REPORT ON DISCUSSIONS, HELD DURING THE PERIOD MARCH 15 – 20, 2018, CONCERNING INTER-UNIVERSAL TEICHMÜLLER THEORY (IUTCH)

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§1. The present document is a report on discussions held during the period March 15 – 20, 2018, concerning **inter-universal Teichmüller theory (IUTch)**. These discussions were held in a seminar room on the fifth floor of Maskawa Hall, Kyoto University, according to the following schedule:

· March 15 (Thurs.): 2PM $\sim$ between 5PM and 6PM,
· March 16 (Fri.): 10AM $\sim$ between 5PM and 6PM,
· March 17 (Sat.): 10AM $\sim$ between 5PM and 6PM,
· March 19 (Mon.): 10AM $\sim$ between 5PM and 6PM,
· March 20 (Tues.): 10AM $\sim$ between 5PM and 6PM.

(On the days when the discussions began at 10AM, there was a lunch break for one and a half to two hours.) Participation in these discussions was restricted to the following mathematicians (listed in order of age): *Peter Scholze*, *Yuichiro Hoshi*, *Jakob Stix*, and *Shinichi Mochizuki*. All four mathematicians participated in all of the sessions listed above (except for Hoshi, who was absent on March 16). The existence of these discussions was kept confidential until the conclusion of the final session. From an organizational point of view, the discussions took the form of "negotations" between two "teams": one team (HM), consisting of *Hoshi* and *Mochizuki*, played the role of explaining various aspects of IUTch; the other team (SS), consisting of *Scholze* and *Stix*, played the role of challenging various aspects of the explanations of HM. Most of the sessions were conducted in the following format: Mochizuki would stand and explain various aspects of IUTch, often supplementing oral explanations by writing on whiteboards using markers in various colors; the other participants remained seated, for the most part, but would, at times, make questions or comments or briefly stand to write on the whiteboards.

§2. Scholze has, for some time, taken a somewhat *negative* position concerning IUTch, and his position, and indeed the position of SS, remained negative even after the March discussions. My own conclusion, and indeed the conclusion of HM, after engaging in the March discussions, is as follows:

> *The negative position of SS is a consequence of certain* **fundamental misunderstandings** *(to be explained in more detail in the remainder of*

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the present report — cf. §17 for a brief summary) on the part of SS concerning IUTch, and, in particular, does **not imply** *the existence of any* **flaws** *whatsoever in IUTch.*

The **essential gist** of these misunderstandings — many of which center around **erroneous attempts** to **"simplify"** IUTeich — may be summarized *very roughly* as follows:

(Smm) Suppose that A and B are *positive real numbers*, which are *defined* so as to satisfy the *relation*

$$-2B = -A$$

(which corresponds to the **Θ-link**). One then proves a *theorem*

$$-2B \leq -2A + 1$$

(which corresponds to the **multiradial representation** of [IUTchIII], Theorem 3.11). This *theorem*, together with the above *defining relation*, implies a *bound on A*

$$-A \leq -2A + 1, \quad \text{i.e.,} \quad A \leq 1$$

(which corresponds to [IUTchIII], Corollary 3.12). From the point of view of this (*very rough*!) summary of IUTch, the *misunderstandings* of SS amount to the assertion that

the theory remains *essentially unaffected even if one takes* $A = B$,

which implies (in light of the above *defining relation*) that $A = B = 0$, in *contradiction* to the initial assumption that A and B are *positive* real numbers. In fact, however, the essential content (i.e., main results) of IUTch **fail(s) to hold** under the assumption "$A = B$"; moreover, the *"contradiction"* $A = B = 0$ is nothing more than a superficial consequence of the *extraneous assumption* "$A = B$" and, in particular, does **not imply** the existence of any **flaws** whatsoever in IUTch. (We refer to (SSIdEx), (ModEll), (HstMod) below for a "slightly less rough" explanation of the *essential logical structure* of an issue that is closely related to the *extraneous assumption* "$A = B$" in terms of

· *complex structures on real vector spaces*

or, alternatively (and essentially equivalently), in terms of the well-known classical theory of

· *moduli of complex elliptic curves.*

Additional comparisons with well-known classical topics such as

· the *invariance of heights of elliptic curves over number fields with respect to isogeny*,
· *Grothendieck's definition of the notion of a connection*, and
· the *differential geometry surrounding $SL_2(\mathbb{R})$*

may be found in §16.)

Indeed, in the present context, it is perhaps useful to recall the following well-known *generalities* concerning *logical reasoning*:

(GLR1) Given **any mathematical argument**, it is always easy to derive a **contradiction** by arbitrarily **identifying** mathematical objects that must be regarded as **distinct** in the situation discussed in the argument. On the other hand, this does **not**, by any means, **imply** the existence of any **logical flaws** in the original mathematical argument!

(GLR2) Put another way, the **correct interpretation** of the **contradiction** obtained in (GLR1) is — *not* the conclusion that the *original argument*, in which the *arbitrary identifications* of (GLR1) were not in force, has *logical flaws* (!), but rather — the conclusion that the contradiction obtained in (GLR1) implies that the distinct mathematical objects that were arbitrarily identified are *indeed distinct*, i.e., must be treated (in order, for instance, to arrive at an accurate understanding of the original argument!) as **distinct** mathematical objects!

It is most unfortunate indeed that the March discussions were insufficient from the point of view of overcoming these misunderstandings. On the other hand, my own experience over the past six years with regard to exposing IUTch to other mathematicians is that this sort of *short period* (roughly a week) is *never sufficient*, i.e., that

> *substantial progress in understanding IUTch always requires discussions over an* **extended period of time**, *typically on the* **order of months**.

Indeed, the issue of lack of time became especially conspicuous during the afternoon of the final day of discussions. Typically, short periods of interaction center around *reacting in real time* and do not leave participants the time to *reflect deeply* on various aspects of the mathematics under discussion. This sort of *deep reflection*, which is absolutely necessary to achieve fundamental progress in understanding, can only occur in situations where the participants are afforded the opportunity to think at their leisure and forget about any time or deadline factors. (In this context, it is perhaps of interest to note that Scholze contacted me in May 2015 by e-mail concerning a question he had regarding the non-commutativity of the log-theta-lattice in IUTch (i.e., in effect, "(Ind3)"). This contact resulted in a short series of e-mail exchanges in May 2015, in which I addressed his (somewhat vaguely worded) question as best I could, but this did not satisfy him at the time. On the other hand, the March 2018 discussions centered around *quite different issues*, such as (Ind1, 2), as will be described in detail below.)

§3. On the other hand, it seems that the March discussions may in fact be regarded as constituting *substantial progress* in the following sense. Prior to the March discussions, (at least to my knowledge)

> *negative positions concerning IUTch were always discussed in highly* **non-mathematical** *terms, i.e., by focusing on various aspects of the situation that were quite far removed from any sort of* **detailed, well-defined, mathematically substantive content**.

That is to say, although it is most regrettable that it was not possible to resolve the fundamental misunderstandings of SS during the March discussions, nevertheless,

> *the March discussions were highly meaningful in that, to my knowledge, they constitute the* **first detailed, well-defined, mathematically substantive** *discussions concerning* **negative positions** *with regard to IUTch.*

Put another way, as a result of the *absence*, during the past six years, of such detailed, well-defined, mathematically substantive discussions concerning negative positions with regard to IUTch, the highly **non-mathematical tone** that has appeared, up till now, in statements by mathematicians critical of IUTch has had the effect of giving the impression that

> *any existing criticism of IUTch is* **entirely devoid of any substantive mathematical content***, i.e., based on entirely non-mathematical considerations.*

On the other hand, as I have emphasized on numerous occasions throughout the past six years (cf., e.g., [Rpt2014], (7); [QC2016], (4), (5)),

> *the* **only** *way to make meaningful, substantive progress with regard to differences of opinion concerning IUTch is by means of discussions concerning* **detailed, well-defined, mathematically substantive content***.*

From this point of view, it seems most desirable that the mathematical content discussed in the present report be made available for *further discussion by all interested mathematicians* (i.e., not just the participants in the March discussions!), in the hope that the present report might play the role of serving to stimulate further *detailed, well-defined, mathematically substantive dialogue* concerning issues related to IUTch. In this context, it seems also of *fundamental importance* to keep the following *historical point of view* in mind:

> It is only by supplementing negative positions concerning IUTch with **detailed, mathematically substantive, accessible records** of the mathematical content underlying such negative positions that humanity can avoid creating an *unfortunate situation* — i.e., of the sort that arose concerning the "proof" asserted by Fermat of "Fermat's Last Theorem"! — in which an accurate evaluation of the substantive mathematical content underlying such negative positions will **remain impossible indefinitely**.

Further remarks concerning the involvement of other mathematicians may be found in §18 below.

§4. Before proceeding to our exposition of the mathematical content of the March discussions, we pause to list briefly the various topics that seem to have been the **main themes** of the March discussions:

(T1) the treatment, in IUTch, of **histories** of various operations performed on mathematical objects;

(T2) the treatment, in IUTch, of **types of mathematical objects** (i.e., *"species"*, in the sense of [IUTchIV], §3);

(T3) the **"id-version"**, i.e., a variant of IUTch obtained by **identifying various copies** (such as *Frobenius-like* and *étale-like* versions, as well as copies appearing in *different Hodge theaters*) of familiar objects by means of the **"identity morphism"** (cf. the discussion of §2!);

(T4) opposition by SS to the use of **poly-morphisms** in IUTch

(T4-1) as a **matter of taste**,

(T4-2) on the grounds that the introduction of indeterminacies such as (Ind1, 2) seemed to SS to be **logically unnecessary** or **"meaningless"**;

(T5) opposition by SS to the use of **labels** in IUTch to distinguish **distinct copies** of various familiar objects

(T5-1) as a **matter of taste**,

(T5-2) on the grounds that the use of such labels seemed to SS to be **logically unnecessary** or **"meaningless"**;

(T6) **refusal, on the part of SS, to consider** various key ideas and notions of IUTch such as *distinct arithmetic holomorphic structures* (i.e., in essence, *distinct ring structures*);

(T7) the issue of **simplification**;

(T8) **occasional misinterpretation by SS** of statements by HM of the form (T8-1) as definitive declarations of (T8-2):

(T8-1) "you may consider such and such a modified version of IUTch if you wish, but it is by no means clear that the essential content of IUTch is valid for such a modified version",

(T8-2) "you may consider such and such a modified version of IUTch if you wish, and, moreover, I affirm that the essential content of IUTch is completely valid for such a modified version";

(T9) detailed exposition of the **multiradial representation** of [IUTchIII], Theorem 3.11.

Of these themes, it seems that (T1) and (T2) are, in some sense, the most *essential* or *fundamental*. Then (T3), (T4), (T5) may be understood as being essentially *"corollaries"* of (T1), (T2). Moreover, (T4), (T5) may be understood as particular aspects of (T3). By contrast, (T6), (T7), (T8) refer to certain *procedural aspects* of the March discussions, which, nonetheless, had a substantial influence on the mathematical content of the discussions. Here, it should be mentioned that the issue of *simplification* mentioned in (T7) refers to the general issue of just what sort of simplifications in the mathematical content under discussion are mathematically correct, meaningful, and helpful (from the point of view eliminating details that are unnecessary and irrelevant to the central points at issue). Thus, *one aspect* of this general theme (T7) is the *erroneous attempts by SS to simplify IUTch* (cf. (T3), (T4), (T5), (T6)) to such an extent that it leads to meaningless contradictions (as summarized in (Smm), (GLR1), (GLR2)). On the other hand, (cf. the discussion of §8, §13 below), (T7) also refers to situations where the techniques introduced in IUTch actually do yield simplifications, relative to more naive approaches to various situations that arise in IUTch. Finally, (T9) proceeded relatively smoothly, in the sense that it consisted essentially of a straightforward exposition of the content of the multiradial algorithms of [IUTchIII], Theorem 3.11. That is to say, the position of SS with regard to (T9) was that they did not dispute the *validity* of these algorithms, but rather the *non-triviality*, or *substantive content*, of these algorithms, on account of their positions with regard to (T1), (T2), (T3), (T4), (T5), (T6).

§5. In some sense, it seems that the **logical starting point** of the differences in point of view between SS and HM (and hence of various *fundamental misunderstandings* of SS) may be understood as a consequence of the difference between the following *two approaches* to considering *histories of operations* performed on mathematical objects in a given discussion of mathematics (cf. (T1)):

(H1) **The conventional approach to histories of operations**: The conventional approach that is typically taken, with regard to histories of operations performed on various mathematical objects, consists of regarding all of these operations as being **embedded** within a **single history**.

$$\begin{array}{ccccccccc} & & & & \bullet & \to & \cdots & & \\ & & & \nearrow & & & & & \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \cdots \\ & & & & & \searrow & & \searrow & \\ & & & & & & \ddots & & \ddots \end{array}$$

In this approach, **all previously executed operations** are regarded as being **permanently accessible**, regardless of the content of subsequent operations.

(H2) **The approach taken in IUTch to histories of operations**: By contrast, the approach to treating such histories of operations that is taken throughout IUTch involves the frequent use of **re-initialization** operations (i.e., "$\|$")

$$\begin{array}{ccccccccc} & & & & \bullet & \to & \cdots & & \\ & & & \nearrow & & & & & \\ \bullet & \to & \bullet & \to & \bullet & \| & \bullet & \to & \cdots \\ & & & & & \searrow & & \searrow & \\ & & & & & & \ddots & & \ddots \end{array}$$

— that is to say, situations where one **forgets** the previous history of some object (such as, for instance, some previously endowed *mathematical structure* on the object) and regards this previous history as being **inaccessible** in subsequent discussions. Such re-initialization operations then require the use of **distinct labels** (cf. (T5)) to denote the "versions" of an object that arise *prior to* and *subsequent to* the execution of such re-initialization operations. Another aspect of central importance in the context of such re-initialization operations is the **explicit specification** of the **type of mathematical objects** (i.e., "species", in the terminology of [IUTchIV], §3 — cf. (T2)) that one considers, for instance, *prior to* and *subsequent to* the execution of such re-initialization operations (e.g., *"groups of automorphisms of some specified field"* versus *"abstract topological groups"*).

§6. The two **main examples** in IUTch (cf. §15 below for a slightly more detailed — though still quite brief! — review of certain aspects of IUTch) of the sort of **re-initialization** operation discussed in (H2) occur in the context of the **gluing operations** that arise in the definition of the **$\mathfrak{log}$-** and **Θ-links**:

(HEx1) **The $\mathfrak{log}$-link**: Here, the gluing operation consists of regarding the

$$\text{“}\Pi\text{'s”}$$

(i.e., in the notation of [IUTchI], Fig. I1.2; [IUTchI], Definition 3.1, (e), (f), "$\Pi_{\underline{v}}$'s", for $\underline{v} \in \underline{\mathbb{V}}$) on either side of the $\mathfrak{log}$-link as being known only as **abstract topological groups**, i.e., of **forgetting** the way in which these abstract topological groups are conventionally related to ring/scheme theory, namely, as *groups of field automorphisms*.

(HEx2) **The Θ-link**: Here, the gluing operation consists of portions:

(HEx2-1) regarding the

$$\text{“}G\text{'s”}$$

(i.e., in the notation of [IUTchI], Fig. I1.2; [IUTchI], Definition 3.1, (e), "$G_{\underline{v}}$'s", for $\underline{v} \in \underline{\mathbb{V}}$) on either side of the Θ-link as being known only as **abstract topological groups**, i.e., **forgetting** the way in which these abstract topological groups are conventionally related to ring/scheme theory, namely, as *groups of field automorphisms*;

(HEx2-2) regarding the

$$\text{“}\mathcal{O}^{\times\boldsymbol{\mu}}\text{'s”}$$

(i.e., in the notation of [IUTchI], Fig. I1.2, "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$'s", for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$) on either side of the Θ-link as being known only as **abstract topological monoids** (that are also equipped with certain *$G_{\underline{v}}$-actions*, as well as certain *collections of submodules*, as discussed in [IUTchII], Definition 4.9, (i), (ii), (iii), (iv), (vi), (vii)), i.e., **forgetting** the way in which these abstract topological monoids are conventionally related to ring/scheme theory, namely, as *subquotients of multiplicative groups of certain fields*.

In this context, we note that there are several differences between (HEx1) and (HEx2-1). Indeed, in the case of the **Θ-link**, although "G" (cf. (HEx2-1)) is *not* regarded as a group of field automorphisms, it *is* regarded as a *group of automorphisms of the abstract topological monoid "$\mathcal{O}^{\times\boldsymbol{\mu}}$"* (i.e., equipped with a certain collection of submodules — cf. (HEx2-2)). In particular, the Θ-link is constructed by considering the *full poly-isomorphism* between distinct copies of the data "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" (where $\mathcal{O}^{\times\boldsymbol{\mu}}$ is always regarded as being equipped with a certain collection of submodules). This full poly-isomorphism gives rise to the **indeterminacies** (Ind1, 2), which play a central role in IUTch. By contrast, in the case of the **$\mathfrak{log}$-link**, the gluing between Frobenius-like data in the codomain and domain (of the $\mathfrak{log}$-link) involves a **specific bijection** between **topological sets** (arising, respectively, from *multiplicative* and *additive* structures in the *domain* and *codomain* of the $\mathfrak{log}$-link). That is to say, (unlike the case of the Θ-link) one does *not* perform the gluing in the $\mathfrak{log}$-link by considering, say, *full poly-isomorphisms* between *topological sets*. A closely related fact is the fact that these topological sets *are* **never** *used as* **vertical cores**, i.e., as *invariants* (up to indeterminate isomorphisms of topological sets) with respect to the $\mathfrak{log}$-link. Nevertheless, these topological sets (regarded up to indeterminate isomorphisms of topological sets) may, in some sense, be thought of as appearing *implicitly*, in the sense that the point of view in which one thinks of the "Π's" as *abstract topological groups* is closely related to the point of view in which one thinks of the "Π's" as **groups of automorphisms of these topological sets** that appear in the gluing of Frobenius-like data that occurs in the case of the $\mathfrak{log}$-link. This state of affairs differs somewhat from the situation in the case of the Θ-link, in which the *$\mathcal{O}^{\times\boldsymbol{\mu}}$'s* (regarded up to the indeterminacy (Ind2)) do indeed play an important role as **horizontal cores**, i.e., as *invariants* with respect to the Θ-link.

§7. Before proceeding, it is of interest to note, in the context of §5, §6, that the sort of **re-initialization** operations discussed in (H2) — i.e., operations that consist of *forgetting mathematical structures* in such a way that the forgotten mathematical structures *cannot* (at least in any sort of *a priori*, or *general nonsense*, sense) *be recovered* — in fact occur in various contexts of mathematics other than IUTch. Indeed, the following two *"classical" examples* are of particular interest in the context of IUTch:

(HC1) **Classical complex Teichmüller theory**: The fundamental set-up of classical complex Teichmüller theory consists of considering *distinct holomorphic (i.e., Riemann surface) structures* on a given *topological (or real analytic) surface*, i.e., of *forgetting* the way in which the Riemann surface (i.e., holomorphic structure) gives rise to such a topological (or real analytic) surface.

(HC2) **Anabelian geometry**: The fundamental set-up of anabelian geometry consists of considering various *topological (typically profinite) groups* that arise as various types of arithmetic (i.e., étale) fundamental groups of schemes or Galois groups of fields as *abstract topological groups*, i.e., of *forgetting* the way in which such topological groups arose as arithmetic fundamental groups or Galois groups.

Indeed, it appears that *SS did not dispute the logical feasibility or consistency of the approach of (H2)* in the context of IUTch. Rather, their main objection to the approach of (H2) in the context of IUTch appears to be to the effect that they believed this approach of (H2) in the context of IUTch, i.e., in particular, in the context of (HEx1), (HEx2-1), (HEx2-2), to be *unnecessary/superfluous.* In particular, it appears that one of the *fundamental assertions of SS* is to the effect that

(SSInd) *the essential content (e.g., multiradial algorithms) of IUTch is* **entirely unaffected** *even if the* **indeterminacies** (Ind1, 2) *(cf. the discussion of §6) are* **eliminated**.

§8. The *central reason* for the introduction, in IUTch, of various types of **indeterminacy** lies in the goal (cf. [IUTchIII], Theorem 3.11) of obtaining **multiradial algorithms** for **representing the Θ-pilot object**, i.e., algorithms for representing the Θ-pilot object that are

(SW) **compatible** with — that is to say, **invariant**, up to suitable indeterminacies, with respect to — the operation of **switching/interchanging** corresponding collections of data (e.g., Θ-pilot objects) in the *domain* and *codomain* of the Θ-link, in a fashion that **fixes** the **gluing** of data that constitutes the Θ-link.

This *switching property* (SW), i.e., *multiradiality*, may also be thought of as a sort of **symmetry** between certain data in the *domain* and *codomain* of the Θ-link. This sort of symmetry is achieved precisely by introducing various **indeterminacies** — i.e., via the operation of **"re-initialization"**, or "forgetting certain mathematical structures", as discussed in (H2). That is to say,

(Sym) unlike the very **rigid** history diagrams discussed in (H1), i.e., where one does **not** allow oneself to perform re-initialization operations, history diagrams such as those in (H2), i.e., diagrams that include re-initialization operations "$\|$" (which typically give rise to certain **indeterminacies**!), are **much more flexible/less rigid**, hence have far *fewer obstructions to* **admitting symmetries**.

Put another way, the re-initialization operations "$\|$" in history diagrams such as those in (H2) may be visualized as **flexible/rotary joints** — such as those in the *human skeleton* or in *robot arms* — whose flexibility renders possible various types of *symmetry*. Perhaps the most fundamental example of this sort of phenomenon is the following very classical/elementary example:

(SWC1) Consider the *ordered set* $E = \{0, 1\}$, equipped with the ordering "$0 < 1$". Then the operation

$$E \quad \mapsto \quad S$$

given by *forgetting the ordering*, i.e., passing from *ordered sets* to *underlying sets*, gives rise to an object "S" (i.e., a set of cardinality 2) that **admits symmetries** (i.e., the symmetric group on 2 letters!) that may only be considered if one **completely forgets the ordering** on "E"!

Another *observation of fundamental importance* in the present context is the following: Although at first glance, the introduction of *re-initialization operations* as in (H2), together with the resulting *indeterminacies*, may appear to give rise to mathematical structures that are more complicated (cf. the "issue of *simplification*", i.e., (T7)!) than the mathematical structures that arise in rigid history diagrams of the sort discussed in (H1), in fact,

> re-initialization operations — i.e., operations of **forgetting** *mathematical structures that* **obstruct desired symmetries** — typically yield mathematical structures that are **much simpler/more tractable/more likely to admit symmetries**, in that they allow one to concentrate on structures of interest while carrying around **much less "unnecessary baggage"**!

This sort of phenomenon — i.e., achieving **simplicity by forgetting**! — may be seen in *numerous classical/elementary examples*, such as the following one:

(SWC2) Consider the geometry of *topological manifolds*, i.e., which involves various continuous maps between topological manifolds. At first glance, considering *topological manifolds equipped with atlases*, i.e., equipped with systems of local coordinates that yield local embeddings into some Euclidean space, may appear to be *"simpler"* mathematical structures (i.e., than topological manifolds that are *not* equipped with atlases) in that they contain specific data that relates such a topological manifold to *Eucidean space*, whose geometry is *more explicit* and *easier to grasp* than the geometry of an arbitrary topological manifold. In fact, however, the operation of *forgetting atlases*, i.e., of regarding topological manifolds as not necessarily being equipped with explicit atlases, yields mathematical structures that are much **simpler/more tractable/more likely to admit symmetries**, especially, for instance, when one considers various *continuous maps between topological manifolds* (which are not necessarily compatible with given atlases in the domain and codomain!), than the geometries/mathematical structures that arise if one **insists** (e.g., in the name of *"simplicity"* — cf. (T7)!) on considering *topological manifolds equipped with atlases.*

In the context of IUTch, the *two main examples* of this general mathematical phenomenon are the following (cf. also the discussion of §15 below for more details):

(SWE1) **(Frobenius-like) ring structures surrounding the $\mathfrak{log}$-link (cf. (HEx1))**: The usual *"Galois-theoretic"* relationship (i.e., via the interpretation as a *group of field automorphisms*) with the ring/field structures in the domains and codomains of the $\mathfrak{log}$-links in a *single vertical line* of the log-theta-lattice of the

$$\text{“}\Pi\text{'s”}$$

that appear in such a vertical line is a mathematical structure that is **far** from being **symmetric** with respect to **switching** operations between the two vertical lines (in the log-theta-lattice) that appear on either side of a Θ-*link* (cf. (LbMn) below). On the other hand, such a **switching symmetry** may be achieved precisely by *forgetting* about these (Frobenius-like) ring/field structures, i.e., by thinking of the various "Π's" as **abstract topological groups** — cf. (HEx1), as well as (EtMn) below.

(SWE2) **(Frobenius-like) ring structures surrounding the Θ-link (cf. (HEx2))**: In a similar vein, the usual *"Galois-theoretic"* relationship (i.e., via the interpretation as a *group of field automorphisms*) with the ring/field structures in the domain and codomain of a Θ-link of the

$$\text{“}G\text{'s”}$$

that appear in the domain and codomain of a Θ-link, as well as the usual relationship — i.e., relative to the local and global **value group** portions of the gluing data that appears in a Θ-link — with the ring/field structures in the domain and codomain of a Θ-link of the

$$\text{“}\mathcal{O}^{\times \boldsymbol{\mu}}\text{'s”}$$

that appear in the domain and codomain of a Θ-link, are mathematical structures that are **far** from being **symmetric** with respect to **switching** operations between the domain and codomain of the Θ-*link* (cf. (LbΘ) below). On the other hand, such a **switching symmetry** may be achieved precisely by *forgetting* about these (Frobenius-like) ring/field structures, i.e., by thinking of the various “G's” as **abstract topological groups** and of the various “$\mathcal{O}^{\times \boldsymbol{\mu}}$'s” as **abstract topological monoids** (equipped with certain collections of submodules), that is to say, by introducing the **indeterminacies** (Ind1, 2) — cf. (HEx2), as well as (VUC), (EtΘ) below.

§9. The **gluing (poly-)isomorphism** constituted by the **Θ-link** may be thought of, in essence, as a(n) (poly-)isomorphism between collections of data as follows:

$$\begin{pmatrix} G \\ \curvearrowright \\ \{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot \mathcal{O}^{\times \boldsymbol{\mu}} \end{pmatrix} \quad \begin{matrix} \text{full} \\ \overset{\sim}{\to} \\ \text{poly-} \\ \text{isom.} \end{matrix} \quad \begin{pmatrix} G \\ \curvearrowright \\ \underline{\underline{q}}^{\mathbb{N}} \cdot \mathcal{O}^{\times \boldsymbol{\mu}} \end{pmatrix}$$

$$\Theta\text{-hol. str.} \qquad\qquad\qquad\qquad q\text{-hol. str.}$$

— where the “j” in “$\{-\}$” varies from 1 to $l^* \stackrel{\text{def}}{=} \frac{1}{2}(l-1)$ (cf. the notation of, e.g., [Alien], §3.3, (vii), as well as the notation of the above discussion). One *fundamental aspect* of IUTch is the following observation:

(ΘNR) The gluing isomorphism constituted by the Θ-link is **not compatible** with the **ring structures** — i.e., does *not* arise from a *ring homomorphism* relative to the ring structures — on the algebraic closures of local fields

$$\text{“}\overline{k}\text{'s”}$$

(i.e., in the notation of the discussion surrounding [IUTchI], Fig. I1.2, “$\overline{F}_{\underline{v}}$'s”, for $\underline{v} \in \underline{\mathbb{V}}$) that lie on either side of the Θ-link. We shall refer to these *ring structures on either side of the* Θ-*link* by the terms **“Θ-holomorphic structure”** and **“q-holomorphic structure”** (and abbreviate the lengthy term “arithmetic holomorphic structure” by “hol. str.”). Here, from the point of view of the data in large parentheses “$(-)$”

in the first display of the present §9, the *"ring structures"* under consideration may be thought of as consisting of the following three structures:

(ΘNR1) the *additive structure* on (each) $\overline{k}$,
(ΘNR2) the *multiplicative structure* on (each) $\overline{k}$,
(ΘNR3) the *"Galois-theoretic interpretation"* of each "G" as a group of *field automorphisms* of the corresponding "$\overline{k}$".

On the other hand, in the present context, it is important to recall that

(GIUT) the *central goal* of IUTch is precisely to **compute** the **Θ-hol. str.** in terms of the **q-hol. str.** (where we regard these two hol. strs. — i.e., *ring structures* — as being related via the gluing (poly-)isomorphism constituted by the Θ-link).

From the point of view of *implementing (GIUT) via the* **multiradial representation algorithms** *developed in IUTch* (i.e., [IUTchIII], Theorem 3.11), the following property is of *central importance*:

(ΘCR) The **Θ-link** is defined in such a way as to be **compatible** with as **large** a **portion** of the **ring structures** on either side of the Θ-link (i.e., the Θ-, q-hol. strs.) as is possible. These ring structures are necessary in order to define the respective **$\mathfrak{log}$-links** associated to the Θ-, q-hol. strs.; the use of these $\mathfrak{log}$-links is an *essential portion* of the **multiradial representation algorithms** developed in IUTch. Here, the "portion" of these ring structures that *is* compatible with the Θ-link consists of the following two structures:

(ΘCR1) the **subquotients** (equipped with certain collections of submodules) of the **multiplicative monoid** $\overline{k}^{\times}$ (i.e., of nonzero elements of the *ring* $\overline{k}$) given by the monoids "$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}}$", "$\underline{\underline{q}}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}}$";
(ΘCR2) the *"interpretation"* of each "G" as a group of *automorphisms* of the monoids considered in (ΘCR1) — an "interpretation" which, in this case, just happens to be equivalent to simply thinking of each "G" as an **abstract topological group**.

Note, moreover, that (ΘCR1), (ΘCR2) are *consistent* with the approach to *switching symmetrization* discussed in (SWE2), hence give rise to the **indeterminacies** (Ind1, 2).

Here, it is important to note that, although the **splittings**

$$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}} \ \stackrel{\sim}{\rightarrow}\ \{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \times \mathcal{O}^{\times\boldsymbol{\mu}}; \qquad \underline{\underline{q}}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}} \ \stackrel{\sim}{\rightarrow}\ \underline{\underline{q}}^{\mathbb{N}} \times \mathcal{O}^{\times\boldsymbol{\mu}}$$

might, at first glance, give the impression that the *"value group portions"* $\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$, $\underline{\underline{q}}^{\mathbb{N}}$ and *"unit group portions"* $\mathcal{O}^{\times\boldsymbol{\mu}}$ are treated (in the gluing (poly-)isomorphism that constitutes the Θ-link) as *independent, unrelated objects*, in fact this is *simply not the case*. That is to say:

(VUSQ) These **splittings** exist merely as a consequence of the (simple!) *monoid structure* of the particular monoids involved. The observation of *central* *importance*, from the point of view of (ΘCR), is that the **entire monoid** "$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}}$" (respectively, "$\underline{\underline{q}}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}}$") — i.e., which contains *both* the **"value group portion"** and **"unit group portion"** discussed above — is a **subquotient** of the **single multiplicative monoid** $\overline{k}^{\times}$ (i.e., of nonzero elements) that arises from the *ring* $\overline{k}$.

At a somewhat more concrete level, the above discussion may be summarized as follows:

(VUC) In the following, we use *left-hand superscripts* "0" and "1" to denote, respectively, objects in the *domain* and *codomain* of various *modified versions of the* Θ*-link*:

(VUC1) Consider the *modified version of the* Θ*-link*

$$\{{}^0\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot {}^0\mathcal{O}^{\times\boldsymbol{\mu}} \quad \stackrel{\sim}{\to} \quad {}^1\underline{\underline{q}}^{\mathbb{N}} \cdot {}^1\mathcal{O}^{\times\boldsymbol{\mu}}$$

in which $\{{}^0\underline{\underline{q}}^{j^2}\} \ \mapsto \ {}^1\underline{\underline{q}}$, and one takes the (poly-)isomorphism ${}^0\mathcal{O}^{\times\boldsymbol{\mu}} \ \stackrel{\sim}{\to} \ {}^1\mathcal{O}^{\times\boldsymbol{\mu}}$ to be the **"identity isomorphism"**, i.e., the isomorphism arising from some fixed choice of ("rigidifying") isomorphisms of both sides with some fixed "model" copy of $\mathcal{O}^{\times\boldsymbol{\mu}}$ arising from a fixed "model" copy of $\overline{k}$. These rigidifying isomorphisms define, on (suitably defined equivalence classes of) suitable collections of pairs of elements of (suitably small neighborhoods of the identity of) the *multiplicative* monoid ${}^i\mathcal{O}^{\times\boldsymbol{\mu}}$ (where $i = 0, 1$), an *additive structure*, with respect to which such collections of pairs of elements of ${}^i\mathcal{O}^{\times\boldsymbol{\mu}}$ *generate* suitably small neighborhoods of "0" in ${}^i\overline{k}$, hence determine an *"identity isomorphism"* ${}^0\overline{k} \ \stackrel{\sim}{\to} \ {}^1\overline{k}$. This "identity isomorphism" ${}^0\overline{k} \ \stackrel{\sim}{\to} \ {}^1\overline{k}$ is **incompatible** with the assignment $\{{}^0\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \ \mapsto \ {}^1\underline{\underline{q}}$ in the sense that, for $j = 2, \ldots, l^{*}$, this "identity isomorphism" ${}^0\overline{k} \ \stackrel{\sim}{\to} \ {}^1\overline{k}$ does **not map** ${}^0\overline{k} \ni {}^0\underline{\underline{q}}^{j^2} \ \mapsto {}^1\underline{\underline{q}} \in {}^1\overline{k}$.

(VUC2) The *incompatibility* discussed in (VUC1) may be *eliminated* by regarding ${}^0\underline{\underline{q}}^{j^2}$ (for $j = 1, \ldots, l^{*}$) as belonging to *yet another copy* "${}^{0'}\overline{k}$" of $\overline{k}$ that is regarded as *not being related to* ${}^0\overline{k}$ *via a field isomorphism* (and, for the sake of "symmetry", regarding ${}^1\underline{\underline{q}}$ as belonging to *yet another copy* "${}^{1'}\overline{k}$" of $\overline{k}$ that is regarded as *not being related to* ${}^1\overline{k}$ *via a field isomorphism*) — cf. the discussion above of the issue of treating the *"value group portions"* $\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$, $\underline{\underline{q}}^{\mathbb{N}}$ and *"unit group portions"* $\mathcal{O}^{\times\boldsymbol{\mu}}$ as *independent, unrelated objects*. This approach to eliminating the incompatibility discussed in (VUC1) is **not** the approach **adopted in IUTch**.

(VUC3) On the other hand, one may also *eliminate* the *incompatibility* discussed in (VUC1) by *abandoning the "identity isomorphism"* and considering instead the *("full") poly-isomorphism* given by the collection of *all isomorphisms of topological monoids* (equipped with certain collections of submodules)

$$\{{}^0\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot {}^0\mathcal{O}^{\times\boldsymbol{\mu}} \quad \stackrel{\sim}{\to} \quad {}^1\underline{\underline{q}}^{\mathbb{N}} \cdot {}^1\mathcal{O}^{\times\boldsymbol{\mu}}$$

that are *equivariant* with respect to some (uniquely determined) isomorphism of topological groups ${}^0G \stackrel{\sim}{\to} {}^1G$. The *significance* of this approach (cf. (VUSQ)!) lies in the fact that it involves isomorphisms between a **single** *subquotient* $\{{}^0\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot {}^0\mathcal{O}^{\times\boldsymbol{\mu}}$ of (copies indexed by labels j of) the topological monoid ${}^0\overline{k}^{\times}$ and a **single** *subquotient* ${}^1\underline{\underline{q}}^{\mathbb{N}} \cdot {}^1\mathcal{O}^{\times\boldsymbol{\mu}}$ of the topological monoid ${}^1\overline{k}^{\times}$. This is the approach that is **actually adopted in IUTch**.

§10. During the afternoon session of the final day of the March discussions, SS made the following assertion, which may be thought of as a sort of *concrete realization* of their assertion (SSInd):

(SSId) The multiradial algorithms of IUTch may be applied to relate the Θ-, q-hol. strs. that appear in the following **"id-version"** (cf. (T3)) of the **Θ-link**

$$\begin{array}{ccc}
\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} & \text{“identity”} & \underline{\underline{q}}^{\mathbb{N}} \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,- & \stackrel{\sim}{\to} & -\,-\,-\,-\,-\,-\,-\,-\,-\,- \\
\left(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}\right) & \begin{array}{c}\text{isom. on}\\ G,\, \mathcal{O}^{\times\boldsymbol{\mu}}\end{array} & \left(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}\right) \\
\Theta\text{-hol. str.} & & q\text{-hol. str.}
\end{array}$$

— where the "$--$" on either side of the " $\stackrel{\sim}{\to}$ " are intended as a notational device to document the understanding that the *ring structure that gives rise to the ("value group") monoid "$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$" (respectively, "$\underline{\underline{q}}^{\mathbb{N}}$")* is to be regarded as **distinct** and **unrelated** to the ring str. — i.e., the Θ- (respectively, q-) hol. str. — that gives rise to the data "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" on the same side of the " $\stackrel{\sim}{\to}$ " (cf. (VUC2)!).

Here, we observe that one may introduce *terminology "${}^{\dagger}\Theta$-hol. str."* (respectively, *"${}^{\dagger}q$-hol. str."*) to denote the *ring structure that gives rise to the ("value group") monoid "$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$" (respectively, "$\underline{\underline{q}}^{\mathbb{N}}$")*. (SS *deeply objected to discussing* and, indeed, *refused to consider* (cf. (T6)) these ring structures, on the grounds that they wished to keep the discussion as "simple" (cf. (T7)) as possible — cf. also (T8), the discussion of (T7) in §8.) From the point of view of this terminology, the **"id-version"** of the **Θ-link** discussed in (SSId) may be described via the diagram

$$\begin{array}{ccc}
\left(\begin{array}{c} G \\ \curvearrowright \\ \{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}} \end{array}\right) & & \left(\begin{array}{c} G \\ \curvearrowright \\ \underline{\underline{q}}^{\mathbb{N}} \cdot \mathcal{O}^{\times\boldsymbol{\mu}} \end{array}\right) \\
{}^{\dagger}\Theta\text{-hol. str.} & \text{“identity”} & {}^{\dagger}q\text{-hol. str.} \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,- & \stackrel{\sim}{\to} & -\,-\,-\,-\,-\,-\,-\,-\,-\,- \\
\left(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}\right) & \begin{array}{c}\text{isom. on}\\ G,\, \mathcal{O}^{\times\boldsymbol{\mu}}\end{array} & \left(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}\right) \\
\Theta\text{-hol. str.} & & q\text{-hol. str.}
\end{array}$$

— where the "$\stackrel{\sim}{\to}$" only relates the *value group portions* of the $^\dagger\Theta$-, $^\dagger q$-hol. strs., i.e., it relates (via the "identity" isomorphism) the "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" portions of the Θ-, q-hol. strs., but does *not* relate the "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" portions of the $^\dagger\Theta$-, $^\dagger q$-hol. strs. It appears that SS introduced the (mathematical content represented by the) "$-\!-$'s"

(SSId1) precisely in order to ensure that the *"*id*-version" of the* Θ*-link* discussed in (SSId) satisfies the **switching property** (SW) (cf. also (VUC2)).

Moreover, it appears that SS felt justified in introducing the "$-\!-$'s" precisely as a consequence of the *misunderstanding*, on the part of SS, that

(SSId2) the *"value group portions"* $\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$, $\underline{\underline{q}}^{\mathbb{N}}$ and *"unit group portions"* $\mathcal{O}^{\times\boldsymbol{\mu}}$ are treated (in the gluing (poly-)isomorphism that constitutes the Θ-link) as *independent, unrelated objects* (cf. the discussion immediately preceding (VUSQ); the discussion of (VUC)).

On the other hand, one *fundamental property* of this **"**id**-version"** of the **Θ-link** is the following:

($^\dagger\Theta$CR) The multiradial algorithms of IUTch *only may be applied* to relate the *value group portion of the* $^\dagger\Theta$**-hol. str.** to the value group portion of some other hol. str. that is **linked to the $^\dagger\Theta$-hol. str.** via data that **contains the "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" portion of the $^\dagger\Theta$-hol. str.** (cf. (ΘCR), (VUSQ), (VUC)). That is to say, the multiradial algorithms of IUTch **cannot** *be applied* (cf. (ΘCR), (VUSQ), (VUC)) in such a way that the "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" that appears in these algorithms may be taken to be the *rigidified* "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}$" portion of the Θ-, q-hol. strs. of (SSId) (i.e., which is *not subject to the indeterminacies* (Ind1, 2)!).

In particular,

(SSIdFs) *the assertion (SSId) is* **false**, i.e., in summary, the use of "$-\!-$" to ensure that (SW) is satisfied (cf. (SSId1)) *tautologically* gives rise to **fundamental obstacles** to satisfying (ΘCR) (cf. also (VUSQ), (VUC), (SSId2)), hence also with respect to implementing (GIUT).

In this context, we remark that one way to understand the *essential logical structure* of the problem with the *"*id*-version" of the* Θ*-link* discussed in (SSId) is to consider the following elementary example concerning *complex structures on real vector spaces*:

(SSIdEx) Write $V = \mathbb{R}^2$ (i.e., a copy of two-dimensional Euclidean space), $e_1 \stackrel{\mathrm{def}}{=} (1,0) \in V$, $e_2 \stackrel{\mathrm{def}}{=} (0,1) \in V$, $V_1 \stackrel{\mathrm{def}}{=} \mathbb{R}\cdot e_1$, $V_2 \stackrel{\mathrm{def}}{=} \mathbb{R}\cdot e_2$. Let $t < 1$ be a positive real number. Consider the following two *complex structures* (i.e., structures as a vector space over the field of complex numbers $\mathbb{C}$) on V:

- the *"U-structure"*: $i\cdot e_1 = e_2$, $i\cdot e_2 = -e_1$;
- the *"W-structure"*: $i\cdot e_1 = t\cdot e_2$, $i\cdot e_2 = -t^{-1}\cdot e_1$.

Thus, the U-, W-structures determine, respectively, *complex vector spaces* U, W whose *underlying real vector space* is V. Write $\phi : U \stackrel{\sim}{\to} W$ for the *isomorphism of real vector spaces* arising from the fact that the underlying real vector space of U, W is V. For $i = 1, 2$, write $U_i \subseteq U$, $W_i \subseteq W$ for the real subspaces determined by $V_i \subseteq V$ and $\phi_i : U_i \xrightarrow{\sim} W_i$ for the restriction of ϕ to U_i. In the following, we shall use a subscript "$\mathbb{C}$" to denote the tensor product over $\mathbb{R}$ with $\mathbb{C}$. Write

$$\iota_U : U_2 \xrightarrow{\sim} i \cdot U_1 \subseteq (U_1)_{\mathbb{C}}$$

for the $\mathbb{R}$-linear morphism that maps $e_2 \mapsto i \cdot e_1$,

$$\iota_W : W_2 \xrightarrow{\sim} i \cdot W_1 \subseteq (W_1)_{\mathbb{C}}$$

for the $\mathbb{R}$-linear morphism that maps $t \cdot e_2 \mapsto i \cdot e_1$. Thus, one may think of ι_U as the restriction to U_2 of the unique $\mathbb{C}$-linear isomorphism

$$\zeta_U : U \xrightarrow{\sim} (U_1)_{\mathbb{C}}$$

that restricts to the identity on U_1; one may think of ι_W as the restriction to W_2 of the unique $\mathbb{C}$-linear isomorphism

$$\zeta_W : W \xrightarrow{\sim} (W_1)_{\mathbb{C}}$$

that restricts to the identity on W_1. Next, let us consider the *gluing isomorphism of collections of data*

$$\begin{aligned}\Phi \stackrel{\text{def}}{=} (\phi, &\phi_1, \phi_2, e_1 \mapsto e_1, e_2 \mapsto e_2) : \\ &(U, U_1, U_2, e_1 \in U_1, e_2 \in U_2) \xrightarrow{\sim} (W, W_1, W_2, e_1 \in W_1, e_2 \in W_2)\end{aligned}$$

— where each collection of data consists of the *underlying real vector space associated to some complex vector space*, together with *two real subspaces* of the real vector space and *distinguished elements* of each of the subspaces. Now suppose that one attempts to

> *approach the issue of understanding the* **relationship** *between the* **complex structures** *of* U*,* W *by* **identifying** U*,* W *with* $(U_1)_{\mathbb{C}}$*,* $(W_1)_{\mathbb{C}}$ *via* ζ_U*,* ζ_W*.*

This *approach* then encounters various *fundamental difficulties*, as follows: First of all,

($\ncong$) the collections of data $(U, U_1, U_2, e_1 \in U_1, e_2 \in U_2, \zeta_U)$ and $(W, W_1, W_2, e_1 \in W_1, e_2 \in W_2, \zeta_W)$ — or, essentially equivalently, the collections of data $(U, U_1, U_2, e_1 \in U_1, e_2 \in U_2, \iota_U)$ and $(W, W_1, W_2, e_1 \in W_1, e_2 \in W_2, \iota_W)$ — (where each collection of data consists of the *underlying real vector space associated to some complex vector space*, together with *two real subspaces* of the real vector space, *distinguished elements* of each of the subspace, and a *linear morphism between real vector spaces*) are **not isomorphic**.

(Indeed, $(\phi_1)_{\mathbb{C}} \circ \iota_U \neq \iota_W \circ \phi_2$, where we note that the $\mathbb{R}$-linear morphisms ϕ_1 and ϕ_2 are determined by the respective images, via these morphisms, of the distinguished elements. Also, we recall that $\iota_U = \zeta_U|_{U_2}$, $\iota_W = \zeta_W|_{W_2}$.)

Of course, one can *modify* the collections of data that appear in ($\not\cong$) by regarding, for $i = 1, 2$, U_i, W_i as *independent real vector spaces* — which, to avoid confusion, we shall denote by $[U_i]$, $[W_i]$, respectively — that are *not equipped* with their respective inclusions into U, W. It then follows *tautologically from the definition of "*$[-]$*"* that we obtain an *isomorphism of (ordered) pairs*

$$[\Phi] = ([\phi_1],[\phi_2], e_1 \mapsto e_1, e_2 \mapsto e_2) :$$
$$([U_1], [U_2], e_1 \in [U_1], e_2 \in [U_2]) \ \stackrel{\sim}{\to}\ ([W_1], [W_2], e_1 \in [W_1], e_2 \in [W_2])$$

— where each pair consists of *two real vector spaces*, each of which is equipped with a *distinguished element*. On the other hand,

(Dilat) working with $[\Phi]$ and thinking of U, W as being **identified** with $(U_1)_{\mathbb{C}}$, $(W_1)_{\mathbb{C}}$ by means of ζ_U, ζ_W does **not** allow one to compute the **nonzero dilatation** of the *"quasiconformal map"*

$$\phi : U \stackrel{\sim}{\to} W$$

by *replacing* ϕ by the $\mathbb{C}$-*linear (i.e., complex holomorphic!) map*

$$(\phi_1)_{\mathbb{C}} : (U_1)_{\mathbb{C}} \stackrel{\sim}{\to} (W_1)_{\mathbb{C}}$$

by means of ζ_U, ζ_W, i.e., whose **dilatation** is **zero**.

Relative to the analogy with IUTch:

· V_1 corresponds to the *unit group portion* of the *gluing data* that appears in the Θ-*link*;
· V_2 corresponds to the *value group portion* of the *gluing data* that appears in the Θ-*link*;
· the *"U-structure"* corresponds to the *q-hol. str.*;
· the *"W-structure"* corresponds to the Θ-*hol. str.*;
· the *property* ($\not\cong$) corresponds to (VUC1);
· the *gluing isomorphism* $[\Phi]$ corresponds to the *gluing isomorphism* that appears in the "id-*version" of the* Θ-*link* discussed in (SSId) (cf. (VUC2));
· the *gluing isomorphism* Φ corresponds to the *gluing isomorphism* that appears in the Θ-*link* (cf. (VUC3));
· the *property* (Dilat) corresponds to (SSIdFs).

In the context of IUTch, it is also perhaps of interest to observe that the *essential mathematical content* of this (very elementary!) example (SSIdEx) may, by introducing slightly more sophisticated (but still very classical!) terminology, be interpreted in terms of the well-known classical theory of *moduli of complex elliptic curves*:

(ModEll) **(Holomorphic moduli of elliptic curves vs. Teichmüller moduli of elliptic curves)**: There are two classical approaches to understanding the *moduli of complex elliptic curves*:

(HolMod) One may start with a **fixed copy** "$\mathbb{C}$" of the **field of complex numbers** (or, essentially equivalently, with a one-dimensional $\mathbb{C}$-vector space) and think of an elliptic curve over $\mathbb{C}$ as a *quotient of* $\mathbb{C}$ *by a lattice*

$$\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \cdot \tau)$$

— where the **"period"** $\tau \in \mathbb{C}$ is a complex number whose imaginary part $\mathrm{Im}(\tau) > 0$. This approach to elliptic curves over $\mathbb{C}$ allows one to think of elliptic curves over $\mathbb{C}$ in terms of **holomorphic moduli**. Then:

· From the point of view of the natural interpretation, in terms of infinitesimal moduli, of the first cohomology module of the (trivial!) tangent bundle of an elliptic curve over $\mathbb{C}$, this approach corresponds to computing this first cohomology module via the **holomorphic de Rham complex**.

· From the point of view of the discussion of (SSIdEx), this approach — which centers around **fixing** a copy "$\mathbb{C}$" of the **field of complex numbers**! — corresponds to the *gluing isomorphism* $[\Phi]$, i.e., to thinking in terms of the identifications via the **$\mathbb{C}$-linear** (hence, in particular, **holomorphic**!) isomorphisms $(U_1)_{\mathbb{C}} \xrightarrow{\sim} (V_1)_{\mathbb{C}} \xrightarrow{\sim} (W_1)_{\mathbb{C}}$, together with an *additional, auxiliary datum* "$t^{-1} \cdot i \cdot e_1$" (which corresponds to taking $\tau \stackrel{\mathrm{def}}{=} t^{-1} \cdot i$ in the present discussion).

· In particular, this approach corresponds to the the *gluing isomorphism* that appears in the **"id-version" of the Θ-link** discussed in (SSId) (cf. (VUC2)).

(TchMod) One may start with a **fixed lattice**

$$\Lambda \stackrel{\mathrm{def}}{=} \mathbb{Z} \oplus \mathbb{Z} \quad (\subseteq \Lambda_{\mathbb{R}} \stackrel{\mathrm{def}}{=} \mathbb{R} \oplus \mathbb{R})$$

and think of an elliptic curve over $\mathbb{C}$ as a **holomorphic structure** on this *fixed lattice* Λ (i.e., on the fixed real vector space $\Lambda_{\mathbb{R}}$). Thus, for instance, the holomorphic structure given by $i \cdot (1,0) = (0,t)$ corresponds to taking the "τ" in (HolMod) to be $t^{-1} \cdot i$. This (*non-holomorphic, quasi-conformal/real analytic*!) approach to elliptic curves over $\mathbb{C}$ is the approach taken in classical **complex Teichmüller theory**. Then:

· From the point of view of the natural interpretation, in terms of infinitesimal moduli, of the first cohomology module of the (trivial!) tangent bundle of an elliptic curve over $\mathbb{C}$, this approach corresponds to computing this first cohomology module via the **Dolbeault complex**.

· From the point of view of the discussion of (SSIdEx), this approach — which centers around **fixing** the **lattice** "$\Lambda \subseteq \Lambda_{\mathbb{R}}$"! — corresponds to the *gluing isomorphism* Φ, i.e., to thinking in terms of the identifications via the $\mathbb{R}$**-linear** (i.e., **quasiconformal, non-holomorphic**!) isomorphisms $U \overset{\sim}{\to} V \overset{\sim}{\to} W$ between the underlying real vector spaces of various complex vector spaces.

· In particular, this approach corresponds to the *gluing isomorphism* that appears in the **Θ-link** (cf. (VUC3)).

Before proceeding, it is perhaps worthwhile to recall that, *historically*,

(HstMod) of these *two approaches* discussed in (ModEll), the *"holomorphic approach"* (HolMod) has been the *"preferred approach"* in (scheme-theoretic) *arithmetic geometry*, presumably because **holomorphic structures** may be related *very directly* to **algebraic/scheme-theoretic structures**, whereas the *central role* played in the *"quasiconformal approach"* (TchMod) by consideration of the **two underlying dimensions** of a holomorphic structure did not, prior to the appearance of IUTch (cf. also the Introduction to [AbsTopIII]), have any evident analogue in arithmetic geometry.

Finally, in this context, it should be pointed out that SS only arrived at (SSId), as formulated above, during the afternoon session of the final day of the March discussions *precisely* as a consequence of the fact that the details of the formulation of their "id-version" *evolved substantially* over the course of the March discussions. For instance, the introduction of "−−" in (SSId) arose precisely from the discussion of (SW), (Sym) (cf. (SSId1)), which in turn arose as a consequence of the discussions on previous days summarized in §5, §6, §7. This state of affairs suggests that if the March discussions had continued further (perhaps after a period of rest, to allow the participants to digest the course of the previous discussions), it is quite possible that further discussion of (ΘCR) could have resulted in a *common understanding* (i.e., as summarized in (SSIdFs)) concerning (SSId) (cf. the discussion following (GLR2)).

§11. In the context of the discussion of §10, it is important to remember that:

(ΘFrd) The data that appears in the Θ-link contains **global realified Frobenioids**, i.e., certain globalized versions of the local value group portions "$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$", "$\underline{\underline{q}}^{\mathbb{N}}$" whose purpose is to category-theoretically encode the notion of the **global arithmetic degree** of an arithmetic line bundle (cf., e.g., [Alien], §3.3, (vii)). In particular, the **multiradial representation algorithms** developed in IUTch (i.e., [IUTchIII], Theorem 3.11), as well as the **inequalities** derived by applying these algorithms, are phenomena that are **essentially global** in nature — i.e., *not just a sort of result of summing up essentially local phenomena* (cf. the discussion of [IUTchIII], Remarks 3.6.2; 3.10.1; 3.12.2, (v)).

In particular, the compatibility of the *multiradial representation algorithms* developed in IUTch (i.e., [IUTchIII], Theorem 3.11) with the correspondence in the Θ-link

$$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \quad \stackrel{\sim}{\to} \quad \underline{\underline{q}}^{\mathbb{N}}$$

cannot be ascribed solely to purely local indeterminacies such as (Ind1, 2, 3). For instance:

(ΘInd) *This compatibility cannot be understood as being a consequence of the existence of automorphisms appearing in* (Ind1, 2) *that permute the elements of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" determined by local units of the form*

$$\{\exp(\underline{\underline{q}}^{j^2})\}, \quad \exp(\underline{\underline{q}})$$

(where $j = 1, \ldots, l^*$).

(Indeed, in general, such automorphisms *do not exist*!) This *essentially global nature of the logical structure of IUTch* has apparently been a source of misunderstanding for some mathematicians. In this context, it is perhaps of interest to recall the following well-known examples of **non-isomorphic** *(but comparable!) mathematical structures* that have isomorphic underlying structures, but which *cannot be transformed into one another* by means of isomorphisms of the underlying structures:

(UnEx1) **Classical complex Teichmüller theory**: (cf. (HC1)) It is well-known and easy to see that there exist **non-isomorphic** *compact Riemann surfaces* whose associated underlying *topological (or real analytic) surfaces* are isomorphic, but whose respective *holomorphic structures cannot be transformed* into one another by conjugating by some isomorphism between the associated underlying topological (or real analytic) surfaces.

(UnEx2) **Group structures on a set**: It is easy to see (for instance, in the case of groups of order four!) that there exist **non-isomorphic** *groups* whose associated underlying *sets* are isomorphic (i.e., of the same cardinality), but whose respective *group structures cannot be transformed* into one another by conjugating by some isomorphism (i.e., bijective map) between the associated underlying sets.

§12. During the afternoon session of the final day of the March discussions, SS proceeded to derive from the "id-*version*" of (SSId) a *"contradiction"* concerning arithmetic degrees (cf. (Smm)). This "contradiction" essentially amounts to the following assertion:

(AD) The correspondence that appears in the **Θ-link**

$$\{\underline{\underline{q}}^{j^2}\} \quad \mapsto \quad \underline{\underline{q}}$$

(where $j = 1, \ldots, l^*$) is, when considered within/relative to a **single hol. str.**, i.e., a **single ring/scheme theory**, manifestly **incompatible/inconsistent** with computing **arithmetic degrees**, unless one considers arithmetic degrees up to the *indeterminacy given by possible multiplication by factors "j^2"* (where $j = 1, \ldots, l^*$) — an indeterminacy which then renders the computation *meaningless*.

Before proceeding further, it should be *emphasized* that:

(IUAD) There is **absolutely no dispute** whatsoever (between SS and HM) with respect to the validity of the assertion (AD). Indeed, the point is that the situation in (AD) occurs whenever one works **within a single hol. str./ring theory**. Moreover, this observation (AD) is *by no means a new observation*, but rather constitutes the **starting point of IUTch**. It is precisely for this reason (cf. (GLR2)!) that, in IUTch, one must treat the hol. strs. in the domain and codomain of the Θ-link as **distinct** hol. strs. that are related in a nontrivial way that may be only be elucidated by means of a nontrivial computation (cf. (GIUT)).

Moreover, in the present context, it is of *fundamental importance* to recall that:

(ADItw) Computations of arithmetic degrees of global arithmetic line bundles that are related to one another *via the* Θ*-link* **depend**, *in an essential way*, on the specification of a *particular relationship*, or **"intertwining"**, between the *"value group portions"* $\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}}$, $\underline{\underline{q}}^{\mathbb{N}}$ and *"unit group portions"* $\mathcal{O}^{\times\boldsymbol{\mu}}$ of the data that appears in the gluing (poly-)isomorphism that constitutes the Θ-link. Such *"intertwinings"* arise from the *hol. strs.* (i.e., *ring structures*) in the domain and codomain of the Θ-link, but (since *the* Θ*-link is* **not** *a ring homomorphism!*) the Θ-link **fails**, *a priori*, to be **compatible** with the *distinct ring structures* in its domain and codomain, hence also with **arithmetic degree computations** in its domain and codomain.

When discussing (AD), SS, at times, asserted that:

(SSAD) The "contradiction" of (AD) occurs *even if one does not invoke (SSId), i.e., even if one applies the multiradial algorithms of IUTch as stated in [IUTchIII]* (that is to say, without necessarily invoking the assertion that these algorithms may be applied to some sort of *"*id*-version"* as in (SSId)).

On the other hand, it should be *stated unequivocally* that:

(SSADFs) The assertion of (SSAD) is **false**. That is to say, the "contradiction" of (AD) *only occurs* when one **identifies** the **hol. strs./ring theories** in the domain and codomain of the **Θ-link** (cf. (ADItw)).

Unfortunately, whenever I tried to explain (SSADFs) during the afternoon session of the final day of the March discussions, for instance, by using **distinct labels** to denote the **distinct hol. strs./ring theories** in the domain and codomain of the **Θ-link**, SS stubbornly **refused to allow the use of such distinct labels** in the discussion (cf. (T5); (T6); (T8); the discussion of (T6) in §10). On the other hand, it should be stated clearly that:

(SSDLFs) **Deleting the labels** used in IUTch (i.e., as formulated in [IUTchI], [IUTchII], [IUTchIII]) to denote the **distinct hol. strs./ring theories** in the domain and codomain of the **Θ-link** amounts to **identifying/confusing** these distinct hol. strs./ring theories — an operation that is already **stronger than** (i.e., "implies, in particular, the content of") the operations discussed in (SSInd) (cf. also the discussion of §5, §6, §7) and (SSId), i.e., operations which already lead to situations in which the multiradial algorithms are **no longer applicable** (cf. the discussion of §10, especially, (SSIdFs)).

Here, it is perhaps helpful to recall that the **labels** used in IUTch to denote *distinct hol. strs./ring theories* may be thought of, from the point of view of taking **log-**

volumes of regions in tensor packets of log-shells (i.e., the situation that occurs, for instance, in the final portion of [IUTchIII], Corollary 3.12), as

(LbLV) **markers** that denote whether (a certain suitable "generator" called the *"pilot object"* of) the **value group portion** of the gluing data in the Θ-link is identified with the **log-volume** of "$\underline{\underline{q}}$" (i.e., which corresponds to the case of the q-hol. str.) or with the **log-volume** of "$\{\underline{\underline{q}}^{j^2}\}$" (i.e., which corresponds to the case of the Θ-hol. str.).

(For more details, we refer to, for instance, the discussion of *"toy models"* in [IUTchIII], Remark 3.12.2, (ii).) The multiradial algorithms of [IUTchIII], Theorem 3.11, then provide an **alternative way** to compute the log-volume of the Θ-pilot object in terms of the q-hol. str., up to *certain indeterminacies.* In this context, it is important to remember that

(MlLV) unlike the **linear** correspondence induced by the Θ-*link* — i.e., in effect, a *linear relationship* between copies of the topological group of real numbers "$\mathbb{R}$" — between the *log-volume* of the Θ-*pilot object* relative to the Θ-*hol. str.* and the *log-volume* of the q-*pilot object* relative to the q-*hol. str.*, this computation, via the **multiradial algorithms** of IUTch, of the log-volume of the Θ-pilot object in terms of the q-hol. str. involves *certain indeterminacies*, which give rise to a **highly non-linear** relationship between the *log-volume* of the Θ-*pilot object in the* Θ-*hol. str.* and the *log-volume* of the *multiradial representation of the* Θ-*pilot object.*

This general qualitative phenomenon of *indeterminacies* giving rise to a **highly non-linear** relationship between the log-volume of a (*"rigid"*) region *not subject* to indeterminacies and the log-volume of an *approximation* (say, from above) of the ("rigid") region that *is subject* to indeterminacies may be understood by considering the following elementary example:

(LVEx) Write $V = \mathbb{R}^2$ (i.e., a copy of two-dimensional Euclidean space), $\sigma : V \xrightarrow{\sim} V$ for the automorphism of V given by $V \ni (x, y) \mapsto (-x, y)$. Denote by W the *"stack-theoretic quotient"* of V by the group G of automorphisms of V of order 2 generated by σ. Thus, we have a natural *finite étale morphism of degree* 2

$$\phi : V \to W$$

of "orbispaces". Next, let us consider the following *regions* of V:

$$\begin{aligned} R_{a,b} &\stackrel{\text{def}}{=} \{(x,y) \in V \mid |x| \leq 1,\ 0 \leq y \leq a\} \\ &\qquad \cup\ \{(x,y) \in V \mid 0 \leq x \leq 1,\ a \leq y \leq a + b\}; \\ S_{a,b} &\stackrel{\text{def}}{=} R_{a,b}\ \cup\ \sigma(R_{a,b}) \end{aligned}$$

— where $a, b > 0$ are positive real numbers. Then one computes easily the *log-volume* (i.e., the natural logarithm of the volume relative to the usual Euclidean measure) of these regions as follows:

$$\text{log-vol}(R_{a,b}) = \log(2a + b); \quad \text{log-vol}(S_{a,b}) = \log(2a + 2b).$$

In particular, we observe that

the *ratio* [log-vol($R_{a,b}$) : log-vol($S_{a,b}$)] (which may be thought of as a point of the projective line over $\mathbb{R}$) **varies continuously** — i.e., is **far** from being **constant**, as would be the case if the relationship between the two log-volumes were *linear* — as a, b vary.

Here, since $\sigma(S_{a,b}) = S_{a,b}$, one may think of $S_{a,b}$ as being *"defined over W"* and hence, at a purely formal level, equipped with a *"label"* W. On the other hand, since $\sigma(R_{a,b}) \neq R_{a,b}$, $R_{a,b}$ is only *"defined over V"*, hence, at a purely formal level, may be thought of as being equipped with a *"label"* V. Relative to the analogy with IUTch, $R_{a,b}$ corresponds to the Θ-*pilot* relative to the Θ-*hol. str.*, while $S_{a,b}$ corresponds to the Θ-*pilot* relative to the *multiradial representation*, with *"indeterminacies"* given by the action of the group G.

Another interesting and relatively elementary example of the phenomenon of computing the *"size"* of a *region* subject to *indeterminacies* (cf. the multiradial representation of the Θ-pilot object in IUTch) that *approximates* (say, from above) some *region of interest* that is *"rigid"* and *not subject to indeterminacies* (cf. the Θ-pilot relative to the Θ-hol. str. in IUTch) may be seen in the **displacement estimates** that appear in **Bogomolov's proof** of the *geometric version of the Szpiro Conjecture* (cf. the discussion of (DsInd) below). Finally, in this context, we remark that the importance of working with *distinct hol. strs./ring theories* is discussed in detail — and in a fashion directed toward an audience of non-mathematicians — in, roughly, the final 30 minutes of the video [FKvid].

§13. One topic to which a substantial amount of time and energy was devoted, especially during the first few days of the March discussions, was the topic of **poly-morphisms** in IUTch (cf. (T4)), i.e., sets (that are not necessarily of cardinality 1!) of morphisms between two objects. SS (especially, Scholze) were *substantially opposed* to the use of poly-morphisms in IUTch. This opposition appeared to be based, to a substantial extent, on *"taste/aesthetics"* (cf. (T4-1)). In this context, however, it should be remembered that, although the term *"poly-morphism"* is apparently new, in fact poly-morphisms — i.e., in effect,

(PMInd) situations in which one wishes to regard morphisms/objects of interest up to/modulo some sort of **indeterminacy**, the elimination of which is **not** a matter of interest

— appear quite frequently in mathematics. Well-known examples include

(PMEx1) *homotopy, or stable homotopy,* **classes** of continuous maps between topological spaces;

(PMEx2) *morphisms in derived categories between complexes*, i.e., morphisms that may be related to morphisms between complexes in the naive sense only by considering the **classes** of such naive morphisms obtained by *inverting the quasi-isomorphisms*.

Here, before proceeding, it is perhaps of interest to observe the existence of *"trivial examples"* of poly-morphisms:

(PMTrv) For $i = 1, 2$, suppose that G_i is a group, and that $H_i \subseteq G_i$ is a normal subgroup. Then a group homomorphism $\phi : G_1/H_1 \to G_2/H_2$ between the quotient groups G_1/H_1, G_2/H_2 may be regarded as a *trivial example of a poly-morphism*, i.e., by thinking of ϕ as the *collection of group homomorphisms* $G_1 \to G_2$ that lift ϕ.

A poly-morphism of the sort discussed in (PMTrv) is *"trivial"* in the sense that it may be expressed as a (single!) *morphism* (i.e., *map*!) between *quotient sets* (i.e., the sets "G_i/H_i" for $i = 1, 2$). By contrast:

(PMQut) The examples of poly-morphisms that occur in IUTch — such as those that arise from the indeterminacies (Ind1, 2) — make use of the notion of a poly-morphism in a nontrivial way precisely because they **cannot** be expressed as **single maps** between **quotient sets**. (Here, we note that the examples (PMEx1) and (PMEx2) are also nontrivial in this sense. For instance, there is no meaningful sense in which one can define the *"quotient set"* of a topological space by its group of homotopy self-equivalences.) It is precisely this sort of situation that requires the use of **formal techniques** — such as in the case of *algebraic stacks* (as opposed to schemes or algebraic spaces) or *orbifolds* (as opposed to topological manifolds) — in order to work with **quotients** in some sort of **abstract** or **formal** sense. This is precisely what is obtained in IUTch by the introduction of the "formal technique" constituted by the notion of a **poly-morphism**.

SS at times justified their opposition to the **"apparently substantively meaningless"** (cf. (T4-2)) use of poly-morphisms by invoking the issue of *simplicity* (cf. (T7)), but, as one may see in the well-known examples (PMEx1), (PMEx2), working with *specific morphisms* in situations where it is much more natural to work with *classes of morphisms*, tends to have the effect of rendering arguments much *more complicated*, or *less tractable* (cf. the discussion of §8). Put another way, as was discussed in §5, §6, §7, §8, §9, §10 (cf., especially, the discussion of (SSInd), (SW), (Sym), (SWE1), (SWE2), (VUC), (SSId), (SSIdFs)):

(PMEss) The use of **poly-morphisms** in IUTch — e.g., the introduction of indeterminacies such as (Ind1, 2) — plays the quite **substantive role** of yielding structures that admit certain **symmetries** (cf. the discussion of §8), which form the basis of **multiradiality**, in a fashion that is **compatible** with suitable portions of the **ring structures** in the domain and codomain of the **Θ-link** (cf. the discussion of §9, §10).

Further remarks concerning the use of poly-morphisms in IUTch may be found in [Alien], §4.1, (iv).

§14. Another topic to which a substantial amount of time and energy was devoted, especially during the first few days of the March discussions, was the topic of **labels** in IUTch (cf. (T5)) to distinguish **distinct copies** of various familiar objects that play **substantively different roles** in the various apparatuses treated in IUTch. SS (especially, Scholze) were *substantially opposed* to the use of labels in IUTch. This opposition appeared to be based, to a substantial extent, on *"taste/aesthetics"* (cf. (T5-1)). In this context, however, it should be remembered that in fact "labels" — i.e., in effect,

(LbRl) situations in which one wishes to distinguish **distinct copies** of various familiar objects that play **substantively different roles** within a complicated apparatus

— appear quite frequently in mathematics. Well-known examples include the following:

(LbEx1) The substantive significance of the use of **labels** may be seen in the following *fundamental example*: One considers the *("line segment") graph* Γ_{seg}

$$\bullet - \bullet$$

given by *two vertices* joined via a *single edge*. Write Γ_{loop} for the *("loop") graph* obtained from Γ_{seg} by identifying the *two distinct vertices* of Γ_{seg} — i.e., which may be thought of as *distinct labels* — to a *single vertex*. Thus, Γ_{seg} is *structurally different* from Γ_{loop} in the sense that Γ_{seg} is **simply connected**, while Γ_{loop} is **not**. This difference gives rise to *very substantive consequences* in many situations, e.g., situations where some sort of **"parallel transport"** or **"analytic continuation"** along the **loop** of Γ_{loop} gives rise to some sort of **nontrivial monodromy operator**. This sort of nontrivial monodromy may be **resolved** precisely by **distinguishing** the situations that arise *prior to* and *subsequent to* the application of the monodromy operator, i.e., *by working over* Γ_{seg} *as opposed to* Γ_{loop}. When, moreover, one wishes to distinguish situations that arise *prior to* and *subsequent to* multiple applications of the monodromy operator, it is natural to consider the *universal covering* Γ_{uni}

$$\dots \quad \bullet - \bullet - \bullet - \bullet \quad \dots$$

of Γ_{loop}, which may be thought of as the result of *concatenating distinct copies of* Γ_{seg} *labeled by elements* $\in \mathbb{Z}$. One *fundamental example* of this sort of situation, i.e., of *nontrivial monodromy around a loop*, is the (angular portion of the) **logarithm function** in one-variable complex analysis. This example is, moreover, reminiscent of the situation surrounding the *vertical columns of* $\mathfrak{log}$*-links* that appear in the *log-theta-lattice* of IUTch (cf. (LbLp) below). Alternatively, one may understand

> the **incompatibility** of the **Θ-link** (i.e., a horizontal arrow in the *log-theta-lattice*) with **arithmetic degrees** (cf. (AD)) as a sort of **nontrivial monodromy** around the **loop** (i.e., a special case of "Γ_{loop}"!) that arises as soon as one *identifies* the *domain* and *codomain* of the Θ-link — an incompatibility, i.e., **"contradiction"**, that is in fact **avoided** in IUTch (cf. (IUAD)) precisely by **distinguishing** *the (ring structures in the) domain and codomain of the* Θ*-link*, i.e., by working with "Γ_{seg}", and then applying the **multiradial algorithms** of IUTch to obtain an alternative way to **compute this nontrivial monodromy** (cf. (GIUT)).

(LbEx2) Suppose that A is a *commutative domain with unity*, $f \in A$ a *nonzero, non-unit*. Then the **localization** A_f of A with respect to f may be constructed as the *inductive limit* of the *inductive system*

$$\dots \quad \xrightarrow{f\cdot} \quad A_{n-1} \quad \xrightarrow{f\cdot} \quad A_n \quad \xrightarrow{f\cdot} \quad A_{n+1} \quad \xrightarrow{f\cdot} \quad \dots$$

given by *copies "A_n" of A* **labeled** *by elements* $n \in \mathbb{Z}$ and *transition morphisms* given by multiplication f. Here, we observe that **forgetting** the choice of a *"specific basepoint label* $0 \in \mathbb{Z}$*"* does not have any substantive effect on the construction if one is only interested in the *A-module structure of A_f*. If, on the other hand, one is interested in constructing the **ring structure** on A_f, then it is of *crucial importance* to specify a **"specific basepoint label $0 \in \mathbb{Z}$"**, i.e., a specific copy for which the unity element $1 \in A$ of A serves as the *unity element for the newly constructed* A_f. Thus, in summary:

> **Omitting the labels** leads to **confusion** concerning *which* copy of the unity element $1 \in A$ is to be regarded as the *unity element for "A_f"*. Such confusion may, of course, be *misinterpreted* as an **"internal contradiction"** in the theory of localizations of commutative rings with unity. In fact, however, there is **no "internal contradiction"** in the theory of such localizations; the apparent "internal contradiction" is nothing more than a superficial consequence of the **erroneous operation** of *omitting the labels.*

(LbEx3) Consider the *polynomial ring* $A \stackrel{\mathrm{def}}{=} \mathbb{F}_p[t]$ *in one variable over the field of cardinality* p, for some prime number p. Write $\phi : A \to A$ for the *Frobenius endomorphism*, i.e., the ring endomorphism given by sending $t \mapsto t^p$. Then the **perfection** A_∞ of A — i.e., $\mathbb{F}_p[\{t^{1/p^n}\}_{n\geq 1}]$ — may be constructed as the *inductive limit* of the *inductive system*

$$\dots \xrightarrow{\phi} A_{n-1} \xrightarrow{\phi} A_n \xrightarrow{\phi} A_{n+1} \xrightarrow{\phi} \dots$$

given by *copies "A_n" of A* **labeled** *by elements* $n \in \mathbb{Z}$ and *transition morphisms* given by (copies of) ϕ. Here, we observe that, if one **forgets** *the labels* $n \in \mathbb{Z}$, i.e., and just considers the inductive system given by a *single copy* of A and the transition morphism $\phi : A \to A$, then one computes easily that the resulting *inductive limit* is isomorphic to a *direct sum* $\oplus\ \mathbb{F}_p$ of p copies of $\mathbb{F}_p$ (i.e., and hence *not isomorphic* to A_∞!). Thus, in summary:

> **Omitting the labels** leads to **confusion** between mutually *non-isomorphic inductive limits* (i.e., A_∞, $\oplus\ \mathbb{F}_p$). Such confusion may, of course, be *misinterpreted* as an **"internal contradiction"** in the theory of perfections in positive characteristic. In fact, however, there is **no "internal contradiction"** in the theory of perfections in positive characteristic; the apparent "internal contradiction" is nothing more than a superficial consequence of the **erroneous operation** of *omitting the labels.*

For more detailed discussions of phenomena arising from the *Frobenius morphism in positive characteristic* that are more directly related to IUTch, we refer to (EtFrEx) below, as well as [Alien], Example 2.6.1; the discussion of [Alien], §2.7.

(LbEx4) Write $\lambda \in \mathbb{C}$ for the *unique positive real root* $\in \mathbb{C}$ of the equation $X^3 -$

$2 = 0$, $\omega \stackrel{\mathrm{def}}{=} \exp(\frac{2}{3}\pi i) \in \mathbb{C}$, $\mathbb{Q} \subseteq \mathbb{C}$ for the field of rational numbers, $F_1 \stackrel{\mathrm{def}}{=} \mathbb{Q}(\lambda) \subseteq \mathbb{C}$, $F_2 \stackrel{\mathrm{def}}{=} \mathbb{Q}(\omega \cdot \lambda) \subseteq \mathbb{C}$, $K \stackrel{\mathrm{def}}{=} \mathbb{Q}(\omega, \lambda) \subseteq \mathbb{C}$. Thus, we have a diagram of *field extensions*

$$\begin{array}{ccccc} & & K & & \\ & \diagup & & \diagdown & \\ F_1 & & & & F_2 \\ & \diagdown & & \diagup & \\ & & \mathbb{Q} & & \end{array}$$

such that $\mathbb{Q} = F_1 \cap F_2$, $K = F_1 \cdot F_2$. In particular, one may interpret the portion of the above diagram involving F_1, F_2, $\mathbb{Q}$ as a situation in which one has **two distinct abstract fields** F_1, F_2 that are **glued together along the subfields** $\mathbb{Q} \subseteq F_1$, $\mathbb{Q} \subseteq F_2$ (via the *unique isomorphism* between these subfields) — cf. the description of the **Θ-link** at the beginning of §9; (ΘNR). Observe that there exists a **unique isomorphism of fields** $\psi : F_1 \stackrel{\sim}{\to} F_2$, which, of course, is **not compatible** with the respective embeddings $F_1 \hookrightarrow K$, $F_2 \hookrightarrow K$. Then:

> If one **identifies** F_1 and F_2 by means of ψ and then uses this identification to **omit the labels** "1", "2" that are used to distinguish the fields F_1, F_2, then it is easy to obtain a *"contradiction"* concerning the structure of the field K. Such a "contradiction" may, of course, be *misinterpreted* as an **"internal contradiction"** in the elementary theory of field extensions. In fact, however, there is **no "internal contradiction"** in the elementary theory of field extensions; the apparent "internal contradiction" is nothing more than a superficial consequence of the **erroneous operation** of *omitting the labels* used to distinguish the fields F_1, F_2.

(LbEx5) (This example may, roughly speaking, be thought of as a sort of generalization of the example discussed in (LbEx1).) Consider the theory of (topological or differential) **manifolds** of dimension n, for n a positive integer. Manifolds (of dimension n) are, by definition, collections of *distinct copies* — i.e., copies equipped with **distinct labels** — of (n-dimensional) *Euclidean space* glued together by means of various *gluing isomorphisms*. In particular, it is quite possible that the use of **numerous distinct copies of Euclidean space** in this theory might impress a newcomer to this theory (such as a student) as being entirely **superfluous**. Such a newcomer might feel motivated to attempt to achieve a **"dramatic simplification"** of this theory by **identifying** these copies of Euclidean space with one another and hence to assert that the geometry of *arbitrary* (topological or differential) *manifolds* (of dimension n) may be **"reduced"** to the geometry of (n-dimensional) *Euclidean space*. Moreover:

> If one **identifies** these **distinct copies of Euclidean space** that appear in the theory of (topological or differential) manifolds with one another, then it is easy to obtain a *"contradiction"*. Such a "contradiction" may, of course, be *misinterpreted* as an **"internal contradiction"** in the theory of (topological or differential) manifolds. In fact, however, there is **no "internal contradiction"** in the theory of (topological or differential) manifolds; the apparent "internal contradiction" is nothing more than a superficial consequence of the **erroneous operation** of *omitting the labels* used to distinguish the distinct copies of Euclidean space.

SS at times justified their opposition to the **"apparently substantively meaningless"** (cf. (T5-2)) use of "labels" by invoking the issue of *simplicity* (cf. (T7)). At other times, SS asserted that the omission of labels could be justified simply by **"remembering"** the fact that some arrow (corresponding to the *"monodromy operator"* in (LbEx1), "$f\cdot$" in (LbEx2), "ϕ" in (LbEx3), "ψ" in (LbEx4), or the "gluing isomorphisms" in (LbEx5)) is not compatible with certain structures in its domain and codomain. On the other hand, as one may see in the above examples (LbEx1), (LbEx2), (LbEx3), (LbEx4), (LbEx5), *omitting the labels* may, depending on the *specifics* of the situation, easily give rise to a **"contradiction"** (which is in fact *meaningless*!) that does *not* occur if one *respects the labels*, i.e., if one *respects the distinct roles played by distinct copies*. In particular, *no matter "how good" a particular mathematician's memory may be*,

(DfLb) relying, in mathematical discussions, on declarations of "remembering" that are *not accompanied by precise, explicit documentation of the labeling apparatuses that are employed* incurs the risk that **different people** will "remember" **different labeling appartuses**, which result in **structurally non-equivalent mathematical structures**.

That is to say, as one may see in the above examples (LbEx1), (LbEx2), (LbEx3), (LbEx4), (LbEx5), the **substantive mathematical content** of a mathematical construction is only completely determined once one **makes completely explicit the labeling apparatus** that is adopted in the construction.

§15. In light of the general considerations concerning the use of **labels** discussed in §14, it is of interest to review the way in which labels for distinct copies of various familiar objects are employed in IUTch in order to construct apparatuses that play various **substantive roles** in IUTch that *cannot be achieved if the labels are deleted*. One *fundamental example* of this phenomenon is

(LbHT) the bookkeeping apparatus for **labels for evaluation points** within a **single Hodge theater** — an apparatus which plays the very substantive role of allowing one to **simulate global multiplicative subspaces** and **canonical generators**.

This phenomenon is discussed in detail in [IUTchI], §I1 (and indeed throughout [IUTchI]!). On the other hand, such labels within a single Hodge theater were only mentioned very briefly during the March discussions. The "label issues" that were discussed in substantial detail during the March discussions concern the **labels** $\in \mathbb{Z} \times \mathbb{Z}$ that correspond to the "$\bullet$'s" in the **log-theta-lattice**. Here, we begin our discussion of these labels by recalling the *(highly noncommutative!) diagram* that is used to denote the *entire log-theta-lattice* (i.e., the LHS of the "$\supseteq$"), together with the portion of the log-theta-lattice (i.e., the RHS of the "$\supseteq$", which consists of the *vertical arrows* in the 0- *and* 1-*columns*, together with the *single horizontal arrow* between the $\bullet$'s labeled $(0, 0)$ and $(1, 0)$) that is *actually used* in the main results of IUTch:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}
\qquad \supseteq \qquad
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & \xrightarrow{\Theta} & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\bullet & & \bullet \\
\uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\vdots & & \vdots
\end{array}
$$

We recall briefly that each $\bullet$ denotes a *single Hodge theater* (labeled by an element $\in \mathbb{Z} \times \mathbb{Z}$), which one thinks of as a *single model of the conventional ring/scheme theory* surrounding the elliptic curve over a number field under consideration. One then considers *two types of gluing* (denoted by the *vertical* and *horizontal arrows* in the diagrams) between certain portions of the Hodge theaters in the domain and codomain of each arrow. The *vertical arrows* denote **$\mathfrak{log}$-links**, while the *horizontal arrows* denote **Θ-links**. The following example of the significance of the labels $\in \mathbb{Z} \times \mathbb{Z}$ for distinct $\bullet$'s is already immediate from the structure of the above diagrams:

(LbLp) Write

$$\vec{\Gamma}_{\mathbb{Z}}$$

for the oriented graph given by the portion of the log-theta-lattice consisting of the *vertical arrows* in the 0-*column* and the *horizontal arrows* originating from the 0-*column*. Write

$$\vec{\Gamma}_0$$

for the oriented subgraph of $\vec{\Gamma}_{\mathbb{Z}}$ consisting of the *vertical arrows* in the 0-*column* and the *single horizontal arrow* between the $\bullet$'s labeled $(0, 0)$ and $(1, 0)$. Then the operation of **forgetting the vertical labels** amounts to *replacing* $\vec{\Gamma}_0$ by the oriented graph

$$\vec{\Gamma}_{\text{loop}}$$

given by the *single horizontal arrow* of $\vec{\Gamma}_0$, together with an *oriented loop* originating from and terminating at the domain of this *single horizontal arrow.* Alternatively, one may think of $\vec{\Gamma}_{\text{loop}}$ as the *quotient* "$\vec{\Gamma}_{\mathbb{Z}}/\mathbb{Z}$" of $\vec{\Gamma}_{\mathbb{Z}}$ by the action of $\mathbb{Z}$ on $\vec{\Gamma}_{\mathbb{Z}}$ via *vertical translations.* On the other hand, *purely at the level of oriented graphs*, it is immediate that $\vec{\Gamma}_{\mathbb{Z}}, \vec{\Gamma}_{\text{loop}} = \vec{\Gamma}_{\mathbb{Z}}/\mathbb{Z}$ are **structurally non-equivalent** to $\vec{\Gamma}_0$. For instance, (unlike $\vec{\Gamma}_{\mathbb{Z}}$!) $\vec{\Gamma}_0$ **fails** to admit **vertical symmetries**; (unlike $\vec{\Gamma}_{\text{loop}}$!) $\vec{\Gamma}_0$ (as well as $\vec{\Gamma}_{\mathbb{Z}}$) is **simply connected** (cf. the discussion of (LbEx1)). These *fundamental structural differences* between $\vec{\Gamma}_0$, on the one hand, and $\vec{\Gamma}_{\mathbb{Z}}$, $\vec{\Gamma}_{\text{loop}}$, on the other, have very **substantive consequences** in the context of the **definition** of the **Θ-link** (i.e., the horizontal arrow of $\vec{\Gamma}_0$), which depends, in an essential way, on **fixing** the **multiplicative** structures of the rings involved, i.e., is **not invariant** with respect to the **"rotations/juggling"** *of the additive and multiplicative structures* that occur as one executes various iterates of the $\mathfrak{log}$-link (cf. the discussion, in the latter portion of [Alien], §3.3, (ii), of the importance, in the context of the *definition of the* Θ*-link*, of *distinguishing* the *ring structures* in the *domain* and *codomain* of the $\mathfrak{log}$-link).

In order to discuss further **symmetry** properties (cf. (SWE1), (SWE2)) of the log-theta-lattice, it is useful to apply the notation "Π", "G", "$\mathcal{O}^{\times\boldsymbol{\mu}}$", "$\overline{k}$", and "$\overline{k}^{\times}$" of (SWE1), (SWE2), (ΘNR), (ΘCR). Here, we recall that $\overline{k}$ is an algebraically closed field equipped with a *p-adic valuation* and a natural action by the topological group G, which, in turn, may be regarded as a quotient of the topological group Π, and that $\overline{k}^{\times} \subseteq \overline{k}$ is the multiplicative monoid of nonzero elements of $\overline{k}$. Write $\mathcal{O}^{\times} \subseteq \overline{k}$ for the multiplicative group of units (with respect to the valuation) of $\overline{k}$. (Thus, $\overline{k}$, $\overline{k}^{\times}$, and $\mathcal{O}^{\times}$ are equipped with natural G-actions, and the object "$\mathcal{O}^{\times\boldsymbol{\mu}}$" may be regarded as the quotient of $\mathcal{O}^{\times}$ by its torsion subgroup.) Then we observe the following **non-symmetry** properties of the log-theta-lattice (cf. (SWE1), (SWE2)):

(LbΘ) **No horizontal arrow** *(i.e.,* Θ*-link) of the log-theta-lattice admits a* **symmetry** *that permutes its domain and codomain.* Indeed, this may be seen by considering the portion of a • (i.e., a Hodge theater) consisting of a single copy of $\overline{k}$. That is to say, such a symmetry would imply the existence of a symmetry that *permutes* the domain and codomain of the diagram

$$\overline{k}^{\times} \ni \underline{\underline{q}}^{j^2} \;\mapsto\; \underline{\underline{q}} \in \overline{k}^{\times}$$

(where $j = 1, \ldots, l^{*}$; $\underline{\underline{q}}$ and the $\underline{\underline{q}}^{j^2}$'s are regarded up to multiplication by possibly distinct roots of unity), i.e., the existence of a pair of isomorphisms of topological fields $\alpha : \overline{k} \overset{\sim}{\to} \overline{k}$, $\beta : \overline{k} \overset{\sim}{\to} \overline{k}$ such that $\alpha(\underline{\underline{q}}^{j^2}) = \underline{\underline{q}}$, $\beta(\underline{\underline{q}}) = \underline{\underline{q}}^{j^2}$ (up to multiplication by possibly distinct roots of unity). One verifies immediately that *such a pair of isomorphisms does not exist.*

(Lb$\mathfrak{log}$) **No vertical arrow** *(i.e.,* $\mathfrak{log}$*-link) of the log-theta-lattice admits a* **symmetry** *that permutes its domain and codomain.* Indeed, this may be seen by considering the portion of a • (i.e., a Hodge theater) consisting of a single copy of $\overline{k}$. That is to say, such a symmetry would imply the existence of a symmetry that *permutes* the domain and codomain of the diagram

$$\overline{k} \supseteq \mathcal{O}^{\times} \xrightarrow{\log} \overline{k}$$

(where log denotes the *p-adic logarithm* defined on $\mathcal{O}^{\times}$), i.e., the existence of a pair of isomorphisms of topological fields $\alpha : \overline{k} \xrightarrow{\sim} \overline{k}$, $\beta : \overline{k} \xrightarrow{\sim} \overline{k}$ such that, if we write $M \stackrel{\mathrm{def}}{=} \log^{-1}(\mathcal{O}^{\times}) \subseteq \mathcal{O}^{\times}$, then $\alpha|_M = \log \circ (\beta|_{\mathcal{O}^{\times}}) \circ (\log|_M)$. One verifies immediately that *such a pair of isomorphisms does not exist.*

(LbMn) Write

$$\vec{\Gamma}_{\mathrm{main}}$$

for the oriented graph given by the portion of the log-theta-lattice that is *actually used* in the main results of IUTch, i.e., the portion consisting of the *vertical arrows* in the 0- *and* 1-*columns* and the *single horizontal arrow* between the $\bullet$'s labeled $(0,0)$ and $(1,0)$. Then one verifies immediately (cf. (LbΘ), (Lb$\mathfrak{log}$)!) that *the portion determined by* $\vec{\Gamma}_{\mathrm{main}}$ *of the log-theta-lattice does* **not** *admit a* **symmetry** *that permutes the* 0- *and* 1-*columns.*

It is *precisely* because of these *non-symmetry* properties (LbΘ), (Lb$\mathfrak{log}$), (LbMn) that (at least from an *a priori* point of view!)

> it is of *crucial importance* to **specify the labels** $\in \mathbb{Z} \times \mathbb{Z}$ when discussing various portions (such as "$\overline{k}$", "$\overline{k}^{\times}$", "$\mathcal{O}^{\times}$") of the $\bullet$'s (i.e., Hodge theaters) in the log-theta-lattice.

The portions of the $\bullet$'s (i.e., Hodge theaters) that (at least from an *a priori* point of view!) **depend on the labels** $\in \mathbb{Z} \times \mathbb{Z}$ are referred to, in IUTch, as **Frobenius-like**. On the other hand, objects arising from certain Galois groups (such as G) or arithmetic fundamental groups (such as Π) are referred to as **étale-like**. If one restricts oneself to considering **isomorphisms** between objects, for instance, in situations that often arise in **anabelian geometry**, then there is not so much of a difference between *Frobenius-like* and *étale-like* versions of various objects. On the other hand, when one considers the relationship between objects on *opposite sides* of the Θ- and $\mathfrak{log}$-*links*, there is a very *fundamental difference* between *Frobenius-like* and *étale-like* versions of various objects, namely:

(EtFr) Whereas **Frobenius-like** objects on **opposite sides** of the **Θ- and $\mathfrak{log}$**-*links* are (typically) related to one another by means of some sort of relationship that is **far** from being an **isomorphism** (e.g., far from being **compatible** with the respective **ring structures**!), **étale-like** objects on *opposite sides* of the Θ- and $\mathfrak{log}$-*links* are (typically) related to one another by means of an **isomorphism**.

Perhaps the most *fundamental classical example* of this sort of situation is the situation that arises in the case of the **anabelian geometry** of **one-dimensional function fields over finite fields**, in the context of the **Frobenius morphism** in positive characteristic [cf. the discussion of [Alien], §2.6, §2.7, §2.8, for more details]:

(EtFrEx) Consider the field $F \stackrel{\mathrm{def}}{=} \mathbb{F}_p(t)$ (i.e., a purely transcendental extension of the field of cardinality p), for some prime number p. Then the *Frobenius* *morphism* $\phi : F \to F$, i.e., the field homomorphism that maps $t \mapsto t^p$, induces an **isomorphism**

$$\mathrm{Aut}(F) \xrightarrow{\sim} \mathrm{Aut}(F)$$

between the respective *groups of field automorphisms*, despite the fact that ϕ is **far** from being an **isomorphism**.

(Indeed, it is precisely this fundamental classical example that gave rise to the terminology *"Frobenius-like"* and *"étale-like"*.) Unlike the various *Frobenius-like* objects discussed in (LbΘ), (Lb$\mathfrak{log}$), (LbMn), *étale-like* objects admit important **symmetry** properties as follows (cf. (SWE1), (SWE2)):

(EtΘ) It is a *tautological consequence of the definitions* that the "G's" in the domain and codomain of the Θ-link — i.e., which act on the copies of $\mathcal{O}^\times \subseteq \overline{k}^\times \subseteq \overline{k}$ in the domain and codomain of the Θ-link — map isomorphically to one another via the Θ-link and, moreover, when regarded as **abstract topological groups** (i.e., which are *not equipped* with any data arising from the usual actions of G on some copy of $\mathcal{O}^\times \subseteq \overline{k}^\times \subseteq \overline{k}$!), may be **permuted** with one another.

(Et$\mathfrak{log}$) It is a *tautological consequence of the definitions* that the "Π's" in the domain and codomain of the $\mathfrak{log}$-link — i.e., which act on the copies of $\mathcal{O}^\times \subseteq \overline{k}^\times \subseteq \overline{k}$ in the domain and codomain of the $\mathfrak{log}$-link — map isomorphically to one another via the $\mathfrak{log}$-link and, moreover, when regarded as **abstract topological groups** (i.e., which are *not equipped* with any data arising from the usual actions of Π on some copy of $\mathcal{O}^\times \subseteq \overline{k}^\times \subseteq \overline{k}$!), may be **permuted** with one another.

(EtMn) It is a *tautological consequence of the definitions* that the diagram

$$\Pi \twoheadrightarrow G \twoheadleftarrow \Pi$$

— where the Π's and G are regarded as **abstract topological groups** (which are only known up to isomorphisms of topological groups); the $\twoheadrightarrow$, $\twoheadleftarrow$ are the *poly-morphisms* (i.e., sets of morphisms) given by the composites of the natural surjection $\Pi \twoheadrightarrow G$ with arbitrary automorphisms of the domain and codomain; the Π on the LHS (respectively, RHS) arises from the 0- (respectively, 1-)column of $\vec{\Gamma}_{\mathrm{main}}$ (cf. (LbMn), (Et$\mathfrak{log}$)); the G arises from the unique horizontal arrow of $\vec{\Gamma}_{\mathrm{main}}$ (cf. (LbMn), (EtΘ)) — admits a **symmetry** that **permutes** the Π's on the LHS and RHS and fixes the unique G of the diagram.

As discussed in §8, these symmetry properties (EtΘ), (Et$\mathfrak{log}$), (EtMn) are the *fundamental properties* that underlie the **multiradial algorithms** of IUTch (i.e., [IUTchIII], Theorem 3.11) and, in particular, play a *central role* in IUTch.

§16. One sort of *undercurrent theme* that underlies the various *misunderstandings* (cf. §2) discussed so far in the present report is

(DsGl) the **deep sense of discomfort** that some mathematicians feel with regard to the **gluing/identification** given by the value group portion of the Θ-link (cf. §9)

$$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \quad \stackrel{\sim}{\to} \quad \underline{\underline{q}}^{\mathbb{N}}$$

— a sense of discomfort that is perhaps partially related to the discussion surrounding (HstMod).

One possible approach to overcoming this sense of discomfort is to recall various *classical examples* of **closely related phenomena**.

Perhaps the *most fundamental* such "classical example" is

(GlMt) the special case of the property of *invariance of heights with respect to isogeny* (established by Faltings) in the case of **global multiplicative subspaces** of modules of torsion points of elliptic curves over number fields, i.e., subspaces that *coincide*, at all nonarchimedean primes of bad multiplicative reduction, with the subspace arising from the copy of the *multiplicative group scheme* "$\mathbb{G}_{\mathrm{m}}$" that occurs in the *Tate uniformization* (cf. [Alien], §2.3, for more details).

That is to say, such isogenies arising from global multiplicative subspaces correspond (say, in the case of l-torsion points, for l a prime number) locally at nonarchimedean primes of bad reduction to the operation of *raising q-parameters to the l-th power*

$$q \quad \mapsto \quad q^l$$

(cf. (DsGl)). In this context, it is useful to recall that:

(DFL) As discussed in [Alien], §2.3, §2.4, §2.5, this example (GlMt) is closely related to the technique of taking **derivatives of Frobenius liftings**, which, as discussed in [Alien], §2.6, §2.7, motivated the development of the technique of **mono-anabelian transport**, which plays a *central role* in IUTch.

One way to summarize the *essential mechanism* involved here is as follows:

(DiIsm) On the one hand, the *assignment*

$$q \quad \mapsto \quad q^l$$

(cf. (DsGl)) is, in essence, a **"dilation"**, i.e., multiplies heights/degrees/log-volumes by a *factor of l*. On the other hand, this assignment *induces*

$$d\log(q) \quad \mapsto \quad l \cdot d\log(q)$$

(essentially, since the *factor of l* preceding the "$d\log(q)$" can typically, for various technical reasons, be *ignored*!) an **"isometry"**. Moreover, since the arithmetic line bundle "ω" of invariant differentials on each of the elliptic curves under consideration arises from an **ample** line bundle on the moduli stack of elliptic curves, its arithmetic degree (i.e., the height of the elliptic curve) is *sufficient to control the moduli of the elliptic curve* (up to finitely many possibilities). It is precisely this **"tension"** between the *"dilation"* and *"isometry"* aspects of the assignment $q \mapsto q^l$, together with the *ampleness* just mentioned, that gives rise to *nontrivial consequences* concerning the elliptic curves involved.

Relative to the analogy with the technique of **mono-anabelian transport** (and hence with IUTch!):

(MAT1) the assignment "$q \mapsto q^l$" corresponds to the **link** involved, i.e., such as the **Θ-link** of IUTch;

(MAT2) the **dilation** aspect of this assignment corresponds to the effect of the link on **Frobenius-like** objects;

(MAT3) the **isometry** aspect of this assignment corresponds to the effect of the link on **étale-like** objects (cf., e.g., the **switching symmetry** discussed in (EtMn), which plays a crucial role in establishing the *multiradial algorithms* of IUTch!);

(MAT4) the **ampleness** of "ω", which may thought of as a sort of manifestation of **hyperbolicity**, corresponds to the **mono-anabelian** properties that are applied in the technique of mono-anabelian transport, i.e., properties that ensure that the isometry (e.g., étale-like) aspect is *"sufficiently rich"* to capture the objects of interest in their entirety.

Another fundamental "classical example" is the **analogy** between the **multiradial representation algorithms** developed in IUTch (i.e., [IUTchIII], Theorem 3.11) and the classical theory of **connections**, which we consider here from the point of view of *Grothendieck's definition of the notion of a connection*, i.e., the point of view that gave rise to the notion of a **crystal**. A detailed discussion of this analogy — which includes a discussion of the analogy between the *main results* of IUTch (i.e., [IUTchIII], Theorem 3.11; [IUTchIII], Corollary 3.12) and the *classical argument* used to prove the *geometric version of the Szpiro Conjecture* by considering the *Kodaira-Spencer morphism* — may be found in [Alien], §3.1, (v). That is to say,

(MlCr1) just as the structure of a **crystal** allows one to relate liftings of an object to **distinct** *nilpotent thickenings* of a **given (say, reduced) scheme** that *coincide* over the given scheme,

(MlCr2) the **multiradial representation algorithms** developed in IUTch (i.e., [IUTchIII], Theorem 3.11) allow one to relate **distinct** *ring-theoretic realizations*, i.e., via $\{\underline{\underline{q}}^{j^2}\}$ or $\underline{\underline{q}}$ (which correspond to *distinct thickenings*), of the **gluing data** that appears in the Θ-link, i.e., *"prime-strips"* of a certain type (which correspond to the object associated by a crystal to the *given reduced scheme*, i.e., for instance, a *positive characteristic scheme*).

Here, we recall (cf. the discussion at the beginning of [Alien], §3.1, (v)) that this analogy — which involves a comparison between degrees of ample line bundles (i.e., *"heights"*) and "0" (i.e., as opposed to some large positive multiple of the "height") — may be thought of as a sort of *limiting version* of a transformation of the form

$$q \quad \mapsto \quad q^N$$

(cf. (DsGl)) as $N \to +\infty$. The analogy between (MlCr1) and (MlCr2) is also of interest from a *historical point of view*:

(MlCrH1) just as the classical scheme-theoretic argument of [Alien], §3.1, (v), which presents the Kodaira-Spencer morphism from the point of view of (log) crystals and nilpotent thickenings of (log) schemes, *only makes sense* if one considers **distinct non-reduced schemes**, i.e., **distinct thickenings** *of a given reduced scheme* — notions that were *unfamiliar* to many algebraic geometers in the 1960's, who were accustomed only to working with *more classical notions of "varieties"*, e.g., notions that revolve around considering *collections of subrings of some fixed field*,

(MlCrH2) the theory surrounding the *main results* of IUTch (i.e., [IUTchIII], Theorem 3.11; [IUTchIII], Corollary 3.12) *only makes sense* if one considers **distinct copies of ring/scheme theory** (cf. (Smm), (GLR2), (IUAD)!) glued together along data of the sort that appears in the Θ-link (i.e., *"prime-strips"* of a certain type) — notions that are *unfamiliar* to many arithmetic geometers today, who are accustomed only to working *within a fixed copy of scheme theory*.

Yet another fundamental "classical example" may be seen in **Bogomolov's proof** of the *geometric version of the Szpiro Conjecture*. We refer to [BogIUT], as well as [Alien], §4.3, (iii), for a detailed discussion of the analogy between Bogomolov's proof and IUTch. One way to understand the **essential mathematical phenomenon** that underlies Bogomolov's proof is to consider the following *elementary observation*:

(SL$\mathbb{Z}\mathbb{R}$) The *unipotent matrices*

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

— where m ranges over the positive integers — are **conjugate** to one another in $SL_2(\mathbb{R})$, but are **not conjugate** to one another in $SL_2(\mathbb{Z})$.

Here, we recall that, from the point of view of the classical geometry of the upper half-plane, the unipotent matrix in (SL$\mathbb{Z}\mathbb{R}$) may be thought of as the **monodromy operator** associated to the *m-th power q^m* of the q-parameter $q = \exp(2\pi i z)$ (where z denotes the standard complex coordinate on the upper half-plane). Bogomolov's proof centers around a

(BgCp) **comparison** between a certain collection of 2×2 matrices regarded as *elements of $SL_2(\mathbb{Z})$* and the same collection of matrices regarded as *elements of $SL_2(\mathbb{R})$*, i.e., a comparison (cf. (SL$\mathbb{Z}\mathbb{R}$)!) between

some **fixed/rigid** (positive integer) power "q^{m_0}" of q

and

an **indeterminate** power "q^m" of q

(i.e., where m ranges over the positive integers) — cf. (DsGl).

More precisely, Bogomolov's proof involves liftings of elements in $SL_2(\mathbb{R})$ (or $SL_2(\mathbb{Z})$) to the *canonical extension*

$$1 \quad \to \quad \mathbb{Z} \quad \to \quad SL_2(\mathbb{R})^{\sim} \quad \to \quad SL_2(\mathbb{R}) \quad \to \quad 1$$

(or the restriction $SL_2(\mathbb{Z})^{\sim}$ of this extension to $SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{R})$). From a more scheme-theoretic point of view, the comparison in Bogomolov's proof may be thought of as a comparison between various *"realizations of the first Chern class"* of the Hodge bundle ω (i.e., arising from the invariant differentials on the tautological elliptic curve) on the moduli stack of elliptic curves $\mathcal{M}_{\mathrm{ell}}$ (and its canonical compactification $\overline{\mathcal{M}}_{\mathrm{ell}}$). That is to say:

(Chrn) The *"$SL_2(\mathbb{Z})^{\sim}$ side"* of the comparison in Bogomolov's proof corresponds to a sort of geometric version of the **complex holomorphic realization** of the first Chern class of ω (with compact supports relative to the compactification $\mathcal{M}_{\mathrm{ell}} \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$), while the *"$SL_2(\mathbb{R})^{\sim}$ side"* of the comparison in Bogomolov's proof corresponds to a sort of **metric/differential-geometric realization** of the first Chern class of ω, via various *canonical metrics arising from the hyperbolic geometry of the upper half-plane.*

At a more concrete level, the comparison — between the point of view of working inside $SL_2(\mathbb{R})^{\sim}$ and the point of view of working inside $SL_2(\mathbb{Z})^{\sim}$ — in Bogomolov's proof centers around consideration of a certain *relation*

$$[\widetilde{\alpha}_1, \widetilde{\beta}_1] \cdot \ldots \cdot [\widetilde{\alpha}_g, \widetilde{\beta}_g] \cdot \widetilde{\gamma}_1 \cdot \ldots \cdot \widetilde{\gamma}_r \;=\; (\widetilde{\tau}^{|\measuredangle|})^{2n^{\measuredangle}}$$

(cf. the notation of the discussion of [BogIUT] preceding (B4)) between elements $\in SL_2(\mathbb{Z})^{\sim} \subseteq SL_2(\mathbb{R})^{\sim}$. This **"linking data"** (i.e., consisting of the *elements* that appear in the above *relation*, together with the *relation* itself) between the *"$SL_2(\mathbb{R})^{\sim}$ side"* and *"$SL_2(\mathbb{Z})^{\sim}$ side"* of the comparison in Bogomolov's proof is interesting, relative to the analogy with IUTch, in that the "$\widetilde{\gamma}_i$'s" (for $i = 1, \ldots, r$) correspond to the *local q-parameters* of the family of elliptic curves under consideration, while (the integer power that appears in) "$(\widetilde{\tau}^{|\measuredangle|})^{2n^{\measuredangle}}$" corresponds to the *(global) height* of the family of elliptic curves under consideration. That is to say,

(LkDt) The *local* "$\widetilde{\gamma}_i$'s" (for $i = 1, \ldots, r$) may be understood as corresponding to the **local value group portion** (at, say, *bad primes*) of the gluing data (i.e., *"prime-strips"* of a certain type) that appears in the **Θ-link**, while the *global* "$(\widetilde{\tau}^{|\measuredangle|})^{2n^{\measuredangle}}$" may be understood as corresponding to the **global realified Frobenioid portion** (i.e., global value group portion) of the gluing data (i.e., *"prime-strips"* of a certain type) that appears in the **Θ-link**. (Here, we note in passing that one may also interpret the "$[\widetilde{\alpha}_j, \widetilde{\beta}_j]$'s" (for $j = 1, \ldots, g$) in this context as corresponding to the **local unit group portion** (at, say, *good primes*) of the gluing data (i.e., *"prime-strips"* of a certain type) that appears in the **Θ-link**, where we recall that this local unit group portion gives rise, via *log-volume computations of log-shells*, to the *log-different* term (i.e., an arithmetic analogue of the *Euler characteristic* $2 - 2g$) in the bound for the height of an elliptic curve over a number field obtained in [IUTchIV], Theorem 1.10.)

Bogomolov's proof then proceeds by *comparing*

(Bg1) a certain **displacement estimate** — which is *invariant* with respect to *independent $SL_2(\mathbb{R})^{\sim}$-conjugation indeterminacies* acting on each of the linking data elements — that arises from considering the above *linking data* in $SL_2(\mathbb{R})^{\sim}$ (cf. the discussion preceding [BogIUT], (B5)) with

(Bg2) a certain **computation of the height** of the family of elliptic curves under consideration in terms of the *local q-parameters* (cf. [BogIUT], (B4), together with the discussion immediately preceding [BogIUT], (B4)) that arises from considering the above *linking data* in $SL_2(\mathbb{Z})^\sim$.

Relative to the analogy with IUTch,

(BgIUT1) the **displacement estimate** in Bogomolov's proof — which arises from working with **indeterminate** *powers of q-parameters* (cf. the *independent* $SL_2(\mathbb{R})^\sim$*-conjugation indeterminacies*!), relative to $SL_2(\mathbb{R})^\sim$ — corresponds to the **multiradial representation** of IUTch (i.e., [IUTchIII], Theorem 3.11), while

(BgIUT2) the **computation of the height** in terms of local q-parameters in Bogomolov's proof — which arises from working with **rigid** *powers of q-parameters*, relative to $SL_2(\mathbb{Z})^\sim$ — corresponds to the argument of [IUTchIII], Corollary 3.12.

Moreover, in this context, it is of interest to observe, relative to the analogy with IUTch, that:

(DsInd) This computation via **displacement estimates** in Bogomolov's proof furnishes a *fascinating* and *relatively elementary* example of the phenomenon of computing the *"size"* of a *region* subject to **indeterminacies** (cf. (Bg1), (BgIUT1), i.e., the displacement estimates in Bogomolov's proof, which are *invariant* with respect to the *independent* $SL_2(\mathbb{R})^\sim$*-conjugation indeterminacies* acting on each of the linking data elements; the multiradial representation of the Θ-pilot object in IUTch) that **approximates** (say, from above) some *region of interest* that is **"rigid"** and *not subject to indeterminacies* (cf. (Bg2), (BgIUT2), i.e., the linking data in Bogomolov's proof considered in $SL_2(\mathbb{Z})^\sim$; the Θ-pilot relative to the Θ-hol. str. in IUTch).

Finally, we observe that the partition of Bogomolov's proof mentioned above into (Bg1)/(BgIUT1) and (Bg2)/(BgIUT2) is of interest not only from the point of view of the correspondence with [IUTchIII], Theorem 3.11, and [IUTchIII], Corollary 3.12, but also from a *historical point of view*. Indeed, (Bg1) (though apparently rediscovered by Bogomolov around the year 2000) was *already known to Milnor in the 1950's*, while (Bg2) was, apparently, *not known until Bogomolov's work* around the year 2000.

§17. The *fundamental misunderstandings of IUTch* discussed in the present report may be *summarized* as a failure to understand the following central aspects of IUTch:

(FM1) the significance of the operation of **re-initialization of histories** (cf. (T1), (H2)), the **explicit specification** of **types of mathematical objects** (cf. (T2), (H2)), and the use of **poly-morphisms** (cf. (T4), (PMInd), (PMEss)) in the construction of mathematical structures (e.g., in IUTch) that admit **switching symmetries** (cf. (SW), (Sym), (SWC1), (SWC2), (SWE1), (SWE2), (LbΘ), (Lb$\mathfrak{log}$), (LbMn), (EtΘ), (Et$\mathfrak{log}$), (EtMn));

(FM2) the significance of **explicit specification** of **label apparatuses** (cf. (T3), (T5), (IUAD), (SSDLFs), (LbRl), (LbEx1), (LbEx2), (LbEx3), (LbEx4), (LbEx5), (LbHT), (LbΘ), (Lb$\mathfrak{log}$), (LbMn)) to distinguish **distinct copies** of various familiar objects that play **substantively different roles** within a complicated mathematical structure;

(FM3) the significance of the *requirement* that the various data (i.e., *value group portion* and *unit group portion*) that occurs in the **domain** of the **Θ-link** arise from a **single ring str.** (cf. (ΘCR), (VUSQ), (VUC)) — a requirement that must be satisfied in order to apply the **multiradial representation algorithms** *developed in IUTch*, i.e., [IUTchIII], Theorem 3.11 (cf. (T3), (SSInd), (GIUT), (SSId), ($^{\dagger}\Theta$CR), (SSIdFs), (SSADFs)).

In this context, we observe that the point of view of (FM1), i.e., of **"forgetting histories"**, may, at first glance, appear to lie in a *contradictory direction* to the point of view of (FM2), i.e., of **"remembering labels"**. In fact, however, although, depending upon the content of various *technical goals*, it may be necessary to *"remember"* some mathematical structures in some situations and to *"forget"* other mathematical structures in other situations,

> the *common thread* that unites (FM1) and (FM2) is the principle that it is always of fundamental importance to **specify explicitly** *which structures one wishes to* **remember** *or* **forget**.

One way to summarize (FM1), (FM2), (FM3) is as follows:

(MVEx) Each of (FM1), (FM2), (FM3) centers around the issue of experimenting with various **modified versions of IUTch** to examine to what extent the essential content of IUTch continues to hold in such modified versions.

Since, typically, research papers in mathematics (such as, for instance, the papers [IUTchI], [IUTchII], [IUTchIII], [IUTchIV]!) do *not contain any exercises*, experimenting with various modified versions of a mathematical theory, i.e., such as those that appear in (FM1), (FM2), (FM3), is an *important step* in acquiring a deep understanding of the theory. From this point of view, the experimental versions of IUTch that appear (either explicitly or implicitly!) in (FM1), (FM2), (FM3) may be regarded as a **valuable pedagogical tool** for (both present and future) students of IUTch and, in particular, as an important step forward (cf. the discussion of §3) towards the goal of

> *completely diagnosing and explicating the logical structure of all misunderstandings concerning IUTch.*

In particular, one might hope (cf. the discussion of §3) that (FM1), (FM2), (FM3) may serve *to stimulate the further examination and development of such exercises/experimental versions of IUTch.*

§18. In the context of the present report, it is important to recall that

(Vrf1) IUTch has been *checked*, *verified*, *read and reread*, and *orally exposed in detail in seminars in its entirety* **countless times** since the release of preprints on IUTch in August 2012 by a collection of mathematicians (not including myself) involved in this line of research. (For instance, Fesenko estimates, in the most recent updated version of §3.1 of his survey [Fsk], that IUTch has been verified at least 30 times.) This collection of mathematicians has (together with me) also been actively involved in detailed discussions and dialogues with mathematicians who have any questions concerning IUTch.

Such discussions have been useful in enhancing the level of understanding of IUTch by many mathematicians, as well as in finding minor inessential inaccuracies, misprints, etc. in the preprints on IUTch. Such discussions have also served to confirm countless times that

(Vrf2) there is **no substantive mathematical reason** whatsoever to suspect the **existence of any oversights**, i.e., any flaws or incompleteness of a mathematically substantive nature, in the numerous verifications that have been conducted so far concerning IUTch.

In light of this state of affairs, it should not be surprising that my oral explanations, over the past few months, to various colleagues involved in (Vrf1), (Vrf2) of the misunderstandings summarized in §17 were met with a **remarkably unanimous response** of **utter astonishment** and even **disbelief** (at times accompanied by bouts of laughter!) that such manifestly erroneous misunderstandings could have occurred. At times, this sort of response put me in the somewhat paradoxical position of having to "justify" or "defend" the line of reasoning of SS that apparently led to such misunderstandings. Put another way,

(CMA) the flaws in IUTch that are alleged in (SSId), (SSAD) — i.e., which correspond to the most **central** aspects of the **misunderstandings** summarized in §17 — are so **utterly pronounced** and **conspicuous** in their **absurdity** that it is very difficult to believe that such flaws could have remained *undetected by any competent mathematician for even a few minutes*, let alone throughout the duration of the *intensive verification activities* (cf. (Vrf1), (Vrf2)) by quite a number of talented mathematicians that have been carried out over the years since August 2012.

Careful reflection concerning this state of affairs over the months since the March discussions has led me to the following conclusion:

(DfMth) Perhaps the most reasonable explanation for the state of affairs summarized above in (Vrf1), (Vrf2), (CMA) is that the *mathematics that SS perceive as being referred to by the term "IUTch"* is simply **substantially different** from the *mathematics that HM* (as well as other mathematicians who have been deeply involved in the various activities concerning IUTch described in (Vrf1), (Vrf2)) *understand by the term "IUTch"* (cf. (Smm), (GLR2), (T6), (T7), (T8), (SSId), (SSAD), (DfLb)).

Indeed, at numerous points in the March discussions, I was often tempted to issue a response of the following form to various assertions of SS (but typically refrained from doing so!):

> *Yes! Yes! Of course, I completely agree that the theory that you are discussing is completely absurd and meaningless, but* **that** *theory is completely different from IUTch!*

Nevertheless, the March discussions were productive in the sense that they yielded a *valuable first glimpse* at the *mathematical content of the misunderstandings* that underlie criticism of IUTch (cf. the discussion of §3). In the present report, we considered various *possible causes for these misunderstandings*, namely:

(PCM1) lack of sufficient **time** to **reflect deeply** on the mathematics under discussion (cf. the discussion in the final portions of §2, §10);

(PCM2) **communication issues** and related **procedural irregularities** (cf. (T6), (T7), (T8));

(PCM3) a **deep sense of discomfort**, or **unfamiliarity**, with *new ways of thinking* about familiar mathematical objects (cf. the discussion of §16; [Rpt2014], (T2); [Fsk], §3.3).

On the other hand, the March discussions were, unfortunately, **by no means sufficient** to yield a **complete elucidation** of the logical structure of the **causes** underlying the misunderstandings summarized in §17.

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Updated versions of preprints are available at the following webpage:

`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`