COMMENTS ON "SEMI-GRAPHS OF ANABELIOIDS"

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(1.) In the third sentence of Example 2.10, the phrase "Now observe that a hyperbolic Riemann surface of finite type" should read *"Now observe that a hyperbolic Riemann surface of finite type of genus* ≥ 1*"*.

(2.) With regard to the proof of Corollary 3.11:

(i) In the first line of the proof, it should be stipulated that the set Σ be *nonempty.*

(ii) The phrase *"as in (ii)"* in line 2 of observation (iv) should read *"as in (iii)"*.

(iii) A more detailed version of the argument used to verify observation (iv) is given in [1], Corollary 2.11.

(3.) In the discussion of the *"pro-*Σ *version" of Corollary 3.11* in Remark 3.11.1,

one should *assume* that $p_\alpha, p_\beta \in \Sigma$.

In fact, this assumption is, in some sense, *implicit* in the phraseology that appears in the first two lines of Remark 3.11.1, but it should have been stated *explicitly.*

(4.) Note that in Theorem 5.4, the case where $\mathcal{A}$ is *trivial* [i.e., is equal to the anabelioid associated to the *trivial group* $\{1\}$] is *not excluded.* Thus, suppose that, in Theorem 5.4, we assume further that $\mathcal{A}$ is *trivial.* Then let us *observe* that this implies that the underlying graph of $\overline{\mathcal{G}}$ [or $\overline{\mathcal{H}}$] consists of a *single vertex* and *no edges.* [Indeed, if the underlying graph of $\overline{\mathcal{G}}$ has at least one edge, then since $\overline{\mathcal{G}}$ is assumed to be *totally elevated*, it follows from the assumption that $\overline{\mathcal{G}}$ is *totally arithmetically estranged* [cf. Definition 5.3, (ii)] that $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ admits a closed subgroup that *fails to be arithmetically ample*, hence that $\Pi_{\mathcal{A}} = \{1\}$ contains a closed subgroup which is *not open* — a *contradiction.*] Thus, $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ itself is a *verticial subgroup* of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$, hence *compact.* In particular, $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$ is the *unique maximal compact subgroup* of $\Pi^{\mathrm{temp}}_{\overline{\mathcal{G}}}$, so assertions (i), (ii), and (iii) of Theorem 5.4 are, in essence, *vacuous.*

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(5.) In the line 2 of Example 5.6, the phrase "Also, Suppose..." should read "Also, suppose...". In lines 7–8 of Example 5.6, one should also assume that M_i was chosen so that the resulting Galois action on the dual semi-graph with compact structure of the special fiber of the stable model is *trivial* [i.e., so as to ensure that the assumption of Theorem 5.4 concerning switching the branches of edges is satisfied].

(6.) Some readers may find the argument given in the third and fourth paragraphs of the proof of Theorem 3.7, (iii), to be a bit confusing in its brevity. A more detailed argument may be given as follows. For $i \in I$, let us write

$$\mathbb{V}_i, \ \mathbb{E}_i$$

for the sets of *vertices* and *closed edges*, respectively, of $\mathbb{G}_{i,\infty}$ that are *fixed* by the action of H. Thus, for $i \geq j \in I$, we have natural maps $\mathbb{V}_i \to \mathbb{V}_j$, $\mathbb{E}_i \to \mathbb{E}_j$; let us write

$$\mathbb{E}_{j,i} \subseteq \mathbb{E}_j$$

for the image of $\mathbb{E}_i$ in $\mathbb{E}_j$. Thus, for $i_1, i_2 \in I$ such that $i_1 \geq i_2$, we have $\mathbb{E}_{j,i_1} \subseteq \mathbb{E}_{j,i_2} \subseteq \mathbb{E}_j$. Also, we recall that, by the argument given in the second paragraph of the proof, we have $\#\mathbb{V}_i \geq 1$ [where we use the notation "#" to denote the cardinality of a set], for all $i \in I$. For simplicity, in the following, we assume that the semi-graph $\mathbb{G}_i$ is *untangled*, for all $i \in I$. Now:

(a) Suppose that for some cofinal subset $J \subseteq I$, we have $\#\mathbb{V}_j = 1$, for all $j \in J$. Then the unique elements of the $\mathbb{V}_j$, for $j \in J$, form a *compatible system of vertices fixed by* H. Thus, we conclude that H is contained in some *verticial subgroup* of $\pi_1^{\text{temp}}(\mathcal{G})$.

(b) Suppose that for some cofinal subset $J \subseteq I$, we have $\#\mathbb{V}_j \geq 2$, for all $j \in J$. Then it follows from Lemma 1.8, (ii), (b), that $\#\mathbb{E}_j \geq 1$, for all $j \in J$. Now I *claim* that for each $j \in J$, the following condition holds:

$(*_j)$ there exists an $i \in J$ such that $i \geq j$ and $\#\mathbb{E}_{j,i} = 1$.

Indeed, suppose that $(*_j)$ *fails to hold*. Then for each $i \geq j$ in J, there exists a pair of *distinct edges* $e_i, e'_i \in \mathbb{E}_i$ whose respective images $e_{j,i}, e'_{j,i} \in \mathbb{E}_j$ are *distinct*. By Lemma 1.8, (ii), (b), we may assume without loss of the generality that the pair $\{e_i, e'_i\}$, hence also the pair $\{e_{j,i}, e'_{j,i}\}$, forms a *subjoint*. Then since $\mathbb{G}_j$ is *untangled*, it follows that the respective images $f_{j,i}$, $f'_{j,i}$ of $e_{j,i}$, $e'_{j,i}$ in $\mathbb{G}_j$ also form a *subjoint*. Write f_i, f'_i for the respective images of e_i, e'_i in $\mathbb{G}_i$. Thus, it follows from the fact that the pair $(f_{j,i}, f'_{j,i})$ forms a subjoint (of $\mathbb{G}_j$) that the pair (f_i, f'_i) forms a subjoint (of $\mathbb{G}_i$). Moreover, for some cofinal subset $J^* \subseteq J$, the subjoints (f_i, f'_i), where $i \in J^*$, converge, in the profinite topology, to some *profinite subjoint*. As discussed in the third paragraph, this leads to a *contradiction*, in light of our assumption that $\mathcal{G}$ is *totally estranged*. This completes the proof of the *claim*. Now it follows from $(*_j)$ that each of the *nonempty* sets $\mathbb{E}_{j,i}$, for $i, j \in J$ such that i is *"sufficiently large"* relative to j, is of cardinality 1. But this implies that each intersection

$$\mathbb{E}_{j,\infty} \stackrel{\text{def}}{=} \bigcap_{i \geq j} \mathbb{E}_{j,i}$$

is of cardinality 1. Thus, the unique elements of the $\mathbb{E}_{j,\infty}$, for $j \in J$, form a *compatible system of closed edges fixed by* H. In particular, we conclude that H is contained in some *edge-like subgroup*, hence also in *two distinct verticial subgroups*, of $\pi_1^{\text{temp}}(\mathcal{G})$.

(c) Now it follows formally from (a), (b) that H is *always* contained in *some verticial subgroup* of $\pi_1^{\text{temp}}(\mathcal{G})$. If H is contained in *three distinct verticial subgroups*, then it follows immediately from Lemma 1.8, (ii), (b), that one obtains a *contradiction* to the condition $(*_j)$ of (b). This completes the proof of assertion (iii) of Theorem 3.7.

Bibliography

[1] S. Mochizuki, *Topics in Absolute Anabelian Geometry II: Decomposition Groups*, RIMS Preprint **1625** (March 2008).