COMMENTS ON "THE ÉTALE THETA FUNCTION AND ITS FROBENIOID-THEORETIC MANIFESTATIONS"

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(i) The first sentence of Definition 3.1, (ii) [i.e., the definition of the term *"log-meromorphic"*], should be replaced by the following text:

> A *log-meromorphic function* on $\mathfrak{Z}_\infty^{\log}$ is defined to be a nonzero meromorphic function f on $\mathfrak{Z}_\infty^{\log}$ such for every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$. [Thus, it follows immediately, by considering the ramification divisors of such tempered coverings that arise from extracting roots of f, that the divisor of zeroes and poles of f is a *log-divisor*.]

That is to say, the class of meromorphic functions that are "log-meromorphic" in the sense of this modified definition is contained in the class of meromorphic functions that are "log-meromorphic" in the sense of the original definition. In light of the content of this modified definition, perhaps a better term for this class of meromorphic functions would be *"tempered-meromorphic"*.

(ii) In order to understand the relationship between the modified definition of (i) and the original definition, it is useful to consider the following conditions on a nonzero meromorphic function f on $\mathfrak{Z}_\infty^{\log}$:

(a) For every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(b) For every $N \in \mathbb{N}_{\geq 1}$ which is *prime to* p, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(c) The divisor of zeroes and poles of f is a *log-divisor*.

Thus, (a) is the condition of the modified definition of (i); (c) is the condition of the original definition. It is immediate that (a) implies (b). Moreover, [cf. (i)] one verifies immediately, by considering the ramification divisors of the tempered coverings that arise from extracting roots of f, that (b) implies (c). When N is *prime to* p, if f satisfies (c), then it follows immediately from the theory of *admissible coverings* [cf., e.g., [1], §2, §8] that there exists a *finite* log étale covering $Y^{\log} \to Z^{\log}$ whose pull-back $Y_\infty^{\log} \to Z_\infty^{\log}$ to $Z_\infty^{\log}$ is sufficient

(R1) to annihilate all ramification over the cusps or special fiber of $\mathfrak{Z}_\infty^{\log}$ that might arise from extracting an N-th root of f, as well as

(R2) to split all extensions of the function fields of irreducible components of the special fiber of $\mathfrak{Z}_\infty^{\log}$ that might arise from extracting an N-th root of f.

That is to say, in this situation, it follows that f *admits an N-root over the tempered covering of $Z^{\log}$ given by the "universal combinatorial covering" of $Y^{\log}$*. In particular, it follows that (c) implies (b). Thus, in summary, we have:

$$\text{(a)} \implies \text{(b)} \iff \text{(c)}.$$

On the other hand, unfortunately, it is not clear to the author at the time of writing *whether or not (c) [or (b)] implies (a).*

(iii) Observe that it follows from the theory of §1 [cf., especially, Proposition 1.3] that the *theta function* that forms the main topic of interest of the present paper *satisfies condition (a).* Indeed, the only instance occurring in the remainder of the text where the modified definition of (i) makes a difference is the proof of Proposition 4.2, (iii). That is to say, in this proof, it is necessary to use property (a) of (ii) [i.e., as opposed to just properties (b) or (c)]. Thus, this situation is remedied [without any affect on the remainder of the text] by taking property (a) to be the definition of "log-meromorphic". The author apologizes for any confusion caused by this oversight on his part.

(iv) An alternative approach to the approach of (i) above [i.e., of modifying the definition of the term *"log-meromorphic"*] is the following. One may leave Definition 3.1, (ii), *unchanged*, if one modifies Definition 4.1, (i), by assuming further that the meromorphic function "$f \in \mathcal{O}^\times(A^{\text{birat}})$" of *loc. cit.* satisfies the following "Frobenioid-theoretic version" of condition (a):

(d) For every $N \in \mathbb{N}_{\geq 1}$, there exists a linear morphism $A' \to A$ in $\mathcal{C}$ such that the pull-back of f to A' admits an N-th root.

[Here, we recall that, as discussed in (iii), the Frobenioid-theoretic theta functions that appear in the present paper satisfy (d).] Note that since the rational function monoid of the Frobenioid $\mathcal{C}$, as well as the linear morphisms of $\mathcal{C}$, are *category-theoretic* [cf. [2], Theorem 3.4, (iii), (v); [2], Corollary 4.10], this condition (d) is *category-theoretic.* Thus, if one modifies Definition 4.1, (i), in this way, then the remainder of the text goes through without change, except that one must replace the reference to the definition of "log-meromorphic" [i.e., Definition 3.1, (ii)] that occurs in the proof of Proposition 4.2, (iii), by a reference to condition (d) [i.e., in the modified version of Definition 4.1, (i)].

(v) In the discussion preceding Definition 2.1, one must in fact assume that the integer l is *odd* in order for the quotient $\overline{\Delta}_X$ to be *well-defined.* Since, ultimately, in the present paper [cf. the discussion following Remark 5.7.1], this is the only case that is of interest, this oversight does not affect the bulk of the remainder of the present paper. Indeed, the only places where the case of *even* l is used are Remark 2.2.1 and the application of Remark 2.2.1 in the proof of Proposition 2.12 for the orbicurves "$\underline{\underline{\dot{C}}}$". Thus, Remark 2.2.1 must be *deleted*; in Proposition 2.12, one must in fact *exclude* the case where the orbicurve under consideration is "$\underline{\underline{\dot{C}}}$". On the other hand, this theory involving Proposition 2.12 [cf., especially, Corollaries 2.18, 2.19] is only applied *after* the discussion following Remark 5.7.1, i.e., which only treats the curves "$\underline{\underline{X}}$". That is to say, ultimately, in the present paper, one is only interested in the curves "$\underline{\underline{X}}$", whose treatment only requires the case of *odd* l.

Bibliography

[1] S. Mochizuki, The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields, *J. Math. Sci. Univ. Tokyo* **3** (1996), pp. 571-627.

[2] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.