ON THE COMBINATORIAL ANABELIAN GEOMETRY OF NODALLY NONDEGENERATE OUTER REPRESENTATIONS

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Abstract. Let Σ be a nonempty set of prime numbers. In the present paper, we continue the study, initiated in a previous paper by the second author, of the *combinatorial anabelian geometry* of *semi-graphs of anabelioids of pro-*Σ *PSC-type*, i.e., roughly speaking, semi-graphs of anabelioids associated to pointed stable curves. Our *first main result* is a partial generalization of one of the main combinatorial anabelian results of this previous paper to the case of *nodally nondegenerate outer representations*, i.e., roughly speaking, a sort of abstract combinatorial group-theoretic generalization of the scheme-theoretic notion of a family of pointed stable curves over the spectrum of a discrete valuation ring. We then apply this result to obtain a generalization, to the case of *proper* hyperbolic curves, of a certain *injectivity* result, obtained in another paper by the second author, concerning outer automorphisms of the pro-Σ fundamental group of a configuration space associated to a hyperbolic curve, as the dimension of this configuration space is lowered from two to one. This injectivity allows one to generalize a certain well-known *injectivity theorem of Matsumoto* to the case of *proper hyperbolic curves.*

Contents

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Introduction

Let Σ be a nonempty set of prime numbers. In the present paper, we continue the study, initiated in [Mzk4] by the second author, of the *combinatorial anabelian geometry* of *semi-graphs of anabelioids of pro-*Σ *PSC-type*, i.e., roughly speaking, semi-graphs of anabelioids associated to pointed stable curves. In particular, it was shown in [Mzk4] (cf. [Mzk4], Corollary 2.7, (iii)) that in the case of a semi-graph of anabelioids of pro-Σ PSC-type that arises from a *stable log curve* over a *log point* (i.e., the spectrum of an algebraically closed field k of characteristic $p \notin \Sigma$ equipped with the log structure determined by the morphism of monoids $\mathbb{N} \ni 1 \mapsto 0 \in k$), the semi-graph of anabelioids in question may be **reconstructed group-theoretically** from the *outer action* of the pro-Σ logarithmic fundamental group of the log point (which is noncanonically isomorphic to the maximal pro-Σ quotient $\widehat{\mathbb{Z}}^{\Sigma}$ of $\widehat{\mathbb{Z}}$) on the pro-Σ fundamental group of the semi-graph of anabelioids. As discussed in the introduction to [Mzk4], this result may be regarded as a substantial refinement of the pro-l criterion of Takayuki Oda for a proper hyperbolic curve over a discretely valued field to have *good reduction* (i.e., a special fiber whose associated semi-graph consists of a single vertex and no edges). We shall refer to an outer action of the type just described as an *outer representation of IPSC-type* (cf. Definition 2.4, (i)).

In the present paper, the theory of [Mzk4] is generalized to the case of *nodally nondegenerate outer representations*, or *outer representations of NN-type*, for short (cf. Definition 2.4, (iii)). Indeed, our *first main result* (cf. Corollary 4.2; Remark 4.2.1) is the following partial generalization of [Mzk4], Corollary 2.7, (iii).

Theorem A (**Graphicity of certain group-theoretically cuspidal isomorphisms**)**.** *Let* Σ *be a nonempty set of prime numbers,* $\mathcal{G}$ *and* $\mathcal{H}$ *semi-graphs of anabelioids of pro-*Σ *PSC-type (cf.* [Mzk4]*, Definition 1.1, (i)),* $\Pi_{\mathcal{G}}$ *(respectively,* $\Pi_{\mathcal{H}}$*) the pro-*Σ *fundamental group of* $\mathcal{G}$ *(respectively,* $\mathcal{H}$*),* $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ *an isomorphism of profinite groups,* I *and* J *profinite groups,* $\rho_I\colon I \to \mathrm{Aut}(\mathcal{G})$ *and* $\rho_J\colon J \to \mathrm{Aut}(\mathcal{H})$ *continuous homomorphisms, and* $\beta\colon I \xrightarrow{\sim} J$ *an isomorphism of profinite groups. Suppose that the following three conditions are satisfied:*

(i) *The diagram*

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ J & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the homomorphism induced by α*; the upper and lower horizontal arrows are the homomorphisms determined by* ρ_I *and* ρ_J*, respectively — commutes.*

(ii) ρ_I, ρ_J *are* **of NN-type** *(cf. Definition 2.4, (iii)).*

(iii) $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$*, and the isomorphism* α *is* **group-theoretically cuspidal** *(i.e., roughly speaking, preserves cuspidal inertia groups — cf.* [Mzk4]*, Definition 1.4, (iv)).*

Then the isomorphism α *is* **graphic** *(i.e., roughly speaking, is compatible with the respective semi-graph structures — cf.* [Mzk4]*, Definition 1.4, (i)).*

The notion of an outer representation of NN-type may be regarded as a natural outgrowth of the philosophy pursued in [Mzk4] of reducing (various aspects of) the *classical pro-*Σ **scheme-theoretic** *arithmetic geometry* of stable curves over a discrete valuation ring whose residue characteristic is not contained in Σ to a matter of **combinatorics**. Ideally, one would like to reduce the entire profinite classical scheme-theoretic arithmetic geometry of hyperbolic curves over number fields or p-adic local fields to a matter of combinatorics, but since this task appears to be too formidable at the time of writing, we concentrate on the pro-prime-to-p aspects of stable log curves over a log point. On the other hand, whereas the outer representations of IPSC-type studied in [Mzk4] literally arise from (log) scheme theory (i.e., a stable log curve over a log point), the outer representations of NN-type studied in the present paper are defined in *purely combinatorial terms*, without reference to any scheme-theoretic family of stable log curves. If one thinks of a stable log curve as a sort of "rational point" of the moduli stack of stable curves, then this point of view may be thought of as a sort of *abandonment* of the point of view implicit in the so-called *"Section Conjecture"*: that is to say, instead of concerning oneself with the issue of *precisely which* group-theoretic objects arise from a scheme-theoretic rational point (as is the case with the Section Conjecture),

> one takes the *definition of group-theoretic objects via purely* **combinatorial/group-theoretic** *conditions* — i.e., group-theoretic objects which do *not* necessarily arise from scheme theory — as the **starting point** of

> one's research, and one regards as the goal of one's research the study of the **intrinsic combinatorial geometry** of such group-theoretic objects (i.e., without regard to the issue of the extent to which these objects arise from scheme theory).

This point of view may be seen throughout the development of the theory of the present paper, as well as in the theory of [Mzk6].

On the other hand, from a more concrete point of view, the theory of the present paper was motivated by the goal of generalizing the *injectivity* portion of [Mzk7], Theorem A, (i), to *proper hyperbolic curves* in the case of the homomorphism induced by the projection from two-dimensional to one-dimensional configuration spaces (cf. Theorem B below). The main injectivity result that *was* proven in [Mzk7] (namely, [Mzk7], Corollary 2.3) was obtained by applying the *combinatorial anabelian result* given in [Mzk4], Corollary 2.7, (iii). On the other hand, this result of [Mzk4] is *insufficient* in the case of *proper* hyperbolic curves. To see why this is so, we begin by recalling that this result of [Mzk4] is applied in [Mzk7] (cf. the discussion of *"canonical splittings"* in the Introduction to [Mzk7]) to study the degenerations of families of hyperbolic curves that arise when

(a) a moving point on an affine hyperbolic curve collides with a **cusp**.

On the other hand, since proper hyperbolic curves have *no cusps*, in order to apply the techniques for proving injectivity — involving *"canonical splittings"* — developed in [Mzk7], it is necessary to consider the degenerations of families of hyperbolic curves that arise when

(b) a moving point on a (not necessarily affine) "degenerate hyperbolic curve" (i.e., a stable curve) collides with a **node**.

Since the local pro-Σ fundamental group in a neighborhood of a cusp or a node — i.e., the profinite group that corresponds to the "fundamental group of the base space of the degenerating family of hyperbolic curves under consideration" — is isomorphic (in both the cuspidal and nodal cases!) to the (same!) profinite group $\widehat{\mathbb{Z}}^{\Sigma}$, *one might at first glance think that the situation of (b) may also be analyzed via the results of* [Mzk7]. Put another way, both (a) and (b) involve a continuous action of a profinite group isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}$ on a semi-graph of anabelioids of pro-Σ PSC-type. On the other hand, closer inspection reveals that there is a **fundamental intrinsic difference** between the situations of (a) and (b). Indeed, in the situation of (a), we apply the reconstruction algorithms developed in [Mzk4], which depend in an essential way on a certain **positivity**, namely, the *positivity of the period matrix* — which implies, in particular, the *nondegeneracy* of this period matrix — of the

Jacobians of the various coverings of the degenerating family of curves under consideration (cf. the proof of [Mzk4], Proposition 2.6). By contrast, one verifies easily that

> the **symmetry** in a neighborhood of a node induced by **switching the two branches of the node** implies that an analogous "positivity of the period matrix" of the Jacobians of the various coverings of the degenerating family of curves under consideration can only hold in the sitation of (b) if this "positivity" satisfies the property of being *invariant with respect to multiplication by* -1 — which is absurd!

In particular, one concludes that the situation of (b) can *never* be "abstractly group-theoretically isomorphic" to the situation of (a). This was what led the second author to seek, in cooperation with the first author, a (partial) *generalization* (cf. Theorem A) of [Mzk4], Corollary 2.7, (iii), to the case of arbitrary *nodally nondegenerate outer representations* (which includes the situation of (b) — cf. Proposition 2.14, as it is applied in the proof of Corollary 5.3).

In passing, we note that the *sense* in which Theorem A is only a *partial* generalization (cf. Remark 4.2.1) of [Mzk4], Corollary 2.7, (iii), is interesting in light of the above discussion of *positivity*. Indeed, in the case of [Mzk4], Corollary 2.7, (iii), it is *not necessary* to assume that the semi-graph of anabelioids of pro-Σ PSC-type under consideration has any cusps. On the other hand, in the case of Theorem A, it is necessary to assume that the semi-graph of anabelioids of pro-Σ PSC-type under consideration has *at least one cusp* (cf. condition (iii) of Theorem A). That is to say, this state of affairs suggests that perhaps there is some sort of *"general principle"* underlying these results — which, at the time of writing, the authors have yet to succeed in making explicit — that requires the existence of **at least one cusp**, whether that cusp lie in the "*base* of the degenerating family of curves under consideration" (cf. (a); [Mzk4], Corollary 2.7, (iii)) or in the "*fibers* of this degenerating family" (cf. (b); condition (iii) of Theorem A).

The content of the various sections of the present paper may be summarized as follows. In §1, we review various "well-known" aspects of the *combinatorial group-theoretic geometry* of semi-graphs of anabelioids of pro-Σ PSC-type — i.e., without considering any continuous action of a profinite group on the semi-graph of anabelioids under consideration. In §2, we define and develop the basic theory surrounding *nodally nondegenerate outer representations*. In §3, we discuss various analogues of the *combinatorial group-theoretic geometry* reviewed in §1 in the case of nodally nondegenerate outer representations. In §4, we observe that the theory developed in §1, §2, and §3 is sufficient to prove the analogue discussed above (i.e., Theorem A) of the combinatorial anabelian result given in [Mzk4], Corollary 2.7, (iii), in the case of nodally nondegenerate outer representations. In §5, we apply this result (i.e., Theorem A) to generalize (cf. the above discussion) [Mzk7], Corollary 2.3, to the case of not necessarily affine curves (cf. Corollary 5.3). Finally, in §6, we discuss various consequences of the injectivity result proven in §5. The first of these is the following partial generalization (cf. Theorem 6.1) of [Mzk7], Theorem A.

Theorem B (**Partial profinite combinatorial cuspidalization).** *Let* Σ *be a set of prime numbers which is either* **of cardinality one** *or* **equal to the set of all prime numbers**, n *a positive integer,* X *a hyperbolic curve of type* (g, r) *over an algebraically closed field of characteristic* $\notin \Sigma$, X_n *the* n*-th* **configuration space** *of* X *(i.e., roughly speaking, the complement of the diagonals in the product of* n *copies of* X *— cf.* [MzTa], *Definition 2.1, (i)),* Π_n *the maximal pro-*Σ *quotient of the fundamental group of* X_n, *and* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *the subgroup of the group* $\mathrm{Out}(\Pi_n)$ *consisting of the outomorphisms (cf. the discussion entitled "Topological groups" in §0) of* Π_n *which are* **FC-admissible** *(i.e., roughly speaking, preserve fiber subgroups and cuspidal inertia groups — cf.* [Mzk7], *Definition 1.1, (ii)). Set* $n_0 \stackrel{\mathrm{def}}{=} 2$ *if* X *is* **affine**, *i.e.,* $r \geq 1$; $n_0 \stackrel{\mathrm{def}}{=} 3$ *if* X *is* **proper**, *i.e.,* $r = 0$ *(cf.* [Mzk7], *Theorem A). Then the natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

induced by the projection $X_{n+1} \to X_n$ *obtained by forgetting the* $(n+1)$*-st factor is* **injective** *if* $n \geq 1$ *and* **bijective** *if* $n \geq n_0 + 1$. *Moreover, the image of the natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

— where we write $\mathfrak{S}_n$ *for the symmetric group on* n *letters — obtained by permuting the various factors of the configuration space* X_n *is contained in the* **centralizer** $Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n))$.

In Corollary 6.6, we also give a *discrete analogue* of the profinite result constituted by Theorem B.

In passing, we observe that the *injectivity portion of the pro-*l *case* of Theorem B may be derived from the *Lie-theoretic version* of Theorem B that was obtained (in the mid-1990's!) by Naotake Takao (cf. [Tk], Corollary 2.7). In this context, we note that the point of view of [Tk] differs quite substantially from the point of view of the present paper and is motivated by the goal of completing the proof of a certain *conjecture of Takayuki Oda* concerning pro-l outer Galois actions associated to various moduli stacks of stable curves. Nevertheless, this point of view of [Tk] is interesting in light of the point of view discussed above to the effect that the content of [Mzk4] — and hence also of Theorem A above — may be thought of as a sort of substantial refinement of Oda's good reduction criterion.

Theorem B allows one to obtain the following generalization (cf. Corollaries 6.2; 6.3, (i)) to not necessarily affine hyperbolic curves of a well-known *injectivity result of Matsumoto* (cf. [Mts], Theorems 2.1, 2.2).

Theorem C (Kernels of outer representations arising from hyperbolic curves). *Let Σ be a set of prime numbers which is either* **of cardinality one** *or* **equal to the set of all prime numbers***, X a hyperbolic curve over a perfect field k such that every element of Σ is* **invertible** *in k, $\overline{k}$ an algebraic closure of k, n a positive integer, X_n the n-th configuration space of X, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, Δ_{X_n} the maximal pro-Σ quotient of the fundamental group of $X_n \otimes_k \overline{k}$, and $\Delta_{\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}}$ the maximal pro-Σ quotient of the fundamental group of $\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}$. Then the following hold:*

(i) *The kernel of the natural outer representation*

$$\rho^{\Sigma}_{X_n/k} \colon G_k \longrightarrow \mathrm{Out}(\Delta_{X_n})$$

is **independent** *of n and* **contained** *in the kernel of the natural outer representation*

$$\rho^{\Sigma}_{\mathbb{P}^1_k \setminus \{0,1,\infty\}/k} \colon G_k \longrightarrow \mathrm{Out}(\Delta_{\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}}) .$$

(ii) *Suppose that Σ is the set of* **all prime numbers***. (Thus, k is necessarily of characteristic zero.) Write $\overline{\mathbb{Q}}$ for the algebraic closure of $\mathbb{Q}$ determined by $\overline{k}$ and $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the kernel of the homomorphism $\rho^{\Sigma}_{X_n/k}$ is* **contained** *in the kernel of the outer homomorphism*

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

In particular, if k is a **number field** *or* **p-adic local field** *(cf. the discussion entitled "Numbers" in §0), and Σ is the set of* **all prime numbers***, then the outer representation*

$$\rho^{\Sigma}_{X/k} \colon G_k \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$$

determined by the natural exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1$$

is **injective***.*

Finally, we remark that in [Bg], a result that corresponds to a certain special case of Theorem C, (i), is asserted (cf. [Bg], Theorem 2.5). At the time of writing, the authors of the present paper were not able to follow the proof of this result given in [Bg]. Nevertheless, in a sequel to the present paper, we hope to discuss in more detail the relationship between the theory of the present paper and the interesting geometric ideas of [Bg] concerning the issue of *"canonical liftings" of cycles on a Riemann surface.*

0. Notations and Conventions

Sets: If S is a set, then we shall denote by 2^S the *power set* of S and by $S^\sharp$ the *cardinality* of S.

Numbers: The notation $\mathbb{N}$ will be used to denote the set or (additive) monoid of nonnegative rational integers. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. If p is a prime number, then the notation $\mathbb{Z}_p$ (respectively, $\mathbb{Q}_p$) will be used to denote the p-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$).

A finite extension field of $\mathbb{Q}$ will be referred to as a *number field.* If p is a prime number, then a finite extension field of $\mathbb{Q}_p$ will be referred to as a *p-adic local field.*

Monoids: We shall write M^{gp} for the *groupification* of a monoid M.

Topological groups: Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$; respectively, $C_G(H)$) the *centralizer* (respectively, *normalizer*; respectively, *commensurator*) of H in G, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{ g \in G \mid ghg^{-1} = h \text{ for any } h \in H \},$$

$$N_G(H) \stackrel{\mathrm{def}}{=} \{ g \in G \mid g \cdot H \cdot g^{-1} = H \},$$

$$C_G(H) \stackrel{\mathrm{def}}{=} \{ g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1} \};$$

we shall refer to $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ as the *center* of G. It is immediate from the definitions that

$$Z_G(H) \subseteq N_G(H) \subseteq C_G(H) \ ; \ \ H \subseteq N_G(H) \, .$$

We shall say that the subgroup H is *commensurably terminal* in G if $H = C_G(H)$.

We shall say that a profinite group G is *slim* if $Z_G(H) = \{1\}$ for any open subgroup H of G.

Let Σ be a set of prime numbers, l a prime number, and G a profinite group. Then we shall write G^Σ for the *maximal pro-Σ quotient* of G and $G^{(l)} \stackrel{\mathrm{def}}{=} G^{\{l\}}$.

We shall write G^{ab} for the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G.

If G is a profinite group, then we shall denote the group of automorphisms of G by $\mathrm{Aut}(G)$ and the group of inner automorphisms of G by $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. Conjugation by elements of G determines a surjection of groups $G \twoheadrightarrow \mathrm{Inn}(G)$. Thus, we have a homomorphism of groups $G \to \mathrm{Aut}(G)$ whose image is $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. We shall denote by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ and refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. In particular, if G is *center-free*, then the natural homomorphism $G \to \mathrm{Inn}(G)$ is an *isomorphism*; thus, we have an exact sequence of groups

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1\,.$$

If, moreover, G is *topologically finitely generated*, then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ with respect to which the above exact sequence determines an exact sequence of *profinite groups*. If $\rho\colon J \to \mathrm{Out}(G)$ is a continuous homomorphism, then we shall denote by

$$G \overset{\mathrm{out}}{\rtimes} J$$

the *profinite group* obtained by pulling back the above exact sequence of profinite groups via ρ. Thus, we have a *natural exact sequence* of profinite groups

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1\,.$$

One verifies easily (cf. [Hsh], Lemma 4.10) that if an automorphism α of $G \overset{\mathrm{out}}{\rtimes} J$ preserves the subgroup $G \subseteq G \overset{\mathrm{out}}{\rtimes} J$ and induces the *identity automorphisms* of the subquotients G and J, then the automorphism α is the *identity automorphism* of $G \overset{\mathrm{out}}{\rtimes} J$.

If M and N are topological modules, then we shall refer to a homomorphism of topological modules $\phi\colon M \to N$ as a *split injection* if there exists a homomorphism of topological modules $\psi\colon N \to M$ such that $\psi \circ \phi\colon M \to M$ is the identity automorphism of M.

Semi-graphs: Let Γ be a connected semi-graph. Then we shall say that Γ is *untangled* if every closed edge of Γ abuts to two *distinct* vertices.

Log stacks: Let $X^{\log}$ and $Y^{\log}$ be log stacks whose underlying (algebraic) stacks we denote by X and Y, respectively; $\mathcal{M}_X$ and $\mathcal{M}_Y$ the respective sheaves of monoids on X and Y defining the log structures of $X^{\log}$ and $Y^{\log}$; $f^{\log}\colon X^{\log} \to Y^{\log}$ a morphism of log stacks. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the *relative characteristic sheaf* of

$f^{\log}$; we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ (where, by abuse of notation, we write X for the log stack obtained by equipping X with the trivial log structure) induced by the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

Curves: We shall use the terms "*hyperbolic curve*", "*cusp*", "*stable log curve*", "*smooth log curve*", and "*tripod*" as they are defined in [Mzk4], §0; [Hsh], §0. If (g, r) is a pair of natural numbers such that $2g-2+r > 0$, then we shall denote by $\overline{\mathcal{M}}_{g,r}$ the *moduli stack of* r*-pointed stable curves* of genus g over $\mathbb{Z}$ whose r marked points are *equipped with an ordering*, $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ *parametrizing smooth curves*, and $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$.

Let n be a positive integer and $X^{\log}$ a stable log curve of type (g, r) over a log scheme $S^{\log}$. Then we shall refer to the log scheme obtained by pulling back the (1-)morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ given by forgetting the last n points via the classifying (1-)morphism $S^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ of $X^{\log}$ as the *n-th log configuration space* of $X^{\log}$.

1. Some complements concerning semi-graphs of anabelioids of PSC-type

In this section, we give some complements to the theory of *semi-graphs of anabelioids of PSC-type* developed in [Mzk4].

A basic reference for the theory of *semi-graphs of anabelioids of PSC-type* is [Mzk4]. We shall use the terms "*semi-graph of anabelioids of PSC-type*", "*PSC-fundamental group of a semi-graph of anabelioids of PSC-type*", "*finite étale covering of semi-graphs of anabelioids of PSC-type*", "*vertex*", "*edge*", "*cusp*", "*node*", "*verticial subgroup*", "*edge-like subgroup*", "*nodal subgroup*", "*cuspidal subgroup*", and "*sturdy*" as they are defined in [Mzk4], Definition 1.1. Also, we shall refer to the "PSC-fundamental group of a semi-graph of anabelioids of PSC-type" simply as the "fundamental group" (of the semi-graph of anabelioids of PSC-type). That is to say, we shall refer to the maximal pro-Σ quotient of the fundamental group of a semi-graph of anabelioids of PSC-type (as a semi-graph of anabelioids!) as the "fundamental group of the semi-graph of anabelioids of PSC-type". In this section, let Σ be a nonempty set of prime numbers, $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type, and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. (In particular, $\mathbb{G}$ is a *finite* semi-graph.) Also, let us fix a universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ with underlying projective system of semi-graphs $\widetilde{\mathbb{G}}$ (i.e., the projective system consisting of the underlying graphs $\mathbb{G}'$ of the connected finite étale subcoverings $\mathcal{G}'$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$) and denote by $\Pi_{\mathcal{G}}$ the (pro-Σ) fundamental group of $\mathcal{G}$.

Definition 1.1.

(i) We shall denote by $\mathrm{Vert}(\mathcal{G})$ (respectively, $\mathrm{Cusp}(\mathcal{G})$; $\mathrm{Node}(\mathcal{G})$) the set of the vertices (respectively, cusps; nodes) of $\mathcal{G}$.

(ii) We shall write

$$\mathrm{Vert}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{Vert}(\mathcal{G}') \ ;$$

$$\mathrm{Cusp}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{Cusp}(\mathcal{G}') \ ;$$

$$\mathrm{Node}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{Node}(\mathcal{G}')$$

— where the projective limits are over all connected finite étale subcoverings $\mathcal{G}' \to \mathcal{G}$ of the fixed universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$.

(iii) We shall write

$$\mathrm{VCN}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\mathcal{G}) \sqcup \mathrm{Cusp}(\mathcal{G}) \sqcup \mathrm{Node}(\mathcal{G}) \ ;$$

$$\mathrm{Edge}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G}) \sqcup \mathrm{Node}(\mathcal{G}) \ ;$$

$$\mathrm{VCN}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\widetilde{\mathcal{G}}) \sqcup \mathrm{Cusp}(\widetilde{\mathcal{G}}) \sqcup \mathrm{Node}(\widetilde{\mathcal{G}}) \ ;$$

$$\mathrm{Edge}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\widetilde{\mathcal{G}}) \sqcup \mathrm{Node}(\widetilde{\mathcal{G}}) \, .$$

(iv) Let

$$\mathcal{V}\colon \mathrm{Edge}(\mathcal{G}) \longrightarrow 2^{\mathrm{Vert}(\mathcal{G})}$$

$$(\text{respectively, } \mathcal{C}\colon \mathrm{Vert}(\mathcal{G}) \longrightarrow 2^{\mathrm{Cusp}(\mathcal{G})};$$

$$\mathcal{N}\colon \mathrm{Vert}(\mathcal{G}) \longrightarrow 2^{\mathrm{Node}(\mathcal{G})};$$

$$\mathcal{E}\colon \mathrm{Vert}(\mathcal{G}) \longrightarrow 2^{\mathrm{Edge}(\mathcal{G})})$$

be the map obtained by sending $e \in \mathrm{Edge}(\mathcal{G})$ (respectively, $v \in \mathrm{Vert}(\mathcal{G})$; $v \in \mathrm{Vert}(\mathcal{G})$; $v \in \mathrm{Vert}(\mathcal{G})$) to the set of vertices (respectively, cusps; nodes; edges) of $\mathcal{G}$ to which e abuts (respectively, which abut to v; which abut to v; which abut to v). Also, we shall write

$$\mathcal{V}\colon \mathrm{Edge}(\widetilde{\mathcal{G}}) \longrightarrow 2^{\mathrm{Vert}(\widetilde{\mathcal{G}})}$$

$$(\text{respectively, } \mathcal{C}\colon \mathrm{Vert}(\widetilde{\mathcal{G}}) \longrightarrow 2^{\mathrm{Cusp}(\widetilde{\mathcal{G}})};$$

$$\mathcal{N}\colon \mathrm{Vert}(\widetilde{\mathcal{G}}) \longrightarrow 2^{\mathrm{Node}(\widetilde{\mathcal{G}})};$$

$$\mathcal{E}\colon \mathrm{Vert}(\widetilde{\mathcal{G}}) \longrightarrow 2^{\mathrm{Edge}(\widetilde{\mathcal{G}})})$$

for the map induced by the various $\mathcal{V}$'s (respectively, $\mathcal{C}$'s; $\mathcal{N}$'s; $\mathcal{E}$'s) involved.

(v) Let $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Suppose that $\mathcal{G}' \to \mathcal{G}$ is a connected finite étale subcovering of $\widetilde{\mathcal{G}} \to \mathcal{G}$. Then we shall denote by $\widetilde{z}(\mathcal{G}') \in \mathrm{VCN}(\mathcal{G}')$ the image of $\widetilde{z}$ in $\mathrm{VCN}(\mathcal{G}')$.

(vi) Let $v \in \mathrm{Vert}(\mathcal{G})$, $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be such that $\widetilde{v}(\mathcal{G}) = v$. Then it is easily verified that there exists a *unique* verticial subgroup $\Pi_{\widetilde{v}}$ of $\Pi_{\mathcal{G}}$ associated to the vertex v such that for every connected finite étale subcovering $\mathcal{G}' \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$, it holds that the subgroup $\Pi_{\widetilde{v}} \cap \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}'}$ — where we write $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$ for the open subgroup corresponding to $\mathcal{G}' \to \mathcal{G}$ — is a verticial subgroup of $\Pi_{\mathcal{G}'}$ associated to $\widetilde{v}(\mathcal{G}') \in \mathrm{Vert}(\mathcal{G}')$; thus, the element $\widetilde{v}$ determines a *particular* verticial subgroup of $\Pi_{\mathcal{G}}$ associated to the vertex v. We shall refer to this verticial subgroup of $\Pi_{\mathcal{G}}$ determined by $\widetilde{v}$ as *the verticial subgroup of* $\Pi_{\mathcal{G}}$ *associated to* $\widetilde{v}$ and denote it by $\Pi_{\widetilde{v}}$.

In a similar vein, for $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ (respectively, $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$; $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$), by a similar argument to the argument just applied to define the verticial subgroup of $\Pi_{\mathcal{G}}$ associated to $\widetilde{v}$, the element $\widetilde{e}$ determines a *particular* cuspidal (respectively, nodal; edge-like) subgroup of $\Pi_{\mathcal{G}}$ associated to the cusp (respectively, node; edge) $\widetilde{e}(\mathcal{G})$ of $\mathcal{G}$. We shall refer to this cuspidal (respectively, nodal; edge-like) subgroup of $\Pi_{\mathcal{G}}$ as *the cuspidal* (respectively, *nodal*; *edge-like*) *subgroup of* $\Pi_{\mathcal{G}}$ *associated to* $\widetilde{e}$ and denote it by $\Pi_{\widetilde{e}}$.

(vii) Let n be a natural number, and v, $w \in \mathrm{Vert}(\mathcal{G})$. Then we shall write $\delta(v, w) \leq n$ if the following conditions are satisfied:

(1) If $n = 0$, then $v = w$.

(2) If $n \geq 1$, then there exist n nodes $e_1, \cdots, e_n \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$ and $n-1$ vertices $v_1, \cdots, v_{n-1} \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$ such that, for $1 \leq i \leq n$, it holds that $\mathcal{V}(e_i) = \{v_{i-1}, v_i\}$ — where we write $v_0 \stackrel{\mathrm{def}}{=} v$ and $v_n \stackrel{\mathrm{def}}{=} w$.

Moreover, we shall write $\delta(v, w) = n$ if $\delta(v, w) \leq n$ and $\delta(v, w) \not\leq n-1$. If $\delta(v, w) = n$, then we shall say that the *distance between* v *and* w is equal to n.

(viii) Let $\widetilde{v}$, $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then we shall write

$$\delta(\widetilde{v}, \widetilde{w}) \stackrel{\mathrm{def}}{=} \sup_{\mathcal{G}'} \{\delta(\widetilde{v}(\mathcal{G}'), \widetilde{w}(\mathcal{G}'))\} \in \mathbb{N} \cup \{\infty\}$$

— where $\mathcal{G}'$ ranges over the connected finite étale subcoverings $\mathcal{G}' \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$. If $\delta(\widetilde{v}, \widetilde{w}) = n \in \mathbb{N} \cup \{\infty\}$, then we shall say that the *distance between* $\widetilde{v}$ *and* $\widetilde{w}$ is equal to n.

Remark 1.1.1. Let $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $z \stackrel{\mathrm{def}}{=} \widetilde{z}(\mathcal{G}) \in \mathrm{VCN}(\mathcal{G})$. Then whereas $\widetilde{z}$ *completely determines* the subgroup $\Pi_{\widetilde{z}}$, z *only determines* the $\Pi_{\mathcal{G}}$-conjugacy class of the subgroup $\Pi_{\widetilde{z}}$.

Definition 1.2. We shall say that the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}$ is *untangled* if the underlying semi-graph of $\mathcal{G}$ is untangled (cf. the discussion entitled "*Semi-graphs*" in §0).

Remark 1.2.1.

(i) It follows from a similar argument to the argument in the discussion entitled "*Curves*" in [Mzk6], §0, that there exists a connected finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}$ such that $\mathcal{G}'$ is *untangled*.

(ii) It is easily verified that if $\mathcal{G}$ is *untangled*, then every finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}$ is *untangled*.

(iii) It follows from (i) and (ii) that for every $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, we have $\mathcal{V}(\widetilde{e})^{\sharp} = 2$.

Definition 1.3.

(i) We shall denote by $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ the quotient of $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ by the closed subgroup generated by the images in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ of the edge-like subgroups of $\Pi_{\mathcal{G}}$.

(ii) Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then we shall denote by $\Pi_{\widetilde{v}}^{\mathrm{ab/edge}}$ the quotient of the abelianization $\Pi_{\widetilde{v}}^{\mathrm{ab}}$ by the closed subgroup generated by the images in $\Pi_{\widetilde{v}}^{\mathrm{ab}}$ of $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$ — where $\widetilde{e}$ ranges over elements of $\mathcal{E}(\widetilde{v})$. (Here, we note that it follows from [Mzk4], Proposition 1.5, (i), that for $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$, it holds that $\widetilde{e} \in \mathcal{E}(\widetilde{v})$ if and only if $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$.)

(iii) Let $v \in \mathrm{Vert}(\mathcal{G})$. Then observe that conjugation by elements of $\Pi_{\mathcal{G}}$ determines natural isomorphisms between the various $\Pi_{\widetilde{v}}^{\mathrm{ab/edge}}$, as $\widetilde{v}$ ranges over the elements of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $v = \widetilde{v}(\mathcal{G})$. We shall denote the resulting profinite group by $\Pi_{v}^{\mathrm{ab/edge}}$.

Lemma 1.4 (**Verticial decompositions inside ab/edge-quotients**). *The natural homomorphism*

$$\bigoplus_{v \in \mathrm{Vert}(\mathcal{G})} \Pi_{v}^{\mathrm{ab/edge}} \longrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$$

is a **split injection** *(cf. the discussion entitled "Topological groups" in §0) whose image is a free* $\widehat{\mathbb{Z}}^{\Sigma}$*-module of finite rank (cf.* [Mzk4]*, Remark 1.1.4).*

Proof. It follows immediately from the well-known structure of the maximal pro-Σ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\notin \Sigma$ that the quotient by the image of the natural homomorphism in question is a *free* $\widehat{\mathbb{Z}}^{\Sigma}$-*module*. Therefore, to verify Lemma 1.4, it suffices to verify that the natural homomorphism in question is *injective*. Now suppose that we have been given, for each $v \in \mathrm{Vert}(\mathcal{G})$, a connected finite étale covering $\mathcal{H}_v \to \mathcal{G}_v$ of the anabelioid $\mathcal{G}_v$ corresponding to $v \in \mathrm{Vert}(\mathcal{G})$ which arises from an open subgroup of $\Pi_v^{\mathrm{ab/edge}}$. Then to verify the desired *injectivity*, it suffices to verify that there exists a connected finite étale covering $\mathcal{F} \to \mathcal{G}$ of $\mathcal{G}$ which arises from an open subgroup of $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ such that, for each $v \in \mathrm{Vert}(\mathcal{G})$, any connected component of the restriction of $\mathcal{F} \to \mathcal{G}$ to $\mathcal{G}_v$ is isomorphic to $\mathcal{H}_v$ over $\mathcal{G}_v$. To this end, for $v \in \mathrm{Vert}(\mathcal{G})$, write $(\Pi_v^{\mathrm{ab/edge}} \twoheadrightarrow)\ A_v$ for the Galois group of the connected finite étale covering $\mathcal{H}_v \to \mathcal{G}_v$,

$$A_{\neq v} \stackrel{\mathrm{def}}{=} \prod_{w \in \mathrm{Vert}(\mathcal{G}) \setminus \{v\}} A_w \subseteq A \stackrel{\mathrm{def}}{=} \prod_{w \in \mathrm{Vert}(\mathcal{G})} A_w \,,$$

$\mathcal{F}_v \to \mathcal{G}_v$ for the (not necessarily connected) finite étale covering of $\mathcal{G}_v$ obtained as the disjoint union of copies of $\mathcal{H}_v$ indexed by the elements of $A_{\neq v}$, and, for $e \in \mathcal{E}(v)$, $\mathcal{F}_v|_{\mathcal{G}_e} \to \mathcal{G}_e$ for the finite étale covering of $\mathcal{G}_e$ obtained as the restriction of $\mathcal{F}_v \to \mathcal{G}_v$ to the anabelioid $\mathcal{G}_e$ corresponding to $e \in \mathcal{N}(v)$. Then the natural action of $A_{\neq v}$ on $A_{\neq v}$ and the tautological action of A_v on $\mathcal{H}_v$ over $\mathcal{G}_v$ naturally determine an action of A on $\mathcal{F}_v$ over $\mathcal{G}_v$. Moreover, one verifies immediately that this A-action determines a structure of A-*torsor* on the covering $\mathcal{F}_v \to \mathcal{G}_v$. Therefore, by gluing the various $\mathcal{F}_v$ (for $v \in \mathrm{vert}(\mathcal{G})$) by A-*equivariant* isomorphisms between the various $\mathcal{F}_v|_{\mathcal{G}_e}$ (for $e \in \mathrm{Node}(\mathcal{G})$), we obtain a finite étale covering $\mathcal{F} \to \mathcal{G}$, any connected component of which satisfies the desired condition. This completes the proof of the *injectivity* of the homomorphism in question. □

Remark 1.4.1. The following two assertions follow immediately from Lemma 1.4.

(i) If $\Pi_v \subseteq \Pi_{\mathcal{G}}$ is a verticial subgroup of $\Pi_{\mathcal{G}}$, then the natural homomorphism $\Pi_v^{\mathrm{ab/edge}} \to \Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is *injective*.

(ii) If $v_1, v_2 \in \mathrm{Vert}(\mathcal{G})$ are *distinct*, then for any verticial subgroups $\Pi_{v_1}, \Pi_{v_2} \subseteq \Pi_{\mathcal{G}}$ associated to v_1, v_2, the intersection of the images of Π_{v_1} and Π_{v_2} in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is *trivial*.

Lemma 1.5 (Intersections of edge-like subgroups). *Let* $\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. *Then the following conditions are* **equivalent***:*

(i) $\widetilde{e}_1 = \widetilde{e}_2$.

(ii) $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$.

In particular, if $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$*, then* $\Pi_{\widetilde{e}_1} = \Pi_{\widetilde{e}_2}$.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; thus, to verify Lemma 1.5, it suffices to prove the implication

$$\text{(ii)} \Longrightarrow \text{(i)}\,.$$

To this end, let us assume that $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$. Since $\Pi_{\mathcal{G}}$ is *torsion-free* (cf. [Mzk4], Remark 1.1.3), by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved, we may assume without loss of generality that $\Sigma = \{l\}$. In particular, since $\Pi_{\widetilde{e}_1}$ and $\Pi_{\widetilde{e}_2}$ are isomorphic to $\mathbb{Z}_l$, we may assume without loss of generality that $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2}$ is *open* in $\Pi_{\widetilde{e}_1}$ and $\Pi_{\widetilde{e}_2}$. Thus, by replacing $\mathcal{G}$ by a connected finite étale covering of $\mathcal{G}$, we may assume without loss of generality that $\Pi_{\widetilde{e}_1} = \Pi_{\widetilde{e}_2}$. Then condition (i) follows from [Mzk4], Proposition 1.2, (i). ☐

Lemma 1.6 (**Group-theoretic characterization of subgroups of edge-like subgroups**). *Let* $J \subseteq \Pi_{\mathcal{G}}$ *be a* **nontrivial procyclic** *closed subgroup of* $\Pi_{\mathcal{G}}$*. Then the following conditions are* **equivalent***:*

(i) *J is contained in a(n) — necessarily* **unique** *(cf. Lemma 1.5) —* **edge-like subgroup**.

(ii) *There exists a connected finite étale covering $\mathcal{G}^\dagger \to \mathcal{G}$ of $\mathcal{G}$ such that for any connected finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}$ that factors through $\mathcal{G}^\dagger \to \mathcal{G}$, the image of the composite*

$$J \cap \Pi_{\mathcal{G}'} \hookrightarrow \Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{\mathcal{G}'}^{\text{ab/edge}}$$

is **trivial**.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; thus, to verify Lemma 1.6, it suffices to prove the implication

$$\text{(ii)} \Longrightarrow \text{(i)}\,.$$

To this end, let us assume that condition (ii) holds. Now since edge-like subgroups are *commensurably terminal* (cf. [Mzk4], Proposition 1.2, (ii)), it suffices to verify condition (i) under the further assumption that $\mathcal{G}^\dagger = \mathcal{G}$ (cf. the *uniqueness* portion of condition (i)). Moreover, since $\Pi_{\mathcal{G}}$ is *torsion-free* (cf. [Mzk4], Remark 1.1.3), to verify condition (i), we may assume without loss of generality (cf. the *uniqueness* portion of condition (i)) — by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved — that $\Sigma = \{l\}$.

If $H \subseteq \Pi_{\mathcal{G}}$ is an open subgroup of $\Pi_{\mathcal{G}}$, then let us denote by $\mathcal{G}_H \to \mathcal{G}$ the connected finite étale covering of $\mathcal{G}$ corresponding to the open subgroup $H \subseteq \Pi_{\mathcal{G}}$ (i.e., $\Pi_{\mathcal{G}_H} = H \subseteq \Pi_{\mathcal{G}}$). Now we *claim* that

> $(*)$ for any normal open subgroup $N \subseteq \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$, there exists an edge of $\mathcal{G}_{J \cdot N}$ at which the connected finite étale covering $\mathcal{G}_N \to \mathcal{G}_{J \cdot N}$ is *totally ramified*, i.e., there exists an edge $e \in \mathrm{Edge}(\mathcal{G}_{J \cdot N})$ such that the composite of natural homomorphisms
>
> $$\Pi_e \hookrightarrow \Pi_{\mathcal{G}_{J \cdot N}} = J \cdot N \twoheadrightarrow (J \cdot N)/N$$
>
> is *surjective*.

Indeed, since J is *procyclic*, it follows that $(J \cdot N)/N$ is *cyclic*; in particular, we obtain a natural surjection $\Pi^{\mathrm{ab}}_{\mathcal{G}_{J \cdot N}} \twoheadrightarrow (J \cdot N)/N$. Moreover, since $(J \cdot N)/N$ is generated by the image of J, it follows from condition (ii) that the composite of natural homomorphisms

$$\bigoplus_{e' \in \mathrm{Edge}(\mathcal{G}_{J \cdot N})} \Pi_{e'} \to \Pi^{\mathrm{ab}}_{\mathcal{G}_{J \cdot N}} \twoheadrightarrow (J \cdot N)/N$$

is *surjective*. Therefore, it follows from the fact that $(J \cdot N)/N$ is a *cyclic l-group* that there exists an edge e of $\mathcal{G}_{J \cdot N}$ such that the composite of the natural homomorphisms $\Pi_e \hookrightarrow \Pi_{\mathcal{G}_{J \cdot N}} = J \cdot N \twoheadrightarrow (J \cdot N)/N$ is surjective, as desired. This completes the proof of $(*)$.

If $N \subseteq \Pi_{\mathcal{G}}$ is a normal open subgroup, then let us denote by $E_N \subseteq \mathrm{Edge}(\mathcal{G}_N)$ the subset of $\mathrm{Edge}(\mathcal{G}_N)$ consisting of edges which are *fixed* by the natural action of J on $\mathcal{G}_N$. Then it follows from $(*)$ that for any normal open subgroup $N \subseteq \Pi_{\mathcal{G}}$, it holds that E_N is *nonempty*; thus, since E_N is *finite*, the projective limit $\varprojlim_N E_N$ — where N ranges over the normal open subgroups of $\Pi_{\mathcal{G}}$ — is *nonempty*. Note that since $\bigcap_N N$ — where N ranges over the normal open subgroups of $\Pi_{\mathcal{G}}$ — is $\{1\}$, it follows that each element of the projective limit $\varprojlim_N E_N$ naturally determines an element of $\mathrm{Edge}(\widetilde{\mathcal{G}})$. Let $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ be an element of $\mathrm{Edge}(\widetilde{\mathcal{G}})$ determined by an element of $\varprojlim_N E_N \neq \emptyset$. Then it follows from the various definitions involved that $J \subseteq \Pi_{\widetilde{e}}$. This completes the proof of the implication

$$\text{(ii)} \Longrightarrow \text{(i)}.$$

□

Remark 1.6.1. When $\mathcal{G}^{\dagger} = \mathcal{G}$ and $\mathrm{Node}(\mathcal{G}) = \emptyset$, Lemma 1.6 follows immediately from [Naka], Lemma 2.1.4.

Lemma 1.7 (**Intersections of verticial and edge-like subgroups**). *Let* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*, and* $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$*. Then the following conditions are* **equivalent***:*

(i) $\widetilde{e} \in \mathcal{E}(\widetilde{v})$.

(ii) $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{e}} \neq \{1\}$.

In particular, if $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{e}} \neq \{1\}$*, then* $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$*.*

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; thus, to verify Lemma 1.7, it suffices to prove the implication

$$\text{(ii)} \Longrightarrow \text{(i)}\,.$$

To this end, let us assume that $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{e}} \neq \{1\}$. Since $\Pi_{\mathcal{G}}$ is *torsion-free* (cf. [Mzk4], Remark 1.1.3), by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved, we may assume without loss of generality that $\Sigma = \{l\}$. In particular, since $\Pi_{\widetilde{e}}$ is isomorphic to $\mathbb{Z}_l$, we may assume without loss of generality that $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{e}}$ is *open* in $\Pi_{\widetilde{e}}$. Thus, by replacing $\mathcal{G}$ by a connected finite étale covering of $\mathcal{G}$, we may assume without loss of generality that $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$. Then condition (i) follows from [Mzk4], Proposition 1.5, (i) (cf. also [Mzk4], Proposition 1.2, (i)). ☐

Lemma 1.8 (**Nonexistence of loops**). *Let* $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *be such that* $\widetilde{v}_1 \neq \widetilde{v}_2$*. Then*

$$(\mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2))^\sharp \leq 1\,;$$
$$\{\, \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \mid \delta(\widetilde{v}, \widetilde{v}_1) = \delta(\widetilde{v}, \widetilde{v}_2) = 1 \,\}^\sharp \leq 1\,.$$

Proof. If the cardinality of either of the sets equipped with a superscript "$\sharp$" is ≥ 2, then the *offending* edges or vertices give rise to a *loop* of $\widetilde{\mathbb{G}}$, i.e., a projective system of *loops* (that map isomorphically to one another) in the various semi-graphs that appear in the projective system $\widetilde{\mathbb{G}}$. On the other hand, since $\widetilde{\mathcal{G}}$ is a *universal covering* of $\mathcal{G}$, one verifies immediately that no such projective system of loops exists. Thus, we obtain a contradiction. This completes the proof of Lemma 1.8. ☐

Remark 1.8.1.

(i) Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Recall that if $\widetilde{e} \in \mathcal{N}(\widetilde{v})$, then the inclusion $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$ is *strict* (i.e., $\Pi_{\widetilde{e}} \neq \Pi_{\widetilde{v}}$). In particular, it follows immediately that either $\mathcal{N}(\widetilde{v}) = \emptyset$ or $\mathcal{N}(\widetilde{v})^\sharp \geq 2$.

(ii) Let $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then, in light of (i), it follows immediately from Lemma 1.8 that the following assertion holds:

$$\widetilde{v}_1 = \widetilde{v}_2 \text{ if and only if } \mathcal{N}(\widetilde{v}_1) = \mathcal{N}(\widetilde{v}_2).$$

(iii) Let $\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Node}(\widetilde{\mathcal{G}})$. Then it follows immediately from Lemma 1.8 that the following assertion holds:

$$\text{If } \widetilde{e}_1 \neq \widetilde{e}_2, \text{ then } (\mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2))^\sharp \leq 1.$$

In particular, it follows from Remark 1.2.1, (iii), that the following assertion holds:

$$\widetilde{e}_1 = \widetilde{e}_2 \text{ if and only if } \mathcal{V}(\widetilde{e}_1) = \mathcal{V}(\widetilde{e}_2).$$

Lemma 1.9 (**Graph-theoretic geometry via verticial subgroups**). *For $i = 1, 2$, let $\widetilde{v}_i \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then the following hold:*

(i) *If $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$, then either $\Pi_{\widetilde{v}_1} = \Pi_{\widetilde{v}_2}$ or $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2}$ is a* **nodal subgroup** *of $\Pi_{\mathcal{G}}$.*

(ii) *Consider the following three (mutually exclusive) conditions:*

(1) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 0$.

(2) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 1$.

(3) $\delta(\widetilde{v}_1, \widetilde{v}_2) \geq 2$.

Then we have **equivalences**

$$(1) \Longleftrightarrow (1') \ ; \ (2) \Longleftrightarrow (2') \Longleftrightarrow (2'') \ ; \ (3) \Longleftrightarrow (3')$$

with the following four conditions:

(1′) $\Pi_{\widetilde{v}_1} = \Pi_{\widetilde{v}_2}$.

(2′) $\Pi_{\widetilde{v}_1} \neq \Pi_{\widetilde{v}_2}$; $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$.

(2″) $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2}$ *is a nodal subgroup of* $\Pi_{\mathcal{G}}$.

(3′) $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} = \{1\}$.

Proof. First, we consider assertion (i). Suppose that $H \stackrel{\mathrm{def}}{=} \Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$, and $\Pi_{\widetilde{v}_1} \neq \Pi_{\widetilde{v}_2}$ (so $\widetilde{v}_1 \neq \widetilde{v}_2$ — cf. [Mzk4], Proposition 1.2, (i)). Note that to verify assertion (i), it suffices to show that H is a *nodal subgroup* of $\Pi_{\mathcal{G}}$. Also, we observe that since nodal and verticial subgroups of $\Pi_{\mathcal{G}}$ are *commensurably terminal* in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)), it follows that we may assume without loss of generality — by replacing $\mathcal{G}$ by a connected finite étale covering of $\mathcal{G}$ — that $\widetilde{v}_1(\mathcal{G}) \neq \widetilde{v}_2(\mathcal{G})$.

Let $J \subseteq H$ be a nontrivial *procyclic* closed subgroup of H. Then we *claim* that J is contained in an *edge-like* subgroup of $\Pi_{\mathcal{G}}$. Indeed, since $J \subseteq H = \Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2}$ — where $\widetilde{v}_1 \neq \widetilde{v}_2$ — it follows from Remark 1.4.1, (ii), together with our assumption that $\widetilde{v}_1(\mathcal{G}) \neq \widetilde{v}_2(\mathcal{G})$, that the image of J in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is *trivial*. Thus, by applying this observation to the various connected finite étale coverings of $\mathcal{G}$ involved, we conclude that

J satisfies condition (ii) in the statement of Lemma 1.6. In particular, it follows from Lemma 1.6 that J is contained in an edge-like subgroup. This completes the proof of the above *claim*. On the other hand, if $\Pi_{\widetilde{e}}$ is an edge-like subgroup of $\Pi_{\mathcal{G}}$ such that $J \subseteq \Pi_{\widetilde{e}}$, then it follows from Lemma 1.7 that the inclusion $J \subseteq \Pi_{\widetilde{e}}$ implies that $\Pi_{\widetilde{e}}$ is in fact *nodal*, and, moreover, that $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} = H$.

By the above discussion, it follows that

$$H = \bigcup_{\widetilde{e} \in \mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2)} \Pi_{\widetilde{e}} .$$

On the other hand, it follows from Lemma 1.8 that the cardinality of the intersection $\mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2)$ is ≤ 1. Therefore, it follows that H is a nodal subgroup of $\Pi_{\mathcal{G}}$. This completes the proof of assertion (i).

Next, we consider assertion (ii). The equivalence

$$(1) \iff (1')$$

follows from [Mzk4], Proposition 1.2, (i). In light of this equivalence, the implications

$$(2) \Longrightarrow (2') \Longrightarrow (2'')$$

follow from assertion (i), while the implication

$$(2'') \Longrightarrow (2)$$

follows from Lemma 1.7. The equivalence

$$(3) \iff (3')$$

then follows from the equivalences

$$(1) \iff (1') \ ; \ \ (2) \iff (2') .$$

□

Remark 1.9.1. It follows immediately from the various definitions involved that for any semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}$, there exists, in the terminology of [Mzk6], Definition 1.2, (ii), an IPSC-extension

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_I \longrightarrow I \longrightarrow 1 .$$

Therefore, Lemma 1.9 may also be obtained as a consequence of [Mzk6], Proposition 1.3, (iv).

Lemma 1.10 (Conjugates of verticial subgroups). *Suppose that* $\mathcal{G}$ *is* **untangled**. *Let* $\widetilde{v}$, $\widetilde{v}' \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *be such that* $\widetilde{v}(\mathcal{G}) = \widetilde{v}'(\mathcal{G})$. *Then* $\widetilde{v} \neq \widetilde{v}'$ *if and only if* $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{v}'} = \{1\}$.

Proof. The *sufficiency* of the condition is immediate; thus, to prove Lemma 1.10, it suffices to verify the *necessity* of the condition. To this end, let us assume that $\widetilde{v} \neq \widetilde{v}'$. Then there exists a connected finite étale subcovering $\mathcal{G}' \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ such that $\widetilde{v}(\mathcal{G}') \neq \widetilde{v}'(\mathcal{G}')$. On the other hand, since $\mathcal{G}$ is *untangled*, and $\widetilde{v}(\mathcal{G}) = \widetilde{v}'(\mathcal{G})$, it follows that $\mathcal{N}(\widetilde{v}(\mathcal{G}')) \cap \mathcal{N}(\widetilde{v}'(\mathcal{G}')) = \emptyset$. Thus, $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{v}'} \cap \Pi_{\mathcal{G}'} = \{1\}$ by Lemma 1.9, (ii); in particular, since $\Pi_{\widetilde{v}}$ is *torsion-free* (cf. [Mzk4], Remark 1.1.3), we obtain that $\Pi_{\widetilde{v}} \cap \Pi_{\widetilde{v}'} = \{1\}$, as desired. □

Remark 1.10.1. It follows immediately from Lemma 1.10 that the following assertion holds:

> Suppose that $\mathcal{G}$ is *untangled*. Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex of $\mathcal{G}$, $\Pi_v \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup associated to v, and $\gamma \in \Pi_{\mathcal{G}} \setminus \Pi_v$. Then $\Pi_v \cap \gamma \cdot \Pi_v \cdot \gamma^{-1} = \{1\}$.

Definition 1.11. Suppose that $\mathcal{G}$ is *sturdy*. Then by eliminating the cusps (i.e., the open edges) of the semi-graph $\mathbb{G}$, and, for each vertex v of $\mathcal{G}$, replacing the anabelioid $\mathcal{G}_v$ corresponding to v by the anabelioid $\overline{\mathcal{G}}_v$ of finite étale coverings of $\mathcal{G}_v$ that restrict to a trivial covering over the cusps of $\mathcal{G}$ that abut to v, we obtain a semi-graph of anabelioids of pro-Σ *PSC-type* $\overline{\mathcal{G}}$. (Thus, the pro-Σ fundamental group of $\overline{\mathcal{G}}_v$ may be naturally identified, up to inner automorphism, with the quotient of Π_v by the subgroup of Π_v topologically normally generated by the $\Pi_e \subseteq \Pi_v$, for $e \in \mathcal{C}(v)$.) We shall refer to $\overline{\mathcal{G}}$ as the *compactification* of $\mathcal{G}$ (cf. [Mzk4], Remark 1.1.6).

Remark 1.11.1. It follows immediately from the definition of the compactification that the quotient of $\Pi_{\mathcal{G}}$ by the closed subgroup of $\Pi_{\mathcal{G}}$ topologically normally generated by the *cuspidal* subgroups of $\Pi_{\mathcal{G}}$ is naturally isomorphic, up to inner automorphism, to the fundamental group $\Pi_{\overline{\mathcal{G}}}$ of $\overline{\mathcal{G}}$. In particular, we have a natural outer surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\overline{\mathcal{G}}}$.

By analogy to the terms *"group-theoretically verticial"* and *"group-theoretically cuspidal"* introduced in [Mzk4] (cf. [Mzk4], Definition 1.4, (iv)), we make the following definition.

Definition 1.12. Let $\mathcal{H}$ be a semi-graph of anabelioids of pro-Σ PSC-type, $\Pi_{\mathcal{H}}$ the (pro-Σ) fundamental group of $\mathcal{H}$, and $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ an isomorphism of profinite groups. Then we shall say that α is *group-theoretically nodal* if, for any $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, the image $\alpha(\Pi_{\widetilde{e}}) \subseteq \Pi_{\mathcal{H}}$ is a nodal subgroup of $\Pi_{\mathcal{H}}$, and, moreover, every nodal subgroup of $\Pi_{\mathcal{H}}$ arises in this fashion.

Proposition 1.13 (Group-theoretical verticiality and nodality). *Let $\mathcal{H}$ be a semi-graph of anabelioids of pro-Σ PSC-type, $\Pi_{\mathcal{H}}$ the fundamental group of $\mathcal{H}$, and $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ a* **group-theoretically verticial** *isomorphism. Then α is* **group-theoretically nodal**.

Proof. This follows immediately from Lemma 1.9, (i). □

Lemma 1.14 (**Graphicity of certain group-theoretically cuspidal and verticial isomorphisms).** *Let $\mathcal{H}$ be a semi-graph of anabelioids of pro-Σ PSC-type, and $\Pi_{\mathcal{H}}$ the fundamental group of $\mathcal{H}$. If an isomorphism $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ satisfies the following two conditions, then α is* **graphic** *(cf.* [Mzk4]*, Definition 1.4, (i)):*

(i) *α is* **group-theoretically cuspidal***.*

(ii) *For any* **sturdy** *connected finite étale covering $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}$ such that the corresponding covering $\mathcal{H}' \to \mathcal{H}$ of $\mathcal{H}$ (relative to the isomorphism α) is* **sturdy***, the induced isomorphism (cf. (i), Remark 1.11.1)*

$$\Pi_{\overline{\mathcal{G}}'} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}'}$$

— where we write $\overline{\mathcal{G}}'$ (respectively, $\overline{\mathcal{H}}'$) for the semi-graph of anabelioids of PSC-type obtained as the **compactification** *(cf. Definition 1.11) of $\mathcal{G}'$ (respectively, $\mathcal{H}'$) — is* **group-theoretically verticial***.*

Proof. Since the isomorphism $\Pi_{\overline{\mathcal{G}}'} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}'}$ is *group-theoretically verticial* (cf. condition (ii)), it follows from Proposition 1.13 that the isomorphism $\Pi_{\overline{\mathcal{G}}'} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}'}$ is *group-theoretically nodal.* Therefore, it follows immediately from (i) that α is *graphically filtration-preserving* (cf. [Mzk4], Definition 1.4, (iii)). Thus, it follows from [Mzk4], Theorem 1.6, (ii), that α is *graphic*, as desired. □

Lemma 1.15 (**Chains of length two lifting adjacent vertices).** *Let $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be such that if we write $v_i \stackrel{\mathrm{def}}{=} \widetilde{v}_i(\mathcal{G})$, then $\delta(v_1, v_2) = 1$. Then there exist $\widetilde{w}_1$, $\widetilde{u}_1$, $\widetilde{w}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ which satisfy the following conditions (which imply that $\delta(\widetilde{w}_1, \widetilde{u}_1) = 2$):*

(i) *$v_1 = \widetilde{w}_1(\mathcal{G}) = \widetilde{u}_1(\mathcal{G})$; $v_2 = \widetilde{w}_2(\mathcal{G})$.*

(ii) *$\delta(\widetilde{w}_1, \widetilde{u}_1) \geq 2$.*

(iii) *$\delta(\widetilde{w}_2, \widetilde{w}_1) = \delta(\widetilde{w}_2, \widetilde{u}_1) = 1$.*

Proof. First, we observe that by replacing $\mathcal{G}$ by a connected finite étale covering of $\mathcal{G}$, we may assume without loss of generality that $\mathcal{G}$ is *sturdy* (cf. [Mzk4], Remark 1.1.5) and *untangled* (cf. Remark 1.2.1, (i)). Then it is easily verified that there exists a *nontrivial* connected finite étale covering of the anabelioid $\mathcal{G}_{v_2}$ corresponding to v_2 which is *unramified* over the nodes and cusps of $\mathcal{G}$ which abut to v_2. In light of the unramified nature of this connected finite étale covering of $\mathcal{G}_{v_2}$, by gluing this covering to a *split* covering over the remaining portion of $\mathcal{G}$, we obtain a connected finite étale covering $\mathcal{H} \to \mathcal{G}$. Then it follows immediately from the various definitions involved that the set $\mathcal{V}_1$ (respectively, $\mathcal{V}_2$) of vertices of $\mathcal{H}$ which lie over v_1 (respectively, v_2) is of cardinality ≥ 2 (respectively, of cardinality 1). Thus, there exist vertices w_1, $u_1 \in \mathcal{V}_1$, $w_2 \in \mathcal{V}_2$ such that $w_1 \neq u_1$ (which, since $\mathcal{G}$ is *untangled*, implies that $\delta(w_1, u_1) \geq 2$ — cf. condition (ii)), and, moreover, $\delta(w_2, w_1) = \delta(w_2, u_1) = 1$ (cf. condition (iii)). In particular, it follows immediately that there exist elements $\widetilde{w}_1$, $\widetilde{u}_1$, $\widetilde{w}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ which satisfy the three conditions in the statement of Lemma 1.15. This completes the proof of Lemma 1.15. □

2. Nodally nondegenerate outer representations

In this section, we define the notion of an *outer representation of NN-type* and verify various fundamental properties of such outer representations.

If $\mathcal{G}$ is a semi-graph of anabelioids of pro-Σ PSC-type for some nonempty set of prime numbers Σ, then since the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ is *topologically finitely generated*, the profinite topology of $\Pi_{\mathcal{G}}$ induces (profinite) topologies on $\mathrm{Aut}(\Pi_{\mathcal{G}})$ and $\mathrm{Out}(\Pi_{\mathcal{G}})$ (cf. the discussion entitled "*Topological groups*" in §0). Moreover, if we write

$$\mathrm{Aut}(\mathcal{G})$$

for the group of automorphisms of $\mathcal{G}$, then by the discussion preceding [Mzk4], Lemma 2.1, the natural homomorphism

$$\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$$

is an *injection with closed image*. (Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph — cf. [Mzk3], Definition 2.1; [Mzk3], Remark 2.4.2.) Thus, by equipping $\mathrm{Aut}(\mathcal{G})$ with the topology induced via this homomorphism by the topology of $\mathrm{Out}(\Pi_{\mathcal{G}})$, we may regard $\mathrm{Aut}(\mathcal{G})$ as being equipped with the structure of a *profinite group*.

Definition 2.1.

(i) Let I be a profinite group, Σ a nonempty set of prime numbers, $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type, $\Pi_{\mathcal{G}}$ the fundamental group of $\mathcal{G}$, and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ a homomorphism of profinite groups. Then we shall refer to the pair

$$\Big(\mathcal{G}, \rho\colon I \to \mathrm{Aut}(\mathcal{G})\ \ (\hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}))\Big)$$

as an *outer representation of pro-Σ PSC-type*. Moreover, we shall refer to an outer representation of pro-Σ PSC-type for some nonempty set of prime numbers Σ as an *outer representation of PSC-type*. For simplicity, we shall also refer to the underlying homomorphism "ρ" of an outer representation of pro-Σ PSC-type (respectively, of PSC-type) as an *outer representation of pro-Σ PSC-type* (respectively, *outer representation of PSC-type*).

(ii) Let $(\mathcal{G}, \rho_I \colon I \to \mathrm{Aut}(\mathcal{G}))$, $(\mathcal{H}, \rho_J \colon J \to \mathrm{Aut}(\mathcal{H}))$ be outer representations of PSC-type. Then we shall refer to a pair

$$(\alpha \colon \mathcal{G} \xrightarrow{\sim} \mathcal{H},\ \beta \colon I \xrightarrow{\sim} J)$$

consisting of an isomorphism α of semi-graphs of anabelioids and an isomorphism β of profinite groups such that the diagram

$$\begin{array}{ccc} I & \xrightarrow{\rho_I} & \mathrm{Aut}(\mathcal{G}) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathrm{Aut}(\alpha)} \\ J & \xrightarrow[\rho_J]{} & \mathrm{Aut}(\mathcal{H}) \end{array}$$

— where the right-hand vertical arrow is the isomorphism induced by α — *commutes* as an *isomorphism of outer representations of PSC-type.*

Remark 2.1.1. It follows immediately that a "*pro-Σ IPSC-extension*"

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_I \longrightarrow I \longrightarrow 1$$

(i.e., roughly speaking, an extension that arises from a stable log curve over a log point — cf. [Mzk6], Definition 1.2, (ii)) gives rise to an outer representation $I \to \mathrm{Out}(\Pi_{\mathcal{G}})$ that factors through $\mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$; in particular, we obtain an outer representation of pro-Σ PSC-type $I \to \mathrm{Aut}(\mathcal{G})$.

In the following, let us fix a nonempty set of prime numbers Σ and an outer representation of pro-Σ PSC-type

$$\Big(\mathcal{G}, \rho_I \colon I \to \mathrm{Aut}(\mathcal{G})\ \ (\hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}))\Big)$$

and write $\Pi_I \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$ (cf. the discussion entitled "*Topological groups*" in §0); thus, we have an exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_I \longrightarrow I \longrightarrow 1\,.$$

Definition 2.2.

(i) Let $v \in \mathrm{Vert}(\mathcal{G})$ be a *vertex* of $\mathcal{G}$ and $\Pi_v \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup of $\Pi_{\mathcal{G}}$ associated to v. Then we shall write

$$D_v \stackrel{\mathrm{def}}{=} N_{\Pi_I}(\Pi_v) \subseteq \Pi_I \text{ (respectively, } I_v \stackrel{\mathrm{def}}{=} Z_{\Pi_I}(\Pi_v) \subseteq D_v)$$

and refer to D_v (respectively, I_v) as *a decomposition* (respectively, *an inertia*) *subgroup* of Π_I associated to the *vertex* v, or, alternatively, *the decomposition* (respectively, *inertia*) *subgroup* of Π_I associated to the *verticial subgroup* $\Pi_v \subseteq \Pi_{\mathcal{G}}$. If, moreover, the verticial subgroup Π_v is the verticial subgroup associated to an element $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (cf. Definition 1.1, (vi)), then we shall write $D_{\widetilde{v}} \stackrel{\mathrm{def}}{=} D_v$ (respectively, $I_{\widetilde{v}} \stackrel{\mathrm{def}}{=} I_v$) and refer to $D_{\widetilde{v}}$ (respectively, $I_{\widetilde{v}}$) as *the decomposition* (respectively, *inertia*) *subgroup of* Π_I *associated to* $\widetilde{v}$.

(ii) Let $e \in \mathrm{Cusp}(\mathcal{G})$ be a *cusp* of $\mathcal{G}$ and $\Pi_e \subseteq \Pi_{\mathcal{G}}$ an edge-like subgroup of $\Pi_{\mathcal{G}}$ associated to e. Then we shall write

$$D_e \stackrel{\mathrm{def}}{=} N_{\Pi_I}(\Pi_e) \subseteq \Pi_I \text{ (respectively, } I_e \stackrel{\mathrm{def}}{=} \Pi_e \subseteq D_e)$$

and refer to D_e (respectively, I_e) as *a decomposition* (respectively, *an inertia*) *subgroup* of Π_I associated to the *cusp* e, or, alternatively, *the decomposition* (respectively, *inertia*) *subgroup* of Π_I associated to the *edge-like subgroup* $\Pi_e \subseteq \Pi_{\mathcal{G}}$. If, moreover, the edge-like subgroup Π_e is the edge-like subgroup associated to an element $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ (cf. Definition 1.1, (vi)), then we shall write $D_{\widetilde{e}} \stackrel{\mathrm{def}}{=} D_e$ (respectively, $I_{\widetilde{e}} \stackrel{\mathrm{def}}{=} I_e$) and refer to $D_{\widetilde{e}}$ (respectively, $I_{\widetilde{e}}$) as *the decomposition* (respectively, *inertia*) *subgroup of* Π_I *associated to* $\widetilde{e}$.

(iii) Let $e \in \mathrm{Node}(\mathcal{G})$ be a *node* of $\mathcal{G}$ and $\Pi_e \subseteq \Pi_{\mathcal{G}}$ an edge-like subgroup of $\Pi_{\mathcal{G}}$ associated to e. Then we shall write

$$D_e \stackrel{\mathrm{def}}{=} N_{\Pi_I}(\Pi_e) \subseteq \Pi_I \text{ (respectively, } I_e \stackrel{\mathrm{def}}{=} Z_{\Pi_I}(\Pi_e) \subseteq D_e)$$

and refer to D_e (respectively, I_e) as *a decomposition* (respectively, *an inertia*) *subgroup* of Π_I associated to the *node* e, or, alternatively, *the decomposition* (respectively, *inertia*) *subgroup* of Π_I associated to the *edge-like subgroup* $\Pi_e \subseteq \Pi_{\mathcal{G}}$. If, moreover, the edge-like subgroup Π_e is the edge-like subgroup associated to an element $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ (cf. Definition 1.1, (vi)), then we shall write $D_{\widetilde{e}} \stackrel{\mathrm{def}}{=} D_e$ (respectively, $I_{\widetilde{e}} \stackrel{\mathrm{def}}{=} I_e$) and refer to $D_{\widetilde{e}}$ (respectively, $I_{\widetilde{e}}$) as *the decomposition* (respectively, *inertia*) *subgroup of* Π_I *associated to* $\widetilde{e}$.

Lemma 2.3 (**Basic properties of inertia subgroups**).

(i) *Let* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. *Then* $\{1\} = I_{\widetilde{v}} \cap \Pi_{\mathcal{G}}$; *in particular, the homomorphism* $I_{\widetilde{v}} \to I$ *induced by the surjection* $\Pi_I \twoheadrightarrow I$ *is* **injective**.

(ii) *Let* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, $\widetilde{v} \in \mathcal{V}(\widetilde{e})$. *Then* $I_{\widetilde{v}} \subseteq I_{\widetilde{e}}$.

Proof. Assertion (i) follows from the *commensurable terminality* of $\Pi_{\widetilde{v}}$ in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)), together with the *slimness* of $\Pi_{\widetilde{v}}$ (cf. [Mzk4], Remark 1.1.3). Assertion (ii) follows from the fact that

$\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$, together with the definitions of inertia subgroups of vertices and nodes. ☐

The following definition will play a *central role* in the present paper.

Definition 2.4.

(i) We shall say that the outer representation of pro-Σ PSC-type ρ_I is *of IPSC-type* (where the "IPSC" stands for "inertial pointed stable curve") if ρ_I is isomorphic, as an outer representation of PSC-type (cf. Definition 2.1, (ii)), to the outer representation of PSC-type determined by (cf. Remark 2.1.1) an "IPSC-extension" (i.e., roughly speaking, an extension that arises from a stable log curve over a log point — cf. [Mzk6], Definition 1.2, (ii)).

(ii) We shall say that the outer representation of pro-Σ PSC-type ρ_I *of VA-type* (where the "VA" stands for "verticially admissible") if the following two conditions are satisfied:

(1) I is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}$ as an abstract profinite group.

(2) For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the image of the injection $I_{\widetilde{v}} \hookrightarrow I$ (cf. Lemma 2.3, (i)) is *open* in I.

We shall say that the outer representation of pro-Σ PSC-type ρ_I is *of SVA-type* (where the "SVA" stands for "strictly verticially admissible") if, in addition to the above condition (1), the following condition is satisfied:

(2′) For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the injection $I_{\widetilde{v}} \hookrightarrow I$ is *bijective*.

(iii) We shall say that the outer representation of pro-Σ PSC-type ρ_I is *of NN-type* (where the "NN" stands for "nodally nondegenerate") if ρ_I is of VA-type, and, moreover, the following condition is satisfied:

(3) For every $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, the homomorphism $I_{\widetilde{v}_1} \times I_{\widetilde{v}_2} \to I_{\widetilde{e}}$ — where we write $\{\widetilde{v}_1, \widetilde{v}_2\} = \mathcal{V}(\widetilde{e}) \subseteq \mathrm{Vert}(\widetilde{\mathcal{G}})$ — induced by the inclusions $I_{\widetilde{v}_1}, I_{\widetilde{v}_2} \subseteq I_{\widetilde{e}}$ (cf. Lemma 2.3, (ii)) is *injective*, and its image is *open* in $I_{\widetilde{e}}$.

We shall say that the outer representation of pro-Σ PSC-type ρ_I is *of SNN-type* (where the "SNN" stands for "strictly nodally nondegenerate") if ρ_I is of SVA-type and of NN-type.

Remark 2.4.1. Note that it is *not* the case that condition (2) of Definition 2.4 is implied by conditions (1) and (3) of Definition 2.4. Indeed, it is easily verified that if $\mathrm{Vert}(\mathcal{G}) = \{v\}$, and $\mathrm{Node}(\mathcal{G}) = \emptyset$ (so $\Pi_v = \Pi_{\mathcal{G}}$), then any *injection* $\widehat{\mathbb{Z}}^{\Sigma} \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$ satisfies conditions (1) and (3), but *fails* to satisfy condition (2). (Moreover, it is also easily verified that such an injection exists.) On the other hand, when $\mathrm{Node}(\mathcal{G}) \neq \emptyset$, it is not clear to the authors at the time of writing whether or not condition (2) of Definition 2.4 is implied by conditions (1) and (3) of Definition 2.4.

Remark 2.4.2. It follows from [Mzk6], Proposition 1.3, (ii), (iii), that if ρ_I is of IPSC-type, then ρ_I is of SNN-type, i.e.,

$$\begin{array}{ccccc} \text{IPSC-type} & \Longrightarrow & \text{SNN-type} & \Longrightarrow & \text{NN-type} \\ & & \Downarrow & & \Downarrow \\ & & \text{SVA-type} & \Longrightarrow & \text{VA-type}. \end{array}$$

Lemma 2.5 (**Group structure of inertia subgroups**). *If ρ_I is* **of VA-type***, then the following hold:*

(i) *Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then as an abstract profinite group, $I_{\widetilde{v}}$ is* **isomorphic** *to $\widehat{\mathbb{Z}}^{\Sigma}$.*

(ii) *Let $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$. Then as an abstract profinite group, $I_{\widetilde{e}}$ is* **isomorphic** *to $\widehat{\mathbb{Z}}^{\Sigma}$.*

(iii) *Let $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$. Then as an abstract profinite group, $I_{\widetilde{e}}$ is* **isomorphic** *to $\widehat{\mathbb{Z}}^{\Sigma} \times \widehat{\mathbb{Z}}^{\Sigma}$.*

(iv) *Let $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$. Then $\Pi_{\widetilde{e}} = I_{\widetilde{e}} \cap \Pi_{\mathcal{G}}$; thus, we have an exact sequence*

$$1 \longrightarrow \Pi_{\widetilde{e}} \longrightarrow I_{\widetilde{e}} \longrightarrow \mathrm{Im}(I_{\widetilde{e}} \to I) \longrightarrow 1$$

— where we write $\mathrm{Im}(I_{\widetilde{e}} \to I)$ for the image of the composite $I_{\widetilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$. Moreover, the subgroup $\mathrm{Im}(I_{\widetilde{e}} \to I) \subseteq I$ is **open** *in I.*

In particular, for $\widetilde{v} \in \mathcal{V}(\widetilde{e})$, the image of the homomorphism $I_{\widetilde{v}} \times \Pi_{\widetilde{e}} \to I_{\widetilde{e}}$ induced by the natural inclusions $I_{\widetilde{v}}, \Pi_{\widetilde{e}} \subseteq I_{\widetilde{e}}$ is **open** *in $I_{\widetilde{e}}$. If, moreover, for $\widetilde{v} \in \mathcal{V}(\widetilde{e})$, the composite $I_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is* **surjective** *(or, equivalently,* **bijective***), then the homomorphism $I_{\widetilde{v}} \times \Pi_{\widetilde{e}} \to I_{\widetilde{e}}$ induced by the natural inclusions $I_{\widetilde{v}}, \Pi_{\widetilde{e}} \subseteq I_{\widetilde{e}}$ is* **bijective***.*

Proof. Assertion (i) (respectively, (ii)) follows from conditions (1) and (2) of Definition 2.4 (respectively, from Definition 2.2, (ii)). Assertion (iv) follows from the *commensurable terminality* of $\Pi_{\widetilde{e}}$ in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)), together with condition (2) of Definition 2.4. Assertion (iii) follows from the fact that $I_{\widetilde{e}}$ is an *extension* of $\widehat{\mathbb{Z}}^{\Sigma}$ by $\widehat{\mathbb{Z}}^{\Sigma}$ and *abelian* (cf. assertion (iv)). □

Lemma 2.6 (**Stability of verticial admissibility and nodal nondegeneracy**). *Suppose that* ρ_I *is* **of VA-type** *(respectively,* **of NN-type***). Then the following hold:*

(i) *Let* $\Pi_{I'} \subseteq \Pi_I$ *be an* **open** *subgroup of* Π_I, $\Pi_{\mathcal{G}'} \stackrel{\text{def}}{=} \Pi_{I'} \cap \Pi_{\mathcal{G}}$*, and* I' *the image of the composite* $\Pi_{I'} \hookrightarrow \Pi_I \twoheadrightarrow I$*. Thus, we have an exact sequence*

$$1 \longrightarrow \Pi_{\mathcal{G}'} \longrightarrow \Pi_{I'} \longrightarrow I' \longrightarrow 1\,;$$

the open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$ *determines a covering* $\mathcal{G}' \to \mathcal{G}$ *of* $\mathcal{G}$*; the outer representation* $I' \to \mathrm{Out}(\Pi_{\mathcal{G}'})$ *determined by* $\Pi_{I'}$ *factors through* $\rho_{I'} \colon I' \to \mathrm{Aut}(\mathcal{G}')$*. Then* $\rho_{I'}$ *is* **of VA-type** *(respectively,* **of NN-type***).*

(ii) *Suppose that* $\mathcal{G}$ *is* **sturdy**. *Then the outer representation of pro-*Σ *PSC-type* $\overline{\rho}_I \colon I \to \mathrm{Aut}(\overline{\mathcal{G}})$ *— where we write* $\overline{\mathcal{G}}$ *for the* **compactification** *of* $\mathcal{G}$ *— induced by* ρ_I *is* **of VA-type** *(respectively,* **of NN-type***).*

Proof. First, we prove assertion (i). It follows immediately from the various definitions involved that $\rho_{I'}$ is of VA-type. Moreover, it follows from Lemma 2.5, (i), (iv), that the various "$I_{\widetilde{v}}$" (respectively, "$I_{\widetilde{e}}$") are *torsion-free*, and, moreover, that the *commensurability class* of the subgroup "$I_{\widetilde{v}}$" (respectively, "$I_{\widetilde{e}}$") is *unaffected by passing from* Π_I *to* $\Pi_{I'}$. Thus, condition (3) of Definition 2.4 for $\rho_{I'}$ follows from condition (3) of Definition 2.4 for ρ_I. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us observe that condition (1) of Definition 2.4 for $\overline{\rho}_I$ follows from condition (1) of Definition 2.4 for ρ_I. Next, let us observe that it follows from Lemma 2.3, (i) (respectively, Lemma 2.5, (iv)), that for $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (respectively, $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$), the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\overline{\mathcal{G}}}$ induces an *open injection* between the respective subgroups "$I_{\widetilde{v}}$" (respectively, "$I_{\widetilde{e}}$"). Thus, condition (2) (respectively, (3)) of Definition 2.4 for $\overline{\rho}_I$ follows from condition (2) (respectively, (3)) of Definition 2.4 for ρ_I. This completes the proof of assertion (ii). □

Lemma 2.7 (**Group structure of decomposition subgroups**). *If* ρ_I *is* **of VA-type***, then the following hold:*

(i) *Let* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*. Then* $\Pi_{\widetilde{v}} = D_{\widetilde{v}} \cap \Pi_{\mathcal{G}}$*; thus, we have an exact sequence*

$$1 \longrightarrow \Pi_{\widetilde{v}} \longrightarrow D_{\widetilde{v}} \longrightarrow \mathrm{Im}(D_{\widetilde{v}} \to I) \longrightarrow 1$$

— where we write $\mathrm{Im}(D_{\widetilde{v}} \to I)$ *for the image of the composite* $D_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$*. Moreover, the subgroup* $\mathrm{Im}(D_{\widetilde{v}} \to I) \subseteq I$ *is* **open** *in* I*.*

In particular, the image of the homomorphism $I_{\tilde{v}} \times \Pi_{\tilde{v}} \to D_{\tilde{v}}$ *induced by the natural inclusions* $I_{\tilde{v}}, \Pi_{\tilde{v}} \subseteq D_{\tilde{v}}$ *is* **open** *in* $D_{\tilde{v}}$*. If, moreover, the composite* $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ *is* **surjective** *(or, equivalently,* **bijective***), then the homomorphism* $I_{\tilde{v}} \times \Pi_{\tilde{v}} \to D_{\tilde{v}}$ *is* **bijective***.*

(ii) *Let* $\tilde{e} \in \mathrm{Cusp}(\tilde{\mathcal{G}})$*. Then* $\Pi_{\tilde{e}} = D_{\tilde{e}} \cap \Pi_{\mathcal{G}}$*; thus, we have an exact sequence*

$$1 \longrightarrow \Pi_{\tilde{e}} \longrightarrow D_{\tilde{e}} \longrightarrow \mathrm{Im}(D_{\tilde{e}} \to I) \longrightarrow 1$$

— where we write $\mathrm{Im}(D_{\tilde{e}} \to I)$ *for the image of the composite* $D_{\tilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$*. Moreover, the subgroup* $\mathrm{Im}(D_{\tilde{e}} \to I) \subseteq I$ *is* **open** *in* I*.*

In particular, for $\tilde{v} \in \mathcal{V}(\tilde{e})$*, the image of the homomorphism* $I_{\tilde{v}} \times \Pi_{\tilde{e}} \to D_{\tilde{e}}$ *induced by the natural inclusions* $I_{\tilde{v}}, \Pi_{\tilde{e}} \subseteq D_{\tilde{e}}$ *is* **open** *in* $D_{\tilde{e}}$*. If, moreover, for* $\tilde{v} \in \mathcal{V}(\tilde{e})$*, the composite* $I_{\tilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ *is* **surjective** *(or, equivalently,* **bijective***), then the homomorphism* $I_{\tilde{v}} \times \Pi_{\tilde{e}} \to D_{\tilde{e}}$ *induced by the natural inclusions* $I_{\tilde{v}}, \Pi_{\tilde{e}} \subseteq D_{\tilde{e}}$ *is* **bijective***.*

(iii) *Let* $\tilde{e} \in \mathrm{Node}(\tilde{\mathcal{G}})$*. Then* $\Pi_{\tilde{e}} = D_{\tilde{e}} \cap \Pi_{\mathcal{G}}$*; thus, we have an exact sequence*

$$1 \longrightarrow \Pi_{\tilde{e}} \longrightarrow D_{\tilde{e}} \longrightarrow \mathrm{Im}(D_{\tilde{e}} \to I) \longrightarrow 1$$

— where we write $\mathrm{Im}(D_{\tilde{e}} \to I)$ *for the image of the composite* $D_{\tilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$*. Moreover, the subgroup* $\mathrm{Im}(D_{\tilde{e}} \to I) \subseteq I$ *is* **open** *in* I*.*

In particular, the image of the natural inclusion $I_{\tilde{e}} \hookrightarrow D_{\tilde{e}}$ *is* **open** *in* $D_{\tilde{e}}$*. If, moreover, for* $\tilde{v} \in \mathcal{V}(\tilde{e})$*, the composite* $I_{\tilde{v}} \to I$ *is* **surjective** *(or, equivalently,* **bijective***), then the natural inclusion* $I_{\tilde{e}} \hookrightarrow D_{\tilde{e}}$ *is* **bijective***.*

Proof. The computation of the intersection with $\Pi_{\mathcal{G}}$ in assertion (i) (respectively, (ii); (iii)) follows from the *commensurable terminality* of $\Pi_{\tilde{v}}$ (respectively, $\Pi_{\tilde{e}}$; $\Pi_{\tilde{e}}$) in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)). The fact that the images of the respective decomposition subgroups in I are open follows from condition (2) of Definition 2.4. The final portion of assertion (i) (respectively, (ii); (iii)) then follows immediately from Lemma(s) 2.3, (i) (respectively, 2.3, (i); 2.3, (i), and 2.5, (iv)) ☐

Remark 2.7.1. It follows immediately from Lemmas 2.5, 2.7 that the following assertion holds:

Let $\tilde{v} \in \mathrm{Vert}(\tilde{\mathcal{G}})$ (respectively, $\tilde{e} \in \mathrm{Cusp}(\tilde{\mathcal{G}})$; $\tilde{e} \in \mathrm{Node}(\tilde{\mathcal{G}})$). If ρ_I is *of SVA-type*, then

$$D_{\tilde{v}} = I_{\tilde{v}} \cdot \Pi_{\tilde{v}} = I_{\tilde{v}} \times \Pi_{\tilde{v}}$$

(respectively, $D_{\tilde{e}} = I_{\tilde{v}} \cdot \Pi_{\tilde{e}} = I_{\tilde{v}} \times \Pi_{\tilde{e}}$, for any $\tilde{v} \in \mathcal{V}(\tilde{e})$;

$$D_{\widetilde{e}} = I_{\widetilde{e}} = I_{\widetilde{v}} \cdot \Pi_{\widetilde{e}} = I_{\widetilde{v}} \times \Pi_{\widetilde{e}}\,, \text{ for any } \widetilde{v} \in \mathcal{V}(\widetilde{e})\,).$$

Remark 2.7.2. Let $\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. If ρ_I is *of VA-type*, then the following three conditions are equivalent:

(i) $\widetilde{e}_1 = \widetilde{e}_2$.

(ii) $I_{\widetilde{e}_1} = I_{\widetilde{e}_2}$.

(iii) $D_{\widetilde{e}_1} = D_{\widetilde{e}_2}$.

Indeed, the implications

$$\text{(i)} \Longrightarrow \text{(ii)}\ ;\ \ \text{(i)} \Longrightarrow \text{(iii)}$$

are immediate. On the other hand, if condition (ii) (respectively, (iii)) is satisifed, then $\Pi_{\widetilde{e}_1} = I_{\widetilde{e}_1} \cap \Pi_{\mathcal{G}} = I_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{e}_2}$ [cf. Definition 2.2, (ii); Lemma 2.5, (iv)] (respectively, $\Pi_{\widetilde{e}_1} = D_{\widetilde{e}_1} \cap \Pi_{\mathcal{G}} = D_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{e}_2}$ [cf. Lemma 2.7, (ii), (iii)]). Thus, it follows from [Mzk4], Proposition 1.2, (i), that $\widetilde{e}_1 = \widetilde{e}_2$.

Definition 2.8. Suppose that ρ_I is *of SVA-type*. Then we shall denote by

$$\mathcal{G}[\rho_I]$$

the connected semi-graph of anabelioids (cf. [Mzk3], Definition 2.1) defined as follows: The underlying graph of $\mathcal{G}[\rho_I]$ is the underlying graph of $\mathcal{G}$. The anabelioid corresponding to a vertex $v \in \mathrm{Vert}(\mathcal{G})$ (respectively, an edge $e \in \mathrm{Edge}(\mathcal{G})$) is the connected anabelioid determined by the decomposition subgroup, regarded up to inner automorphism, $D_v \subseteq \Pi_I$ (respectively, $D_e \subseteq \Pi_I$) associated to v (respectively, e); for $v \in \mathcal{V}(e)$, the associated morphism of anabelioids is the morphism determined by the natural inclusion $D_e\ (= I_v \cdot \Pi_e) \hookrightarrow D_v\ (= I_v \cdot \Pi_v)$ (cf. Remark 2.7.1).

Remark 2.8.1.

(i) Note that the fundamental group of the anabelioid corresponding to a vertex of $\mathcal{G}[\rho_I]$ (i.e., the decomposition subgroup, regarded up to inner automorphism, associated to the vertex) is *not* center-free (cf. Lemma 2.7, (i)). In particular, the semi-graph of anabelioids $\mathcal{G}[\rho_I]$ is *not* of *PSC-type*.

(ii) Let $\Pi_{\mathcal{G}[\rho_I]}$ be the pro-Σ fundamental group (i.e., the maximal pro-Σ quotient of the fundamental group) of the connected semi-graph of anabelioids $\mathcal{G}[\rho_I]$ (cf. the discussion following [Mzk3], Definition 2.1). Then it follows from the definition of $\Pi_{\mathcal{G}[\rho_I]}$ that the *inductive system of homomorphisms* determined by the natural outer inclusions $D_v \hookrightarrow \Pi_I$ and $D_e \hookrightarrow \Pi_I$ gives rise to a natural outer homomorphism

$$\Pi_{\mathcal{G}[\rho_I]} \longrightarrow \Pi_I \, .$$

Lemma 2.9 (An isomorphism of fundamental groups). *Suppose that* ρ_I *is* **of SVA-type**. *Let* $\Pi_{\mathcal{G}[\rho_I]}$ *be the pro-*Σ *fundamental group of the connected semi-graph of anabelioids* $\mathcal{G}[\rho_I]$. *Then the homomorphism* $\Pi_{\mathcal{G}[\rho_I]} \to \Pi_I$ *defined in Remark 2.8.1, (ii), is an* **isomorphism**.

Proof. First, we observe (cf. Remark 2.7.1) that the decomposition subgroup D_z — where $z \in \mathrm{VCN}(\mathcal{G})$ — is an extension of I by Π_z. Now it is easily verified that the *profinite* Galois covering of $\mathcal{G}[\rho_I]$ determined by the various quotients $D_z \twoheadrightarrow I$ (i.e., that arise as composites $D_z \hookrightarrow \Pi_I \twoheadrightarrow I$) is isomorphic to $\mathcal{G}$; thus, we obtain an exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{G}[\rho_I]} \longrightarrow I \longrightarrow 1 \, .$$

On the other hand, it follows from the construction of this profinite covering $\mathcal{G} \to \mathcal{G}[\rho_I]$, together with the definition of the homomorphism $\Pi_{\mathcal{G}[\rho_I]} \to \Pi_I$, that the composite $\Pi_{\mathcal{G}} \to \Pi_{\mathcal{G}[\rho_I]} \to \Pi_I$ coincides with the natural inclusion $\Pi_{\mathcal{G}} \hookrightarrow \Pi_I$. Thus, the bijectivity of the homomorphism $\Pi_{\mathcal{G}[\rho_I]} \to \Pi_I$ follows from the "Five Lemma". This completes the proof of Lemma 2.9. ◻

Definition 2.10. Let (g, r) be a pair of natural numbers such that $2g - 2 + r > 0$, k an algebraically closed field of characteristic $\notin \Sigma$, $s \in \overline{\mathcal{M}}_{g,r}(k)$ a k-valued geometric point of $\overline{\mathcal{M}}_{g,r}$ (cf. the discussion entitled "*Curves*" in §0), and $s^{\log}\colon \mathrm{Spec}\,(k)^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ the *strict* morphism of log stacks whose underlying morphism of stacks is the morphism corresponding to s.

(i) We shall denote by $X_s^{\log}$ the stable log curve determined by $s^{\log}$.

(ii) We shall denote by $\mathcal{G}_s$ the semi-graph of anabelioids of pro-Σ PSC-type determined by the stable log curve $X_s^{\log}$ (cf. [Mzk4], Example 2.5).

(iii) Write Q_s for the monoid obtained as the stalk of the *characteristic sheaf* (cf. the discussion entitled "*Log stacks*" in §0) of $\overline{\mathcal{M}}_{g,r}^{\log}$ at s, and

$$I_s \stackrel{\mathrm{def}}{=} \mathrm{Hom}(Q_s^{\mathrm{gp}}, \widehat{\mathbb{Z}}(1)^{\Sigma})$$

— where the "(1)" denotes a "Tate twist". Recall (cf. [Knud], Theorem 2.7) that it follows from the well-known geometry of the irreducible components of the divisor that defines the log structure of $\overline{\mathcal{M}}_{g,r}^{\log}$ that we have a natural decomposition

$$Q_s \simeq \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_s)} \mathbb{N}_e$$

— where we write $\mathbb{N}_e$ for a copy of $\mathbb{N}$ indexed by $e \in \mathrm{Node}(\mathcal{G}_s)$; thus, we obtain a decomposition

$$I_s \simeq \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_s)} \Lambda[e]$$

— where we write $\Lambda[e]$ for a copy of $\widehat{\mathbb{Z}}(1)^{\Sigma}$ indexed by $e \in \mathrm{Node}(\mathcal{G}_s)$.

(iv) It follows from the various definitions involved that, if we write $\pi_1^{\Sigma}(X_s^{\log})$ for the maximal pro-Σ quotient of the logarithmic fundamental group of $X_s^{\log}$, then we have a natural exact sequence of profinite group

$$1 \longrightarrow \Pi_{\mathcal{G}_s} \longrightarrow \pi_1^{\Sigma}(X_s^{\log}) \longrightarrow I_s \longrightarrow 1$$

— which gives rise to an outer representation $I_s \to \mathrm{Out}(\Pi_{\mathcal{G}_s})$ that factors through $\mathrm{Aut}(\mathcal{G}_s) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_s})$. Write

$$\rho_s \colon I_s \longrightarrow \mathrm{Aut}(\mathcal{G}_s)$$

for the resulting homomorphism of profinite groups and $\Pi_{I_s} \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}_s} \stackrel{\mathrm{out}}{\rtimes} I_s$. Thus, we have a natural isomorphism of profinite groups $\pi_1^{\Sigma}(X_s^{\log}) \xrightarrow{\sim} \Pi_{I_s}$.

(v) Let $s' \in \overline{\mathcal{M}}_{g,r+1}(k)$ be a k-valued geometric point of $\overline{\mathcal{M}}_{g,r+1}$ that corresponds to a node of X_s. Then it follows immediately from the various definitions involved that the quotient of $\Pi_{\mathcal{G}_{s'}}$ — where we use the notation obtained by applying (ii) to s' — by the closed subgroup of $\Pi_{\mathcal{G}_{s'}}$ topologically normally generated by the edge-like subgroups of $\Pi_{\mathcal{G}_{s'}}$ associated to the $(r+1)$-st cusp is naturally isomorphic to $\Pi_{\mathcal{G}_s}$; in particular, we have a natural surjection $\Pi_{\mathcal{G}_{s'}} \twoheadrightarrow \Pi_{\mathcal{G}_s}$. We shall denote by

$$\mathfrak{N}_{s'/s} \colon \mathrm{Node}(\mathcal{G}_{s'}) \longrightarrow \mathrm{Node}(\mathcal{G}_s)$$

the map which — as is easily verified — is *uniquely determined* by the following condition:

> If $e \in \mathrm{Node}(\mathcal{G}_{s'})$, and $\Pi_e \subseteq \Pi_{\mathcal{G}_{s'}}$ is an edge-like subgroup associated to e, then the image of Π_e via the above surjection $\Pi_{\mathcal{G}_{s'}} \twoheadrightarrow \Pi_{\mathcal{G}_s}$ is an edge-like subgroup associated to $\mathfrak{N}_{s'/s}(e) \in \mathrm{Node}(\mathcal{G}_{s'})$.

Lemma 2.11 (Log fundamental groups in a neighborhood of a node). *In the notation of Definition 2.10, let* $e \in \mathrm{Node}(\mathcal{G}_s)$ *be a node* *of* $\mathcal{G}_s$*, and* $s' \in \overline{\mathcal{M}}_{g,r+1}(k)$ *a* k*-valued geometric point that corresponds to the node of* X_s *determined by* e*. Then the following hold:*

(i) *The inverse image* $\mathfrak{N}_{s'/s}^{-1}(e)$ *consists of* **precisely two elements** $e_1,\ e_2 \in \mathrm{Node}(\mathcal{G}_{s'})$*; the map*

$$\mathrm{Node}(\mathcal{G}_{s'}) \setminus \{e_1, e_2\} \longrightarrow \mathrm{Node}(\mathcal{G}_s) \setminus \{e\}$$

determined by $\mathfrak{N}_{s'/s}$ *is* **bijective***.*

(ii) *Write* $I_{s'}$ *for the result of applying Definition 2.10, (iii), to* s'*. Then the homomorphism* $I_{s'} \to \Pi_{I_s}$ *induced on maximal pro-*Σ *quotients of log fundamental groups by (the strict morphism of log schemes whose underlying morphism of schemes is the morphism corresponding to)* s' *is* **injective***, and its image is an* **inertia subgroup** I_e *of* Π_{I_s} *associated to* e*. Moreover, if, in the notation of (i), we write*

$$M_{\neq e_1,e_2} \stackrel{\mathrm{def}}{=} \bigoplus_{f \in \mathrm{Node}(\mathcal{G}_{s'}) \setminus \{e_1,e_2\}} \Lambda[f] \subseteq I_{s'}\,,$$

then for $i = 1,\ 2$*, there exists a vertex* $v_i \in \mathcal{V}(e) \subseteq \mathrm{Vert}(\mathcal{G}_{\mathrm{s}})$ *of* $\mathcal{G}_s$ *such that the subgroup obtained as the image of the composite of the injections*

$$\Lambda[e_i] \oplus M_{\neq e_1,e_2} \hookrightarrow I_{s'} \hookrightarrow \Pi_{I_s}$$

is an **inertia subgroup** I_{v_i} *of* Π_{I_s} *associated to* v_i*. In this situation, we shall refer to* v_i *as the* **vertex associated to** e_i*.*

(iii) *Let*

$$M_{\neq e} \stackrel{\mathrm{def}}{=} \bigoplus_{f \in \mathrm{Node}(\mathcal{G}_s) \setminus \{e\}} \Lambda[f] \subseteq I_s\,.$$

Then the homomorphism $I_{s'} \to I_s$ *induced by* $\overline{\mathcal{M}}_{g,r+1}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ *(i.e., the composite* $I_{s'} \hookrightarrow \Pi_{I_s} \twoheadrightarrow I_s$*)* **coincides** *with the homomorphism*

$$\Lambda[e_1] \oplus \Lambda[e_2] \oplus M_{\neq e_1,e_2} = I_{s'} \longrightarrow I_s = \Lambda[e] \oplus M_{\neq e}$$

determined by the homomorphism

$$\begin{array}{ccc} \Lambda[e_1] \oplus \Lambda[e_2] & \longrightarrow & \Lambda[e] \\ (a,b) & \mapsto & a+b \end{array}$$

and the isomorphism

$$M_{\neq e_1,e_2} \stackrel{\sim}{\longrightarrow} M_{\neq e}$$

induced by the bijective portion of $\mathfrak{N}_{s'/s}$ *(cf. (i)).*

Proof. Assertion (i) follows immediately from the various definitions involved. Assertions (ii) and (iii) follow by computing the log structures involved by means of a chart for the morphism $X_s^{\log} \to \mathrm{Spec}\,(k)^{\log}$ at the k-valued point s' of X_s. $\square$

Remark 2.11.1. In the notation of Definition 2.10, let $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}}_s)$, $e \stackrel{\mathrm{def}}{=} \widetilde{e}(\mathcal{G}_s) \in \mathrm{Node}(\mathcal{G}_s)$, $s' \in \overline{\mathcal{M}}_{g,r+1}(k)$ a k-valued geometric point that corresponds to the node of X_s determined by e, and $\{e_1, e_2\} = \mathfrak{N}^{-1}_{s'/s}(e) \subseteq \mathrm{Node}(\mathcal{G}_{s'})$ (cf. Lemma 2.11, (i)). Moreover, for $i = 1, 2$, let us denote by $\widetilde{v}_i$ the (unique!) element of $\mathrm{Vert}(\widetilde{\mathcal{G}}_s)$ such that $\widetilde{v}_i \in \mathcal{V}(\widetilde{e})$, and, moreover, $\widetilde{v}_i(\mathcal{G}_s)$ is the vertex associated to e_i (cf. Lemma 2.11, (ii)). (Thus, $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}_1, \widetilde{v}_2\}$.) Then it follows from Lemma 2.11 that if we identify $M_{\neq e}$ with $M_{\neq e_1.e_2}$ via the isomorphism of Lemma 2.11, (iii), then the following assertion holds:

> The isomorphisms $I_{\widetilde{v}_i} \xrightarrow{\sim} \Lambda[e_i] \oplus M_{\neq e}$ (cf. Lemma 2.11, (ii)), $I_{\widetilde{e}} \xrightarrow{\sim} I_{s'} \xrightarrow{\sim} \Lambda[e_1] \oplus \Lambda[e_2] \oplus M_{\neq e}$ (cf. Lemma 2.11, (ii), (iii); Definition 2.10, (iii)), and $I_s \xrightarrow{\sim} \Lambda[e] \oplus M_{\neq e}$ (cf. Lemma 2.11, (iii); Definition 2.10, (iii)) fit into the following *commutative diagram*
>
> $$\begin{array}{ccccc} I_{\widetilde{v}_1} \times I_{\widetilde{v}_2} & \longrightarrow & I_{\widetilde{e}} & \longrightarrow & I_s \\ \wr\downarrow & & \wr\downarrow & & \downarrow\wr \\ \Big(\Lambda[e_1] \oplus M_{\neq e}\Big) \times \Big(\Lambda[e_2] \oplus M_{\neq e}\Big) & \longrightarrow & \Lambda[e_1] \oplus \Lambda[e_2] \oplus M_{\neq e} & \longrightarrow & \Lambda[e] \oplus M_{\neq e} \\ (a, m, b, n) & \mapsto & (a, b, m+n) & & \\ & & (a, b, m) & \mapsto & (a+b, m) \end{array}$$
>
> — where the upper left-hand horizontal arrow $I_{\widetilde{v}_1} \times I_{\widetilde{v}_2} \to I_{\widetilde{e}}$ is the homomorphism induced by the natural inclusions $I_{\widetilde{v}_1}, I_{\widetilde{v}_2} \subseteq I_{\widetilde{e}}$ (cf. Lemma 2.3, (ii)), and the upper right-hand horizontal arrow $I_{\widetilde{e}} \to I_s$ is the composite $I_{\widetilde{e}} \hookrightarrow \Pi_{I_s} \twoheadrightarrow I_s$.

Lemma 2.12 (The invertibility of a certain homomorphism of free modules). *Let A be a commutative ring with unity, M a free A-module of finite rank, N a free A-module of rank 1, $\rho\colon N \to N \oplus M$ a homomorphism of A-modules, $\rho_1\colon N \to N$ the composite of ρ and the first projection $N \oplus M \twoheadrightarrow N$, and*

$$N_1 \times N_2 \stackrel{\mathrm{def}}{=} \Big\{(N \oplus M) \times_{N \oplus M} N\Big\} \times \Big\{(N \oplus M) \times_{N \oplus M} N\Big\}$$

$$\longrightarrow N_0 \stackrel{\mathrm{def}}{=} (N \oplus N \oplus M) \times_{N \oplus M} N \longrightarrow (N \oplus M) \times_{N \oplus M} N$$

— where the definition of "N_1" (respectively, "N_2") is to be understood as the first (respectively, second) module in brackets "$\{-\}$"; the notation "$(-) \times_{N \oplus M} (-)$" denotes the fiber product of modules over $N \oplus M$ — the diagram obtained via ρ from the diagram

$$\begin{array}{ccccc} (N \oplus M) \times (N \oplus M) & \longrightarrow & N \oplus N \oplus M & \longrightarrow & N \oplus M \\ (a, m, b, n) & \mapsto & (a, b, m+n) & & \end{array}$$

$$(a,b,m) \mapsto (a+b,m).$$

Then the following hold:

(i) *N_1 and N_2 are free A-modules of rank 1, and N_0 is a free A-module of rank 2.*

(ii) *If ϕ is a homomorphism of free A-modules of rank 1, then let us denote by $D(\phi) \subseteq \mathrm{Spec}(A)$ the open subscheme of $\mathrm{Spec}(A)$ on which (the homomorphism of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules determined by) ϕ is an isomorphism. Then $D(\rho_1) = D(\det(N_1 \times N_2 \to N_0))$ (cf. (i)).*

Proof. Assertion (i) is immediate from the definition of N_1, N_2, and N_0. Thus, to complete the proof of Lemma 2.12, it suffices to verify assertion (ii). To this end, since the various definitions of modules and homomorphisms in the statement of Lemma 2.12 are *compatible with base-change*, we may assume without loss of generality that A is a *field*. On the other hand, if A is a field, then ρ_1 is either *zero* or an *isomorphism*, so it follows immediately from an easy computation that $D(\rho_1)$ and $D(\det(N_1 \times N_2 \to N_0))$ coincide. This completes the proof of assertion (ii). $\square$

Lemma 2.13 (**Injectivity and images of homomorphisms of $\widehat{\mathbb{Z}}^{\boldsymbol{\Sigma}}$-modules**). *In the notation of Definition 2.10, let $\rho \colon I \stackrel{\mathrm{def}}{=} \widehat{\mathbb{Z}}^{\Sigma} \to I_s$ be a homomorphism of profinite groups, and*

$$J_{\widetilde{v}_1} \times J_{\widetilde{v}_2} \stackrel{\mathrm{def}}{=} \left(I_{\widetilde{v}_1} \times_{I_s} I\right) \times \left(I_{\widetilde{v}_2} \times_{I_s} I\right) \longrightarrow J_{\widetilde{e}} \stackrel{\mathrm{def}}{=} I_{\widetilde{e}} \times_{I_s} I \longrightarrow I$$

the diagram of homomorphisms of profinite groups obtained via ρ from the upper row of the diagram in Remark 2.11.1. Then the following conditions are equivalent:

(i) *The image of the composite*

$$I \xrightarrow{\rho} I_s \simeq \Lambda[e] \oplus M_{\neq e} \xrightarrow{\mathrm{pr}} \Lambda[e]$$

is **open** *in $\Lambda[e]$.*

(ii) *The first arrow $J_{\widetilde{v}_1} \times J_{\widetilde{v}_2} \to J_{\widetilde{e}}$ of the above sequence is* **injective**, *and its image is* **open** *in $J_{\widetilde{e}}$.*

Proof. It follows immediately from the various definitions involved that the implication

$$\text{(i)} \Longrightarrow \text{(ii)} \quad (\text{respectively, } \text{(ii)} \Longrightarrow \text{(i)})$$

follows from the inclusion "$D(\rho) \subseteq D(\det(N_1 \times N_2 \to N_0))$" (respectively, "$D(\det(N_1 \times N_2 \to N_0)) \subseteq D(\rho)$") implicit in Lemma 2.12, (ii). Here, we consider the case of "$D(-)$" that arise from an *open* ideal of the topological ring $\widehat{\mathbb{Z}}^{\Sigma}$, i.e., an ideal generated by a nonzero element of $\mathbb{Z}$. $\square$

Proposition 2.14 (**Nodal nondegeneracy of certain outer representations**)**.** *In the notation of Definition 2.10, let* $\rho\colon I \stackrel{\text{def}}{=} \widehat{\mathbb{Z}}^{\Sigma} \to I_s$ *be a homomorphism of profinite groups,* $\rho_I\colon I \to \mathrm{Aut}(\mathcal{G}_s)$ *the outer representation of pro-*Σ *PSC-type obtained as the composite*

$$I \xrightarrow{\rho} I_s \xrightarrow{\rho_s} \mathrm{Aut}(\mathcal{G}_s)\,,$$

and $\Pi_I \stackrel{\text{def}}{=} \Pi_{\mathcal{G}_s} \stackrel{\text{out}}{\rtimes} I$*. In the following, for* $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}}_s)$*, we shall write* $J_{\widetilde{z}}$ *for the inertia subgroup "*$I_{\widetilde{z}}$*" of* Π_I *(i.e., to avoid confusion with the corresponding inertia subgroups of* Π_{I_s}*). Then the following hold:*

(i) ρ_I *is* **of SVA-type***.*

(ii) *If* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}}_s)$*, then the following two conditions are* **equivalent***:*

(1) *The image of the composite*

$$I \xrightarrow{\rho} I_s \simeq \bigoplus_{e\in \mathrm{Node}(\mathcal{G}_s)} \Lambda[e] \xrightarrow{\mathrm{pr}_{\widetilde{e}(\mathcal{G}_s)}} \Lambda[\widetilde{e}(\mathcal{G}_s)]$$

is **open** *in* $\Lambda[\widetilde{e}(\mathcal{G}_s)]$*.*

(2) *If* $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}_1, \widetilde{v}_2\}$*, then the homomorphism* $J_{\widetilde{v}_1} \times J_{\widetilde{v}_2} \to J_{\widetilde{e}}$ *induced by the inclusions* $J_{\widetilde{v}_1}, J_{\widetilde{v}_2} \subseteq J_{\widetilde{e}}$ *is* **injective***, and its image is* **open** *in* $J_{\widetilde{e}}$*.*

In particular, if the image of the composite

$$I \xrightarrow{\rho} I_s \simeq \bigoplus_{e\in \mathrm{Node}(\mathcal{G}_s)} \Lambda[e] \xrightarrow{\mathrm{pr}_f} \Lambda[f]$$

is **open** *in* $\Lambda[f]$ *for every* $f \in \mathrm{Node}(\mathcal{G})$*, then* ρ_I *is* **of SNN-type***.*

Proof. The various assertions of Proposition 2.14 follow immediately from the various definitions involved, together with Lemma 2.13. $\square$

Remark 2.14.1. In the notation of Proposition 2.14, it is not difficult to show — by applying various well-known group-theoretic constructions of certain *natural isomorphisms* between the various copies of $\widehat{\mathbb{Z}}(1)^{\Sigma}$ involved — that the condition on the homomorphism $\rho\colon I \to I_s$ that the composite $\rho_s \circ \rho$ be *of IPSC-type* is equivalent to the condition on ρ that there exists an isomorphism

$$I \simeq \mathrm{Hom}(\mathbb{N}^{\mathrm{gp}}, \widehat{\mathbb{Z}}(1)^{\Sigma})$$

with respect to which ρ is *positive definite* in the sense that it arises (by applying the functor "$\mathrm{Hom}(-, \widehat{\mathbb{Z}}(1)^{\Sigma})$") from a homomorphism of monoids $Q_s \to \mathbb{N}$ such that for any $f \in \mathrm{Node}(\mathcal{G}_s)$, the composite

$$\mathbb{N}_f \hookrightarrow \bigoplus_{e\in \mathrm{Node}(\mathcal{G}_s)} \mathbb{N}_e \simeq Q_s \to \mathbb{N}$$

is *nonzero*. On the other hand, it follows from Proposition 2.14 that the (necessarily strict) *nodal nondegeneracy* of ρ_I is equivalent to the *nondegeneracy* of ρ, i.e., the condition that the image of the composite

$$I \xrightarrow{\rho} I_s \simeq \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_s)} \Lambda[e] \xrightarrow{\mathrm{pr}_f} \Lambda[f]$$

be *open* for every $f \in \mathrm{Node}(\mathcal{G}_s)$. That is to say,

$$\begin{array}{ccc} \text{IPSC-type} & \Longrightarrow & \text{(S)NN-type} \\ \Updownarrow & & \Updownarrow \\ \text{positive definite} & \Longrightarrow & \text{nondegenerate.} \end{array}$$

3. Group-theoretic aspects of the geometry of the underlying semi-graphs

In this section, we consider the geometry of the underlying semi-graph associated to a semi-graph of anabelioids of PSC-type from a group-theoretic point of view in the context of outer representations of NN-type (cf. [Mzk6], Proposition 1.3, for an analogous discussion in the case of outer representations of IPSC-type).

In this section, let Σ be a nonempty set of prime numbers, $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type, $\Pi_\mathcal{G}$ the fundamental group of $\mathcal{G}$, $\rho_I \colon I \to \mathrm{Aut}(\mathcal{G})$ an outer representation *of NN-type*, and $\Pi_I \stackrel{\mathrm{def}}{=} \Pi_\mathcal{G} \stackrel{\mathrm{out}}{\rtimes} I$.

Lemma 3.1 (**Contagious conditions**). *Let* (**C**) *be a condition on an element of* $\mathrm{Vert}(\widetilde{\mathcal{G}})$ *which satisfies the following property* $(*)$*:*

> $(*)$: *Let* $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *be such that* $\delta(\widetilde{v}_1(\mathcal{G}), \widetilde{v}_2(\mathcal{G})) \leq 1$*. Then* $\widetilde{v}_1$ *satisfies the condition* (**C**) *if and only if* $\widetilde{v}_2$ *satisfies the condition* (**C**)*.*

Suppose that there exists an element of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ *which satisfies the condition* (**C**)*. Then every element of* $\mathrm{Vert}(\widetilde{\mathcal{G}})$ *satisfies the condition* (**C**)*.*

Proof. This follows immediately from the *connectedness* of the underlying semi-graph of a semi-graph of anabelioids of PSC-type. □

Lemma 3.2 (**Verticial decompositions inside ab/(edge+iner)-quotients**). *Let* $\Pi_I^{\mathrm{ab/edge}}$ *be the quotient of the abelianization* Π_I^{ab} *by the closed subgroup generated by the images in* Π_I^{ab} *of the edge-like subgroups of* $\Pi_\mathcal{G}$*. Suppose that* ρ_I *is* **of SNN-type***. Then the following hold:*

(i) *For* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*, write* $M_{\widetilde{v}}$ *for the image of the composite* $I_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow \Pi_I^{\mathrm{ab/edge}}$*. Then the closed subgroup* $M_{\widetilde{v}} \subseteq \Pi_I^{\mathrm{ab/edge}}$ *is*

independent *of the choice of the element* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. *Denote this closed subgroup by* M. *In the following, we shall write*

$$\Pi_I^{\mathrm{ab/(edge+iner)}} \stackrel{\mathrm{def}}{=} \Pi_I^{\mathrm{ab/edge}}/M\,.$$

(ii) *The composite of the injection of Lemma 1.4 with the natural inclusion* $\Pi_{\mathcal{G}} \hookrightarrow \Pi_I$ *induces a* **split injection** *(cf. the discussion entitled "Topological groups" in §0)*

$$\bigoplus_{v \in \mathrm{Vert}(\mathcal{G})} \Pi_v^{\mathrm{ab/edge}} \hookrightarrow \Pi_I^{\mathrm{ab/(edge+iner)}}$$

(cf. Definition 1.3, (i)) whose image is a free $\widehat{\mathbb{Z}}^{\Sigma}$*-module of finite rank.*

Proof. First, we verify assertion (i). If $\mathrm{Node}(\mathcal{G}) = \emptyset$, then assertion (i) is immediate; thus, assume that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Next, let us fix an element $\widetilde{v}_0 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. For $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, we shall say that $\widetilde{v}$ satisfies the condition $(*^{\mathrm{triv}})$ if the image of $I_{\widetilde{v}}$ in the quotient $\Pi_I^{\mathrm{ab/edge}}/M_{\widetilde{v}_0}$ is *trivial*. To verify assertion (i), it is immediate that it suffices to show that any $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfies $(*^{\mathrm{triv}})$. Therefore, to verify assertion (i), it follows from Lemma 3.1 that it suffices to show that the condition $(*^{\mathrm{triv}})$ satisfies the property $(*)$ in the statement of Lemma 3.1. To this end, let $\widetilde{v}$, $\widetilde{v}' \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be such that $\delta(\widetilde{v}(\mathcal{G}), \widetilde{v}'(\mathcal{G})) \leq 1$, and $\widetilde{v}$ satisfies $(*^{\mathrm{triv}})$. Let $D_e \subseteq \Pi_I$ be a decomposition subgroup associated to $e \in \mathcal{N}(\widetilde{v}(\mathcal{G})) \cap \mathcal{N}(\widetilde{v}'(\mathcal{G}))$. Then since the image of $I_{\widetilde{v}}$ in $\Pi_I^{\mathrm{ab/edge}}/M_{\widetilde{v}_0}$ is *trivial*, and D_e is generated by an edge-like subgroup and a conjugate of $I_{\widetilde{v}}$ (cf. Remark 2.7.1), it follows that the image of D_e in $\Pi_I^{\mathrm{ab/edge}}/M_{\widetilde{v}_0}$ is *trivial*. Therefore, since there exists a conjugate of $I_{\widetilde{v}'}$ contained in D_e, we conclude that the image of $I_{\widetilde{v}'}$ in $\Pi_I^{\mathrm{ab/edge}}/M_{\widetilde{v}_0}$ is *trivial*; in particular, $\widetilde{v}'$ satisfies $(*^{\mathrm{triv}})$. This completes the proof of assertion (i).

Finally, we observe that assertion (ii) follows from a similar argument involving coverings — this time of $\mathcal{G}[\rho_I]$ (cf. Definition 2.8) as opposed to $\mathcal{G}$ — to the argument applied in the proof of Lemma 1.4. □

Remark 3.2.1. Suppose that ρ_I is *of SNN-type*. Let $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then it follows immediately from Remark 1.4.1, (ii); Lemma 3.2 that the following assertion holds:

If $\widetilde{v}_1(\mathcal{G}) \neq \widetilde{v}_2(\mathcal{G})$, then the image of the intersection

$$(I_{\widetilde{v}_1} \cdot D_{\widetilde{v}_2}) \cap \Pi_{\widetilde{v}_1} \subseteq \Pi_{\mathcal{G}}$$

in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is *trivial*.

Indeed, it follows from Lemma 3.2, (i); Remark 2.7.1, together with the various definitions involved, that the image of $I_{\widetilde{v}_1}$ (respectively, $D_{\widetilde{v}_2}$) in $\Pi_I^{\mathrm{ab/(edge+iner)}}$ is trivial (respectively, coincides with the image of $\Pi_{\widetilde{v}_2} \subseteq \Pi_{\mathcal{G}}$). But, by Lemmas 1.4; 3.2, (ii), this implies that the image of $(I_{\widetilde{v}_1} \cdot D_{\widetilde{v}_2}) \cap \Pi_{\widetilde{v}_1}$ in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is contained in the intersection of the images of $\Pi_{\widetilde{v}_2}$ and $\Pi_{\widetilde{v}_1}$ in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$. Therefore, the above assertion follows from Remark 1.4.1, (ii).

Remark 3.2.2. In fact, it is not difficult to verify that both the statement and the proof of Lemma 3.2 remain valid even under the *weaker assumption* that ρ_I is *of SVA-type*.

Lemma 3.3 (**Submodules of free $\mathbb{Z}_l$-modules**)**.** *Let l be a prime number, r a positive integer; also, for $1 \leq j \leq r$, let $c_j \in \mathbb{Z}_l \setminus \{0\}$. For $1 \leq i \leq l$, $1 \leq j \leq r$, set $M_{i,j} \stackrel{\mathrm{def}}{=} \mathbb{Z}_l$, $M_0 \stackrel{\mathrm{def}}{=} \mathbb{Z}_l$; write $\iota_{i,j} \in M_{i,j}$, $\iota_0 \in M_0$ for the generators corresponding to the element "1". Next, let us write $M_{\mathrm{diag}} \subseteq \bigoplus_{i,j} M_{i,j}$ for the submodule obtained as the image of the diagonal homomorphism $\mathbb{Z}_l \hookrightarrow \bigoplus_{i,j} M_{i,j}$,*

$$N \stackrel{\mathrm{def}}{=} \left\{ \Big(\bigoplus_{i,j} M_{i,j} \Big) / M_{\mathrm{diag}} \right\} \oplus M_0$$

and regard M_0 as a submodule of N via the inclusion $M_0 \simeq 0 \oplus M_0 \hookrightarrow N$. Then if we denote by H the submodule of N generated by the elements of N determined by the $(lr+1)$-tuples of the form

$$(0, \cdots, 0, c_j \cdot \iota_{i,j}, 0, \cdots, 0, \iota_0)$$

— where (i,j) ranges over pairs of natural numbers such that $1 \leq i \leq l$, $1 \leq j \leq r$ — then $H \cap M_0 \subseteq l \cdot M_0$.

Proof. Suppose that the element $h \in H$ determined by

$$\sum_{i,j} d_{i,j}(0, \cdots, 0, c_j \cdot \iota_{i,j}, 0, \cdots, 0, \iota_0)$$

$$= (\cdots, d_{i,j}c_j \cdot \iota_{i,j}, \cdots, \sum_{i,j} d_{i,j} \cdot \iota_0) \in \Big(\bigoplus_{i,j} M_{i,j} \Big) \oplus M_0$$

— where $d_{i,j} \in \mathbb{Z}_l$ — is contained in M_0. Now let us observe that the homomorphism ϕ

$$\begin{array}{ccc} N = \left\{ \Big(\bigoplus_{i,j} M_{i,j} \Big) / M_{\mathrm{diag}} \right\} \oplus M_0 & \longrightarrow & \Big(\bigoplus_{(i,j) \neq (1,1)} M_{i,j} \Big) \oplus M_0 \\ ([(\lambda_{i,j})_{i,j}], \lambda_0) & \mapsto & ((\lambda_{i,j} - \lambda_{1,1})_{(i,j) \neq (1,1)}, \lambda_0) \end{array}$$

— where we write "[?]" for the image of "?" in the module "$\{-\}$", and "$\lambda_{(-)}$" is an element of "$M_{(-)}$" — is an *isomorphism*. Thus, by applying ϕ to $h \in H \cap M_0$, we conclude that $d_{i,j}c_j - d_{1,1}c_1 = 0$, for $1 \leq i \leq l$, $1 \leq j \leq r$; in particular, it follows that $d_{i,j}c_j$ is *independent* of the pair (i,j), hence that $(d_{i,j} - d_{i',j})c_j = 0$. But, since $c_j \neq 0$, this implies that $d_{i,j}$ is *independent* of i, hence — since i ranges over the integers from 1 to l — that $\sum_{i,j} d_{i,j} \in l \cdot \mathbb{Z}_l$, as desired. $\square$

Lemma 3.4 (**Existence of certain coverings).** *Suppose that the following two conditions are satisfied:*

(a) ρ_I *is* **of SNN-type**.

(b) $\mathcal{G}$ *is* **sturdy** *and* **untangled** *(cf. Definition 1.2).*

If, by abuse of notation, we write $\mathbb{G}$ *for the underlying semi-graph of* $\mathcal{G}[\rho_I]$ *(cf. Definition 2.8), then, for a vertex* v *(respectively, an edge* e*) of* $\mathbb{G}$*, let us write* $\mathcal{D}_v$ *(respectively,* $\mathcal{D}_e$*) for the connected anabelioid corresponding to* v *(respectively,* e*), and* $\Pi_{\mathcal{D}_v}$ *(respectively,* $\Pi_{\mathcal{D}_e}$*) for the fundamental group of the connected anabelioid* $\mathcal{D}_v$ *(respectively,* $\mathcal{D}_e$*) [so it follows from the definition of* $\mathcal{G}[\rho_I]$ *that* $\Pi_{\mathcal{D}_v}$, $\Pi_{\mathcal{D}_e}$ *are naturally isomorphic, up to inner automorphism, to* D_v, D_e*, respectively]. Fix a vertex* $v_0 \in \mathrm{Vert}(\mathcal{G})$*. Then there exists a* **connected covering** *of semi-graphs of anabelioids (cf.* [Mzk3]*, Definition 2.2, (i))*

$$\mathcal{H} \longrightarrow \mathcal{G}[\rho_I]$$

of $\mathcal{G}[\rho_I]$ *such that if we denote the underlying semi-graph of* $\mathcal{H}$ *by* $\mathbb{H}$ *and use analogous notation for* $\mathcal{H}$ *to the notation introduced above for* $\mathcal{G}[\rho_I]$*, then the following conditions are satisfied:*

(1) *The set of vertices of* $\mathbb{H}$ *which lie over* v_0 *consists of* **precisely one** *element* w_0*, and the image of the outer injection* $\Pi_{\mathcal{D}_{w_0}} \hookrightarrow \Pi_{\mathcal{D}_{v_0}} \simeq D_{v_0}$ *induced by the morphism* $\mathcal{D}_{w_0} \to \mathcal{D}_{v_0}$ **does not contain** *the normal subgroup* $I_{v_0} \subseteq D_{v_0}$*, i.e., "*$I_{v_0} \not\subseteq \Pi_{\mathcal{D}_{w_0}}$*".*

(2) *For any* $v_1 \in \mathrm{Vert}(\mathcal{G})$ *such that* $\delta(v_0, v_1) = 1$*, the set of vertices of* $\mathbb{H}$ *which lie over* v_1 *consists of* **precisely one** *element* w_1*, and the image of the outer injection* $\Pi_{\mathcal{D}_{w_1}} \hookrightarrow \Pi_{\mathcal{D}_{v_1}} \simeq D_{v_1}$ *induced by the morphism* $\mathcal{D}_{w_1} \to \mathcal{D}_{v_1}$ **contains** *the normal subgroup* $I_{v_1} \subseteq D_{v_1}$*, i.e., "*$I_{v_1} \subseteq \Pi_{\mathcal{D}_{w_1}}$*".*

(3) *For any* $v \in \mathrm{Vert}(\mathcal{G})$ *such that* $\delta(v_0, v) \geq 2$*, and any vertex* w *of* $\mathbb{H}$ *which lies over* v*, the morphism* $\mathcal{D}_w \to \mathcal{D}_v$ *is an* **isomorphism**.

(4) *For any* $e \in \mathrm{Node}(\mathcal{G})$ *such that* $v_0 \notin \mathcal{V}(e)$*, and any closed edge* f *of* $\mathbb{H}$ *which lies over* e*, the morphism* $\mathcal{D}_f \to \mathcal{D}_e$ *is an* **isomorphism**.

(5) *For any* $e \in \mathrm{Cusp}(\mathcal{G})$*, and any open edge* f *of* $\mathbb{H}$ *which lies over* e*, the image of the outer injection* $\Pi_{\mathcal{D}_f} \hookrightarrow \Pi_{\mathcal{D}_e} \simeq D_e$ *induced by the morphism* $\mathcal{D}_f \to \mathcal{D}_e$ **contains** *the normal subgroup* $\Pi_e \subseteq D_e$*, i.e., "*$\Pi_e \subseteq \Pi_{\mathcal{D}_f}$*".*

Proof. To verify Lemma 3.4, by replacing $\mathcal{G}$ by the compactification of $\mathcal{G}$ (cf. Definition 1.11), we may assume without loss of generality that $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, and hence that condition (5) is satisfied automatically. Moreover, by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of the various pro-Σ groups involved, to verify Lemma 3.4, we may assume without loss of generality that $\Sigma = \{l\}$.

Write

$$\mathcal{V}_{\leq 1} \subseteq \mathrm{Vert}(\mathcal{G})$$

for the set of vertices v of $\mathcal{G}$ such that $\delta(v_0, v) \leq 1$,

$$\mathcal{V}_{=1} \subseteq \mathcal{V}_{\leq 1}$$

for the set of vertices v of $\mathcal{G}$ such that $\delta(v_0, v) = 1$ (i.e., $\mathcal{V}_{=1} = \mathcal{V}_{\leq 1} \setminus \{v_0\}$),

$$\mathcal{N}_0 \subseteq \mathrm{Node}(\mathcal{G})$$

for the set of nodes of $\mathcal{G}$ which abut to v_0 (i.e., $\mathcal{N}_0 \stackrel{\mathrm{def}}{=} \mathcal{N}(v_0)$), and

$$\mathcal{N}_1 \subseteq \mathrm{Node}(\mathcal{G})$$

for the set of nodes e of $\mathcal{G}$ such that $\mathcal{V}(e) \cap \mathcal{V}_{=1} \neq \emptyset$ and $v_0 \notin \mathcal{V}(e)$. Then we *claim* that there exists a connected finite étale covering of semi-graphs of anabelioids $\mathcal{F} \to \mathcal{G}[\rho_I]$ of $\mathcal{G}[\rho_I]$ such that if we denote the underlying semi-graph of $\mathcal{F}$ by $\mathbb{F}$ and use analogous notation for $\mathcal{F}$ to the notation introduced in the statement of Lemma 3.4, then the following conditions are satisfied:

(i) The connected finite étale covering of semi-graphs of anabelioids $\mathcal{F} \to \mathcal{G}[\rho_I]$ of $\mathcal{G}[\rho_I]$ is *Galois*, and its Galois group is isomorphic to $\mathbb{Z}/l\mathbb{Z}$.

(ii) For any $v \in \mathcal{V}_{\leq 1}$, the set of vertices of $\mathbb{F}$ which lie over v consists of *precisely one* element u, and the image of the outer injection $\Pi_{\mathcal{D}_u} \hookrightarrow \Pi_{\mathcal{D}_v} \simeq D_v$ induced by the morphism $\mathcal{D}_u \to \mathcal{D}_v$ *contains* the normal subgroup $I_v \subseteq D_v$, i.e., "$I_v \subseteq \Pi_{\mathcal{D}_u}$".

(iii) For any $v \in \mathrm{Vert}(\mathcal{G})$ such that $\delta(v_0, v) \geq 2$, and any vertex u of $\mathbb{F}$ which lies over v, the morphism $\mathcal{D}_u \to \mathcal{D}_v$ is an *isomorphism*.

(iv) For any $e \in \mathrm{Node}(\mathcal{G})$, and any edge h of $\mathbb{F}$ which lies over e, the morphism $\mathcal{D}_h \to \mathcal{D}_e$ is an *isomorphism*.

Indeed, since $\mathcal{G}$ is *sturdy* (cf. condition (b)), it follows that $\Pi_v^{\mathrm{ab/edge}} \neq \{1\}$ (cf. Definition 1.3, (iii)) for any $v \in \mathrm{Vert}(\mathcal{G})$. Thus, the above *claim* follows immediately from the existence of the natural *split injection*

$$\bigoplus_{v \in \mathrm{Vert}(\mathcal{G})} \Pi_v^{\mathrm{ab/edge}} \hookrightarrow \Pi_I^{\mathrm{ab/(edge+iner)}}$$

of Lemma 3.2, (ii).

In light of the above *claim*, to complete the proof of Lemma 3.4, we may *replace* $\mathcal{G}[\rho_I]$ by $\mathcal{F}$ and assume in the following that

($*_1$) there exists an action of a group Φ isomorphic to $\mathbb{Z}/l\mathbb{Z}$ on $\mathcal{G}[\rho_I]$ such that the induced action of Φ on

$\mathrm{Vert}(\mathcal{G})$ *fixes every element of* $\mathcal{V}_{\leq 1}$, and the induced action of Φ on $\mathcal{N}_0$ is *free*.

Now let $\Pi_{v_0} \subseteq \Pi_{\mathcal{G}}$ be a verticial subgroup associated to v_0; write $D_{v_0} \subseteq \Pi_I$ (respectively, $I_{v_0} \subseteq \Pi_I$) for the decomposition (respectively, inertia) subgroup associated to Π_{v_0}. Next, for $e \in \mathcal{N}_0$, write v_e for the *unique* element of $\mathcal{V}(e) \setminus \{v_0\} \subseteq \mathcal{V}_{=1}$ (cf. assumption (b)); let $\Pi_e \subseteq \Pi_{\mathcal{G}}$ be an edge-like subgroup associated to e such that $\Pi_e \subseteq \Pi_{v_0}$; write I_e for the inertia subgroup associated to Π_e. Next, let $I_{v_e} \subseteq \Pi_I$ be an inertia subgroup associated to v_e such that $I_{v_e} \subseteq I_e$. (Here, we note that it is easily verified that such an I_{v_e} exists.) Thus, $I_e = \Pi_e \times I_{v_0} \subseteq \Pi_{v_0} \times I_{v_0} = D_{v_0}$ (cf. Remark 2.7.1); in particular, $I_{v_e} \subseteq D_{v_0}$.

Next, write H for the $\mathbb{Z}_l$-submodule of the free $\mathbb{Z}_l$-module $D_{v_0}^{\mathrm{ab}}$ ($\simeq \Pi_{v_0}^{\mathrm{ab}} \times I_{v_0}$) generated by the images of the composite homomorphisms

$$I_{v_e} \hookrightarrow D_{v_0} \twoheadrightarrow D_{v_0}^{\mathrm{ab}}$$

— where e ranges over elements of $\mathcal{N}_0$. Then we *claim* that

$$(*_2)\ H \cap \mathrm{Im}(I_{v_0}) \subseteq l \cdot \mathrm{Im}(I_{v_0})$$

— where we write $\mathrm{Im}(I_{v_0})$ for the image of the composite $I_{v_0} \hookrightarrow D_{v_0} \twoheadrightarrow D_{v_0}^{\mathrm{ab}}$. Indeed, by the well-known structure of the maximal pro-l quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\neq l$, there exists a topological generator $\iota_e \in \Pi_e$ of Π_e such that the inclusions $I_{v_0} \hookrightarrow D_{v_0}^{\mathrm{ab}}$ and $\Pi_e \hookrightarrow D_{v_0}^{\mathrm{ab}}$ determine a *split injection*

$$\Big\{\Big(\bigoplus_{e \in \mathcal{N}_0} \Pi_e\Big)/\mathbb{Z}_l \cdot (\iota_e)_{e \in \mathcal{N}_0}\Big\} \oplus I_{v_0} \hookrightarrow D_{v_0}^{\mathrm{ab}}$$

into $D_{v_0}^{\mathrm{ab}}$. Now let us fix a topological generator $\iota_{v_0} \in I_{v_0}$ and denote by $\iota_{v_e} \in I_{v_e}$ the topological generator of I_{v_e} obtained as the image of $\iota_{v_0} \in I_{v_0}$ via the composite isomorphism $I_{v_0} \xrightarrow{\sim} I \xleftarrow{\sim} I_{v_e}$ (cf. Definition 2.4, (2′)). Then it follows from condition (3) of Definition 2.4 that the natural inclusions $I_{v_0}, I_{v_e} \hookrightarrow I_e$ determine an *open* subgroup $I_{v_0} \times I_{v_e} \subseteq I_e$ (cf. condition (a)); in particular, there exists an element $c_{v_e} \in \mathbb{Z}_l \setminus \{0\}$ such that $\iota_{v_e} = c_{v_e}\iota_e + \iota_{v_0}$. Moreover, since we have an action of Φ on $\mathcal{G}[\rho_I]$ as in $(*_1)$, we obtain, for any $e \in \mathcal{N}_0$ and $\sigma \in \Phi$, that $c_{v_e} = c_{v_{e^\sigma}}$. Therefore, $(*_2)$ follows immediately from Lemma 3.3.

In light of $(*_2)$, there exists an open subgroup $H' \subseteq D_{v_0}^{\mathrm{ab}}$ such that $H \subseteq H'$ and $\mathrm{Im}(I_{v_0}) \not\subseteq H'$. Thus, since H is stabilized by the action of Φ on $D_{v_0}^{\mathrm{ab}}$, it follows (for instance, by replacing H' by the intersection of the translates of H' by the action of Φ) that we may assume that H' is stabilized by the action of Φ on $D_{v_0}^{\mathrm{ab}}$. Write $D_{w_0} \subseteq D_{v_0}$ for the inverse image of $H' \subseteq D_{v_0}^{\mathrm{ab}}$ via the natural surjection $D_{v_0} \twoheadrightarrow D_{v_0}^{\mathrm{ab}}$. Then it follows immediately from the definition of D_{w_0} that the following hold:

(v) D_{w_0} is *open* and *normal* in D_{v_0}, and, moreover, D_{w_0} is *stabilized* by the induced outer action of Φ on D_{v_0}.

(vi) For any $e \in \mathcal{N}_0$, we have $I_{v_e} \subseteq D_{w_0}$; in particular, by (v), for any $e \in \mathcal{N}_0$, every D_{v_0}-conjugate of I_{v_e} is contained in D_{w_0}.

(vii) $I_{v_0} \nsubseteq D_{w_0}$.

Write $\mathcal{D}_{w_0} \to \mathcal{D}_{v_0}$ for the connected finite étale covering of anabelioids corresponding to the open subgroup $D_{w_0} \subseteq D_{v_0}$ of D_{v_0};

$$\mathbb{G}^{\mathrm{sub}}$$

for the connected sub-semi-graph of $\mathbb{G}$ whose set of vertices is $\mathcal{V}_{\leq 1} \subseteq \mathrm{Vert}(\mathcal{G})$, and whose set of edges is $\mathcal{N}_0 \cup \mathcal{N}_1$; and

$$\mathcal{G}[\rho_I]^{\mathrm{sub}}$$

for the semi-graph of anabelioids determined by restricting $\mathcal{G}[\rho_I]$ to $\mathbb{G}^{\mathrm{sub}}$ (cf. the discussion preceding [Mzk3], Definition 2.2). Then since we have an action of Φ on $\mathcal{G}[\rho_I]$ as in $(*_1)$, it follows from (v) that for any $e \in \mathcal{N}_0$ and $\sigma \in \Phi$, the ramification indices of this covering $\mathcal{D}_{w_0} \to \mathcal{D}_{v_0}$ at the cusps of $\mathcal{D}_{v_0}$ determined by e and e^σ *coincide*. Thus, it follows from (vi) (together with the elementary fact that there exist $l-1$ elements $a_i \in \mathbb{Z}$ — where $1 \leq i \leq l-1$ — such that the a_i's and $\sum_{i=1}^{l-1} a_i$ are *prime* to l) that one may *extend* this covering $\mathcal{D}_{w_0} \to \mathcal{D}_{v_0}$ to a connected finite étale covering $\mathcal{H}^{\mathrm{sub}} \to \mathcal{G}[\rho_I]^{\mathrm{sub}}$ which satisfies the following conditions:

(viii) The set of vertices of $\mathbb{H}^{\mathrm{sub}}$ (i.e., the underlying semi-graph of $\mathcal{H}^{\mathrm{sub}}$) which lie over an element of $\mathcal{V}_{\leq 1}$ consists of *precisely one* element.

(ix) For any $e \in \mathcal{N}_0$, if we denote by w_e the — necessarily unique (cf. (viii)) — vertex of $\mathbb{H}^{\mathrm{sub}}$ which lies over $v_e \in \mathcal{V}_{=1}$, by $\mathcal{D}_{w_e}$ the anabelioid corresponding to w_e, and by $\Pi_{\mathcal{D}_{w_e}}$ the fundamental group of $\mathcal{D}_{w_e}$, then the image of the outer injection $\Pi_{\mathcal{D}_{w_e}} \hookrightarrow \Pi_{\mathcal{D}_{v_e}} \simeq D_{v_e}$ *contains* the normal subgroup $I_{v_e} \subseteq D_{v_e}$.

(x) $\mathcal{H}^{\mathrm{sub}} \to \mathcal{G}[\rho_I]^{\mathrm{sub}}$ restricts to the *trivial* covering over every edge corresponding to an element of $\mathcal{N}_1$.

Moreover, it follows immediately from (x) that one may extend the covering $\mathcal{H}^{\mathrm{sub}} \to \mathcal{G}[\rho_I]^{\mathrm{sub}}$ obtained above to a connected finite étale covering $\mathcal{H} \to \mathcal{G}[\rho_I]$ of $\mathcal{G}[\rho_I]$ such that

(xi) $\mathcal{H} \to \mathcal{G}[\rho_I]$ restricts to the *trivial* covering over the vertices v of $\mathcal{G}$ such that $\delta(v_0, v) \geq 2$.

Now by (vii) and (viii) (respectively, (viii) and (ix); (xi); (x) and (xi)), this covering $\mathcal{H} \to \mathcal{G}[\rho_I]$ satisfies condition (1) (respectively, (2); (3); (4)). This completes the proof of Lemma 3.4. $\square$

Remark 3.4.1. In light of the isomorphism of Lemma 2.9, the content of Lemma 3.4 admits the following interpretation:

> Suppose that ρ_I is *of SNN-type*, and that $\mathcal{G}$ is *sturdy* and *untangled*. Let $\widetilde{v}_0 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then there exists an open subgroup $\Pi \subseteq \Pi_I$ of Π_I which satisfies the following conditions:
>
> (i) If $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfies $\delta(\widetilde{v}_0(\mathcal{G}), \widetilde{v}(\mathcal{G})) = 0$ (i.e., $v_0(\mathcal{G}) = v(\mathcal{G})$), then $I_{\widetilde{v}} \not\subseteq \Pi$.
>
> (ii) If $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfies $\delta(\widetilde{v}_0(\mathcal{G}), \widetilde{v}(\mathcal{G})) = 1$, then $I_{\widetilde{v}} \subseteq \Pi$.
>
> (iii) If $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ satisfies $\delta(\widetilde{v}_0(\mathcal{G}), \widetilde{v}(\mathcal{G})) \geq 2$, then $D_{\widetilde{v}} \subseteq \Pi$.
>
> (iv) If $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ satisfies $\widetilde{e}(\mathcal{G}) \notin \mathcal{E}(\widetilde{v}_0(\mathcal{G}))$, then $D_{\widetilde{e}} \subseteq \Pi$.
>
> (v) If $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$, then $\Pi_{\widetilde{e}} \subseteq \Pi$.

Remark 3.4.2. Let $\Pi \subseteq \Pi_I$ be the open subgroup of Remark 3.4.1. Then the following assertion holds:

> For $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, consider the following conditions:
>
> (i) $\widetilde{e} \in \mathcal{N}(\widetilde{v}_0)$.
>
> (i′) $\widetilde{e}(\mathcal{G}) \in \mathcal{N}(\widetilde{v}_0(\mathcal{G}))$.
>
> (ii) $\Pi_{\widetilde{e}} \not\subseteq \Pi$.
>
> (ii′) $\exists\, \gamma \in \Pi_{\mathcal{G}}$ such that $\gamma \cdot \Pi_{\widetilde{e}} \cdot \gamma^{-1} \not\subseteq \Pi$.
>
> Then
> $$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(ii}'\text{)} \Longleftrightarrow \text{(i}'\text{)}\,.$$

Indeed, if condition (i) is satisfied, but condition (ii) is *not* satisfied, then it follows from condition (ii) in Remark 3.4.1 that $I_{\widetilde{e}} = I_{\widetilde{v}} \cdot \Pi_{\widetilde{e}} \subseteq \Pi$ (cf. Remark 2.7.1), where we write $\widetilde{v}$ for the unique element of $\mathcal{V}(\widetilde{e}) \setminus \{\widetilde{v}_0\}$; thus, since $I_{\widetilde{v}_0} \subseteq I_{\widetilde{e}}$, we obtain that $I_{\widetilde{v}_0} \subseteq \Pi$ — in contradiction to condition (i) in Remark 3.4.1. This completes the proof of the implication
$$\text{(i)} \Longrightarrow \text{(ii)}\,.$$
The implication
$$\text{(ii)} \Longrightarrow \text{(ii}'\text{)}$$
is immediate. Next, if condition (i′) is *not* satisfied, then by applying condition (iv) in Remark 3.4.1 to the $\Pi_{\mathcal{G}}$-conjugates of $\widetilde{e}$, we conclude (since $\Pi_{\widetilde{e}} \subseteq D_{\widetilde{e}}$) that condition (ii′) is *not* satisfied. This completes the proof of the implication
$$\text{(ii}'\text{)} \Longrightarrow \text{(i}'\text{)}\,.$$

Finally, by applying the implication "(i) $\Rightarrow$ (ii)" to a suitable $\Pi_{\mathcal{G}}$-conjugate of $\widetilde{e}$, we obtain the implication

$$(\mathrm{i}') \Longrightarrow (\mathrm{ii}') .$$

Proposition 3.5 (**Graph-theoretic geometry via inertia subgroups).** *Let* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*,* $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$*. Then the following conditions are* **equivalent***:*

(i) $\widetilde{v} \in \mathcal{V}(\widetilde{e})$.

(ii) $I_{\widetilde{v}} \cap D_{\widetilde{e}} \neq \{1\}$.

In particular, if $I_{\widetilde{v}} \cap D_{\widetilde{e}} \neq \{1\}$*, then* $I_{\widetilde{v}} \subseteq D_{\widetilde{e}}$*.*

Proof. The implication

$$(\mathrm{i}) \Longrightarrow (\mathrm{ii})$$

is immediate from the various definitions involved; thus, to complete the proof of Proposition 3.5, it suffices to verify the *implication*

$$(\mathrm{ii}) \Longrightarrow (\mathrm{i}) .$$

To this end, let us assume that condition (ii) is satisfied. Then since $I_{\widetilde{v}}$ is *torsion-free* (cf. Lemma 2.5, (i)), to verify condition (i), by replacing Π_I by an open subgroup of Π_I, we may assume without loss of generality that ρ_I is *of SNN-type*, and that $\mathcal{G}$ is *sturdy* (cf. [Mzk4], Remark 1.1.5) and *untangled* (cf. Remark 1.2.1, (i)); moreover, by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved, to verify condition (i), we may assume without loss of generality that $\Sigma = \{l\}$. On the other hand, since $I_{\widetilde{v}}$ is isomorphic to $\mathbb{Z}_l$ as an abstract profinite group (cf. Lemma 2.5, (i)), by replacing I by an open subgroup of I, to verify condition (i), we may assume without loss of generality that $I_{\widetilde{v}} \subseteq D_{\widetilde{e}}$.

Assume that $\widetilde{v} \notin \mathcal{V}(\widetilde{e})$, i.e., that condition (i) is *not* satisfied. Then by applying Remark 3.4.1, where we take "$\widetilde{v}_0$" to be $\widetilde{v}$, there exists an open subgroup $\Pi \subseteq \Pi_I$ such that $I_{\widetilde{v}} \not\subseteq \Pi$ (cf. condition (i) in Remark 3.4.1), and, moreover, $D_{\widetilde{e}} \subseteq \Pi$ (cf. condition (iv) in Remark 3.4.1); in particular, $I_{\widetilde{v}} \not\subseteq D_{\widetilde{e}}$ — in *contradiction* to our assumption that $I_{\widetilde{v}} \subseteq D_{\widetilde{e}}$. This completes the proof of the *implication* in question. $\square$

Remark 3.5.1. Let $\widetilde{v}_1$, $\widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then it follows immediately from Proposition 3.5 that the following assertion holds:

$$\text{If } I_{\widetilde{v}_1} \cap I_{\widetilde{v}_2} \neq \{1\}, \text{ then } \mathcal{N}(\widetilde{v}_1) = \mathcal{N}(\widetilde{v}_2).$$

Indeed, suppose that $I_{\widetilde{v}_1} \cap I_{\widetilde{v}_2} \neq \{1\}$. Now if $\widetilde{e} \in \mathcal{N}(\widetilde{v}_1)$, then it follows from Proposition 3.5 that $I_{\widetilde{v}_1} \subseteq D_{\widetilde{e}}$; thus, since $I_{\widetilde{v}_1} \cap I_{\widetilde{v}_2} \neq \{1\}$, it follows that $I_{\widetilde{v}_2} \cap D_{\widetilde{e}} \neq \{1\}$. In particular, again by Proposition 3.5, we obtain that $\widetilde{e} \in \mathcal{N}(\widetilde{v}_2)$. This completes the proof of the above assertion.

In particular, it follows from Remark 1.8.1, (ii), that the following assertion holds:

$$\widetilde{v}_1 = \widetilde{v}_2 \text{ if and only if } I_{\widetilde{v}_1} \cap I_{\widetilde{v}_2} \neq \{1\}.$$

Lemma 3.6 (**Centralizers, normalizers, and commensurators of verticial inertia subgroups**). *Let* $J \subseteq I_{\widetilde{v}}$ *be a* **nontrivial** *closed subgroup of* $I_{\widetilde{v}}$*, where* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*. Then the following hold:*

(i) $\Pi_{\widetilde{v}} = Z_{\Pi_I}(J) \cap \Pi_{\mathcal{G}} = N_{\Pi_I}(J) \cap \Pi_{\mathcal{G}} = C_{\Pi_I}(J) \cap \Pi_{\mathcal{G}}$.

(ii) *If* ρ_I *is* **of SNN-type**, *then* $D_{\widetilde{v}} = Z_{\Pi_I}(J) = N_{\Pi_I}(J) = C_{\Pi_I}(J)$.

Proof. First, we prove assertion (i). If $\mathrm{Node}(\mathcal{G}) = \emptyset$, then assertion (i) is immediate from the various definitions involved; thus, assume that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Since it is immediate that $\Pi_{\widetilde{v}} \subseteq Z_{\Pi_I}(J) \cap \Pi_{\mathcal{G}}$, to prove assertion (i), it suffices to verify that $C_{\Pi_I}(J) \cap \Pi_{\mathcal{G}} \subseteq \Pi_{\widetilde{v}}$. To this end, let us assume that $(C_{\Pi_I}(J) \cap \Pi_{\mathcal{G}}) \setminus \Pi_{\widetilde{v}} \neq \emptyset$ (where "$\setminus$" denotes the set-theoretic complement). Let $\gamma \in (C_{\Pi_I}(J) \cap \Pi_{\mathcal{G}}) \setminus \Pi_{\widetilde{v}}$; write $\widetilde{v}^{\gamma}$ for the element of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ that corresponds to the verticial subgroup $\gamma \cdot \Pi_{\widetilde{v}} \cdot \gamma^{-1} \subseteq \Pi_{\mathcal{G}}$. Then since $\gamma \notin \Pi_{\widetilde{v}}$, it follows from the *commensurable terminality* of $\Pi_{\widetilde{v}}$ in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)) that $\Pi_{\widetilde{v}} \neq \gamma \cdot \Pi_{\widetilde{v}} \cdot \gamma^{-1}$; in particular, it follows that $\widetilde{v} \neq \widetilde{v}^{\gamma}$. On the other hand, since $\gamma \in C_{\Pi_I}(J)$, it follows that $J \cap (\gamma \cdot J \cdot \gamma^{-1}) \neq \{1\}$; thus, it follows from Remark 3.5.1 that $\widetilde{v} = \widetilde{v}^{\gamma}$ — a contradiction. This completes the proof of assertion (i).

Next, we prove assertion (ii). Since ρ_I is *of SNN-type*, it follows from Remark 2.7.1 (cf. also Lemma 2.5, (i)) that $D_{\widetilde{v}} \subseteq Z_{\Pi_I}(I_{\widetilde{v}}) \subseteq Z_{\Pi_I}(J)$, and that the composite $D_{\widetilde{v}} \subseteq \Pi_I \twoheadrightarrow I$ is *surjective*. Thus, assertion (ii) follows from assertion (i), together with Remark 2.7.1. □

Lemma 3.7 (**Centralizers, normalizers, and commensurators of edge-like inertia subgroups**). *Let* $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$*. Then the following hold:*

(i) $\Pi_{\widetilde{e}} = Z_{\Pi_I}(I_{\widetilde{e}}) \cap \Pi_{\mathcal{G}} = N_{\Pi_I}(I_{\widetilde{e}}) \cap \Pi_{\mathcal{G}} = C_{\Pi_I}(I_{\widetilde{e}}) \cap \Pi_{\mathcal{G}}$.

(ii) *If* ρ_I *is* **of SNN-type**, *then* $D_{\widetilde{e}} = Z_{\Pi_I}(I_{\widetilde{e}}) = N_{\Pi_I}(I_{\widetilde{e}}) = C_{\Pi_I}(I_{\widetilde{e}})$.

Proof. Assertion (i) in the case where $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ follows from the *commensurable terminality* of $\Pi_{\widetilde{e}}$ in $\Pi_{\mathcal{G}}$ (cf. [Mzk4], Proposition 1.2, (ii)), together with the definition of an inertia subgroup of a *cusp*. Assertion (i) in the case where $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ follows from a similar argument to the argument used in the proof of Lemma 3.6, (i), together with Remark 2.7.2. Assertion (ii) follows from a similar argument to the argument used in the proof of Lemma 3.6, (ii). □

Proposition 3.8 (**Graph-theoretic geometry via edge-like decomposition subgroups).** *For* $i = 1,\ 2$, *let* $\widetilde{e}_i \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. *Then the following hold:*

(i) *Consider the following three (mutually exclusive) conditions:*

(1) $\widetilde{e}_1 = \widetilde{e}_2$.

(2) $\widetilde{e}_1 \neq \widetilde{e}_2$; $\mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2) \neq \emptyset$.

(3) $\mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2) = \emptyset$ *(which implies that* $\widetilde{e}_1 \neq \widetilde{e}_2$*).*

Then we have **equivalences**

$$(1) \iff (1') \ ; \ \ (2) \iff (2') \ ; \ \ (3) \iff (3')$$

with the following three (mutually exclusive [cf. Lemma 1.5]) conditions:

(1′) $D_{\widetilde{e}_1} = D_{\widetilde{e}_2}$ *(so* $\Pi_{\widetilde{e}_1} = D_{\widetilde{e}_1} \cap \Pi_{\mathcal{G}} = D_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{e}_2}$ *— cf. Lemma 2.7, (ii), (iii)).*

(2′) $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2}$ $(= D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}}) = \{1\}$; $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \neq \{1\}$.

(3′) $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} = \{1\}$.

(ii) *Suppose that* ρ_I *is* **of SNN-type**. *Then if condition (2′) is satisfied, then* $\mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2) \neq \emptyset$, *and, moreover,* $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} = I_{\widetilde{v}}$ *— where we write* $\widetilde{v}$ *for the* **unique** *element of* $\mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2)$ *(cf. Lemmas 1.5, 1.8).*

Proof. First, we verify assertion (i). The equivalence

$$(1) \iff (1')$$

follows from [Mzk4], Proposition 1.2, (i). The implication

$$(2) \implies (2')$$

follows from Lemma 1.5, together with the fact that $I_{\widetilde{v}} \subseteq D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$, where $\widetilde{v} \in \mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2)$ (cf. Proposition 3.5). Thus, since it is immediate that the equivalence

$$(3) \iff (3')$$

follows from the equivalences

$$(1) \iff (1') \ ; \ \ (2) \iff (2') \,,$$

together with Lemma 1.5, to complete the proof of assertion (i), it suffices to verify the *implication*

$$(\dagger) \qquad\qquad (2') \implies (2)$$

under the assumption that $\widetilde{e}_1 \neq \widetilde{e}_2$ (cf. Lemma 1.5).

If $\mathrm{Node}(\mathcal{G}) = \emptyset$, then (†) is immediate; thus, assume that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Now if condition (2′) is satisfied, then since $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}} = \{1\}$ — which implies, in particular, that the composite $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \hookrightarrow \Pi_I \twoheadrightarrow I$

is *injective* — and I is *torsion-free*, it follows that the intersection $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$ is *torsion-free*. Thus, to prove (†), by replacing Π_I by an open subgroup of Π_I, we may assume without loss of generality that $\mathcal{G}$ is *sturdy* (cf. [Mzk4], Remark 1.1.5) and *untangled* (cf. Remark 1.2.1, (i)), and that ρ_I is *of SNN-type*; moreover, by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved, to prove (†), we may assume without loss of generality that $\Sigma = \{l\}$. Write $J \stackrel{\text{def}}{=} D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$.

Now we verify (†) in the case where $\{\widetilde{e}_1, \widetilde{e}_2\} \subseteq \mathrm{Cusp}(\widetilde{\mathcal{G}})$. To this end, let us assume that condition (2′) is satisfied. Then it follows from Lemma 2.7, (ii), that the image of the composite $D_{\widetilde{e}_i} \hookrightarrow \Pi_I = \Pi_{\mathcal{G}} \stackrel{\text{out}}{\rtimes} I \twoheadrightarrow \Pi_{\overline{\mathcal{G}}} \stackrel{\text{out}}{\rtimes} I$ — where we write $\overline{\mathcal{G}}$ for the compactification of $\mathcal{G}$ (cf. Definition 1.11) — coincides with the inertia subgroup $I_{\widetilde{v}_i}$ of $\Pi_{\overline{\mathcal{G}}} \stackrel{\text{out}}{\rtimes} I$ associated to the element $\widetilde{v}_i$ of $\mathrm{Vert}((\overline{\mathcal{G}})^{\sim})$ determined by the unique element of $\mathcal{V}(\widetilde{e}_i) \subseteq \mathrm{Vert}(\widetilde{\mathcal{G}})$. Thus, since $J \neq \{1\}$ and $J \cap \Pi_{\mathcal{G}} = \{1\}$ (cf. condition (2′)), it follows that $I_{\widetilde{v}_1} \cap I_{\widetilde{v}_2} \neq \{1\}$; in particular, it follows from Remark 3.5.1 that $\widetilde{v}_1 = \widetilde{v}_2$, hence — by applying this conclusion to the various open subgroups of Π_I — that $\mathcal{V}(\widetilde{e}_1) = \mathcal{V}(\widetilde{e}_2)$. This completes the proof of (†) in the case where $\{\widetilde{e}_1, \widetilde{e}_2\} \subseteq \mathrm{Cusp}(\widetilde{\mathcal{G}})$.

Next, we verify (†) in the case where $\{\widetilde{e}_1, \widetilde{e}_2\} \not\subseteq \mathrm{Cusp}(\widetilde{\mathcal{G}})$. Thus, we may assume without loss of generality that $\widetilde{e}_1 \in \mathrm{Node}(\widetilde{\mathcal{G}})$. Write $\mathcal{V}(\widetilde{e}_1) = \{\widetilde{v}, \widetilde{v}'\}$.

Now we *claim* that if condition (2′) is satisfied (i.e., $\widetilde{e}_1 \neq \widetilde{e}_2$ and $J \neq \{1\}$), and $J \cap I_{\widetilde{v}} = \{1\}$, then condition (2) is satisfied. Indeed, suppose that condition (2′) is satisfied and $J \cap I_{\widetilde{v}} = \{1\}$, but that condition (2) is *not* satisfied. Then since $J \cap I_{\widetilde{v}} = \{1\}$ and $\widetilde{e}_1 \in \mathrm{Node}(\widetilde{\mathcal{G}})$, it follows that $(J \cdot I_{\widetilde{v}}) \cap \Pi_{\mathcal{G}} = (J \times I_{\widetilde{v}}) \cap \Pi_{\mathcal{G}}$ ($\simeq \mathbb{Z}_l$) is an *open* subgroup of $D_{\widetilde{e}_1} \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{e}_1}$ ($\simeq \mathbb{Z}_l$) (cf. Remark 2.7.1; Lemma 3.7); thus, by replacing Π_I by an open subgroup of Π_I, we may assume without loss of generality that $(J \cdot I_{\widetilde{v}}) \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{e}_1}$. In particular, we obtain that $\Pi_{\widetilde{e}_1} \subseteq J \cdot I_{\widetilde{v}} \subseteq D_{\widetilde{e}_2} \cdot I_{\widetilde{v}}$. On the other hand, since $\widetilde{v}' \notin \mathcal{V}(\widetilde{e}_2)$ (by the assumption that condition (2) is *not* satisfied), by applying Remark 3.4.1, where we take "$\widetilde{v}_0$" to be $\widetilde{v}'$, we obtain an open subgroup $\Pi \subseteq \Pi_I$ such that $\Pi_{\widetilde{e}_1} \not\subseteq \Pi$ (cf. the implication "(i) ⇒ (ii)" in Remark 3.4.2), and, moreover, $I_{\widetilde{v}}, D_{\widetilde{e}_2} \subseteq \Pi$ (cf. conditions (ii), (iv) in Remark 3.4.1) — in contradiction to the inclusion $\Pi_{\widetilde{e}_1} \subseteq D_{\widetilde{e}_2} \cdot I_{\widetilde{v}}$. This completes the proof of the above *claim*.

Next, we *claim* that if condition (2′) is satisfied (i.e., $\widetilde{e}_1 \neq \widetilde{e}_2$ and $J \neq \{1\}$), and $J \cap I_{\widetilde{v}} \neq \{1\}$, then condition (2) is satisfied. Indeed, suppose that condition (2′) is satisfied, and $J \cap I_{\widetilde{v}} \neq \{1\}$. Then since $\Sigma = \{l\}$, by replacing I by an open subgroup of I, we may assume that $I_{\widetilde{v}} = J$; thus, $I_{\widetilde{v}} = J \subseteq D_{\widetilde{e}_2}$. Therefore, it follows from Proposition 3.5 that $\widetilde{v} \in \mathcal{V}(\widetilde{e}_2)$; in particular, since $\widetilde{v} \in \mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2)$, condition (2) is satisfied. This completes the proof of the above *claim*, hence also of the proof of (†).

Next, we verify assertion (ii). Since condition (2′) in assertion (i) is satisfied, we have $I_{\widetilde{v}} \subseteq D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$ (cf. Proposition 3.5). Moreover, since $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \cap \Pi_{\mathcal{G}} = \{1\}$, the composite $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \hookrightarrow \Pi_I \twoheadrightarrow I$ is *injective*. On the other hand, since ρ_I is *of SNN-type*, the composite $I_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is *bijective*. Therefore, we obtain that $I_{\widetilde{v}} = D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$, as desired. □

Proposition 3.9 (**Graph-theoretic geometry via verticial decomposition subgroups).** *For $i = 1, 2$, let $\widetilde{v}_i \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then the following hold:*

(i) *Consider the following four (mutually exclusive) conditions:*

(1) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 0$.

(2) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 1$.

(3) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 2$.

(4) $\delta(\widetilde{v}_1, \widetilde{v}_2) \geq 3$.

Then we have **equivalences**

$$(1) \Longleftrightarrow (1') \ ; \ (2) \Longleftrightarrow (2') \ ; \ (3) \Longleftrightarrow (3') \ ; \ (4) \Longleftrightarrow (4')$$

with the following four (mutually exclusive [cf. Lemma 1.9, (ii)]) conditions:

(1′) $\Pi_{\widetilde{v}_1} = \Pi_{\widetilde{v}_2}$ *(so $D_{\widetilde{v}_1} = D_{\widetilde{v}_2}$, $I_{\widetilde{v}_1} = I_{\widetilde{v}_2}$).*

(2′) $\Pi_{\widetilde{v}_1} \neq \Pi_{\widetilde{v}_2}$*;* $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2}$ $(= D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \cap \Pi_{\mathcal{G}}) \neq \{1\}$.

(3′) $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2}$ $(= D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \cap \Pi_{\mathcal{G}}) = \{1\}$*;* $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \neq \{1\}$.

(4′) $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} = \{1\}$.

(ii) *Suppose that ρ_I is* **of SNN-type**. *Then if condition (2′) is satisfied, then $\mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2) \neq \emptyset$, and, moreover, $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} = D_{\widetilde{e}}$ — where we write $\widetilde{e}$ for the* **unique** *element of $\mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2)$ (cf. Lemmas 1.8; 1.9, (ii)).*

(iii) *Suppose that ρ_I is* **of SNN-type**. *Then if condition (3′) is satisfied, then there exists a(n) — necessarily* **unique** *(cf. Lemmas 1.8; 1.9, (ii)) — element of $\widetilde{v}_3 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\delta(\widetilde{v}_1, \widetilde{v}_3) = \delta(\widetilde{v}_2, \widetilde{v}_3) = 1$, and, moreover, $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} = I_{\widetilde{v}_3}$.*

Proof. First, we verify assertion (ii). To this end, suppose that ρ_I is of SNN-type, and that condition (2′) is satisfied. Then it follows from Lemma 1.9, (ii), that there exists an element $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ such that $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}_1, \widetilde{v}_2\}$ (i.e., $\widetilde{e} \in \mathcal{N}(\widetilde{v}_1) \cap \mathcal{N}(\widetilde{v}_2)$). Thus, it follows from Remark 2.7.1 that $D_{\widetilde{e}} \subseteq D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2}$. Therefore, since $\Pi_{\widetilde{e}} = D_{\widetilde{e}} \cap \Pi_{\mathcal{G}} =$

$D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \cap \Pi_{\mathcal{G}}$ (cf. Lemmas 1.9, (ii); 2.7, (i), (iii)), and the composite $D_{\widetilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$ is *surjective* (since ρ_I is *of SNN-type*), it follows immediately that $D_{\widetilde{e}} = D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2}$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). To this end, suppose that ρ_I is of SNN-type, and that condition (3′) is satisfied, i.e., that $J \stackrel{\text{def}}{=} D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \neq \{1\}$, and $J \cap \Pi_{\mathcal{G}} = \{1\}$. Note that it follows from Lemma 1.9, (ii), that $\widetilde{v}_1 \neq \widetilde{v}_2$; in particular, $\mathrm{Node}(\mathcal{G}) \neq \emptyset$.

Now we *claim* that

$$(*_1)\ J \cap I_{\widetilde{v}_1} = J \cap I_{\widetilde{v}_2} = \{1\}.$$

Indeed, if $J \cap I_{\widetilde{v}_1} \neq \{1\}$, then (since $I_{\widetilde{v}_1}$ is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}$ — cf. Lemma 2.5, (i)) by projecting to the maximal pro-l quotients, for some $l \in \Sigma$, of suitable open subgroups of the various pro-Σ groups involved, we may assume without loss of generality that $J = I_{\widetilde{v}_1}$. But this implies that $I_{\widetilde{v}_1} = J \subseteq D_{\widetilde{v}_2} = Z_{\Pi_I}(I_{\widetilde{v}_2})$ (cf. Lemma 3.6, (ii)), hence that $I_{\widetilde{v}_2} \subseteq Z_{\Pi_I}(I_{\widetilde{v}_1}) = D_{\widetilde{v}_1}$ (cf. Lemma 3.6, (ii)). Therefore, we obtain that $I_{\widetilde{v}_2} \subseteq D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} = J = I_{\widetilde{v}_1}$; in particular, it follows from Remark 3.5.1 that $\widetilde{v}_1 = \widetilde{v}_2$ — a contradiction. This completes the proof of $(*_1)$.

Next, for $i = 1, 2$, let us write $J_i \stackrel{\text{def}}{=} (I_{\widetilde{v}_i} \cdot J) \cap \Pi_{\mathcal{G}}$ $(= (I_{\widetilde{v}_i} \times J) \cap \Pi_{\mathcal{G}}$ — cf. $(*_1)$). Then for any pair of integers i, j such that $\{i, j\} = \{1, 2\}$, since $J \subseteq D_{\widetilde{v}_j}$, it follows that $J_i = (I_{\widetilde{v}_i} \cdot J) \cap \Pi_{\mathcal{G}} \subseteq (I_{\widetilde{v}_i} \cdot D_{\widetilde{v}_j}) \cap \Pi_{\mathcal{G}}$; since, moreover, $J \subseteq D_{\widetilde{v}_i}$, it follows that $J_i = (I_{\widetilde{v}_i} \cdot J) \cap \Pi_{\mathcal{G}} \subseteq (I_{\widetilde{v}_i} \cdot D_{\widetilde{v}_i}) \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{v}_i}$ (cf. Lemma 2.7, (i)). In particular, it follows that for any pair of integers i, j such that $\{i, j\} = \{1, 2\}$, we have $J_i \subseteq (I_{\widetilde{v}_i} \cdot D_{\widetilde{v}_j}) \cap \Pi_{\widetilde{v}_i}$. On the other hand, it follows immediately from $(*_1)$ that $J_i \neq \{1\}$.

Next, we *claim* that

> $(*_2)$ for $i = 1, 2$, there exists an element $\widetilde{e}_i \in \mathcal{E}(\widetilde{v}_i)$ such that $J_i \subseteq \Pi_{\widetilde{e}_i}$.

Indeed, let us first *observe* that, for any pair of integers i, j such that $\{i, j\} = \{1, 2\}$, since $J_i \subseteq (I_{\widetilde{v}_i} \cdot D_{\widetilde{v}_j}) \cap \Pi_{\widetilde{v}_i}$, it follows immediately from Remark 3.2.1 that if $\widetilde{v}_1(\mathcal{G}) \neq \widetilde{v}_2(\mathcal{G})$, then the image of J_i in $\Pi_{\mathcal{G}}^{\mathrm{ab/edge}}$ is *trivial*. Moreover, by applying this *observation* to arbitrary open subgroups $H \subseteq \Pi_I$ corresponding to connected finite étale coverings of $\mathcal{G}[\rho_I]$ that determine outer representations of SNN-type, we conclude that, if we write $\mathcal{G}' \to \mathcal{G}$ for the connected finite étale covering of $\mathcal{G}$ determined by H, then the image of

$$\Big((I_{\widetilde{v}_i} \cap H) \cdot (J \cap H)\Big) \cap \Pi_{\mathcal{G}'}$$

in $\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}}$ is *trivial*; but since, for a suitable positive integer n (that depends on H!), we have

$$(J_i \cap \Pi_{\mathcal{G}'})^n \subseteq \Big(I_{\widetilde{v}_i}^n \cdot J^n\Big) \cap \Pi_{\mathcal{G}'} \subseteq \Big((I_{\widetilde{v}_i} \cap H) \cdot (J \cap H)\Big) \cap \Pi_{\mathcal{G}'}\,,$$

it follows from the fact that $\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}}$ is *torsion-free* (cf. [Mzk4], Remark 1.1.4) that the image of $J_i \cap \Pi_{\mathcal{G}'}$ in $\Pi_{\mathcal{G}'}^{\mathrm{ab/edge}}$ is *trivial*. Thus, we may apply Lemma 1.6, together with Lemma 1.7, to conclude the existence of an $\widetilde{e}_i \in \mathcal{E}(\widetilde{v}_i)$, as desired. This completes the proof of $(*_2)$. Note that since $\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2) = \emptyset$ (cf. Lemma 1.9, (ii)), it follows that $\widetilde{e}_1 \neq \widetilde{e}_2$.

Now it follows immediately from the definition of J_i that $J \subseteq J_i \cdot I_{\widetilde{v}_i}$. Thus, by $(*_2)$, we obtain that $J \subseteq J \cdot I_{\widetilde{v}_i} \subseteq J_i \cdot I_{\widetilde{v}_i} \subseteq \Pi_{\widetilde{e}_i} \cdot I_{\widetilde{v}_i} = D_{\widetilde{e}_i}$ (cf. Remark 2.7.1); in particular, $J \subseteq D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$. Now since $J \neq \{1\}$, and $\widetilde{e}_1 \neq \widetilde{e}_2$, it follows from Proposition 3.8, (i), (ii), that there exists an element $\widetilde{v}_3 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\widetilde{v}_3 \in \mathcal{V}(\widetilde{e}_1) \cap \mathcal{V}(\widetilde{e}_2)$, and, moreover, $(J \subseteq)\ D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} = I_{\widetilde{v}_3}$. Moreover, since $\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2) = \emptyset$, it follows that $\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Node}(\widetilde{\mathcal{G}})$, and that $\widetilde{v}_3 \neq \widetilde{v}_1, \widetilde{v}_2$; in particular, it follows that $\delta(\widetilde{v}_3, \widetilde{v}_1) = \delta(\widetilde{v}_3, \widetilde{v}_2) = 1$. Thus, since $I_{\widetilde{v}_3} \subseteq I_{\widetilde{e}_i} \subseteq D_{\widetilde{e}_i} \subseteq D_{\widetilde{v}_i}$ (cf. Remark 2.7.1) for $i = 1, 2$, it follows that $I_{\widetilde{v}_3} \subseteq D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} = J$. This completes the proof of assertion (iii).

Finally, we verify assertion (i). First, let us observe that the equivalences

$$(1) \iff (1')\ ;\ \ (2) \iff (2')$$

follow from Lemma 1.9, (ii). Now since the equivalence

$$(4) \iff (4')$$

follows from the equivalences

$$(1) \iff (1')\ ;\ \ (2) \iff (2')\ ;\ \ (3) \iff (3')$$

— together with the *mutual exclusivity* observed in the statement of Proposition 3.9, (i) — to complete the proof of assertion (i), it suffices to verify the equivalence

$$(3) \iff (3')\,.$$

To this end, assume that condition (3) is satisfied. Then it follows from Lemma 1.9, (ii), that $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \cap \Pi_{\mathcal{G}} = \{1\}$. Now to verify condition (3′), by replacing I by an open subgroup of I, we may assume without loss of generality that ρ_I is *of SNN-type* (so that we may apply Remark 2.7.1). Since condition (3) is satisfied, there exists an element $\widetilde{v}_3 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\delta(\widetilde{v}_1, \widetilde{v}_3) = \delta(\widetilde{v}_2, \widetilde{v}_3) = 1$. For $i = 1, 2$, let $\widetilde{e}_i \in \mathcal{N}(\widetilde{v}_i) \cap \mathcal{N}(\widetilde{v}_3)$. Then it follows that $I_{\widetilde{v}_3} \subseteq I_{\widetilde{e}_i} \subseteq D_{\widetilde{e}_i} \subseteq D_{\widetilde{v}_i}$ (cf. Remark 2.7.1); in particular, $I_{\widetilde{v}_3} \subseteq D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2}$. Thus, condition (3′) is satisfied.

Next, let us assume that condition (3′) is satisfied. Then since $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \cap \Pi_{\mathcal{G}} = \{1\}$ — which implies, in particular, that the composite $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2} \hookrightarrow \Pi_I \twoheadrightarrow I$ is *injective* — and I is *torsion-free* (cf. condition (1) of Definition 2.4), it follows that $D_{\widetilde{v}_1} \cap D_{\widetilde{v}_2}$ is *torsion-free*. Therefore, to verify condition (3), by replacing I by an open subgroup of I, we may assume without loss of generality that ρ_I is *of SNN-type*. Then it follows immediately from assertion (iii), together with Lemma 1.9, (ii), that condition (3) is satisfied. This completes the proof of assertion (i). □

4. A combinatorial anabelian theorem for nodally nondegenerate outer representations

In this section, we prove two combinatorial anabelian results in the style of [Mzk4] for outer representations of NN-type.

Theorem 4.1 (Group-theoretic verticiality and nodality of certain isomorphisms). *Let Σ be a nonempty set of prime numbers, $\mathcal{G}$ and $\mathcal{H}$ semi-graphs of anabelioids of pro-Σ PSC-type, $\widetilde{v}_{\mathcal{G}} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, $\widetilde{v}_{\mathcal{H}} \in \mathrm{Vert}(\widetilde{\mathcal{H}})$, $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) the fundamental group of $\mathcal{G}$ (respectively, $\mathcal{H}$), $\alpha \colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ an isomorphism of profinite groups, I and J profinite groups, $\rho_I \colon I \to \mathrm{Aut}(\mathcal{G})$ and $\rho_J \colon J \to \mathrm{Aut}(\mathcal{H})$ continuous homomorphisms, and $\beta \colon I \xrightarrow{\sim} J$ an isomorphism of profinite groups. Suppose that the following three conditions are satisfied:*

(i) *The diagram*

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ J & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the homomorphism induced by α; the upper and lower horizontal arrows are the homomorphisms determined by ρ_I and ρ_J, respectively — commutes.

(ii) *ρ_I, ρ_J are* **of NN-type**.

(iii) *$\alpha(\Pi_{\widetilde{v}_{\mathcal{G}}}) = \Pi_{\widetilde{v}_{\mathcal{H}}}$.*

Then the isomorphism α is **group-theoretically verticial**, *hence, in particular,* **group-theoretically nodal** *(cf. Proposition 1.13).*

Proof. Note that to verify Theorem 4.1, it is immediate that by replacing I by an open subgroup of I, we may assume without loss of generality that ρ_I and ρ_J are *of SNN-type*. Let us denote by $\widetilde{\alpha} \colon \Pi_I \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I \xrightarrow{\sim} \Pi_J \overset{\mathrm{def}}{=} \Pi_{\mathcal{H}} \overset{\mathrm{out}}{\rtimes} J$ (cf. the discussion entitled "*Topological groups*" in §0) the isomorphism determined by α and β (cf. assumption (i)).

For $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, we shall say that $\widetilde{v}$ satisfies the condition $(*^{\mathrm{pres}})$ if $\alpha(\Pi_{\widetilde{v}}) \subseteq \Pi_{\mathcal{H}}$ is a *verticial subgroup of* $\Pi_{\mathcal{H}}$. First, we *claim* that this condition $(*^{\mathrm{pres}})$ satisfies the property $(*)$ in the statement of Lemma 3.1. To this end, let $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be such that $\delta(\widetilde{v}_1(\mathcal{G}), \widetilde{v}_2(\mathcal{G})) \leq 1$ and, moreover, $\widetilde{v}_1$ satisfies the condition $(*^{\mathrm{pres}})$. Now if $\widetilde{v}_1(\mathcal{G}) = \widetilde{v}_2(\mathcal{G})$, then it is immediate that $\widetilde{v}_2$ satisfies the condition $(*^{\mathrm{pres}})$; thus, we may assume that $\widetilde{v}_1(\mathcal{G}) \neq \widetilde{v}_2(\mathcal{G})$. Then it follows from Lemma 1.15 that there exist $\widetilde{w}_1$, $\widetilde{u}_1$, $\widetilde{w}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ which satisfy the following conditions:

(1) $\widetilde{v}_1(\mathcal{G}) = \widetilde{w}_1(\mathcal{G}) = \widetilde{u}_1(\mathcal{G})$; $\widetilde{v}_2(\mathcal{G}) = \widetilde{w}_2(\mathcal{G})$.

(2) $\delta(\widetilde{w}_1, \widetilde{u}_1) \geq 2$.

(3) $\delta(\widetilde{w}_2, \widetilde{w}_1) = \delta(\widetilde{w}_2, \widetilde{u}_1) = 1$.

Now it follows from condition (1), together with the assumption that $\widetilde{v}_1$ satisfies the condition $(*^{\mathrm{pres}})$, that $\widetilde{w}_1$ and $\widetilde{u}_1$ also satisfy $(*^{\mathrm{pres}})$; in particular, there exist $\widetilde{w}'_1$, $\widetilde{u}'_1 \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $\widetilde{\alpha}(D_{\widetilde{w}_1}) = D_{\widetilde{w}'_1}$ and $\widetilde{\alpha}(D_{\widetilde{u}_1}) = D_{\widetilde{u}'_1}$. Moreover, it follows from Proposition 3.9, (iii), together with conditions (2), (3), that $D_{\widetilde{w}_1} \cap D_{\widetilde{u}_1} = I_{\widetilde{w}_2}$; in particular, it follows that $D_{\widetilde{w}'_1} \cap D_{\widetilde{u}'_1} \neq \{1\}$ and $D_{\widetilde{w}'_1} \cap D_{\widetilde{u}'_1} \cap \Pi_{\mathcal{H}} = \{1\}$. Thus, again by Proposition 3.9, (iii), there exists an element $w'_2 \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $D_{\widetilde{w}'_1} \cap D_{\widetilde{u}'_1} = I_{\widetilde{w}'_2}$. Now since $\widetilde{\alpha}(D_{\widetilde{w}_1}) = D_{\widetilde{w}'_1}$ and $\widetilde{\alpha}(D_{\widetilde{u}_1}) = D_{\widetilde{u}'_1}$, it follows that $\widetilde{\alpha}(I_{\widetilde{w}_2}) = I_{\widetilde{w}'_2}$; thus, it follows from Lemma 3.6, (i), that $\alpha(\Pi_{\widetilde{w}_2}) = \Pi_{\widetilde{w}'_2}$. In particular, it follows from condition (1) that $\widetilde{v}_2$ satisfies the condition $(*^{\mathrm{pres}})$. This completes the proof of the above *claim*.

Now in light of the above *claim*, together with assumption (iii), we may apply Lemma 3.1 to conclude that the isomorphism α is group-theoretically verticial. This completes the proof of Theorem 4.1. $\square$

Corollary 4.2 (**Graphicity of certain group-theoretically cuspidal isomorphisms**). *Let Σ be a nonempty set of prime numbers, $\mathcal{G}$ and $\mathcal{H}$ semi-graphs of anabelioids of pro-Σ PSC-type (cf.* [Mzk4]*, Definition 1.1, (i)), $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) the pro-Σ fundamental group of $\mathcal{G}$ (respectively, $\mathcal{H}$), $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ an isomorphism of profinite groups, I and J profinite groups, $\rho_I\colon I \to \mathrm{Aut}(\mathcal{G})$ and $\rho_J\colon J \to \mathrm{Aut}(\mathcal{H})$ continuous homomorphisms, and $\beta\colon I \xrightarrow{\sim} J$ an isomorphism of profinite groups. Suppose that the following three conditions are satisfied:*

(i) *The diagram*

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ J & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the homomorphism induced by α; the upper and lower horizontal arrows are the homomorphisms determined by ρ_I and ρ_J, respectively — commutes.

(ii) *ρ_I, ρ_J are* **of NN-type** *(cf. Definition 2.4, (iii)).*

(iii) $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$, *and the isomorphism* α *is* **group-theoretically cuspidal** *(cf.* [Mzk4]*, Definition 1.4, (iv)).*

Then the isomorphism α *is* **graphic** *(cf.* [Mzk4]*, Definition 1.4, (i)).*

Proof. It is immediate that to verify Corollary 4.2, by replacing Π_I by an open subgroup of Π_I, we may assume without loss of generality that $\mathcal{G}$ and $\mathcal{H}$ are *sturdy* (cf. [Mzk4], Remark 1.1.5), and that ρ_I and ρ_J are *of SNN-type*. Let us denote by $\widetilde{\alpha}\colon \Pi_I \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I \xrightarrow{\sim} \Pi_J \overset{\mathrm{def}}{=} \Pi_{\mathcal{H}} \overset{\mathrm{out}}{\rtimes} J$ (cf. the discussion entitled "*Topological groups*" in §0) the isomorphism determined by α and β (cf. assumption (i)).

Now it follows from Lemma 1.14 that to prove the graphicity of α, it suffices to show that the isomorphism α satisfies condition (ii) in the statement of Lemma 1.14. Moreover, by replacing $\mathcal{G}$ by the "$\mathcal{G}'$" in the statement of Lemma 1.14, it suffices to show that the isomorphism $\Pi_{\overline{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}}$ — where we write $\overline{\mathcal{G}}$ (respectively, $\overline{\mathcal{H}}$) for the compactification (cf. Definition 1.11) of $\mathcal{G}$ (respectively, $\mathcal{H}$) — induced by α is *group-theoretically verticial*. The rest of the proof of Corollary 4.2 is devoted to the proof of the *fact* that the isomorphism $\Pi_{\overline{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}}$ induced by α is *group-theoretically verticial*.

Write $I \to \mathrm{Out}(\Pi_{\overline{\mathcal{G}}})$ (respectively, $J \to \mathrm{Out}(\Pi_{\overline{\mathcal{H}}})$) for the outer representation of pro-Σ PSC-type determined by ρ_I (respectively, ρ_J) and $\overline{\Pi}_I \overset{\mathrm{def}}{=} \Pi_{\overline{\mathcal{G}}} \overset{\mathrm{out}}{\rtimes} I$ (respectively, $\overline{\Pi}_J \overset{\mathrm{def}}{=} \Pi_{\overline{\mathcal{H}}} \overset{\mathrm{out}}{\rtimes} J$). Let $\widetilde{e}_{\mathcal{G}} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ (cf. assumption (iii)). Then it follows from assumption (iii) that there exists an element $\widetilde{e}_{\mathcal{H}} \in \mathrm{Cusp}(\widetilde{\mathcal{H}})$ such that $\widetilde{\alpha}(D_{\widetilde{e}_{\mathcal{G}}}) = D_{\widetilde{e}_{\mathcal{H}}}$. Moreover, if we denote by $\widetilde{v}_{\mathcal{G}}$ (respectively, $\widetilde{v}_{\mathcal{H}}$) the unique element of $\mathcal{V}(e_{\mathcal{G}})$ (respectively, $\mathcal{V}(e_{\mathcal{H}})$), then it follows from Remark 2.7.1 that the image of the composite $D_{\widetilde{e}_{\mathcal{G}}} \hookrightarrow \Pi_I \twoheadrightarrow \overline{\Pi}_I$ (respectively, $D_{\widetilde{e}_{\mathcal{H}}} \hookrightarrow \Pi_J \twoheadrightarrow \overline{\Pi}_J$) coincides with $I_{\widetilde{v}_{\mathcal{G}}}$ (respectively, $I_{\widetilde{v}_{\mathcal{H}}}$). Therefore, it follows from Lemma 3.6, (i), that $\alpha(\Pi_{\widetilde{v}_{\mathcal{G}}}) = \Pi_{\widetilde{v}_{\mathcal{H}}}$. In particular, we may apply Theorem 4.1 to conclude that the isomorphism $\Pi_{\overline{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\overline{\mathcal{H}}}$ is group-theoretically verticial. This completes the proof of Corollary 4.2. □

Remark 4.2.1. One may verify the following *assertion* by applying [Mzk4], Corollary 2.7, (iii), as in the proof of [Mzk4], Corollary 2.8:

> In the notation of Corollary 4.2, if the following three conditions are satisfied, then α is *graphic*:
>
> (i) The diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ J & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the homomorphism induced by α; the upper and lower horizontal arrows are the homomorphisms determined by ρ_I and ρ_J, respectively — commutes.

(ii) ρ_I, ρ_J are *of IPSC-type*.

(iii) The isomorphism α is *group-theoretically cuspidal*.

That is to say, one may think of Corollary 4.2 as a *partial* (cf. the condition "$\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$" of Corollary 4.2, (iii)) *generalization* of the above *assertion* — whose proof is *independent* of the methods of [Mzk4].

5. Injectivity via nodally nondegenerate degenerations

In this section, we apply Corollary 4.2, together with a similar argument to the argument used in the proof of [Mzk7], Corollary 2.3, to prove a certain *injectivity* result concerning FC-admissible outomorphisms (cf. the discussion entitled "*Topological groups*" in §0) of pro-Σ fundamental groups of configuration spaces (cf. Corollary 5.3).

Definition 5.1. Let Σ be a set of prime numbers which is either *of cardinality one* or *equal to the set of all prime numbers*, (g, r) a pair of natural numbers such that $2g - 2 + r > 0$, n a natural number, $S^{\log}$ an fs log scheme whose underlying scheme is the spectrum of an algebraically closed field of characteristic $\notin \Sigma$, and $X^{\log}$ an r-pointed stable log curve of genus g over $S^{\log}$, i.e., the log scheme obtained by pulling back the universal r-pointed stable log curve of genus g over $\overline{\mathcal{M}}_{g,r}^{\log}$ (cf. the discussion entitled "*Curves*" in §0) via a (1-)morphism $S^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$.

(i) We shall denote by $X_n^{\log}$ the n-th log configuration space of $X^{\log}$ (cf. the discussion entitled "*Curves*" in §0).

(ii) We shall denote by Π_n the maximal pro-Σ quotient of the kernel of the surjection $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$.

(iii) For $i = 1, 2$, we shall denote by

$$\mathrm{pr}_i^{\log} \colon X_2^{\log} \longrightarrow X_1^{\log} = X^{\log}$$

the projection to the factor labeled i, and by

$$p_i \colon \Pi_2 \twoheadrightarrow \Pi_1$$

the surjection induced by $\mathrm{pr}_i^{\log}$.

(iv) We shall denote by $\Pi_{2/1}$ the kernel of the surjection $p_1 \colon \Pi_2 \twoheadrightarrow \Pi_1$.

(v) We shall denote by $\mathcal{G}$ the connected semi-graph of anabelioids of pro-Σ PSC-type arising from the pointed stable curve determined by the stable log curve $X^{\log}$ over $S^{\log}$ (cf. [Mzk4], Example 2.5), and by $\Pi_{\mathcal{G}}$ the fundamental group of $\mathcal{G}$. Note that by the various definitions involved, there exists a natural isomorphism $\Pi_1 \simeq \Pi_{\mathcal{G}}$. In the following, we shall *assume* that

$$(\mathrm{Vert}(\mathcal{G})^{\sharp}, \mathrm{Node}(\mathcal{G})^{\sharp}) = (2, 1)$$

(cf. Remark 5.1.1 below) and write

$$\mathrm{Vert}(\mathcal{G}) = \{v_1, v_2\} \ ; \ \ \mathrm{Node}(\mathcal{G}) = \{e\}$$

(cf. Figure 1). Also, we observe that (it follows immediately from the various definitions involved that) we have $\mathrm{Cusp}(\mathcal{G})^{\sharp} = r$.

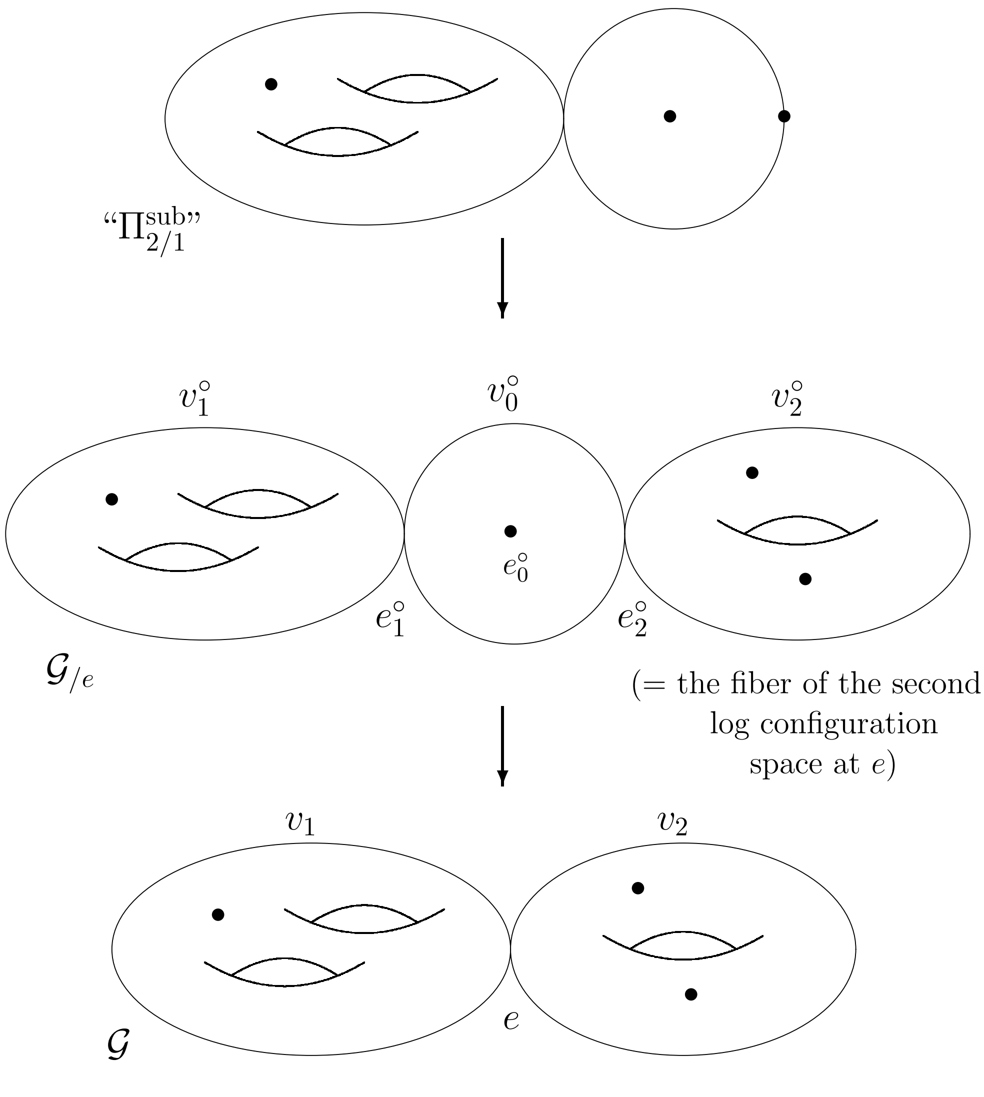

Figure 1: $\mathcal{G}$, $\mathcal{G}_{/e}$, and “$\Pi_{2/1}^{\mathrm{sub}}$”

(vi) We shall denote by $\mathcal{G}_{/e}$ the connected semi-graph of anabelioids of pro-Σ PSC-type arising from the pointed stable curve determined by the fiber of $\mathrm{pr}_1^{\log}\colon X_2^{\log} \to X^{\log}$ at the *unique* node e of $X^{\log}$ (cf. Figure 1), and by $\Pi_{\mathcal{G}_{/e}}$ the fundamental group of $\mathcal{G}_{/e}$. Note that by the various definitions involved, we have

$$(\mathrm{Vert}(\mathcal{G}_{/e})^\sharp, \mathrm{Cusp}(\mathcal{G}_{/e})^\sharp, \mathrm{Node}(\mathcal{G}_{/e})^\sharp) = (3, r+1, 2)\,;$$

moreover, there exists a natural isomorphism $\Pi_{2/1} \simeq \Pi_{\mathcal{G}_{/e}}$.

(vii) For $i = 1$, 2, there exists a *unique* vertex of $\mathcal{G}_{/e}$ such that the image via the surjection $\Pi_{\mathcal{G}_{/e}} \simeq \Pi_{2/1} \twoheadrightarrow \Pi_1 \simeq \Pi_{\mathcal{G}}$ induced by p_2 of a verticial subgroup of $\Pi_{\mathcal{G}_{/e}}$ associated to the vertex is a *verticial* subgroup of $\Pi_{\mathcal{G}}$ associated to $v_i \in \mathrm{Vert}(\mathcal{G})$. We shall denote this vertex by $v_i^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$. On the other hand, there exists a *unique* vertex of $\mathcal{G}_{/e}$ such that the image via the surjection $\Pi_{\mathcal{G}_{/e}} \simeq \Pi_{2/1} \twoheadrightarrow \Pi_1 \simeq \Pi_{\mathcal{G}}$ induced by p_2 of a verticial subgroup of $\Pi_{\mathcal{G}_{/e}}$ associated to the vertex is an *edge-like* subgroup of $\Pi_{\mathcal{G}}$ associated to the unique node $e \in \mathrm{node}(\mathcal{G})$. We shall denote this vertex by $v_0^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$. Thus, in summary, we have

$$\mathrm{Vert}(\mathcal{G}_{/e}) = \{v_1^\circ, v_2^\circ, v_0^\circ\}\,.$$

(viii) For $i = 1$, 2, there exists a *unique* node of $\mathcal{G}_{/e}$ such that the subset of vertices of $\mathcal{G}_{/e}$ to which the node abuts is $\{v_i^\circ, v_0^\circ\}$. We shall denote this node by $e_i^\circ \in \mathrm{Node}(\mathcal{G}_{/e})$, i.e., $\mathcal{V}(e_i^\circ) = \{v_i^\circ, v_0^\circ\}$. On the other hand, there exists a *unique* cusp which abuts to v_0°. We shall denote this cusp by $e_0^\circ \in \mathrm{Cusp}(\mathcal{G}_{/e})$. Thus, in summary, we have

$$\mathrm{Node}(\mathcal{G}_{/e}) = \{e_1^\circ, e_2^\circ\}\ ;\ \ \mathcal{V}(e_i^\circ) = \{v_i^\circ, v_0^\circ\}\ ;\ \ \mathcal{V}(e_0^\circ) = \{v_0^\circ\}\,.$$

(ix) Let $Y \subseteq X$ be the irreducible component of the underlying scheme X of $X^{\log}$ corresponding to v_1, $U_Y \subseteq Y$ the open subscheme of Y obtained as the complement of the nodes and cusps which abut to v_1, and $Y^{\log}$ the *smooth log curve* (whose underlying scheme is Y) over $S^{\log}$ determined by the hyperbolic curve U_Y. (Thus, $U_Y \subseteq Y$ is the open subscheme of points at which the log structure of $Y^{\log}$ coincides with the pull-back of the log structure of $S^{\log}$.) Write g_Y for the genus of U_Y and r_Y for the number of cusps of U_Y. Let $Y_n^{\log}$ be the n-th log configuration space of $Y^{\log}$ (cf. the discussion entitled "*Curves*" in §0). Note that the natural closed immersion $Y \hookrightarrow X$ induces a commutative diagram

$$\begin{array}{ccc} Y_2 & \longrightarrow & X_2 \\ \big\downarrow & & \big\downarrow \mathrm{pr}_1 \\ Y & \longrightarrow & X \end{array}$$

— where the left-hand vertical arrow is the morphism induced by pr_1 (cf. the discussion of [Mzk7], Definition 2.1, (iii)).

(x) We shall denote by Π_n^{sub} the maximal pro-Σ quotient of the kernel of the surjection $\pi_1(Y_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$, and by $\Pi_{2/1}^{\mathrm{sub}}$ the kernel of the surjection $\Pi_2^{\mathrm{sub}} \twoheadrightarrow \Pi_1^{\mathrm{sub}}$ induced by the first projection $Y_2^{\log} \to Y^{\log}$. Note that if we denote by $U_{Y_n} \subseteq Y_n$ the open subscheme of points at which the log structure of $Y_n^{\log}$ coincides with the pull-back of the log structure of $S^{\log}$, then recall that by the log purity theorem (cf. [Mzk7], the discussion of §0), the inclusion $U_{Y_n} \hookrightarrow Y_n$ induces a natural isomorphism $\pi_1(U_{Y_n})^{(\Sigma)} \xrightarrow{\sim} \Pi_n^{\mathrm{sub}}$. Thus, by restricting coverings of $X_n^{\log}$ to U_{Y_n} for $n = 1, 2$, we obtain a commutative diagram (cf. the discussion of [Mzk7], Definition 2.1, (vi))

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{2/1}^{\mathrm{sub}} & \longrightarrow & \Pi_2^{\mathrm{sub}} & \longrightarrow & \Pi_1^{\mathrm{sub}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{2/1}\,(\simeq \Pi_{\mathcal{G}_{/e}}) & \longrightarrow & \Pi_2 & \underset{p_1}{\longrightarrow} & \Pi_1\,(\simeq \Pi_{\mathcal{G}}) & \longrightarrow & 1 \end{array}$$

— where the right-hand upper horizontal arrow is the surjection induced by p_1, the vertical arrows are *injective outer homomorphisms*, the horizontal sequences are *exact*, and the image of the right-hand vertical arrow is a verticial subgroup of $\Pi_{\mathcal{G}}$ associated to v_1 (cf. Figure 1).

Remark 5.1.1. One verifies easily that if $2g - 2 + r > 1$ (i.e., $(g, r) \neq (0, 3), (1, 1)$), then there exists a stable log curve $X^{\log}$ as in Definition 5.1 (cf., especially, the *assumption* in Definition 5.1, (v)).

Lemma 5.2 (**Basic properties of verticial subgroups**). *In the notation of Definition 5.1: For $i = 1, 2$, let us fix an edge-like subgroup $\Pi_{e_i^\circ} \subseteq \Pi_{\mathcal{G}_{/e}}$ associated to $e_i^\circ \in \mathrm{Node}(\mathcal{G}_{/e})$ (Definition 5.1, (viii)). Then the following hold:*

(i) *There exists a* **unique** *verticial subgroup $\Pi_{v_i^\circ}$ (respectively, $\Pi_{v_0^\circ}$) of $\Pi_{\mathcal{G}_{/e}}$ associated to $v_i^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$ (respectively, $v_0^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$) [cf. Definition 5.1, (vii)] that* **contains** $\Pi_{e_i^\circ}$.

(ii) *There exists a* **unique** *$\Pi_{2/1}$-conjugate of the image of $\Pi_{2/1}^{\mathrm{sub}}$ via the left-hand vertical arrow in the diagram of Definition 5.1, (x), that* **contains** *and is* **topologically generated** *by the verticial subgroups $\Pi_{v_1^\circ}$, $\Pi_{v_0^\circ} \subseteq \Pi_{\mathcal{G}_{/e}}$ obtained in (i) in the case where $i = 1$. By abuse of notation, we shall denote this particular $\Pi_{2/1}$-conjugate of the image of $\Pi_{2/1}^{\mathrm{sub}}$ by means of the notation "$\Pi_{2/1}^{\mathrm{sub}}$".*

(iii) *Suppose that* $\Pi_{e_2^\circ}$ *was chosen so that (in the notation of (ii)) we have* $\Pi_{e_2^\circ} \subseteq \Pi_{2/1}^{\mathrm{sub}}$. *Then (in the notation of (i) and (ii))* $\Pi_{2/1}$ *is* **topologically generated** *by* $\Pi_{v_2^\circ}$ *and* $\Pi_{2/1}^{\mathrm{sub}}$.

Proof. These assertions follow from similar arguments to the arguments used in the proofs of [Mzk7], Proposition 2.2, (ii), (iii). □

The following result is the main result of the present section.

Corollary 5.3 (**Injectivity for not necessarily affine hyperbolic curves**)**.** *In the notation of Definition 5.1, the natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

— where we write "$\mathrm{Out}^{\mathrm{FC}}(-)$*" for the subgroup of the group "*$\mathrm{Out}(-)$*" of outomorphisms (cf. the discussion entitled "Topological groups" in §0) of "*$(-)$*" defined in* [Mzk7]*, Definition 1.1, (ii) — induced by* p_1 *is* **injective**.

Proof. If $2g-2+r=1$, then Corollary 5.3 follows from [Mzk7], Corollary 2.3, (ii); thus, to verify Corollary 5.3, we may assume without loss of generality that $2g-2+r>1$. Note that since $2g-2+r>1$, there exists a stable log curve $X^{\log}$ as in Definition 5.1 (cf. Remark 5.1.1). Thus, in the following, we assume that we are in the situation described in Definition 5.1.

To complete the proof of Corollary 5.3, it suffices, by [Mzk7], Proposition 1.2, (iii), to verify the *assertion* that if an automorphism α of Π_2 is *IFC-admissible* (cf. [Mzk7], Definition 1.1, (ii)), i.e., α satisfies the following three conditions (i), (ii), and (iii), then the automorphism α is a Ξ_2*-inner automorphism* — where we write $\Xi_2 \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p_1)\cap\mathrm{Ker}(p_2) \subseteq \Pi_2$ (cf. Definition 5.1, (iii)):

(i) α *preserves* $\mathrm{Ker}(p_1)$ $(=\Pi_{2/1})$ and $\mathrm{Ker}(p_2)$.

(ii) The automorphism of $\Pi_{\mathcal{G}_{/e}}$ $(\simeq \Pi_{2/1})$ obtained as the restriction $\alpha|_{\Pi_{2/1}}$ of α (cf. (i)) is *group-theoretically cuspidal.*

(iii) The automorphism of the quotient $(p_1,p_2)\colon \Pi_2 \twoheadrightarrow \Pi_1\times\Pi_1$ of Π_2 induced by α (cf. (i)) is the *identity automorphism* of $\Pi_1\times\Pi_1$.

The rest of the proof of Corollary 5.3 is devoted to verifying this *assertion.*

It follows immediately from (i) and (iii) that we have a commutative diagram

$$\begin{array}{ccc} \Pi_1 & \xrightarrow{\rho} & \mathrm{Out}(\Pi_{2/1}) \\ \| & & \downarrow{\scriptstyle \mathrm{Out}(\alpha|_{\Pi_{2/1}})} \\ \Pi_1 & \xrightarrow[\rho]{} & \mathrm{Out}(\Pi_{2/1}) \end{array}$$

— where the right-hand vertical arrow is the homomorphism induced by $\alpha|_{\Pi_{2/1}}$, and we write ρ for the outer representation determined by the exact sequence

$$1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_2 \xrightarrow{p_1} \Pi_1 \longrightarrow 1 .$$

Let $\Pi_e \subseteq \Pi_1$ be an edge-like subgroup of Π_1 ($\simeq \Pi_{\mathcal{G}}$) associated to the unique node e of $\mathcal{G}$. Then it follows immediately from the various definitions involved that the composite $\Pi_e \hookrightarrow \Pi_1 \xrightarrow{\rho} \mathrm{Out}(\Pi_{2/1})$ factors through $\mathrm{Aut}(\mathcal{G}_{/e}) \subseteq \mathrm{Out}(\Pi_{2/1})$; moreover, in light of the well-known local structure of $X^{\log}$ in a neighborhood of the node corresponding to e, it follows immediately from Proposition 2.14 that the resulting outer representation of pro-Σ PSC-type $\Pi_e \to \mathrm{Aut}(\mathcal{G}_{/e})$ is *of SNN-type*. In particular, it follows immediately from (ii), together with the fact that $\mathcal{G}_{/e}$ has *at least one cusp* (cf. Definition 5.1, (vi)), that we may apply Corollary 4.2 to conclude that the restriction $\alpha|_{\Pi_{2/1}}$ is *graphic*.

Next, let us fix an edge-like subgroup $\Pi_{e_1^\circ} \subseteq \Pi_{2/1}$ associated to $e_1^\circ \in \mathrm{Node}(\mathcal{G}_{/e})$ (cf. Definition 5.1, (viii)). Then we *claim* that there exists an element $\gamma \in \Xi_2$ such that $\alpha(\Pi_{e_1^\circ}) = \gamma \cdot \Pi_{e_1^\circ} \cdot \gamma^{-1}$. Indeed, it follows from the *graphicity* of $\alpha|_{\Pi_{2/1}}$, together with (iii), that $\alpha|_{\Pi_{2/1}}$ induces the *identity automorphism* of the underlying semi-graph of $\mathcal{G}_{/e}$ (cf. Definition 5.1, (vii), (viii)), hence that there exists an element $\gamma' \in \Pi_{2/1}$ such that $\alpha(\Pi_{e_1^\circ}) = \gamma' \cdot \Pi_{e_1^\circ} \cdot \gamma'^{-1}$; in particular, again by (iii), we obtain that $p_2(\Pi_{e_1^\circ}) = p_2(\gamma') \cdot p_2(\Pi_{e_1^\circ}) \cdot p_2(\gamma'^{-1})$. On the other hand, it follows immediately from the various definitions involved that $p_2(\Pi_{e_1^\circ}) \subseteq \Pi_1$ is an *edge-like subgroup* of Π_1 associated to $e \in \mathrm{Node}(\mathcal{G})$. Thus, it follows from the *commensurably terminality* of $p_2(\Pi_{e_1^\circ})$ in Π_1 (cf. [Mzk4], Proposition 1.2, (ii)) that $p_2(\gamma') \in p_2(\Pi_{e_1^\circ})$. In particular, by multiplying γ' by an appropriate element of $\Pi_{e_1^\circ}$, we obtain an element γ, as desired. This completes the proof of the above *claim*.

In light of the above *claim*, we may assume without loss of generality — by composing α with an appropriate Ξ_2-inner automorphism — that $\alpha(\Pi_{e_1^\circ}) = \Pi_{e_1^\circ}$. Let $\Pi_{v_1^\circ}$, $\Pi_{v_0^\circ} \subseteq \Pi_{2/1}$ be the *unique* verticial subgroups associated, respectively, to v_1°, $v_0^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$ that *contain* the fixed edge-like subgroup $\Pi_{e_1^\circ}$ (cf. Lemma 5.2, (i)); $\Pi_{2/1}^{\mathrm{sub}} \subseteq \Pi_{2/1}$ the *unique* $\Pi_{2/1}$-conjugate of the image of the left-hand vertical arrow in the diagram in Definition 5.1, (x), that contains and is topologically generated by these verticial subgroups $\Pi_{v_1^\circ}$, $\Pi_{v_0^\circ}$. (cf. Lemma 5.2, (ii)). Then in light of the *graphicity* of α, it follows from the fact that $\alpha(\Pi_{e_1^\circ}) = \Pi_{e_1^\circ}$, together with Lemma 5.2, (i), (ii), that $\alpha(\Pi_{v_1^\circ}) = \Pi_{v_1^\circ}$, $\alpha(\Pi_{v_0^\circ}) = \Pi_{v_0^\circ}$, and $\alpha(\Pi_{2/1}^{\mathrm{sub}}) = \Pi_{2/1}^{\mathrm{sub}}$.

Next, let us observe that $\Pi_{2/1}^{\mathrm{sub}}$ is *commensurably terminal* in $\Pi_{2/1}$. (Indeed, this follows by applying [Mzk4], Proposition 1.2, (ii) — where we think of the fiber of $Y_2^{\log} \to Y^{\log}$ over e [by, for instance, *deforming* the unique node of this fiber] as a *single irreducible component* of the fiber of $X_2^{\log} \to X^{\log}$.) Note that in light of this commensurable terminality, the compatibility of $\alpha_{2/1}$ with the outer action of Π_1 on $\Pi_{2/1}$ (relative to the identity automorphism of Π_1 — cf. condition (iii)) implies the compatibility of $\alpha_{2/1}|_{\Pi^{\rm sub}_{2/1}}$ with the outer action of $\Pi_1^{\rm sub}$ on $\Pi^{\rm sub}_{2/1}$ (relative to the identity automorphism of $\Pi_1^{\rm sub}$). Thus, it follows from the commutative diagram in Definition 5.1, (x) (i.e., by applying the natural isomorphism $\Pi_2^{\rm sub} \simeq \Pi^{\rm sub}_{2/1} \overset{\rm out}{\rtimes} \Pi_1^{\rm sub}$ [cf. the discussion entitled "*Topological groups*" in §0]), that the automorphism $\alpha_{2/1}|_{\Pi^{\rm sub}_{2/1}}$ arises from an automorphism $\alpha^{\rm sub}$ of $\Pi_2^{\rm sub}$. Moreover, it follows immediately from the construction of $\alpha^{\rm sub}$ (cf. also [Mzk4], Proposition 1.5, (i)) that $\alpha^{\rm sub}$ is *IFC-admissible* (cf. [Mzk7], Definition 1.1, (ii)), i.e., that $\alpha^{\rm sub}$ satisfies the analogue for $\Pi_2^{\rm sub}$ of the above three conditions (i), (ii), and (iii). Therefore, since the stable log curve $Y^{\log}$ [unlike the stable log curve $X^{\log}$!] necessarily has *at least one cusp*, we may apply [Mzk7], Corollary 1.12, (i), and [Mzk7], Corollary 2.3, (i) (cf. also Remark 5.3.1 below), to conclude that $\alpha^{\rm sub}$ is a $\Xi_2^{\rm sub}$*-inner automorphism* — where we write $\Xi_2^{\rm sub} \overset{\rm def}{=} \Xi_2 \cap \Pi_2^{\rm sub}$ for the analogue of "Ξ_2" for $\Pi_2^{\rm sub}$. In particular, it follows that $\alpha_{2/1}|_{\Pi^{\rm sub}_{2/1}}$ is a Ξ_2*-inner automorphism.*

Now from the point of view of verifying the assertion that α is a Ξ_2-inner automorphism, we may assume without loss of generality — by composing with an appropriate Ξ_2-inner automorphism — that α *stabilizes* and restricts to the *identity automorphism* of $\Pi^{\rm sub}_{2/1}$; in particular, since $\Pi_{v_0^\circ} \subseteq \Pi^{\rm sub}_{2/1}$, it follows that α *stabilizes* and restricts to the *identity automorphism* of $\Pi_{v_0^\circ}$.

Let $\Pi_{e_2^\circ} \subseteq \Pi_{2/1}$ be an edge-like subgroup associated to $e_2^\circ \in \mathrm{Node}(\mathcal{G}_{/e})$ which *is contained in* $\Pi_{v_0^\circ}$, and $\Pi_{v_2^\circ} \subseteq \Pi_{2/1}$ the *unique* (cf. Lemma 5.2, (i)) verticial subgroup associated to $v_2^\circ \in \mathrm{Vert}(\mathcal{G}_{/e})$ that contains $\Pi_{e_2^\circ}$. Now since α *stabilizes* and restricts to the *identity automorphism* of $\Pi_{v_0^\circ}$, it follows that $\alpha(\Pi_{e_2^\circ}) = \Pi_{e_2^\circ}$. Thus, in light of the *graphicity* of α, we may apply Lemma 5.2, (i), to conclude that $\alpha(\Pi_{v_2^\circ}) = \Pi_{v_2^\circ}$. Next, let us observe that the surjection $\Pi_{v_2^\circ} \twoheadrightarrow p_2(\Pi_{v_2^\circ})$ determined by p_2 is an isomorphism. Thus, it follows immediately from condition (iii) that $\alpha|_{\Pi_{v_2^\circ}}$ is the *identity automorphism.*

Since $\Pi_{2/1}$ is topologically generated by $\Pi^{\rm sub}_{2/1}$ and $\Pi_{v_2^\circ}$ (cf. Lemma 5.2, (iii)), the fact (cf. the above discussion) that $\alpha|_{\Pi^{\rm sub}_{2/1}}$ and $\alpha|_{\Pi_{v_2^\circ}}$ are equal to the respective *identity automorphisms* on $\Pi^{\rm sub}_{2/1}$ and $\Pi_{v_2^\circ}$ implies that $\alpha|_{\Pi_{2/1}}$ is the *identity automorphism.* But this implies that α is the *identity automorphism* (cf. the discussion entitled "*Topological groups*" in §0). This completes the proof of Corollary 5.3. □

Remark 5.3.1.

(i) An alternative approach to the portion in the latter half of the proof of Corollary 5.3 where one applies [Mzk7], Corollary 2.3, (i), may be given, at least when $(g, r) \neq (2, 0)$, as follows. One verifies easily that $3g_Y - 3 + r_Y < 3g - 3 + r$, and, moreover that, at least when $(g, r) \neq (2, 0)$, one may always choose $X^{\log}$ (in the situation of Definition 5.1, so $(g, r) \neq (0, 3)$, $(1, 1)$ — cf. Remark 5.1.1) so that $(g_Y, r_Y) \neq (1, 1)$. Thus, by applying *induction* on $3g - 3 + r$, one may reduce this portion of the proof of Corollary 5.3 to the case where $3g - 3 + r = 0$, i.e., the case of a *tripod*. That is to say, instead of applying [Mzk7], Corollary 2.3, (i), it suffices to apply [Mzk7], Corollary 1.12, (i). In particular, this alternative approach yields a *new proof* — at least in the case of $(g, r) \neq (1, 1)$ — of [Mzk7], Corollary 2.3, (ii) (i.e., via Corollary 4.2, as opposed to [Mzk4], Corollary 2.7, (iii) — cf. Remark 4.2.1).

(ii) In passing, we recall that [Mzk4], Corollary 2.7, (iii), is applied in various situations throughout [Mzk7]. In fact, however, (cf. the discussion of (i)) it is not difficult to verify that the partial generalization of [Mzk4], Corollary 2.7, (iii), constituted by Corollary 4.2 (cf. Remark 4.2.1) is *sufficient* (i.e., in the sense that the condition "$\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$" of Corollary 4.2, (iii), is always satisfied) for verifying the various assertions in [Mzk7] (cf. the proof of [Mzk7], Proposition 1.3, (iv)) that are derived from [Mzk4], Corollary 2.7, (iii).

6. Consequences of injectivity

In this section, we discuss various consequences of the injectivity result proven in §5.

The following theorem is a generalization of [Mzk7], Theorem A, (i), (ii).

Theorem 6.1 (**Partial profinite combinatorial cuspidalization).** *Let* Σ *be a set of prime numbers which is either* **of cardinality one** *or* **equal to the set of all prime numbers***,* n *a positive integer,* X *a hyperbolic curve of type* (g, r) *over an algebraically closed field of characteristic* $\notin \Sigma$*,* X_n *the* n*-th* **configuration space** *of* X *(cf.* [MzTa]*, Definition 2.1, (i)),* Π_n *the maximal pro-*Σ *quotient of the fundamental group of* X_n*, and* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *the subgroup of the group* $\mathrm{Out}(\Pi_n)$ *consisting of the outomorphisms (cf. the discussion entitled "Topological groups" in §0) of* Π_n *which are* **FC-admissible** *(cf.* [Mzk7]*, Definition 1.1, (ii)). Set* $n_0 \stackrel{\mathrm{def}}{=} 2$ *if* X *is* **affine***, i.e.,* $r \geq 1$*;* $n_0 \stackrel{\mathrm{def}}{=} 3$ *if* X *is* **proper***, i.e.,* $r = 0$ *(cf.* [Mzk7]*, Theorem A). Then the* *natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

induced by the projection $X_{n+1} \to X_n$ *obtained by forgetting the* $(n+1)$*-st factor is* **injective** *if* $n \geq 1$ *and* **bijective** *if* $n \geq n_0 + 1$. *Moreover, the image of the natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

— where we write $\mathfrak{S}_n$ *for the symmetric group on* n *letters — obtained by permuting the various factors of the configuration space* X_n *is contained in the* **centralizer** $Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n))$.

Proof. First, we consider the *surjectivity* portion of the *bijectivity* assertion in the statement of Theorem 6.1. This surjectivity already follows from [Mzk7], Theorem A, (i), if $n \geq 4$. Thus, we may assume that $n = 3$, which implies that $r \geq 1$. Now by [Mzk7], Lemma 2.4; [Mzk7], Theorem 4.1, (ii), (a), it suffices (cf. the proof of the surjectivity portion of [Mzk7], Theorem 4.1, (i)) to verify (in the notation of [Mzk7]) that $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{cusp}} = \mathrm{Out}^{\mathrm{FCP}}(\Pi_3)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\Delta+}$ — where the first equality follows from [Mzk7], Theorem A, (ii). But this follows from a similar argument to the argument applied to prove [Mzk7], Corollary 3.4, (iii), by taking the section "$\xi \in X_2(X)$" of *loc. cit.* to be the section determined by the *diagonal* and applying the *symmetry* observed in the proof of [Mzk7], Corollary 3.4, (i).

Next, we observe that the assertion concerning the *centralizer* follows immediately from the *injectivity* assertion, together with [Mzk7], Theorem A, (ii); [Mzk7], Proposition 1.2, (iii). Thus, to complete the proof of Theorem 6.1, it suffices to verify the *injectivity* assertion. To this end, write $\Pi_2^{\dagger}$ (respectively, $\Pi_1^{\dagger}$) for the kernel of the surjection $\Pi_{n+1} \twoheadrightarrow \Pi_{n-1}$ (respectively, $\Pi_n \twoheadrightarrow \Pi_{n-1}$) induced by the projection obtained by forgetting the n-th and $(n+1)$-st factors (respectively, the n-th factor). Here, if $n = 1$, then we set $\Pi_{n-1} = \Pi_0 \stackrel{\mathrm{def}}{=} \{1\}$. Then recall (cf. e.g., the proof of [Mzk7], Theorem 4.1, (i)) that we have natural isomorphisms

$$\Pi_{n+1} \simeq \Pi_2^{\dagger} \stackrel{\mathrm{out}}{\rtimes} \Pi_{n-1} \ ; \ \ \Pi_n \simeq \Pi_1^{\dagger} \stackrel{\mathrm{out}}{\rtimes} \Pi_{n-1}$$

(cf. the discussion entitled "*Topological groups*" in §0). Also, we recall (cf. [MzTa], Proposition 2.4, (i)) that one may *interpret* the surjection $\Pi_2^{\dagger} \twoheadrightarrow \Pi_1^{\dagger}$ induced by the surjection $\Pi_{n+1} \twoheadrightarrow \Pi_n$ in question as the surjection "$\Pi_2 \twoheadrightarrow \Pi_1$" of Definition 5.1 (i.e., the surjection that arises from the projection $\mathrm{pr}_2 \colon X_2^{\log} \to X^{\log}$) in the case of an "$X^{\log}$" of type $(g, r + n - 1)$. Moreover, one verifies easily that this *interpretation* is compatible with the definition of the various "Out$(-)$'s" involved. Thus, the above *natural isomorphisms* allow one to reduce the injectivity in question to the case where $n = 1$ (cf. the discussion entitled "*Topological groups*" in §0), which follows immediately from

Corollary 5.3 when $2g-2+r>1$ (cf. Remark 5.1.1) and from [Mzk7], Theorem A, (i), when $2g-2+r=1$. This completes the proof of Theorem 6.1. $\square$

The following corollary is a generalization of [Mts], Theorem 2.2. Note that [Mts], Theorem 2.2, corresponds to the following corollary in the case where k is a subfield of the field of complex numbers, and, moreover, X is a curve of *positive genus* that has *at least one cusp* defined over k.

Corollary 6.2 (**Kernels of outer representations arising from hyperbolic curves**). *Let Σ be a set of prime numbers which is either* **of cardinality one** *or* **equal to the set of all prime numbers**, *X a hyperbolic curve over a perfect field k such that every element of Σ is* **invertible** *in k, $\overline{k}$ an algebraic closure of k, n a positive integer, X_n the n-th configuration space of X, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, Δ_{X_n} the maximal pro-Σ quotient of the fundamental group of $X_n \otimes_k \overline{k}$, and $\Delta_{\mathbb{P}^1_{\overline{k}}\setminus\{0,1,\infty\}}$ the maximal pro-Σ quotient of the fundamental group of $\mathbb{P}^1_{\overline{k}}\setminus\{0,1,\infty\}$. Then the following hold:*

(i) *The kernel of the natural outer representation*

$$\rho^{\Sigma}_{X_n/k} \colon G_k \longrightarrow \mathrm{Out}(\Delta_{X_n})$$

is **independent** *of n and* **contained** *in the kernel of the natural outer representation*

$$\rho^{\Sigma}_{\mathbb{P}^1_k\setminus\{0,1,\infty\}/k} \colon G_k \longrightarrow \mathrm{Out}(\Delta_{\mathbb{P}^1_{\overline{k}}\setminus\{0,1,\infty\}})\,.$$

(ii) *Suppose that Σ is the set of* **all prime numbers**. *(Thus, k is necessarily of characteristic zero.) Write $\overline{\mathbb{Q}}$ for the algebraic closure of $\mathbb{Q}$ determined by $\overline{k}$ and $G_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the kernel of the homomorphism $\rho^{\Sigma}_{X_n/k}$ is* **contained** *in the kernel of the outer homomorphism*

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

Proof. Assertion (ii) follows immediately from assertion (i), together with a well-known *injectivity result of Belyi* (cf., e.g., the discussion surrounding [Mts], Theorem 2.2). Thus, to complete the proof of Corollary 6.2, it suffices to verify assertion (i). It follows immediately from Theorem 6.1 that the kernel of $\rho^{\Sigma}_{X_n/k}$ is independent of n. Moreover, if we denote by $k' \subseteq \overline{k}$ the minimal Galois extension of k over which every cusp of X is defined, then by considering the action of G_k on the set of conjugacy classes of edge-like subgroups of Δ_X associated to cusps of X — a set which admits a natural bijection with the set of cusps of X (cf. e.g., [Mzk4], Proposition 1.2, (i)) — it follows immediately from the various definitions involved that for any n, the kernel of the homomorphism $\rho^{\Sigma}_{X_n/k}$ is contained in $G_{k'} \subseteq G_k$, and that the restriction of $\rho^{\Sigma}_{X_n/k}$ to $G_{k'}$ factors through the subgroup

$$\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n})^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n})$$

defined in [Mzk7], Definition 1.1, (v). Thus, we have continuous homomorphisms

$$(\mathrm{Ker}(\rho^{\Sigma}_{X_n/k}) \subseteq)\, G_{k'} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_3})^{\mathrm{cusp}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}})$$

— where the first arrow is the homomorphism induced by $\rho^{\Sigma}_{X_3/k}$, and the second arrow is the homomorphism determined by the diagonal in X_2 (cf. [Mzk7], Theorem A, (iii)). Moreover, one verifies easily that the *composite* of these homomorphisms *coincides* with $\rho^{\Sigma}_{\mathbb{P}^1_{k'} \setminus \{0,1,\infty\}/k'}$ (cf. the construction of the homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_3})^{\mathrm{cusp}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\}})$$

in [Mzk7]). Now assertion (i) follows immediately. □

The injectivity portion of assertion (i) (in the case where $n = 1$) of the following corollary is a generalization of [Mts], Theorem 2.1. Note that [Mts], Theorem 2.1, corresponds to the following corollary in the case where X is *affine*.

Corollary 6.3 (**Injectivity and commensurable terminality for outer representations arising from hyperbolic curves).** *In the situation of Corollary 6.2, suppose that k is a* **number field** *or* ***p*-adic local field** *(cf. the discussion entitled "Numbers" in §0), and that Σ is the set of* **all prime numbers**. *Write $k_0 \stackrel{\mathrm{def}}{=} \mathbb{Q}$ if k is a number field; $k_0 \stackrel{\mathrm{def}}{=} \mathbb{Q}_p$ if k is a p-adic local field; $\mathrm{Aut}(X_{\overline{k}}/k_0)$ for the group of k_0-linear automorphisms of the scheme $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \otimes_k \overline{k}$; $\rho_n \stackrel{\mathrm{def}}{=} \rho^{\Sigma}_{X_n/k}$. Then the following hold:*

(i) *The outer representation*

$$\rho_n : G_k \longrightarrow \mathrm{Out}(\Delta_{X_n})$$

is **injective**. *Moreover, the outer representations ρ_{n+1} and ρ_n are* **compatible**, *in the evident sense, with the injection $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_{n+1}}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n})$ of Theorem 6.1.*

(ii) *Every $\alpha \in \mathrm{Aut}(X_{\overline{k}}/k_0)$ induces a k_0-linear automorphism of $(X_n)_{\overline{k}} \stackrel{\mathrm{def}}{=} X_n \otimes_k \overline{k}$. In particular, we have a natural outer representation*

$$\rho_{n/0} : \mathrm{Aut}(X_{\overline{k}}/k_0) \longrightarrow \mathrm{Out}(\Delta_{X_n})$$

which **factors** *through* $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n}) \subseteq \mathrm{Out}(\Delta_{X_n})$ *and is* **compatible** *with* ρ_n *relative to the natural injection* $G_k \hookrightarrow \mathrm{Aut}(X_{\overline{k}}/k_0)$ *determined by taking the fiber product over* $\mathrm{Spec}\,(k)$ *with* X*. Moreover, the outer representations* $\rho_{n+1/0}$ *and* $\rho_{n/0}$ *are* **compatible***, in the evident sense, with the injection* $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_{n+1}}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n})$ *of Theorem 6.1.*

(iii) *The outer representation* $\rho_{n/0}$ *of (ii) is* **injective***.*

(iv) *Suppose that the hyperbolic curve* X *is of* **quasi-Belyi type** *[cf.* [Mzk5]*, Definition 2.3, (iii)] (respectively,* **affine***;* **proper***). Set* $n_0 \stackrel{\mathrm{def}}{=} 1$ *(respectively,* $n_0 \stackrel{\mathrm{def}}{=} 2$*;* $n_0 \stackrel{\mathrm{def}}{=} 3$*). Then the image of* $\rho_{n/0}$ *is* **commensurably terminal** *in* $\mathrm{Out}^{\mathrm{F}}(\Delta_{X_n})$ *(cf.* [Mzk7]*, Definition 1.1, (ii)) for all* $n \geq n_0$*.*

Proof. The *injectivity* portion of assertion (i) follows immediately from Corollary 6.2, (ii), together with the injectivity of the outer homomorphism "$G_k \to G_{\mathbb{Q}}$" in the statement of Corollary 6.2, (ii), when k is a number field or p-adic local field. The *compatibility* portion of assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from the various definitions involved. Next, we consider assertion (iii). In light of the *compatibility* portion of assertion (ii), it suffices to verify assertion (iii) in the case where $n = 1$. Write $\mathrm{Aut}(X_{\overline{k}}/\overline{k}) \subseteq \mathrm{Aut}(X_{\overline{k}}/k_0)$ for the subgroup of $\overline{k}$-linear automorphisms. Then the injectivity of the restriction of $\rho_{1/0}$ to $\mathrm{Aut}(X_{\overline{k}}/\overline{k})$ is *well-known* (cf. e.g., the injectivity portion of [Mzk1], Theorem A). On the other hand, one verifies immediately that by restricting an automorphism $\alpha \in \mathrm{Aut}(X_{\overline{k}}/k_0)$ to the base field $\overline{k}$, one obtains a *natural exact sequence*

$$1 \longrightarrow \mathrm{Aut}(X_{\overline{k}}/\overline{k}) \longrightarrow \mathrm{Aut}(X_{\overline{k}}/k_0) \longrightarrow \mathrm{Gal}(\overline{k}/k_0)$$

such that the image of the homomorphism $\mathrm{Aut}(X_{\overline{k}}/k_0) \to \mathrm{Gal}(\overline{k}/k_0)$ contains $G_k = \mathrm{Gal}(\overline{k}/k)$, hence is *open*. Thus, it follows immediately from the *injectivity* portion of assertion (i) (cf. also the first compatibility discussed in assertion (ii)) that the kernel of $\rho_{1/0}$ maps isomorphically to a *finite normal closed subgroup* of some open subgroup of the *slim* profinite group $\mathrm{Gal}(\overline{k}/k_0)$ (cf. e.g., [Mzk2], Theorem 1.1.1, (ii)), hence is *trivial*, as desired. This completes the proof of assertion (iii). Finally, we consider assertion (iv). First, let us observe that it follows immediately from [Mzk4], Corollary 2.7, (i) (cf. also [Mzk7], Remark 1.1.3), that the commensurator of the image of $\rho_{n/0}$ in $\mathrm{Out}^{\mathrm{F}}(\Delta_{X_n})$ is in fact *contained in* $\mathrm{Out}^{\mathrm{FC}}(\Delta_{X_n})$. Thus, it suffices to verify assertion (iv) with "$\mathrm{Out}^{\mathrm{F}}(-)$" replaced by "$\mathrm{Out}^{\mathrm{FC}}(-)$". Next, let us observe that by the *injectivity* portion of Theorem 6.1, it suffices to verify assertion (iv) in the case where $n = n_0$. Thus, let us assume that $n = n_0$. Then in light of assertion (iii), together with the fact that Δ_{X_n} is *slim* (cf. the discussion entitled *"Topological Groups"* in §0; [MzTa], Proposition 2.2, (ii)), assertion (iv) follows immediately — in the case where the hyperbolic curve X is of *quasi-Belyi type* (respectively,is *affine*; *proper*) — from the *"Grothendieck Conjecture-type result"* given in [Mzk5], Corollary 2.3 (respectively, [Mzk6], Corollary 1.11, (iii), (iv); [Mzk6], Corollary 1.11, (iii), (iv)). ☐

The following corollary is a generalization of [MtTa], Theorem 1.1. Note that [MtTa], Theorem 1.1, corresponds to the following corollary in the case where $r \geq 1$.

Corollary 6.4 (**Triviality of simultaneously arithmetic-geometric actions**). *Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, (g, r) a pair of natural numbers such that $2g - 2 + r > 0$, $(\mathcal{M}_{g,r})_k$ the moduli stack of r-pointed smooth curves of genus g over k whose marked points are equipped with an ordering, $\Pi_{\mathcal{M}_{g,r}}$ the profinite fundamental group of the stack $(\mathcal{M}_{g,r})_k$, and $\Delta_{\mathcal{M}_{g,r}}$ the profinite fundamental group of the stack $(\mathcal{M}_{g,r})_k \otimes_k \overline{k}$; thus, we have an exact sequence*

$$1 \longrightarrow \Delta_{\mathcal{M}_{g,r}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1\,.$$

Moreover, let X be a hyperbolic curve of type (g, r) over k,

$$\rho_{X/k} \colon \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$$

the **outer representation arising from the hyperbolic curve** X *over k, i.e., the outer representation arising from the natural exact sequence*

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1\,,$$

and

$$\rho_{g,r} \colon \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$$

the **profinite universal monodromy outer representation** *over k, i.e., the outer representation arising from the natural exact sequence*

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \Pi_{\mathcal{M}_{g,r+1}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow 1\,.$$

Then the subgroup

$$\rho_{X/k}^{-1}(\rho_{g,r}(\Delta_{\mathcal{M}_{g,r}})) \subseteq \mathrm{Gal}(\overline{k}/k)$$

of $\mathrm{Gal}(\overline{k}/k)$ *is* **contained** *in the kernel of the outer homomorphism*

$$\mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

In particular, if k is a **number field** *or* ***p*-adic local field**, *then the intersection of the image of the outer representation*

$$\rho_{X/k} \colon \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$$

and the image of the restriction

$$\rho_{g,r}|_{\Delta_{\mathcal{M}_{g,r}}}: \Delta_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$$

of $\rho_{g,r}$ *to* $\Delta_{\mathcal{M}_{g,r}} \subseteq \Pi_{\mathcal{M}_{g,r}}$ *is* **trivial**.

Proof. The various assertions of Corollary 6.4 follow from Theorem 6.1 via a similar argument to the argument used in the proof of Corollary 6.2, (i), (ii). Alternatively, one may derive Corollary 6.4 directly from Corollary 6.2, (ii) — where we take "k" to be the *function field* of $\mathcal{M}_{g,r}$ — via a similar argument to the argument used in the proof of [MtTa], Theorem 1.1, i.e., by considering, in effect, the *semi-direct product decomposition* $\Pi_{\mathcal{M}_{g,r}} \simeq \Delta_{\mathcal{M}_{g,r}} \rtimes \mathrm{Gal}(\overline{k}/k)$ determined by the k-valued point of $\mathcal{M}_{g,r}$ corresponding to X. □

Corollary 6.5 (**Outer representations arising from moduli stacks of stable curves).** *Let* k *be a* **number field** *or* **p-adic local field**, $\overline{k}$ *an algebraic closure of* k, (g, r) *a pair of natural numbers such that* $2g - 2 + r > 0$, $(\mathcal{M}_{g,r})_k$ *the moduli stack of* r*-pointed smooth curves of genus* g *over* k *whose marked points are equipped with an ordering,* $\Pi_{\mathcal{M}_{g,r}}$ *the profinite fundamental group of the stack* $(\mathcal{M}_{g,r})_k$, $\Delta_{\mathcal{M}_{g,r}}$ *the profinite fundamental group of the stack* $(\mathcal{M}_{g,r})_k \otimes_k \overline{k}$, $\Pi_{g,r}$ *the profinite completion of the surface group of type* (g, r) *(i.e., the topological fundamental group of the complement of* r *distinct points in a compact oriented topological surface of genus* g*), and*

$$\rho_{g,r}: \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\Pi_{g,r})$$

the **profinite universal monodromy outer representation** *over* k. *Then the congruence subgroup problem for the pair* (g, r) *may be resolved in the* **affirmative** *(i.e., the restriction of* $\rho_{g,r}$ *to* $\Delta_{\mathcal{M}_{g,r}} \subseteq \Pi_{\mathcal{M}_{g,r}}$ *is* **injective***)* **if and only if** *the homomorphism* $\rho_{g,r}$ *is* **injective**.

Proof. This follows immediately from Corollary 6.4, by considering a hyperbolic curve "X" of type (g, r) that is defined over k (as in the statement of Corollary 6.4). Alternatively, one may deduce Corollary 6.5 directly from Corollary 6.2, (ii), by applying Corollary 6.2, (ii), to the *function field* of $\mathcal{M}_{g,r}$. □

The following corollary is a generalization of [Mzk7], Corollary 5.1, (ii), (iv).

Corollary 6.6 (**Discrete combinatorial cuspidalization).** *Let* (g, r) *be a pair of natural numbers such that* $2g - 2 + r > 0$, n *a positive integer,* $\mathcal{X}$ *a topological surface of type* (g, r) *(i.e., the complement of* r *distinct points in a compact oriented topological surface of genus* g*),* $\mathcal{X}_n$ *the* n*-th configuration space of* $\mathcal{X}$, Π_n *the topological fundamental group of* $\mathcal{X}_n$, *and* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *the subgroup of the group* $\mathrm{Out}(\Pi_n)$ *of outomorphisms (cf. the discussion entitled "Topological groups" in*

§0) of Π_n *defined in the statement of* [Mzk7], *Corollary 5.1. Then the natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

induced by the projection $\mathcal{X}_{n+1} \to \mathcal{X}_n$ *obtained by forgetting the* $(n+1)$*-st factor is* **bijective**. *Moreover, the image of the natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

— where we write $\mathfrak{S}_n$ *for the symmetric group on n letters — obtained by permuting the various factors of the configuration space* $\mathcal{X}_n$ *is contained in the* **centralizer** $Z_{\mathrm{Out}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n))$.

Proof. The assertion concerning the *centralizer* follows immediately from the *bijectivity* assertion, together with [Mzk7], Corollary 5.1, (iv), and the easily verified *discrete analogue* of [Mzk7], Proposition 1.2, (iii) (which may be verified, for instance, by applying [Mzk7], Corollary 5.1, (i); [Mzk7], Proposition 1.2, (iii)). Thus, to complete the proof of Theorem 6.1, it suffices to verify the *bijectivity* assertion. Moreover, it follows from [Mzk7], Corollary 5.1, (ii), that to complete the proof of the *bijectivity* assertion, it suffices to verify the *injectivity* portion of this bijectivity assertion. On the other hand, this injectivity follows from Theorem 6.1, together with [Mzk7], Theorem 5.1, (i). That is to say, the injectivity of the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ follows from the commutativity of the diagram of natural homomorphisms

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\widehat{\Pi}_{n+1}) \\ \downarrow & & \downarrow \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\widehat{\Pi}_n) \end{array}$$

— where we write "$\widehat{\Pi}_{(-)}$" for the profinite completion of "$\Pi_{(-)}$" — together with the *injectivity* of the upper horizontal and right-hand vertical arrows of the diagram. □

Remark 6.6.1. Just as in the case of [Mzk7], Corollary 5.1, there is a partial overlap between the content of Corollary 6.6 above and Theorems 1, 2 of [IIM].

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