COMMENTS ON "THE ÉTALE THETA FUNCTION AND ITS FROBENIOID-THEORETIC MANIFESTATIONS"

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(i) The first sentence of Definition 3.1, (ii) [i.e., the definition of the term *"log-meromorphic"*], should be replaced by the following text:

> A *log-meromorphic function* on $\mathfrak{Z}_\infty^{\log}$ is defined to be a nonzero meromorphic function f on $\mathfrak{Z}_\infty^{\log}$ such for every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$. [Thus, it follows immediately, by considering the ramification divisors of such tempered coverings that arise from extracting roots of f, that the divisor of zeroes and poles of f is a *log-divisor*.]

That is to say, the class of meromorphic functions that are "log-meromorphic" in the sense of this modified definition is contained in the class of meromorphic functions that are "log-meromorphic" in the sense of the original definition. In light of the content of this modified definition, perhaps a better term for this class of meromorphic functions would be *"tempered-meromorphic"*.

(ii) In order to understand the relationship between the modified definition of (i) and the original definition, it is useful to consider the following conditions on a nonzero meromorphic function f on $\mathfrak{Z}_\infty^{\log}$:

(a) For every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(b) For every $N \in \mathbb{N}_{\geq 1}$ which is *prime to* p, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(c) The divisor of zeroes and poles of f is a *log-divisor*.

Thus, (a) is the condition of the modified definition of (i); (c) is the condition of the original definition. It is immediate that (a) implies (b). Moreover, [cf. (i)] one verifies immediately, by considering the ramification divisors of the tempered coverings that arise from extracting roots of f, that (b) implies (c). When N is *prime to* p, if f satisfies (c), then it follows immediately from the theory of *admissible coverings* [cf., e.g., [1], §2, §8] that there exists a *finite* log étale covering $Y^{\log} \to Z^{\log}$ whose pull-back $Y_\infty^{\log} \to Z_\infty^{\log}$ to $Z_\infty^{\log}$ is sufficient

(R1) to annihilate all ramification over the cusps or special fiber of $\mathfrak{Z}_\infty^{\log}$ that might arise from extracting an N-th root of f, as well as

(R2) to split all extensions of the function fields of irreducible components of the special fiber of $\mathfrak{Z}^{\log}_{\infty}$ that might arise from extracting an N-th root of f.

That is to say, in this situation, it follows that f *admits an N-root over the tempered covering of $Z^{\log}$ given by the "universal combinatorial covering" of $Y^{\log}$*. In particular, it follows that (c) implies (b). Thus, in summary, we have:

$$\text{(a)} \implies \text{(b)} \iff \text{(c)}.$$

On the other hand, unfortunately, it is not clear to the author at the time of writing *whether or not (c) [or (b)] implies (a).*

(iii) Observe that it follows from the theory of §1 [cf., especially, Proposition 1.3] that the *theta function* that forms the main topic of interest of the present paper *satisfies condition (a).* Indeed, the only instance occurring in the remainder of the text where the modified definition of (i) makes a difference is the proof of Proposition 4.2, (iii). That is to say, in this proof, it is necessary to use property (a) of (ii) [i.e., as opposed to just properties (b) or (c)]. Thus, this situation is remedied [without any affect on the remainder of the text] by taking property (a) to be the definition of "log-meromorphic". The author apologizes for any confusion caused by this oversight on his part.

(iv) An alternative approach to the approach of (i) above [i.e., of modifying the definition of the term *"log-meromorphic"*] is the following. One may leave Definition 3.1, (ii), *unchanged*, if one modifies Definition 4.1, (i), by assuming further that the meromorphic function "$f \in \mathcal{O}^{\times}(A^{\mathrm{birat}})$" of *loc. cit.* satisfies the following "Frobenioid-theoretic version" of condition (a):

(d) For every $N \in \mathbb{N}_{\geq 1}$, there exists a linear morphism $A' \to A$ in $\mathcal{C}$ such that the pull-back of f to A' admits an N-th root.

[Here, we recall that, as discussed in (iii), the Frobenioid-theoretic theta functions that appear in the present paper satisfy (d).] Note that since the rational function monoid of the Frobenioid $\mathcal{C}$, as well as the linear morphisms of $\mathcal{C}$, are *category-theoretic* [cf. [2], Theorem 3.4, (iii), (v); [2], Corollary 4.10], this condition (d) is *category-theoretic*. Thus, if one modifies Definition 4.1, (i), in this way, then the remainder of the text goes through without change, except that one must replace the reference to the definition of "log-meromorphic" [i.e., Definition 3.1, (ii)] that occurs in the proof of Proposition 4.2, (iii), by a reference to condition (d) [i.e., in the modified version of Definition 4.1, (i)].

(v) In the discussion preceding Definition 2.1, one must in fact assume that the integer l is *odd* in order for the quotient $\overline{\Delta}_X$ to be *well-defined*. Since, ultimately, in the present paper [cf. the discussion following Remark 5.7.1], this is the only case that is of interest, this oversight does not affect the bulk of the remainder of the present paper. Indeed, the only places where the case of *even* l is used are Remark 2.2.1 and the application of Remark 2.2.1 in the proof of Proposition 2.12 for the orbicurves "$\underline{\underline{\dot{C}}}$". Thus, Remark 2.2.1 must be *deleted*; in Proposition 2.12, one must in fact *exclude* the case where the orbicurve under consideration is "$\underline{\underline{\dot{C}}}$". On the other hand, this theory involving Proposition 2.12 [cf., especially, Corollaries 2.18, 2.19] is only applied *after* the discussion following Remark 5.7.1, i.e., which only treats the curves “$\underline{\underline{X}}$”. That is to say, ultimately, in the present paper, one is only interested in the curves “$\underline{\underline{X}}$”, whose treatment only requires the case of *odd* l.

(vi) The phrase “the *unique* value $\in \mathcal{O}_K^\times$” in the first line of Definition 1.9, (ii), should read “the *unique* value $\in K^\times$”.

(vii) The following text should be added after the second paragraph of §1:

> Let $\mathfrak{X}^{\log}$ be the formal log scheme obtained by *p-adically completing* the log scheme defined by equipping the spectrum of the ring of integers of a finite extension of $\mathbb{Q}_p$ with the log structure determined by the closed point. In the discussion to follow concerning various formal schemes that are Zariski locally isomorphic to the underlying formal scheme of some stable log curve over $\mathfrak{X}^{\log}$ [for varying $\mathfrak{X}^{\log}$], we shall frequently have occasion to work with *“divisors”* on such formal schemes. Such “divisors” are to be understood in the following sense: An *effective Cartier divisor* is a formal closed subscheme that is defined by a coherent sheaf of ideals $\mathcal{I}$ which is an invertible sheaf. An *effective divisor* is a formal closed subscheme that is defined by a coherent sheaf of ideals $\mathcal{I}$ which is an invertible sheaf away from the nodes of the special fiber and, moreover, satisfies the following condition at each node ν: if we write $\widehat{\mathcal{O}}$ for the completion of the structure sheaf of the formal scheme under consideration at ν, $\mathcal{I} \cdot \widehat{\mathcal{O}}$ for the ideal of $\widehat{\mathcal{O}}$ generated by $\mathcal{I}$, and $\mathfrak{m} \subseteq \widehat{\mathcal{O}}$ for the maximal ideal of $\widehat{\mathcal{O}}$, then $V(\mathcal{I} \cdot \widehat{\mathcal{O}}) \subseteq \mathrm{Spec}(\widehat{\mathcal{O}})$ is the schematic closure of an effective divisor [in the usual sense!] on the one-dimensional regular scheme $\mathrm{Spec}(\widehat{\mathcal{O}}) \setminus \{\mathfrak{m}\}$. A [not necessarily effective] *divisor* is a fractional ideal of the form $\mathcal{I} \cdot \mathcal{J}^{-1}$, where $\mathcal{I}$ is a coherent sheaf of ideals that determines an effective divisor, and $\mathcal{J}$ is a coherent sheaf of ideals that determines an effective Cartier divisor; if $\mathcal{I}$ may also be taken to be a coherent sheaf of ideals that determines an effective Cartier divisor, then we shall say that the divisor given by the fractional ideal $\mathcal{I} \cdot \mathcal{J}^{-1}$ is *Cartier*.

(viii) In the discussion following the proof of Proposition 1.1, the notation $\log(q_X)$ is to be understood as a *formal symbol* which is used in situations in which we wish to write the multiplication operation on the multiplicative monoid of regular functions to which q_X belongs *additively*.

(ix) In the final sentence of Remark 1.10.4, (i), the phrase “*divisor zeroes*” should read “*divisor of zeroes*”.

(x) In Proposition 1.5, (i), (ii), the three instances of the notation “$\underline{\Delta}^{\mathrm{tp}}_{(-)})^{\mathrm{ell}}/\underline{\Delta}_\Theta$”, where “$(-)$” is either Y or $\ddot{Y}$, should be replaced by the notation “$\underline{\Delta}^{\mathrm{tp}}_{(-)})^{\Theta}/\underline{\Delta}_\Theta$”.

(xi) In Proposition 5.2, (iii), the phrase “bi-Kummer N-th root of the N-th root of (i)” should read “bi-Kummer N-th root of (i)”.

(xii) The phrase "a ...-multiple" should be replaced by the phrase "an ...-multiple" in the second paragraph of the proof of Theorem 1.6 [one instance]; the discussion following Remark 2.6.1 [two instances].

(xiii) In the discussion following Remark 2.6.1, the phrase "determines a class" should be replaced by the phrase "arises from a class".

(xiv) In the first display of Corollary 2.18, (ii), the notation "$(l \cdot \underline{\Delta}_{\Theta})$" should read "$(l \cdot \underline{\Delta}^{\bullet}_{\Theta})$".

(xv) In the discussion of Example 3.9, (iii), the various *"perf-saturations"* that occur may be replaced simply by *"perfections"*. That is to say, the notion of "perf-saturation in a monoid that is *already perfect*" is entirely *equivalent* to the usual notion of the "perfection" of a monoid. In particular, although there is *no inaccuracy* in the description of the relevant monoids as "perf-saturations", the notion of a "perf-saturation" [which is not applied elsewhere in the present paper] is, in fact, *unnecessary* in the present paper.

(xvi) In Definition 3.3, (i), (c), the assertion that "$i_H \in I$ is necessarily unique" is *false*, in general. The *intended assertion* here is the assertion [which is immediate from the definitions involved!] that "$\Delta^{\mathrm{fil},\infty}_{i_H} \subseteq H$ is necessarily unique". Moreover, this uniqueness of $\Delta^{\mathrm{fil},\infty}_{i_H}$ is entirely *sufficient*, from the point of view of concluding that the notion of the "Δ^{fil}-closure of H in Δ" is *well-defined*.

(xvii) In Proposition 1.3; Proposition 1.4, (iii); Theorem 1.6, (iii); Remark 1.6.4, the notation "$\in$" applied to collections of cohomology classes should, strictly speaking, be a "$\subseteq$".

(xviii) In the explanation immediately following the display of Proposition 1.5, (iii), it should also have been noted that the notation "$\log(\ddot{U})$" is used to denote the *Kummer class*, written *additively*, of the meromorphic function $\ddot{U}$ on $\ddot{\mathfrak{Y}}$.

(xix) In the discussion immediately following the display of the paragraph immediately preceding Definition 2.13, the slightly rough explanation constituted by the phrase

"of $K^{\times}$ on $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$, which induces ... and the kernel of this quotient."

should be replaced by the following more precise description:

"of $K^{\times}$ on $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$; that is to say, each outer automorphism in the image of $K^{\times}$ lifts to an automorphism of $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ that induces the identity automorphism of both the quotient $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \twoheadrightarrow \underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}$ and the kernel of this quotient."

(xx) Strictly speaking, the definition of the monoid "Φ^{ell}_W" given in Example 3.9, (iii), leads to certain technical difficulties, which are, in fact, *entirely irrelevant* to the theory of the present paper. These technical difficulties may be averted by making the following slight modifications to the text of Example 3.9, as follows:

(xx-1) In the discussion following the first display of Example 3.9, (i), the phrase "$Y^{\log}$ is of *genus* 1" should be replaced by the phrase "$Y^{\log}$ is of *genus* 1 and has either *precisely one cusp* or *precisely two cusps* whose difference is a 2-torsion element of the underlying elliptic curve".

(xx-2) In the discussion following the first display of Example 3.9, (i), the phrase

$$\text{the lower arrow of the diagram to be "}\underline{\dot{X}}^{\log} \to \underline{\dot{C}}^{\log}\text{"}$$

should be replaced by the phrase

$$\text{the lower arrow of the diagram to be "}\dot{X}^{\log} \to \dot{C}^{\log}\text{".}$$

(xx-3) In the discussion following the first display of Example 3.9, (ii), the phrase "*unramified over the cusps* of ..." should be replaced by the phrase "*unramified over the cusps* as well as *over the generic points* of the irreducible components of the special fibers of the stable models of ...". Also, the phrase "tempered coverings of the underlying ..." should be replaced by the phrase "tempered admissible coverings of the underlying ...".

In a word, the thrust of both the original text and the slight modifications just discussed is that the monoid "Φ_W^{ell}" is to be defined to be just large enough to include precisely those divisors which are necessary in order to treat the *theta functions* that appear in the present paper.

(xxi) In the second paragraph of §1, it should have been mentioned explicitly that $\mathfrak{X}$ denotes the *underlying formal scheme* of the formal log scheme $\mathfrak{X}^{\log}$. In a similar vein, in the third paragraph of §1, it should have been mentioned explicitly that X denotes the *underlying scheme* of the log scheme $X^{\log}$.

(xxii) In the final sentence of Remark 2.6.1, the phrase "by taken" should read "by taking".

(xxiii) In Remark 2.18.2, the phrase "this may" should read "that may".

(xxiv) In Corollary 2.19, (ii), the notation "$\alpha_M : \mathbb{M}_M \xrightarrow{\sim} \mathbb{M}_M$ " should read "$\alpha_M : \mathbb{M}_M \xrightarrow{\sim} \mathbb{M}_M^{\bullet}$".

(xxv) In the discussion preceding Definition 3.3, the phrase "of the *p-adic completion*" should read "on the *p-adic completion*".

(xxvi) In Remark 3.6.4, the phrase "of a tempered Frobenioids" should read "of a tempered Frobenioid".

(xxvii) In the first paragraph of §4, the phrase

"bi-Kummer theory" theory developed here

should read as follows:

"bi-Kummer theory" developed here

(xxviii) In the first paragraph of the proof of Proposition 4.3, the phrase "the fact the monoid" should read "the fact that the monoid".

(xxix) In Remark 5.12.2, the phrase "given given collection" should read "given collection"; the phrase "the fact there is" should read "the fact that there is".

(xxx) Concerning the *classical theory of theta functions on Tate curves*, some readers have remarked that the exposition that may be found in "[Mumf], pp. 306-307" is not sufficiently detailed. One reader has remarked in this context that he found the exposition given in [3], Chapter I, §2, and [3], Chapter II, §5, to be helpful.

(xxxi) In Proposition 1.3, the text "*whose restriction to ... Moreover,*" surrounding the third to last display should read as follows:

> *whose restriction*
>
> $$H^1(\underline{\Delta}_\Theta, \frac{1}{2}\underline{\Delta}_\Theta) = \mathrm{Hom}(\underline{\Delta}_\Theta, \frac{1}{2}\underline{\Delta}_\Theta)$$
>
> *to* $\underline{\Delta}_\Theta \subseteq (\Delta_Y^{\mathrm{tp}})^\Theta \subseteq (\Pi_Y^{\mathrm{tp}})^\Theta$ *is given by the natural inclusion* $\underline{\Delta}_\Theta \hookrightarrow \frac{1}{2}\underline{\Delta}_\Theta$. *Moreover,*

(xxxii) In the second display of Corollary 2.19, (iii), the notation "$H^1(\underline{\underline{\ddot{Y}}}, (l \cdot \underline{\Delta}_\Theta))$" should read as follows:

$$H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, (l \cdot \underline{\Delta}_\Theta))$$

(xxxiii) We remark that in the paragraph preceding Corollary 2.9, the *"labels"* referred to in the phrase

"we thus obtain labels $\in \mathbb{Z}$ for the cusps of $\underline{\underline{\ddot{Y}}}^{\log}$"

should be understood as consisting of some *map* — i.e., from the set of cusps of $\underline{\underline{\ddot{Y}}}^{\log}$ to $\mathbb{Z}$ — which is *not necessarily injective*!

(xxxiv) In Theorem 3.7, (ii), the phrase "Suppose $\mathcal{D}$" should read "Suppose that $\mathcal{D}$".

(xxxv) In Proposition 2.4, it should also have been stated that the notation "$\ddot{Y}^{\log}_{\square}$" is used to denote the covering associated to the curve "$X^{\log}_{\square}$" of Proposition 2.4 as in the discussion of §1 [i.e., the discussion preceding Lemma 1.2, applied in the case where "$X^{\log}$" is taken to be the "$X^{\log}_{\square}$" of Proposition 2.4].

(xxxvi) In the second and third paragraphs of §1, "$\mathfrak{X}$" and "X" denote the respective underlying objects associated to $\mathfrak{X}^{\log}$ and $X^{\log}$, i.e., by *forgetting the log structure.*

(xxxvii) In the explanation immediately following the first display of Definition 2.10, the phrase "*cyclotomic envelope*" should read "*[mod N] cyclotomic envelope*".

(xxxviii) In the first sentence of Definition 2.13, (ii), "folows" should read "follows".

(xxxix) In Remark 2.18.2, the phrase "appears as an object this may" should read "appears as an object that may".

(xl) In Definition 3.3, (ii), the phrase "$Z^{\log}_{\infty} \to Z^{\log}$ corresponds to the subgroup $\Delta_i^{\mathrm{fil},\infty} \subseteq \underline{\Delta}_X^{\mathrm{tp}}$" should read "the coverings $Z^{\log}_{\infty} \to Z^{\log} \to X^{\log}$ correspond to the filtration of subgroups $\Delta_i^{\mathrm{fil},\infty} \subseteq \Delta_i^{\mathrm{fil}} \subseteq \underline{\Delta}_X^{\mathrm{tp}}$".

(xli) In Example 3.9, (i), it should be noted that the "$X^{\log}$" and "$Y^{\log}$" of Example 3.9 *differ* from the "$X^{\log}$" and "$Y^{\log}$" of §1, §2.

(xlii) The following sentence should be inserted immediately following the first sentence of Example 3.9, (iii):

> [Here, we note that one verifies immediately [cf. the discussion of Definition 3.3, (i), (ii)] that there exists a tempered filter on $Y^{\log}$.]

(xliii) In the first paragraph of the proof of Proposition 4.3, the phrase "together with the fact the monoid" should read "together with the fact that the monoid".

(xliv) The following sentence [is, in fact, *implicit*, but, for the sake of clarity] could be inserted at the beginning of the discussion immediately following Remark 2.6.1:

> In the following discussion, we assume that the hypotheses on K and l made at the beginning of Definition 2.5 are in force, i.e., that l is *odd*, that K is a *finite extension of* $\mathbb{Q}_p$ of *odd residue characteristic*, and that $K = \ddot{K}$.

(xlv) The data that constitutes the *third* and [when it exists] *fourth member(s)* of the collection of data used to specify the *model mono- and bi-theta environments* in the first sentence of Proposition 2.14, (iii), and the fifth display of Corollary 2.18 is a *section* [i.e., as opposed to a "$\boldsymbol{\mu}_N$-conjugacy class of subgroups determined by the image of a section", as stipulated in Definition 2.13, (ii), (c), and Definition 2.13, (iii), (c), (d)]. Thus, in order for this sort of collection of data to conform to the requirements of the definition of a model mono- or bi-theta environment, one should understand the notation of these *sections* as a sort of *shorthand* for the phrase "the $\boldsymbol{\mu}_N$-conjugacy class of subgroups determined by the image of the section ...".

(xlvi) At the beginning of the proof of Lemma 2.17, the phrase "a set of generators of H" should read "a set of free generators of [the free discrete group] H".

Bibliography

[1] S. Mochizuki, The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields, *J. Math. Sci. Univ. Tokyo* **3** (1996), pp. 571-627.

[2] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[3] A. Robert, *Elliptic curves. Notes from postgraduate lectures given in Lausanne 1971/72*, *Lecture Notes in Mathematics* **326**, Springer-Verlag (1973).