COMMENTS ON THE MAIN THEOREM OF POP-STIX

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Let k be a finite extension of $\mathbb{Q}_p$, $\overline{k}$ an algebraic closure of k, and X a *proper hyperbolic curve* over k. Write Π_X for the *étale fundamental group* of X [relative to some basepoint], Π_X^{tp} for the *tempered fundamental group* of X [relative to some basepoint], and

$$\Pi_X \twoheadrightarrow G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k), \quad \Pi_X^{\text{tp}} \twoheadrightarrow G_k$$

for the natural projections. Also, we write K for the function field of X and $\mathcal{O}_k$ for the ring of integers of k. Then:

(1) During the conference on Galois-Teichmüller theory held at RIMS, Kyoto University, in October 2010, it was suggested by Yves André that the Main Theorem of [PS] should imply the following assertion:

(A1) Every section of $\Pi_X \twoheadrightarrow G_k$ arises, up to Π_X-conjugacy, from a section of $\Pi_X^{\text{tp}} \twoheadrightarrow G_k$.

We refer to (7) below for a more detailed formulation. One consequence of this assertion (A1) is the following assertion:

(A2) The *"Profinite p-adic Section Conjecture"* for hyperbolic curves may be reduced to the *"Tempered p-adic Section Conjecture"* for hyperbolic curves.

As far as I can see, (A1) and (A2) may be verified immediately.

(2) In light of the discussion of (1), it is natural to ask the question (cf. also the discussion of (8) below):

(Q1) If one *starts* with a section $s : G_k \to \Pi_X^{\text{tp}}$ of $\Pi_X^{\text{tp}} \twoheadrightarrow G_k$, then does the Main Theorem of [PS] tell you anything *new* concerning the section s?

In this context, we observe that, even *without applying the theory of [PS]*, it follows immediately from [Semi], Theorem 5.4, (i) — cf. also the discussion of [Semi], Example 5.6; [Semi], Remark 6.9.1 — that the section s determines a *system of vertices*

$$\dots \rightsquigarrow v_{i+1} \rightsquigarrow v_i \rightsquigarrow \dots$$

of any cofinal system of finite étale connected Galois coverings of X with stable reduction

$$\dots \rightarrow X_{i+1} \rightarrow X_i \rightarrow \dots$$

of X — i.e., each v_i is an irreducible component of the special fiber of the stable model of X_i that is *fixed* by the natural action of the image $\mathrm{Im}(s)$ of the section s. Here, i ranges over the positive integers, and, after possibly passing to a cofinal subsystem, the notation "$\rightsquigarrow$" may be interpreted as the statement that one of the following two [mutually exclusive] conditions holds:

(C1) For each positive integer i, the irreducible component v_{i+1} maps *quasi-finitely* to the irreducible component v_i.

(C2) For each positive integer i, the irreducible component v_{i+1} maps to a *closed point* x_i of the irreducible component v_i.

In the case of (C1), the v_i determine a *discrete valuation* of K that contains $\mathcal{O}_k$, but not k. In the case of (C2), the v_i determine a system of *closed points*

$$\dots \mapsto x_{i+1} \mapsto x_i \mapsto \dots$$

and hence a corresponding inductive system of normal local rings

$$\dots \hookrightarrow R_i \hookrightarrow R_{i+1} \hookrightarrow \dots$$

— each of which contains $\mathcal{O}_k$, but not k — whose union R_∞ is, consequently, a *normal local ring* that is *not necessarily noetherian.* That is to say, in either of the two cases (C1), (C2):

(A3) One obtains a *normal local ring* $R \subseteq K$ that contains $\mathcal{O}_k$, but not k, such that the section $s : G_k \rightarrow \Pi_X^{\mathrm{tp}}$ is contained in [one of the Π_X^{tp}-conjugates of] the *decomposition group* associated to R.

The statement of (A3) is *reminiscent* of the Main Theorem of [PS] in the sense that, like the Main Theorem of [PS], the conclusion (A3) may be thought of as a *reduction of the Section Conjecture to an essentially local problem.* On the other hand,

(Q2) it is not clear, in the case of (C2), whether or not the normal local ring R that is obtained in (A3) is necessarily a *valuation ring.*

Perhaps a definitive answer to (Q2) would require results along the lines of Tamagawa's "resolution of nonsingularities" (cf. [Tama]). It would be interesting, however, if one could give a definitive answer to (Q2) via an elementary argument. Although I have not studied the proof of the Main Theorem of [PS] in detail, at least at a naive level, it is difficult to believe that one must apply [Tama] in order to give an affirmative answer to (Q2): Indeed, at least at a naive level, it it difficult to believe that the essential consequence of [Tama] in the present context should be any *stronger* than the existence of a system of vertices $\{v_i\}_i$ as above.

(3) Thus, in summary, the theory of [Semi] already implies the statement (A3) concerning sections $s : G_k \to \Pi_X^{\mathrm{tp}}$ of the tempered fundamental group. From this point of view:

(A4) In the tempered case, the Main Theorem of [PS] may be regarded as a strengthening of (A3) to the effect that the ring R of (A3) may be taken to be a *valuation ring* (that may possibly contain k).

That is to say, (A4) may be thought of as being at least a partial answer to (Q1).

(4) Another consequence of [Semi], Theorem 5.4, (i), and [Semi], Lemma 5.5, is the conclusion that, after possibly passing to a cofinal subsystem, *one* of the following holds:

(U1) The system of vertices $\{v_i\}_i$ obtained in (3) is "essentially" *unique.*

(U2) There exists an "essentially" *unique system of edges* $\{e_i\}_i$ — i.e., each e_i is a node of the special fiber of the stable model of X_i such that $e_{i+1} \mapsto e_i$ — that are *fixed* by the natural action of $\mathrm{Im}(s)$.

Here, the term "essentially" is used in the evident sense; we leave the routine task of giving a precise formulation to the reader. In particular, in the case that (U1) holds, one concludes that the system of vertices $\{v_i\}_i$ *coincides* with the system of vertices determined by any *valuation* obtained as in the Main Theorem of [PS].

(5) In fact, unlike the case with [PS], by combining the analysis of verticial decomposition groups given in [NodNon], Proposition 3.9, (i), with a similar argument to the argument used (in [Semi]) to prove [Semi], Theorem 5.4, (i), one concludes that the consequence (A3) obtained above from [Semi], Theorem 5.4, (i), continues to hold if one takes k to be an *arbitrary complete discrete valuation field of mixed characteristic.* (Here, we note in passing that one verifies easily — by considering the *pro-p Kummer map* — that, for any such k, the resulting absolute Galois group G_k continues to be *center-free*, hence *slim.*) In this situation, the analysis of verticial decomposition groups given in [NodNon], Proposition 3.9, (i), also yields the following version of "uniqueness":

(U3) For each i, any two possibilities for the vertex v_i either *coincide*, are *adjacent*, or *admit a common adjacent vertex.*

Here, we note that (U3) is weaker that the version of "uniqueness" involving (U1), (U2) [cf. (4)] precisely because one does not have an analogue of [Semi], Lemma 5.5, in the case of more general k.

(6) In a similar vein, unlike the case with [PS], the consequences (A3), (U3) obtained above continue to hold, even in the more general setting discussed in (5), if, for Σ

a *set of primes* that contains a prime $l \neq p$, one replaces Π_X by the *geometrically pro-*Σ *quotient*

$$\Pi_X \twoheadrightarrow \Pi_X^{(\Sigma)}$$

of Π_X and Π_X^{tp} by the *"*Σ*-tempered fundamental group"*

$$\Pi_X^{\mathrm{tp},(\Sigma)} \subseteq \Pi_X^{(\Sigma)}$$

given by the image of Π_X^{tp} in $\Pi_X^{(\Sigma)}$.

(7) In the notation of (6), we observe that another important consequence of [Semi], Theorem 5.4, (i), in the present context is the following:

(A5) The natural map

$$\mathrm{Sect}(\Pi_X^{\mathrm{tp},(\Sigma)}/G_k) \ \to \ \mathrm{Sect}(\Pi_X^{(\Sigma)}/G_k)$$

— i.e., from $\Pi_X^{\mathrm{tp},(\Sigma)}$-conjugacy classes of sections of $\Pi_X^{\mathrm{tp},(\Sigma)} \to G_k$ to $\Pi_X^{(\Sigma)}$-conjugacy classes of sections of $\Pi_X^{(\Sigma)} \to G_k$ — is *injective*.

In particular, we obtain the following consequence of the Main Theorem of [PS] (cf. (A1)):

(A6) If Σ is the set of all primes, and k is a finite extension of $\mathbb{Q}_p$, then the natural map

$$\mathrm{Sect}(\Pi_X^{\mathrm{tp}}/G_k) \ \to \ \mathrm{Sect}(\Pi_X/G_k)$$

is *bijective*.

A proof of (A5) may be sketched as follows. Consider two sections $s, t : \Pi_X^{\mathrm{tp},(\Sigma)} \to G_k$ such that $s^\gamma = t$ for some $\gamma \in \Pi_X^{(\Sigma)}$ (where the superscript "γ" denotes conjugation by γ). Then, by applying (A3), we conclude that the sections s, t give rise, respectively, to systems $\{v_i\}_i$, $\{w_i\}_i$ of vertices, which (cf. (U3); [NodNon], Proposition 3.9, (i)) are, in some appropriate sense, *"essentially unique"*. In particular, we obtain that, for each i, the vertices w_i and $\gamma(v_i)$ either *coincide*, are *adjacent*, or *admit a common adjacent vertex*. But this implies that the systems $\{v_i\}$ and $\{\gamma(v_i)\}$ correspond to the *same "tempered basepoint"*, i.e., that $\gamma \in \Pi_X^{\mathrm{tp},(\Sigma)}$, so s and t are $\Pi_X^{\mathrm{tp},(\Sigma)}$-conjugate, as desired.

(8) One reason why I was interested in, for instance, the pSC (i.e., the "(profinite/pro-Σ) p-adic Section Conjecture for hyperbolic curves") was because I was able to show in [AbsTopII], Corollary 2.9, that

$$p\mathrm{SC} \Longrightarrow \mathrm{abs}\ p\mathrm{GC}$$

— i.e., that the pSC implies the "absolute (profinite/pro-Σ) p-adic version of the Grothendieck Conjecture" (cf. [AbsTopII], Corollary 2.9, for more details). On the other hand, from the point of view of verifying the abs pGC, results such as (A2) or (A6) are *not so interesting*, since one already knows (cf., e.g., [Semi], Theorem 6.6) that the absolute *profinite* p-adic version of the Grothendieck Conjecture may be reduced to the absolute *tempered* p-adic version of the Grothendieck Conjecture. It is precisely this state of affairs that prompted me, upon hearing of (A2), to pose the question (Q1).

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