(i) The first sentence of Definition 3.1, (ii) [i.e., the definition of the term “log-meromorphic”], should be replaced by the following text:

A log-meromorphic function on $\mathbb{Z}_\log^\infty$ is defined to be a nonzero meromorphic function $f$ on $\mathbb{Z}_\log^\infty$ such for every $N \in \mathbb{N}_{\geq 1}$, it holds that $f$ admits an $N$-th root over some tempered covering of $\mathbb{Z}_\log$. [Thus, it follows immediately, by considering the ramification divisors of such tempered coverings that arise from extracting roots of $f$, that the divisor of zeroes and poles of $f$ is a log-divisor.]

That is to say, the class of meromorphic functions that are “log-meromorphic” in the sense of this modified definition is contained in the class of meromorphic functions that are “log-meromorphic” in the sense of the original definition. In light of the content of this modified definition, perhaps a better term for this class of meromorphic functions would be “tempered-meromorphic”.

(ii) In order to understand the relationship between the modified definition of (i) and the original definition, it is useful to consider the following conditions on a nonzero meromorphic function $f$ on $\mathbb{Z}_\log^\infty$:

(a) For every $N \in \mathbb{N}_{\geq 1}$, it holds that $f$ admits an $N$-th root over some tempered covering of $\mathbb{Z}_\log$.

(b) For every $N \in \mathbb{N}_{\geq 1}$ which is prime to $p$, it holds that $f$ admits an $N$-th root over some tempered covering of $\mathbb{Z}_\log$.

(c) The divisor of zeroes and poles of $f$ is a log-divisor.

Thus, (a) is the condition of the modified definition of (i); (c) is the condition of the original definition. It is immediate that (a) implies (b). Moreover, [cf. (i)] one verifies immediately, by considering the ramification divisors of the tempered coverings that arise from extracting roots of $f$, that (b) implies (c). When $N$ is prime to $p$, if $f$ satisfies (c), then it follows immediately from the theory of admissible coverings [cf., e.g., [1], §2, §8] that there exists a finite log étale covering $Y_\log \to \mathbb{Z}_\log$ whose pull-back $Y_\infty^{\log} \to Z_\infty^{\log}$ to $Z_\infty^{\log}$ is sufficient.
(R1) to annihilate all ramification over the cusps or special fiber of $Z^\log_\infty$ that might arise from extracting an $N$th root of $f$, as well as

(R2) to split all extensions of the function fields of irreducible components of the special fiber of $Z^\log_\infty$ that might arise from extracting an $N$th root of $f$.

That is to say, in this situation, it follows that $f$ admits an $N$-root over the tempered covering of $Z^\log$ given by the “universal combinatorial covering” of $Y^\log$. In particular, it follows that (c) implies (b). Thus, in summary, we have:

$$(a) \implies (b) \iff (c).$$

On the other hand, unfortunately, it is not clear to the author at the time of writing whether or not (c) [or (b)] implies (a).

(iii) Observe that it follows from the theory of §1 [cf., especially, Proposition 1.3] that the theta function that forms the main topic of interest of the present paper satisfies condition (a). Indeed, the only instance occurring in the remainder of the text where the modified definition of (i) makes a difference is the proof of Proposition 4.2, (iii). That is to say, in this proof, it is necessary to use property (a) of (ii) [i.e., as opposed to just properties (b) or (c)]. Thus, this situation is remedied [without any affect on the remainder of the text] by taking property (a) to be the definition of “log-meromorphic”. The author apologizes for any confusion caused by this oversight on his part.

(iv) An alternative approach to the approach of (i) above [i.e., of modifying the definition of the term “log-meromorphic”] is the following. One may leave Definition 3.1, (ii), unchanged, if one modifies Definition 4.1, (i), by assuming further that the meromorphic function $f \in \mathcal{O}^\times(A_{\text{birat}})$ of loc. cit. satisfies the following “Frobenioid-theoretic version” of condition (a):

(d) For every $N \in \mathbb{N}_{\geq 1}$, there exists a linear morphism $A' \to A$ in $C$ such that the pull-back of $f$ to $A'$ admits an $N$-th root.

[Here, we recall that, as discussed in (iii), the Frobenioid-theoretic theta functions that appear in the present paper satisfy (d).] Note that since the rational function monoid of the Frobenioid $C$, as well as the linear morphisms of $C$, are category-theoretic [cf. [2], Theorem 3.4, (iii), (v); [2], Corollary 4.10], this condition (d) is category-theoretic. Thus, if one modifies Definition 4.1, (i), in this way, then the remainder of the text goes through without change, except that one must replace the reference to the definition of “log-meromorphic” [i.e., Definition 3.1, (ii)] that occurs in the proof of Proposition 4.2, (iii), by a reference to condition (d) [i.e., in the modified version of Definition 4.1, (i)].

(v) In the discussion preceding Definition 2.1, one must in fact assume that the integer $l$ is odd in order for the quotient $\overline{\Delta}_X$ to be well-defined. Since, ultimately, in the present paper [cf. the discussion following Remark 5.7.1], this is the only case
that is of interest, this oversight does not affect the bulk of the remainder of the present paper. Indeed, the only places where the case of even \( l \) is used are Remark 2.2.1 and the application of Remark 2.2.1 in the proof of Proposition 2.12 for the orbicurves \( \tilde{\mathcal{C}} \). Thus, Remark 2.2.1 must be deleted; in Proposition 2.12, one must in fact exclude the case where the orbicurve under consideration is \( \tilde{\mathcal{C}} \). On the other hand, this theory involving Proposition 2.12 [cf., especially, Corollaries 2.18, 2.19] is only applied after the discussion following Remark 5.7.1, i.e., which only treats the curves \( \tilde{X} \). That is to say, ultimately, in the present paper, one is only interested in the curves \( \tilde{X} \), whose treatment only requires the case of odd \( l \).

(vi) The phrase “the unique value \( \in \mathcal{O}_K^\times \)” in the first line of Definition 1.9, (ii), should read “the unique value \( \in K^\times \)”.

(vii) The following text should be added after the second paragraph of §1:

Let \( \mathfrak{T}^{\log} \) be the formal log scheme obtained by \textit{\( p \)-adically completing} the log scheme defined by equipping the spectrum of the ring of integers of a finite extension of \( \mathbb{Q}_p \) with the log structure determined by the closed point. In the discussion to follow concerning various formal schemes that are Zariski locally isomorphic to the underlying formal scheme of some stable log curve over \( \mathfrak{T}^{\log} \) [for varying \( \mathfrak{T}^{\log} \)], we shall frequently have occasion to work with “divisors” on such formal schemes. Such “divisors” are to be understood in the following sense: An \textit{effective Cartier divisor} is a formal closed subscheme that is defined by a coherent sheaf of ideals \( \mathcal{I} \) which is an invertible sheaf. An \textit{effective divisor} is a formal closed subscheme that is defined by a coherent sheaf of ideals \( \mathcal{I} \) which is an invertible sheaf away from the nodes of the special fiber and, moreover, satisfies the following condition at each node \( \nu \): if we write \( \hat{\mathcal{O}} \) for the completion of the structure sheaf of the formal scheme under consideration at \( \nu \), \( \mathcal{I} \cdot \hat{\mathcal{O}} \) for the ideal of \( \hat{\mathcal{O}} \) generated by \( \mathcal{I} \), and \( \mathfrak{m} \subseteq \hat{\mathcal{O}} \) for the maximal ideal of \( \hat{\mathcal{O}} \), then \( V(\mathcal{I} \cdot \hat{\mathcal{O}}) \subseteq \text{Spec}(\hat{\mathcal{O}}) \) is the schematic closure of an effective divisor [in the usual sense!] on the one-dimensional regular scheme \( \text{Spec}(\hat{\mathcal{O}}) \setminus \{ \mathfrak{m} \} \). A [not necessarily effective] \textit{divisor} is a fractional ideal of the form \( \mathcal{I} \cdot J^{-1} \), where \( \mathcal{I} \) is a coherent sheaf of ideals that determines an effective divisor, and \( J \) is a coherent sheaf of ideals that determines an effective Cartier divisor; if \( \mathcal{I} \) may also be taken to be a coherent sheaf of ideals that determines an effective Cartier divisor, then we shall say that the divisor given by the fractional ideal \( \mathcal{I} \cdot J^{-1} \) is \textit{Cartier}.

(viii) In the discussion following the proof of Proposition 1.1, the notation \( \log(q_X) \) is to be understood as a \textit{formal symbol} which is used in situations in which we wish to write the multiplication operation on the multiplicative monoid of regular functions to which \( q_X \) belongs \textit{additively}.

(ix) In the final sentence of Remark 1.10.4, (i), the phrase “divisor zeroes” should read “divisor of zeroes”.


(x) In Proposition 1.5, (i), (ii), the three instances of the notation “Δ_{(\cdot)}^{\text{ell}} / \Delta_\Theta”, where “(\cdot)” is either \( Y \) or \( \check{Y} \), should be replaced by the notation “\( \Delta_{(\cdot)}^{\text{tp}} \Theta / \Delta_\Theta \)”.

(xi) In Proposition 5.2, (iii), the phrase “bi-Kummer \( N \)-th root of the \( N \)-th root of (i)” should read “bi-Kummer \( N \)-th root of (i)”.

(xii) The phrase “a ...-multiple” should be replaced by the phrase “an ...-multiple” in the second paragraph of the proof of Theorem 1.6 [one instance]; the discussion following Remark 2.6.1 [two instances].

(xiii) In the discussion following Remark 2.6.1, the phrase “determines a class” should be replaced by the phrase “arises from a class”.

(xiv) In the first display of Corollary 2.18, (ii), the notation “(l \cdot \Delta_{(\cdot)}”) should read “(l \cdot \Delta_{(\cdot)}^\bullet)”.

(xv) In the discussion of Example 3.9, (iii), the various “perf-saturations” that occur may be replaced simply by “perfections”. That is to say, the notion of “perf-saturation in a monoid that is already perfect” is entirely equivalent to the usual notion of the “perfection” of a monoid. In particular, although there is no inaccuracy in the description of the relevant monoids as “perf-saturations”, the notion of a “perf-saturation” [which is not applied elsewhere in the present paper] is, in fact, unnecessary in the present paper.

(xvi) In Definition 3.3, (i), (c), the assertion that “\( i_H \in I \) is necessarily unique” is false, in general. The intended assertion here is the assertion [which is immediate from the definitions involved!] that “\( \Delta_{i_H}^{\text{fil}, \infty} \subseteq H \) is necessarily unique”. Moreover, this uniqueness of \( \Delta_{i_H}^{\text{fil}, \infty} \) is entirely sufficient, from the point of view of concluding that the notion of the “\( \Delta_{\text{fil}} \)-closure of \( H \) in \( \Delta \)” is well-defined.

(xvii) In Proposition 1.3; Proposition 1.4, (iii); Theorem 1.6, (iii); Remark 1.6.4, the notation “\( \in \)” applied to collections of cohomology classes should, strictly speaking, be a “\( \subseteq \)”.

(xviii) In the explanation immediately following the display of Proposition 1.5, (iii), it should also have been noted that the notation “log(\( \check{U} \))” is used to denote the Kummer class, written additively, of the meromorphic function \( \check{U} \) on \( \check{Y} \).

(xix) In the discussion immediately following the display of the paragraph immediately preceding Definition 2.13, the slightly rough explanation constituted by the phrase

“of \( K^\times \) on \( \Pi_Y^{\text{tp}} \mu_N \), which induces ... and the kernel of this quotient.”
should be replaced by the following more precise description:

“of $K^\times$ on $\prod_{\Sigma}^{\text{tp}}[\mu_N]$; that is to say, each outer automorphism in the image of $K^\times$ lifts to an automorphism of $\prod_{Y}^{\text{tp}}[\mu_N]$ that induces the identity automorphism of both the quotient $\prod_{Y}^{\text{tp}}[\mu_N] \rightarrow \prod_{\Sigma}^{\text{tp}}$ and the kernel of this quotient.”

(xx) Strictly speaking, the definition of the monoid “$\Phi^{\text{ell}}_{W}$” given in Example 3.9, (iii), leads to certain technical difficulties, which are, in fact, entirely irrelevant to the theory of the present paper. These technical difficulties may be averted by making the following slight modifications to the text of Example 3.9, as follows:

(xx-1) In the discussion following the first display of Example 3.9, (i), the phrase “$Y^{\log}$ is of genus 1” should be replaced by the phrase “$Y^{\log}$ is of genus 1 and has either precisely one cusp or precisely two cusps whose difference is a 2-torsion element of the underlying elliptic curve”.

(xx-2) In the discussion following the first display of Example 3.9, (i), the phrase the lower arrow of the diagram to be “$\hat{X}^{\log} \rightarrow \hat{C}^{\log}$” should be replaced by the phrase the lower arrow of the diagram to be “$\hat{X}^{\log} \rightarrow \hat{C}^{\log}$”.

(xx-3) In the discussion following the first display of Example 3.9, (ii), the phrase “unramified over the cusps of ...” should be replaced by the phrase “unramified over the cusps as well as over the generic points of the irreducible components of the special fibers of the stable models of ...”. Also, the phrase “tempered coverings of the underlying ...” should be replaced by the phrase “tempered admissible coverings of the underlying ...”.

In a word, the thrust of both the original text and the slight modifications just discussed is that the monoid “$\Phi^{\text{ell}}_{W}$” is to be defined to be just large enough to include precisely those divisors which are necessary in order to treat the theta functions that appear in the present paper.

(xxii) In the second paragraph of §1, it should have been mentioned explicitly that $\mathcal{X}$ denotes the underlying formal scheme of the formal log scheme $\mathcal{X}^{\log}$. In a similar vein, in the third paragraph of §1, it should have been mentioned explicitly that $X$ denotes the underlying scheme of the log scheme $X^{\log}$.

(xxii) In the final sentence of Remark 2.6.1, the phrase “by taken” should read “by taking”.

(xxiii) In Remark 2.18.2, the phrase “this may” should read “that may”.
(xxiv) In Corollary 2.19, (ii), the notation “$\alpha_M : M \sim M \rightarrow M$” should read “$\alpha_M : M \sim M \rightarrow M$”.

(xxv) In the discussion preceding Definition 3.3, the phrase “of the $p$-adic completion” should read “on the $p$-adic completion”.

(xxvi) In Remark 3.6.4, the phrase “of a tempered Frobenioids” should read “of a tempered Frobenoid”.

(xxvii) In the first paragraph of §4, the phrase

“bi-Kummer theory” theory developed here

should read as follows:

“bi-Kummer theory” developed here

(xxviii) In the first paragraph of the proof of Proposition 4.3, the phrase “the fact the monoid” should read “the fact that the monoid”.

(xxix) In Remark 5.12.2, the phrase “given given collection” should read “given collection”; the phrase “the fact there is” should read “the fact that there is”.

(XXX) Concerning the classical theory of theta functions on Tate curves, some readers have remarked that the exposition that may be found in “[Mumf], pp. 306-307” is not sufficiently detailed. One reader has remarked in this context that he found the exposition given in [3], Chapter I, §2, and [3], Chapter II, §5, to be helpful.

(XXxi) In Proposition 1.3, the text “whose restriction to ... Moreover,” surrounding the third to last display should read as follows:

whose restriction

$$H^1(\Delta_\Theta, \frac{1}{2} \Delta_\Theta) = \text{Hom}(\Delta_\Theta, \frac{1}{2} \Delta_\Theta)$$

to $\Delta_\Theta \subseteq (\Delta_{\text{Y}})^\Theta \subseteq (\Pi_{\text{Y}}^\Theta)^\Theta$ is given by the natural inclusion $\Delta_\Theta \hookrightarrow \frac{1}{2} \Delta_\Theta$.

Moreover,

(XXXii) In the second display of Corollary 2.19, (iii), the notation “$H^1(\tilde{Y}, (l \cdot \Delta_\Theta))$” should read as follows:

$$H^1(\Pi_{\text{Y}}^\Theta, (l \cdot \Delta_\Theta))$$

(XXXiii) We remark that in the paragraph preceding Corollary 2.9, the “labels” referred to in the phrase
“we thus obtain labels ∈ ℤ for the cusps of $\tilde{Y}^{\log}$.”

should be understood as consisting of some map — i.e., from the set of cusps of $\tilde{Y}^{\log}$ to ℤ — which is not necessarily injective!

(xxiv) In Theorem 3.7, (ii), the phrase “Suppose $D$” should read “Suppose that $D$”.

(xxv) In Proposition 2.4, it should also have been stated that the notation “$\tilde{Y}^{\log}$” is used to denote the covering associated to the curve “$X^{\log}$” of Proposition 2.4 as in the discussion of §1 [i.e., the discussion preceding Lemma 1.2, applied in the case where “$X^{\log}$” is taken to be the “$X^{\log}$” of Proposition 2.4].

Bibliography

