models — consisting of “sets” and a relation “∈” — of the standard ZFC axioms of axiomatic set theory [i.e., the nine axioms of Zermelo-Fraenkel, together with the axiom of choice — cf., e.g., [Dk], Chapter 1, §3]. We shall refer to such models as ZFC-models. Recall that a (Grothendieck) universe \( V \) is a set satisfying the following axioms [cf. [McLn], p. 194]:

(i) \( V \) is transitive, i.e., if \( y \in x, x \in V \), then \( y \in V \).

(ii) The set of natural numbers \( \mathbb{N} \in V \).

(iii) If \( x \in V \), then the power set of \( x \) also belongs to \( V \).

(iv) If \( x \in V \), then the union of all members of \( x \) also belongs to \( V \).

(v) If \( x \in V, y \subseteq V \), and \( f: x \to y \) is a surjection, then \( y \in V \).

We shall say that a set \( E \) is a \( V \)-set if \( E \in V \).

The various ZFC-models that we work with may be thought of as [but are not restricted to be!] the ZFC-models determined by various universes that are sets relative to some ambient ZFC-model which, in addition to the standard axioms of ZFC set theory, satisfies the following existence axiom [attributed to the “Grothendieck school” — cf. the discussion of [McLn], p. 193]:

\((\dagger_G)\) Given any set \( x \), there exists a universe \( V \) such that \( x \in V \).

We shall refer to a ZFC-model that also satisfies this additional axiom of the Grothendieck school as a ZFCG-model. This existence axiom \((\dagger_G)\) implies, in particular, that:

Given a set \( I \) and a collection of universes \( V_i \) (where \( i \in I \)), indexed by \( I \) [i.e., a “function” \( I \ni i \mapsto V_i \)], there exists a [larger] universe \( V \) such that \( V_i \in V, \) for \( i \in I \).

Indeed, since the graph of the function \( I \ni i \mapsto V_i \) is a set, it follows that \( \{V_i\}_{i \in I} \) is a set. Thus, it follows from the existence axiom \((\dagger_G)\) that there exists a universe \( V \) such that \( \{V_i\}_{i \in I} \in V \). Hence, by condition (i), we conclude that \( V_i \in V, \) for
all \( i \in I \), as desired. Note that this means, in particular, that there exist infinite ascending chains of universes

\[
V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V
\]

— where \( n \) ranges over the natural numbers. On the other hand, by the axiom of foundation, there do not exist infinite descending chains of universes

\[
V_0 \ni V_1 \ni V_2 \ni V_3 \ni \ldots \ni V_n \ni \ldots
\]

— where \( n \) ranges over the natural numbers.

Although we shall not discuss in detail here the quite difficult issue of whether or not there actually exist ZFCG-models, we remark in passing that it may be possible to justify the stance of ignoring such issues in the context of the present series of papers — at least from the point of view of establishing the validity of various “final results” that may be formulated in ZFC-models — by invoking the work of Feferman [cf. [Ffmn]]. Precise statements concerning such issues, however, lie beyond the scope of the present paper [as well as of the level of expertise of the author!].

In the following discussion, we use the phrase “set-theoretic formula” as it is conventionally used in discussions of axiomatic set theory [cf., e.g., [Drk], Chapter 1, §2], with the following proviso: In the following discussion, it should be understood that every set-theoretic formula that appears is “absolute” in the sense that its validity for a collection of sets contained in some universe \( V \) relative to the model of set theory determined by \( V \) is equivalent, for any universe \( W \) such that \( V \subseteq W \), to its validity for the same collection of sets relative to the model of set theory determined by \( W \) [cf., e.g., [Drk], Chapter 3, Definition 4.2].

**Definition 3.1.**

(i) A 0-species \( S_0 \) is a collection of conditions given by a set-theoretic formula

\[
\Phi_0(\mathcal{E})
\]

involving an ordered collection \( \mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_{n_0}) \) of sets \( \mathcal{E}_1, \ldots, \mathcal{E}_{n_0} \) [which we think of as “indeterminates”], for some integer \( n_0 \geq 1 \); in this situation, we shall refer to \( \mathcal{E} \) as a collection of species-data for \( S_0 \). If \( S_0 \) is a 0-species given by a set-theoretic formula \( \Phi_0(\mathcal{E}) \), then a 0-specimen of \( S_0 \) is a specific ordered collection of \( n_0 \) sets \( E = (E_1, \ldots, E_{n_0}) \) in some specific ZFC-model that satisfies \( \Phi_0(E) \). If \( E \) is a 0-specimen of a 0-species \( S_0 \), then we shall write \( E \in S_0 \). If, moreover, it holds, in any ZFC-model, that the 0-specimens of \( S_0 \) form a set, then we shall refer to \( S_0 \) as 0-small.

(ii) Let \( S_0 \) be a 0-species. Then a 1-species \( S_1 \) acting on \( S_0 \) is a collection of set-theoretic formulas \( \Phi_1, \Phi_{101} \) satisfying the following conditions:

(a) \( \Phi_1 \) is a set-theoretic formula

\[
\Phi_1(\mathcal{E}, \mathcal{E}', \mathcal{F})
\]
We shall refer to a species-isomorphism whose domain and codomain are equal as composites (respectively, 1-specimen) of in this situation, we shall say that involving two collections of species-data \( \mathcal{E}, \mathcal{E}' \) for \( \mathcal{S}_0 \) [i.e., the conditions \( \Phi_0(\mathcal{E}), \Phi_0(\mathcal{E}') \) hold] and an ordered collection \( \mathfrak{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1}) \) of \([\text{"indeterminate"}]\) sets \( \mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1} \), for some integer \( n_1 \geq 1 \); in this situation, we shall refer to \((\mathcal{E}, \mathcal{E}', \mathfrak{F})\) as a collection of species-data for \( \mathcal{S}_1 \) and write \( \mathfrak{F} : \mathcal{E} \to \mathcal{E}' \). If, moreover, it holds, in any ZFC-model, \( E, E' \in \mathcal{S}_0 \), and \( F \) is a specific ordered collection of \( n_1 \) sets that satisfies the condition \( \Phi_1(E, E', F) \), then we shall refer to the data \((E, E', F)\) as a 1-specimen of \( \mathcal{S}_1 \) and write \((E, E', F) \in \mathcal{S}_1\); alternatively, we shall denote a 1-specimen \((E, E', F)\) via the notation \( F : E \to E' \) and refer to \( E \) (respectively, \( E' \)) as the domain (respectively, codomain) of \( F : E \to E' \).

(b) \( \Phi_{101} \) is a set-theoretic formula

\[
\Phi_{101}(\mathcal{E}, \mathcal{E}', \mathcal{E}'', \mathfrak{F}, \mathfrak{F}', \mathfrak{F}'')
\]

involving three collections of species-data \( \mathfrak{F} : \mathcal{E} \to \mathcal{E}' \), \( \mathfrak{F}' : \mathcal{E}' \to \mathcal{E}'' \), \( \mathfrak{F}'' : \mathcal{E} \to \mathcal{E}'' \) for \( \mathcal{S}_1 \) [i.e., the conditions \( \Phi_0(\mathcal{E}), \Phi_0(\mathcal{E}'), \Phi_0(\mathcal{E}''), \Phi_1(\mathcal{E}, \mathcal{E}', \mathfrak{F}), \Phi_1(\mathcal{E}', \mathcal{E}'', \mathfrak{F}'); \Phi_1(\mathcal{E}, \mathcal{E}', \mathfrak{F}); \Phi_1(\mathcal{E}', \mathcal{E}'', \mathfrak{F}) \) hold]; in this situation, we shall refer to \( \mathfrak{F}'' \) as a composite of \( \mathfrak{F} \) with \( \mathfrak{F}' \) and write \( \mathfrak{F}'' = \mathfrak{F}' \circ \mathfrak{F} \) [which is, \text{a priori}, an abuse of notation, since there may exist many composites of \( \mathfrak{F} \) with \( \mathfrak{F}' \) — cf. (c) below]; we shall use similar terminology and notation for 1-specimens in specific ZFC-models.

(c) Given a pair of 1-specimens \( F : E \to E', F' : E' \to E'' \) of \( \mathcal{S}_1 \) in some ZFC-model, there exists a unique composite \( F'' : E \to E'' \) of \( F \) with \( F' \) in the given ZFC-model.

(d) Composition of 1-specimens \( F : E \to E', F' : E' \to E'' \), \( F'' : E'' \to E''' \) of \( \mathcal{S}_1 \) in a ZFC-model is associative.

(e) For any 0-specimen \( E \) of \( \mathcal{S}_0 \) in a ZFC-model, there exists a [necessarily unique] 1-specimen \( F : E \to E \) of \( \mathcal{S}_1 \) [in the given ZFC-model] — which we shall refer to as the identity 1-specimen \( \text{id}_E \) of \( E \) — such that for any 1-specimens \( F' : E' \to E, F'' : E \to E'' \) of \( \mathcal{S}_1 \) [in the given ZFC-model] we have \( F \circ F' = F', F'' \circ F = F'' \).

If, moreover, it holds, in any ZFC-model, that for any two 0-specimens \( E, E' \) of \( \mathcal{S}_0 \), the 1-specimens \( F : E \to E' \) of \( \mathcal{S}_1 \) [i.e., the 1-specimens of \( \mathcal{S}_1 \) with domain \( E \) and codomain \( E' \)] form a set, then we shall refer to \( \mathcal{S}_1 \) as 1-small.

(iii) A species \( \mathcal{S} \) is defined to be a pair consisting of a 0-species \( \mathcal{S}_0 \) and a 1-species \( \mathcal{S}_1 \) acting on \( \mathcal{S}_0 \). Fix a species \( \mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1) \). Let \( i \in \{0, 1\} \). Then we shall refer to an \( i \)-specimen of \( \mathcal{S}_i \) as an \( i \)-specimen of \( \mathcal{S} \). We shall refer to a 0-specimen (respectively, 1-specimen) of \( \mathcal{S} \) as a species-object (respectively, a species-morphism) of \( \mathcal{S} \). We shall say that \( \mathcal{S} \) is \( i \)-small if \( \mathcal{S}_i \) is \( i \)-small. We shall refer to a species-morphism \( F : E \to E' \) as a species-isomorphism if there exists a species-morphism \( F' : E' \to E \) such that the composites \( F \circ F', F' \circ F \) are identity species-morphisms; in this situation, we shall say that \( E, E' \) are species-isomorphic. [Thus, one verifies immediately that composites of species-isomorphisms are species-isomorphisms.] We shall refer to a species-isomorphism whose domain and codomain are equal as
a *species-automorphism*. We shall refer to as *model-free* [cf. Remark 3.1.1 below] an *i*-specimen of \( \mathcal{S} \) equipped with a description via a *set-theoretic formula* that is “independent of the ZFC-model in which it is given” in the sense that for any pair of universes \( V_1, V_2 \) of some ZFC-model such that \( V_1 \in V_2 \), the set-theoretic formula determines the *same* *i*-specimen of \( \mathcal{S} \), whether interpreted relative to the ZFC-model determined by \( V_1 \) or the ZFC-model determined by \( V_2 \).

(iv) We shall refer to as the *category determined by* \( \mathcal{S} \) in a ZFC-model the category whose objects are the *species-objects* of \( \mathcal{S} \) in the given ZFC-model and whose arrows are the *species-morphisms* of \( \mathcal{S} \) in the given ZFC-model. [One verifies immediately that this description does indeed determine a category.]

**Remark 3.1.1.** We observe that any of the familiar descriptions of \( \mathbb{N} \) [cf., e.g., [Drk], Chapter 2, Definition 2.3], \( \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_p \), or \( \mathbb{R} \), for instance, yield *species* [all of whose species-morphisms are identity species-morphisms] each of which has a *unique* species-object in any given ZFC-model. Such species are *not to be confused* with such species as the species of “monoids isomorphic to \( \mathbb{N} \) and monoid isomorphisms”, which admits *many species-objects* [all of which are species-isomorphic] in any ZFC-model. On the other hand, the set-theoretic formula used, for instance, to define the former “species \( \mathbb{N} \)” may be applied to define a “*model-free species-object* \( \mathbb{N} \)” of the latter “species of monoids isomorphic to \( \mathbb{N} \)”.

**Remark 3.1.2.**

(i) It is important to remember when working with species that

the *essence* of a *species* lies *not in the specific sets* that occur as species-objects or species-morphisms of the species in various ZFC-models, but rather in the *collection of rules*, i.e., *set-theoretic formulas*, that govern the construction of such sets in an *unspecified, “indeterminate” ZFC-model*.

Put another way, the emphasis in the theory of species lies in the *programs* — i.e., “*software*” — that yield the desired output data, *not on the output data itself*. From this point of view, one way to describe the various set-theoretic formulas that constitute a species is as a “*deterministic algorithm*” [a term suggested to the author by M. Kim] for constructing the sets to be considered.

(ii) One interesting point of view that arose in discussions between the author and F. Kato is the following. The relationship between the classical approach to discussing mathematics relative to a *fixed model of set theory* — an approach in which *specific sets* play a central role — and the “*species-theoretic*” approach considered here — in which the *rules*, given by set-theoretic formulas for constructing the sets of interest [i.e., not specific sets themselves!], play a central role — may be regarded as *analogous* to the relationship between *classical approaches to algebraic varieties* — in which specific sets of solutions of polynomial equations in an algebraically closed field play a central role — and *scheme theory* — in which the
functor determined by a scheme, i.e., the polynomial equations, or “rules”, that determine solutions, as opposed to specific sets of solutions themselves, play a central role. That is to say, in summary:

\[
\text{[fixed model of set theory approach : species-theoretic approach]} \\
\leftrightarrow \\
\text{[varieties : schemes]}
\]

A similar analogy — i.e., of the form

\[
\text{[fixed model of set theory approach : species-theoretic approach]} \\
\leftrightarrow \\
\text{[groups of specific matrices : abstract groups]}
\]

— may be made to the notion of an “abstract group”, as opposed to a “group of specific matrices”. That is to say, just as a “group of specific matrices may be thought of as a specific representation of an “abstract group”, the category of objects determined by a species in a specific ZFC-model may be thought of as a specific representation of an “abstract species”.

(iii) If, in the context of the discussion of (i), (ii), one tries to form a sort of quotient, in which “programs” that yield the same sets as “output data” are identified, then one must contend with the resulting indeterminacy, i.e., working with programs is only well-defined up to internal modifications of the programs in question that does not affect the final output. This leads to somewhat intractable problems concerning the internal structure of such programs — a topic that lies well beyond the scope of the present work.

Remark 3.1.3.

(i) Typically, in the discussion to follow, we shall not write out explicitly the various set-theoretic formulas involved in the definition of a species. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the conventional descriptions of the mathematical objects involved. When applying such conventional descriptions, however, it is important to check that they are well-defined and do not depend upon the use of arbitrary choices that are not describable via well-defined set-theoretic formulas.

(ii) The fact that the data involved in a species is given by abstract set-theoretic formulas imparts a certain canonicality to the mathematical notion constituted by the species, a canonicality that is not shared, for instance, by mathematical objects whose construction depends on an invocation of the axiom of choice in some particular ZFC-model [cf. the discussion of (i) above]. Moreover, by furnishing a stock of such “canonical notions”, the theory of species allows one, in effect, to compute the extent of deviation of various “non-canonical objects” [i.e., whose construction depends upon the invocation of the axiom of choice!] from a sort of “canonical norm”.

Remark 3.1.4. Note that because the data involved in a species is given by abstract set-theoretic formulas, the mathematical notion constituted by the species is immune to, i.e., unaffected by, extensions of the universe — i.e., such as
the ascending chain $V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$ that appears in the discussion preceding Definition 3.1 — in which one works. This is the sense in which we apply the term “inter-universal”. That is to say, “inter-universal geometry” allows one to relate the “geometries” that occur in distinct universes.

**Remark 3.1.5.** Similar remarks to the remarks made in Remarks 3.1.2, 3.1.3, and 3.1.4 concerning the significance of working with set-theoretic formulas may be made with regard to the notions of mutations, morphisms of mutations, mutation-histories, observables, and cores to be introduced in Definition 3.2 below.

One fundamental example of a species is the following.

**Example 3.2. Categories.** The notions of a [small] category and an isomorphism class of [covariant] functors between two given [small] categories yield an example of a species. That is to say, at a set-theoretic level, one may think of a [small] category as, for instance, a set of arrows, together with a set of composition relations, that satisfies certain properties; one may think of a [covariant] functor between [small] categories as the set given by the graph of the map on arrows determined by the functor [which satisfies certain properties]; one may think of an isomorphism class of functors as a collection of such graphs, i.e., the graphs determined by the functors in the isomorphism class, which satisfies certain properties. Then one has “dictionaries”

\[
\begin{align*}
0\text{-species} & \quad \longleftrightarrow \quad \text{the notion of a category} \\
1\text{-species} & \quad \longleftrightarrow \quad \text{the notion of an isomorphism class of functors}
\end{align*}
\]

at the level of notions and

\[
\begin{align*}
a \ 0\text{-specimen} & \quad \longleftrightarrow \quad \text{a particular [small] category} \\
a \ 1\text{-specimen} & \quad \longleftrightarrow \quad \text{a particular isomorphism class of functors}
\end{align*}
\]

at the level of specific mathematical objects in a specific ZFC-model. Moreover, one verifies easily that species-isomorphisms between 0-species correspond to isomorphism classes of equivalences of categories in the usual sense.

**Remark 3.2.1.** Note that in the case of Example 3.2, one could also define a notion of “2-species”, “2-specimens”, etc., via the notion of an “isomorphism of functors”, and then take the 1-species under consideration to be the notion of a functor [i.e., not an isomorphism class of functors]. Indeed, more generally, one could define a notion of “$n$-species” for arbitrary integers $n \geq 1$. Since, however, this approach would only serve to add an unnecessary level of complexity to the theory, we choose here to take the approach of working with “functors considered up to isomorphism”.

**Definition 3.3.** Let $\mathcal{S} = (S_0, S_1)$; $\mathcal{S} = (S_0, S_1); \mathcal{S} = (S_0, S_1)$ be species.

(i) A mutation $\mathcal{M}: \mathcal{S} \leadsto \mathcal{S}$ is defined to be a collection of set-theoretic formulas $\Psi_0, \Psi_1$ satisfying the following properties:
(a) $\Psi_0$ is a set-theoretic formula

$$\Psi_0(\mathcal{E}, \mathcal{E})$$

involving a collection of species-data $\mathcal{E}$ for $\mathcal{G}_0$ and a collection of species-data $\mathcal{E}'$ for $\mathcal{G}_0$; in this situation, we shall write $\mathcal{M}(\mathcal{E})$ for $\mathcal{E}$. Moreover, if, in some ZFC-model, $E \in \mathcal{G}_0$, then we require that there exist a unique $E \in \mathcal{G}_0$ such that $\Psi_0(E, E)$ holds; in this situation, we shall write $\mathcal{M}(E)$ for $E$.

(b) $\Psi_1$ is a set-theoretic formula

$$\Psi_1(\mathcal{E}, \mathcal{E}', \mathcal{F}, \mathcal{F}')$$

involving a collection of species-data $\mathcal{F} : \mathcal{E} \to \mathcal{E}'$ for $\mathcal{G}_1$ and a collection of species-data $\mathcal{F} : \mathcal{E} \to \mathcal{E}'$ for $\mathcal{G}_1$, where $\mathcal{E} = \mathcal{M}(\mathcal{E})$, $\mathcal{E}' = \mathcal{M}(\mathcal{E}')$; in this situation, we shall write $\mathcal{M}(\mathcal{F})$ for $\mathcal{F}$. Moreover, if, in some ZFC-model, $(F : E \to E') \in \mathcal{G}_1$, then we require that there exist a unique $(F : E \to E') \in \mathcal{G}_1$ such that $\Psi_0(E, E', F, F)$ holds; in this situation, we shall write $\mathcal{M}(F)$ for $F$. Finally, we require that the assignment $F \mapsto \mathcal{M}(F)$ be compatible with composites and map identity species-morphisms of $\mathcal{G}$ to identity species-morphisms of $\mathcal{G}$. In particular, if one fixes a ZFC-model, then $\mathcal{M}$ determines a functor from the category determined by $\mathcal{G}$ in the given ZFC-model to the category determined by $\mathcal{G}$ in the given ZFC-model.

There are evident notions of "composition of mutations" and "identity mutations".

(ii) Let $\mathcal{M}, \mathcal{M}' : \mathcal{G} \to \mathcal{G}$ be mutations. Then a morphism of mutations $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ is defined to be a set-theoretic formula $\Xi$ satisfying the following properties:

(a) $\Xi$ is a set-theoretic formula

$$\Xi(\mathcal{E}, \mathcal{F})$$

involving a collection of species-data $\mathcal{E}$ for $\mathcal{G}_0$ and a collection of species-data $\mathcal{F} : \mathcal{M}(\mathcal{E}) \to \mathcal{M}'(\mathcal{E})$ for $\mathcal{G}_1$; in this situation, we shall write $\mathcal{F}(\mathcal{E})$ for $\mathcal{F}$. Moreover, if, in some ZFC-model, $E \in \mathcal{G}_0$, then we require that there exist a unique $E \in \mathcal{G}_1$ such that $\Xi(E, E)$ holds; in this situation, we shall write $\mathcal{F}(E)$ for $\mathcal{F}$.

(b) Suppose, in some ZFC-model, that $F : E_1 \to E_2$ is a species-morphism of $\mathcal{G}$. Then one has an equality of composite species-morphisms $\mathcal{M}'(F) \circ \mathcal{F}(E_1) = \mathcal{F}(E_2) \circ \mathcal{M}(E_1) : \mathcal{M}(E_1) \to \mathcal{M}'(E_2)$. In particular, if one fixes a ZFC-model, then a morphism of mutations $\mathcal{M} \to \mathcal{M}'$ determines a natural transformation between the functors determined by $\mathcal{M}, \mathcal{M}'$ in the ZFC-model — cf. (i).

There are evident notions of "composition of morphisms of mutations" and "identity morphisms of mutations". If it holds that for every species-object $E$ of $\mathcal{G}$, $\mathcal{F}(E)$ is
a species-isomorphism, then we shall refer to $\mathcal{F}$ as an isomorphism of mutations. In particular, one verifies immediately that $\mathcal{F}$ is an isomorphism of mutations if and only if there exists a morphism of mutations $\mathcal{F}' : \mathcal{M}' \to \mathcal{M}$ such that the composite morphisms of mutations $\mathcal{F}' \circ \mathcal{F} : \mathcal{M} \to \mathcal{M}$, $\mathcal{F} \circ \mathcal{F}' : \mathcal{M}' \to \mathcal{M}'$ are the respective identity morphisms of the mutations $\mathcal{M}, \mathcal{M}'$.

(iii) Let $\mathcal{M} : \mathcal{S} \rightsquigarrow \mathcal{S}$ be a mutation. Then we shall say that $\mathcal{M}$ is a mutation-equivalence if there exists a mutation $\mathcal{M}' : \mathcal{S} \rightsquigarrow \mathcal{S}$, together with isomorphisms of mutations between the composites $\mathcal{M} \circ \mathcal{M}'$, $\mathcal{M}' \circ \mathcal{M}$ and the respective identity mutations. In this situation, we shall say that $\mathcal{M}, \mathcal{M}'$ are mutation-quasi-inverses to one another. Note that for any two given species-objects in the domain species of a mutation-equivalence, the mutation-equivalence induces a bijection between the collection of species-isomorphisms between the two given species-objects [of the domain species] and the collection of species-isomorphisms between the two species-objects [of the codomain species] obtained by applying the mutation-equivalence to the two given species-objects.

(iv) Let $\vec{\Gamma}$ be an oriented graph, i.e., a graph $\Gamma$, which we shall refer to as the underlying graph of $\vec{\Gamma}$, equipped with the additional data of a total ordering, for each edge $e$ of $\Gamma$, on the set [of cardinality 2] of branches of $e$ [cf., e.g., [AbsTopIII], §0]. Then we define a mutation-history $\mathcal{H} = (\vec{\Gamma}, \mathcal{S}^*, \mathcal{M}^*)$ [indexed by $\vec{\Gamma}$] to be a collection of data as follows:

(a) for each vertex $v$ of $\vec{\Gamma}$, a species $\mathcal{S}^v$;

(b) for each edge $e$ of $\vec{\Gamma}$, running from a vertex $v_1$ to a vertex $v_2$, a mutation $\mathcal{M}^e : \mathcal{S}^{v_1} \rightsquigarrow \mathcal{S}^{v_2}$.

In this situation, we shall refer to the vertices, edges, and branches of $\vec{\Gamma}$ as vertices, edges, and branches of $\mathcal{H}$. Thus, the notion of a “mutation-history” may be thought of as a species-theoretic version of the notion of a “diagram of categories” given in [AbsTopIII], Definition 3.5, (i).

(v) Let $\mathcal{H} = (\vec{\Gamma}, \mathcal{S}^*, \mathcal{M}^*)$ be a mutation-history; $\mathcal{S}$ a species. For simplicity, we assume that the underlying graph of $\vec{\Gamma}$ is simply connected. Then we shall refer to as a(n) [$\mathcal{S}$-valued] covariant (respectively, contravariant) observable $\mathcal{V}$ of the mutation-history $\mathcal{H}$ a collection of data as follows:

(a) for each vertex $v$ of $\vec{\Gamma}$, a mutation $\mathcal{V}^v : \mathcal{S}^v \to \mathcal{S}$, which we shall refer to as the observation mutation at $v$;

(b) for each edge $e$ of $\vec{\Gamma}$, running from a vertex $v_1$ to a vertex $v_2$, a morphism of mutations $\mathcal{V}^e : \mathcal{V}^{v_1} \to \mathcal{V}^{v_2} \circ \mathcal{M}^e$ (respectively, $\mathcal{V}^e : \mathcal{V}^{v_2} \circ \mathcal{M}^e \to \mathcal{V}^{v_1}$).

If $\mathcal{V}$ is a covariant observable such that all of the morphisms of mutations “$\mathcal{V}^e$” are isomorphisms of mutations, then we shall refer to the covariant observable $\mathcal{V}$ as a core. Thus, one may think of a core $\mathcal{C}$ of a mutation-history as lying “under” the entire mutation-history in a “uniform fashion”. Also, we shall refer to the “property [of an observable] of being a core” as the “coricity” of the observable. Finally, we
note that the notions of an “observable” and a “core” given here may be thought of as simplified, species-theoretic versions of the notions of “observable” and “core” given in [AbsTopIII], Definition 3.5, (iii).

**Remark 3.3.1.**

(i) One well-known consequence of the *axiom of foundation* of axiomatic set theory is the assertion that “$\in$-loops”

\[ a \in b \in c \in \ldots \in a \]

can *never occur* in the set theory in which one works. On the other hand, there are many situations in mathematics in which one wishes to somehow “identify” mathematical objects that arise at higher levels of the $\in$-structure of the set theory under consideration with mathematical objects that arise at lower levels of this $\in$-structure. In some sense, the notions of a “set” and of a “bijection of sets” allow one to achieve such “identifications”. That is to say, the mathematical objects at both higher and lower levels of the $\in$-structure constitute examples of the *same mathematical notion of a “set”*, so that one may consider “bijections of sets” between those sets without violating the axiom of foundation. In some sense, the notion of a *species* may be thought of as a natural *extension* of this observation. That is to say,

the notion of a “species” allows one to consider, for instance, *species-isomorphisms* between species-objects that occur at different levels of the $\in$-structure of the set theory under consideration — i.e., roughly speaking, to “simulate $\in$-loops” — without violating the axiom of foundation.

Moreover, typically the species-objects at higher levels of the $\in$-structure occur as the result of executing the *mutations* that arise in some sort of *mutation-history*

\[ \ldots \rightsquigarrow \mathcal{G} \rightsquigarrow \mathcal{G}' \rightsquigarrow \mathcal{G}'' \rightsquigarrow \ldots \rightsquigarrow \mathcal{G} \rightsquigarrow \ldots \]

— e.g., the “output species-objects” of the “$\mathcal{G}$” on the right that arise from applying various mutations to the “input species-objects” of the “$\mathcal{G}$” on the left.

(ii) In the context of constructing “loops” in a mutation-history as in the final display of (i), we observe that

the *simpler* the structure of the *species* involved, the *easier* it is to construct “loops”.

It is for this reason that species such as the species determined by the notion of a *category* [cf. Example 3.2] are easier to work with, from the point of view of constructing “loops”, than more complicated species such as the species determined by the notion of a *scheme*. This is one of the *principal motivations* for the “geometry of categories” — of which “*absolute anabelian geometry*” is the special case that arises when the categories involved are Galois categories — i.e., for the theory of *representing scheme-theoretic geometries via categories* [cf., e.g., the Introductions...
of [LgSch], [ArLgSch], [SemiAnbd], [Cusp], [FrdI]]. At a more concrete level, the utility of working with categories to reconstruct objects that occurred at lower levels of some sort of “series of constructions” [cf. the mutation-history of the final display of (i)!] may be seen in the “reconstruction of the underlying scheme”, given in [LgSch], Corollary 2.15, from a certain category constructed from a log scheme, as well as in the theory of “slim exponentiation” discussed in the Appendix to [FrdI].

(iii) Again in the context of mutation-histories such as the one given in the final display of (i), although one may, on certain occasions, wish to apply various mutations that fundamentally alter the structure of the mathematical objects involved and hence give rise to “output species-objects” of the “$S$” on the right that are related in a highly nontrivial fashion to the “input species-objects” of the “$S$” on the left, it is also of interest to consider

“portions” of the various mathematical objects that occur that are left unaltered by the various mutations that one applies.

This is precisely the reason for the introduction of the notion of a core of a mutation-history. One important consequence of the construction of various cores associated to a mutation-history is that often

one may apply various cores associated to a mutation-history to describe, by means of non-coric observables, the portions of the various mathematical objects that occur which are altered by the various mutations that one applies in terms of the unaltered portions, i.e., cores.

Indeed, this point of view plays a central role in the theory of the present series of papers — cf. the discussion of Remark 3.6.1, (ii), below.

**Remark 3.3.2.** One somewhat naive point of view that constituted one of the original motivations for the author in the development of theory of the present series of papers is the following. In the classical theory of schemes, when considering local systems on a scheme, there is no reason to restrict oneself to considering local systems valued in, say, modules over a finite ring. If, moreover, there is no reason to make such a restriction, then one is naturally led to consider, for instance, local systems of schemes [cf., e.g., the theory of the “Galois mantle” in [$p$Teich]], or, indeed, local systems of more general collections of mathematical objects. One may then ask what happens if one tries to consider local systems on the schemes that occur as fibers of a local system of schemes. [More concretely, if $X$ is, for instance, a connected scheme, then one may consider local systems $\mathcal{X}$ over $X$ whose fibers are isomorphic to $X$; then one may repeat this process, by considering such local systems over each fiber of the local system $\mathcal{X}$ on $X$, etc.] In this way, one is eventually led to the consideration of “systems of nested local systems” — i.e., a local system over a local system over a local system, etc. It is precisely this point of view that underlies the notion of “successive iteration of a given mutation-history”, relative to the terminology formulated in the present §3. If, moreover, one thinks of such “successive iterates of a given mutation-history” as being a sort of abstraction of the naive idea of a “system of nested local systems”,

then the notion of a core may be thought of as a sort of mathematical object that is invariant with respect to the application of the operations that gave rise to the “system of nested local systems”.

Example 3.4. Topological Spaces and Fundamental Groups.

(i) One verifies easily that the notions of a topological space and a continuous map between topological spaces determine an example of a species $\mathcal{S}^{\text{top}}$. In a similar vein, the notions of a universal covering $\tilde{X} \to X$ of a pathwise connected topological space $X$ and a continuous map between such universal coverings $\tilde{X} \to X, \tilde{Y} \to Y$ [i.e., a pair of compatible continuous maps $\tilde{X} \to \tilde{Y}, X \to Y$], considered up to composition with a deck transformation of the universal covering $\tilde{Y} \to Y$, determine an example of a species $\mathcal{S}^{\text{u-top}}$. We leave to the reader the routine task of writing out the various set-theoretic formulas that define the species structures of $\mathcal{S}^{\text{top}}, \mathcal{S}^{\text{u-top}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathcal{S}^{\text{u-top}}$ are collections of continuous maps [between two given universal coverings], any two of which differ from one another by composition with a deck transformation.

(ii) One verifies easily that the notions of a group and an outer homomorphism between groups [i.e., a homomorphism considered up to composition with an inner automorphism of the codomain group] determine an example of a species $\mathcal{S}^{\text{gp}}$. We leave to the reader the routine task of writing out the various set-theoretic formulas that define the species structure of $\mathcal{S}^{\text{gp}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathcal{S}^{\text{gp}}$ are collections of homomorphisms [between two given groups], any two of which differ from one another by composition with an inner automorphism.

(iii) Now one verifies easily that the assignment

$$(\tilde{X} \to X) \mapsto \text{Aut}(\tilde{X}/X)$$

— where $(\tilde{X} \to X)$ is a species-object of $\mathcal{S}^{\text{u-top}}$, and $\text{Aut}(\tilde{X}/X)$ denotes the group of deck transformations of the universal covering $\tilde{X} \to X$ — determines a mutation $\mathcal{S}^{\text{u-top}} \sim \mathcal{S}^{\text{gp}}$. That is to say, the “fundamental group” may be thought of as a sort of mutation.

Example 3.5. Absolute Anabelian Geometry.

(i) Let $\mathcal{S}$ be a class of connected normal schemes that is closed under isomorphism [of schemes]. Suppose that there exists a set $E_\mathcal{S}$ of schemes describable by a set-theoretic formula with the property that every scheme of $\mathcal{S}$ is isomorphic to some scheme belonging to $E_\mathcal{S}$. Then just as in the case of universal coverings of topological spaces discussed in Example 3.4, (i), one verifies easily, by applying the set-theoretic formula describing $E_\mathcal{S}$, that the universal pro-finite étale coverings $\tilde{X} \to X$ of schemes $X$ belonging to $\mathcal{S}$ and isomorphisms of such coverings considered up to composition with a deck transformation give rise to a species $\mathcal{S}^{\mathcal{S}}$.

(ii) Let $\mathcal{G}$ be a class of topological groups that is closed under isomorphism [of topological groups]. Suppose that there exists a set $E_\mathcal{G}$ of topological groups
describable by a set-theoretic formula with the property that every topological group of $\mathcal{G}$ is isomorphic to some topological group belonging to $E_{\mathcal{G}}$. Then just as in the case of abstract groups discussed in Example 3.4, (ii), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{G}}$, that topological groups belonging to $\mathcal{G}$ and [bi-continuous] outer isomorphisms between such topological groups give rise to a species $\mathcal{S}^{\mathcal{G}}$.

(iii) Let $\mathcal{S}$ be as in (i). Then for an appropriate choice of $\mathcal{G}$, by associating to a universal pro-finite étale covering the resulting group of deck transformations, one obtains a mutation

$$\Pi : \mathcal{S}^{\mathcal{S}} \sim \sim \mathcal{S}^{\mathcal{G}}$$

[cf. Example 3.4, (iii)]. Then one way to define the notion that the schemes belonging to the class $\mathcal{S}$ are “[absolute] anabelian” is to require the specification of a mutation

$$\mathbb{A} : \mathcal{S}^{\mathcal{G}} \sim \sim \mathcal{S}^{\mathcal{S}}$$

which forms a mutation-quasi-inverse to $\Pi$. Here, we note that the existence of the bijections [i.e., “fully faithfulness”] discussed in Definition 3.3, (iii), is, in essence, the condition that is usually taken as the definition of “anabelian”. By contrast, the species-theoretic approach of the present discussion may be thought of as an explicit mathematical formulation of the algorithmic approach to [absolute] anabelian geometry discussed in the Introduction to [AbsTopI].

(iv) The framework of [absolute] anabelian geometry [cf., e.g., the framework discussed above] gives a good example of the importance of specifying precisely what species one is working with in a given “series of constructions” [cf., e.g., the mutation-history of the final display of Remark 3.3.1, (i)]. That is to say, there is a quite substantial difference between working with a

profinite group in its sole capacity as a profinite group

and working with the same profinite group — which may happen to arise as the étale fundamental group of some scheme! —

regarded as being equipped with various data that arise from the construction of the profinite group as the étale fundamental group of some scheme.

It is precisely this sort of issue that constituted one of the original motivations for the author in the development of the theory of species presented here.

Example 3.6. The Étale Site and Frobenius.

(i) Let $p$ be a prime number. If $S$ is a reduced scheme over $\mathbb{F}_p$, then denote by $S^{(p)}$ the scheme with the same topological space as $S$, but whose structure sheaf is given by the subsheaf

$$\mathcal{O}_{S^{(p)}} \overset{\text{def}}{=} (\mathcal{O}_S)^p \subseteq \mathcal{O}_S$$

of $p$-th powers of sections of $S$. Thus, the natural inclusion $\mathcal{O}_{S^{(p)}} \hookrightarrow \mathcal{O}_S$ induces a morphism $\Phi_S : S \rightarrow S^{(p)}$. Moreover, “raising to the $p$-th power” determines a
natural isomorphism $\alpha_S : S^{(p)} \xrightarrow{\sim} S$ such that the resulting composite $\alpha_S \circ \Phi_S : S \to S$ is the Frobenius morphism of $S$. Write

$$\mathcal{S}^{p\text{-sch}}$$

for the species of reduced schemes over $\mathbb{F}_p$ and morphisms of schemes. Note that by considering, for instance, [necessarily quasi-affine!] étale morphisms of finite presentation $T \to S$ equipped with factorizations $T|_U \subseteq \mathbb{A}^N_U \to U$ for each affine open $U \subseteq S$ [where $\mathbb{A}^N_U$ denotes a “standard copy of affine $N$-space over $U$”, for some integer $N \geq 1$; the “$\subseteq$” exhibits $T|_U$ as a finitely presented subscheme of $\mathbb{A}^N_U$], one may construct an assignment

$$S \mapsto S_{\text{ét}}$$

that maps a species-object $S$ of $\mathcal{S}^{p\text{-sch}}$ to the category $S_{\text{ét}}$ of such étale morphisms of finite presentation $T \to S$ and $S$-morphisms — i.e., “the small étale site of $S$” — in such a way that the assignment $S \mapsto S_{\text{ét}}$ is contravariantly functorial with respect to species-morphisms $S_1 \to S_2$ of $\mathcal{S}^{p\text{-sch}}$, and, moreover, may be described via set-theoretic formulas. Thus, such an assignment determines an “étale site mutation”

$$\mathcal{M}_{\text{ét}} : \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{\text{cat}}$$

— where we write $\mathcal{S}^{\text{cat}}$ for the species of categories and isomorphism classes of contravariant functors [cf. Example 3.2]. Another natural assignment in the present context is the assignment

$$S \mapsto S^{\text{pf}}$$

which maps $S$ to its perfection $S^{\text{pf}}$, i.e., the scheme determined by taking the inverse limit of the inverse system $\cdots \to S \to S \to S$ obtained by considering iterates of the Frobenius morphism of $S$. Thus, by considering the final copy of “$S$” in this inverse system, one obtains a natural morphism $\beta_S : S^{\text{pf}} \to S$. Finally, one obtains a “perfection mutation”

$$\mathcal{M}^{\text{pf}} : \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{p\text{-sch}}$$

by considering the set-theoretic formulas underlying the assignment $S \mapsto S^{\text{pf}}$.

(ii) Write

$$\mathcal{F}^{p\text{-sch}} : \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{p\text{-sch}}$$

for the “Frobenius mutation” obtained by considering the set-theoretic formulas underlying the assignment $S \mapsto S^{(p)}$. Thus, one may formulate the well-known “invariance of the étale site under Frobenius” [cf., e.g., [FK], Chapter I, Proposition 3.16] as the statement that the “étale site mutation” $\mathcal{M}_{\text{ét}}$ exhibits $\mathcal{S}^{\text{cat}}$ as a core — i.e., an “invariant piece” — of the “Frobenius mutation-history”

$$\cdots \rightsquigarrow \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{p\text{-sch}} \rightsquigarrow \mathcal{S}^{p\text{-sch}} \rightsquigarrow \cdots$$

determined by the “Frobenius mutation” $\mathcal{F}^{p\text{-sch}}$. In this context, we observe that the “perfection mutation” $\mathcal{M}^{\text{pf}}$ also yields a core — i.e., another “invariant piece” — of the Frobenius mutation-history. On the other hand, the natural morphism $\Phi_S : S \to S^{(p)}$ may be interpreted as a covariant observable
of this mutation-history whose observation mutations are the identity mutations $\text{id} : S^{\text{p-sch}} \rightarrow S^{\text{p-sch}}$. Since $\Phi_S$ is not, in general, an isomorphism, it follows that this observable constitutes an example of a non-coric observable. Nevertheless, the natural morphism $\beta_S : S^{\text{pf}} \rightarrow S$ may be interpreted as a morphism of mutations $\mathcal{M}^{\text{pf}} \rightarrow \text{id} : S^{\text{p-sch}} \rightarrow S^{\text{p-sch}}$ that serves to relate the non-coric observable just considered to the coric observable arising from $\mathcal{M}^{\text{pf}}$.

(iii) One may also develop a version of (i), (ii) for log schemes; we leave the routine details to the interested reader. Here, we pause to mention that the theory of log schemes motivates the following “combinatorial monoid-theoretic” version of the non-coric observable on the Frobenius mutation-history of (ii). Write

$$S^{\text{mon}}$$

for the species of torsion-free abelian monoids and morphisms of monoids. If $M$ is a species-object of $S^{\text{mon}}$, then write $M(p) \overset{\text{def}}{=} p \cdot M \subseteq M$. Then the assignment $M \mapsto M(p)$ determines a “monoid-Frobenius mutation”

$$\mathfrak{F}^{\text{mon}} : S^{\text{mon}} \rightarrow S^{\text{mon}}$$

and hence a “monoid-Frobenius mutation-history”

$$\ldots \rightarrow S^{\text{mon}} \rightarrow S^{\text{mon}} \rightarrow \ldots$$

which is equipped with a non-coric contravariant observable determined by the natural inclusion morphism $M(p) \rightarrow M$ and the observation mutations given by the identity mutations $\text{id} : S^{\text{mon}} \rightarrow S^{\text{mon}}$. On the other hand, the $p$-perfection $M^{\text{pf}}$ of $M$, i.e., the inductive limit of the inductive system $M \rightarrow M \rightarrow M \rightarrow \ldots$ obtained by considering the inclusions given by multiplying by $p$, gives rise to a “monoid-$p$-perfection mutation”

$$\mathfrak{F}^{\text{pf}} : S^{\text{mon}} \rightarrow S^{\text{mon}}$$

which may be interpreted as a core of the monoid-Frobenius mutation-history. Finally, the natural inclusion of monoids $M \rightarrow M^{\text{pf}}$ may be interpreted as a morphism of mutations $\text{id} : S^{\text{mon}} \rightarrow \mathfrak{F}^{\text{pf}}$ that serves to relate the non-coric observable just considered to the coric observable arising from $\mathfrak{F}^{\text{pf}}$.

**Remark 3.6.1.**

(i) The various constructions of Example 3.6 may be thought of as providing, in the case of the phenomena of “invariance of the étale site under Frobenius” and “invariance of the perfection under Frobenius”, a “species-theoretic interpretation” — i.e., via consideration of “coric” versus “non-coric” observables — of the difference between “étale-type” and “Frobenius-type” structures [cf. the discussion of [FrI], §14]. This sort of approach via “combinatorial patterns” to expressing the difference between “étale-type” and “Frobenius-type” structures
plays a central role in the theory of the present series of papers. Indeed, the mutation-histories and cores considered in Example 3.6, (ii), (iii), may be thought of as the underlying motivating examples for the theory of both

- the vertical lines, i.e., consisting of log-links, and
- the horizontal lines, i.e., consisting of \( \Theta^{\times \mu_{\text{gau}}}/\Theta_{\text{LGP}}^{\times \mu}/\Theta_{\text{lgp}}^{\times \mu} \)-links,

of the log-theta-lattice [cf. [IUTchIII], Definitions 1.4, 3.8]. Finally, we recall that this approach to understanding the log-links may be seen in the introduction of the terminology of "observables" and "cores" in [AbsTopIII], Definition 3.5, (iii).

(ii) Example 3.6 also provides a good example of the important theme [cf. the discussion of Remark 3.3.1, (iii)] of describing non-coric data in terms of coric data — cf. the morphism \( \beta_S : S^{\text{pf}} \to S \) of Example 3.6, (ii); the natural inclusion \( M \to M^{\text{pf}} \) of Example 3.6, (iii). From the point of view of the vertical and horizontal lines of the log-theta-lattice [cf. the discussion of (i)], this theme may also be observed in the vertically coric log-shells that serve as a common receptacle for the various arrows of the log-Kummer correspondences of [IUTchIII], Theorem 3.11, (ii), as well as in the multiradial representations of [IUTchIII], Theorem 3.11, (i), which describe [certain aspects of] the arithmetic holomorphic structure on one vertical line of the log-theta-lattice in terms that may be understood relative to an alien arithmetic holomorphic structure on another vertical line — i.e., separated from the first vertical line by horizontal arrows — of the log-theta-lattice [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)].

Remark 3.6.2.

(i) In the context of the theme of "coric descriptions of non-coric data" discussed in Remark 3.6.1, (ii), it is of interest to observe the significance of the use of set-theoretic formulas [cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4] to realize such descriptions. That is to say, descriptions in terms of arbitrary choices that depend on a particular model of set theory [cf. Remark 3.1.3] do not allow one to calculate in terms that make sense in one universe the operations performed in an alien universe! This is precisely the sort of situation that one encounters when one considers the vertical and horizontal arrows of the log-theta-lattice [cf. (ii) below], where distinct universes arise from the distinct scheme-theoretic basepoints on either side of such an arrow that correspond to distinct ring theories, i.e., ring theories that cannot be related to one another by means of a ring homomorphism — cf. the discussion of Remark 3.6.3 below. Indeed,

it was precisely the need to understand this sort of situation that led the author to develop the "inter-universal" version of Teichmüller theory exposed in the present series of papers.

Finally, we observe that the algorithmic approach [i.e., as opposed to the "fully faithfulness/Grothendieck Conjecture-style approach" — cf. Example 3.5, (iii)] to
reconstruction issues via set-theoretic formulas plays an essential role in this context. That is to say, although different algorithms, or software, may yield the same output data, it is only by working with specific algorithms that one may understand the delicate inter-relations that exist between various components of the structures that occur as one performs various operations [i.e., the mutations of a mutation-history]. In the case of the theory developed in the present series of papers, one central example of this phenomenon is the cyclotomic rigidity isomorphisms that underlie the theory of $\Theta^{x\mu}_{\text{LGP}}$-link compatibility discussed in [IUTchIII], Theorem 3.11, (iii), (c), (d) [cf. also [IUTchIII], Remarks 2.2.1, 2.3.2].

(ii) The algorithmic approach to reconstruction that is taken throughout the present series of papers, as well as, for instance, in [FrDI], [EtTh], and [AbsTopIII], was conceived by the author in the spirit of the species-theoretic formulation exposed in the present §3. Nevertheless, [cf. Remark 3.1.3, (i)] we shall not explicitly write out the various set-theoretic formulas involved in the various species, mutations, etc. that are implicit throughout the theory of the present series of papers. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the conventional descriptions that are given of the mathematical objects involved. When applying such conventional descriptions, however, the reader is obliged to check that they are well-defined and do not depend upon the use of arbitrary choices that are not describable via well-defined set-theoretic formulas.

(iii) The sharp contrast between

\[ \cdot \text{ the canonicality imparted by descriptions via set-theoretic formulas in the context of extensions of the universe in which one works} \]

[cf. Remarks 3.1.3, 3.1.4] and

\[ \cdot \text{ the situation that arises if one allows, in one’s descriptions, the various arbitrary choices arising from invocations of the axiom of choice} \]

may be understood somewhat explicitly if one attempts to “catalogue the various possibilities” corresponding to various possible choices that may occur in one’s description. That is to say, such a “cataloguing operation” typically obligates one to work with “sets of very large cardinality”, many of which must be constructed by means of set-theoretic exponentiation [i.e., such as the operation of passing from a set $E$ to the set $2^E$ of all subsets of $E$]. Such a rapid outbreak of “unwieldy large sets” is reminiscent of the rapid growth, in the $p$-adic crystalline theory, of the $p$-adic valuations of the denominators that occur when one formally integrates an arbitrary connection, as opposed to a “canonical connection” of the sort that arises from a crystalline representation. In the $p$-adic theory, such “canonical connections” are typically related to “canonical liftings”, such as, for instance, those that occur in $p$-adic Teichmüller theory [cf. [pOrd], [pTeich]]. In this context, it is of interest to recall that the canonical liftings of $p$-adic Teichmüller theory may, under certain conditions, be thought of as liftings “of minimal complexity” in the sense that their Witt vector coordinates are given by polynomials of minimal degree — cf. the computations of [Finot].
Remark 3.6.3.

(i) In the context of Remark 3.6.2, it is useful to recall the fundamental reason for the need to pursue “inter-universality” in the present series of papers [cf. the discussion of [IUTchIII], Remark 1.2.4; [IUTchIII], Remark 1.4.2], namely,

since étale fundamental groups — i.e., in essence, Galois groups — are defined as certain automorphism groups of fields/rings, the definition of such a Galois group as a certain automorphism group of some ring structure is fundamentally incompatible with the vertical and horizontal arrows of the log-theta-lattice [i.e., which do not arise from ring homomorphisms]!

In this respect, “transformations” such as the vertical and horizontal arrows of the log-theta-lattice differ, quite fundamentally, from “transformations” that are compatible with the ring structures on the domain and codomain, i.e., morphisms of rings/schemes, which tautologically give rise to functorial morphisms between the respective étale fundamental groups. Put another way, in the notation of [IUTchI], Definition 3.1, (e), (f), for, say, \( v \in V^{\text{non}} \),

the only natural correspondence that may be described by means of set-theoretic formulas between the isomorphs of the local base field Galois groups \( "G_v" \) on either side of a vertical or horizontal arrow of the log-theta-lattice is the correspondence constituted by an indeterminate isomorphism of topological groups.

A similar statement may be made concerning the isomorphs of the geometric fundamental group \( \Delta_{\bar{\nu}} \defeq \text{Ker}(\Pi_{\bar{\nu}} \rightarrow G_{\bar{\nu}}) \) on either side of a horizontal — but not vertical! — cf. the discussion of (ii) below — arrow of the log-theta-lattice — that is to say,

the only natural correspondence that may described by means of set-theoretic formulas between these isomorphs is the correspondence constituted by an indeterminate isomorphism of topological groups equipped with some outer action by the respective isomorph of \( "G_{\bar{\nu}}" \)

— cf. the discussion of [IUTchIII], Remark 1.2.4. Here, again we recall from the discussion of Remark 3.6.2, (i), (ii), that it is only by working with such correspondences that may be described by means of set-theoretic formulas that one may obtain descriptions that allow one to calculate the operations performed in one universe from the point of view of an alien universe.

(ii) One fundamental difference between the vertical and horizontal arrows of the log-theta-lattice is that whereas, for, say, \( v \in V^{\text{bad}} \),

(V1) one identifies, up to isomorphism, the isomorphs of the full arithmetic fundamental group \( "\Pi_{\bar{\nu}}" \) on either side of a vertical arrow,

(H1) one distinguishes the \( \Delta_{\bar{\nu}}'s \) on either side of a horizontal arrow, i.e., one only identifies, up to isomorphism, the local base field Galois groups \( "G_{\bar{\nu}}" \) on either side of a horizontal arrow.
— cf. the discussion of [IUTchIII], Remark 1.4.2. One way to understand the fundamental reason for this difference is as follows.

(V2) In order to construct the log-link — i.e., at a more concrete level, the power series that defines the $p_v$-adic logarithm at $v$ — it is necessary to avail oneself of the local ring structures at $v$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the full “$\Pi_v$” [i.e., not from “$G_v$ stripped of its structure as a quotient of $\Pi_v$” — cf. the discussion of [IUTchIII], Remark 1.4.1, (i); [IUTchIII], Remark 2.1.1, (ii); [AbsTopIII], §13].

(H2) In order to construct the $\Theta^\times_{\mathrm{gau}}/\Theta^\times_{\mathrm{LGP}}/\Theta^\times_{\mathrm{lgp}}$-links — i.e., at a more concrete level, the correspondence

$$q \mapsto \left\{ q^{j^2} \right\}_{j=1,...,l^*}$$

[cf. [IUTchII], Remark 4.11.1] — it is necessary, in effect, to construct an “isomorphism” between a mathematical object [i.e., the theta values “$q^j$”] that depends, in an essential way, on regarding the various “$j$” as distinct labels [which are constructed from “$\Delta_v$”!] and a mathematical object [i.e., “$q$”] that is independent of these labels; it is then a tautology that such an “isomorphism” may only be achieved if the labels — i.e., in essence, “$\Delta_v$” — on either side of the “isomorphism” are kept distinct from one another.

Here, we observe in passing that the “apparently horizontal arrow-related” issue discussed in (H2) of simultaneous realization of “label-dependent” and “label-free” mathematical objects is reminiscent of the vertical arrow portion of the bicoricty theory of [IUTchIII], Theorem 1.5 — cf. the discussion of [IUTchIII], Remark 1.5.1, (i), (ii); Step (vii) of the proof of [IUTchIII], Corollary 3.12.
Bibliography


