The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields

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Section 0: Introduction

In [Tama], a proof of the Grothendieck Conjecture (reviewed below) was given for smooth affine hyperbolic curves over finite fields (and over number fields). The purpose of this paper is to show how one can derive the Grothendieck Conjecture for arbitrary (i.e., not necessarily affine) smooth hyperbolic curves over number fields from the results of [Tama] for affine hyperbolic curves over finite fields.

We obtain three types of results: one over number fields, one over finite fields, and one over local fields. We remark here that when this paper was first written (October 1995), Theorems A and C below were the strongest known results of their respective kinds. Since then, the author wrote [Mzk2] (November 1995), which gives rise to much stronger results than Theorems A or C of the present paper. Moreover, the proofs of [Mzk2] are completely different from (and, in particular, do not rely on) the proofs of the present paper. Nevertheless, it seems to the author that the present paper still has some marginal interest, partly because most of the present paper is devoted to the proof of Theorem B below (which is *not* implied by any result of [Mzk2]), and partly because it is in some sense of interest to see how Theorems A or C can be derived within the context of the theory of [Tama].

Our main result over number fields (Theorem 10.2 in the text) is as follows:

Theorem A: *Let K be a finite extension of $\mathbf{Q}$; let $\overline{K}$ be an algebraic closure of K. Let $X_K \rightarrow Spec(K)$ and $X'_K \rightarrow Spec(K)$ be smooth, geometrically connected, proper curves over K, of genus ≥ 2. Let Δ_{X_K} (respectively, $\Delta_{X'_K}$) be the geometric fundamental group of X_K (respectively, X'_K). Then the natural map*

$$Isom_K(X_K, X'_K) \rightarrow Out_\rho(\Delta_{X_K}, \Delta_{X'_K})$$

is bijective. Here, "Out_ρ" refers to outer isomorphisms that respect the natural outer representations of $Gal(\overline{K}/K)$ on Δ_{X_K} and $\Delta_{X'_K}$.

The statement of this Theorem is commonly referred to as "the Grothendieck Conjecture." In [Tama], a theorem similar to Theorem A, except that X is replaced by a hyperbolic *affine* curve, is proven. It is a simple exercise to derive the affine case from the proper case. On the other hand, to derive the proper case from the affine case is by no means straightforward; in this paper, we derive the proper case over *number fields* from the affine case over *finite fields.*

In fact, to be more precise, we shall derive the proper case over *number fields* from a certain "logarithmic Grothendieck Conjecture" for singular (proper) stable curves over finite fields. This "logarithmic Grothendieck conjecture," which is our main result over finite fields (Theorem 7.4), is as follows:

Theorem B: *Let S^{log} be a log scheme such that S is the spectrum of a finite field k, and the log structure is isomorphic to the one associated to the chart $\mathbf{N} \rightarrow k$ given by the zero map. Let $X^{log} \rightarrow S^{log}$ and $(X')^{log} \rightarrow S^{log}$ be stable log-curves such that at least one of X or X' is not smooth over k. Let $\Delta_{X^{log}}$ (respectively, $\Delta_{(X')^{log}}$) be the geometric fundamental group of X^{log} (respectively, $(X')^{log}$) obtained by considering log admissible coverings of X^{log} (respectively, $(X')^{log}$) (as in [Mzk], §3). Then the natural map*

$$Isom_{S^{log}}(X^{log}, (X')^{log}) \rightarrow Out^{D}_{\rho}(\Delta_{X^{log}}, \Delta_{(X')^{log}})$$

is bijective. Here, the "D" stands for "degree one (outer isomorphisms)."

This Theorem is derived directly from Tamagawa's results on affine hyperbolic curves over finite fields. Its proof occupies the bulk of the present paper.

Finally, by supplementing Theorem B with various arguments concerning the fundamental groups of curves over local fields, we obtain the following local result (Theorem 9.8):

Theorem C: *Let K be a finite extension of $\mathbf{Q}_p$; let $A \subseteq K$ be its ring of integers; and let k be its residue field. Let X_K be a smooth, geometrically connected, proper curve of genus $g \geq 2$ over K. Assume that X_K admits a stable extension $X \rightarrow Spec(A)$ such that the abelian variety portion of $Pic^0(X_k)$ (where $X_k = X \otimes_A k$) is ordinary. Then the natural outer representation*

$$\rho_X : \Gamma_K \rightarrow Out(\Delta_{X_K})$$

of the absolute Galois group of K on the geometric fundamental group of X_K completely determines the isomorphism class of X_K.

Even though this is a rather weak version of the Grothendieck Conjecture (compared to the results we obtain over finite and global fields), this sort of result is interesting in the sense that it shows that curves behave somewhat differently from abelian varieties (cf. the Remark following Theorem 9.8).

Now we discuss the contents of the paper in more detail. Sections 1 through 7 are devoted to deriving Theorem B from the results of [Tama]. In Section 1, we show how to recover the set of irreducible components of a stable curve from its fundamental group. In Section 2, we review various facts from [Mzk] concerning log admissible coverings, and show how one can define an "admissible fundamental group" of a stable log-curve. In Section 3, we show how one can group-theoretically characterize the quotient of the admissible fundamental group corresponding to étale coverings. In Section 4, we show that the tame fundamental group of each connected component of the smooth locus of a stable log-curve is contained inside the admissible fundamental group of the stable log-curve. In Section 5, we show how to recover the set of nodes (including the information of which irreducible components each node sits on) of a stable log-curve from its admissible fundamental group. In Section 6, we show how the log structure at a node of a stable log-curve can be recovered from the admissible fundamental group of the stable log-curve. In Section 7, we put all of this information together and show how one can derive Theorem B from the results of [Tama].

In Sections 8 and 9, we shift from studying curves over finite fields to studying curves over local fields. In order to do this, it is necessary first to characterize (group-theoretically) the quotient of the (characteristic zero) geometric fundamental group of a curve over a local field which corresponds to admissible coverings. This is done in Section 8. In Section 9, we first show (Lemma 9.1) that the degree of an isomorphism between the arithmetic fundamental groups of two curves over a local field is necessarily one. This is important because one cannot apply Theorem 7.4 to an arbitrary isomorphism of fundamental groups: one needs to know first that the degree is equal to one. Then, by means of a certain trick which allows one to reduce the study of curves over local fields with smooth reduction to the study of curves over local fields with singular reduction, we show (Theorem 9.2) that one can recover the reduction (over the residue field) of a given smooth, proper, hyperbolic curve over a local field group-theoretically. The rest of Section 9 is devoted to curves with ordinary reduction, culminating in the proof of Theorem C. Finally, in Section 10, we observe that Theorem A follows formally from Theorem 9.2.

The author would like to thank A. Tamagawa for numerous fruitful discussions concerning the contents of [Tama], as well as the present paper. In some sense, the present paper is something of a long appendix to [Tama]: That is to say, several months after the author learned of the results of [Tama], it dawned upon the author that by using admissible coverings, Theorem A follows "trivially" from the results of [Tama]. On the other hand, since many people around the author were not so familiar with admissible coverings or log structures, it seemed to the author that it might be useful to write out a detailed version of this "trivial argument." The result is the present paper.

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Section 1: The Set of Irreducible Components

Let k be the finite field of $q = p^f$. Fix an algebraic closure $\overline{k}$ of k. Let Γ be the absolute Galois group of k. Let $X \to Spec(k)$ be a morphism of schemes.

Definition 1.1: *We shall call X a multi-stable curve of genus g if $dim_k(H^1(X, \mathcal{O}_X)) = g$, and $X_{\overline{k}} \stackrel{\text{def}}{=} X \otimes_k \overline{k}$ is a finite disjoint union of stable curves over $\overline{k}$ of genus ≥ 2. If X is multi-stable, then we shall call X sturdy if every irreducible component of the normalization of $X_{\overline{k}}$ has genus ≥ 2.*

Thus, in particular, a curve is stable if and only if it is geometrically connected and multi-stable. Moreover, a finite étale covering of a multi-stable (respectively, sturdy) curve is multi-stable (respectively, sturdy).

Suppose that X is stable of genus $g \geq 2$. Fix a base-point $x \in X(k)$. Then we may form the (algebraic) fundamental group $\Pi \stackrel{\text{def}}{=} \pi_1(X, x_{\overline{k}})$ of X. Let $X_{\overline{k}} \stackrel{\text{def}}{=} X \otimes_k \overline{k}$; $\Delta \stackrel{\text{def}}{=} \pi_1(X_{\overline{k}}, x_{\overline{k}})$. Then we have a natural exact sequence of groups

$$1 \to \Delta \to \Pi \to \Gamma \to 1$$

induced by the structure morphism $X \to Spec(k)$.

The purpose of this Section is to show how the set of irreducible components of X can be canonically recovered from the morphism $\Pi \to \Gamma$. Fix a prime l different from p. Let us consider the étale cohomology group $H^e \stackrel{\text{def}}{=} H^1_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_l)$. Let $\psi : \widetilde{X} \to X$ be the normalization of X. Then we can also consider $H^n \stackrel{\text{def}}{=} H^1_{\acute{e}t}(\widetilde{X}_{\overline{k}}, \mathbf{Z}_l)$. By considering the long exact cohomology sequence in étale cohomology associated to

$$0 \to \mathbf{Z}_l \to \psi_* \psi^* \mathbf{Z}_l \to (\psi_* \psi^* \mathbf{Z}_l)/\mathbf{Z}_l \to 0$$

we obtain a surjection $H^e \to H^n$. Let us write H^c for the kernel of this surjection. (Here, "e" (respectively, "n"; "c") stands for étale (respectively, normalization; combinatorial).) Note that H^c is a free $\mathbf{Z}_l$-module of rank $N_X - I_X + 1$, where N_X (respectively, I_X) is the number nodes (respectively, irreducible components) of $X_{\overline{k}}$. Moreover, H^n is a free $\mathbf{Z}_l$-module of rank equal to twice the sum of the genera of the connected components of $\widetilde{X}_{\overline{k}}$.

Thus, we obtain a natural exact sequence of Γ-modules

$$0 \to H^c \to H^e \to H^n \to 0$$

Let $\phi \in \Gamma$ be the automorphism of $\overline{k}$ given by raising to the q^{th} power. Then one sees easily that some finite power of ϕ acts trivially on H^c. On the other hand, by the Weil conjectures (applied to the various geometric connected components of $\widetilde{X}$), no power of ϕ acts with eigenvalue 1 on H^n. We thus obtain the following

Proposition 1.2: *The natural exact sequence* $0 \to H^c \to H^e \to H^n \to 0$ *can be recovered entirely from* $\Pi \to \Gamma$.

Proof: Indeed, $H^e = Hom(\Delta, \mathbf{Z}_l)$, while H^c can be recovered by looking at the maximal $\mathbf{Z}_l$-submodule of H^e on which some power of $\phi \in \Gamma$ acts trivially. ◯

Let $L^e = H^e \otimes \mathbf{F}_l$; $L^c = H^c \otimes \mathbf{F}_l$; $L^n = H^n \otimes \mathbf{F}_l$. Thus, $L^e = H^1_{ét}(X_{\overline{k}}, \mathbf{F}_l)$, and we have an exact sequence of Γ-modules

$$0 \to L^c \to L^e \to L^n \to 0$$

Moreover, elements of L^e correspond to étale, abelian coverings of $X_{\overline{k}}$ of degree l. Let $L^* \subseteq L^e$ be the subset of elements whose image in L^n is *nonzero.*

Suppose that $\alpha \in L^*$. Let $Y_\alpha \to X_{\overline{k}}$ be the corresponding covering. Then $N_{Y_\alpha} = l \cdot N_X$. Thus, we obtain a morphism $\epsilon : L^* \to \mathbf{Z}$ that maps $\alpha \mapsto I_{Y_\alpha}$. Since L^* is a finite set, the image of ϵ is finite. Let $M \subseteq L^*$ be the subset of elements α on which ϵ attains its maximum. Let us define a pre-equivalence relation "$\sim$" on M as follows:

> If $\alpha, \beta \in M$, then we write $\alpha \sim \beta$ if, for every $\lambda, \mu \in \mathbf{F}_l^\times$ for which $\lambda \cdot \alpha + \mu \cdot \beta \in L^*$, we have $\lambda \cdot \alpha + \mu \cdot \beta \in M$.

Now we have the following result:

Proposition 1.3: *Suppose that X is stable and sturdy. Then "$\sim$" is, in fact, an equivalence relation, and moreover, $C_X \stackrel{\text{def}}{=} M/\sim$ is naturally isomorphic to the set of irreducible components of $X_{\overline{k}}$.*

Proof: First, let us observe, that I_{Y_α} is maximal (equal to $l(I_X - 1) + 1$) if and only if there exists a unique irreducible component Z_α of $\widetilde{X}_{\overline{k}}$ over which the covering $Y_\alpha \to X_{\overline{k}}$ is nontrivial. Now, if Z is a connected component of $\widetilde{X}_{\overline{k}}$, let $L_Z \stackrel{\text{def}}{=} H^1_{ét}(Z, \mathbf{F}_l)$. Thus,

$$L^n = \bigoplus_Z L_Z$$

where the direct sum is over the connected components of $\widetilde{X}_{\overline{k}}$. Then it follows immediately from the definitions that M consists precisely of those elements $\alpha \in L^*$ whose image in L^n has (relative to the above direct sum decomposition) exactly one nonzero component (namely, in L_{Z_α}). Moreover, $\alpha \sim \beta$ is equivalent to $Z_\alpha = Z_\beta$. Finally, that every Z appears as a Z_α follows from the sturdiness assumption. This completes the proof. ○

Remark: Note that although at first glance the set $C_X = M/\sim$ appears to depend on the choice of prime l, it is not difficult to see that in fact, if one chooses another prime l, and hence obtains a resulting $C'_X = M'/\sim'$, one obtains a natural isomorphism $C_X \cong C'_X$ (compatible with the isomorphisms just obtained of C_X and C'_X to the set of irreducible components of $X_{\overline{k}}$) as follows: If $\alpha \in M$ and $\alpha' \in M'$, let us consider the product $Y_{\alpha\alpha'} = Y_\alpha \times_X Y'_\alpha$. Thus, we have a cyclic étale covering $Y_{\alpha\alpha'} \to X$ of degree $l \cdot l'$. Then one checks easily that α and α' correspond to the same irreducible component if and only if $(Y_{\alpha\alpha'})_{\overline{k}}$ has precisely $l \cdot l'(I_X - 1) + 1$ irreducible components.

Proposition 1.4: *Suppose that X is stable and sturdy. Then the set of irreducible components of $X_{\overline{k}}$ (together with its natural Γ-action) can be recovered entirely from $\Pi \to \Gamma$.*

Proof: Indeed, it follows from Proposition 1.2 that L^* can be recovered from $\Pi \to \Gamma$. Moreover, we claim that M can be recovered, as well. Indeed, the maximality of I_{Y_α} is equivalent to the minimality of $N_{Y_\alpha} - I_{Y_\alpha} + 1 = l \cdot N_X - I_{Y_\alpha} + 1$, which is equal to the dimension over $\mathbf{F}_l$ of the "L^c" of Y. Once one has M, it follows that one can also recover "$\sim$," hence by Proposition 1.3, one can recover the set of irreducible components of $X_{\overline{k}}$. Finally, by the above Remark, the set that one recovers is independent of the choice of l. ○

Corollary 1.5: *Suppose that X is stable and sturdy. Let $H \subseteq \Pi$ be an open subgroup. Let $Y_H \to X$ be the corresponding étale covering. Then the set of irreducible components of Y_H can be recovered from $\Pi \to \Gamma$ and H.*

Proof: Let k' be the (finite) extension of k which is the subfield of $\overline{k}$ stabilized by the image of H in Γ. Then Y_H is geometrically connected, hence stable and sturdy over k'. Thus, we reduce to the case $H = \Pi$, $Y_H = X$. But then the set of irreducible components of X is the set of Γ-orbits of the set of irreducible components of $X_{\overline{k}}$. Thus, the Corollary follows from Proposition 1.4. ○

Looking back over what we have done, one sees that in fact, we have proven a stronger result that what is stated in Corollary 1.5. Indeed, fix an irreducible component $I \subseteq X$. Then let $J_l(I)$ be the set of $\overline{k}$-valued l-torsion points of the Jacobian of the normalization of $I_{\overline{k}}$. Then not only have we recovered set of all irreducible components I, we have also recovered, for each I, the set $J_l(I)$ (with its natural Frobenius action). We state this as a Corollary:

Corollary 1.6: *Suppose that X is stable and sturdy, and l is a prime number different from p. Then for each irreducible component I of X, the set $J_l(I)$ (with its natural Frobenius action) can be recovered naturally from $\Pi \to \Gamma$.*

Section 2: The Admissible Fundamental Group

Let r and g be nonnegative integers such that $2g-2+r \geq 1$. If $(\mathcal{C} \to \overline{\mathcal{M}}_{g,r}; \sigma_1, \ldots, \sigma_r : \overline{\mathcal{M}}_{g,r} \to \mathcal{C})$ is the universal r-pointed stable curve of genus g over the moduli stack, then $\mathcal{C}$ and $\overline{\mathcal{M}}_{g,r}$ have natural log structures defined by the respective divisors at infinity and the images of the σ_i. Denote the resulting log morphism by $\mathcal{C}^{log} \to \overline{\mathcal{M}}_{g,r}^{log}$. Let $X \to S$ be the underlying curve associated to an r-pointed stable curve of genus g over a scheme S (where S is the underlying scheme of some log scheme S^{log}). Suppose that X is equipped with the log structure (call the resulting log scheme X^{log}) obtained by pulling back $\mathcal{C}^{log} \to \overline{\mathcal{M}}_{g,r}^{log}$ via some log morphism $S^{log} \to \overline{\mathcal{M}}_{g,r}^{log}$ whose underlying non-log morphism $S \to \overline{\mathcal{M}}_{g,r}$ is the classifying morphism of X (equipped with its marked points). In this case, we shall call $X^{log} \to S^{log}$ an *r-pointed stable log-curve of genus g*. Similarly, we have *r-pointed multistable log-curves of genus g*: that is, $X^{log} \to S^{log}$ such that over some finite étale covering $S' \to S$, $X^{log} \times_S S'$ becomes a finite union of stable pointed log-curves.

Let k be as in the preceding section. Let S^{log} be a log scheme whose underlying scheme is $Spec(k)$ and whose log structure is (noncanonically!) isomorphic to the log structure associated to the morphism $\mathbf{N} \to k$, where $1 \in \mathbf{N} \mapsto 0 \in k$. Let $X^{log} \to S^{log}$ be a stable log-curve of genus $g \geq 2$.

Next, we would like to consider liftings of $X^{log} \to S^{log}$. Let A be a complete discrete valuation ring which is finite over $\mathbf{Z}_p$ and has residue field equal to k. Let T^{log} be a log scheme whose underlying scheme is $Spec(A)$ and whose log structure is that defined by the special point $S = Spec(k) \subseteq T$. Also, let us assume that S^{log} is equal to the restriction of the log structure of T^{log} to $S = Spec(k) \subseteq T$. Let $Y^{log} \to T^{log}$ be a stable log-curve of genus g whose restriction to S^{log} is $X^{log} \to S^{log}$. In this case, we shall say that $Y^{log} \to T^{log}$ *lifts* $X^{log} \to S^{log}$. It is well-known (from the log-smoothness of the moduli stack of stable curves equipped with its natural log structure) that such log-curves $Y^{log} \to T^{log}$ always exist.

Next, we would like to consider *log admissible coverings*

$$Z^{log} \to Y^{log}$$

of Y^{log}. We refer to [Mzk], §3.5, for the rather lengthy and technical definition and first properties of such coverings. It follows in particular from the definition that Z is a stable curve over T. In fact, (as is shown in [Mzk], Proposition 3.11), one can define such coverings without referring to log structures. That is, there is a notion of an *admissible covering* ([Mzk]. §3.9) $Z \to Y$ (which can be defined without using log structures). Moreover, $Z \to Y$ is admissible if and only if Z admits a log structure such that $Z^{log} \to Y^{log}$ is log admissible. In [Mzk], we dealt strictly with the case where Z is geometrically connected over T. Here, we shall call $Z \to Y$ *multi-admissible* if Z is a disjoint union of connected components Z_i such that each $Z_i \to Y$ is admissible.

Let η be the generic point of T. Let $Y_\eta = Y \times_T \eta$. If $Z \to Y$ is multi-admissible, then it will always be the case that the restriction $Z_\eta \to Y_\eta$ of this covering to the generic fiber is finite étale. Now suppose that $\psi_\eta : Z_\eta \to Y_\eta$ is a finite étale covering. If ψ_η extends to an multi-admissible covering $Z \to Y$, then this extension is unique ([Mzk], §3.13).

Definition 2.1: *We shall call ψ_η pre-admissible if it extends to an multi-admissible covering $\psi : Z \to Y$. We shall call ψ_η potentially pre-admissible if it becomes pre-admissible after a tamely ramified base-change (i.e., replacing A by a tamely ramified extension of A).*

Thus, in particular, if A' is a tamely ramified extension of A, then $Y_\eta \otimes_A A' \to Y_\eta$ is potentially pre-admissible. If ψ_η is potentially pre-admissible and Z_η is geometrically connected over η, then it is pre-admissible if and only if Z_η has stable reduction over A.

Lemma 2.2 : *Suppose that $Z_\eta \to Y_\eta$ and $Z'_\eta \to Y_\eta$ are pre-admissible. Let $Z''_\eta \stackrel{\text{def}}{=} Z_\eta \times_{Y_\eta} Z'_\eta$. Then $Z''_\eta \to Y_\eta$ is pre-admissible.*

Proof: Let $Z \to Y$ and $Z' \to Y$ be the respective multi-admissible extensions. Let Z'' be the normalization of Y in Z''_η. Thus, we have a natural morphism $Z'' \to Z \times_Y Z'$ which is an isomorphism at height one primes. In particular, Z'' is étale over Y at all height one primes. It thus follows from Lemma 3.12 of [Mzk] that $Z'' \to Y$ is multi-admissible. ◯

Lemma 2.3 : *Suppose that $Z_\eta \to Y_\eta$ is pre-admissible, and that $Z_\eta \to Y_\eta$ factors through finite étale surjections $Z_\eta \to Z'_\eta$ and $Z'_\eta \to Y_\eta$. Then $Z'_\eta \to Y_\eta$ is pre-admissible.*

Proof: Similar to that of Lemma 2.2. ◯

Let K be the quotient field of A. Fix an algebraic closure $\overline{K}$ of K. Suppose that Y is equipped with a base-point $y \in Y(A)$ such that the corresponding morphism $T \to Y$ avoids the nodes of the special fiber of Y. Write Π_Y for $\pi_1(Y_K, y_{\overline{K}})$. Thus, we have a natural surjection $\Pi_Y \to Gal(\overline{K}/K)$, whose kernel is a group $\Delta_Y \subseteq \Pi_Y$.

Definition 2.4: *We shall call an open subgroup $H \subseteq \Pi_Y$ co-admissible if the corresponding finite étale covering $Z_\eta \to Y_\eta$ is potentially pre-admissible. Let Π_Y^{adm} be the quotient of Π_Y by the intersection $\bigcap H$ of all co-admissible $H \subseteq \Pi_Y$.*

Remark: The admissible fundamental group Π_Y^{adm} has already been defined and studied by K. Fujiwara ([Fuji]). Moreover, the author learned much about Π_Y^{adm} (as well as about the theory of log structures in general) by means of oral communication with K. Fujiwara.

It is easy to see that the intersection $\bigcap H$ of Definition 2.4 is a normal subgroup of Π_Y. Thus, Π_Y^{adm} is a group. Moreover, by Lemmas 2.2 and 2.3, it follows that an open subgroup $H \subseteq \Pi_Y$ is co-admissible if and only if $Ker(\Pi_Y \to \Pi_Y^{adm}) \subseteq H$. Finally, it is immediate from the definitions that the subfield of $\overline{K}$ stabilized by the image of $\bigcap H$ in $Gal(\overline{K}/K)$ is the maximal tamely ramified extension K_∞ of K. Thus, we have a surjection

$$\Pi_Y^{adm} \to Gal(K_\infty/K)$$

whose kernel $\Delta_Y^{adm} \subseteq \Pi_Y^{adm}$ is a quotient of Δ_Y.

Definition 2.5: *We shall refer to as orderly coverings of Y_η those coverings $Z_\eta \to Y_\eta$ which are Galois and factor as $Z_\eta \to Y_\eta \times_T U \to Y_\eta$, where the first morphism is pre-admissible; the second morphism is the natural projection; $U = Spec(B)$; and B is a tamely ramified finite extension of A. We shall refer to as orderly quotients of Π_Y^{adm} those quotients of Π_Y^{adm} that give rise to orderly coverings of Y_η.*

It is easy to see that orderly quotients of Π_Y^{adm} are cofinal among all quotients of Π_Y^{adm}.

Let $A_\infty \subseteq K_\infty$ be the normalization of A in K_∞. Let k_∞ (respectively, m_∞) be the residue field (respectively, maximal ideal) of A_∞. Let $T_\infty = Spec(A_\infty)$, and let us endow T_∞ with the log structure given by the multiplicative monoid $\mathcal{O}_{T_\infty} - \{0\}$ (equipped with the natural morphism into $\mathcal{O}_{T_\infty}$). We call the resulting log scheme T_∞^{log}. Let S_∞^{log} be the log scheme whose underlying scheme is $Spec(k_\infty)$ and whose log structure is pulled back from T_∞^{log}. Thus, the log structure on S_∞^{log} is (noncanonically!) isomorphic to the log structure defined by the zero morphism $(\mathbf{Z}_{(p)})_{\geq 0} \to k_\infty$. (Here "$(\mathbf{Z}_{(p)})_{\geq 0}$" denotes the set of nonnegative rational numbers whose denominators are prime to p.) Note that $Gal(K_\infty/K)$ induces S^{log}-automorphisms of S_∞^{log}. In fact, it is easy to see that this correspondence defines a natural isomorphism

$$Gal(K_\infty/K) \cong Aut_{S^{log}}(S_\infty^{log})$$

Now we have the following important

Lemma 2.6 : *Suppose that we are given:*

(1) *another lifting $(Y')^{log} \to (T')^{log}$ (where $T' = Spec(A')$) of $X^{log} \to S^{log}$*

(2) *an algebraic closure $\overline{K}'$ of $K' = Q(A')$ (hence a resulting $K'_\infty \subseteq \overline{K}'$; $(S')_\infty^{log}$);*

(3) *an* S^{log}*-isomorphism* $\gamma : S^{log}_\infty \cong (S')^{log}_\infty$;

(4) *a base-point* $y' \in Y'(A')$ *such that* $y|_S = y'|_S$ *in* $X(k)$.

Then there is a natural isomorphism between the surjections $\Pi^{adm}_Y \to Gal(K_\infty/K)$ *and* $\Pi^{adm}_{Y'} \to Gal(K'_\infty/K')$.

Proof: Let us first consider coverings of Y_η obtained by pulling back tamely ramified Galois coverings of K. Thus, if $U = Spec(B) \to T$ is finite, Galois, and tamely ramified (obtained from some field extension $K \subseteq L$), let U^{log} be the log scheme obtained by equipping U with the log structure defined by the special point u of U. Thus, we obtain a finite, log étale morphism $U^{log} \to T^{log}$. By base-changing to S^{log}, we then obtain a finite, log étale morphism $V^{log} \to S^{log}$. On the other hand, by the definition of "log étaleness," this morphism then necessarily lifts to a finite, log étale morphism $(U')^{log} \to (T')^{log}$, whose underlying morphism $U' \to T'$ is a Galois, tamely ramified finite extension. Thus, if we pass to the limit, and apply this construction to the extension $L = K_\infty$ of K, we end up with some maximal tamely ramified extension L' of K'. By the functoriality of this construction, we have a natural isomorphism $Gal(L/K) \cong Gal(L'/K')$. Now observe that there is a unique K'-isomorphism $L' \cong K'_\infty$ which (relative to this construction) is compatible with γ. Thus, we get an isomorphism $Gal(L'/K') = Gal(K'_\infty/K')$, hence an isomorphism $Gal(K_\infty/K) \cong Gal(K'_\infty/K')$, as desired.

Now let us consider orderly coverings $Z_\eta \to Y_\eta$. Thus, we have a factorization $Z_\eta \to Y_\eta \times_T U \to Y_\eta$. Let Z be the normalization of Y_η in Z. Then $Z \to Y \times_T U$ is multi-admissible. Thus, Z admits a log structure such that we have a log multi-admissible covering $Z^{log} \to Y^{log} \times_{T^{log}} U^{log}$. Base-changing, we obtain a log multi-admissible covering $Z^{log}|_{S^{log}} \to X^{log} \times_{S^{log}} V^{log}$. But since log multi-admissible coverings are log étale, it thus follows that this covering lifts uniquely to a log multi-admissible covering $(Z')^{log} \to (Y')^{log} \times_{(T')^{log}} (U')^{log}$. Similarly, we obtain a bijective correspondence between Y_η-automorphisms of Z_η and $Y'_{\eta'}$-automorphisms of $Z'_{\eta'}$. Now observe further that K_∞-valued points of Z_η over y_{K_∞} define A_∞-valued points of Z over y (since Z is proper over A). Moreover, these points define T^{log}_∞-valued points of Z^{log} over $y_{T^{log}_\infty}$, hence (by reducing modulo m_∞) S^{log}_∞-valued points of $Z^{log}|_{S^{log}}$ over $y_{S^{log}}$, hence (using γ) $(S')^{log}_\infty$-valued points of $Z^{log}|_{S^{log}}$ over $y_{S^{log}} = y'_{S^{log}}$, hence (by log étaleness) $(T')^{log}_\infty$-valued points of $(Z')^{log}$ over $y'_{T^{log}}$, which, finally, give rise to K'_∞-valued points of $Z'_{\eta'}$ over $y'_{K'_\infty}$.

Thus, in summary, we have defined a natural equivalence of categories between orderly coverings of Y_η and orderly coverings of $Y'_{\eta'}$. Moreover, this equivalence is compatible with the fiber functors defined by the base-points y_{K_∞} and $y'_{K'_\infty}$. Thus, we obtain an isomorphism between the surjections $\Pi^{adm}_Y \to Gal(K_\infty/K)$ and $\Pi^{adm}_{Y'} \to Gal(K'_\infty/K')$, as desired. ○

Now let us interpret Lemma 2.6. In summary, what Lemma 2.6 says is the following: Suppose we start with the following data:

(1) a log scheme S^{log}, where $S = Spec(k)$, and the log structure is (noncanonically!) isomorphic to the log structure associated to the zero morphism $\mathbf{N} \to k$;

(2) a log scheme S^{log}_{∞} over S^{log} which is (noncanonically!) k-isomorphic to $Spec(\overline{k})$ equipped with the log structure associated to the zero morphism $(\mathbf{Z}_{(p)})_{\geq 0} \to \overline{k}$ (where the $\mathbf{N} \subseteq (\mathbf{Z}_{(p)})_{\geq 0}$ is pulled back from a chart for S^{log} as in (1));

(3) a stable log-curve $X^{log} \to S^{log}$ of genus g;

(4) a base-point $x \in X(k)$ which is not a node.

Then, to this data, we can naturally associate an "admissible fundamental group" Π_X^{adm} with augmentation $\Pi_X^{adm} \to \Pi_{S^{log}} \stackrel{\mathrm{def}}{=} Aut_{S^{log}}(S^{log}_{\infty})$. That is to say, by choosing a lifting of the above data, we may take $\Pi_X^{adm} = \Pi_Y^{adm}$, and the augmentation to be $\Pi_Y^{adm} \to Gal(K_{\infty}/K)$. Then Lemma 2.6 says that, up to canonical isomorphism, Π_X^{adm} and its augmentation do not depend on the choice of lifting.

Definition 2.7: *We shall refer to the data (1) through (4) above as admissible data of genus g. Given admissible data as above, we shall write $\pi_1(X^{log}, x_{S^{log}_{\infty}})$ for Π_X^{adm} and $\pi_1(S^{log}, S^{log}_{\infty})$ for $\Pi_{S^{log}}$. We shall refer to Π_X^{adm} as the admissible fundamental group of X. Write $\Delta_{X^{log}} \subseteq \Pi_X^{adm}$ for the kernel of the augmentation. We shall refer to $\Delta_{X^{log}}$ as the geometric admissible fundamental group of X.*

Next, let us observe that $\Pi_{S^{log}}$ admits a natural surjection onto $\Pi_S \stackrel{\mathrm{def}}{=} Gal(\overline{k}/k)$. We shall denote the kernel of this surjection by $I_S \subseteq \Pi_{S^{log}}$, and refer to I_S as the *inertia subgroup of* $\Pi_{S^{log}}$. Note that I_S is isomorphic to the inverse limit of the various $(k')^{\times}$ (for finite extensions $k' \subseteq \overline{k}$ of k), where the transition morphisms in the inverse limit are given by taking the norm. Or, in other words, $I_S = \widehat{\mathbf{Z}}'(1)$, where $\widehat{\mathbf{Z}}'$ is the inverse limit of the quotients of $\mathbf{Z}$ of order prime to p, and the "(1)" is a Tate twist. Thus, we have a natural exact sequence

$$1 \to I_S = \widehat{\mathbf{Z}}'(1) \to \Pi_{S^{log}} \to \Pi_S = Gal(\overline{k}/k) \to 1$$

Suppose that we are given a continuous action of $\Pi_{S^{log}}$ on a finite set Σ. Then we can associate a geometric object to Σ as follows. Without loss of generality, we can assume that the action on Σ is transitive. If we choose a lifting T^{log} of S^{log} (where $T = Spec(A)$), then Σ corresponds to some finite, tamely ramified extension L of K. Let B be the normalization of A in L. Equip $U \stackrel{\mathrm{def}}{=} Spec(B)$ with the log structure defined by the special point u of U. Thus, we obtain U^{log}. Equip $Spec(k(u))$ with the log structure induced by that of U^{log}. Then the geometric object associated to Σ is the finite, log étale morphism

$Spec(k(u))^{log} \to S^{log}$. We shall call such morphisms *finite, tamely ramified coverings of* S^{log}.

Now suppose that we have an open subgroup $H \subseteq \Pi_X^{adm}$ that surjects onto $\Pi_{S^{log}}$. Let $\Delta_H \stackrel{\text{def}}{=} H \bigcap \Delta_{X^{log}}$. In terms of liftings, H corresponds to a finite étale covering $Z_\eta \to Y_\eta$, where Z_η is geometrically connected over η. Note that by a well-known criterion ([SGA7]), Z_η has stable reduction over A if and only if I_S acts unipotently on $Hom(\Delta_H, \mathbf{Z}_l)$ (for some prime l distinct from p). But, as noted above, in this situation, Z_η has stable reduction if and only if $Z_\eta \to Y_\eta$ is pre-admissible. If $Z_\eta \to Y_\eta$ is pre-admissible, it extends to some $Z^{log} \to Y^{log}$, which we can base-change via $S^{log} \to T^{log}$ to obtain a log admissible covering $Z^{log}|_{S^{log}} \to X^{log}$. Conversely, every log admissible covering of X^{log} can be obtained in this manner. Thus, in summary, we have the following result:

Corollary 2.8: *The open subgroups of* Π_X^{adm} *that correspond to orderly coverings can be characterized entirely group-theoretically by means of* $\Pi_X^{adm} \to \Pi_{S^{log}}$ *(and* $\Pi_{S^{log}} \to \Pi_S$*). Moreover, these subgroups can be interpreted in terms of geometric coverings of* X^{log} *(namely, base-change via a finite, tamely ramified covering of* S^{log}*, followed by a log admissible covering).*

Remark: Note that this Corollary thus allows us to speak of "orderly coverings of X^{log}," i.e., coverings that arise from orderly quotients (Definition 2.5) of $\Pi_X^{adm} = \Pi_Y^{adm}$.

Before continuing, let us make the following useful technical observation:

Lemma 2.9 : *Given any stable* X *over* k*, there always exists an admissible covering* $Z \to X$ *such that* Z *is (multi-stable and) sturdy.*

Indeed, this follows from the definition of an admissible covering, plus elementary combinatorial considerations. Moreover, an admissible covering of a sturdy curve is always sturdy. Thus, if one wishes to work only with sturdy curves, one can always pass to such a situation by replacing our original X by some suitable admissible covering of X.

Finally, although most of this paper deals with the case of nonpointed stable curves, it turns out that we will need to deal with pointed stable curves a bit later on. In fact, it will suffice to consider pointed smooth curves. Thus, let g and r be nonnegative integers such that $2g-2+r \geq 1$. Let $S_\infty^{log} \to S^{log}$ be as above, and let $X^{log} \to S^{log}$ be an r-pointed stable log-curve of genus g such that X is k-smooth. Also, let $x \in X(k)$ be a nonmarked point. Then it is easy to see that, just as above, we can define (by considering various liftings to some A, then showing that what we have done does not depend on the lifting) an *admissible fundamental group* Π_X^{adm} (with base-point $x_{S_\infty^{log}}$), together with a natural surjection

$$\Pi_X^{adm} \to \Pi_{S^{log}}$$

Moreover, the kernel $\Delta_{X^{log}} \subseteq \Pi_X^{adm}$ of this surjection is naturally isomorphic to the tame fundamental group of $X_{\overline{k}}$ (with base-point $x_{\overline{k}}$). Unlike the singular case, we don't particular gain anything new by doing this, but what will be important is that we still nonetheless obtain a *natural* surjection $\Pi_X^{adm} \to \Pi_{S^{log}}$ which arises functorially from the same framework as the nontrivial $\Pi_X^{adm} \to \Pi_{S^{log}}$ that appears in the case of singular curves.

Section 3: Characterization of the Étale Fundamental Group

We maintain the notation of the preceding Section. Thus, in particular, we have a stable log-curve $X^{log} \to S^{log}$, together with a choice of S^{log}_∞, and a base-point $x \in X(k)$ (which is not a node). Then note that we have a natural morphism of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X^{log}} & \longrightarrow & \Pi_X^{adm} & \longrightarrow & \Pi_{S^{log}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & \Pi_S & \longrightarrow & 1 \end{array}$$

Here the vertical arrows are all surjections. The goal of this Section is to show how one can recover the quotient $\Pi_X^{adm} \to \Pi_X$ group-theoretically from $\Pi_X^{adm} \to \Pi_{S^{log}}$.

Let $Y^{log} \to X^{log}$ be a log admissible covering which is abelian, with Galois group equal to $\mathbf{F}_l$, where l is a prime number (which is not necessarily distinct from p). Let us consider the following condition on this covering:

> (*) Over $\overline{k}$, there is an infinite log admissible covering $Z^{log} \to X^{log}_{\overline{k}}$ which is abelian with Galois group $\mathbf{Z}_l$ such that the intermediate covering corresponding to $\mathbf{Z}_l \to \mathbf{F}_l$ is $Y^{log}_{\overline{k}} \to X^{log}_{\overline{k}}$.

Here, by "infinite log admissible covering," we mean an inverse limit of log admissible coverings in the usual finite sense. Suppose that $Y^{log} \to X^{log}$ satisfies (*). Then we claim that $Y \to X$ is, in fact, *étale*. Indeed, if $p = l$, then every abelian log admissible covering of degree l is automatically étale, so there is nothing to prove. If $p \neq l$, then we can fix a node $\nu \in X$, and consider the ramification over the two branches of X at ν. Considering this ramification gives rise to an inertia subgroup $H \subseteq \mathbf{Z}_l$. If $Y^{log} \to X^{log}$ is ramified at ν, then H surjects onto $\mathbf{F}_l$, so $H = \mathbf{Z}_l$. On the other hand, by the definition of a log admissible covering, in order to have infinite ramification occuring over the geometric branches of X at ν, we must also have infinite ramification over the base S^{log}. But, by (*), $Z^{log} \to X^{log}$ is already log admissible over $S^{log} \otimes_k \overline{k}$ (which is, of course, étale over S^{log}). This contradiction shows that Z^{log}, and hence Y^{log}, are unramified over X^{log} at ν. Thus, we see that (*) implies that $Y \to X$ is étale, as claimed. Note that conversely, if we know that $Y \to X$ is étale to begin with, then it is easy to see that (*) is satisfied. Thus, for an abelian log admissible covering $Y^{log} \to X^{log}$ of prime degree l, (*) is equivalent to the étaleness of $Y \to X$.

Now observe that the kernel of $\Delta_{X^{log}} \to \Delta_X$ is normal not just in $\Delta_{X^{log}}$, but also in Π_X^{adm}. Let $\Pi'_X = \Pi_X^{adm}/Ker(\Delta_{X^{log}} \to \Delta_X)$. Let Δ'_X be the kernel of $\Pi'_X \to \Pi_S$. Thus, $\Delta_X \subseteq \Delta'_X \subseteq \Pi'_X$. Then we have the following result:

Proposition 3.1: *The quotient $\Pi_X^{adm} \to \Pi'_X$ can be recovered entirely group theoretically from $\Pi_X^{adm} \to \Pi_{S^{log}}$.*

Proof: It suffices to characterize subgroups $H \subseteq \Pi_X^{adm}$ of finite index that contain $Ker(\Delta_{X^{log}} \to \Delta_X)$. Without loss of generality, we may assume that H is normal in Π_X^{adm} and (by Corollary 2.8) corresponds to an orderly covering. Since an orderly covering may be factored as a composite of a log multi-admissible covering followed by a tamely ramified covering of S^{log}, one sees immediately that we may reduce to the case where H corresponds to a log admissible covering. Let $G = \Pi_X^{adm}/H$. Let $Y^{log} \to X^{log}$ be the corresponding covering. For every subgroup $N \subseteq G$, denote by $Y^{log} \to Y_N^{log}$ the corresponding intermediate covering. By considering ramification at the nodes, one sees immediately that $Y \to X$ is étale if and only for every cyclic $N \subseteq G$ of prime order, $Y \to Y_N$ is étale. But for such N, the étaleness of $Y \to Y_N$ is equivalent to the condition (*) discussed above. Moreover, it is clear that (*) can be phrased in entirely group theoretic terms, using only $\Pi_X^{adm} \to \Pi_{S^{log}}$ (and $\Pi_{S^{log}} \to \Pi_S$). This completes the proof. ○

Now suppose that $Y^{log} \to X^{log}$ is an *abelian orderly covering of prime order* l obtained from a quotient of Π'_X such that Y is geometrically connected. Assume $l \neq p$. Consider the following condition on $Y^{log} \to X^{log}$:

> (†) There do not exist any infinite abelian orderly coverings $Z^{log} \to X_{\overline{k}}^{log}$ with Galois group $\mathbf{Z}_l$ that satisfy both of the following two conditions:
> (i) the intermediate covering corresponding to $\mathbf{Z}_l \to \mathbf{F}_l$ is $Y_{\overline{k}}^{log} \to X_{\overline{k}}^{log}$;
> (ii) some finite power ϕ^M of the Frobenius morphism $\phi \in \Gamma = \Pi_S$ stabilizes $Z^{log} \to X_{\overline{k}}^{log}$ and acts on the Galois group $\mathbf{Z}_l$ with eigenvalue q^M.

Because $Y^{log} \to X^{log}$ is abelian of prime order, it follows that one of the following holds:

(1) $Y^{log} \to X^{log}$ is obtained from an étale covering $Y \to X$ (where Y is geometrically connected) base-changed by $S^{log} \to S$.

(2) $Y^{log} \to X^{log}$ is obtained by base-change via $X^{log} \to S^{log}$ from some totally (tamely) ramified covering of S^{log}.

Suppose that (†) is satisfied. Then we claim that (1) holds. Indeed, if this were false, then (2) would hold, but it is clear that if (2) holds, then one can easily construct $Z^{log} \to X_{\overline{k}}^{log}$ that contradict (†) (by pulling back via $X^{log} \to S^{log}$ an infinite ramified covering of S^{log}).

This proves the claim. Now suppose that (1) holds. Then we claim that (†) is satisfied. To prove this, suppose that there exists an offending $Z^{log} \to X^{log}_{\overline{k}}$. This offending covering defines an injection $\mathbf{Z}_l \hookrightarrow Hom(\Delta'_X, \mathbf{Z}_l) \to Hom(\Delta_X, \mathbf{Z}_l) = H^1_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_l)$. On the other hand, as we saw in Section 1, by the Weil conjectures, no power ϕ^M of ϕ acts with eigenvalue q^M on $H^1_{\acute{e}t}(X_{\overline{k}}, \mathbf{Z}_l)$. This contradiction completes the proof of the claim. Thus, in summary, (1) is equivalent to (†). In other words, we have essentially proven the following result:

Proposition 3.2: *The quotient $\Pi_X^{adm} \to \Pi_X$ can be recovered entirely group theoretically from $\Pi_X^{adm} \to \Pi_{S^{log}}$.*

Proof: It suffices to characterize finite index subgroups $H \subseteq \Pi'_X$ that contain $Ker(\Pi'_X \to \Pi_X)$. Without loss of generality, we may assume that H is normal in Π'_X and (by Corollary 2.8) corresponds to an orderly covering. Let $G = \Pi'_X/H$. Let $Y^{log} \to X^{log}$ be the corresponding covering. Again, without loss of generality, we may assume that Y is geometrically connected over k. For every normal subgroup $N \subseteq G$, denote by $Y_N^{log} \to X^{log}$ the corresponding intermediate covering. Next, we observe the following: $Y^{log} \to X^{log}$ arises from an étale covering of X if and only if, for every normal subgroup $N \subseteq G$ such that $Y_N^{log} \to X^{log}$ is orderly and G/N is cyclic of prime order, $Y_N \to X$ arises from an étale covering of X. (This equivalence follows immediately from the definitions and the fact that Δ'_X/Δ_X is abelian.) But for such $Y_N^{log} \to X^{log}$, we can apply the criterion (†) discussed above. Moreover, it is clear that (†) can be phrased in entirely group theoretic terms, using only $\Pi'_X \to \Pi_{S^{log}}$ (and $\Pi_{S^{log}} \to \Pi_S$). This completes the proof. ◯

Section 4: The Decomposition Group of an Irreducible Component

We maintain the notation of the preceding Section. Fix an irreducible component $I \subseteq X$. Then, corresponding to I, there is a unique (up to conjugacy) *decomposition subgroup*

$$\Delta_I^{adm} \subseteq \Pi_X^{adm}$$

which may be defined as follows. Let $Z^{log} \to X^{log}$ be the log scheme obtained by taking the inverse limit of the various $Y_H^{log} \to X^{log}$ corresponding to open orderly subgroups $H \subseteq \Pi_X^{adm}$. Choose an "irreducible component" $J \subseteq Z$ that maps down to $I \subseteq X$. Here, by "irreducible component of Z," we mean a compatible system of irreducible components $I_H \subseteq Y_H$. Then $\Delta_I^{adm} \subseteq \Pi_X^{adm}$ is the subgroup of elements that take the irreducible component J to itself. We can also define an *inertia subgroup*

$$\Delta_I^{in} \subseteq \Delta_I^{adm}$$

as follows: Namely, we let Δ_I^{in} be the subgroup of elements of Δ_I^{adm} that act trivially on J. (That is to say, elements of Δ_I^{in} will, in general, act nontrivially on the log structure of

J, but trivially on the underlying scheme J.) By Proposition 3.2 and Corollaries 1.5 and 1.6, we thus obtain the following:

Proposition 4.1: *Suppose that X is stable and sturdy. Then one can recover the set of irreducible components of X from $\Pi_X^{adm} \to \Pi_{S^{log}}$. Moreover, for each irreducible component I of X, one can recover the corresponding inertia and decomposition subgroups $\Delta_I^{in} \subseteq \Delta_I^{adm} \subseteq \Pi_X^{adm}$ entirely from $\Pi_X^{adm} \to \Pi_{S^{log}}$.*

Proof: That one can recover Δ_I^{adm} follows formally from Proposition 3.2 and Corollary 1.5. Now observe that (as is well known – see, e.g., [DM]) any automorphism of I that acts trivially on $J_l(I)$ (where $l \geq 5$) is the identity. This observation, coupled with Corollary 1.6, allows one to recover Δ_I^{in}. ◯

Now let us suppose that the base-point $x \in X(k)$ is contained in I. Let $\widetilde{I}$ be the normalization of I. Then one can also define Δ_I^{adm} as follows. Let $\breve{I} \subseteq I$ be the open subset which is the complement of the nodes. We give $\breve{I}$ a log structure by restricting to $\breve{I}$ the log structure of X^{log}. Denote the resulting log scheme by $\breve{I}^{log}$. Now let us regard the points of $\widetilde{I}$ that map to nodes of I as *marked points of* $\widetilde{I}$. This gives $\widetilde{I}$ the structure of a smooth, pointed curve over k. Because $\widetilde{I} \to Spec(k)$ is smooth, it follows that there exists a unique multistable pointed log-curve $\widetilde{I}^{log} \to S^{log}$ whose underlying curve is $\widetilde{I}$ and whose marked points are as just specified. Since $x \in \widetilde{I}(k)$, by using $S_\infty^{log} \to S^{log}$, we can define (as in the discussion as the end of Section 2) the *admissible fundamental group* Π_I^{adm} of $\widetilde{I}^{log}$. Moreover, we have natural log morphisms

$$\begin{array}{ccc} \breve{I}^{log} & \longrightarrow & X^{log} \\ \downarrow & & \\ \widetilde{I}^{log} & & \end{array}$$

where the vertical morphism is an open immersion. Now observe that if we restrict (say, orderly) coverings of X^{log} to $\breve{I}^{log}$, such a covering extends naturally to an orderly covering of $\widetilde{I}^{log}$. Thus, we obtain a natural morphism

$$\zeta_I : \Pi_I^{adm} \to \Pi_X^{adm}$$

It is immediate from the definitions that the subgroup $\zeta_I(\Pi_I^{adm}) \subseteq \Pi_X^{adm}$ is a "Δ_I^{adm}."

Proposition 4.2: *Suppose that X is stable and sturdy. Then the morphism ζ_I is injective.*

Proof: Let $\Pi_{S^{log}}^I$ be the image of Π_I^{adm} in $\Pi_{S^{log}}$. Then let us note that we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{\widetilde{I}^{log}} & \longrightarrow & \Pi_I^{adm} & \longrightarrow & \Pi_{S^{log}}^I & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{X^{log}} & \longrightarrow & \Pi_X^{adm} & \longrightarrow & \Pi_{S^{log}} & \longrightarrow & 1 \end{array}$$

where the vertical arrow on the right is the natural inclusion. Thus, it suffices to prove that $\Delta_{\widetilde{I}^{log}} \to \Delta_{X^{log}}$ is injective. In particular, we are always free to replace S^{log} by a finite, tamely ramified covering of S^{log}.

Now it suffices to show that (up to base-changing, when necessary, by finite, tamely ramified coverings of S^{log}) we can obtain every log admissible covering of $\widetilde{I}^{log}$ by pulling back a log admissible covering of X^{log}. Let us call X *untangled at* I if every node of X that lies on I also lies on an irreducible component of X distinct from I. In general, we can form a ("combinatorial") étale covering of X as follows: Write $X = I \bigcup J$, where J is the union of the irreducible components of X other than I. Let $\widetilde{I}_1$ and $\widetilde{I}_2$ (respectively, J_1 and J_2) be copies of $\widetilde{I}$ (respectively, J). For $i = 1, 2$, let us glue $\widetilde{I}_i$ to J_i at every node of I that also lies on J. If ν is a node of I that only lies on I, let α and β be the points of $\widetilde{I}$ that lie over ν. Then glue $\alpha_1 \in \widetilde{I}_1$ to $\beta_2 \in \widetilde{I}_2$, and $\beta_1 \in \widetilde{I}_1$ to $\alpha_2 \in \widetilde{I}_2$. With these various gluings, $\widetilde{I}_1 \bigcup \widetilde{I}_2 \bigcup J_1 \bigcup J_2$ forms a curve Y which is finite étale over X. Moreover, Y is untangled at $\widetilde{I}_1$ and $\widetilde{I}_2$, and $\Pi_I^{adm} = \Pi_{I_i}^{adm}$ (for $i = 1, 2$). Thus, it suffices to prove the Proposition under the assumption that X is untangled at I. Therefore, for the remainder of the proof, *we shall assume that X is untangled at I.*

Now we would like to construct another double étale covering of X. For convenience, we will assume that $p \geq 3$. (The case $p = 2$ is only combinatorially a bit more difficult.) Write $X = I \bigcup J$, as above. Since X is sturdy, it follows that (after possibly enlarging k), there exists an étale covering $\widetilde{J} \to J$ of degree two such that for any irreducible component $C \subseteq J$, the restriction of $\widetilde{J} \to J$ to C is nontrivial. Let I_1 and I_2 be copies of I. If ν is a node on I and J, let α (respectively, β) be the corresponding point on I (respectively, J). (After possibly enlarging k) we may assume that $\widetilde{J}$ has two k-rational points β_1 and β_2 over $\beta \in J$. Now, for $i = 1, 2$, glue $\alpha_i \in I_i$ to $\beta_i \in \widetilde{J}$. We thus obtain a double étale covering $Y = I_1 \bigcup I_2 \bigcup \widetilde{J} \to X$. Endow Y with the log structure obtained by pulling back the log structure of X^{log}. One can then define various log structures on the irreducible components of Y, analogously to the way in which various log structures were defined on an irreducible component I of X above. We will then use similar notation for the log structures thus obtained on irreducible components of Y.

Now let $L^{log} \to I^{log}$ be a Galois log admissible covering of $I^{log} \stackrel{\text{def}}{=} \widetilde{I}^{log}$ (recall that $I = \widetilde{I}$) of degree d. Let $M^{log} \to \widetilde{J}^{log}$ be an abelian log multi-admissible covering of degree d with the following property:

> (*) For each node ν on I and J, suppose that over the corresponding $\alpha \in I$, L has n geometric points, each ramified with index e over I.

Then, we stipulate that for $i = 1, 2$, over $\beta_i \in \widetilde{J}$, M has n geometric points, each ramified with index e over I.

Note that such an $M^{log} \to \widetilde{J}^{log}$ exists precisely because the β's on $\widetilde{J}$ *come in pairs.* Now let L_1^{log} and L_2^{log} be copies of L^{log}. Then, for each $i = 1, 2$, let us glue the geometric points of $L_i|_{\alpha_i}$ to those of $M|_{\beta_i}$. This gives us (after possibly replacing S^{log} by a tamely ramified covering of S^{log}) a log admissible covering $Z^{log} \to Y^{log}$, where $Z = L_1 \bigcup L_2 \bigcup M$. Moreover, the restriction of $Z^{log} \to Y^{log}$ to I_i^{log} (for $i = 1, 2$) is $L^{log} \to I^{log}$. This completes the proof of the Proposition. ○

Section 5: The Set of Nodes

We continue with the notation of the preceding Section. Thus, $X^{log} \to S^{log}$ is a stable log-curve of genus g. Let us also assume that X is *sturdy.* In this Section, we would like to show how (by a technique similar to, but slightly more complicated than that employed in Section 1) we can recover the set of nodes of X. This, in turn, will allow us to recover the decomposition group of a node. In the following Section, we shall then show how the log structure at a node can be reconstructed from the decomposition group at the node.

Let l and n be prime numbers distinct from each other and from p. We assume moreover that $l \equiv 1$ $(mod\ n)$. This means that all n^{th} roots of unity are contained in $\mathbf{F}_l$. Let us write $G_n \subseteq \mathbf{F}_l^{\times}$ for the subgroup of n^{th} roots of unity. Next, let us fix a G_n-torsor over $X_{\overline{k}}$

$$Y \to X$$

which is nontrivial over the generic point of every irreducible component of $X_{\overline{k}}$. (Here, by G_n-torsor, we mean a cyclic étale covering of X of degree n whose Galois group is equipped with an isomorphism with G_n.) Equip Y with the log structure pulled back from that of X^{log}. Let us consider the admissible fundamental group Π_Y^{adm} of Y^{log}. Let $H^1_{adm}(Y^{log}_{\overline{k}}, \mathbf{F}_l) \stackrel{\text{def}}{=} Hom(\Delta_{Y^{log}}, \mathbf{F}_l)$. Note that we have a natural injection $L^e \stackrel{\text{def}}{=} H^1_{\acute{e}t}(Y_{\overline{k}}, \mathbf{F}_l) \hookrightarrow L^a \stackrel{\text{def}}{=} H^1_{adm}(Y^{log}_{\overline{k}}, \mathbf{F}_l)$. Let us write L^r for the cokernel of this injection. (Here, "e" (respectively, "a"; "r") stands for "étale" (respectively, "admissible"; "ramification"). Thus, we have an exact sequence

$$0 \to L^e \to L^a \to L^r \to 0$$

which may (by Proposition 3.2) be recovered from $\Pi_X^{adm} \to \Pi_{S^{log}}$ and the subgroup of Π_X^{adm} that defines $Y \to X$. Note, moreover, that G_n acts on the above exact sequence. Let $L^r_G \subseteq L^r$ be the subset of elements on which G_n acts via the character $G_n \hookrightarrow \mathbf{F}_l^{\times}$. Let $L^* \subseteq L^a$ be the subset of elements that map to nonzero elements of L^r_G.

We would like to analyze L^r_G. First of all, let us consider L^r. For each node $\nu \in Y(\overline{k})$, write Y_ν for the completion of Y at ν, and let γ_ν and δ_ν be the two irreducible components of Y_ν. Let D'_Y (respectively, E_Y) be the free $\mathbf{F}_l$-module which is the direct sum of copies of $\mathbf{F}_l(-1)$ (where the "(-1)" is a Tate twist) generated by the symbols γ_ν, δ_ν (respectively, I), as ν (respectively, I) ranges over all the nodes of Y_ν (respectively, irreducible components of $Y_{\overline{k}}$). Let $D_Y \subseteq D'_Y$ be the submodule generated by $(\gamma_\nu - \delta_\nu) \cdot \mathbf{F}_l(-1)$ (where ν ranges over all the nodes of Y_ν). Note that we have a natural morphism $D'_Y \to E_Y$ given by assigning to the symbol γ_ν (respectively, δ_ν) the unique irreducible component I in which γ_ν (respectively, δ_ν) is contained. In particular, restricting to D_Y, we obtain a morphism

$$D_Y \to E_Y$$

Let $K_Y \subseteq D_Y$ be the kernel of this morphism. Now let us note that we have a natural morphism

$$\lambda : L^r \to D'_Y$$

given by restricting an admissible covering of $Y_{\overline{k}}$ to the various γ_ν and δ_ν. It follows immediately from the definition of an admissible covering that $Im(\lambda) \subseteq D_Y$. Moreover, by considering the Leray-Serre spectral sequence in étale cohomology for the morphism $\breve{I} \hookrightarrow \widetilde{I}$ (where $\widetilde{I} \subseteq Y_{\overline{k}}$ is the normalization of an irreducible component of $Y_{\overline{k}}$, and $\breve{I}$ is the complement of the points that map to nodes), plus the definition of an admissible covering, one sees easily that, in fact, $\lambda(L^r) = K_Y$. Finally, by counting dimensions, we see that λ is injective. Thus, we see that λ defines a natural isomorphism of L^r with K_Y. In the following, we shall identify L^r and K_Y by means of λ.

Now let us consider the subset $L^r_G \subseteq L^r$. Let $\mu \in X(\overline{k})$ be a node. For each such μ, let us fix a node $\nu \in Y(\overline{k})$ over μ. If $\sigma \in G_n$ (regarded as the Galois group of $Y \to X$), we shall write $a_\sigma \in \mathbf{F}_l^\times$ for σ regarded as an element of $\mathbf{F}_l^\times$. Fix a generator $\omega \in \mathbf{F}_l(-1)$. Let

$$\omega_\mu \stackrel{\text{def}}{=} \sum_{\sigma \in G_n} (a_\sigma^{-1} \cdot \omega)(\sigma(\gamma_\nu) - \sigma(\delta_\nu)) \in D_Y$$

One checks easily that ω_μ is, in fact, an element of $K_Y = L^r$. Moreover, by calculating $\tau(\omega_\mu)$ (for $\tau \in G_n$), one sees that ω_μ is manifestly an element of $L^r_G \subseteq L^r$. Finally, it is routine to check that, in fact, L^r_G is freely generated by the ω_μ (as μ ranges over the nodes of $X(\overline{k})$ – but ω is fixed). This completes our analysis of L^r_G.

Suppose that $\alpha \in L^*$. Let $Z_\alpha \to Y_{\overline{k}}$ be the corresponding covering. Let $\epsilon : L^* \to \mathbf{Z}$ be the morphism that maps α to N_{Z_α} (i.e., the number of nodes of Z_α). Let $M \subseteq L^*$ be the subset of elements α on which ϵ attains its maximum. Let us define a pre-equivalence relation "$\sim$" on M as follows:

> If $\alpha, \beta \in M$, then we write $\alpha \sim \beta$ if, for every $\lambda, \mu \in \mathbf{F}_l^\times$ for which $\lambda \cdot \alpha + \mu \cdot \beta \in L^*$, we have $\lambda \cdot \alpha + \mu \cdot \beta \in M$.

Now we have the following result:

Proposition 5.1: *Suppose that X is stable and sturdy. Then "$\sim$" is, in fact, an equivalence relation, and moreover, $M/\sim$ is naturally isomorphic to the set of nodes of $X_{\overline{k}}$.*

Proof: Suppose that $\alpha \in L^*$ maps to a linear combination (with nonzero coefficients) of precisely $c \geq 1$ of the elements $\omega_\mu \in L^r_G$. Then one calculates easily that Z_α has precisely $\epsilon(\alpha) = l(N_Y - cn) + cn = l \cdot N_Y + cn(1-l)$ nodes. Thus, $\epsilon(\alpha)$ attains its maximum precisely when $c = 1$. Thus, $M \subseteq L^*$ consists of those α which map to a nonzero multiple of one of the ω_μ's. It is thus easy to see (as in the proof of Proposition 1.3) that $M/\sim$ is naturally isomorphic to the set of nodes $\mu \in X(\overline{k})$. ○

Remark: Note that at first glance the set $M/\sim$ appears to depend on the choice of n, l, and $Y \to X$. However, it is not difficult to see that in fact, if one chooses different data $n' \neq n$, $l' \neq l$, and $Y' \to X$, and hence obtains a resulting $M'/\sim'$, then there is a natural isomorphism $(M/\sim) \cong (M'/\sim')$ (compatible with the isomorphisms just obtained of $M/\sim$ and $M'/\sim'$ to the set of nodes of $X_{\overline{k}}$) as follows: If $\alpha \in M$ and $\alpha' \in M'$, let us consider the product $Z_{\alpha\alpha'} = Z_\alpha \times_X Z'_\alpha$. Thus, we have an admissible covering $Z_{\alpha\alpha'} \to X$ of degree $(ln)(l'n')$. Then one checks easily that α and α' correspond to the same node if and only if $(Z_{\alpha\alpha'})_{\overline{k}}$ has precisely $nn'\{l \cdot l'(N_X - 1) + 1\}$ nodes.

Proposition 5.2: *Suppose that X is stable and sturdy. Then the set of nodes of $X_{\overline{k}}$ (together with its natural Π_S-action) can be recovered entirely from $\Pi_X^{adm} \to \Pi_{S^{log}}$. Moreover, (relative to Proposition 1.4) for each node μ of $X_{\overline{k}}$, the set of irreducible components of $X_{\overline{k}}$ containing μ can also be recovered entirely from $\Pi_X^{adm} \to \Pi_{S^{log}}$.*

Proof: Indeed, (after possibly replacing k by a finite extension of k) one can always choose l, n, and $Y \to X$ as above. Then one can recover L^e and L^a from $\Pi_X^{adm} \to \Pi_{S^{log}}$ and the subgroup of Π_X^{adm} defined by $Y \to X$. Thus, one can also recover L^r. We saw in Section 1 that for any Z_α, $N_{Z_\alpha} - I_{Z_\alpha}$, as well as I_{Z_α}, may be recovered group-theoretically. In particular, N_{Z_α} can also be recovered group-theoretically. Thus, M and $\sim$ can also be recovered group-theoretically. Moreover, by the above Remark, $M/\sim$ is independent of the choice of n, l, and $Y \to X$. (That is to say, the isomorphism $(M/\sim) \cong (M'/\sim')$ of the above Remark can clearly be recovered group-theoretically.) This completes the proof that the nodes can be recovered group-theoretically.

Now let us consider the issue of determining which irreducible components of $X_{\overline{k}}$ (relative to the reconstruction of the set of irreducible components of $X_{\overline{k}}$ given in Proposition 1.4) μ lies on. To this end, note first that (by Corollary 1.6) the genus g_I of each irreducible component I of $X_{\overline{k}}$ can also be recovered group-theoretically. (Indeed, $J_l(I)$ has precisely l^{2g_I} elements.) Thus, if $\alpha \in M$ corresponds to the node μ, the irreducible components I

of $X_{\overline{k}}$ containing μ are precisely those that are the image of irreducible components J of Z_α such that $(g_J - 1) > l \cdot n \cdot (g_I - 1)$. ○

Corollary 5.3: *Suppose that X is stable and sturdy. Let $H \subseteq \Pi_X^{adm}$ be an open orderly subgroup. Let $Y_H^{log} \to X^{log}$ be the corresponding covering. Then the sets of nodes $\mathcal{N}_{Y_H}$ and $\mathcal{N}_X$ of Y_H and X, respectively, as well as the natural morphism $\mathcal{N}_{Y_H} \to \mathcal{N}_X$ can be recovered from $\Pi_X^{adm} \to \Pi_{S^{log}}$ and H. Moreover, the set of irreducible components of $(Y_H)_{\overline{k}}$ on which each node of $\mathcal{N}_{Y_H}$ lies can also be recovered from $\Pi_X^{adm} \to \Pi_{S^{log}}$ and H.*

Proof: In this case, in order to obtain the morphism $\mathcal{N}_{Y_H} \to \mathcal{N}_X$ it is useful to choose l and n prime to the index of H in Π_X^{adm}, and to choose the G_n-torsor "Y" over Y_H to be the pull-back of a G_n-torsor on X_H. The rest of the proof is straightforward. ○

Now let μ be a node of X. Let $I \subseteq X$ be an irreducible component of X on which μ sits. Then there is a unique (up to conjugacy) *decomposition subgroup*

$$\Delta_\mu^{adm} \subseteq \Pi_X^{adm}$$

which may be defined as follows. Let $Z^{log} \to X^{log}$ be the log scheme obtained by taking the inverse limit of the various $Y_H^{log} \to X^{log}$ corresponding to open orderly subgroups $H \subseteq \Pi_X^{adm}$. Choose a "node" $\nu \in Z$ that maps down to $\mu \in X$. Here, by "node of Z," we mean a compatible system of nodes $\nu_H \in Y_H$. Then $\Delta_\mu^{adm} \subseteq \Pi_X^{adm}$ is the subgroup of elements that take the node ν to itself. If ν sits on an irreducible component J of Z which lies over I, then we can also form $\Delta_I^{adm}, \Delta_I^{in} \subseteq \Pi_X^{adm}$, hence $\Delta_{\mu,I}^{adm} \stackrel{\text{def}}{=} \Delta_I^{adm} \bigcap \Delta_\mu^{adm}$, and $\Delta_{\mu,I}^{in} \stackrel{\text{def}}{=} \Delta_I^{in} \bigcap \Delta_\mu^{adm}$. Up to conjugacy, $\Delta_{\mu,I}^{adm}$ and $\Delta_{\mu,I}^{in}$ are independent of all the choices made. Now, it follows formally from Corollary 5.3 that

Corollary 5.4: *Given μ and I as above, one can recover $\Delta_\mu^{adm}; \Delta_{\mu,I}^{adm}; \Delta_{\mu,I}^{in} \subseteq \Pi_X^{adm}$ entirely from $\Pi_X^{adm} \to \Pi_{S^{log}}$.*

Section 6: The Log Structure at a Node

We maintain the notation of the preceding Section. In addition to assuming that X is stable and sturdy, let us assume that it is *untangled* (i.e., every node lies on two distinct irreducible components). Let $\mu \in X(k)$ be a node of X. Let I, I' be the two irreducible components of X on which μ lies. Let $Z^{log} \to X^{log}$ be the covering corresponding to the trivial subgroup of Π_X^{adm}. Let ν be a node of Z lying over μ. Let J (respectively, J') be the irreducible component of Z that touches ν and lies over I (respectively, I'). Thus, as in the preceding Section, we have various subgroups, such as $\Delta_\mu^{adm}, \Delta_I^{adm}, \Delta_{I'}^{adm} \subseteq \Pi_X^{adm}$. Note that (since X is untangled) elements of Δ_μ^{adm} fix J and J'. Thus, $\Delta_\mu^{adm} \subseteq \Delta_I^{adm}, \Delta_{I'}^{adm}$.

Next, consider the natural morphism $\Delta_\mu^{adm} \subseteq \Pi_X^{adm} \to \Pi_S$. Since μ is k-rational, this morphism is surjective. Let us denote the kernel of this morphism by Δ_μ^{in}. Thus, we have an exact sequence

$$1 \to \Delta_\mu^{in} \to \Delta_\mu^{adm} \to \Pi_S \to 1$$

Moreover, by sorting through the definitions, it is clear that $\Delta_I^{in}, \Delta_{I'}^{in} \subseteq \Delta_\mu^{in}$.

Next, let us consider the natural morphism $\Delta_I^{in} \subseteq \Pi_X^{adm} \to \Pi_{S^{log}}$. It is clear that the image of this morphism is contained in the inertia subgroup $I_S \subseteq \Pi_{S^{log}}$. Thus, we obtain a natural morphism $\Delta_I^{in} \to I_S$. By using the fact that the restriction of the log structure of X^{log} to the generic point of I is the pull-back (to the generic point of I) of the log structure of S^{log}, it is then easy to see that this morphism $\Delta_I^{in} \to I_S$ is an isomorphism. Thus, we see that we obtain a natural isomorphism

$$\Delta_I^{in} \cong I_S = \widehat{\mathbf{Z}}'(1)$$

In the sequel, we shall identify Δ_I^{in} with $\widehat{\mathbf{Z}}'(1)$ via this isomorphism. Similarly, we have $\Delta_{I'}^{in} \cong \widehat{\mathbf{Z}}'(1)$.

In order to understand these various groups better, it is helpful to think in terms of a local model, as follows: If $e \geq 1$ is an integer, let $R \stackrel{\text{def}}{=} \overline{k}[[x, y, t]]/(xy - t^e)$, $A \stackrel{\text{def}}{=} \overline{k}[[t]] \subseteq R$. Let $\mathcal{X} \stackrel{\text{def}}{=} Spec(R)$, $T \stackrel{\text{def}}{=} Spec(A)$. Endow T with the log structure defined by the divisor $t = 0$. Let N_{node} be the monoid given by taking the quotient of the free monoid on the symbols $log(x), log(y), log(t)$ by the relation $log(x) + log(y) \sim e \cdot log(t)$. Map $N_{node} \to R$ by letting $log(?) \mapsto ?$, for $? = x, y, t$. Endow $\mathcal{X}$ with the log structure associated to $N_{node} \to R$. Thus, we obtain a morphism of log schemes $\mathcal{X}^{log} \to T^{log}$. Let us denote by $\tau : Spec(\overline{k}) \hookrightarrow T$ the special point of T. Moreover, there is a unique e such that the completion of $X_{\overline{k}}$ at $\mu_{\overline{k}}$ is equal to $\mathcal{X}^{log}|_\tau$. We shall call this e the *order of the node* μ.

Let $\mathcal{I} = V(y, t) = Spec(\overline{k}[[x]])$; $\mathcal{I}' = V(x, t) = Spec(\overline{k}[[y]])$; $\mathcal{X}_\tau = \mathcal{X} \times_T \tau$. Let $\mathcal{U} = \mathcal{X} - \mathcal{I} - \mathcal{I}'$. Let us denote by $\Pi_{\mathcal{X}}^{adm}$ the quotient of the fundamental group of $\mathcal{U}$ which is tamely ramified over the divisors $\mathcal{I}$ and $\mathcal{I}'$. Note that $\Pi_{\mathcal{X}}^{adm}$ is *abelian* (thus eliminating the need to choose a base point). Indeed, this follows by noting that $\Pi_{\mathcal{X}}^{adm}$ for $e \geq 1$ injects into $\Pi_{\mathcal{X}}^{adm}$ for $e = 1$; but when $e = 1$, $\mathcal{I} \bigcup \mathcal{I}'$ is a divisor with normal crossings, so $\Pi_{\mathcal{X}}^{adm} \cong \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1)$. Let us denote by Π_T^{adm} the tame fundamental group of $T - \tau$. Thus, $\Pi_T^{adm} = \widehat{\mathbf{Z}}'(1)$. Note (for instance, by reducing to the case $e = 1$) that the decomposition groups of the divisors $\mathcal{I}$ and $\mathcal{I}'$ are both equal to $\Pi_{\mathcal{X}}^{adm}$. Let us denote by $\Delta_{\mathcal{I}}^{in}, \Delta_{\mathcal{I}'}^{in} \subseteq \Pi_{\mathcal{X}}^{adm}$ the inertia groups of the divisors $\mathcal{I}$ and $\mathcal{I}'$. As above, it is easy to see that the natural map $\Delta_{\mathcal{I}}^{in} \subseteq \Pi_{\mathcal{X}}^{adm} \to \Pi_T^{adm}$ is an isomorphism. Thus, we have isomorphisms $\Delta_{\mathcal{I}}^{in} \cong \widehat{\mathbf{Z}}'(1)$; $\Delta_{\mathcal{I}'}^{in} \cong \widehat{\mathbf{Z}}'(1)$. Observe, moreover, (for instance, by reducing to the case $e = 1$) that $\Delta_{\mathcal{I}}^{in} \bigcap \Delta_{\mathcal{I}'}^{in} = \{1\}$.

Let $\xi \stackrel{\mathrm{def}}{=} V(x, y, t) \in X$. Let us denote by $\Pi^{adm}_{\mathcal{I}}$ the tame fundamental group of $\mathcal{I} - \xi$. Thus, $\Pi^{adm}_{\mathcal{I}} = \widehat{\mathbf{Z}}'(1)$. Similarly, we have $\Pi^{adm}_{\mathcal{I}'} = \widehat{\mathbf{Z}}'(1)$. Note, moreover, that we have a natural morphism $\Pi^{adm}_{\mathcal{X}}/\Delta^{in}_{\mathcal{I}} \hookrightarrow \Pi^{adm}_{\mathcal{I}}$. Thus, we obtain a morphism

$$\widehat{\mathbf{Z}}'(1) = \Delta^{in}_{\mathcal{I}'} \to \Pi^{adm}_{\mathcal{X}}/\Delta^{in}_{\mathcal{I}} \to \Pi^{adm}_{\mathcal{I}} = \widehat{\mathbf{Z}}'(1)$$

This morphism corresponds to multiplication by some element $\epsilon \in \widehat{\mathbf{Z}}'$. We claim that $\epsilon = e$. Indeed, this follows from the fact that taking roots of the function $x|_{\mathcal{I}}$ on $\mathcal{I}$ corresponds to taking roots of $t^e = y^{-1} \cdot x$ over $\mathcal{I}' - \xi$ (since y is invertible on $\mathcal{I}' - \xi$). Thus, we see that we have injections

$$\widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta^{in}_{\mathcal{I}} \times \Delta^{in}_{\mathcal{I}'} \hookrightarrow \Pi^{adm}_{\mathcal{X}} \subseteq \Pi^{adm}_{\mathcal{I}'} \times \Pi^{adm}_{\mathcal{I}} = \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1)$$

Here the composite $\widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) \to \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1)$ is given by multiplication by e.

Now let us translate what we have learned locally back into information concerning our original X. First of all, let us observe that $\Pi^{adm}_{\mathcal{X}}$ (respectively, $\Delta^{in}_{\mathcal{I}}$; $\Delta^{in}_{\mathcal{I}'}$) corresponds to Δ^{in}_{μ} (respectively, Δ^{in}_{I}; $\Delta^{in}_{I'}$). Thus, in particular, we obtain that $\Delta^{in}_{I} \bigcap \Delta^{in}_{I'} = \{1\}$. Next, let us observe that $\Delta^{adm}_{I}/\Delta^{in}_{I}$ may be identified with $\Pi^{adm}_{\breve{I}}$, i.e., the tame fundamental group of $\breve{I}$. Let us denote by $\Delta^{in}_{\mu}[I] \subseteq \Pi^{adm}_{\breve{I}} = \Delta^{adm}_{I}/\Delta^{in}_{I}$ the subgroup of elements that fix ν and act trivially on the residue field of ν. That is to say, $\Delta^{in}_{\mu}[I]$ is the inertia group of μ in $\Pi^{adm}_{\breve{I}}$. In particular, we have a natural isomorphism

$$\Delta^{in}_{\mu}[I] \cong \widehat{\mathbf{Z}}'(1)$$

Similarly, we have an isomorphism $\Delta^{in}_{\mu}[I'] \cong \widehat{\mathbf{Z}}'(1)$. Then $\Delta^{in}_{\mu}[I]$ (respectively, $\Delta^{in}_{\mu}[I']$) corresponds to $\Pi^{adm}_{\mathcal{I}}$ (respectively, $\Pi^{adm}_{\mathcal{I}'}$).

Now let us observe that, by the theory developed thus far, all the subgroups and isomorphisms of the above discussion may be recovered from $\Pi^{adm}_{X} \to \Pi_{S^{log}}$, *except* (possibly) the injections

$$\iota_{\mu} : \widehat{\mathbf{Z}}'(1) \cong \Delta^{in}_{\mu}[I] \hookrightarrow \Pi^{adm}_{\breve{I}}; \qquad \iota'_{\mu} : \widehat{\mathbf{Z}}'(1) \cong \Delta^{in}_{\mu}[I'] \hookrightarrow \Pi^{adm}_{\breve{I}'}$$

(or, equivalently, except (possibly) the isomorphisms $\widehat{\mathbf{Z}}'(1) \cong \Delta^{in}_{\mu}[I]$, $\widehat{\mathbf{Z}}'(1) \cong \Delta^{in}_{\mu}[I']$). Thus, we have the following result

Proposition 6.1: *Suppose that X is stable, sturdy, and untangled, and that $\mu \in X(k)$ is a node. Then the order e of the node μ may be recovered entirely from $\Pi^{adm}_{X} \to \Pi_{S^{log}}$, ι_{μ}, and ι'_{μ}.*

Next we would like to consider the issue of reconstructing the log structure of X at μ group-theoretically. Let S_μ^{log} be the log scheme whose underlying scheme is $S = Spec(k)$, and whose log structure is that obtained by pulling back the log structure of X^{log} via $\mu : S \hookrightarrow X$. Thus, we have a natural structure morphism $S_\mu^{log} \to S^{log}$. Let M_S (respectively, M_μ) be the monoid defining the log structure of S^{log} (respectively, S_μ^{log}). Thus, we have $k^\times \subseteq M_S$; $k^\times \subseteq M_\mu$. Moreover, $M_S/k^\times = \mathbf{N}$; $M_\mu/k^\times = N_{node}$ (where N_{node} is the monoid introduced above). The structure morphism $S_\mu^{log} \to S^{log}$ defines a morphism $M_S \to M_\mu$ (which is the identity on $k^\times$), hence an inclusion $\mathbf{N} \to N_{node}$. Let us denote by $\epsilon \in N_{node}$ the image of $e \in \mathbf{N}$ in N_{node}. Let us denote by $\mathcal{L}_e \subseteq M_S$ (respectively, $\mathcal{L}_\epsilon \subseteq M_\mu$) the inverse image of $e \in \mathbf{N}$ (respectively, $\epsilon \in N_{node}$) in M_S (respectively, M_μ). Thus, we obtain an isomorphism of $k^\times$-torsors

$$\zeta_\mu : \mathcal{L}_e \to \mathcal{L}_\epsilon$$

We would like to reconstruct ζ_μ group-theoretically.

First, recall that we may naturally regard $k^\times$ as a quotient of $\widehat{\mathbf{Z}}'(1)$. Now it follows from Kummer theory that

Lemma 6.2 : *There is a natural one-to-one correspondence between elements of $\mathcal{L}_\epsilon$ and morphisms $\psi : \Delta_\mu^{adm} \to k^\times$ whose restriction to $\widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta_I^{in} \times \Delta_{I'}^{in} \subseteq \Delta_\mu^{adm}$ is the composite of the morphism $(e, e) : \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) \to \widehat{\mathbf{Z}}'(1)$ (i.e., multiplication by e on both factors) with the natural quotient $\widehat{\mathbf{Z}}'(1) \to k^\times$.*

Note that here, the $k^\times$-torsor structure on the set of such ψ is given by observing that the difference between two such ψ is a morphism $Hom(\Pi_S, k^\times)$, which may be identified with $k^\times$ by means of the Frobenius element $\phi \in \Gamma = \Pi_S$.

Similarly, we have

Lemma 6.3 : *There is a natural one-to-one correspondence between elements of $\mathcal{L}_e$ and morphisms $\psi : \Pi_{S^{log}} \to k^\times$ whose restriction to $\widehat{\mathbf{Z}}'(1) = I_S \subseteq \Pi_{S^{log}}$ is the composite of the morphism $e\cdot : \widehat{\mathbf{Z}}'(1) \to \widehat{\mathbf{Z}}'(1)$ (i.e., multiplication by e) with the natural quotient $\widehat{\mathbf{Z}}'(1) \to k^\times$.*

Moreover, the correspondence induced by $\zeta_\mu : \mathcal{L}_e \cong \mathcal{L}_\epsilon$ between the ψ of Lemma 6.2 and the ψ of Lemma 6.3 is the correspondence obtained by composing $\Pi_{S^{log}} \to k^\times$ with $\Delta_\mu^{adm} \hookrightarrow \Pi_X^{adm} \to \Pi_{S^{log}}$.

We thus conclude the following

Proposition 6.4: *Suppose that X is stable, sturdy, and untangled, and that $\mu \in X(k)$ is a node. Then the morphism $\zeta_\mu : \mathcal{L}_e \cong \mathcal{L}_\epsilon$ may be recovered entirely from $\Pi_X^{adm} \to \Pi_{S^{log}}$, ι_μ, and ι'_μ.*

Section 7: The Main Result over Finite Fields

We are now ready to put everything together and prove the main result over finite fields. The point is that the theory developed thus far in this paper will allow us to reduce the Grothendieck conjecture for singular stable log-curves over finite fields to the Grothendieck conjecture for smooth, affine, hyperbolic curves over finite fields (which is already proven in [Tama]).

Let $S^{log}_{\infty} \to S^{log}$ be as in Definition 2.7. Let $X^{log} \to S^{log}$ and $(X')^{log} \to S^{log}$ be stable log-curves of genus g (equipped with base-points $x \in X(k)$ and $x' \in X'(k)$), such that *neither* X *nor* X' *is smooth over* k. Let us assume that we are given a commutative diagram of continuous group homomorphisms:

$$\begin{array}{ccc} \Pi^{adm}_X & \stackrel{\alpha^{\Pi}}{\longrightarrow} & \Pi^{adm}_{X'} \\ \downarrow & & \downarrow \\ \Pi_{S^{log}} & \stackrel{id}{\longrightarrow} & \Pi_{S^{log}} \end{array}$$

where the vertical morphisms are the natural ones, and the horizontal morphisms are isomorphisms. The goal of this Section is to show that (under a certain technical assumption on the "RT-degree") α^{Π} arises (up to conjugation by an element of $\Delta_{X^{log}}$) from a geometric S^{log}-isomorphism of X^{log} with $(X')^{log}$.

We begin by proving the result under the following simplifying assumption on X and X':

> (*) X and X' are sturdy and untangled, and their nodes are rational over k.

By Proposition 4.1, α^{Π} induces a natural isomorphism between the sets of irreducible components of X and X'. Let $I \subseteq X$ be an irreducible component. Then there is a corresponding irreducible component $I' \subseteq X'$. Moreover, by Proposition 4.1, α^{Π} necessarily maps decomposition (respectively, inertia) subgroups of Π^{adm}_X to similar subgroups of $\Pi^{adm}_{X'}$. Thus, we may choose $\Delta^{adm}_I \subseteq \Pi^{adm}_X$ such that α^{Π} maps Δ^{adm}_I onto $\Delta^{adm}_{I'} \subseteq \Pi^{adm}_{X'}$. Moreover, we also have $\alpha^{\Pi}(\Delta^{in}_I) = \Delta^{in}_{I'}$. Thus, since $\Pi^{adm}_{\breve{I}} \stackrel{\text{def}}{=} \Delta^{adm}_I / \Delta^{in}_I$, we get a natural isomorphism

$$\alpha^{\Pi}_{\breve{I}} : \Pi^{adm}_{\breve{I}} \cong \Pi^{adm}_{\breve{I}'}$$

Now as we saw at the beginning of Section 6, the natural morphism $\Delta^{in}_I \to I_S$ is an isomorphism. Hence, the quotient of $\Pi^{adm}_{\breve{I}}$ induced by $\Pi^{adm}_X \to \Pi_{S^{log}}$ is simply $\Pi^{adm}_{\breve{I}} \to \Pi_S$. Thus, we see that $\alpha^{\Pi}_{\breve{I}}$ fits into a diagram:

$$\begin{array}{ccc} \Pi_{\breve{I}}^{adm} & \stackrel{\alpha_{\breve{I}}^{\Pi}}{\longrightarrow} & \Pi_{\breve{I}'}^{adm} \\ \downarrow & & \downarrow \\ \Pi_S & \stackrel{id}{\longrightarrow} & \Pi_S \end{array}$$

Let $\Delta_{\breve{I}} = Ker(\Pi_{\breve{I}}^{adm} \to \Pi_S)$. Since $\breve{I}$ is a smooth affine hyperbolic curve over k, we are now in a position to apply the theory of [Tama]. The only two consequences of the theory of [Tama] that we will use in this paper are the following:

(1) $\alpha_{\breve{I}}^{\Pi}$ induces a commutative diagram

$$\begin{array}{ccc} \breve{I} & \stackrel{\alpha_{\breve{I}}}{\longrightarrow} & \breve{I}' \\ \downarrow & & \downarrow \\ S & \stackrel{\gamma_I}{\longrightarrow} & S \end{array}$$

of morphisms of schemes. Here the vertical morphisms are the natural ones, and the horizontal morphisms are isomorphisms. The morphism γ_I, however, need not be the identity.

(2) For each node μ of X lying on I, the morphisms $\iota_\mu : \widehat{\mathbf{Z}}'(1) \hookrightarrow \Pi_{\breve{I}}^{adm}$ (well-defined up to conjugation by an element of $\Delta_{\breve{I}}$) of Proposition 6.1 are taken to each other by α^{Π}, *up to multiplication by some unit* $\theta_I \in (\widehat{\mathbf{Z}}')^{\times}$. Here, the automorphism induced by θ_I on the quotient $\widehat{\mathbf{Z}}'(1) \to k^{\times}$ is equal to the automorphism of $k^{\times}$ induced by γ_I.

It follows, in particular, that $\theta_I \in p^{\mathbf{Z}} \subseteq \mathbf{Q}$ (i.e., θ_I is a rational number which is a power of p). In fact, as we shall see below (Lemma 7.1), θ_I is *independent* of I. Thus, we shall write

$$deg_{RT}(\alpha^{\Pi}) \stackrel{\text{def}}{=} \theta_I \in p^{\mathbf{Z}} \in \mathbf{Q}$$

and we shall refer to this number as the *RT-degree of* α^{Π} *on* I (where "RT" stands for "ramification-theoretic," as opposed to another type of degree that will be discussed in Section 9).

Lemma 7.1 : *The rational number θ_I is independent of the irreducible component I.*

Proof: Suppose that there is another irreducible component J of X that touches I. Let $J' \subseteq X'$ be the corresponding component of X'. Let us focus our attention at a node μ lying on I and J. By Corollary 5.3, there is a corresponding node $\mu' \in X'(k)$. Let e

(respectively, e') be the order of the node μ (respectively, μ'). Now we have a commutative diagram:

$$\begin{array}{ccccc}
\widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta^{in}_{\mu}[I] \times \Delta^{in}_{\mu}[J] & \stackrel{(e,e)}{\longleftarrow} & \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta^{in}_{J} \times \Delta^{in}_{I} & \stackrel{(1,1)}{\longrightarrow} & \widehat{\mathbf{Z}}'(1) = I_S \\
\big\downarrow \theta_I \times \theta_J & & \big\downarrow & & \big\downarrow id \\
\widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta^{in}_{\mu'}[I'] \times \Delta^{in}_{\mu'}[J'] & \stackrel{(e',e')}{\longleftarrow} & \widehat{\mathbf{Z}}'(1) \times \widehat{\mathbf{Z}}'(1) = \Delta^{in}_{J'} \times \Delta^{in}_{I'} & \stackrel{(1,1)}{\longrightarrow} & \widehat{\mathbf{Z}}'(1) = I_S
\end{array}$$

where the vertical map in the middle is that induced by α^{Π}, and θ_I and θ_J are the morphisms obtained from the theory of [Tama] (cf., item (2) in the list given above). It follows immediately from the commutativity of this diagram that $\theta_I = e' \cdot e^{-1} = \theta_J$. (Thus, we also obtain a new proof of the rationality of θ_I.) This completes the proof. ○

Let us assume henceforth that

(†) $deg_{RT}(\alpha^{\Pi}) = 1$

Thus, $\alpha_{\check{I}}$ is an S-isomorphism, and the ι_μ are taken to each other *precisely* (not just up to some multiple) by α^{Π}. By the theory of [Tama], this means that $\alpha_{\check{I}}$ induces an S-isomorphism $I \cong I'$ that respects nodes. We thus obtain an S-isomorphism

$$\alpha_X : X \cong X'$$

Thus, it remains to consider log structures. Let $\widetilde{X} \to X$ be the normalization of X. Let us equip $\widetilde{X}$ with the log structure defined by the divisor consisting of the points of $\widetilde{X}$ that map to nodes of X. Thus, we obtain a log scheme $\widetilde{X}^{log}$ whose log structure is generically trivial. Similarly, we have $(\widetilde{X}')^{log}$. Note that α_X already induces an S-isomorphism

$$\alpha_{\widetilde{X}^{log}} : \widetilde{X}^{log} \cong (\widetilde{X}')^{log}$$

Now let us concentrate on a single node $\mu \in X(k)$ (which corresponds to $\mu' \in X'(k)$). It is not difficult to see that given $\widetilde{X}^{log}$, in order to recover the S^{log}-log scheme X^{log} in a neighborhood of μ, it suffices to know the isomorphism

$$\zeta_\mu : \mathcal{L}_e \cong \mathcal{L}_\epsilon$$

But by Proposition 6.4, this morphism may be recovered from $\Pi^{adm}_X \to \Pi_{S^{log}}$, plus item (2) of the above review of the theory of [Tama] (now that we know/are assuming that all the $\theta_I = 1$). Thus, we see that α_X extends naturally to a morphism

$$\alpha_{X^{log}} : X^{log} \cong (X')^{log}$$

as desired.

The next step is to check that the morphism induced by α_X^{log} between Π_X^{adm} and $\Pi_{X'}^{adm}$ agrees (up to conjugation by an element of $\Delta_{X^{log}}$) with the original α^{Π}. But this follows by using a similar argument to that of [Tama]: Namely, first we observe that it is clear from the construction of $\alpha_{X^{log}}$ that if we start with an α^{Π} that arises from some $\beta : X^{log} \cong (X')^{log}$, then $\beta = \alpha_{X^{log}}$. Then we note that for each orderly covering $Y^{log} \to X^{log}$, there is a corresponding orderly covering $(Y')^{log} \to (X')^{log}$, together with an isomorphism $\Pi_Y^{adm} \cong \Pi_{Y'}^{adm}$ induced by α^{Π}. Moreover, this isomorphism gives rise to an isomorphism $\alpha_{Y^{log}}$ that is compatible with $\alpha_{X^{log}}$. It thus follows formally from the general theory of the algebraic fundamental group that the isomorphism of Π_X^{adm} with $\Pi_{X'}^{adm}$ induced by $\alpha_{X^{log}}$ differs from α^{Π} by conjugation by some element $\eta \in \Pi_X^{adm}$. On the other hand, since the automorphism of Π_X^{adm} given by conjugation by η must induce the identity on $\Pi_{S^{log}}$, and $\Pi_{S^{log}}$ clearly has trivial center, it thus follows that the image of η in $\Pi_{S^{log}}$ is trivial. Thus, $\eta \in \Delta_{X^{log}}$, as desired.

Now let us denote by $Isom_{S^{log}}(X^{log}, (X')^{log})$ the set of S^{log}-isomorphisms of log schemes between X^{log} and $(X')^{log}$. Next, let us consider the set $Isom_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$ of isomorphisms $\Pi_X^{adm} \cong \Pi_{X'}^{adm}$ that preserve and induce the identity on the quotient $\Pi_{S^{log}}$. Then given $\alpha, \alpha' \in Isom_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$, we regard $\alpha \sim \alpha'$ if and only if there exists an $\eta \in \Delta_{X^{log}}$ such that

$$\alpha(\eta \cdot \pi \cdot \eta^{-1}) = \alpha'(\pi)$$

for all $\pi \in \Pi_X^{adm}$. The resulting set of equivalence classes will be denoted

$$Isom^{GO}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$$

Here, "*GO*" stands for "geometrically outer." Now observe that it is clear that inner automorphisms induced by elements of $\Delta_{X^{log}}$ do not affect the RT-degree. Thus, it makes sense to consider

$$Isom^{GORT}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm}) \subseteq Isom^{GO}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$$

that is, the classes of isomorphisms α whose RT-degree is equal to 1.

Thus, in summary, we have proven (at least under the assumption (*)) the following result:

Theorem 7.2: *Let $S^{log}_{\infty} \to S^{log}$ be as in Definition 2.7. Let $X^{log} \to S^{log}$ and $(X')^{log} \to S^{log}$ be stable log-curves (equipped with base-points $x \in X(k)$ and $x' \in X'(k)$), such that at least one of X or X' is not smooth over k. Then the natural map*

$$Isom_{S^{log}}(X^{log}, (X')^{log}) \to Isom^{GORT}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$$

is bijective.

Proof: It remains to deal with the case where the simplifying assumption (*) is not satisfied. But let us note that given an arbitrary X^{log} as in the statement of the Theorem, there is always a finite orderly covering $Y^{log} \to X^{log}$ such that Y satisfies (*). Moreover, any admissible covering (with rational nodes) of Y will clearly still satisfy (*). Thus, it is clear that we may take $Y^{log} \to X^{log}$ such that the corresponding (via $\alpha^{\Pi} : \Pi_X^{adm} \cong \Pi_{X'}^{adm}$) $(Y')^{log} \to (X')^{log}$ is such that Y' also satisfies (*). Then one concludes the Theorem by descent. ◯

Just as in [Tama], if $\Delta_{X^{log}}$ is center-free, then one can rewrite Theorem 7.2 in terms of outer automorphisms. For this, we need the following

Lemma 7.3 : *The group $\Delta_{X^{log}}$ is center-free.*

Proof: The proof is formally the same as that given in [Tama], §1, for the case where X is smooth over k. The only facts that one needs to check are:

(1) The pro-p-quotient $\Delta^p_{X^{log}}$ of $\Delta_{X^{log}}$ is free.

(2) There exists an open subgroup $H \subseteq \Delta_{X^{log}}$ for which H^p is nonabelian.

We begin by checking (1). First note that $H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{F}_p) = 0$. Indeed, this follows immediately from writing out the long exact sequence associated to

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathbf{G}_a \xrightarrow{1-F} \mathbf{G}_a \longrightarrow 0$$

(where "F" is the Frobenius morphism). Thus, for all finite étale coverings $Y \to X_{\overline{k}}$, we also have $H^2_{\acute{e}t}(Y, \mathbf{F}_p) = 0$. It thus follows that $H^2(\Delta^p_{X^{log}}, \mathbf{F}_p) \cong H^2(\Delta^p_X, \mathbf{F}_p) = H^2_{\acute{e}t}(X_{\overline{k}}, \mathbf{F}_p) = 0$. (Here we use the elementary fact that the natural surjection $\Delta^p_{X^{log}} \to \Delta^p_X$ is an isomorphism.) On the other hand, it is a well-known fact from group-theory ([Shatz], Chapter III, §3, Proposition 2.3) that this implies that $\Delta^p_{X^{log}}$ is free. This completes the verification of (1). As for (2), since the smooth case is already discussed in [Tama], §1, we shall concentrate here on the case when X is singular. Then it suffices to note the existence of an orderly covering of $X^{log}_{\overline{k}}$ whose dual graph has a nonabelian fundamental group. But the existence of such a covering follows immediately from simple combinatorial considerations. ◯

If G_1 and G_2 are two topological groups, let us denote by $Isom(G_1, G_2)$ the set of continuous isomorphisms $G_1 \cong G_2$. Let us denote by $Out(G_1, G_2)$ the set of equivalence classes of $Isom(G_1, G_2)$, where we consider two isomorphisms equivalent if they differ by an inner automorphism. If $G_1 = G_2$, then we shall denote $Out(G_1, G_2)$ by $Out(G_1)$.

If $\alpha \in Isom(G_1, G_2)$, then α induces an isomorphism $Out(\alpha) : Out(G_1) \cong Out(G_2)$. Moreover, $Out(\alpha)$ depends only the class $[\alpha] \in Out(G_1, G_2)$ defined by α.

Suppose that X^{log} and $(X')^{log}$ are as in Theorem 7.2. Let

$$\rho_X : \Pi_{S^{log}} \to Out(\Delta_{X^{log}})$$

be the representation arising from the extension $1 \to \Delta_{X^{log}} \to \Pi_X^{adm} \to \Pi_{S^{log}} \to 1$. Note that ρ_X is independent of the choice of base point x. Similarly, we have $\rho_{X'}$. Let us denote by

$$Out_\rho(\Delta_{X^{log}}, \Delta_{(X')^{log}})$$

the set of $[\alpha] \in Out(\Delta_{X^{log}}, \Delta_{(X')^{log}})$ such that $Out(\alpha) \circ \rho_X = \rho_{X'}$. Now note that we have a natural map

$$Isom^{GO}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm}) \to Out_\rho(\Delta_{X^{log}}, \Delta_{(X')^{log}})$$

Then it follows group-theoretically (cf. [Tama]) from the fact that $\Delta_{X^{log}}$ has trivial center that this map is a bijection. Let us denote by

$$Out^D_\rho(\Delta_{X^{log}}, \Delta_{(X')^{log}})$$

the image under this bijection of

$$Isom^{GORT}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm}) \subseteq Isom^{GO}_{\Pi_{S^{log}}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$$

(Here, the "D" stands for "degree one.") Thus, one can rephrase Theorem 7.2 in the following (seemingly weaker) form:

Theorem 7.4: *Let $S^{log}_\infty \to S^{log}$ be as in Definition 2.7. Let $X^{log} \to S^{log}$ and $(X')^{log} \to S^{log}$ be stable log-curves such that at least one of X or X' is not smooth over k. Then the natural map*

$$Isom_{S^{log}}(X^{log}, (X')^{log}) \to Out^D_\rho(\Delta_{X^{log}}, \Delta_{(X')^{log}})$$

is bijective.

Section 8: Characterization of Admissible Coverings

So far, we have been working with stable curves over a finite field. In this Section and the next, we shift gears and consider stable curves over local fields. The purpose of this Section is to show that given such a curve, one can characterize the quotient of the fundamental group (of the generic curve) corresponding to admissible coverings in entirely group-theoretic terms. The technique of proof is similar to that of Proposition 3.1, where we characterized étale coverings among all admissible coverings.

Let K be a finite extension of $\mathbf{Q}_p$. Let $\overline{K}$ be an algebraic closure of K; let $\Gamma_K \stackrel{\text{def}}{=} Gal(\overline{K}/K)$. Let $K^{unr} \subseteq \overline{K}$ be the maximal unramified extension of K in $\overline{K}$; let $\Gamma_K^{unr} \subseteq \Gamma_K$ be the corresponding closed subgroup. Let $A \subseteq K$ (respectively, $A^{unr} \subseteq K^{unr}$) be the ring of integers; k be the residue field of A; $S = Spec(A)$. Let us endow S with the log structure defined by the closed point. Let $X^{log} \to S^{log}$ be a stable log-curve of genus g which is generically smooth. Thus, $X_K \to Spec(K)$ is a smooth curve of genus g. Choose a base-point $x \in X(K)$. Let $\Pi_{X_K} \stackrel{\text{def}}{=} \pi_1(X_K, x_{\overline{K}})$ be the resulting fundamental group. Thus, we have an exact sequence

$$1 \to \Delta_{X_K} \to \Pi_{X_K} \to \Gamma_K \to 1$$

where $\Delta_{X_K} \stackrel{\text{def}}{=} \pi_1(X_{\overline{K}}, x_{\overline{K}})$. Let $\Pi_{X_K} \to \Pi_X^{adm}$ be the quotient (as in Definition 2.4) of Π_{X_K} by the intersection $\bigcap H$ of all the co-admissible open subgroups $H \subseteq \Pi_{X_K}$.

Let $Y_K \to X_K$ be an abelian étale covering of degree p. Let us assume that Y_K is geometrically connected over K. Now let us consider the following condition on $Y_K \to X_K$:

> (*) Over K^{unr}, there is an infinite abelian étale covering $Z_{K^{unr}} \to X_{K^{unr}} = X_K \otimes_K K^{unr}$ with Galois group $\mathbf{Z}_p$ such that the intermediate covering corresponding to $\mathbf{Z}_p \to \mathbf{F}_p$ is $Y_{K^{unr}} \to X_{K^{unr}}$.

In the next few paragraphs, we would like to show that condition (*) is equivalent to the statement that $Y_K \to X_K$ extends to a finite abelian étale covering $Y \to X$. Indeed, the necessity of condition (*) follows easily from well-known facts concerning the fundamental group of $X_{\overline{k}} = X \otimes_A \overline{k}$. Thus, it remains to show that condition (*) is sufficient.

To prove the sufficiency of (*), we will need to review certain basic facts from [FC] concerning the p-adic Tate module of a semi-abelian scheme over S. Let $J_K \to Spec(K)$ be the Jacobian of X_K. We shall always regard J_K as equipped with its usual principal polarization. By [FC], Chapter I, Theorem 2.6 and Proposition 2.7, it follows that J_K extends uniquely to a semi-abelian scheme $J \to S$ over S. Let $V_J \stackrel{\text{def}}{=} Hom(\mathbf{Q}_p/\mathbf{Z}_p, J(\overline{K}))$ be the p-adic Tate module of J_K. Thus, V_J is a free $\mathbf{Z}_p$-module of rank $2g$, equipped with a natural Γ_K-action. In Chapter III of [FC], one finds a theory of degenerations of semi-abelian varieties. According to this theory (more precisely: the equivalence "M_{pol}" of Corollary 7.2 of [FC], Chapter III), there exists an abelian scheme $G \to S$, together with a torus $T \to S$, and an extension

$$0 \to T \to \widetilde{J} \to G \to 0$$

over S, such that "roughly speaking," J is obtained as a rigid analytic quotient of $\widetilde{J}$ by an étale sheaf P of free abelian groups of rank $r \stackrel{\mathrm{def}}{=} dim(T/S)$ on S. (Caution: Our choice of notation differs somewhat from that of [FC].) Let $V_{\widetilde{J}} \stackrel{\mathrm{def}}{=} Hom(\mathbf{Q}_p/\mathbf{Z}_p, \widetilde{J}(\overline{K}))$ be the p-adic Tate module of $\widetilde{J}_K$. Thus, if $d = dim(G/S)$, then $V_{\widetilde{J}}$ is a free $\mathbf{Z}_p$-module of rank $r + 2d$. Note that the étale sheaf P may also be regarded as a free abelian group of rank r equipped with a Γ_K-action. Let $P_{\mathbf{Z}_p} \stackrel{\mathrm{def}}{=} P \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Then, according to Corollary 7.3 of [FC], Chapter III, we have an exact sequence of Γ_K-modules:

$$0 \to V_{\widetilde{J}} \to V_J \to P_{\mathbf{Z}_p} \to 0$$

Let $V_G \stackrel{\mathrm{def}}{=} Hom(\mathbf{Q}_p/\mathbf{Z}_p, G(\overline{K}))$ (respectively, $V_T \stackrel{\mathrm{def}}{=} Hom(\mathbf{Q}_p/\mathbf{Z}_p, T(\overline{K}))$) be the p-adic Tate module of G_K (respectively, T_K). Then $V_{\widetilde{J}}$ itself fits into another exact sequence of Γ_K-modules:

$$0 \to V_T \to V_{\widetilde{J}} \to V_G \to 0$$

Next let us consider the Γ_K-module V_G. Let a be the p-rank of the abelian variety $G_k \stackrel{\mathrm{def}}{=} G \otimes_A k$ over k. Then, as is well-known, there is a free $\mathbf{Z}_p$-module $V_{G^{ord}}$ of rank a equipped with an unramified Γ_K-action such that we have exact sequences of Γ_K-modules:

$$0 \to V'_G \to V_G \to V_{G^{ord}} \to 0$$

and

$$0 \to V^{\vee}_{G^{ord}}(1) \to V'_G \to V_{G^{ss}} \to 0$$

Here, the "(1)" is a Tate twist, and by "unramified action" we mean that Γ_K^{unr} acts trivially. Thus, $V_{G^{ss}}$ is the "supersingular part" of the representation V_G. Let us denote by $V_{J^{et}}$ the quotient of V_J by $Ker(V_{\widetilde{J}} \to V_{G^{ord}}) \subseteq V_{\widetilde{J}} \subseteq V_J$. Thus, $V_{J^{et}}$ is a free $\mathbf{Z}_p$-module of rank $r + a$, equipped with a natural unramified Γ_K-action. Let $V_{J^{mt}} \subseteq V_{\widetilde{J}} \subseteq V_J$ be the inverse image of $V^{\vee}_{G^{ord}}(1)$ ($\subseteq V_G$) under the projection $V_{\widetilde{J}} \to V_G$. Note that the submodule $V_{J^{mt}} \subseteq V_J$ is dual to the quotient $V_J \to V_{J^{et}}$ under the bilinear form on V_J arising from the canonical polarization of J.

Now we would like to take a closer look at the Γ_K-module $V_{G^{ss}}$. First, recall that the Γ_K-module V_G is *crystalline*. This fact is well-known from the general theory of Galois representations arising from p-adic étale cohomology groups of varieties (see, e.g., [FC], Chapter VI, §6, for a brief review of this theory, as well as a list of further references). Let

$K_0 \subseteq K$ be the maximal unramified extension of $\mathbf{Q}_p$ in K. Then V_G is (say, covariantly) "associated" to a filtered module with Frobenius action $(M, F^{\cdot}(M), \Phi_M)$, where M is a K_0-vector space, $\Phi_M : M \to M$ is a semilinear automorphism, and $F^{\cdot}(M)$ is a filtration of $M \otimes_{K_0} K$. It then follows from the above exact sequences that $V_{G^{ss}}$ is also crystalline, and is associated to a filtered module with Frobenius action $(M^{ss}, F^{\cdot}(M^{ss}), \Phi_{M^{ss}})$ which is a subquotient of $(M, F^{\cdot}(M), \Phi_M)$. Moreover, since $V_{G^{ss}}$ was constructed as the "supersingular part of V_G," it follows that the Frobenius action $\Phi_{M^{ss}}$ on M^{ss} is *topologically nilpotent*. Now we have the following crucial

Lemma 8.1 : *Any Γ_K^{unr}-equivariant $\mathbf{Z}_p$-linear morphism $\psi : V_{\widetilde{J}} \to \mathbf{Z}_p$ (where $\mathbf{Z}_p$ is equipped with the trivial Γ_K^{unr}-action) factors through the quotient $V_{\widetilde{J}} \to V_{G^{ord}}$.*

Proof: First, note that since T is a torus, and K^{unr} contains only finitely many p^{th} power roots of unity, the restriction of ψ to V_T must be zero. Thus, ψ factors through V_G. Denote the resulting morphism $V_G \to \mathbf{Z}_p$ by ψ'. By the same argument, the restriction of ψ' to $V_{G^{ord}}^{\vee}(1)$ must be zero. Thus, we obtain a Γ_K^{unr}-equivariant morphism $\psi_{G^{ss}} : V_{G^{ss}} \to \mathbf{Z}_p$. But since $V_{G^{ss}}$ is *crystalline*, it follows from the basic theory of crystalline representations that $\psi_{G^{ss}}$ defines a Frobenius-equivariant, K^{unr}-linear morphism $\psi_{M^{ss}} : M^{ss} \otimes_{K_0} K^{unr} \to K^{unr}$. (Here, K^{unr} is equipped with the trivial Frobenius action.) On the other hand, it follows from the topological nilpotence of $\Phi_{M^{ss}}$ that $\psi_{M^{ss}}$ must be zero. Thus, $\psi_{G^{ss}}$ is also zero. This completes the proof. ○

Now let us return to the abelian covering $Y_K \to X_K$ of degree p discussed above. Suppose that this covering satisfies condition (*). Then there exists a covering $Z_{K^{unr}} \to X_{K^{unr}}$ as in (*). Moreover, since $V_J = Hom(\Delta_{X_K}, \mathbf{Z}_p(1))$, it follows that $Z_{K^{unr}} \to X_{K^{unr}}$ defines a $\mathbf{Z}_p$-linear morphism $\kappa : \mathbf{Z}_p(1) \to V_J$ which is Γ_K^{unr}-equivariant. By taking the dual to κ (and using the fact that the polarization of J gives an isomorphism of V_J with its Cartier dual), we obtain a Γ_K^{unr}-equivariant morphism $\kappa^{\vee} : V_J \to \mathbf{Z}_p$. Thus, by Lemma 8.1, it follows that $\kappa^{\vee}$ factors through $V_{J^{et}}$. In particular, κ factors through $V_{J^{mt}}$. On the other hand, it is clear that $\mathbf{Z}_p$-coverings arising from morphisms $\mathbf{Z}_p(1) \to V_{J^{mt}}$ extend to étale coverings of $X_{A^{unr}}$. Thus, in particular, it follows that $Y_K \to X_K$ extends to a finite étale covering $Y \to X$. Thus, we have proven the following result:

Lemma 8.2 : *Let $Y_K \to X_K$ be an abelian étale covering of degree p. Let us assume that Y_K is geometrically connected over K. Then $Y_K \to X_K$ extends to a finite étale covering $Y \to X$ if and only if $Y_K \to X_K$ satisfies condition (*) above.*

Next, we would like to consider more general coverings $Y_K \to X_K$. We begin with the following

Lemma 8.3 : *Let $\psi_K : Y_K \to X_K$ be a finite étale covering such that Y_K is geometrically connected over K. Suppose, moreover, that Y_K arises from a stable curve $Y \to S$ over S. Then ψ_K extends to a proper surjective morphism $\psi : Y \to X$.*

Proof: Note that if $\psi \otimes_A K = \psi_K$, then ψ will automatically be proper and dominant, hence surjective. Thus, it suffices to show the existence of such a ψ. Let us first show the existence of such a ψ under the additional assumption that X_k *is sturdy*. Now it follows from the general theory of stable curves that there exists a semi-stable curve $Y' \to S$ equipped with a birational (blow-up) morphism $Y' \to Y$ such that Y' is *regular*. Moreover, it follows from the general theory of "elimination of indeterminacy" for regular two-dimensional schemes ([Lipman], Lemma 3.1) that the rational map from Y' to X defined by ψ_K becomes a morphism over some Y'', where Y'' is obtained from Y' by blowing up points. Denote the resulting morphism by $\psi'' : Y'' \to X$. On the other hand, for $E \subseteq Y''_k \subseteq Y''$ such that $E \cong \mathbf{P}^1_k$, it follows from the sturdiness assumption on X_k that $\psi''|_E$ is *constant*. It thus follows that ψ'' factors through Y, as desired.

Now let us prove the result in general (without the assumption that X_k is sturdy). First, observe that by descent, it suffices to prove the result after replacing K by a finite extension of K. Thus, we may assume that there exists a Galois admissible covering $\theta_X : X' \to X$ (over S) such that X'_k *is sturdy*. Pulling back $(\theta_X)_K$ to Y_K, one sees that one obtains a finite étale covering $Y'_K \to Y_K$ which extends to a Galois multi-admissible covering $\theta_Y : Y' \to Y$. On the other hand, if we pull-back ψ_K to X', we obtain a morphism $\psi'_K : Y'_K \to X'_K$. Moreover, since X'_k is *sturdy*, it follows from the first paragraph of this proof that ψ'_K extends to a morphism $\psi' : Y' \to X'$. Thus, by composing ψ' with θ_X, we obtain a morphism $\phi : Y' \to X$ whose restriction to Y'_K is given by taking the composing of $(\theta_Y)_K : Y'_K \to Y_K$ with ψ_K. Now, let us note that since $\theta_Y : Y' \to Y$ is an admissible covering, it is, in particular, *finite*. Thus, if G is the Galois group of Y'_K over Y_K, it follows (from Zariski's main theorem) that $\mathcal{O}_Y$ is obtained from $\mathcal{O}_{Y'}$ by taking G-invariants. Hence, $\phi : Y' \to X$ factors through a morphism $Y \to X$ which extends ψ_K, as desired. ○

Remark: As pointed out by the referee, one can also prove this Lemma (without reducing to the sturdy case) by considering the number of points at which a (-1)-curve in Y'' (notation of the above proof) intersects other irreducible components of the special fiber of Y''. The author finds the proof involving sturdiness to be more transparent, but this is a matter of taste.

Let $\psi_K : Y_K \to X_K$ be a finite Galois étale covering (with Galois group G) such that Y_K is geometrically connected over K, and Y_K has a stable extension $Y \to S$ over S. Let us also assume that ψ_K satisfies the following condition:

> (†) Let $H \subseteq G$ be any subgroup with $H \cong \mathbf{Z}/p\mathbf{Z}$, and let $Y_K \to Z_K$ be the subcovering corresponding to H. Then Z_K has a stable extension $Z \to S$ over S, and, moreover, $Y_K \to Z_K$ extends to a finite étale covering $Y \to Z$ over S.

Let $\wp \in Y$ be a height one prime arising from an irreducible component of the special fiber Y_k. Then we claim that the inertia subgroup $I_\wp \subseteq G$ is trivial. (Here, by the inertia subgroup, we mean the subgroup of elements of G that fix $\wp$ and act trivially on $k(\wp)$.)

Indeed, first note that since Y is smooth over S at $\wp$, there cannot be any tame ramification, so $I_\wp$ must be a p-group. If $I_\wp$ is nontrivial, then it contains a subgroup $H \subseteq I_\wp$ such that $H \cong \mathbf{Z}/p\mathbf{Z}$. But then the statement that H acts trivially on $k(\wp)$ clearly contradicts condition (†). Thus, $I_\wp$ must be trivial, as claimed.

Now let $\wp \in X$ be a height one prime arising from an irreducible component of the special fiber X_k. It follows from the preceding paragraph that $\wp$ is unramified in $K(Y)$ (the function field of Y). Thus, if we let Y' be the normalization of X in Y_K, it follows that $Y' \to X$ is finite over all of X, and étale over the complement of the nodes of X (by purity). Then it follows by the same argument as that used in the proof of Lemma 3.12 of [Mzk] that $Y' \to X$ must be an admissible covering. In particular, Y' is a stable curve over S, so $Y = Y'$. Thus, it follows from Lemma 8.2 and the above discussion that we have proven the following:

Proposition 8.4: *The admissible quotient $\Pi_{X_K} \to \Pi_X^{adm}$ can be recovered entirely group-theoretically from $\Pi_{X_K} \to \Gamma_K$.*

Proof: Indeed, clearly (by replacing K by a finite, tamely ramified extension of K) it suffices to show that given an open normal subgroup $H \subseteq \Pi_{X_K}$ that surjects onto Γ_K, the issue of whether or not the resulting covering $Y_K \to X_K$ is pre-admissible can be settled group-theoretically. But, by the above discussion, $Y_K \to X_K$ is pre-admissible if and only if Y_K and X_K admit stable extensions over S (which is well-known, by the criterion of Serre-Tate, to be a group-theoretic condition), and, moreover, (†) is satisfied. On the other hand, by Lemma 8.2, (†) is also a group-theoretic condition. This completes the proof. ○

Section 9: Consequences for Curves over Local Fields

We retain the notation of the preceding Section. Thus, in particular, we have a stable log-curve $X^{log} \to S^{log}$. Let $K_\infty \subseteq \overline{K}$ be the maximal tamely ramified extension of K; let $\Gamma_K^{tm} \stackrel{\text{def}}{=} Gal(K_\infty/K)$. Thus, we have a natural surjection $\Gamma_K \to \Gamma_K^{tm}$. We saw in Proposition 8.4 that one can recover the admissible quotient $\Pi_{X_K} \to \Pi_X^{adm}$ (hence also the surjection $\Pi_X^{adm} \to \Gamma_K^{tm}$) group-theoretically from $\Pi_{X_K} \to \Gamma_K$. Suppose that $(X')^{log} \to S^{log}$ is also a stable log-curve over S^{log}. Let us denote (as in Theorem 7.2) by

$$Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K})$$

the set of equivalence classes of isomorphisms $\Pi_{X_K} \to \Pi_{X'_K}$ that are compatible with the surjections to Γ_K, modulo inner automorphisms arising from elements of the geometric fundamental group $\Delta_{X_K} \subseteq \Pi_{X_K}$. Finally, let us denote by s^{log} the special point $Spec(k) \subseteq S$ equipped with the log structure pulled back from S^{log}. We would like to use Proposition 8.4 and Theorem 7.2 to obtain information on the special fiber of X^{log}, but before we can do this, we need to clear up some technical issues concerning the notion of "degree."

Thus, suppose that we are given an isomorphism $\alpha : \Pi_{X_K} \to \Pi_{X'_K}$ over Γ_K. By Proposition 8.4, this isomorphism induces an isomorphism $\alpha^{adm} : \Pi_X^{adm} \to \Pi_{X'}^{adm}$ over $\Pi_{s^{log}}$. Let us denote the RT-degree of α^{adm} (as discussed in Section 7) by d_{RT}. We would like to show that in this case (i.e., when α^{adm} arises from α), d_{RT} is automatically equal to 1. In order to do this, we will have to look at several other notions of "degree" and show that they all coincide. From this, it will follow that $d_{RT} = 1$.

First, let us consider the p-adic fundamental class of X_K:

$$\eta_{X,p} \in H^2(\Delta_{X_K}, \mathbf{Z}_p(1))$$

(i.e., the first Chern class of a line bundle of degree one on $X_{\overline{K}}$). We also have the l-adic fundamental class of X_K:

$$\eta_{X,l} \in H^2(\Delta_{X_K}, \mathbf{Z}_l(1))$$

for primes l different from p. Also, we have similar classes $\eta_{X',p}$ and $\eta_{X',l}$ for X'. Let us denote by

$$d_p \in \mathbf{Z}_p{}^\times \quad (\textit{respectively, } d_l \in \mathbf{Z}_l{}^\times)$$

the unique unit such that $\eta_{X,p}$ (respectively, $\eta_{X,l}$) is taken to $d_p \cdot \eta_{X',p}$ (respectively, $d_l \cdot \eta_{X',l}$). Now we propose to prove that

$$d_p = d_{RT} = d_l \in \mathbf{Q}$$

Since $d_{RT} \in p^{\mathbf{Z}}$, and $d_p \in \mathbf{Z}_p{}^\times$, it will follow immediately that $d_{RT} = 1$.

Note that from the point of view of showing $d_p = d_{RT} = d_l$, we may always replace X_K by some finite étale covering of X_K, since we know how the fundamental class behaves with respect to coverings (namely, it simply gets multiplied by the degree of the covering). We shall see below in the proof of Theorem 9.2 that by replacing X_K by such a covering, we may assume that X_k *is singular, and, moreover, that its graph is not a tree.* We shall assume this until the end of the following proof and statement of Lemma 9.1.

Let us first consider the l-adic theory. Let $H_{\mathbf{Z}}$ be the first singular cohomology group of the *dual graph* of $X_{\overline{k}}$. (Recall that the *dual graph* is the graph whose vertices (respectively, edges) are the irreducible components (respectively, nodes) of $X_{\overline{k}}$.) Note that $H_{\mathbf{Z}}$ is equipped with a natural $Gal(\overline{k}/k)$-action, hence a natural Γ_K-action. Let $H_{\mathbf{Z}_l} \stackrel{\text{def}}{=} H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Z}_l$. Now in Section 8, we considered the p-adic Tate module of J_K, but since the theory of [FC] applies to the l-adic Tate modules as well, we can define similar exact sequences to those discussed in Section 8 in the l-adic case. Thus, we let V_J^l be the l-adic Tate module of J_K. Moreover, we obtain a Γ_K-equivariant quotient $V_J^l \to P_{\mathbf{Z}_l} \stackrel{\text{def}}{=} P \otimes_{\mathbf{Z}} \mathbf{Z}_l$, as well as a submodule $V^l_T \subseteq V^l_J$. Here, we shall not regard $P_{\mathbf{Z}_l}$ as the tensor product of some $\mathbf{Z}$-module P with $\mathbf{Z}_l$, but rather *solely in its capacity as a quotient of* V^l_J. By the theory of [FC], Chapter III, it follows that as Γ_K-modules, we have a natural isomorphism

$$\zeta_l : H_{\mathbf{Z}_l}(1) \cong V^l_T$$

We shall identify these two modules via ζ_l. This identification is justified by the following observation: From our theory of irreducible components and nodes (Sections 1 and 5), it follows that α (or, more precisely, α^{adm}) induces a natural isomorphism $\alpha_{H_{\mathbf{Z}}} : H_{\mathbf{Z}} \cong H'_{\mathbf{Z}}$ (where $H'_{\mathbf{Z}}$ is the object obtained from X' that corresponds to $H_{\mathbf{Z}}$). Moreover, if we use primes to denote objects obtained from X' that correspond to various objects obtained from X, then we have a commutative diagram

$$\begin{array}{ccc} H_{\mathbf{Z}_l}(1) & \xrightarrow{\zeta_l} & V^l_T \\ \downarrow{\scriptstyle \alpha_{H_{\mathbf{Z}}}\otimes\mathbf{Z}_l(1)} & & \downarrow \\ H'_{\mathbf{Z}_l}(1) & \xrightarrow{\zeta'_l} & (V^l_T)' \end{array}$$

where the vertical arrows are those naturally induced by α. It is the commutativity of this diagram that justifies the identification (based on ζ_l) proposed above.

Now we have an exact sequence $0 \to V^l_{\tilde{J}} \to V^l_J \to P_{\mathbf{Z}_l} \to 0$ (analogous to the p-adic version of this exact sequence which was reviewed in Section 8). Clearly, it may be recovered group-theoretically from $\Pi_{X_K} \to \Gamma_K$. In particular, $P_{\mathbf{Z}_l}$ may be recovered group-theoretically from $\Pi_{X_K} \to \Gamma_K$. Next recall from the theory of [FC], Chapter III, that there is a $\mathbf{Z}$-bilinear pairing $B : H_{\mathbf{Z}} \times Hom_{\mathbf{Z}}(P, \mathbf{Z}) \to K^\times/A^\times = \mathbf{Z}$ which induces an injection of modules $H_{\mathbf{Z}} \hookrightarrow P$, which becomes an isomorphism over $\mathbf{Q}$. By Corollary 7.3 of [FC], Chapter III, by considering the various extension classes involved and applying Kummer theory, the pairing

$$B_l \stackrel{\text{def}}{=} B \otimes \mathbf{Z}_l : H_{\mathbf{Z}_l} \times P^\vee_{\mathbf{Z}_l} \to \mathbf{Z}_l$$

may be recovered from the extension $0 \to V^l_{\tilde{J}} \to V^l_J \to P_{\mathbf{Z}_l} \to 0$. This pairing determines an injection $H_{\mathbf{Z}_l} \hookrightarrow P_{\mathbf{Z}_l}$, which becomes an isomorphism over $\mathbf{Q}_l$. Thus, we may recover (group-theoretically from $\Pi_{X_K} \to \Gamma_K$) the injection $H_{\mathbf{Z}} \hookrightarrow P_{\mathbf{Z}_l}$.

Now, let us note that $V^l_J = Hom(\Delta_{X_K}, \mathbf{Z}_l(1)) = H^1_{\acute{e}t}(\Delta_{X_K}, \mathbf{Z}_l(1))$. Thus, to summarize, we have an injection

$$H_{\mathbf{Z}_l}(1) \hookrightarrow H^1_{\acute{e}t}(\Delta_{X_K}, \mathbf{Z}_l(1))$$

as well as a surjection (obtained by using the isomorphism $H_{\mathbf{Q}_l} \cong P_{\mathbf{Q}_l}$ derived above from B)

$$H^1_{ét}(\Delta_{X_K}, \mathbf{Q}_l) \to H_{\mathbf{Q}_l}(-1)$$

Now, by using the cup product operation in group cohomology, we obtain

$$H_{\mathbf{Q}_l} \otimes_{\mathbf{Q}_l} H_{\mathbf{Q}_l} \to H^2_{ét}(\Delta_{X_K}, \mathbf{Q}_l(1))$$

Composing this with the natural inclusion $H_{\mathbf{Q}} \otimes H_{\mathbf{Q}} \hookrightarrow H_{\mathbf{Q}_l} \otimes_{\mathbf{Q}_l} H_{\mathbf{Q}_l}$, we thus obtain

$$\mu_l : H_{\mathbf{Q}} \otimes H_{\mathbf{Q}} \to H^2_{ét}(\Delta_{X_K}, \mathbf{Q}_l(1))$$

Moreover, it follows from the theory of [FC], Chapter III, that there exists a nondegenerate bilinear form $< -, - >: H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$ such that for all $h_1, h_2 \in H_{\mathbf{Q}}$, we have $\mu_l(h_1, h_2) =< h_1, h_2 > \eta_{X,l}$. Here, the bilinear form $< -, - >$ is *independent of* l.

Let us denote analogous objects associated to X' by means of primes. Thus, it follows immediately that we have a commutative diagram

$$\begin{array}{ccccc}
H_{\mathbf{Q}} \otimes H_{\mathbf{Q}} & \xrightarrow{\mu_l} & H^2_{ét}(\Delta_{X_K}, \mathbf{Q}_l(1)) & \xleftarrow{\cdot\eta_{X,l}} & \mathbf{Z}_l \\
\downarrow \alpha_{H_{\mathbf{Q}}} \otimes \alpha_{H_{\mathbf{Q}}} & & \downarrow H^2_{ét}(\alpha) & & \downarrow d_l \\
H'_{\mathbf{Q}} \otimes H'_{\mathbf{Q}} & \xrightarrow{\mu'_l} & H^2_{ét}(\Delta_{X'_K}, \mathbf{Q}_l(1)) & \xleftarrow{\cdot\eta_{X',l}} & \mathbf{Z}_l
\end{array}$$

whose vertical morphisms are those naturally induced by α. In particular, we obtain that (up to identifying $H_{\mathbf{Q}}$ with $H'_{\mathbf{Q}}$ via $\alpha_{H_{\mathbf{Q}}}$) $d_l\cdot < -, - >=< -, - >'$. Moreover, the l-adic theory of the last few paragraphs goes through entirely without change in the p-adic case, as well. Thus, we obtain $d_p\cdot < -, - >=< -, - >'$. In particular, (since the graph of X_k is not a tree, $H_{\mathbf{Z}} \neq 0$, so) $d_p = d_l \in \mathbf{Q}$ for every prime l different from p. Since $d_l \in {\mathbf{Z}_l}^{\times}$ and $d_p \in {\mathbf{Z}_p}^{\times}$, we thus obtain that $d_p = d_l = \pm 1$.

On the other hand, the relationship between d_l and d_{RT} can be established as follows. First note that, as we saw in Section 5, there is a natural *combinatorial* (perfect) duality (unrelated to the duality defined by the form B) between $H_{\mathbf{Z}_l}$ and $P_{\mathbf{Z}_l}$. (Indeed, in Section 5, the discussion concerning "L^r" and "K_Y" shows that $P_{\mathbf{Z}_l}$ is the first *homology* group of the dual graph of $X_{\overline{k}}$ (with $\mathbf{Z}_l$-coefficients), whereas $H_{\mathbf{Z}_l}$ is – by definition – the first *cohomology* group of the dual graph of $X_{\overline{k}}$ (with $\mathbf{Z}_l$-coefficients).) Let us denote this duality by $D^{com}_l : H_{\mathbf{Z}_l} \otimes_{\mathbf{Z}_l} P_{\mathbf{Z}_l} \to \mathbf{Z}_l$. Moreover, relative to the natural combinatorial isomorphisms $H_{\mathbf{Z}_l} \cong H'_{\mathbf{Z}_l}$ and $P_{\mathbf{Z}_l} \cong P'_{\mathbf{Z}_l}$ (obtained from considering the isomorphisms between the graphs of $X_{\overline{k}}$ and $X'_{\overline{k}}$ induced by α^{adm}), we have $D^{com}_l = (D^{com}_l)'$ (since "everything is combinatorial"). Thus, we shall identify $H_{\mathbf{Z}_l}$, $P_{\mathbf{Z}_l}$ and D^{com}_l with their primed counterparts in what follows.

Now recall the natural inclusion $H_{\mathbf{Z}_l}(1) \subseteq H^1(\Delta_{X_K}, \mathbf{Z}_l(1))$, and the natural surjection $H^1(\Delta_{X_K}, \mathbf{Z}_l) \to P_{\mathbf{Z}_l}(-1)$. Using the cup product in group cohomology, we thus obtain a pairing $H_{\mathbf{Z}_l}(1) \otimes P_{\mathbf{Z}_l}(-1) = H_{\mathbf{Z}_l} \otimes P_{\mathbf{Z}_l} \to \mathbf{Z}_l \cdot \eta_{X,l}$. It is tautological that this pairing is simply $D_l^{com}(-,-) \cdot \eta_{X,l}$. On the other hand, by recalling the definition of d_{RT} (in terms of inertia groups) and the fact that the quotient $V_J^l \to P_{\mathbf{Z}_l}$ pertains to the inertia, it follows immediately that we have a commutative diagram

$$\begin{array}{ccccc} H_{\mathbf{Z}_l}(1) \otimes P_{\mathbf{Z}_l}(-1) & \stackrel{\cup}{\longrightarrow} & H^2_{\acute{e}t}(\Delta_{X_K}, \mathbf{Q}_l(1)) & \stackrel{\cdot \eta_{X,l}}{\longleftarrow} & \mathbf{Z}_l \\ \downarrow id_{H_{\mathbf{Z}_l}} \otimes (d_{RT} \cdot id_{P_{\mathbf{Z}_l}}) & & \downarrow H^2_{\acute{e}t}(\alpha) & & \downarrow d_l \\ H_{\mathbf{Z}_l}(1) \otimes P_{\mathbf{Z}_l}(-1) & \stackrel{\cup}{\longrightarrow} & H^2_{\acute{e}t}(\Delta_{X'_K}, \mathbf{Q}_l(1)) & \stackrel{\cdot \eta_{X',l}}{\longleftarrow} & \mathbf{Z}_l \end{array}$$

It thus follows that $d_l = d_{RT}$. Since d_{RT} is positive, we thus obtain the following

Lemma 9.1 : *We have $d_p = d_l = d_{RT} = 1$. In particular, $\eta_{X,p}$ (respectively, $\eta_{X,l}$) is taken to $\eta_{X',p}$ (respectively, $\eta_{X',l}$) by α.*

We are now ready to prove the following result:

Theorem 9.2: *Let K be a finite extension of $\mathbf{Q}_p$; let $A \subseteq K$ be its ring of integers; and let S^{log} be $Spec(A)$ equipped with the log structure defined by the closed point. Let $X^{log} \to S^{log}$ and $(X')^{log} \to S^{log}$ be stable log-curves (equipped with base-points $x \in X(K)$ and $x' \in X'(K)$). Then there exists a morphism*

$$\xi_k : Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K}) \to Isom_{s^{log}}(X_k^{log}, (X')_k^{log})$$

that makes the following diagram commute

$$\begin{array}{ccc} Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K}) & \longrightarrow & Isom^{GO}_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_{X'}^{adm}) \\ \downarrow \xi_k & & \downarrow id \\ Isom_{s^{log}}(X_k^{log}, (X')_k^{log}) & \longrightarrow & Isom^{GO}_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_{X'}^{adm}) \end{array}$$

(where the horizontal morphisms are the natural ones).

Proof: This Theorem follows immediately from Theorem 7.2 (together with Proposition 8.4 and Lemma 9.1) if it is the case that *at least one of X_k or X'_k is not smooth over k*. Thus, it remains to consider the case when both X_k and X'_k are smooth over k. In this case, let us (after possibly replacing K by a finite extension of K, which won't affect the final result) assume that there exists a finite, abelian, étale covering $Y_K \to X_K$ of degree p such that Y_K extends to a stable curve over S. Let G be the Galois group of Y_K over X_K. By Lemma 8.3, $Y_K \to X_K$ extends to a morphism $Y \to X$. Moreover, by the review of the structure of V_J given in Section 8, it is easy to see that one can always choose $Y_K \to X_K$ so that $Y \to X$ is *not étale*.

Then I claim that Y_k is singular. Indeed, suppose that Y_k is smooth over k. Let $\wp \in X$ be the prime defined by the special fiber. Then $\wp$ must be ramified in $K(Y)$ (the function field of Y), for if it were not ramified, it is easy to see that $Y \to X$ would be étale. On the other hand, since X and Y are both smooth over A, the statement that $\wp$ is ramified in $K(Y)$ means that $K(Y_k)$ is an inseparable extension of $K(X_k)$ (of degree p). But this means that the genus of Y_k (and hence of Y_K) is the same as that of X_k. But since $Y_K \to X_K$ is étale of degree p, this is absurd. This completes the proof of the claim.

Now suppose that we are given an isomorphism $\phi_\Pi : \Pi_{X_K} \to \Pi_{X'_K}$ that respects the surjections to Γ_K. Then the covering $Y_K \to X_K$ corresponds, via ϕ, to some covering $Y'_K \to X'_K$. Now we can apply the part of the Theorem that has already been established to Y^{log} and $(Y')^{log}$. We thus obtain an isomorphism $\phi_Y : Y_k^{log} \cong (Y')_k^{log}$ (over s^{log}). Now it is easy to see that the irreducible component C of Y_k^{log} that maps finitely to X_k^{log} can be characterized group-theoretically as the unique component such that abelian étale coverings of X_k^{log} of degree l (where l is prime to p) pull back to nontrivial coverings of C. Moreover, note that nonsplit admissible coverings of X_k^{log} pull-back to nonsplit admissible coverings of Y_k^{log}. Thus, one sees that ϕ_Y induces a unique isomorphism $\phi_X : X_k^{log} \cong (X')_k^{log}$ such that the morphism induced by ϕ_X on Π^{adm}'s is compatible with the morphism induced on Π^{adm}'s by ϕ_Π. This completes the proof of the Theorem. ○

Once Theorem 9.2 is in hand, the next natural step is to try to show that the isomorphism of $X_k^{log} \cong (X')_k^{log}$ obtained from some $\phi_\Pi : \Pi_{X_K} \to \Pi_{X'_K}$ lifts to an isomorphism $X^{log} \cong (X')^{log}$ over S^{log}. Unfortunately, we do not succeed in doing this in general. The problem is as follows: Let $\mathcal{G}_K$ be the p-divisible group over K defined by the Γ_K-module $Hom(\mathbf{Q}_p/\mathbf{Z}_p, V_J)$. Then the exact sequence $0 \to V_{\widetilde{J}} \to V_J \to P \otimes (\mathbf{Q}_p/\mathbf{Z}_p) \to 0$ of Γ_K modules gives rise to an exact sequence of p-divisible groups over K:

$$0 \to \widetilde{\mathcal{G}}_K \to \mathcal{G}_K \to \mathcal{G}_K^P \to 0$$

Moreover, it follows from the theory of [FC], Chapter III, that $\widetilde{\mathcal{G}}_K$ and $\mathcal{G}_K^P$ extend, respectively, to p-divisible groups $\widetilde{\mathcal{G}}$ and $\mathcal{G}^P$ over S. (Here, $\widetilde{\mathcal{G}}$ is the p-divisible group obtained from the semi-abelian scheme $\widetilde{J}$.) For readers used to this language, we note that it also follows from the theory of [FC], Chapter III, that $\mathcal{G}_K$ extends to a *log p-divisible group* $\mathcal{G}^{log}$ over S^{log}. Finally, we also have primed objects $\widetilde{\mathcal{G}}'$, $(\mathcal{G}^P)'$, etc. arising from $(X')^{log}$.

Now let us observe that any isomorphism $\phi_\Pi : \Pi_{X_K} \to \Pi_{X'_K}$ that respects the surjections to Γ_K defines an isomorphism $\phi_{\mathcal{G}_K} : \mathcal{G}_K \cong \mathcal{G}'_K$ of p-divisible groups over K. Note that $\phi_{\mathcal{G}_K}$ maps $\widetilde{\mathcal{G}}_K$ into $\widetilde{\mathcal{G}}'_K$, hence induces an isomorphism $\phi_{\widetilde{\mathcal{G}}_K} : \widetilde{\mathcal{G}}_K \cong \widetilde{\mathcal{G}}'_K$. Moreover, it follows from a theorem of Tate ([Tate]) that $\phi_{\widetilde{\mathcal{G}}}$ extends uniquely to an isomorphism $\phi_{\widetilde{\mathcal{G}}} : \widetilde{\mathcal{G}} \cong \widetilde{\mathcal{G}}'$. If we tensor this isomorphism with k, we thus obtain an isomorphism

$$\phi_{\widetilde{\mathcal{G}}_k} : \widetilde{\mathcal{G}}_k \cong \widetilde{\mathcal{G}}'_k$$

On the other hand, note that $\widetilde{\mathcal{G}}_k$ is the p-divisible group associated to $\widetilde{J}_k$. Moreover, since $\widetilde{J}_k$ is the identity component of the Picard scheme of X_k, the isomorphism $\phi_X : X_k^{log} \cong (X')_k^{log}$ thus induces an isomorphism $Pic^0(X_k) = \widetilde{J}_k \cong Pic^0(X'_k) = \widetilde{J}'_k$, hence an isomorphism

$$\phi_{Pic^0(X)} : \widetilde{\mathcal{G}}_k \cong \widetilde{\mathcal{G}}'_k$$

Then the following fundamental question arises:

Question 9.3: *Is* $\phi_{\widetilde{\mathcal{G}}_k}$ *equal to* $\phi_{Pic^0(X)}$*?*

Remark: If one can prove that the answer to Question 9.3 is affirmative, then it follows formally from the techniques discussed in this paper that one can prove a version of the Grothendieck Conjecture for hyperbolic curves over *local fields*. Unfortunately, however, because we are only able to settle Question 9.3 in the affirmative under the additional assumption that the abelian variety G_k is *ordinary* (see Lemma 9.4 below), we are only able to prove (in the context of this paper) a relatively weak version of the Grothendieck Conjecture for hyperbolic curves over local fields (see Theorem 9.7 below). In fact, a very strong local version of the Grothendieck Conjecture is proven in [Mzk2]. The existence of such a local result implies *a posteriori* that the answer to Question 9.3 is always affirmative. Nevertheless, it is still of interest to what extent Question 9.3 can be settled within the context of the present paper, and so we proceed to do this below.

Now we would like to settle Question 9.3 in the affirmative under the assumption that G_k *is an ordinary abelian variety.*

Lemma 9.4 : *Suppose that in the exact sequence of groups* $0 \to T \to \widetilde{J} \to G \to 0$ *associated to the Jacobian* J *of* X*, the abelian variety* G_k *is ordinary. Then we have* $\phi_{\widetilde{\mathcal{G}}_k} = \phi_{Pic^0(X)}$.

Proof: Indeed, in this case, the p-divisible group $\widetilde{\mathcal{G}}_k$ admits a canonical splitting

$$\widetilde{\mathcal{G}}_k = \widetilde{\mathcal{G}}_k^{mt} \oplus \widetilde{\mathcal{G}}_k^{et}$$

into multiplicative and étale parts. Since both $\phi_{\widetilde{\mathcal{G}}_k}$ and $\phi_{Pic^0(X)}$ clearly respect this splitting, it suffices to show that they agree on each of the direct summands. Note, moreover, that $\widetilde{\mathcal{G}}_k^{mt}$ lifts naturally to a multiplicative p-divisible group $\widetilde{\mathcal{G}}^{mt} \subseteq \widetilde{\mathcal{G}}$. Also, the submodule of $V_{\widetilde{J}} \subseteq V_J$ defined by $\widetilde{\mathcal{G}}_K^{mt}$ is simply $V_{J^{mt}}$. But $V_{J^{mt}} \subseteq V_J = H^1(\Delta_{X_K}, \mathbf{Z}_p(1))$ is the portion of $H^1(\Delta_{X_K}, \mathbf{Z}_p(1))$ that arises from étale coverings of X (or, equivalently, X_k). Thus, the fact that $\phi_{\widetilde{\mathcal{G}}_k} = \phi_{Pic^0(X)}$ on $\widetilde{\mathcal{G}}_k^{mt}$ follows from the commutativity of the diagram in the statement of Theorem 9.2.

Let $\mathcal{G}$ be the p-divisible group over S arising from the abelian scheme $G \to S$. Thus, we have a surjection $\widetilde{\mathcal{G}} \to \mathcal{G}$. Moreover, $\phi_{\widetilde{\mathcal{G}}_k}$ and $\phi_{Pic^0(X)}$ induce isomorphisms $\mathcal{G}_k \cong \mathcal{G}'_k$. Let us refer to an isomorphism of an object with its Cartier as dual as a *polarization of the object.* Now observe that we have two polarizations of $\mathcal{G}_k$, one arising from the polarization of V_J, and the other arising from regarding $\widetilde{J}_k$ as $Pic^0(X_k)$. By the general theory of semi-abelian schemes (as discussed in [FC]), it follows that these polarizations of $\mathcal{G}_k$ coincide. Moreover, by Lemma 9.1, $\phi_{\widetilde{\mathcal{G}}_K}$ is compatible with the first of the two (coinciding) polarizations of $\mathcal{G}_k$, while, by definition, $\phi_{Pic^0(X)}$ is compatible with the second of the two (coinciding) polarizations of $\mathcal{G}_k$. In particular, the fact that $\phi_{\widetilde{\mathcal{G}}_k} = \phi_{Pic^0(X)}$ on $\widetilde{\mathcal{G}}_k^{mt}$ implies that $\phi_{\widetilde{\mathcal{G}}_k} = \phi_{Pic^0(X)}$ on $\widetilde{\mathcal{G}}_k^{et}$, hence on all of $\widetilde{\mathcal{G}}_k$. This completes the proof of the Lemma. ○

Now let us suppose that we are in a situation where $\phi_{\widetilde{\mathcal{G}}_k} = \phi_{Pic^0(X)}$. Thus, it follows that we have an isomorphism $\phi_{\widetilde{J}_k} : \widetilde{J}_k \cong \widetilde{J}'_k$ such that the resulting isomorphism on p-divisible groups lifts to an isomorphism $\phi_{\widetilde{\mathcal{G}}} : \widetilde{\mathcal{G}} \cong \widetilde{\mathcal{G}}'$ over S. By "Grothendieck-Messing theory" (see, e.g., [FC], Chapter I, §3, for a review), it thus follows that $\phi_{\widetilde{J}_k}$ extends to a unique isomorphism $\phi_{\widetilde{J}} : \widetilde{J} \cong \widetilde{J}'$ compatible with $\phi_{\widetilde{\mathcal{G}}}$. Moreover, one checks easily that the rest of the "semi-abelian degeneration data" for J (as in [FC], Chapter III, §2) is determined by the extension $0 \to V_{\widetilde{J}} \to V_J \to P_{\mathbf{Z}_p} \to 0$ (and its l-adic counterparts, for $l \neq p$), as well as other data that we have already seen to be group-theoretically characterizable. Thus, we conclude (by the natural categorical equivalences of [FC], Chapter III, Corollary 7.2) that we have an isomorphism

$$\phi_J : J \cong J'$$

which is uniquely determined by the condition that it is compatible with $\phi_{\mathcal{G}_K}$. Moreover, ϕ_J is compatible (by Lemma 9.1) with the canonical polarizations on J and J'. Thus, by Torelli's theorem ([Milne], Theorem 12.1), we conclude that there is an isomorphism

$$\psi : X \cong X'$$

such that the isomorphism induced by ψ on Jacobians is $\pm\phi_J$. Note that ψ always extends to a unique log-isomorphism $\psi^{log} : X^{log} \cong (X')^{log}$.

Definition 9.5: *Let us call X ordinary if G_k is an ordinary abelian variety. Let us call X equi-hyperelliptic if either (i) X_K is hyperelliptic; or (ii) X_k^{log} does not admit an s^{log}-automorphism that induces -1 on J_k.*

Note that if p is *odd*, then (ii) is equivalent to the condition that X_k^{log} not arise as a *logarithmic degeneration* of a smooth hyperelliptic curve. (Here, by "logarithmic degeneration," we mean that it arises as the special fiber of some generically smooth, generically hyperelliptic log-curve over a trait equipped with the log structure defined by the special point.) Indeed, that (ii) implies this condition is clear. On the other hand, suppose that this condition is satisfied, but that there exists an automorphism α of X_k^{log} that induces -1 on J_k. Thus, $\alpha^2 = id$. But then it is easy to see that by forming the quotient of X_k^{log} by the group $< 1, \alpha >$, we can exhibit X_k^{log} (up to adding some more marked points to X_k^{log} and modifying the log structure accordingly) as an admissible double covering of a stable log-curve curve of genus 0. In particular, X_k^{log} will then be a logarithmic degeneration of a smooth hyperelliptic curve, as desired.

Moreover, we have the following

Lemma 9.6 : *If X is equi-hyperelliptic, then we can choose ψ to be compatible with ϕ_J.*

Proof: Indeed, this is clear in case (i) (of the definition of "equi-hyperelliptic") since then the hyperelliptic involution of X_K (which induces -1 on the Jacobian) extends to an automorphism of X, so we can always adjust ψ accordingly. On the other hand, suppose that we are in case (ii), and that ψ is compatible with $-\phi_J$. Then ψ_k differs from the isomorphism $\phi_X : X_k^{log} \cong (X')_k^{log}$ by an automorphism α of X_k^{log} that induces -1 on J_k, thus violating the assumption that X is equi-hyperelliptic. This completes the proof. $\bigcirc$

If X_k^{log} does not satisfy condition (ii) of Definition 9.5, let α be the (necessarily unique) offending automorphism. Let $\iota \in Isom^{GO}_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_X^{adm})$ be the (equivalence class of) isomorphism(s) induced by α. Note that by Theorem 7.2, if $Isom_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$ is nonempty, then X_k^{log} has an offending automorphism if and only if $(X')_k^{log}$ does. Now let us define

$$Isom^{GOH}_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$$

to be the set of equivalence classes of elements of $Isom^{GO}_{\Gamma_K^{tm}}(\Pi_X^{adm}, \Pi_{X'}^{adm})$, where two classes of isomorphisms $\Pi_X^{adm} \cong \Pi_{X'}^{adm}$ are considered equivalent

(1) if they are equal or differ at most by composition with ι (when α exists);

(2) if they are equal (when α does not exist).

It is easy to show that this equivalence relation is well-defined, and compatible with the equivalence relation used to define $Isom^{GO}$ from $Isom$.

Now we are ready to state the strongest version of the Grothendieck Conjecture that we are able to prove (in the context of the present paper) in the local case.

Theorem 9.7: *Let K be a finite extension of $\mathbf{Q}_p$; let $A \subseteq K$ be its ring of integers; and let S^{log} be $Spec(A)$ equipped with the log structure defined by the closed point. Let $X^{log} \to S^{log}$ and $(X')^{log} \to S^{log}$ be stable log-curves (equipped with base-points $x \in X(K)$ and $x' \in X'(K)$). Suppose that at least one of X and X' is ordinary. Then there exists a (not necessarily unique) morphism*

$$\xi_S : Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K}) \to Isom_{S^{log}}(X^{log}, (X')^{log})$$

that makes the following diagram commute

$$\begin{array}{ccc} Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K}) & \longrightarrow & Isom^{GO}_{\Gamma^{tm}_K}(\Pi^{adm}_X, \Pi^{adm}_{X'}) \\ \downarrow \xi_S & & \downarrow \\ Isom_{S^{log}}(X^{log}, (X')^{log}) & \longrightarrow & Isom^{GOH}_{\Gamma^{tm}_K}(\Pi^{adm}_X, \Pi^{adm}_{X'}) \end{array}$$

(where the morphisms other than ξ_S are the natural ones).

Moreover, if either X or X' is equi-hyperelliptic, *then one can choose ξ_S uniquely such that the above diagram commutes when the projection on the right is replaced by the identity on $Isom^{GO}_{\Gamma^{tm}_K}(\Pi^{adm}_X, \Pi^{adm}_{X'})$.*

Finally, just as we derived Theorem 7.4 from Theorem 7.2, we have the following "outer automorphism version" of Theorem 9.7: First, let us denote by

$$\rho_X : \Gamma_K \to Out(\Delta_{X_K})$$

the representation derived from the extension $1 \to \Delta_{X_K} \to \Pi_{X_K} \to \Gamma_K \to 1$. Then we have the following

Theorem 9.8: *Let K be a finite extension of $\mathbf{Q}_p$; let $A \subseteq K$ be its ring of integers; and let S^{log} be $Spec(A)$ equipped with the log structure defined by the closed point. Let $X^{log} \to S^{log}$ be an* ordinary stable log-curve of genus g. *Then the isomorphism class of X^{log} is completely determined by the isomorphism class of the representation*

$$\rho_X : \Gamma_K \to Out(\Delta_{X_K})$$

Remark: Note that Theorem 9.8 shows that ordinary hyperbolic curves over local fields behave quite differently from ordinary abelian varieties over local fields. Indeed, given an ordinary elliptic curve $E_k \to Spec(k)$, it is easy to see (using the Serre-Tate theory of liftings of ordinary abelian varieties) that there exist many mutually nonisogenous liftings $E \to Spec(A)$ (where $A = W(k)$) of E_k to A, for which the representations

$$\rho_E : \Gamma_K \to Out(\pi_1(E_{\overline{K}})) = Aut_{\widehat{\mathbf{Z}}}(\pi_1(E_{\overline{K}}))$$

are isomorphic.

Section 10: The Main Result over Number Fields

In this Section, we prove the Grothendieck Conjecture for closed hyperbolic curves over number fields. The only result from Section 9 (the local theory) that we will use is Theorem 9.2; the rest of Section 9 is unnecessary. We will concentrate here on the closed case, since the open case has already been proven in [Tama].

Let K be a finite extension of $\mathbf{Q}$. Choose an algebraic closure $\overline{K}$ of K, and write Γ_K for $Gal(\overline{K}/K)$. Let $X_K \to Spec(K)$ be a smooth hyperbolic (i.e., of genus ≥ 2) curve. (By this, we shall always mean that X_K is geometrically connected over K.) Suppose that we are given a base-point $x \in X(K)$. Let $\Pi_{X_K} \stackrel{\mathrm{def}}{=} \pi_1(X_K, x_{\overline{K}})$; $\Delta_{X_K} \stackrel{\mathrm{def}}{=} \pi_1(X_{\overline{K}}, x_{\overline{K}})$. Thus, we have a natural exact sequence

$$1 \to \Delta_{X_K} \to \Pi_{X_K} \to \Gamma_K \to 1$$

Then we have the following result

Theorem 10.1: *Let K be a finite extension of $\mathbf{Q}$. Let $X_K \to Spec(K)$ and $X'_K \to Spec(K)$ be smooth hyperbolic curves over K, equipped with base-points $x \in X(K)$ and $x' \in X'(K)$). Then the natural map*

$$Isom_K(X_K, X'_K) \to Isom^{GO}_{\Gamma_K}(\Pi_{X_K}, \Pi_{X'_K})$$

is bijective.

Proof: Pick an isomorphism $\alpha : \Pi_{X_K} \cong \Pi_{X'_K}$ compatible with the surjections to Γ_K. Let A be a localization of the ring of integers of K over which X_K and X'_K extend to smooth curves $X \to S$, $X' \to S$ (where $S = Spec(A)$). Let $I \to S$ be the scheme $Isom_S(X, X')$ of isomorphisms of X with X' over S. It is well-known that I is finite and unramified over S. By localizing A further, we may assume that I is étale over S. Fix a prime number l. By localizing A further, we may assume that $l \in A^\times$. Let $\wp$ be a finite prime of A; let $A_\wp$ be the completion of A at $\wp$. By base-changing to $A_\wp$ and applying Theorem 9.2, we conclude that there exists a unique $\beta \in I(A_\wp)$ such that the isomorphism defined by β induces the same isomorphism as α on the first l-adic cohomology groups of X and X'. Note that β is, in fact, defined over some finite Galois extension L of K. Denote the resulting point of $I(L)$ by γ. To see that γ descends to K, it suffices to note that γ is the *unique* point of $I(L)$ that induces the right isomorphism on the $H^1(-, \mathbf{Z}_l)$'s. Thus, γ

descends to K. Thus, we obtain a K-isomorphism $\psi : X_K \cong X'_K$ corresponding to γ. To see that the isomorphism $\Pi_{X_K} \cong \Pi'_X$ induced by ψ coincides with the original α (up to an inner automorphism induced by an element of Δ_{X_K}), we apply the same argument as that given in [Tama] or the discussion preceding Theorem 7.2. (Note that here, we also use the well-known fact that Γ_K has trivial center.) $\bigcirc$

Now, as usual, we denote by

$$\rho_X : \Gamma_K \to Out(\Delta_{X_K})$$

the representation derived from the extension $1 \to \Delta_{X_K} \to \Pi_{X_K} \to \Gamma_K \to 1$. Just as in the discussion preceding Theorem 7.4, we may form $Out_\rho(\Delta_{X_K}, \Delta_{X'_K})$. Then we have the following

Theorem 10.2: *Let K be a finite extension of $\mathbf{Q}$. Let $X_K \to Spec(K)$ and $X'_K \to Spec(K)$ be smooth hyperbolic curves over K. Then the natural map*

$$Isom_K(X_K, X'_K) \to Out_\rho(\Delta_{X_K}, \Delta_{X'_K})$$

is bijective.

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