INVITATION TO INTER-UNIVERSAL TEICHMÜLLER THEORY

Shinichi Mochizuki (RIMS, Kyoto University)

http://www.kurims.kyoto-u.ac.jp/~motizuki
"Travel and Lectures"

§1. Hodge-Arakelov-theoretic Motivation

For $l \geq 5$ a prime number, the module of l-torsion points associated to a Tate curve $E \stackrel{\mathrm{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ (over, say, a p-adic field or $\mathbb{C}$) fits into a natural exact sequence:

$$0 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0$$

That is to say, one has canonical objects as follows:

a "multiplicative subspace" $\boldsymbol{\mu}_l \subseteq E[l]$ and "generators" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$

In the following, we fix an elliptic curve E over a number field F and a prime number $l \geq 5$. Also, we suppose that E has stable reduction at all finite places of F.

Then, in general, $E[l]$ does not admit

a global "multiplicative subspace" and "generators"

that coincide with the above canonical "multiplicative subspace" and "generators" at all finite places where E has bad mult. reduction! Nevertheless, let us suppose (!!) that such global objects do in fact exist. Then the Fundamental Theorem of Hodge-Arakelov Theory may be formulated as follows:

$$\Gamma(E^{\dagger}, \mathcal{L})^{<l} \quad \stackrel{\sim}{\to} \quad \bigoplus_{j=-l^*}^{l^*} \underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F$$

— where

- $E^{\dagger} \to E$ is the "universal vectorial extension" of E;
- "$< l$" is the "relative degree" w.r.t. this extension; $l^* \stackrel{\mathrm{def}}{=} (l-1)/2$;
- $\mathcal{L}$ is a line bundle that arises from a (nontrivial) 2-torsion point;
- "q" is the q-parameter at bad places of F; $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} q^{1/2l}$;
- LHS admits a Hodge filtration F^{-i} s.t. F^{-i}/F^{-i+1} is (roughly)

  $\stackrel{\sim}{\to} \ \omega_E^{\otimes(-i)} \qquad (i = 0, 1, \ldots, l-1;\ \omega_E =$ cotang. bun. at the origin$)$;

- RHS admits a natural Galois action compatible with "$\bigoplus$".

This isom. is, *a priori*, only defined/F, but is in fact (essentially) compatible with the natural integral structures/metrics at all places of F.

A similar isom. may be considered over the moduli stack of elliptic curves. The proof of such an isom. is based on a computation, which shows that the degrees $[-]$ of the vector bundles on either side of the isom. coincide:

$$\frac{1}{l} \cdot \mathrm{LHS} \approx -\frac{1}{l} \cdot \sum_{i=0}^{l-1} i \cdot [\omega_E] \approx -\frac{l}{2} \cdot [\omega_E]$$

$$\frac{1}{l} \cdot \mathrm{RHS} \approx -\frac{1}{l^2} \cdot \sum_{j=1}^{l^*} j^2 \cdot [\log(\underline{\underline{q}})] \approx -\frac{l}{24} \cdot [\log(\underline{\underline{q}})] = -\frac{l}{2} \cdot [\omega_E]$$

On the other hand, returning to the situation over number fields, since F^i is not compatible with the above direct sum decomposition, it follows that, by projecting to the factors of this direct sum decomp., one may construct a sort of relative of the so-called "arithmetic Kodaira-Spencer morphism", i.e., for (most) j, a (nonzero) morphism of line bundles

$$(\mathcal{O}_F \approx) \; F^0 \hookrightarrow \underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F.$$

Since, moreover, $\deg_{\mathrm{arith}}(F^0) \approx 0$, it follows that, if we denote the "height" determined by the log. diffs. $\Omega^{\log}_{\mathcal{M}}|_E$ associated to the moduli stack of elliptic curves by $\mathrm{ht}_E \stackrel{\mathrm{def}}{=} 2 \cdot \deg_{\mathrm{arith}}(\omega_E) = \deg_{\mathrm{arith}}(\Omega^{\log}_{\mathcal{M}}|_E)$, then we obtain an inequality (!) as follows:

$$\frac{1}{6} \cdot \deg_{\mathrm{arith}}(\log(\underline{\underline{q}})) = \mathrm{ht}_E < \text{constant}$$

In fact, of course, since the global multiplicative subspace and generators which play an essential role in the above argument do not, in general, exist, this argument cannot be applied immediately in its present form.

This state of affairs motivates the following approach, which may appear somewhat far-fetched at first glance! Suppose that the assignment

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*} \mapsto \underline{\underline{q}}$$

somehow determines an automorphism of the number field F! Such an "automorphism" necessarily preserves degrees of arithmetic line bundles. Thus, since the absolute value of the degree of the RHS of the above assignment is "small" by comparison to the absolute value of the (average!) degree of the LHS, we thus conclude that a similar inequality (!) holds:

$$\frac{1}{6} \cdot \deg_{\mathrm{arith}}(\log(\underline{\underline{q}})) = \mathrm{ht}_E < \text{constant}$$

Of course, such an autom. of a NF does not in fact exist!! On the other hand, what happens if we regard the "$\{\underline{\underline{q}}^{j^2}\}$" on the LHS and the "$\underline{\underline{q}}$" on the RHS as belonging to distinct copies of "conventional ring/scheme theory" = "arithmetic holomorphic structures", and we think of the assignment under consideration

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$

— i.e., which may be regarded as a sort of "tautological solution" to the

"obstruction to applying HA theory to diophantine geometry"

— as a sort of quasiconformal map between Riemann surfaces equipped with distinct holomorphic structures?

That is to say, this approach allows us to realize the assignment under consideration, albeit at the cost of partially dismantling conventional ring/sch. theory. On the other hand, this approach requires us

to compute just how much of a distortion occurs

as a result of dismantling = deforming conventional ring/scheme theory. This vast computation is the content of IUTeich.

In conclusion, at a concrete level, the "distortion" that occurs at the portion labeled by the index j is (roughly)

$$\leq \; j \cdot \text{log-diff}_F.$$

In particular, by the exact same computation (i.e., of the "leading term" of the average over j) as the computation discussed above in the case of the moduli stack of elliptic curves, we obtain the following inequality:

$$\tfrac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \;=\; \text{ht}_E \;\leq\; (1+\epsilon)\text{log-diff}_F \;+\; \text{constant}$$

This inequality is the content of the so-called

Szpiro Conjecture ($\Longleftrightarrow$ ABC Conjecture).

§2. Teichmüller-theoretic Deformations

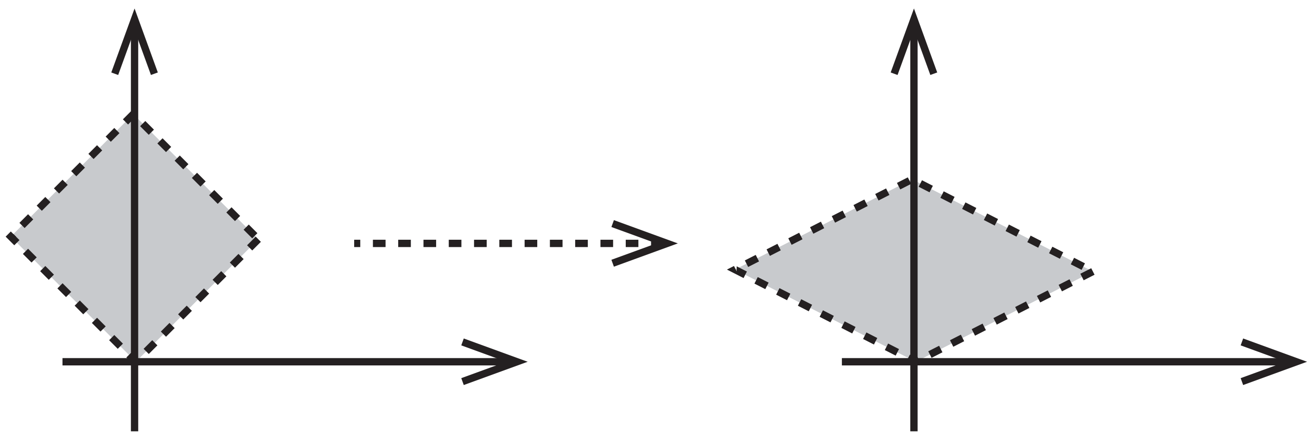
Classical Teichmüller theory over $\mathbb{C}$:

Relative to the canonical coordinate $z = x + iy$ (associated to a square differential) on the Riemann surface, Teichmüller deformations are given by

$$z \mapsto \zeta = \xi + i\eta = Kx + iy$$

— where $1 < K < \infty$ is the dilation factor.

Key point: one holomorphic dimension, but two underlying real dimensions, of which one is dilated/deformed, while the other is left fixed/undeformed!



p-adic Teichmüller theory:

· p-adic canonical liftings of a hyperbolic curve in positive characteristic equipped with a nilpotent indigenous bundle

· Frobenius liftings over the ordinary locus of the moduli stack of curves, as well as over tautological curve — cf. the metric on the Poincaré upper half-plane, Weil-Petersson metric in the theory/$\mathbb{C}$.

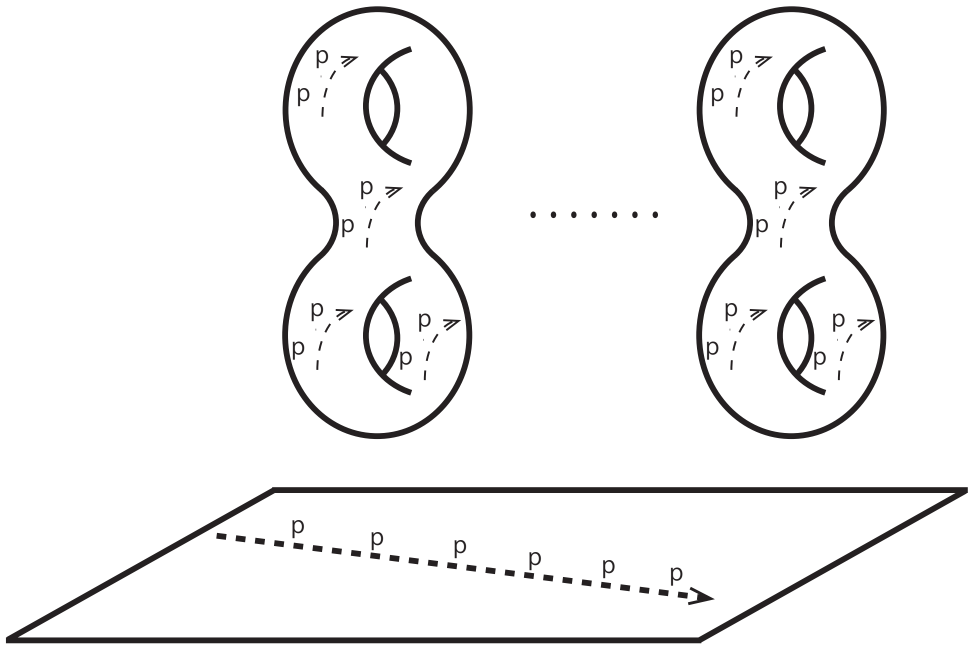
Analogy between IUTeich and pTeich:

conventional scheme theory/$\mathbb{Z}$ $\longleftrightarrow$ scheme theory/$\mathbb{F}_p$

number field (+ fin. many places) $\longleftrightarrow$ hyperbolic curve in pos. char.

once-punctured elliptic curve/NF $\longleftrightarrow$ nilpotent indigenous bundle

log-Θ-lattice $\longleftrightarrow$ p-adic canonical lifting + Frob. lifting



The arithmetic case: addition and multiplication, cohomological dim.:

Regard the ring structure of rings such as $\mathbb{Z}$ as a

one-dimensional "arithmetic holomorphic structure"!

— which has

two underlying combinatorial dimensions!

"addition"	and	"multiplication"
$(\mathbb{Z}, +)$	$\curvearrowleft$	$(\mathbb{Z}, \times)$
one combinatorial dim.		one combinatorial dim.

— cf. the two cohomological dims. of the absolute Galois group of

- a (totally imaginary) number field $F/\mathbb{Q} < \infty$,
- a p-adic local field $k/\mathbb{Q}_p < \infty$,

as well as the two underlying real dims. of

- $\mathbb{C}^\times$.

Units and value groups:

In case of a p-adic local field $k/\mathbb{Q}_p < \infty$, one may also think of these two underlying combinatorial dimensions as follows:

$$\begin{array}{ccccc} \mathcal{O}_k^\times & \subseteq & k^\times & \twoheadrightarrow & k^\times/\mathcal{O}_k^\times \ (\cong \mathbb{Z}) \\ \text{one combinatorial dim.} & & & & \text{one combinatorial dim.} \end{array}$$

— cf. the direct product decomp. in the complex case: $\mathbb{C}^\times = \mathbb{S}^1 \times \mathbb{R}_{>0}$.

In IUTeich, we shall deform the holomorphic str. of the number field by

dilating the value groups via the theta function, while

leaving the units undilated

§3. The Log-theta-lattice

Noncommutative (!) 2-dim. diagram of Hodge theaters:

2 dims. of the diagram $\longleftrightarrow$ 2 comb. dims. of a p-adic local field!

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

Analogy between IUTeich and pTeich:

each Hodge theater $\bullet$ $\longleftrightarrow$ a copy of scheme theory/$\mathbb{F}_p$

$\uparrow$ = $\mathfrak{log}$-link $\longleftrightarrow$ the Frobenius morphism in pos. char.

$\longrightarrow$ = Θ-link $\longleftrightarrow$ $\left(p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}\right)$

$\mathfrak{log}$-Link:

At nonarchimedean v of the number field F, the ring structures on either side of the $\mathfrak{log}$-link are related by a non-ring-homomorphism (!)

$$\log_v : \overline{k}^{\times} \quad \to \quad \overline{k}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\mathrm{def}}{=} F_v$; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Key point: The $\mathfrak{log}$-link is compatible with the isomorphism

$$\Pi_v \ \stackrel{\sim}{\to} \ \Pi_v$$

between the arithmetic fundamental groups Π_v on either side of the $\mathfrak{log}$-link, relative to the natural actions via $\Pi_v \twoheadrightarrow G_v$. Moreover, if one allows v to vary, the $\mathfrak{log}$-link is also compatible with the action of the global absolute Galois groups. Finally, at archimedean v of F, one has an analogous theory.

Θ-Link:

At bad nonarchimedean v of the number field F, the ring structures on either side of the Θ-link are related by a non-ring-homomorphism (!)

$$\mathcal{O}^{\times}_{\overline{k}} \ \stackrel{\sim}{\to} \ \mathcal{O}^{\times}_{\overline{k}}; \qquad \Theta|_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \ \mapsto \ \underline{\underline{q}}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\mathrm{def}}{=} F_v$; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Key point: The Θ-link is compatible with the isomorphism

$$G_v \ \stackrel{\sim}{\to} \ G_v$$

between the Galois groups G_v on either side of the Θ-link, relative to the natural actions on $\mathcal{O}^{\times}_{\overline{k}}$. At good nonarchimedean/archimedean v of F, one can give an analogous definition, by applying the product formula.

Remark: It is only possible to define the "walls/barriers" (i.e., from the point of view of the ring structure of conventional ring/scheme theory) constituted by the $\mathfrak{log}$-, Θ-links by working with abstract monoids/... !

Remark: By contrast, the objects that appear in the étale-picture (cf. the diagram below!) — i.e., the portion of the log-theta-lattice constituted by the arithmetic fundamental groups/Galois groups — have the power to slip through these "walls"!

Various versions of "Kummer theory" — which allow us to relate the following two types of mathematical objects:

abstract monoids = Frobenius-like objects

and

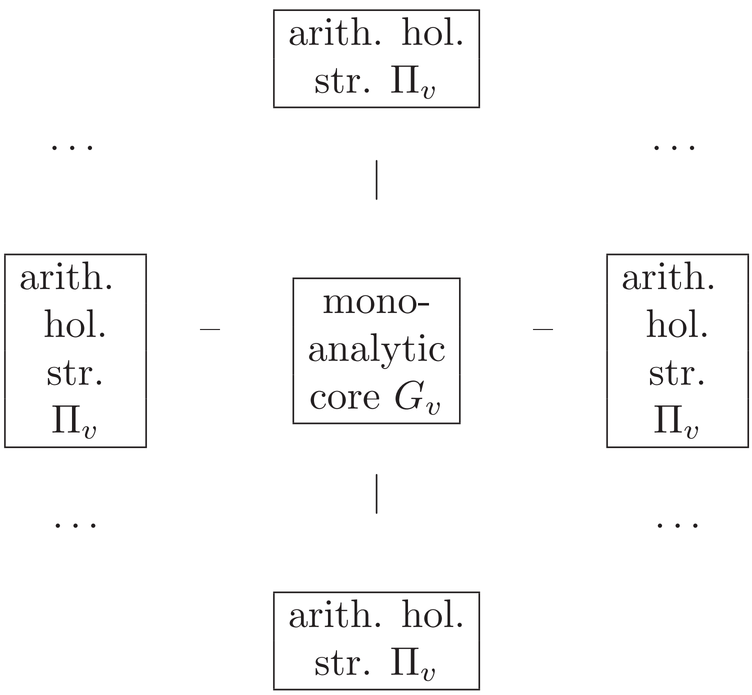
arithmetic fundamental groups/Galois groups = étale-like objects

— play a very important role throughout IUTeich! Moreover, the transition

「Frobenius-like ⇝ étale-like」

may be regarded as a global analogue over number fields of the computation — i.e., via "cartesian coords. ⇝ polar coords." — of the classical Gaussian integral

$$\int_0^\infty e^{-x^2} dx \quad !$$



§4. Inter-universality and Anabelian Geometry

Note that the $\mathfrak{log}$-, Θ-links are not compatible with the ring structures

$$\log_v : \overline{k}^{\times} \;\to\; \overline{k}, \qquad\qquad \Theta|_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \;\mapsto\; \underline{\underline{q}}$$

in their domains and codomains, hence are not compatible, in a quite essential way, with the scheme-theoretic "basepoints" and

$$\text{Galois groups} \qquad (\quad \subseteq \mathrm{Aut}_{\text{field}}(\overline{k}) \;\; !! \quad)$$

that arise from ring homomorphisms! That is to say, when one passes to the "opposite side" of the $\mathfrak{log}$-, Θ-links,

"Π_v" and "G_v"

only make sense in their capacity as abstract topological groups!

$\Longrightarrow$ As a consequence, in order to compute the relationship between the ring structures in the domain and codomain of the $\mathfrak{log}$-, Θ-links, it is necessary to apply anabelian geometry! At the level of previous papers by the author, we derive the following Main Theorem by applying the results and theory of

- Semi-graphs of Anabelioids
- The Geometry of Frobenioids I, II
- The Étale Theta Function ...
- Topics in Absolute Anab. Geo. III

concerning

absolute anabelian geometry and

various rigidity properties of the étale theta function.

Main Theorem : One can give an explicit, algorithmic description, up to mild indeterminacies, of the LHS of the Θ-link in terms of the "alien" ring structure on the RHS of the Θ-link.

Key points:

· the coricity (i.e., coric nature) of $G_v \curvearrowright \mathcal{O}_{\overline{k}}^{\times}$!

· various versions of "Kummer theory" which allow us to relate the following two types of mathematical objects:

$$\underline{\text{abstract monoids}} = \underline{\text{Frobenius-like}} \text{ objects}$$

and

$$\underline{\text{arithmetic fundamental groups/Galois groups}} = \underline{\text{étale-like}} \text{ objects.}$$

Here, we recall the analogy with the computation of the Gaussian integral:

definition of $\mathfrak{log}$-, Θ-link, log-theta-lattice $\longleftrightarrow$ cartesian coords.

algorithmic description via abs. anabelian geometry $\longleftrightarrow$ polar coords.

essential rigidity of cyclotomes ($\cong \widehat{\mathbb{Z}}(1)$) $\longleftrightarrow$ coord. trans. via $\mathbb{S}^1 \curvearrowright$

By performing a volume computation, as discussed in §1, concerning the output of the algorithms of the above Main Theorem, one obtains:

Corollary: The "Szpiro Conjecture" ($\Longleftrightarrow$ "ABC Conjecture").

Relative to the analogy with the classical theory over p-adic local fields and $\mathbb{C}$, the corresponding inequalities are as follows:

· the degree $= (2g-2)(1-p) \leq 0$ of the

$$\text{“}\underline{\text{Hasse invariant}} = \frac{1}{p} \cdot d(\underline{\text{Frob. lift.}})\text{”}$$

in pTeich,

· the Gauss-Bonnet Theorem for a hyperbolic Riemann surface S

$$0 > -\int_S (\text{Poincaré metric}) = 4\pi(1-g).$$