The Hodge-Arakelov Theory of Elliptic Curves

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§1. Main Results
(Comparison Isomorphisms and Arithmetic Kodaira-Spencer Morphism)

§2. Philosophy: In Search of an Absolute Derivative

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§1. Main Results

(A.) Simple Version of the Main Comparison Theorem

\( K \): a field of characteristic 0
\( E \): an elliptic curve/\( K \)
\( E^\dagger \): its \underline{universal extension}

\[
\begin{align*}
\text{char } 0 &= H^1_{\text{DR}}(E, \mathcal{O}_E) \\
\text{Over } \mathbb{C} &\quad E^\dagger = H^1_{\text{DR}}(E, \mathcal{O}_E)/\Lambda \\
\text{where } \Lambda &= H^1_{\text{sing}}(E, 2\pi i \mathbb{Z}) \cong \mathbb{Z}^2 \\
\text{In general:} &\quad \text{Tang. sp. to } E^\dagger = H^1_{\text{DR}}(E, \mathcal{O}_E)
\end{align*}
\]
Char. 0: \( dE^\dagger \cong dE \stackrel{\text{def}}{=} \ker [d] : E \rightarrow E \)
\((d: \text{a positive integer})\)

(in mixed char., denominators arise)

\( \eta \in E(K) \): torsion point of order \( m \), s. t. \( d \) does not divide \( m \)

\( \mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta]) \)

**Theorem:** The restriction morphism

\[ \Gamma(E^\dagger, \mathcal{L}) \xleftarrow{d} \sim \mathcal{L}|_{dE^\dagger} \]

is an isomorphism.
Note:

(1.) “< d” denotes torsorial degree (relative degree: $E^\dagger/E$) < d.

(2.) Both sides are $K$-vector spaces of dimension $d^2$.

(3.) Theorem false if d divides m.
   (e.g., if $d = m = 1$, then $\Gamma(E, \mathcal{O}_E([0_E]) = \mathcal{L} \rightarrow \mathcal{L}|_{0_E}$ is 0)

(4.) Proof:
   Mumford’s algebraic theta functions + Zhang’s theory of admiss. metrics
   + complicated degree computations
(B.) Integral Structures at “Arithmetic” Primes

In general:

\[ 0 \rightarrow \omega_E \rightarrow E^\dagger \rightarrow E \rightarrow 0 \]

\( (\omega_E = \text{invariant diffs. on } E) \)

Near “point at infinity” \( \infty \):

\[ E = \mathbb{G}_m/q^\mathbb{Z} \]

(“Tate curve”)

\[ \Rightarrow \text{Over power series in } q \text{ (‘hat’):} \]

\[ \hat{E} = \hat{\mathbb{G}}_m \]

\[ \hat{E}^\dagger = \hat{\mathbb{G}}_m \times \hat{\omega}_E = \hat{\mathbb{G}}_m \times \left\langle \frac{dg}{q} \right\rangle \]
Integral structure at finite primes (mixed char.):

\[ \mathcal{O}_E[T] = \bigoplus \mathcal{O}_E \cdot T^j \implies \bigoplus \mathcal{O}_E \cdot \frac{1}{j!} \cdot T^j \]

where \( T \): coord. on \( \omega_E \), def’d by \( \frac{dq}{q} \)

...(p-adic analytically) extends over all \( \overline{\mathcal{M}}_{1,0} \), not just near \( \infty \)

Integral Structure Near \( \infty \):

\[ \bigoplus \mathcal{O}_E \cdot \frac{1}{j!} \cdot T^j \implies \bigoplus \mathcal{O}_E \cdot \frac{1}{j!} \cdot q \approx -\frac{j^2}{8d} \cdot T^j \]

“Gaussian poles” (cf. \( e^{-x^2} \))
Important Theme:
Gaussian and its derivatives
  (cf. Hermite polynomials)
...also, main obstruct. to Dio. applies.

Integral Structure at Arch. Primes:
To relate ‘DR metric’ to ‘étale metric’
⇒ approximate by comparison to special functions — models:
Hermite polys. (slope = $\frac{1}{2}$)
Legendre polys. (slope = 1)
  = lim (disc. Tchebycheff polys.)
Binomial polys. = ($T_r$) (slope = 0)
slope = scaling factor as $d \to \infty$
  (cf. Frobenius on cryst. coh.)
(C.) Arithmetic Kodaira-Spencer Morphism

Main Theorem is a sort of function-theoretic comp. isom.: linear fns. + completion of tors. pts. \( \Rightarrow \) get classical comp. isoms.:

Over \( \mathbb{C} \):

\[
\begin{align*}
H^1_{DR}(E, \mathcal{O}_E) & \supseteq H^1_{\text{sing}}(E, 2\pi i \cdot \mathbb{R}) \\
\downarrow & \\
E^\dagger & \supseteq E_{\mathbb{R}}
\end{align*}
\]
Over $p$-adics:

Hodge-Tate, DR comp. isoms:

\[ H^1_{\text{DR}} \cong H^1_{\text{\acute{e}t}} \]

also def’able by rest. to $p^\infty$ tors. pts.

In general (global, $\mathbb{C}$, $p$-adics):

\[ \{ \text{DR coh.} \} \xrightarrow{\sim} \{ \text{\acute{e}t. coh.} \} \curvearrowright \text{Galois} \]

\[ \implies \text{Galois acts on DR coh.!!} \]

\[ \implies \text{Look at effect on Hodge filtr.} \]

\[ \implies \text{Kodaira-Spencer morphism} \]

motion in base

\[ \xrightarrow{\text{induced motion of Hodge filtr.}} \]
Over $\mathbb{C}$:

“Galois” $= SL_2(\mathbb{R})$ on upp. half-plane $\implies$ above ‘arith. KS’ $= \text{classical KS}$

Over $p$-adics:

$\text{Gal}(\mathbb{Z}_p[[T]]_{\mathbb{Q}_p}) \approx \text{Tang. bun.}(\mathbb{Z}_p[[T]]_{\mathbb{Q}_p})$

(Faltings’ theory of alm. et. extns.) $\implies$ above ‘arith. KS’ $= \text{classical KS}$

(cf. Serre-Tate theory)

Hodge-Arakelov (global) Case:

$\text{Gal(Base of Fam. of Ell. Curves } \otimes \mathbb{Q})$ $\arw$ $\{\text{Arak.-theoretic flag bun.}\} !!$
§2. Philosophy: In Search of an Absolute Derivative

(A.) From Differentiation to Comparison Isomorphisms

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\( S: \) a scheme; \( E \to S \) fam. of ell. curves

\[ \Longrightarrow \text{classifying morphism} \quad S \to \mathcal{M}_{1,0} \]

\[ \Longrightarrow \text{derivative} \quad (\text{KS}) \quad \Omega_{\mathcal{M}_{1,0}}|_S \to \Omega_S \]

\[ \Downarrow \]

Does \( \exists \) arithmetic/absolute analogue

\[ \left( \Omega_{\mathcal{M}_{1,0}}|_S \to \Omega_{\mathbb{Z}/F_1} \right) \]

(when \( S = \text{Spec}(\mathbb{Z}) \),
or \( \text{Spec}(\mathcal{O}_F) \), \([F : \mathbb{Q}] < \infty\))?
Observe: $\Omega_{\mathcal{M}_{1,0}}|_{S} = \omega_{E}^{\otimes 2}$, and

$$\omega_{E} \hookrightarrow H_{\text{DR}}^1(E) \xrightarrow{\nabla_{\text{GM}}} H_{\text{DR}}^1(E) \otimes \Omega_{S} \xrightarrow{} \tau_{E} \otimes \Omega_{S}$$

$$\implies \Omega_{\mathcal{M}_{1,0}}|_{S} = \omega_{E}^{\otimes 2} \rightarrow \Omega_{S} \text{ (KS)}$$

($\nabla_{\text{GM}}$: Gauss-Manin conn. on $H_{\text{DR}}^1$)

$$\downarrow$$

Since $\exists H_{\text{DR}}^1$, Hodge filtr. ($\omega_{E} \hookrightarrow H_{\text{DR}}^1$) in arith. case, need analogue of $\nabla_{\text{GM}}$

$$\implies \text{Recall de Rham isomorphism} \quad (=\text{comparison isomorphism}/\mathbb{C})$$
$S$: Riemann surface $\implies$

\[ H^1_{\text{DR}}(E/S) \cong H^1_{\text{sing}}(E/S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}_S \]

$\implies$ sections of $H^1_{\text{sing}}(E/S, \mathbb{Z})$ are horizontal for $\nabla_{\text{GM}}$

$\implies \nabla_{\text{GM}}$ is the unique conn. for which sects. of $H^1_{\text{sing}}(E/S, \mathbb{Z})$ are horiz.

$\Downarrow$

Knowledge of comp. isom. $\implies$

Knowledge of $\nabla_{\text{GM}}$

**Conclusion:** To construct arith. KS, suffices to construct arith. comp. isom.
(B.) Function-Theoretic Comparison Isomorphisms

So what form should a (global) arith. comp. isom. (ACI) take?

(e.g., over $\mathbb{C}$: $\otimes \mathbb{C}$; over $p$-adics: $\otimes B_{\text{DR}}, B_{\text{crys}},$ etc.)

In geometric case/\(\mathbb{C}\): one implicit sign of exist. of $\nabla_{GM}$ is a sort of ‘stability’:

$$0 \to \omega_E \to H^1_{\text{DR}}(E/S) \to \tau_E \to 0$$

If this sequ. split — i.e., $H^1_{\text{DR}}$ is ‘unstable’ — then $\exists \nabla$ on $\omega_E$ (= ample l.b.): ABSURD!
Even if can’t translate ‘∇’ into arith. case, can translate stability — i.e., of Arakelov bundles = usual v.b. + metric

‘Stability’ (e.g., over \( \mathbb{Z} \)) = ‘equidistrib. of matter in lattice’

Note: Arakelov degree large (small) \( \iff \) matter dense (sparse)

Expected Form I of ACI:
\[
\begin{align*}
\{ \text{Matter Distrib. in DR coh.} \} \\
\cong \{ \text{Matter Distrib. in étale coh.} \}
\end{align*}
\]
Note: RHS is ‘equidist.’ by ‘Galois’

\[ \implies \text{By ACI, LHS also ‘equidist.’} \]

In no. theory, ‘matter distributions’
typically measured by ‘test fns.’ \[ \implies \]

Expected Form II of ACI:
\[
\{ \text{test fns. on DR coh.} \} \\
\cong \{ \text{test fns. on étale coh.} \}
\]

where ‘\(\cong\)’ is isometry at all primes
of a number field (cf. Arak. theory)

... = the content of the main theorem!!

‘Hodge-Arakelov Comp. Isom.’

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‘Split’ distrib. of matter:

‘Equidist.’ distrib. of matter:
(C.) Discretization and the Meaning of Nonlinearity

Note: To measure distrib. in this case, need nonlinear test fns. — cf. linearity of Hodge theory/\(\mathbb{C}\), \(p\)-adics, additive approach to motive theory.

Reasons for Nonlinearity:

(1.) In Arakelov theory, things tend to become nonlinear (e.g., \(H^0(\mathcal{L})\)).

(2.) Nonlinear symmetries of noncomm. torus \(\simeq\) theta gp. \(\simeq\) Heisenberg alg. (cf. Gaussians!)
Also, related to **discreteness**:  
Hodge-Arakelov Comp. Isom. = ‘discretization of loc. Hodge theories’  
— e.g.,  
Hodge theory/$\mathcal{C} \approx$ ‘calculus on $E_{\mathbb{R}}$’ 
HACI $\approx$ ‘discrete calc. on tors. pts.’ 
$\implies$ periods analogous to 

$$2\pi i = \lim_{d \to \infty} d \cdot (e^{2\pi i/d} - 1)$$

$$= \lim_{d \to \infty} (\text{‘theta fns.’ on } \mathbb{G}_m \text{ evaluated on tors. pts. of } \mathbb{G}_m)$$