§1. Tempered Fundamental Groups of Semi-graphs of Anabelioids

Consider a (connected) semi-graph of Anabelioids (considered geometrically, e.g., allow fractal curving).

\[ \mathcal{N} = (\Pi, \{ \mu_v \}, \{ \mu_e \}, \{ (\nu_v, \nu_e) \} \]

\[ \text{vertices} \quad \text{edges} \quad \text{branches of edges} \]

(cf. "graph of groups")

\[ B^{\text{cov}}(\mathcal{N}) : \text{category of } \{ (S_v, \mu_v) : S_v \in \mathcal{N}_v, \mu_v : \pi_1 S_v \to \pi_1 S_v \} \]

\[ B^{\text{cov}}(\mathcal{N}) \cong k^{\text{temp}}(\mathcal{N}) \cong B(\mathcal{N}) \]

Gal\text{ois category}

\[ \text{distinguished } (\mathcal{N}) \]

\[ \text{dominated by a combinatorial } (= \text{graph-theoretic}) \]

\[ \text{covering of a finite covering (cf. work of André!)} \]

\[ \pi_2^{\text{temp}}(\mathcal{N}) \]

\[ \text{category of etale, discrete sets with continuous } \Pi \text{-action.} \]

\[ \text{Has similar properties to Galois categories!} \]

\[ \text{e.g. } \text{Hom}_{\text{et}}(\mathcal{N}_1, \mathcal{N}_2) \]

\[ \text{pull-back functor that preserves } \text{finite, limits, etc. categories (considered up to isom. of functors)} \]
Example: \( K/\mathbb{Q}_p < \infty \), \( X \): hyperbolic curve \( /K \), \( \tilde{\text{stable model}} \ X \)

\[ x_K \rightarrow \text{semi-graph of anab'lds} \ Y^c_X \]

\((k= \text{res. fld. of } K) \quad \text{(pro-} \Sigma \text{ version)}\)

\[ C \text{ (continuous action)} \]
\[ G_K = \text{Gal} (K/\mathbb{Q}) \]

§2. Maximal Compact S'gps:

\( \Gamma \) 'nice' semi-gr. of anab'do.

(e.g., Example)

Then: \( \text{maximal compact s'gps. of } \pi_1^{\text{temp}} (S) \leftrightarrow \text{vertical s'gps. of } \pi_1^{\text{temp}} (S) \)

\( \left( \frac{\mathbb{Z}}{2} \right)_V \)

\( \text{nontrivial intersections of two distinct max. compact s'gps. of } \pi_1^{\text{temp}} (S) \leftrightarrow \text{edge-like s'gps. of } \pi_1^{\text{temp}} (S) \)

\( \left( \frac{\mathbb{Z}}{2} (\mathbb{Z}_V) \right) \)

\( \text{Apply result of Sene, Trees: finite group } G \text{ tree} \)

\( \Rightarrow \exists \text{ fixed vertex or edge} \)

Corollary: Can reconstruct underlying graph (semi-graph \( \setminus \) open edges) ('anabelian') of \( \Delta \) from \( \text{top. gr. } \pi_1^{\text{temp}} (S) \)
3. Applications.

Prop A: \( X, Y \) hyperbolic curves/hyperarch. local fields, \( \mathbb{A} \) stable model

\[ \Delta_{X} \sim \Delta_{Y} \text{ (geometric tempered fund. gp. - cf. Andre)} \]

\[ \forall \Delta_{X} \sim \Delta_{Y} \quad \mathcal{M}_{X} \sim \mathcal{M}_{Y} \]

Proof: (Cor. of §2) + (wild ram. covering \( \sim \) rigidified comp. / cusp \( \sim \) open edges)

Remark: can combine with results of Tamagawa for \( \pi_{1}^{\text{temp.}} \) curves (\( \pi_{1}^{\text{temp.}} \)).

\( X/K \to Y/L \): hyp. curves; \( K, L \) nonarch. loc. fields.

Thin B: (dominant mors.) \( \Leftrightarrow \) (outer home. of DOF-type.

\[ \left( \Xi_{X} \sim \Xi_{Y} \right) \Rightarrow \left( \Pi_{X}^{\text{temp.}} \to \Pi_{Y}^{\text{temp.}} \right) \text{ over } \mathbb{G}_{k} \to \mathbb{G}_{L} \]

Thin C: (P-adic anal. view) \( \Rightarrow \) (inner home.

Properties fund. gps, abs. p-adic anal. geom.)
Thin D: Suppose that \( X \) 'of strictly Belyi type', defd / as flasque isogenous to genus 0 curve isomorphic to common fn. d. curve.

Let \( T_X \) be an isom. Then:

1. \( X \) preserves decomps. of closed pts.
2. \( Y \) also of strictly Belyi type.

Pf. Uses Frac. galois, geometry of fn. gp. G tree. (cf. 'real section conjecture', geometry of 'straight line spaces')

Thin E: Let \( X, Y, \alpha \) be as in Thin D. Then \( \alpha \) arises geometrically.

Pf. (Thin D + ...) or (Thin D + ... + Thin C)

Rank: (i) first **strong** abs. pGC result!
   (ii) only for ctbly many curves (false for arb. \( X, Y \)??)
   (iii) abs pGC for canonical curves of \( p \)-adic Tech. theory
   - also only ctbly many