

RESPONSE TO A QUESTION OF TAN POSED DURING THE SECOND SKYPE SESSION OF THE DECEMBER 2015 OXFORD WORKSHOP ON INTER-UNIVERSAL TEICHMÜLLER THEORY

SHINICHI MOCHIZUKI

December 2015

In the second skype session of the December 2015 Oxford Workshop on IUTeich (i.e., inter-universal Teichmüller theory), Fucheng Tan posed the following question:

Why, in IUTeich, does one need to consider *arbitrary iterates* of the **log**-link rather than just a *single iterate*?

The reason that one must consider *arbitrary iterates* of the **log**-link is that, in order to construct objects in the *domain* of the Θ -link that may be described in terms of structures that arise in the *codomain* of the Θ -link, it is necessary to construct objects that are

invariant with respect to vertical shifts, i.e., applications of the **log**-link, in the log-theta-lattice.

This issue is discussed and emphasized in numerous places, such as the following: the Introduction to [IUTchIII]; [IUTchIII], Remark 3.11.3; [IF-IUT], §2.8. Here, we remark that references to this phenomenon in [IUTchIII] may be located by performing a search of the PDF file for the string “shift”. The main points may be summarized as follows:

Since the Θ -link is **not compatible** with **ring structures** — which are necessary to define the power series for the *logarithm* that appears in the **log**-link! — one can only see, from the point of view of the *codomain* of the Θ -link, the **unit group** and **value group** portions [i.e., of the data that appears in the *domain* of the Θ -link] associated to the **same** vertical coordinate “ m ” of the vertical line of the log-theta-lattice that corresponds to the *domain* of the Θ -link under consideration. On the other hand, one wants to apply the **log**-link, say, from the vertical coordinate m to the vertical coordinate $m + 1$ in order to give a presentation of the **value group** portion at $m + 1$ by means of an action of this value group portion at $m + 1$ on the *log-shell* that arises from applying the **log**-link to the **unit group** portion at m . That is to say, this presentation of *value group portions acting on log-shells* can only be seen from the *codomain* of the Θ -link if, so to speak, “ $m = m + 1$ ” — a situation that can only be achieved if, so to speak, “ $m = \infty$ ”, i.e., if one considers structures that are **invariant** with respect to **arbitrary vertical shifts** “ $m \mapsto m + 1$ ”.

Bibliography

- [IF-IUT] I. Fesenko, Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta-functions, notes on the work of Shinichi Mochizuki, *Eur. J. Math.* **1** (2015), pp. 405-440.
- [IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, preprint available from the following web page:
<http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-japanese.html>