Abstract. The present paper, which forms the second part of a three-part series in which we study absolute anabelian geometry from an algorithmic point of view, focuses on the study of the closely related notions of decomposition groups and endomorphisms in this anabelian context. We begin by studying an abstract combinatorial analogue of the algebro-geometric notion of a stable polycurve [i.e., a “successive extension of families of stable curves”] and showing that the “geometry of log divisors on stable polycurves” may be extended, in a purely group-theoretic fashion, to this abstract combinatorial analogue; this leads to various anabelian results concerning configuration spaces. We then turn to the study of the absolute pro-$\Sigma$ anabelian geometry of hyperbolic curves over mixed-characteristic local fields, for $\Sigma$ a set of primes of cardinality $\geq 2$ that contains the residue characteristic of the base field. In particular, we prove a certain “pro-$p$ resolution of nonsingularities” type result, which implies a “conditional” anabelian result to the effect that the condition, on an isomorphism of arithmetic fundamental groups, of preservation of decomposition groups of “most” closed points implies that the isomorphism arises from an isomorphism of schemes — i.e., in a word, “point-theoreticity implies geometricity”; a non-conditional” version of this result is then obtained for “pro-curves” obtained by removing from a proper curve some set of closed points which is “$p$-adically dense in a Galois-compatible fashion”. Finally, we study, from an algorithmic point of view, the theory of Belyi and elliptic cuspidalizations, i.e., group-theoretic reconstruction algorithms for the arithmetic fundamental group of an open subscheme of a hyperbolic curve that arise from consideration of certain endomorphisms determined by Belyi maps and endomorphisms of elliptic curves.

Contents:

§0. Notations and Conventions
§1. A Combinatorial Analogue of Stable Polycurves
§2. Geometric Uniformly Toral Neighborhoods
§3. Elliptic and Belyi Cuspidalizations

Introduction

In the present paper, which forms the second part of a three-part series, we continue our discussion of various topics in absolute anabelian geometry from
a “group-theoretic algorithmic” point of view, as discussed in the Introduction to [Mzk15]. The topics presented in the present paper center around the following two themes:

(A) [the subgroups of arithmetic fundamental groups constituted by] decomposition groups of subvarieties of a given variety [such as closed points, divisors] as a crucial tool that leads to absolute anabelian results;

(B) “hidden endomorphisms” — which may be thought of as “hidden symmetries” — of hyperbolic curves that give rise to various absolute anabelian results.

In fact, “decomposition groups” and “endomorphisms” are, in a certain sense, related notions — that is to say, the monoid of “endomorphisms” of a variety may be thought of as a sort of “decomposition group of the generic point”!

With regard to the theme (B), we recall that the endomorphisms of an abelian variety play a fundamental role in the theory of abelian varieties [e.g., elliptic curves!]. Unlike abelian varieties, hyperbolic curves [say, in characteristic zero] do not have sufficient “endomorphisms” in the literal, scheme-theoretic sense to form the basis for an interesting theory. This difference between abelian varieties and hyperbolic curves may be thought of, at a certain level, as reflecting the difference between linear Euclidean geometries and non-linear hyperbolic geometries. From this point of view, it is natural to search for “hidden endomorphisms” that are, in some way, related to the intrinsic non-linear hyperbolic geometry of a hyperbolic curve. Examples [that appear in previous papers of the author] of such “hidden endomorphisms” — which exhibit a remarkable tendency to be related [for instance, via some induced action on the arithmetic fundamental group] to some sort of “anabelian result” — are the following:

(i) the interpretation of the automorphism group \( \text{PSL}_2(\mathbb{R}) \) of the universal covering of a hyperbolic Riemann surface as an object that gives rise to a certain “Grothendieck Conjecture-type result” in the “geometry of categories” [cf. [Mzk11], Theorem 1.12];

(ii) the interpretation of the theory of Teichmüller mappings [a sort of endomorphism — cf. (iii) below] between hyperbolic Riemann surfaces as a “Grothendieck Conjecture-type result” in the “geometry of categories” [cf. [Mzk11], Theorem 2.3];

(iii) the use of the endomorphisms constituted by Frobenius liftings — in the form of \( p \)-adic Teichmüller theory — to obtain the absolute anabelian result constituted by [Mzk6], Corollary 3.8;

(iv) the use of the endomorphism rings of Lubin-Tate groups to obtain the absolute anabelian result constituted by [Mzk15], Corollaries 3.8, 3.9.

The main results of the present paper — in which both themes (A) and (B) play a central role — are the following:

(1) In §1, we develop a purely combinatorial approach to the algebro-geometric notion of a stable polycurve [cf. [Mzk2], Definition 4.5]. This approach may
be thought of as being motivated by the purely combinatorial approach to the notion of a stable curve given in [Mzk13]. Moreover, in §1, we apply the theory of [Mzk13] to give, in effect, “group-theoretic algorithms” for reconstructing the “abstract combinatorial analogue” of the geometry of the various divisors — in the form of inertia and decomposition groups — associated to the canonical log structure of a stable polycurve [cf. Theorem 1.7]. These techniques, together with the theory of [MT], give rise to

various relative and absolute anabelian results concerning configuration spaces associated to hyperbolic curves

[cf. Corollaries 1.10, 1.11]. Relative to the discussion above of “hidden endomorphisms”, we observe that such configuration spaces may be thought of as representing a sort of “tautological endomorphism/correspondence” of the hyperbolic curve in question.

(2) In §2, we study the absolute pro-$\Sigma$ anabelian geometry of hyperbolic curves over mixed-characteristic local fields, for $\Sigma$ a set of primes of cardinality $\geq 2$ that contains the residue characteristic of the base field. In particular, we show that the condition, on an isomorphism of arithmetic fundamental groups, of preservation of decomposition groups of “most” closed points implies that the isomorphism arises from an isomorphism of schemes — i.e., in a word,

“point-theoreticity implies geometricity”

[cf. Corollary 2.9]. This condition may be removed if one works with “pro-curves” obtained by removing from a proper curve some set of closed points which is “$p$-adically dense in a Galois-compatible fashion” [cf. Corollary 2.10]. The key technical result that underlies these anabelian results is a certain “pro-$p$ resolution of nonsingularities” type result [cf. Lemma 2.6; Remark 2.6.1; Corollary 2.11] — i.e., a result reminiscent of the main [profinite] results of [Tama2]. This technical result allows one to apply the theory of uniformly toral neighborhoods developed in [Mzk15], §3. Relative to the discussion above of “hidden endomorphisms”, this technical result is interesting [cf., e.g., (iii) above] in that one central step of the proof of the technical result is quite similar to the well-known classical argument that implies the nonexistence of a Frobenius lifting for stable curves over the ring of Witt vectors of a finite field [cf. Remark 2.6.2].

(3) In §3, we re-examine the theory of [Mzk8], §2, for reconstructing the decomposition groups of closed points from the point of view of the present series of developing “group-theoretic algorithms”. In particular, we observe that these group-theoretic algorithms allow one to use

Belyi maps and endomorphisms of elliptic curves to construct [not only decomposition groups of closed points, but also]

“cuspidalizations”
[i.e., the full arithmetic fundamental groups of the open subschemes obtained by removing various closed points — cf. the theory of [Mzk14] associated to various types of closed points [cf. Corollaries 3.3, 3.4, 3.7, 3.8]. Relative to the discussion above of “hidden endomorphisms”, the theory of Belyi and elliptic cuspidalizations given in §3 illustrates quite explicitly how endomorphisms [arising from Belyi maps or endomorphisms of elliptic curves] can give rise to group-theoretic reconstruction algorithms.

Finally, we remark that although the “algorithmic approach” to stating anabelian results is not carried out very explicitly in §1, §2 [by comparison to §3 or [Mzk15]], the translation into “algorithmic language” of the more traditional “Grothendieck Conjecture-type” statements of the main results of §1, §2 is quite routine. [Here, it should be noted that the results of §1 that depend on “Uchida’s theorem” — i.e., Theorem 1.8, (ii); Corollary 1.11, (iv) — constitute a notatable exception to this “remark”, an exception that will be discussed in more detail in [Mzk16] — cf., e.g., [Mzk16], Remark 1.9.5.] That is to say, this translation was not carried out explicitly by the author solely because of the complexity of the algorithms implicit in §1, §2, i.e., not as a result of any substantive mathematical obstacles.

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Section 0: Notations and Conventions

We shall continue to use the “Notations and Conventions” of [Mzk15], §0. In addition, we shall use the following notation and conventions:

**Topological Groups:**

Let $G$ be a topologically finitely generated, slim profinite group. Thus, $G$ admits a basis of characteristic open subgroups. Any such basis determines a profinite topology on the groups $\text{Aut}(G)$, $\text{Out}(G)$. If $\rho : H \to \text{Out}(G)$ is any continuous homomorphism of profinite groups, then we denote by

$$G \rtimes H$$

the profinite group obtained by pulling back the natural exact sequence of profinite groups $1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1$ via $\rho$. Thus, we have a natural exact sequence of profinite groups $1 \to G \to G \rtimes H \to H \to 1$.

**Semi-graphs:**

Let $\Gamma$ be a connected semi-graph [cf., e.g., [Mzk9], §1, for a review of the theory of semi-graphs]. We shall refer to the [possibly infinite] dimension over $\mathbb{Q}$ of the singular homology module $H_1(\Gamma, \mathbb{Q})$ as the loop-rank $\text{lp-rk}(\Gamma)$ of $\Gamma$. We shall say that $\Gamma$ is loop-ample if, for any edge $e$ of $\Gamma$, the semi-graph obtained from $\Gamma$ by removing $e$ remains connected. We shall say that $\Gamma$ is untangled if every closed edge of $\Gamma$ abuts to two distinct vertices [cf. [Mzk9], §1]. We shall say that $\Gamma$ is edge-paired (respectively, edge-even) if $\Gamma$ is untangled, and, moreover, for any two [not necessarily distinct!] vertices $v, v'$ of $\Gamma$, the set of edges $e$ of $\Gamma$ such that $e$ abuts to a vertex $w$ of $\Gamma$ if and only if $w \in \{v, v'\}$ is either empty or of cardinality $\geq 2$ (respectively, empty or of even cardinality). [Thus, one verifies immediately that if $\Gamma$ is edge-paired (respectively, edge-even), then it is loop-ample (respectively, edge-paired).] We shall refer to as a simple path in $\Gamma$ any connected subgraph $\gamma \subseteq \Gamma$ such that the following conditions are satisfied: (a) $\gamma$ is a finite tree that has at least one edge; (b) given any vertex $v$ of $\gamma$, there exist at most two branches of edges of $\gamma$ that abut to $v$. Thus, [one verifies easily that] a simple path $\gamma$ has precisely two vertices $v$ such that there exists precisely one branch of an edge of $\gamma$ that abuts to $v$; we shall refer to these two vertices as the terminal vertices of the simple path $\gamma$. If $\gamma, \gamma'$ are simple paths in $\Gamma$ such that the terminal vertices of $\gamma, \gamma'$ coincide, then we shall say that $\gamma, \gamma'$ are co-terminal.

**Log Schemes:**

We shall often regard a scheme as a log scheme equipped the trivial log structure. Any fiber product of fs [i.e., fine, saturated] log schemes is to be taken in the category of fs log schemes. In particular, the underlying scheme of such a product is finite over, but not necessarily isomorphic to, the fiber product of the underlying schemes.
Curves:

We shall refer to a hyperbolic orbicurve $X$ as semi-elliptic [i.e., “of type $(1,1)_{\pm}$” in the terminology of [Mzk12], §0] if there exists a finite étale double covering $Y \to X$, where $Y$ is a once-punctured elliptic curve, and the covering is given by the stack-theoretic quotient of $Y$ by the “action of $\pm 1$” [i.e., relative to the group operation on the elliptic curve given by the canonical compactification of $Y$].

For $i = 1, 2$, let $X_i$ be a hyperbolic orbicurve over a field $k_i$. Then we shall say that $X_1, X_2$ are isogenous [cf. [Mzk14], §0] if there exists a hyperbolic orbicurve $X$ over a field $k$, together with finite étale morphisms $X \to X_i$, for $i = 1, 2$. Note that in this situation, the morphisms $X \to X_i$ induce finite separable inclusions of fields $k_i \hookrightarrow k$. [Indeed, this follows immediately from the easily verified fact that every subgroup $G \subseteq \Gamma(X, \mathcal{O}_X^\times)$ such that $G \cup \{0\}$ determines a field is necessarily contained in $k^\times$.]

We shall use the term stable log curve as it was defined in [Mzk9], §0. Let

$$X^{\log} \to S^{\log}$$

be a stable log curve over an fs log scheme $S^{\log}$, where $S = \text{Spec}(k)$ for some field $k$; $\overline{k}$ a separable closure of $k$. Then we shall refer to as the loop-rank $\text{lp-rk}(X^{\log})$ [or $\text{lp-rk}(X)$] of $X^{\log}$ [or $X$] the loop-rank of the dual graph of $X^{\log} \times_k \overline{k}$ [or $X \times_k \overline{k}$].

We shall say that $X^{\log}$ [or $X$] is loop-ample (respectively, untangled; edge-paired; edge-even) if the dual semi-graph with compact structure [cf. [Mzk5], Appendix] of $X^{\log} \times_k \overline{k}$ is loop-ample (respectively, untangled; edge-paired; edge-even) [as a connected semi-graph]. We shall say that $X^{\log}$ [or $X$] is sturdy if the normalization of every irreducible component of $X$ is of genus $\geq 2$ [cf. [Mzk13], Remark 1.1.5].

Observe that for any prime number $l$ invertible on $S$, there exist an fs log scheme $T^{\log}$ over $S^{\log}$, where $T = \text{Spec}(k')$ for some finite separable extension $k'$ of $k$, and a connected Galois log admissible covering $Y^{\log} \to X^{\log} \times_{S^{\log}} T^{\log}$ [cf. [Mzk1], §3] of degree a power of $l$ such that $Y^{\log}$ is sturdy and edge-paired [hence, in particular, untangled and loop-ample]; if, moreover, $l = 2$, then one may also take $Y^{\log}$ to be edge-even. [Indeed, to verify this observation, we may assume that $k = \overline{k}$. Then note that any hyperbolic curve $U$ over $k$ admits a connected finite étale Galois covering $V \to U$ of degree a power of $l$ such that $V$ is of genus $\geq 2$ and ramified with ramification index $l^2$ at each of the cusps of $V$ — cf. the discussion at the end of the present §0. Thus, by gluing together such coverings at the nodes of $X$, one concludes that there exists a connected Galois log admissible covering $Z^{\log}_1 \to X^{\log} \times_{S^{\log}} T^{\log}$ of degree a power of $l$ which is totally ramified over every node of $X$ with ramification index $l^2$ such that every irreducible component of $Z_1$ is of genus $\geq 2$ — i.e., $Z^{\log}_1$ is sturdy. Next, observe that there exists a connected Galois log admissible covering $Z^{\log}_2 \to Z^{\log}_1$ of degree a power of $l$ that arises from a covering of the dual graph of $Z_1$ such that $Z^{\log}_2$ is untangled [and still sturdy]. Finally, observe that there exists a connected Galois log admissible covering $Z^{\log}_3 \to Z^{\log}_2$ of degree a positive power of $l$ which restricts to a connected finite étale covering over every irreducible component of $Z_2$ [hence is unramified at the nodes] such that $Z^{\log}_3$ is edge-paired [and still sturdy and untangled] for arbitrary $l$ and edge-even when $l = 2$. Thus, we may take $Y^{\log} \equiv Z^{\log}_3$.]


Observe that if $X$ is loop-ample, then for every point $x \in X(k)$ which is not a unique cusp of $X$ [i.e., either $x$ is not a cusp or if $x$ is a cusp, then it is not the unique cusp of $X$], the evaluation map
\[ H^0(X, \omega_{X^\text{log}/S^\text{log}}) \to \omega_{X^\text{log}/S^\text{log}}|_x \]
is surjective. Indeed, by considering the long exact sequence associated to the short exact sequence $0 \to \omega_{X^\text{log}/S^\text{log}} \otimes_{\mathcal{O}_X} \mathcal{I}_x \to \omega_{X^\text{log}/S^\text{log}} \to \omega_{X^\text{log}/S^\text{log}}|_x \to 0$, where $\mathcal{I}_x \subseteq \mathcal{O}_X$ is the sheaf of ideals corresponding to $x$, one verifies immediately that it suffices to show that the surjection
\[ \mathcal{H}_x \overset{\text{def}}{=} H^1(X, \omega_{X^\text{log}/S^\text{log}} \otimes_{\mathcal{O}_X} \mathcal{I}_x) \to \mathcal{H} \overset{\text{def}}{=} H^1(X, \omega_{X^\text{log}/S^\text{log}}) \]
is injective. If $x$ is a node, then the fact that $X$ is loop-ample implies [by computing via Serre duality] that either $\dim_k(\mathcal{H}_x) = \dim_k(\mathcal{H}) = 1$ [if $X$ has no cusps] or $\dim_k(\mathcal{H}_x) = \dim_k(\mathcal{H}) = 0$ [if $X$ has cusps]. Thus, we may assume that $x$ is not a node, so the surjection $\mathcal{H}_x \twoheadrightarrow \mathcal{H}$ is dual to the injection
\[ \mathcal{M} \overset{\text{def}}{=} H^0(X, \mathcal{O}_X(-D)) \hookrightarrow \mathcal{M}_x \overset{\text{def}}{=} H^0(X, \mathcal{O}_X(x - D)) \quad (\subseteq \mathcal{N}_x \overset{\text{def}}{=} H^0(X, \mathcal{O}_X(x))) \]
— where we write $D \subseteq X$ for the divisor of cusps of $X$. If $x$ is a cusp, then it follows that $D$ is of degree $\geq 2$, and one computes easily that $\dim_k(\mathcal{M}) = \dim_k(\mathcal{M}_x) = 0$. Thus, we may assume that $x$ is not a cusp. Write $C$ for the irreducible component of $X$ containing $x$. Now suppose that the injection $\mathcal{M} \hookrightarrow \mathcal{M}_x$ is not surjective. Then it follows that $\dim_k(\mathcal{N}_x) = 2$, and that $\mathcal{N}_x$ determines a basepoint-free linear system on $X$. In particular, $\mathcal{N}_x$ determines a morphism $\phi : X \to \mathbb{P}^1_k$ that is of degree $1$ on $C$ — i.e., determines an isomorphism $C \overset{\sim}{\to} \mathbb{P}^1_k$ — and constant on the other irreducible components of $X$. Since $X$ is loop-ample, it follows that the dual graph $\Gamma$ of $X$ either has no edges or admits a loop containing the vertex determined by $C$. On the other hand, the existence of such a loop contradicts the fact that $\phi$ determines an isomorphism $C \overset{\sim}{\to} \mathbb{P}^1_k$. Thus, we conclude that $X = C \cong \mathbb{P}^1_k$, so $D$ is of degree $\geq 3$. But this implies that $\mathcal{M}_x = 0$, a contradiction.

Finally, let $U$ be a hyperbolic curve over an algebraically closed field $k$ and $l$ a prime number invertible in $k$. Suppose that the cardinality $r$ of the set of cusps of $U$ is $\geq 2$, and, moreover, that, if $l = 2$, then $r$ is even. Then observe that it follows immediately from the well-known structure of the maximal pro-$l$ quotient of the abelianization of the étale fundamental group of $U$ that

for every power $l^n$ of $l$, where $n$ is a positive integer, there exists a cyclic covering $V \to U$ of degree $l^n$ that is totally ramified over the cusps of $U$.

Indeed, this observation is an immediate consequence of the elementary fact that, in light of our assumptions on $r$, there always exist $r - 1$ integers prime to $l$ whose sum is also prime to $l$. We shall often make use of the assumption that a stable log curve is edge-paired — or, when $l = 2$, edge-even — by applying the above observation to the various connected components of the complement of the cusps and nodes of the stable log curve.
Section 1: A Combinatorial Analogue of Stable Polycrures

In the present §1, we apply the theory of [Mzk13] to study a sort of purely group-theoretic, combinatorial analogue [cf. Definition 1.5 below] of the notion of a stable polycurve introduced in [Mzk2], Definition 4.5. This allows one to reconstruct the “abstract combinatorial analogue” of the “geometry of log divisors” [i.e., divisors associated to the log structure of a stable polycurve] of such a combinatorial object via group theory [cf. Theorem 1.7]. Finally, we apply the theory of [MT] to obtain various consequences of the theory of the present §1 [cf. Corollaries 1.10, 1.11] concerning the absolute anabelian geometry of configuration spaces.

We begin by recalling the discussion of [Mzk13], Example 2.5.

Example 1.1. Stable Log Curves over a Logarithmic Point (Revisited).

(i) Let \( k \) be a separably closed field; \( \Sigma \) a nonempty set of prime numbers invertible in \( k \); \( M \subseteq \mathbb{Q} \) the monoid of positive rational numbers with denominators invertible in \( k \); \( S_{\log} \) (respectively, \( T_{\log} \)) the log scheme obtained by equipping \( S \) (respectively, \( T \)) with the log structure determined by the chart \( N \ni 1 \mapsto 0 \in k \) (respectively, \( M \ni 1 \mapsto 0 \in k \)); \( T_{\log} \rightarrow S_{\log} \) the morphism determined by the natural inclusion \( N \hookrightarrow M \); \( X_{\log} \) a stable log curve over \( S_{\log} \). Thus, the profinite group \( I_{S_{\log}} \) admits a natural isomorphism \( I_{S_{\log}} \cong \text{Hom}(\mathbb{Q}/\mathbb{Z}, k^*) \) and fits into an natural exact sequence
\[
1 \rightarrow \Delta_{X_{\log}} \xrightarrow{\Delta_{X_{\log}}} \pi_1(X_{\log}) \rightarrow \Pi_{X_{\log}} \xrightarrow{\pi_1(X_{\log})} I_{S_{\log}} \rightarrow 1
\]
where we write “\( \pi_1(\cdot) \)” for the “log fundamental group” of the log scheme in parentheses [which amounts, in this case, to the fundamental group arising from the admissible coverings of \( X_{\log} \)], relative to an appropriate choice of basepoint [cf. [III] for a survey of the theory of log fundamental groups]. In particular, if we write \( I_{S_{\log}} \) for the maximal pro-\( \Sigma \) quotient of \( I_{S_{\log}} \), then as abstract profinite groups, \( I_{S_{\log}}^\Sigma \cong \hat{\mathbb{Z}}^\Sigma \), where we write \( \hat{\mathbb{Z}}^\Sigma \) for the maximal pro-\( \Sigma \) quotient of \( \hat{\mathbb{Z}} \).

(ii) On the other hand, \( X_{\log} \) determines a semi-graph of anabelioids [cf. [Mzk9], Definition 2.1] of pro-\( \Sigma \) PSC-type [cf. [Mzk13], Definition 1.1, (i)], whose underlying semi-graph we denote by \( \mathcal{G} \). Thus, for each vertex \( v \) [corresponding to an irreducible component of \( X_{\log} \)] (respectively, edge \( e \) [corresponding to a node or cusp of \( X_{\log} \)]) of \( \mathcal{G} \), we have a connected anabelioid [i.e., a Galois category] \( \mathcal{G}_v \) (respectively, \( \mathcal{G}_e \)), and for each branch \( b \) of an edge \( e \) abutting to a vertex \( v \), we are given a morphism of anabelioids \( \mathcal{G}_e \rightarrow \mathcal{G}_v \). Then the maximal pro-\( \Sigma \) completion of \( \Delta_{X_{\log}} \) may be identified with the “PSC-fundamental group” \( \Pi_\mathcal{G} \) associated to \( \mathcal{G} \). Also, we recall that \( \Pi_\mathcal{G} \) is slim [cf., e.g., [Mzk13], Remark 1.1.3], and that the groups \( \text{Aut}(\mathcal{G}) \), \( \text{Out}(\Pi_\mathcal{G}) \) may be equipped with profinite topologies in such a way that the natural morphism
\[
\text{Aut}(\mathcal{G}) \rightarrow \text{Out}(\Pi_\mathcal{G})
\]
is a continuous injection [cf. the discussion at the beginning of [Mzk13], §2], which we shall use to identify Aut(\mathcal{G}) with its image in Out(\Pi_\mathcal{G}). In particular, we obtain a natural continuous homomorphism \( I_{S^{\log}_G} \rightarrow Aut(\mathcal{G}) \). Moreover, it follows immediately from the well-known structure of admissible coverings at nodes [cf., e.g., [Mzk1], §3.23] that this homomorphism factors through \( I_{S^{\log}_G}^\Sigma \), hence determines a natural continuous homomorphism \( \rho_I : I_{S^{\log}_G}^\Sigma \rightarrow Aut(\mathcal{G}) \). Also, we recall that each vertex \( v \) (respectively, edge \( e \)) of \( \mathbb{G} \) determines a(n) vertical subgroup \( \Pi_v \subseteq \Pi_\mathcal{G} \) (respectively, edge-like subgroup \( \Pi_e \subseteq \Pi_\mathcal{G} \)), which is well-defined up to conjugation — cf. [Mzk13], Definition 1.1, (ii). Here, the edge-like subgroups \( \Pi_e \) may be either nodal or cuspidal, depending on whether \( e \) corresponds to a node or to a cusp. If an edge \( e \) corresponds to a node (respectively, cusp), then we shall simply say that \( e \) “is” a node (respectively, cusp).

(iii) Let \( e \) be a node of \( X \). Write \( M_e \) for the stalk of the characteristic sheaf of the log scheme \( X^{\log} \) at \( e \); \( M_S \) for the stalk of the characteristic sheaf of the log scheme \( S^{\log} \) at the tautological \( S \)-valued point of \( S \). Thus, \( M_S \cong \mathbb{N} \); we have a natural inclusion \( M_S \hookrightarrow M_e \), with respect to which we shall often [by abuse of notation] identify \( M_S \) with its image in \( M_e \). Write \( \sigma \in M_e \) for the unique generator of [the image of] \( M_S \). Then there exist elements \( \xi, \eta \in M_e \) satisfying the relation

\[
\xi + \eta = i_e \cdot \sigma
\]

for some positive integer \( i_e \), which we shall refer to as the index of the node \( e \), such that \( M_e \) is generated by \( \xi, \eta, \sigma \). Also, we shall write \( i^\Sigma_e \) for the largest positive integer \( j \) such that \( i_e/j \) is a product of primes \( \notin \Sigma \) and refer to \( i^\Sigma_e \) as the \( \Sigma \)-index of the node \( e \). One verifies easily that the set of elements \( \{\xi, \eta\} \) of \( M_e \) may be characterized intrinsically as the set of elements \( \theta \in M_e \setminus M_S \) such that any relation \( \theta = n \cdot \theta' + \theta'' \) for \( n \) a positive integer, \( \theta' \in M_e \setminus M_S, \theta'' \in M_S \) implies that \( n = 1, \theta'' = 0 \). In particular, \( i, i^\Sigma_e \) are well-defined and depend only on the isomorphism class of the pair consisting of the monoid \( M_e \) and the submonoid \( \subseteq \subseteq \) \( M_e \) given by the image of \( M_S \).

Remark 1.1.1. Of course, in Example 1.1, it is not necessary to assume that \( k \) is separably closed [cf. [Mzk13], Example 2.5]. If \( k \) is not separably closed, then one must also contend with the action of the absolute Galois group of \( k \). More generally, for the theory of the present §1, it is not even necessary to assume that an “additional profinite group” acting on \( \mathcal{G} \) arises “from scheme theory”. It is this point of view that formed the motivation for Definition 1.2 below.

Definition 1.2. In the notation of Example 1.1:

(i) Let \( \rho_H : H \rightarrow Aut(\mathcal{G}) \) (\( \subseteq \) Out(\( \Pi_\mathcal{G} \))) be a continuous homomorphism of profinite groups; suppose that \( X^{\log} \) is nonsingular [i.e., has no nodes]. Then we shall refer to as a [pro-\( \Sigma \)] PSC-extension [i.e., “pointed stable curve extension”] any extension of profinite groups that is isomorphic — via an isomorphism which we shall refer to as the “structure of [pro-\( \Sigma \)] PSC-extension” — to an extension of the form

\[
1 \rightarrow \Pi_\mathcal{G} \rightarrow \Pi_H \overset{\text{def}}{=} (\Pi_\mathcal{G} \overset{\text{out}}{\times} H) \rightarrow H \rightarrow 1
\]
[cf. §0 for more on the notation “$\times^\text{out}$”], which we shall refer to as the PSC-extension associated to the construction data ($X^\log \to S^\log, \Sigma, G, \rho_H$). In this situation, each [necessarily cuspidal] edge $e$ of $G$ determines [up to conjugation in $\Pi_G$] a subgroup $\Pi_e \subseteq \Pi_G$, whose normalizer $D_e \overset{\text{def}}{=} N_{\Pi_H}(\Pi_e)$ in $\Pi_H$ we shall refer to as the decomposition group associated to the cusp $e$; we shall refer to $I_e \overset{\text{def}}{=} \Pi_e = D_e \cap \Pi_G \subseteq D_e$ [cf. [Mzk13], Proposition 1.2, (ii)] as the inertia group associated to the cusp $e$. Finally, we shall apply the terminology applied to objects associated to $1 \to \Pi_G \to \Pi_H \to H \to 1$ to the objects associated to an arbitrary PSC-extension via its “structure of PSC-extension” isomorphism.

(ii) Let $\rho_H : H \to \text{Aut}(G)$ ($\subseteq \text{Out}(\Pi_G)$) be a continuous homomorphism of profinite groups; $\iota : I_{G, log}^G \to H$ a continuous injection of profinite groups with normal image such that $\rho_H \circ \iota = \rho_I$. Suppose that $X^\log$ is arbitrary [i.e., $X$ may be singular or nonsingular]. Then we shall refer to as a [pro-$\Sigma$] PSC-extension [i.e., “degenerating pointed stable curve extension”] any extension of profinite groups that is isomorphic — via an isomorphism which we shall refer to as the “structure of [pro-$\Sigma$] PSC-extension” — to an extension of the form

$$1 \to \Pi_G \to \Pi_H \overset{\text{def}}{=} (\Pi_G \times^\text{out} H) \to H \to 1$$

— which we shall refer to as the DPSC-extension associated to the construction data ($X^\log \to S^\log, \Sigma, G, \rho_H, \iota$). In this situation, we shall refer to the image $I \subseteq H$ of $\iota$ as the inertia subgroup of $H$ and to the extension

$$1 \to \Pi_G \to \Pi_I \overset{\text{def}}{=} (\Pi_G \times^\text{out} I) \to I \to 1$$

[so $\Pi_I = \Pi_H \times_H I \subseteq \Pi_H$] as the [pro-$\Sigma$] IPSC-extension [i.e., “inertial pointed stable curve extension”] associated to the construction data ($X^\log \to S^\log, \Sigma, G, \rho_H, \iota$); each vertex $v$ (respectively, edge $e$) of $G$ determines [up to conjugation in $\Pi_G$] a subgroup $\Pi_v \subseteq \Pi_G$ (respectively, $\Pi_e \subseteq \Pi_G$), whose normalizer

$$D_v \overset{\text{def}}{=} N_{\Pi_H}(\Pi_v) \quad (\text{respectively, } D_e \overset{\text{def}}{=} N_{\Pi_H}(\Pi_e))$$

in $\Pi_H$ we shall refer to as the decomposition group associated to $v$ (respectively, $e$); for $v$ arbitrary (respectively, $e$ a node), we shall refer to the centralizer

$$I_v \overset{\text{def}}{=} Z_{\Pi_I}(\Pi_v) \subseteq D_v \quad (\text{respectively, } I_e \overset{\text{def}}{=} Z_{\Pi_I}(\Pi_e) \subseteq D_e)$$

as the inertia group associated to $v$ (respectively, $e$). If $e$ is a cusp of $G$, then we shall refer to $I_e \overset{\text{def}}{=} \Pi_e = D_e \cap \Pi_G \subseteq D_e$ [cf. [Mzk13], Proposition 1.2, (ii)] as the inertia group associated to the cusp $e$. Finally, we shall apply the terminology applied to objects associated to $1 \to \Pi_G \to \Pi_H \to H \to 1$ to the objects associated to an arbitrary DPSC-extension via its “structure of DPSC-extension” isomorphism.

**Remark 1.2.1.** Note that in the situation of Definition 1.2, (i) (respectively, (ii)), any open subgroup of $\Pi_H$ [equipped with the induced extension structure] admits a structure of [pro-$\Sigma$] PSC-extension (respectively, DPSC-extension) for appropriate
construction data that may be derived from the original construction data. On the other hand, in the situation of Definition 1.2, (ii), given an open subgroup of $\Pi_I$ [equipped with the induced extension structure], in order to endow this open subgroup with a structure of IPSC-extension, it may be necessary — even if, for instance, this open subgroup of $\Pi_I$ surjects onto $I$ — to replace the inertia subgroup $I$ of $H$ by some open subgroup of $I$ [i.e., in effect, to replace the given open subgroup of $\Pi_I$ with the intersection of this given open subgroup with the inverse image in $\Pi_I$ of some open subgroup of $I$]. Such replacements may be regarded as a sort of abstract group-theoretic analogue of the operation of passing to a finite extension of a discretely valued field in order to achieve a situation in which a given hyperbolic curve over the original field has **stable reduction**.

**Remark 1.2.2.** Note that in the situation of Definition 1.2, (ii), the inertia subgroup $I \subseteq H$ is *not intrinsically determined* in the sense that any open subgroup of $I$ may also serve as the inertia subgroup of $H$ — cf. the *replacement operation* discussed in Remark 1.2.1.

**Remark 1.2.3.** Recall that for $l \in \Sigma$, one may construct directly from $G$ a pro-$l$ cyclotomic character $\chi_l : \text{Aut}(G) \to \hat{\mathbb{Z}}_l^\times$ [cf. [Mzk13], Lemma 2.1]. In particular, any $\rho_H$ as in Definition 1.2, (i), (ii), determines a pro-$l$ cyclotomic character $\chi_l|_H : H \to \hat{\mathbb{Z}}_l^\times$. The action $\rho_H$ is called *l-cyclotomically full* [cf. [Mzk13], Definition 2.3, (ii)] if the image of $\chi_l|_H$ is open. We shall also apply this terminology “$l$-cyclotomically full” to the corresponding PSC-, DPSC-extensions. In fact, it follows immediately from the first portion of [Mzk13], Proposition 2.4, (iv), that the issue of whether or not $\rho_H$ is $l$-cyclotomically full depends only on the outer representation $H \to \text{Out}(\Pi_G)$ of $H$ on $\Pi_G$ determined by $\rho_H$.

**Proposition 1.3.** *(Basic Properties of Inertia and Decomposition Groups)*

In the notation of Definition 1.2, (ii):

(i) If $e$ is a cusp of $G$, then as abstract profinite groups, $I_e \cong \hat{\mathbb{Z}}^\Sigma$.

(ii) If $e$ is a node of $G$, then we have a natural exact sequence $1 \to \Pi_e \to I_e \to I \to 1$; as abstract profinite groups, $I_e \cong \hat{\mathbb{Z}}^\Sigma \times \hat{\mathbb{Z}}^\Sigma$. If $e$ abuts to vertices $v, v'$, then [for appropriate choices of conjugates of the various inertia groups involved] we have inclusions $I_v, I_{v'} \subseteq I_e$, and the natural morphism $I_v \times I_{v'} \to I_e$ is an open injective homomorphism, with image of index equal to $I_e^\Sigma$.

(iii) If $v$ is a vertex of $G$, then we have a natural isomorphism $I_v \cong I$; $D_v \cap \Pi_I = I_v \times \Pi_v$; as abstract profinite groups, $I_v \cong \hat{\mathbb{Z}}^\Sigma$. If $e$ is a cusp that abuts to $v$, then [for appropriate choices of conjugates of the various inertia and decomposition groups involved] we have inclusions $I_e, I_v \subseteq D_e \cap \Pi_I$, and the natural morphism $I_e \times I_v \to D_e \cap \Pi_I$ is an isomorphism; in particular, as abstract profinite groups, $D_e \cap \Pi_I \cong \hat{\mathbb{Z}}^\Sigma \times \hat{\mathbb{Z}}^\Sigma$, and we have a natural exact sequence $1 \to I_e \to D_e \cap \Pi_I \to I \to 1$.

(iv) Let $v, v'$ be vertices of $G$. If $D_v \cap D_{v'} \cap \Pi_I \neq \{1\}$, then one of the following three [mutually exclusive] properties holds: (1) $v = v'$; (2) $v$ and $v'$ are
distinct, but adjacent [i.e., there exists a node e that abuts to \(v, v'\)]; (3) \(v\) and \(v'\) are distinct and non-adjacent, but there exists a vertex \(v'' \neq v, v'\) of \(G\) such that \(v''\) is adjacent to \(v\) and \(v'\). Moreover, in the situation of (2), we have \(I_v \cap I_v' = \{1\}\), [for appropriate choices of conjugates of the various inertia and decomposition groups involved] \(D_v \cap D_v' \cap I_I = I_v; \) in the situation of (3), we have \(\Pi_v \cap \Pi_v' = I_v \cap I_v' = I_v' \cap I_v' = \{1\}\), [for appropriate choices of conjugates of the various inertia and decomposition groups involved] \(D_v \cap D_v' \cap I_I = I_v'\). In particular, \(I_v \cap I_v' \neq \{1\}\) implies that \(v = v'\).

(v) Let \(v\) be a vertex of \(G\). Then \(D_v = C_{\Pi_H}(I_v) = N_{\Pi_H}(I_v)\) is commensurably terminal in \(\Pi_H\); \(D_v \cap \Pi_I = C_{\Pi_I}(I_v) = N_{\Pi_I}(I_v) = Z_{\Pi_I}(I_v)\) is commensurably terminal in \(\Pi_I\); \(D_v \cap \Pi_G = I_v\) is commensurably terminal in \(\Pi_G\).

(vi) Let \(v\) be a vertex of \(G\). Then the image of \(D_v\) in \(H\) is open; on the other hand, if \(G\) has more than one vertex [i.e., the curve \(X\) is singular], then \(D_v\) is not open in \(\Pi_H\).

(vii) Let \(e\) be an edge of \(G\). Then \(D_e = C_{\Pi_H}(I_e) = N_{\Pi_H}(I_e)\) is commensurably terminal in \(\Pi_H\). If \(e\) is a node, then \(I_v = D_e \cap \Pi_I\).

(viii) Let \(e, e'\) be edges of \(G\). If \(D_e \cap D_e' \cap \Pi_I \neq \{1\}\), then one of the following two [mutually exclusive] properties holds: (1) \(e = e'\); (2) \(e\) and \(e'\) are distinct, but abut to the same vertex \(v\), and \(D_e \cap D_e' \cap \Pi_G = \{1\}\). Moreover, in the situation of (2), [for appropriate choices of conjugates of the various inertia and decomposition groups involved] we have \(I_v = D_e \cap D_e' \cap \Pi_I\).

(ix) Let \(e\) be an edge of \(G\). Then the image of \(D_e\) in \(H\) is open, but \(D_e\) is not open in \(\Pi_H\).

(x) Let \(\tau_I : I \to \Pi_I\) be the [outer] homomorphism that arises [by functoriality!] from a “log point” \(T_S \in X^{\log}(S^{\log})\). Let us call \(\tau_I\) non-vertical (respectively, non-edge-like) if \(\tau_I(I)\) is not contained in \(I_v\) (respectively, \(I_e\)) for any vertex \(v\) (respectively, edge \(e\)) of \(G\). Then if \(\tau_I\) is non-vertical and non-edge-like, then the image of \(T_S\) is the unique cusp \(e_\tau\) of \(X\) such that [for an appropriate choice of conjugate of \(D_e\)] \(\tau_I(I) \subseteq D_e\). Now suppose that the image of \(T_S\) is not a cusp. Then \(\tau_I\) satisfies the condition \(\tau_I(I) = I_v\), for some vertex \(v_\tau\) of \(G\) [and an appropriate choice of conjugate of \(I_v\)] if and only if the image of \(T_S\) is a non-nodal point of the irreducible component of \(X\) corresponding to \(v_\tau\); \(\tau_I\) is non-vertical and satisfies the condition \(\tau_I(I) \subseteq I_e\), for some node \(e_\tau\) of \(G\) [and an appropriate choice of conjugate of \(I_e\)] if and only if the image of \(T_S\) is the node of \(X\) corresponding to \(e_\tau\).

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). Write \(\nu (\cong S)\) for the closed subscheme of \(X\) determined by the node of \(X\) corresponding to \(e\); \(\nu^{\log}\) for the result of equipping \(\nu\) with the log structure pulled back from \(X^{\log}\). Thus, we obtain a natural [outer] homomorphism \(\Pi_\nu \cong \pi_1(\nu^{\log}) \to \pi_1(X^{\log}) = \Pi_X^{\log} \to \Pi_I\). Now in the notation of Example 1.1, (iii) one computes easily [by considering the Galois groups of the various Kummer log étale coverings of \(\nu^{\log}\)] that we have natural isomorphisms \(\Pi_\nu \cong \text{Hom}(M_{k^e}^{\text{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^x)\),
Moreover, if we write $\Pi_v \to \Pi^\Sigma$ for the maximal pro-$\Sigma$ quotient of $\Pi_v$, then one verifies immediately that the isomorphisms induced on maximal pro-$\Sigma$ quotients by these natural isomorphisms are compatible, relative to the surjection

$$(\Pi^\Sigma_v \to ) \quad \text{Hom}(M^\text{gp}_S \otimes Q/Z, k^\times) \otimes \hat{\mathbb{Z}}^\Sigma \to \text{Hom}(M^\text{gp}_S \otimes Q/Z, k^\times) \otimes \hat{\mathbb{Z}}^\Sigma \quad (\cong \ I^\Sigma_{\text{topo}})$$

induced by the inclusion $M_S \hookrightarrow M_e$, with the morphism $\Pi^\Sigma_v \to I^\Sigma_{\text{topo}}$ induced by the composite morphism $\Pi_v \to \Pi_I \to I \cong I^\Sigma_{\text{topo}} \cong \text{Hom}(M^\text{gp}_S \otimes Q/Z, k^\times) \otimes \hat{\mathbb{Z}}^\Sigma$.

The kernel of this surjection $\Pi^\Sigma_v \to \hat{I}_{\text{topo}}$ may be identified with the profinite group $\text{Hom}(M^\text{gp}_S/M^\text{gp}_S \otimes Q/Z, k^\times) \otimes \hat{\mathbb{Z}}^\Sigma$, and one verifies immediately [from the definition of $G$] that this kernel maps isomorphically onto $\Pi_e \subseteq \Pi_I$. In particular, it follows that we obtain an injection $\Pi^\Sigma_v \hookrightarrow \Pi_I$ whose image contains $\Pi_e$ and surjects onto $I$.

Since $\Pi^\Sigma_v$ is abelian, it follows that the image $\text{Im}((\Pi^\Sigma_v)$ of this injection is contained in $I_e$; since $I_e \cap \Pi_G = \Pi_e$ [cf. [Mzk13], Proposition 1.2, (ii)], we thus conclude that $\text{Im}(\Pi^\Sigma_v) = I_e$. Now it follows immediately from the definitions that $I_v, I_v' \subseteq I_e$; moreover, one computes immediately that [in the notation of Example 1.1, (iii)] the subgroups $I_v, I_v' \subseteq I_e$ correspond to the subgroups of $\text{Hom}(M^\text{gp}_S \otimes Q/Z, k^\times) \otimes \hat{\mathbb{Z}}^\Sigma$ consisting of homomorphisms that vanish on $\xi, \eta$, respectively. Now the various assertions contained in the statement of assertion (ii) follow immediately. This completes the proof of assertion (ii).

Next, we consider assertion (iii). Since $\Pi_v$ is slim [cf., e.g., [Mzk13], Remark 1.1.3] and commensurable terminal in $\Pi_G$ [cf. [Mzk13], Proposition 1.2, (ii)], it follows that $D_v \cap \Pi_G = \Pi_v$ and $I_v \cap \Pi_G = \{1\}$, so we obtain a natural injection $I_v \hookrightarrow I$. The fact that this injection is, in fact, surjective is immediate from the definitions when $X$ is smooth over $k$ and follows from the computation of “$I_v$” performed in the proof of assertion (ii) when $X$ is singular. Next, let us observe that since $I_v$ commutes [by definition!] with $\Pi_v$, we obtain a natural morphism $I_v \times \Pi_v \to D_v \cap \Pi_I$, which is both injective [since $I_v \cap \Pi_v = \{1\}$] and surjective [cf. the isomorphism $I_v \cong I$; the fact that $D_v \cap \Pi_G = \Pi_e$]. Now suppose that $e$ is a cusp that abuts to $v$. Then [for appropriate choices of conjugates] it follows immediately from the definitions that we have inclusions $I_e, I_e \subseteq D_v \cap \Pi_I$, and that $I_e$ commutes with $I_v$. Note, moreover, that $D_e \cap \Pi_G = I_e$ [cf. [Mzk13], Proposition 1.2, (ii)]. Thus, the fact that the natural projection $I_v \to I$ is an isomorphism implies that we have a natural exact sequence $1 \to I_v \to D_v \cap \Pi_I \to I \to 1$, and that the natural morphism $I_e \times I_v \to D_v \cap \Pi_I$ is an isomorphism. This completes the proof of assertion (iii).

Next, we consider assertion (vii). Since $D_e \cap \Pi_G = \Pi_e$ [cf. [Mzk13], Proposition 1.2, (ii)], it follows that $D_e \subseteq C_{\Pi_H} (D_e) \subseteq C_{\Pi_H} (\Pi_e)$; on the other hand, by [Mzk13], Proposition 1.2, (i), it follows that $C_{\Pi_H} (\Pi_e) = N_{\Pi_H} (\Pi_e) = D_e$; thus, $D_e = C_{\Pi_H} (D_e) = C_{\Pi_H} (\Pi_e) = N_{\Pi_H} (\Pi_e)$, as desired. Now it remains only to consider the case where $e$ is a node. In this case, since $I_e$ is abelian [cf. assertion (ii)], it follows that $I_e \subseteq Z_{\Pi_H} (I_e) \subseteq C_{\Pi_H} (I_e) \subseteq C_{\Pi_H} (I_e \cap \Pi_G) = C_{\Pi_H} (\Pi_e)$; thus, the fact that $D_e \cap \Pi_G = C_{\Pi_H} (I_e) = C_{\Pi_H} (I_e) = I_e$ follows from the fact that $I_e$ surjects onto $I$ [cf. assertion (ii)], together with the commensurable terminality of $\Pi_e$ in $\Pi_G$ [cf. [Mzk13], Proposition 1.2, (ii)]. This completes the proof of assertion (vii).

Next, we consider assertion (iv). Suppose that (2) holds. Then it follows from assertions (ii), (iii) [and the definitions] that $I_v \cap I_v' = \{1\}$, $I_e = \Pi_e \cdot I_v =$
Write $C_v, C_{v'}$ for the irreducible components of $X$ corresponding to $v, v'$. Suppose that both (1) and (2) are false. Recall that $\Pi_G$ [cf. [MT], Remark 1.2.2] and $I (\cong \hat{\mathbb{Z}}^\Sigma)$, hence also $\Pi_I$, are torsion-free. Thus, $I_v \times \Pi_v = D_v \cap \Pi_I$ [cf. assertion (iii)] is torsion-free, so by replacing $\Pi_H$ by an open subgroup of $\Pi_H$ [cf. Remark 1.2.1], we may assume without loss of generality that $\mathcal{G}$ is edge-paired [cf. §0], and that $\mathcal{G}$ is edge-paired [cf. §0]. Also, by projecting to the maximal pro-$l$ quotients, for some $l \in \Sigma$, of suitable open subgroups [cf. Remark 1.2.1] of the various pro-$\Sigma$ groups involved, one verifies immediately that we may assume without loss of generality [for the remainder of the proof of assertion (iv)] that $\Sigma = \{1\}$. When $l = 2$, we may also assume without loss of generality [by replacing $\Pi_H$ by an open subgroup of $\Pi_H$] that $\mathcal{G}$ is edge-even [cf. §0].

Now I claim that $D_v \cap D_{v'} \cap \Pi_G = \Pi_v \cap \Pi_{v'} = \{1\}$. Indeed, suppose that $\Pi_v \cap \Pi_{v'} \neq \{1\}$. Then one verifies immediately that there exist log admissible coverings [cf. [Mzk1], §3] $Y^{l \log} \rightarrow X^{l \log} \times s^{l \log} T^{l \log}$, corresponding to open subgroups $J \subseteq \mathcal{G}$, which are split over $C_{v'}$ [so $\Pi_{v'} \subseteq J$, $\Pi_v \cap \Pi_{v'} \subseteq \Pi_v \cap J$, but determine arbitrarily small neighborhoods $\Pi_v \cap J$ of the identity element in $\Pi_v$. [Here, we note that the existence of such coverings follows immediately from the fact that $X^{l \log}$ is edge-paired for arbitrary $l$ and edge-even when $l = 2$. That is to say, one starts by constructing the covering over $C_v$ in such a way that the ramification indices at the nodes and cusps of $C_v$ are all equal; one then extends the covering over the irreducible components of $X$ adjacent to $v$ [by applying the fact that $X^{l \log}$ is edge-paired for arbitrary $l$ and edge-even when $l = 2$ — cf. the discussion of §0] in such a way that the covering is unramified over the nodes of these irreducible components that do not abut to $C_v$; finally, one extends the covering to a split covering over the remaining portion of $X$ [which includes $C_{v'}$!] .] But the existence of such $J$ implies that $\Pi_v \cap \Pi_{v'} = \{1\}$, a contradiction. This completes the proof of the claim. Thus, the natural projection $D_v \cap D_{v'} \cap \Pi_I \rightarrow I$ has nontrivial open image [since $\Sigma = \{1\}$], which we denote by $I_{v,v'} \subseteq I$. Moreover, to complete the proof of assertion (iv), it suffices to derive a contradiction under the assumption that (1), (2), and (3) are false. Thus, for the remainder of the proof of assertion (iv), we assume that (1), (2), and (3) are false.

Write $C^+_v \subseteq X$ for the union of $C_v$ and the irreducible components of $X$ that are adjacent to $C_v$. We shall refer to a vertex of $\mathcal{G}$ as a $C^+_v$-vertex if it corresponds to an irreducible component of $C^+_v$. We shall say that a node $e$ is a bridge node if it abuts both to a $C^+_v$-vertex and to a non-$C^+_v$-vertex. Thus, no bridge node abuts to $v$. Now let us write $i_v$ for the least common multiple of the indices $i_e$ of the bridge nodes $e$; $i^\Sigma_v$ for the largest nonnegative power of $l$ dividing $i_v$. Let $d \overset{\text{def}}{=} l \cdot i_v \cdot [I : I_{v,v'}]$; $d^\Sigma \overset{\text{def}}{=} l \cdot i^\Sigma_v \cdot [I : I_{v,v'}]$ [so $d^\Sigma$ is the largest positive power of $l$ dividing $d$]. Here, we observe that for any open subgroup $J_0 \subseteq \Pi_I$ such that
$D_v \cap J_0$ surjects onto $I$ [cf. assertion (iii)], and $D_v' \cap \Pi_I \subseteq J_0$, it holds that

$$[I : I_{v,v'}] = ([D_v \cap J_0] : (\Pi_v \cap J_0) \cdot (D_v \cap D_v' \cap \Pi_I)]$$

[where we note that $D_v \cap D_v' \cap \Pi_I \subseteq D_v \cap J_0$. Thus, it suffices to construct open subgroups $J \subseteq J_0 \subseteq \Pi_I$ such that $D_v \cap J_0$ surjects onto $I$, $\Pi_v \cap J_0 \subseteq J$, and $D_v' \cap \Pi_I \subseteq J$ [which implies that $(\Pi_v \cap J_0) \cdot (D_v \cap D_v' \cap \Pi_I) \subseteq D_v \cap J_0$], but $[D_v \cap J_0 : D_v \cap J] > [I : I_{v,v'}]$.]

To this end, let us first observe that the characteristic sheaf of the log scheme $X^{\log}$ admits a section $\zeta$ over $X$ satisfying the following properties:

(a) $\zeta$ vanishes on the open subscheme of $X$ given by the complement of $C_v^+$ [hence, in particular, on $C_v'$];

(b) $\zeta$ coincides with $i_v \cdot \sigma \in M_\Sigma$ [cf. the notation of Example 1.1, (iii)] at the generic points of $C_v^+$;

(c) $\zeta$ coincides with either $(i_v/i_e) \cdot \xi \in M_e$ or $(i_v/i_e) \cdot \eta \in M_e$ [cf. the notation of Example 1.1, (iii)] at each bridge node $e$.

[Indeed, the existence of such a section $\zeta$ follows immediately from the discussion of Example 1.1, (iii), together with our definition of $i_v$.] Thus, by taking the inverse image of $\zeta$ in the monoid that defines the log structure of $X^{\log}$, we obtain a line bundle $\mathcal{L}$ on $X$. Let $Y \rightarrow X$ be a finite étale cyclic covering of order a positive power of $l$ such that $\mathcal{L}|_Y$ has degree divisible by $d^\Sigma$ on every irreducible component of $Y$, and $Y \rightarrow X$ restricts to a connected covering over every irreducible component of $X$ that is $\neq C_v'$ [e.g., $C_v$], but splits over $C_v'$; $Y^{\log} \overset{\text{def}}{=} X^{\log} \times_X Y$; $C_w \overset{\text{def}}{=} C_v \times_X Y$; $C_w^+ \overset{\text{def}}{=} C_v^+ \times_X Y$. [Note that the fact that such a covering exists follows immediately from our assumption that $G$ is sturdy.] Now let

$$Z^{\log} \rightarrow Y^{\log}$$

be a log étale cyclic covering of degree $d^\Sigma$ satisfying the following properties:

(d) $Z^{\log} \rightarrow Y^{\log}$ restricts to an étale covering of schemes over the complement of $C_w^+$ and splits over the irreducible components of $Y$ that lie over $C_v'$ [cf. (a); the fact that (1), (2), and (3) are assumed to be false!];

(e) $Z^{\log} \rightarrow Y^{\log}$ is ramified, with ramification index $d^\Sigma / i_v^\Sigma$, over the generic points of $C_w^+$, but induces the trivial extension of the function field of $C_w$ [cf. (b)];

(f) for each node $f$ of $Y$ that lies over a bridge node $e$ of $X$, the restriction of $Z^{\log} \rightarrow Y^{\log}$ to the branch of $f$ that does not abut to an irreducible component of $C_w^+$ is ramified, with ramification index $d^\Sigma \cdot i_e^\Sigma / i_v^\Sigma$ [cf. (c)].

Indeed, to construct such a covering $Z^{\log} \rightarrow Y^{\log}$, it suffices to construct a covering satisfying (d), (f) over the complement of $C_w^+$ [which is always possible, by the conditions imposed on $Y$, together with the fact that (1), (2), and (3) are assumed
to be false], and then to glue this covering to a suitable [i.e., such that (e) is satisfied] Kummer log étale covering of \((C_w^+)\) \(\log\) \(\text{def} Y^{\log} \times Y.C^+\) [by an fs log scheme!] obtained by extracting a \(d^E\)-th root of \(L|_{C_w^+}\) [cf. the divisibility condition on the degrees of \(L\) over the irreducible components of \(C_w^+\)]. [Here, we regard the \(G_m\)-torsor determined by \(C_w\) as a subsheaf of the monoid defining the log structure of \((C_w^+)\) \(\log\).] Now if we write \(J_Z \subseteq J_Y \subseteq \Pi_I\) for the open subgroups defined by the coverings \(Z^{\log} \to Y^{\log} \to X^{\log}\), then \(D_v \cap J_Y\) surjects onto \(I\); \(D_v \cap \Pi_I \subseteq J_Y\). On the other hand, \(\Pi_v \cap J_Y \subseteq J_Z\) [cf. (e)] and \(D_v \cap \Pi_I \subseteq J_Z\) [cf. (d)], while [cf. (e)]

\[
[D_v \cap J_Y : D_v \cap J_Z] = d^E / i_v^E > [I : i_v, v']
\]

[since \(d^E = I \cdot i_v^E \cdot [I : i_v, v']\)]. Thus, it suffices to take \(J_0 \text{ def} J_Y, J \text{ def} J_Z\). This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that it follows from assertion (iv) [i.e., by applying assertion (iv) to various open subgroups of \(\Pi_H, \Pi_I\) — cf. also Remark 1.2.1] that if, for \(\gamma \in \Pi_H, I_v \cap (\gamma \cdot I_v \cdot \gamma^{-1}) \neq \{1\}\), then \(\Pi_v = \gamma \cdot \Pi_v \cdot \gamma^{-1}\). Thus, we conclude that \(N_{\Pi_H}(I_v) \subseteq C_{\Pi_H}(I_v) \subseteq N_{\Pi_H}(\Pi_v) = D_v\). On the other hand, since [by definition] \(I_v = Z_{\Pi_H}(\Pi_v)\), and \(I_v\) is normal in \(H\), it follows that \(D_v = N_{\Pi_H}(\Pi_v) \subseteq N_{\Pi_H}(I_v)\), so \(N_{\Pi_H}(I_v) = C_{\Pi_H}(I_v) = D_v\), as desired. In particular, \(D_v \cap \Pi_I = N_{\Pi_H}(I_v) = C_{\Pi_H}(I_v)\). Next, let us observe that \(D_v \cap \Pi_G = \Pi_v\) [cf. [Mzk13], Proposition 1.2, (ii)]. Thus, \(D_v \subseteq C_{\Pi_H}(D_v) \subseteq C_{\Pi_H}(\Pi_v)\). Moreover, by [Mzk13], Proposition 1.2, (i), it follows that \(C_{\Pi_H}(\Pi_v) = N_{\Pi_H}(\Pi_v) = D_v\); thus, we conclude that \(D_v\) (respectively, \(D_v \cap \Pi_I; D_v \cap \Pi_G\)) is commensurably terminal in \(\Pi_H\) (respectively, \(\Pi_I; \Pi_G\)). Finally, by assertion (iii), we have \(D_v \cap \Pi_I = I_v \times \Pi_v \subseteq Z_{\Pi_H}(I_v) \subseteq N_{\Pi_H}(I_v) = D_v \cap \Pi_I\), so \(D_v \cap \Pi_I = Z_{\Pi_H}(I_v)\), as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). The fact that the image of \(D_v\) in \(H\) is open follows immediately from the fact that since the semi-graph \(G\) is finite, some open subgroup of \(H\) necessarily fixes \(v\). On the other hand, if \(G\) admits a vertex \(v' \neq v\), then \(\Pi_v \cap \Pi_{v'}\) is not open in \(\Pi_{v'}\) [cf. [Mzk13], Proposition 1.2, (i)]; since \(D_v \cap \Pi_G = \Pi_v\) [cf. [Mzk13], Proposition 1.2, (ii)], this implies that \(D_v\) is not open in \(\Pi_H\). This completes the proof of assertion (vi).

Next, we consider assertion (viii). First, we observe that if property (2) holds, then by assertions (ii), (iii), [for appropriate choices of conjugates] \((\tilde{Z}^\Sigma \cong I_v \subseteq D_v \cap D_{v'} \cap \Pi_I \lra I, so I_v = D_v \cap D_{v'} \cap \Pi_I\). Thus, it suffices to verify that either (1) or (2) holds. Next, let us observe that, since, as observed above, \(\Pi_I, \Pi_G\), hence also \(D_v \cap D_{v'} \cap \Pi_{v'}\), is torsion-free, by projecting to the maximal pro-\(l\) quotients, for some \(l \in \Sigma\), of suitable open subgroups [cf. Remark 1.2.1] of the various pro-\(\Sigma\) groups involved, one verifies immediately we may assume without loss of generality [for the remainder of the proof of assertion (viii)] that \(\Sigma = \{l\}\). Now if \(D_v \cap D_{v'} \cap \Pi_G \neq \{1\}\), then since \(\Sigma = \{l\}\) \(D_v \cap D_{v'} \cap \Pi_G\) is open in \(D_v \cap \Pi_G\) \((\cong \tilde{Z}^\Sigma), D_{v'} \cap \Pi_G \cong \tilde{Z}^\Sigma\) [cf. assertions (ii), (iii), (vii), so we conclude from [Mzk13], Proposition 1.2, (i), that \(e = e'\). Thus, to complete the proof of assertion (viii), it suffices to derive a contradiction under the further assumption that \(D_v \cap D_{v'} \cap \Pi_G = \{1\}\), and \(e\) and \(e'\) do not abut to a common vertex. Moreover, by replacing \(\Pi_H\) by an open subgroup of \(\Pi_H\) [cf. Remark 1.2.1], we may
assume without loss of generality that \( \mathcal{G} \) [i.e., \( X^\log \)] is sturdy [cf. §0], and that \( \mathcal{G} \) is edge-paired [cf. §0] for arbitrary \( l \) and edge-even [cf. §0] when \( l = 2 \).

Now if, say, \( e \) is a cusp that abuts to a vertex \( v \), then one verifies immediately that there exist log étale cyclic coverings \( Y^\log \to X^\log \) of degree an arbitrarily large power of \( l \) which are totally ramified over \( e \), but unramified over the nodes of \( X \), as well as over the cusps of \( X \) that abut to vertices \( \neq v \). [Indeed, the existence of such coverings follows immediately from the fact that \( X^\log \) is edge-paired for arbitrary \( l \) and edge-even when \( l = 2 \) — cf. the discussion of §0.] In particular, such coverings are unramified over \( e' \), as well as over the generic point of the irreducible component of \( X \) corresponding to \( v \), hence correspond to open subgroups \( J \subseteq \Pi_I \) such that \( D_e \cap \Pi_J \subseteq J \) [so \( D_e \cap D_e' \cap \Pi_J \subseteq J \cap D_e \cap \Pi_J ] \), and, moreover, \( J \) may be chosen so that the subgroup \( J \cap \Pi_J \subseteq D_e \cap \Pi_J = I_v \times I_v \) [cf. assertion (iii)] forms an arbitrarily small neighborhood of \( I_v \). Thus, we conclude that \( D_e \cap D_e' \cap \Pi_J \subseteq I_v \).

On the other hand, if \( e' \) is also a cusp that abuts to a vertex \( v' \), then [by symmetry] we conclude that \( D_e \cap D_e' \cap \Pi_J \subseteq I_v \), hence that \( I_v \cap I_v' \neq \{1\} \). But, by assertion (iv), this implies that \( v = v' \), a contradiction. Thus, we may assume that, say, \( e \) is a node, so \( I_v = D_e \cap \Pi_J \) [cf. assertion (vii)].

Write \( v_1, v_2 \) for the two distinct vertices to which \( e \) abuts; \( C_1, C_2 \) for the irreducible components of \( X \) corresponding to \( v_1, v_2 \); \( C \overset{\text{def}}{=} C_1 \cup C_2 \subseteq X \); \( U_C \subseteq C \) for the open subscheme obtained by removing the nodes that abut to vertices \( \neq v_1, v_2 \). Let us refer to the nodes and cusps of \( U_C \) as inner, to the nodes of \( X \) that were removed from \( C \) to obtain \( U_C \) as bridge nodes, and to the nodes and cusps of \( X \) which are neither inner nodes/cusps nor bridge nodes as external. [Thus, \( e \) is inner; \( e' \) is external.] Observe that the natural projection to \( I \) yields an inclusion \( D_e \cap D_e' \cap \Pi_J \to I \) with open image [since \( \Sigma = \{l\} \)]; denote the image of this inclusion by \( I_C \). Write \( i_C \) for the least common multiple of the indices \( i_f \) of the bridge nodes \( f \); \( i_C^2 \) for the largest nonnegative power of \( l \) dividing \( i_C \). Let \( d \overset{\text{def}}{=} l \cdot i_C \cdot [I : I_C] \); \( d^\Sigma \overset{\text{def}}{=} l \cdot i_C^2 \cdot [I : I_C] \) [so \( d^\Sigma \) is the largest positive power of \( l \) dividing \( d \)]. Here, we observe that 

\[
[I : I_C] = [I_v : \Pi_v \cdot (D_e \cap D_e' \cap \Pi_J)]
\]

[cf. assertion (ii)]. Then it suffices to construct an open subgroup \( J \subseteq \Pi_I \) such that \( \Pi_e \subseteq J \) and \( D_v \cap \Pi_J \subseteq J \) [which implies that \( \Pi_e \cdot (D_v \cap \Pi_J) \subseteq I_v \cap J \)], but \( [I_v : \Pi_v \cdot J] > [I : I_C] \).

To this end, let us first observe that the characteristic sheaf of the log scheme \( X^\log \) admits a section \( \zeta \) over \( X \) satisfying the following properties:

(a) \( \zeta \) vanishes on the open subscheme of \( X \) given by the complement of \( C \);

(b) \( \zeta \) coincides with \( i_C \cdot \sigma \in M_S \) [cf. the notation of Example 1.1, (iii)] at the generic points of \( C \overset{\text{def}}{=} C_1 \cup C_2 \);

(c) \( \zeta \) coincides with either \( (i_C/i_f) \cdot \xi \in M_f \) or \( (i_C/i_f) \cdot \eta \in M_f \) [cf. the notation of Example 1.1, (iii)], where we take “\( e \)” to be \( f \) at each bridge node \( f \).
[Indeed, the existence of such a section \( \zeta \) follows immediately from the discussion of Example 1.1, (iii), together with our definition of \( i_C \).] Thus, by taking the inverse image of \( \zeta \) in the monoid that defines the log structure of \( X^{\log} \), we obtain a line bundle \( \mathcal{L} \) on \( X \). Let \( Y \rightarrow X \) be a \textit{finite étale Galois covering} of order a positive power of \( l \) such that \( \mathcal{L}|_Y \) has \textit{degree divisible by} \( d^\Sigma \) on every irreducible component of \( Y \), and \( Y \rightarrow X \) restricts to a \textit{connected covering} over every irreducible component of \( X \); \( Y^{\log} \) def \( X^{\log} \times_X Y \). [Note that the fact that such a covering exists follows immediately from our assumption that \( \mathcal{G} \) is \textit{sturdy}.] Write \( C_1^Y \), \( C_2^Y \) for the irreducible components of \( Y \) lying over \( C_1 \), \( C_2 \), respectively; we shall apply the terms \"internal\", \"external\", and \"bridge\" to nodes/cusps of \( Y \) that lie over such nodes/cusps of \( X \). Now let

\[
Z^{\log} \rightarrow Y^{\log}
\]

be a \textit{log étale cyclic covering of degree} \( d^\Sigma \) satisfying the following properties:

(d) \( Z^{\log} \rightarrow Y^{\log} \) restricts to an \textit{étale covering} of schemes over the complement of \( C^Y \) def \( C_1^Y \cup C_2^Y \) [cf. (a)], hence, in particular, over the \textit{external nodes/cusps} of \( Y \);

(e) \( Z^{\log} \rightarrow Y^{\log} \) is \textit{ramified}, with ramification index \( d^\Sigma / i_C^Y \), over the generic points of \( C_1^Y \), \( C_2^Y \), and, at each \textit{internal node} of \( Y \) lying over \( e \), determines a covering corresponding to an open subgroup of \( I_e \) that contains \( \Pi_e \) [cf. (b)];

(f) for each \textit{bridge node} \( f \) of \( Y \), the restriction of \( Z^{\log} \rightarrow Y^{\log} \) to the branch of \( f \) that does not abut to \( C^Y \) is \textit{ramified}, with ramification index \( d^\Sigma / i_f^Y / i_C^Y \) [cf. (c)].

Indeed, to construct such a covering \( Z^{\log} \rightarrow Y^{\log} \), it suffices to construct a covering satisfying (d), (f) over the complement of \( C^Y \) [which is always possible, by the conditions imposed on \( Y \)], and then to \textit{glue} this covering to a \textit{suitable} [i.e., such that (e) is satisfied!] \textit{Kummer log étale covering} of \( (C^Y)^{\log} \) def \( Y^{\log} \times_Y C^Y \) [by an fs log scheme!] obtained by \textit{extracting} a \( d^\Sigma \)-th root of \( \mathcal{L}|_{C^Y} \) [cf. the \textit{divisibility} condition on the degrees of \( \mathcal{L}|_{C_1^Y}, \mathcal{L}|_{C_2^Y} \)]. [Here, we regard the \( \mathbb{G}_{m-torsor} \) determined by \( \mathcal{L}|_{C^Y} \) as a \textit{subsheaf} of the monoid defining the log structure of \( (C^Y)^{\log} \).] Now if we write \( J_Z \subset J_Y \subset \Pi_I \) for the open subgroups defined by the coverings \( Z^{\log} \rightarrow Y^{\log} \rightarrow X^{\log} \), then \( I_e, D_e \cap \Pi_I \subset J_Y \). On the other hand, \( \Pi_e \subset J_Z \) [cf. (e)] and \( D_e \cap \Pi_I \subset J_Z \) [cf. (d)], while [cf. (e)]

\[
[I_e : I_e \cap J_Z] = d^\Sigma / i_C^Y > [I : I_C]
\]

[since \( d^\Sigma = l \cdot i_C^Y \cdot [I : I_C] \)]. Thus, it suffices to take \( J \) def \( J_Z \). This completes the proof of assertion (viii).

Next, we consider assertion (ix). The fact that the image of \( D_e \) in \( H \) is \textit{open} follows immediately from the fact that since the semi-graph \( \mathcal{G} \) is \textit{finite}, some open subgroup of \( H \) necessarily \textit{fixes} \( e \). On the other hand, since \( D_e \cap \Pi_\mathcal{G} = N_{\Pi_\mathcal{G}}(\Pi_e) = \Pi_e \) [cf. [Mzk13], Proposition 1.2, (ii)] is \textit{abelian}, hence not \textit{open} in the \textit{slim, nontrivial}
profinite group $\Pi_G$, it follows that $D_e$ is not open in $\Pi_H$. This completes the proof of assertion (ix).

Finally, we consider assertion (x). First, let us observe that an easy computation reveals that if the image of $\tau_S$ is a non-nodal, non-cuspidal point of the irreducible component of $X$ corresponding to a vertex $v_\tau$ of $G$, then $\tau_1(I) = I_{v_\tau}$. Next, let us suppose that the image of $\tau_S$ is the node of $X$ corresponding to some node $e_\tau$ of $G$. Then an easy computation [cf. the computations performed in the proof of assertion (ii)] reveals that $\tau_1(I) \subseteq I_{e_\tau}$, but that $\tau_1(I)$ is not contained in $I_{v'}$ for any vertex $v'$ to which $e_\tau$ abuts. If, moreover, $\tau_1(I) \subseteq I_v$ for some vertex $v$ to which $e_\tau$ does not abut, then [since the very existence of the node $e_\tau$ implies that $X$ is singular] there exists a node $e \neq e_\tau$ that abuts to $v$, so $\tau_1(I) \subseteq I_v \subseteq I_e$ [cf. assertion (ii)]; but this implies that $\tau_1(I) \subseteq I_3 \cap I_{e_\tau}$, so, by assertion (viii), it follows that $\tau_1(I) \subseteq I_{v'}$ for some vertex to which both $e$ and $e_\tau$ abut — a contradiction. Thus, in summary, we conclude that in this case, $\tau_1$ is non-vertical.

Now suppose that $\tau_1$ is non-vertical and non-edge-like. Then the observations of the preceding paragraph imply that the image of $\tau_S$ is a cusp of $X$. Write $e_\tau$ for the corresponding cusp of $G$. Thus, one verifies immediately that $\tau_1(I) \subseteq D_{e_\tau}$. The uniqueness of $e_\tau$ then follows from assertion (viii) [and the fact that $\tau_1$ is non-vertical]. Thus, for the remainder of the proof of assertion (x), we may assume that the image of $\tau_S$ is not a cusp. Now the remainder of assertion (x) follows formally, in light of what of we have done so far, from assertions (iv), (viii). This completes the proof of assertion (x). $\square$

**Corollary 1.4.** (Graphicity of Isomorphisms of (D)PSC-Extensions)

Let $l$ be a prime number. For $i = 1, 2$, let $1 \to \Pi_{G_i} \to \Pi_{H_i} \to H_i \to 1$ be an $l$-cyclotomically full [cf. Remark 1.2.3] DPSC-extension (respectively, PSC-extension), associated to construction data $(X_i^\log \to S_i^\log, \Sigma_i, G_i, \rho_{H_i}, \xi_i)$ (respectively, $(X_i^\log \to S_i^\log, \Sigma_i, G_i, \rho_{H_i})$) such that $l \in \Sigma_i$; in the non-resp’d case, write $I_i \subseteq H_i$ for the inertia subgroup. Let

$$\phi_H : H_1 \simeq H_2; \quad \phi_\Pi : \Pi_{G_1} \simeq \Pi_{G_2}$$

be compatible [i.e., with the respective outer actions of $H_i$ on $\Pi_{G_i}$] isomorphisms of profinite groups; in the non-resp’d case, suppose further that $\phi_H(I_1) = I_2$. Then $\Sigma_1 = \Sigma_2$; $\phi_\Pi$ is graphic [cf. [Mzk13], Definition 1.4, (i)], i.e., arises from an isomorphism of semi-graphs of anabelioids $G_1 \simeq G_2$.

*Proof.* This follows immediately from [Mzk13], Corollary 2.7, (i), (iii). Here, as in the proof of [Mzk13], Corollary 2.8, we first apply [Mzk13], Corollary 2.7, (i) [which suffices to complete the proof of Corollary 1.4 in the resp’d case and allows one to reduce to the noncuspidal case in the non-resp’d case], then apply [Mzk13], Corollary 2.7, (iii), to the compactifications of corresponding sturdy finite étale coverings of the $G_i$. $\square$

We are now ready to define a purely group-theoretic, combinatorial analogue of the notion of a stable polycurve given in [Mzk2], Definition 4.5.
Definition 1.5. We shall refer to an extension of profinite groups as a PPSC-extension [i.e., “poly-PSC-extension”] if, for some positive integer \( n \) and some nonempty set of primes \( \Sigma \), it admits a “structure of pro-\( \Sigma \) PPSC-extension of dimension \( n \)”. Here, for \( n \) a positive integer, \( \Sigma \) a nonempty set of primes, and

\[
1 \to \Delta \to \Pi \to H \to 1
\]

an extension of profinite groups, we define the notion of a structure of pro-\( \Sigma \) PPSC-extension of dimension \( n \) as follows [inductively on \( n \)]:

(i) A structure of pro-\( \Sigma \) PPSC-extension of dimension 1 on the extension \( 1 \to \Delta \to \Pi \to H \to 1 \) is defined to be a structure of pro-\( \Sigma \) PSC-extension. Suppose that the extension \( 1 \to \Delta \to \Pi \to H \to 1 \) is equipped with a structure of pro-\( \Sigma \) PPSC-extension of dimension 1. Thus, we have an associated semi-graph of anabelioids \( G \), together with a continuous action of \( H \) on \( G \), and a compatible isomorphism \( \Delta \cong \Pi \). We define the [horizontal] divisors of this PPSC-extension to be the cusps of the PSC-extension \( 1 \to \Delta \to \Pi \to H \to 1 \). Thus, each divisor \( c \) of the PPSC-extension \( 1 \to \Delta \to \Pi \to H \to 1 \) has associated inertia and decomposition groups \( I_c \subseteq \Delta \) and \( D_c \subseteq \Pi \) [cf. Definition 1.2, (i)]. Moreover, by [Mzk13], Proposition 1.2, (i), a divisor is completely determined by [the conjugacy class of] its inertia group, as well as by [the conjugacy class of] its decomposition group. Finally, we shall refer to the extension \( 1 \to \Delta \to \Pi \to H \to 1 \) [itself] as the fiber extension associated to the PPSC-extension \( 1 \to \Delta \to \Pi \to H \to 1 \) of dimension 1.

(ii) A structure of pro-\( \Sigma \) PPSC-extension of dimension \( n + 1 \) on the extension \( 1 \to \Delta \to \Pi \to H \to 1 \) is defined to be a collection of data as follows:

(a) a quotient \( \Pi \to \Pi^* \) such that \( \Delta^\dagger \overset{\text{def}}{=} \ker(\Pi \to \Pi^*) \subseteq \Delta \); thus, the image \( \Delta^* \subseteq \Pi^* \) of \( \Delta \) in \( \Pi^* \) determines an extension

\[
1 \to \Delta^* \to \Pi^* \to H \to 1
\]

— which we shall refer to as the associated base extension; the subgroup \( \Delta^\dagger \subseteq \Pi \) determines an extension

\[
1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1
\]

— which we shall refer to as the associated fiber extension;

(b) a structure of pro-\( \Sigma \) PPSC-extension of dimension \( n \) on the base extension \( 1 \to \Delta^* \to \Pi^* \to H \to 1 \);

(c) a structure of pro-\( \Sigma \) PPSC-extension of dimension 1 on the fiber extension \( 1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1 \);

(d) for each base divisor [i.e., divisor of the base extension] \( c^* \), a structure of DPSC-extension on the extension

\[
1 \to \Delta^\dagger \to \Pi_c^* \overset{\text{def}}{=} \Pi \times_{\Pi^*} D_c^* \to D_c^* \to 1
\]

— which we shall refer to as the extension at \( c^* \) — which is compatible with the PSC-extension structure on the fiber extension [cf. (c)], in the
Remark 1.5.2. Let \( 1 \to \Delta \to \Pi \to H \to 1 \); we shall refer to as a divisor of the PPSC-extension 
\( 1 \to \Delta \to \Pi \to H \to 1 \) any element of the union of the set of cusps — which we shall refer to as horizontal divisors — of the PSC-extension 
\( 1 \to \Delta \to \Pi \to H \to 1 \) and, for each base divisor \( c^* \), the set of vertices of the DPSC-extension 
\( 1 \to \Delta \to \Pi \to H \to 1 \) — which we shall refer to as vertical divisors [lying over \( c^* \)]. Thus, each divisor \( c \) of the PPSC-extension 
\( 1 \to \Delta \to \Pi \to H \to 1 \) has associated inertia and decomposition groups \( I_c \subseteq D_c \subseteq \Pi \). In particular, whenever \( c \) is vertical and lies over a base divisor \( c^* \), we have \( I_c \subseteq D_c \subseteq \Pi_{c^*} \).

Remark 1.5.1. Thus, [the collection of fiber extensions arising from] any structure of PPSC-extension of dimension \( n \) on an extension 
\( 1 \to \Delta \to \Pi \to H \to 1 \) determine two compatible sequences of surjections

\[
\Delta_n \overset{\text{def}}{=} \Delta \to \Delta_{n-1} \to \ldots \to \Delta_1 \to \Delta_0 \overset{\text{def}}{=} \{1\}
\]

\[
\Pi_n \overset{\text{def}}{=} \Pi \to \Pi_{n-1} \to \ldots \to \Pi_1 \to \Pi_0 \overset{\text{def}}{=} H
\]

such that each [extension determined by a] surjection \( \Pi_m \to \Pi_{m-1} \), for \( m = 1, \ldots, n \), is a fiber extension [hence equipped with a structure of PSC-extension];
\( \Delta_m = \ker(\Pi_m \to H) \). If \( c = c_n \) is a divisor of [the extension determined by] \( \Pi = \Pi_n \), then [cf. Definition 1.5, (ii)] there exists a uniquely determined sequence of divisors

\[
c_n \mapsto c_{n-1} \mapsto \ldots \mapsto c_{n_c-1} \mapsto c_{n_c}
\]

— where \( n_c \leq n \) is a positive integer; for \( m = n_{c}, \ldots, n \), \( c_m \) is a divisor of \( \Pi_m \);
\( c_{n_c} \) is a horizontal divisor; the notation “\( \mapsto \)” denotes the relation of “lying over” [so \( c_{m+1} \) is a vertical divisor that lies over \( c_m \), for \( n_c \leq m < n \)] — together with sequences of [conjugacies classes of] inertia and decomposition groups

\[
I_{c_n} \to I_{c_{n-1}} \to \ldots \to I_{c_{n_c-1}} \to I_{c_{n_c}}
\]

\[
D_{c_n} \to D_{c_{n-1}} \to \ldots \to D_{c_{n_c-1}} \to D_{c_{n_c}}
\]

[i.e., for \( n_c \leq m < n \), \( I_{c_{m+1}} \subseteq \Pi_{m+1} \) maps into \( I_{c_m} \subseteq \Pi_m \), and \( D_{c_{m+1}} \subseteq \Pi_{m+1} \) maps into \( D_{c_m} \subseteq \Pi_m \)].

Remark 1.5.2. Let \( 1 \to \Delta \to \Pi \to H \to 1 \) be a PPSC-extension of dimension \( n \) [where \( n \) is a positive integer]. Then one verifies immediately that if \( \Pi_\bullet \subseteq \Pi \) is any open subgroup of \( \Pi \), then there exists an open subgroup \( \Pi_{\bullet \bullet} \) of \( \Pi_\bullet \) that [when equipped with the induced extension structure] admits a structure of PPSC-extension of dimension \( n \) — cf. Remark 1.2.1. Here, we note that one must, in general, pass to “some open subgroup” \( \Pi_{\bullet \bullet} \) of \( \Pi_\bullet \) in order to achieve a situation in which all of the fiber [PSC]-extensions have “stable reduction” [cf. Remark 1.2.1; Definition 1.5, (ii), (d)].
Remark 1.5.3. For \( l \) a prime number, we shall say that a PPSC-extension \( 1 \to \Delta \to \Pi \to H \to 1 \) of dimension \( n \) is \( l \)-cyclotomically full if each of its \( n \) associated fiber extensions [cf. Remark 1.5.1] is \( l \)-cyclotomically full as a PSC-extension [cf. Remark 1.2.3]. Thus, it follows immediately from the final portion of Remark 1.2.3 that the issue of whether or not the PPSC-extension \( 1 \to \Delta \to \Pi \to H \to 1 \) is \( l \)-cyclotomically full depends only on the sequence of surjections of profinite groups \( \Pi_n \to \Pi_{n-1} \to \ldots \to \Pi_1 \to \Pi_0 \) [cf. Remark 1.5.1].

Remark 1.5.4. Let \( k \) be a field; \( \overline{k} \) a separable closure of \( k \); \( G_k \) def \( = \text{Gal}(\overline{k}/k) \); \( S \) def \( = \text{Spec}(k) \);

\[
Z^\log \to S
\]

the log scheme determined by a stable polycurve over \( S \) — i.e., \( Z^\log \) admits a successive fibration by generically smooth stable log curves [cf. [Mzk2], Definition 4.5, for more details]; \( U_Z \subseteq Z \) the interior of \( Z^\log \); \( D_Z \) def \( = Z \setminus U_Z \) [with the reduced induced structure]; \( n \) the dimension of the scheme \( Z \);

\[
1 \to \Delta_Z \overset{\text{def}}{=} \pi_1(U_Z \times_k \overline{k}) \to \Pi_Z \overset{\text{def}}{=} \pi_1(U_Z) \to G_k \to 1
\]

the exact sequence of étale fundamental groups [well-defined up to inner automorphism] associated to the structure morphism \( U_Z \to S \). Then by repeated application of the discussion of Example 1.1 to the fibers of the successive fibration [mentioned above] of \( Z^\log \) by stable log curves, one verifies immediately that:

(i) If \( k \) is of characteristic zero, then the structure of stable polycurve on \( Z^\log \) determines a structure of profinite PPSC-extension of dimension \( n \) on the extension \( 1 \to \Delta_Z \to \Pi_Z \to G_k \to 1 \).

Moreover, one verifies immediately that:

(ii) In the situation of (i), the divisors of \( \Pi_Z \) [in the sense of Definition 1.5] are in natural bijective correspondence with the irreducible divisors of \( D_Z \) in a fashion that is compatible with the inertia and decomposition groups of divisors of \( \Pi_Z \) [in the sense of Definition 1.5] and of irreducible divisors of \( D_Z \) [in the usual sense].

Finally, whether or not \( k \) is of characteristic zero, depending on the structure of \( Z^\log \) [cf., e.g., Corollary 1.10 below], various quotients of the extension \( 1 \to \Delta_Z \to \Pi_Z \to G_k \to 1 \) may be equipped with a structure of pro-\( \Sigma \) PPSC-extension [induced by the structure of stable polycurve on \( Z \)], for various nonempty sets of prime numbers \( \Sigma \) that are not equal to the set of all prime numbers; a similar observation to (ii) concerning a natural bijective correspondence of “divisors” then applies to such quotients. When considering such quotients \( 1 \to \Delta \to \Pi \to H \to 1 \) of the extension \( 1 \to \Delta_Z \to \Pi_Z \to G_k \to 1 \), it is useful to observe that the slimness of \( \Delta \) [cf. Proposition 1.6, (i), below] implies that such a quotient \( \Pi_Z \to \Pi \) is completely determined by the induced quotients \( \Delta_Z \to \Delta, G_k \to H \) [cf. the discussion of the notation “\( \Gamma^\times \)” in §0]; we shall refer to such a quotient \( 1 \to \Delta \to \Pi \to H \to 1 \) as...
a PPSC-extension arising from $\mathbb{Z}^{\log}, \bar{k}$ — where we write $\bar{k} \subseteq \tilde{k}$ for the subfield fixed by $\text{Ker}(G_k \to H)$.

**Proposition 1.6.** (Basic Properties of PPSC-Extensions) Let

$$1 \rightarrow \Delta \rightarrow \Pi \rightarrow H \rightarrow 1$$

be a pro-$\Sigma$ PPSC-extension of dimension $n$ [where $n$ is a positive integer]; $1 \rightarrow \Delta^* \rightarrow \Pi \rightarrow \Pi^* \rightarrow 1$ the associated fiber extension; $c, c'$ divisors of $\Pi$. Then:

(i) $\Delta$ is slim. In particular, if $H$ is slim, then so is $\Pi$.

(ii) $D_c$ is commensurably terminal in $\Pi$.

(iii) We have: $C_\Pi(I_c) = D_c$. As abstract profinite groups, $I_c \cong \mathbb{Z}^\Sigma$.

(iv) $D_c$ is not open in $\Pi$. The divisor $c$ is horizontal if and only if $D_c$ projects to an open subgroup of $\Pi^*$. If $c$ is vertical and lies over a base divisor $c^*$, then $D_c$ projects onto an open subgroup of $D_{c^*}$.

(v) If $D_c \cap D_{c'}$ is open in $D_c, D_{c'}$, then $c = c'$. In particular, a divisor of $\Pi$ is completely determined by its associated decomposition group.

(vi) If $I_c \cap I_{c'}$ is open in $I_c, I_{c'}$, then $c = c'$. In particular, a divisor of $\Pi$ is completely determined by its associated inertia group.

**Proof.** Assertion (i) follows immediately from the “slimness of $\Pi_c$” discussed in Example 1.1, (ii) [cf. Definition 1.5, (i); Definition 1.5, (ii), (c)]. Next, we consider assertion (ii). We apply induction on $n$. If $c$ is horizontal, then assertion (ii) follows from [the argument applied in] Proposition 1.3, (vii) [cf. also Definition 1.5, (ii), (c)]. If $c$ is vertical, then $c$ lies over some base divisor $c^*$, and we are in the situation of Definition 1.5, (ii), (d). By Proposition 1.3, (vi), it follows that $D_c$ surjects onto some open subgroup of $D_{c^*}$, hence that $C_\Pi(D_c)$ maps into $C_\Pi^*(D_{c^*})$; by the induction hypothesis, $C_\Pi^*(D_{c^*}) = D_{c^*}$, so $C_\Pi(D_c) \subseteq \Pi_{c^*}$. Thus, the fact that $C_\Pi(D_c) = D_c$ follows from Proposition 1.3, (v). This completes the proof of assertion (ii).

Next, we consider assertion (iii). Again we apply induction on $n$. If $c$ is horizontal, then assertion (iii) follows from [the argument applied in] Proposition 1.3, (i), (vii). If $c$ is vertical, then $c$ lies over some base divisor $c^*$, and we are in the situation of Definition 1.5, (ii), (d). By Proposition 1.3, (iii) [cf. Definition 1.5, (ii), (d)], we have isomorphisms $I_c \cong I_{c^*} \cong \mathbb{Z}^\Sigma$. In particular, $C_\Pi(I_c)$ maps into $C_\Pi^*(I_{c^*})$; by the induction hypothesis, $C_\Pi^*(I_{c^*}) = D_{c^*}$. Thus, $C_\Pi(I_c) \subseteq \Pi_{c^*}$, so the fact that $C_\Pi(I_c) = D_c$ follows from Proposition 1.3, (v). This completes the proof of assertion (iii).

Next, we consider assertion (iv). Again we apply induction on $n$. If $c$ is horizontal, then by [the argument applied in] Proposition 1.3, (ix), $D_c$ is not open in $\Pi$, but $D_c$ projects to an open subgroup of $\Pi^*$. If $c$ is vertical, then $c$ lies over some base divisor $c^*$, and we are in the situation of Definition 1.5, (ii), (d); $D_c \subseteq \Pi_{c^*}$. By Proposition 1.3, (vi), $D_c$ projects onto an open subgroup of $D_{c^*}$. By the induction
hypothesis, $D_{c^*}$ is not open in $\Pi^*$, so $\Pi_{c^*}$ is not open in $\Pi$; thus, $D_c$ is not open in $\Pi$, and its image in $\Pi^*$ is not open in $\Pi'$. This completes the proof of assertion (iv).

Next, we consider assertion (v). Again we apply induction on $n$. By assertion (iv), $c$ is horizontal if and only if $c'$ is. If $c, c'$ are horizontal, then the fact that $c = c'$ follows from [Mzk13], Proposition 1.2, (i), (ii). Thus, we may suppose that $c, c'$ are vertical and lie over respective base divisors $c^*, (c')^*$. By assertion (iv), it follows that $D_{c^*} \cap D_{(c')^*}$ is open in $D_{c^*}, D_{(c')^*}$; by the induction hypothesis, this implies that $c^* = (c')^*$. Thus, by intersecting with “$\Pi^\square$” [cf. Proposition 1.3, (v)] and applying [Mzk13], Proposition 1.2, (i), we conclude that $c = c'$. This completes the proof of assertion (v). Finally, we observe that assertion (vi) is an immediate consequence of assertions (iii), (v). ∎

We are now ready to state and prove the main result of the present §1.

**Theorem 1.7.** (Graphicity of Isomorphisms of PPSC-Extensions) Let $l$ be a prime number; $n$ a positive integer. For $\square = \alpha, \beta$, let $\Sigma^{\square}$ be a nonempty set of primes; $1 \to \Delta^{\square} \to \Pi^{\square} \to H^{\square} \to 1$ an $l$-cyclotomically full [cf. Remark 1.5.3] pro-$\Sigma^{\square}$ PPSC-extension of dimension $n$;

$$
\Pi^{\square}_n \overset{\text{def}}{=} \Pi^{\square} \to \Pi^{\square}_{n-1} \to \ldots \to \Pi^{\square}_1 \to \Pi^{\square}_0 \overset{\text{def}}{=} H^{\square}
$$

the sequence of successive fiber extensions associated to $\Pi^{\square}$ [cf. Remark 1.5.1].

Let

$$
\phi : \Pi^\alpha \xrightarrow{\sim} \Pi^\beta
$$

be an isomorphism of profinite groups that induces isomorphisms $\phi_m : \Pi^\alpha_m \xrightarrow{\sim} \Pi^\beta_m$, for $m = 0, 1, \ldots, n$ [so $\phi = \phi_n$]. Then:

(i) We have $\Sigma^\alpha = \Sigma^\beta$.

(ii) For $m \in \{1, \ldots, n\}$, $\phi_m$ induces a bijection between the set of divisors of $\Pi^\alpha_m$ and the set of divisors of $\Pi^\beta_m$.

(iii) For $m \in \{1, \ldots, n\}$, suppose that $c^\alpha, c^\beta$ are divisors of $\Pi^\alpha_m, \Pi^\beta_m$, respectively, that correspond via the bijection of (ii). Then $\phi_m(I_{c^\alpha}) = I_{c^\beta}, \phi_m(D_{c^\alpha}) = D_{c^\beta}$. That is to say, $\phi_m$ is compatible with the inertia and decomposition groups of divisors.

(iv) For $m \in \{0, \ldots, n - 1\}$, the isomorphism

$$
\text{Ker}(\Pi^\alpha_{m+1} \to \Pi^\alpha_m) \xrightarrow{\sim} \text{Ker}(\Pi^\beta_{m+1} \to \Pi^\beta_m)
$$

induced by $\phi_{m+1}$ is graphic [i.e., compatible with the semi-graphs of anabelioids that appear in the respective collections of construction data of the PSC-extensions $\Pi^{\square}_{m+1} \to \Pi^{\square}_m$, for $\square = \alpha, \beta$].

(v) For $m \in \{1, \ldots, n - 1\}$, $c^\alpha, c^\beta$ corresponding divisors of $\Pi^\alpha_m, \Pi^\beta_m$, the isomorphism

$$
\text{Ker}((\Pi^\alpha_{m+1})_{c^\alpha} \to D_{c^\alpha}) \xrightarrow{\sim} \text{Ker}((\Pi^\beta_{m+1})_{c^\beta} \to D_{c^\beta})
$$
induced by $\phi_{m+1}$ is graphic [i.e., compatible with the semi-graphs of anabelioids that appear in the respective collections of construction data of the DPSC-extensions $(\Pi_{m+1})_c \rightarrow D_c$, for $c = \alpha, \beta$].

**Proof.** All of the assertions of Theorem 1.7 follow immediately from [the various definitions involved, together with] repeated application of Corollary 1.4 to the PSC-extensions $\Pi_{m+1} \rightarrow \Pi_m$ [cf. Definition 1.5, (ii), (c)] and the DPSC-extensions $(\Pi_{m+1})_c \rightarrow D_c$ [cf. Definition 1.5, (ii), (d)], for $c = \alpha, \beta$.

**Remark 1.7.1.** In Theorem 1.7, instead of phrasing the result as an assertion concerning the preservation of structures via some isomorphism between two PPSC-extensions, one may instead phrase the result as an assertion concerning the existence of an explicit “group-theoretic algorithm” for reconstructing, from a single given PPSC-extension, the various structures corresponding to graphicity, divisors, and inertia and decomposition groups of divisors — i.e., in the fashion of [Mzk15], Lemma 4.5, for cuspidal decomposition groups; a similar remark may be made concerning Corollary 1.4. [We leave the routine details to the interested reader.] Indeed, both Corollary 1.4 and Theorem 1.7 are, in essence, formal consequences of the “graphicity theory” of [Mzk13], which [just as in the case of [Mzk15], Lemma 4.5] consists precisely of such explicit “group-theoretic algorithms” for reconstructing the various structures corresponding to graphicity in the case of semi-graphs of anabelioids of PSC-type.

Before proceeding, we observe the following result, which is, in essence, independent of the theory of the present §1.

**Theorem 1.8.** (PPSC-Extensions over Galois Groups of Arithmetic Fields) For $c = \alpha, \beta$, let $k_c$ be a field of characteristic zero; $\tilde{k}_c$ a solvably closed [cf. [Mzk15], Definition 1.4] Galois extension of $k_c$; $H_c$ def Gal($\tilde{k}_c/k_c$); $(Z_c)^{\log}$ the log scheme determined by a stable polycurve over $k_c$; $\Sigma_c$ a nonempty set of primes; $n$ a positive integer; $1 \rightarrow \Delta \rightarrow \Pi \rightarrow H \rightarrow 1$ a pro-$\Sigma$ PPSC-extension of dimension $n$ associated to $(Z_c)^{\log}$, $\tilde{k}_c$ [cf. Remark 1.5.4];

$$
\Pi^n_c \overset{\text{def}}{=} \Pi^n \rightarrow \Pi^{n-1} \rightarrow \ldots \rightarrow \Pi_1 \rightarrow \Pi_0 \overset{\text{def}}{=} H
$$

the sequence of successive fiber extensions associated to $\Pi$ [cf. Remark 1.5.1]. Let

$$
\phi : \Pi^\alpha \rightarrow \Pi^\beta
$$

be an isomorphism of profinite groups that induces isomorphisms $\phi_m : \Pi^\alpha_m \rightarrow \Pi^\beta_m$, for $m = 0, 1, \ldots, n$ [so $\phi = \phi_n$]. Then:

(i) (Relative Version of the Grothendieck Conjecture for Stable Polycurves over Generalized Sub-p-adic Fields) Suppose that for $c = \alpha, \beta$, $k_c$ is generalized sub-p-adic [cf. [Mzk4], Definition 4.11] for some prime number $p \in \Sigma^\alpha \cap \Sigma^\beta$, and that the isomorphism of Galois groups $\phi_0 : H^\alpha \rightarrow H^\beta$ arises from a pair of isomorphisms of fields $k_\alpha \rightarrow k_\beta$, $k_\alpha \rightarrow k_\beta$. Then $\Sigma^\alpha = \Sigma^\beta$; there
exists a unique isomorphism of log schemes \((Z_\alpha)_{\log} \sim \to (Z_\beta)_{\log}\) that gives rise to \(\phi\).

(ii) (Absolute Version of the Grothendieck Conjecture for Stable Polycurves over Number Fields) Suppose that for \(\square = \alpha, \beta, k_{\square}\) is a number field. Then \(\Sigma^\alpha = \Sigma^\beta\); there exists a unique isomorphism of log schemes \((Z_\alpha)_{\log} \sim \to (Z_\beta)_{\log}\) that gives rise to \(\phi\).

Proof. Assertions (i), (ii) follow immediately from repeated application of [Mzk4], Theorem 4.12 [cf. also [Mzk2], Corollary 7.4], together with [in the case of assertion (ii)] “Uchida’s theorem” [cf., e.g., [Mzk10], Theorem 3.1].

Finally, we study the consequences of the theory of the present §1 in the case of configuration spaces. We refer to [MT] for more details on the theory of configuration spaces.

Definition 1.9. Let \(l\) be a prime number; \(\Sigma\) a set of primes which is either of cardinality one or equal to the set of all primes; \(X\) a hyperbolic curve of type \((g, r)\) over a field \(k\) of characteristic \(\not\in \Sigma\); \(\overline{k}\) a separable closure of \(k\); \(n \geq 1\) an integer; \(X_n\) the \(n\)-th configuration space associated to \(X\) [cf. [MT], Definition 2.1, (i)]; \(E\) the index set [i.e., the set of factors — cf. [MT], Definition 2.1, (i)] of \(X_n\);

\[\pi_1(X_n \times_k \overline{k}) \to \Theta\]

the maximal pro-\(\Sigma\) quotient of \(\pi_1(X_n \times_k \overline{k})\); \(\Delta \subseteq \Theta\) a product-theoretic open subgroup [cf. [MT], Definition 2.3, (ii)]; \(1 \to \Delta \to \Pi \to H \to 1\) an extension of profinite groups.

(i) We shall refer to as a labeling on \(E\) a bijection \(\Lambda : \{1, 2, \ldots, n\} \sim \to E\). Thus, for each labeling \(\Lambda\) on \(E\), we obtain a structure of hyperbolic polycurve [i.e., a collection of data exhibiting \(X_n\) as a hyperbolic polycurve — cf. [Mzk2], Definition 4.6] on \(X_n\), arising from the various natural projection morphisms associated to \(X_n\) [cf. [MT], Definition 2.1, (ii)], by projecting in the order specified by \(\Lambda\). In particular, for each labeling \(\Delta\) on \(E\), we obtain a structure of PPSC-extension on [the extension \(1 \to \Delta \to \Pi \to H \to 1\) associated to] some open subgroup \(\Delta \subseteq \Delta\) [which may be taken to be arbitrarily small — cf. Remark 1.5.2].

(ii) Let \(\Lambda\) be a labeling on \(E\). Then we shall refer to a structure of PPSC-extension on [the extension \(1 \to \Delta \to \Pi \to H \to 1\) by intersecting with] an open subgroup \(\Pi_\Lambda \subseteq \Pi\) as \(\Lambda\)-admissible if it induces the structure of PPSC-extension on \(\Delta\) discussed in (i).

(iii) We shall refer to as a structure of \([\text{pro-}\Sigma]\) CPSC-extension [of type \((g, r)\) and dimension \(n\), with index set \(E\)] [i.e., “configuration (space) pointed stable curve extension”] on the extension \(1 \to \Delta \to \Pi \to H \to 1\) any collection of data as follows: for each labeling \(\Lambda\) on \(E\), a \(\Lambda\)-admissible structure of PPSC-extension on some open subgroup \(\Pi_\Lambda \subseteq \Pi\) [which may be taken to be arbitrarily small — cf. Remark 1.5.2]. We shall refer to a structure of CPSC-extension on \(\Pi\) as \(l\)-cyclotomically full if, for each labeling \(\Lambda\) on \(E\), the \(\Lambda\)-admissible structure of PPSC-extension that
constitutes the given structure of CPSC-extension is $l$-cyclotomically full. We shall refer to a structure of CPSC-extension on $\Pi$ as \textit{strict} if one may take $\Delta$ to be $\Theta$, and, for each labeling $\Lambda$ on $E$, one may take $\Pi_{\Lambda}$ to be $\Pi$. We shall refer to $1 \to \Delta \to \Pi \to H \to 1$ as a \textit{[pro-$\Sigma$]} CPSC-extension \textit{[of type $(g, r)$ and dimension $n$, with index set $E$]} if it admits a structure of CPSC-extension; if this structure of CPSC-extension may be taken to be $l$-cyclotomically full (respectively, strict), then we shall refer to the CPSC-extension itself as $l$-cyclotomically full (respectively, strict). If $1 \to \Delta \to \Pi \to H \to 1$ is a CPSC-extension, then we shall refer to $(\Sigma, X, k, \Theta)$ as construction data for this CPSC-extension.

(iv) Let $\tilde{k} \subseteq \kbar$ be a solvably closed [cf. [Mzk15], Definition 1.4] Galois extension of $k$; suppose that

$$Z \to X_n$$

is a \textit{finite étale covering} such that $Z \times_k \kbar \to X_n \times_k \kbar$ is the [connected] covering determined by the open subgroup $\Delta \subseteq \Theta$ [so we have a natural surjection $\pi_1(Z \times_k \kbar) \to \Delta$]. Then [cf. the discussion of Remark 1.5.4] we shall refer to a [structure of] CPSC-extension [on] $1 \to \Delta \to \Pi \to H \to 1$ as \textit{arising from} $Z$, $\tilde{k}/k$ if there exist a surjection $\pi_1(Z) \to \Pi$ and an isomorphism $\text{Gal}(\tilde{k}/k) \simeq H$ that are \textit{compatible} with one another as well as with the natural surjections $\pi_1(Z \times_k \kbar) \to \Delta$, $\pi_1(Z) \to \text{Gal}(\tilde{k}/k) \to \text{Gal}(k/k)$ and, moreover, satisfy the property that the \textit{structure of CPSC-extension} on $1 \to \Delta \to \Pi \to H \to 1$ is induced by the various structures of hyperbolic polycurve on $Z$, $X_n$, associated to a suitable \textit{labeling} of $E$ [cf. (i)].

\textbf{Corollary 1.10. (Combinatorial Configuration Spaces)} Let $l$ be a prime number. For $\square = \alpha, \beta$, let

$$1 \to \Delta^\square \to \Pi^\square \to H^\square \to 1$$

be an \textit{extension of profinite groups} equipped with some \textit{[fixed]} $l$-cyclotomically full \textit{structure of CPSC-extension} \textit{of type $(g^\square, r^\square)$} $\notin \{(0, 3), (1, 1)\}$ and dimension $n^\square$, with index set $E^\square$. If this fixed structure of CPSC-extension is \textit{not strict} for either $\square = \alpha$ or $\square = \beta$, then we assume that both $g_\alpha$, $g_\beta$ are $\geq 2$. Let

$$\phi : \Pi^\alpha \simeq \Pi^\beta$$

be an \textit{isomorphism of profinite groups} such that $\phi(\Delta^\alpha) = \Delta^\beta$. Then:

(i) The isomorphism $\phi$ determines a \textit{bijection} $E^\alpha \simeq E^\beta$ of index sets. In particular, $n^\alpha = n^\beta$, so we write $n^\square \overset{\text{def}}{=} n^\alpha = n^\beta$.

(ii) For each pair of \textit{compatible} \textit{[i.e., relative to the bijection of (i)] labelings} $\Lambda = (\Lambda^\alpha, \Lambda^\beta)$ of $E^\alpha$, $E^\beta$, there exist \textit{open subgroups} $\Pi^\square_{\Lambda} \subseteq \Pi^\square$ [for $\square = \alpha, \beta$] such that the following properties hold: (a) $\phi(\Pi^\square_{\Lambda}) = \Pi^\square_{\Lambda}$; (b) for $\square = \alpha, \beta$, the open subgroup $\Pi^\square_{\Lambda}$ admits an $\Lambda^\square$-admissible \textit{structure of PPSC-extension}; (c) if we write

$$\Pi^\square_{\Lambda} \overset{n}{=} (\Pi^\square_{\Lambda}) \to (\Pi^\square_{\Lambda})_{n-1} \to \ldots \to (\Pi^\square_{\Lambda})_1 \to (\Pi^\square_{\Lambda})_0 \overset{\text{def}}{=} H^\square_{\Lambda}$$
for the sequence of successive fiber extensions associated to the structures of PPSC-extension of (b) [cf. Remark 1.5.1], then \( \phi \) induces isomorphisms

\[
(\Pi^\alpha_m) \sim (\Pi^\beta_m)
\]

[for \( m = 0, \ldots, n \)]. In particular, \( \phi \) satisfies the hypotheses of Theorem 1.7.

Proof. By [MT], Corollaries 4.8, 6.3 [cf. our hypotheses on \( (g_\Box, r_\Box) \)], \( \phi \) induces a bijection \( E_\alpha \sim E_\beta \) between the respective index sets, together with compatible isomorphisms between the various fiber subgroups of \( \Delta^\alpha, \Delta^\beta \). [Note that even though these results of [MT] are stated only in the case where the field appearing in the construction data is of characteristic zero, the results generalize immediately to the case where this field is of characteristic invertible in \( \Sigma^\Box \), since any hyperbolic curve in positive characteristic may be lifted to a hyperbolic curve in characteristic zero in a fashion that is compatible with the maximal pro-\( \Sigma^\Box \) quotients of the étale fundamental groups of the associated configuration spaces — cf., e.g., [MT], Proposition 2.2, (v).] To obtain open subgroups \( \Pi^\Box_\Lambda \subseteq \Pi^\Box \) satisfying the desired properties, it suffices to argue by induction on \( n \), by applying Remark 1.5.2. \( \Box \)

Remark 1.10.1. A similar remark to Remark 1.7.1 may be made for Corollary 1.10.

Corollary 1.11. (Configuration Spaces over Arithmetic Fields) For \( \Box = \alpha, \beta \), let \( k_\Box \) be a perfect field; \( \bar{k}_\Box \) a solvably closed [cf. [Mzk15], Definition 1.4] Galois extension of \( k_\Box \); \( X_\Box \) a hyperbolic curve of type \( (g_\Box, r_\Box) \notin \{(0,3), (1,1)\} \) over \( k_\Box \); \( n_\Box \) a positive integer;

\[
Z_\Box \to (X_\Box)_{n_\Box}
\]

a geometrically connected [over \( k_\Box \)] finite étale covering of the \( n_\Box \)-th configuration space \( (X_\Box)_{n_\Box} \) of \( X_\Box \); \( \Sigma^{\Box} \) a nonempty set of primes;

\[
1 \to \Delta^{\Box} \to \Pi^{\Box} \to H^{\Box} \to 1
\]

an extension of profinite groups equipped with some [fixed!] structure of pro-\( \Sigma^{\Box} \) CPSC-extension arising from \( Z_\Box, \bar{k}_\Box \) [cf. Definition 1.9, (iv)]. If this fixed structure of CPSC-extension is not strict for either \( \Box = \alpha \) or \( \Box = \beta \), then we assume that both \( g_\alpha, g_\beta \) are \( \geq 2 \). Let

\[
\phi : \Pi^\alpha \sim \Pi^\beta
\]

be an isomorphism of profinite groups. Then:

(i) (Relative Version of the Grothendieck Conjecture for Configuration Spaces over Generalized Sub-p-adic Fields) Suppose, for \( \Box = \alpha, \beta \), that \( k_\Box \) is generalized sub-p-adic [cf. [Mzk4], Definition 4.11] for some prime number \( p \in \Sigma^\alpha \cap \Sigma^\beta \), and that \( \phi \) lies over an isomorphism of Galois groups \( \phi_0 : H^\alpha \sim H^\beta \) that arises from a pair of isomorphisms of fields \( k_\alpha \sim \bar{k}_\beta, k_\alpha \sim \bar{k}_\beta \). Then
\(\Sigma^\alpha = \Sigma^\beta;\) there exists a unique isomorphism of schemes \(Z_\alpha \xrightarrow{\sim} Z_\beta\) that gives rise to \(\phi\).

(ii) (Strict Semi-absoluteness) Suppose, for \(\Box = \alpha, \beta\), that \(k_\Box\) is either an FF, an MLF, or an NF [cf. [Mzk15], \S 0]. Then \(\phi(\Delta^\alpha) = \Delta^\beta\) [i.e., \(\phi\) is “strictly semi-absolute”].

(iii) (Absolute Version of the Grothendieck Conjecture for Configuration Spaces over MLF’s) Suppose, for \(\Box = \alpha, \beta\), that \(k_\Box\) is an MLF, that \(n_\Box \geq 2\), that \(n_\Box \geq 3\) if \(X_\Box\) is proper, and that \(\Sigma^\Box\) is the set of all primes. Then there exists a unique isomorphism of schemes \(Z_\alpha \xrightarrow{\sim} Z_\beta\) that gives rise to \(\phi\).

(iv) (Absolute Version of the Grothendieck Conjecture for Configuration Spaces over NF’s) Suppose, for \(\Box = \alpha, \beta\), that \(k_\Box\) is an NF. Then \(\Sigma^\alpha = \Sigma^\beta;\) there exists a unique isomorphism of schemes \(Z_\alpha \xrightarrow{\sim} Z_\beta\) that gives rise to \(\phi\).

Proof. Assertion (i) (respectively, (iv)) follows immediately from Corollary 1.10, (ii), and Theorem 1.8, (i) (respectively, Theorem 1.8, (ii)) [applied to the coverings of \(Z_\alpha, Z_\beta\) determined by the open subgroups of Corollary 1.10, (ii)]. Assertion (ii) follows immediately from [Mzk15], Corollary 2.8, (ii). Note that in the situation of assertion (ii), assertion (ii) implies that \(\Sigma^\alpha = \Sigma^\beta\) [since \(\Sigma^\Box\) may be characterized as the unique minimal set of primes \(\Sigma^\Box\) such that \(\Delta^\Box\) is a pro-\(\Sigma^\Box\) group]; moreover, in light of our assumptions on \(k_\Box\), it follows immediately that \(\Pi^\Box\) is \(l\)-cyclotomically full for any \(l \in \Sigma^\alpha \cap \Sigma^\beta = \Sigma^\alpha = \Sigma^\beta\).

Finally, we consider assertion (iii). First, let us observe that by Corollary 1.10, (ii), and Theorem 1.8, (i) [applied to the coverings of \(Z_\alpha, Z_\beta\) determined by the open subgroups of Corollary 1.10, (ii)], it suffices to verify that the isomorphism \(\phi_H : H^\alpha \xrightarrow{\sim} H^\beta\) induced by \(\phi\) [cf. assertion (ii)] arises from an isomorphism of fields \(k_\alpha \xrightarrow{\sim} k_\beta\). To this end, let us observe that by Corollary 1.10, (ii), we may apply Theorem 1.7 to the present situation. Also, by Corollary 1.10, (i), \(n = n_\alpha = n_\beta\) is always \(\geq 2\); moreover, if either of the \(X_\Box\) is proper, then \(n \geq 3\). Next, let us observe that if \(X_\Box\) is proper (respectively, affine), then the stable log curve that appears in the logarithmic compactification of the fibration \((X_\Box)_3 \to (X_\Box)_2\) (respectively, \((X_\Box)_2 \to (X_\Box)_1\)) over the generic point of the diagonal divisor of \((X_\Box)_2\) (respectively, over any cusp of \(X_\Box\)) contains an irreducible component whose interior is a tripod [i.e., a copy of the projective line minus three marked points]. In particular, if we apply Theorem 1.7, (iii), to the vertical divisor determined by such an irreducible component, then we may conclude that \(\phi\) induces an isomorphism between the decomposition groups of these vertical divisors. In particular, [after possibly replacing the given \(k_\alpha, k_\beta\) by corresponding finite extensions of \(k_\alpha, k_\beta\)] we obtain, for \(\Box = \alpha, \beta\), a hyperbolic curve \(C_\Box\) over \(k_\Box\), together with an isomorphism of profinite groups

\[\phi_C : \pi_1(C_\alpha \times_{k_\alpha} \tilde{k}_\alpha) \xrightarrow{\sim} \pi_1(C_\beta \times_{k_\beta} \tilde{k}_\beta)\]

induced by \(\phi\) [so the \(\pi_1(C_\Box \times_{k_\Box} \tilde{k}_\Box)\) correspond to the respective “\(\Pi_\alpha\)’s” of the vertical divisors under consideration] that is compatible with the outer action of
$H$ on $\pi_1(C \times k \tilde{k})$ and the isomorphism $\phi_H$; moreover, here we may assume that, say, $C_\alpha$ is a finite étale covering of a tripod. [In particular, we observe that the existence of the natural outer action of $H$ on $\pi_1(C \times k \tilde{k})$ implies — cf. the argument given in the proof of [Mzk10], Proposition 3.3 — that $\tilde{k}$ is necessarily an algebraic closure of $k$.] On the other hand, since the “absolute $p$-adic version of the Grothendieck Conjecture” is known to hold in this situation [cf. [Mzk14], Corollary 2.3], we thus conclude that $\phi_H$ does indeed arise from an isomorphism of fields $k_\alpha \sim k_\beta$, as desired. This completes the proof of assertion (iii).

Remark 1.11.1. At the time of writing Corollary 1.11, (iii), constitutes the only absolute isomorphism version of the Grothendieck Conjecture over MLF’s [to the knowledge of the author] that may be applied to arbitrary hyperbolic curves.
Section 2: Geometric Uniformly Toral Neighborhoods

In the present §2, we prove a certain “resolution of nonsingularities” type result [cf. Lemma 2.6; Remark 2.6.1; Corollary 2.11] — i.e., a result reminiscent of the main results of [Tama2] [cf. also the techniques applied in the verification of “observation (iv)” given in the proof of [Mzk9], Corollary 3.11] — that allows us to apply the theory of uniformly toral neighborhoods developed in [Mzk15], §3, to prove a certain “conditional absolute p-adic version of the Grothendieck Conjecture” — namely, that “point-theoreticity implies geometricity” [cf. Corollary 2.9]. This condition of point-theoreticity may be removed if, instead of starting with a hyperbolic curve, one starts with a “pro-curve” obtained by removing from a proper curve some [necessarily infinite] set of closed points which is “p-adically dense in a Galois-compatible fashion” [cf. Corollary 2.10].

First, we recall the following “positive slope version of Hensel’s lemma” [cf. [Serre], Chapter II, §2.2, Theorem 1, for a discussion of a similar result].

Lemma 2.1. (Positive Slope Version of Hensel’s Lemma) Let \( k \) be a complete discretely valued field; \( \mathcal{O}_k \subseteq k \) the ring of integers of \( k \) [equipped with the topology determined by the valuation]; \( \mathfrak{m}_k \) the maximal ideal of \( \mathcal{O}_k \); \( \pi \in \mathfrak{m}_k \) a uniformizer of \( \mathcal{O}_k \); \( A \overset{\text{def}}{=} \mathcal{O}_k[[X_1, \ldots, X_m]] \); \( B \overset{\text{def}}{=} \mathcal{O}_k[[Y_1, \ldots, Y_n]] \). Let us suppose that \( A \) (respectively, \( B \)) is equipped with the topology determined by its maximal ideal; write \( \mathcal{X} \overset{\text{def}}{=} \text{Spf}(A) \) (respectively, \( \mathcal{Y} \overset{\text{def}}{=} \text{Spf}(B) \)), \( K_A \) (respectively, \( K_B \)) for the quotient field of \( A \) (respectively, \( B \)), and \( \Omega_A \) (respectively, \( \Omega_B \)) for the module of continuous differentials of \( A \) (respectively, \( B \)) over \( \mathcal{O}_k \) [so \( \Omega_A \) (respectively, \( \Omega_B \)) is a free \( A \)- (respectively, \( B \)-) module of rank \( m \) (respectively, \( n \))]. Let \( \phi : B \to A \) be the continuous \( \mathcal{O}_k \)-algebra homomorphism induced by an assignment

\[
B \ni \beta_0 \mapsto f_j(X_1, \ldots, X_m) \in A
\]

[where \( j = 1, \ldots, n \)]; let us suppose that the induced morphism \( d\phi : \Omega_B \otimes_B A \to \Omega_A \) satisfies the property that the image of \( d\phi \otimes_A K_A \) is a \( K_A \)-subspace of rank \( n \) in \( \Omega_A \otimes_A K_A \) [so \( n \leq m \)]. Then there exists a point \( \beta_0 \in \mathcal{Y}(\mathcal{O}_k) \) and a positive integer \( r \) satisfying the following property: Let \( k' \) be a finite extension of \( k \), with ring of integers \( \mathcal{O}_{k'} \); write \( B(\beta_0, k', r) \) for the “ball” of points \( \beta' \in \mathcal{Y}(\mathcal{O}_{k'}) \) such that \( \beta' \), \( \beta_0 \) map to the same point of \( \mathcal{Y}(\mathcal{O}_{k'}/(\pi^r)) \). Then the image of the map

\[
\mathcal{X}(\mathcal{O}_{k'}) \to \mathcal{Y}(\mathcal{O}_{k'})
\]

induced by \( \phi \) contains the “ball” \( B(\beta_0, k', r) \).

Proof. First, let us observe that by Lemma 2.2 below, after possibly re-ordering the \( X_i \)'s, we may assume that the differentials \( dX_i \in \Omega_A \), where \( i = n + 1, \ldots, m \), together with the differentials \( df_j \in \Omega_A \), where \( j = 1, \ldots, n \), form a \( K_A \)-basis of \( \Omega_A \otimes_A K_A \). Thus, by adding indeterminates \( Y_{n+1}, \ldots, Y_m \) to \( B \) and extending \( \phi \) by sending \( Y_i \mapsto X_i \) for \( i = n + 1, \ldots, m \), we may assume without loss of generality that \( n = m \), \( A = B \), \( \mathcal{X} = \mathcal{Y} \), i.e., that the morphism \( \text{Spf}(\phi) : \mathcal{X} \to \mathcal{Y} = \mathcal{X} \) is “generically formally étale”. 
Write $M$ for the $n$ by $n$ matrix with coefficients in $A$ given by $\{ df_i/dX_j \}_{i,j=1,\ldots,n}$; $g \in A$ for the determinant of $M$. Thus, by elementary linear algebra, it follows that there exists an $n$ by $n$ matrix $N$ with coefficients in $A$ such that $M \cdot N = N \cdot M = g \cdot I$ [where we write $I$ for the $n$ by $n$ identity matrix]. By our assumption concerning the image of $d\phi \otimes_A K_A$, it follows that $g \neq 0$, hence, by Lemma 2.3 below, that there exist elements $x_i \in \mathfrak{m}_k$, where $i = 1, \ldots, n$, such that $g_0 \overset{\text{def}}{=} g(x_1, \ldots, x_n) \in \mathfrak{m}_k$ is nonzero. By applying appropriate “affine translations” to the domain and codomain of $\phi$, we may assume without loss of generality that, for $i = 1, \ldots, n$, we have $x_i = f_i(0, \ldots, 0) = 0 \in O_k$. Write $M_0 \overset{\text{def}}{=} M(0, \ldots, 0), N_0 \overset{\text{def}}{=} N(0, \ldots, 0)$ [so $M_0$, $N_0$ are $n$ by $n$ matrices with coefficients in $O_k$].

Next, suppose that $g_0 \in \mathfrak{m}_k^2 \setminus \mathfrak{m}_k^{s+1}$. In the remainder of the present proof, all “vectors” are to be understood as column vectors with coefficients in $O_{k'}$, where $k'$ is as in the statement of Lemma 2.1. Then I claim that for every vector $\vec{y} = (y_1, \ldots, y_n) \equiv 0 \pmod{\pi^{2s}}$, there exists a vector $\vec{x} = (x_1, \ldots, x_n) \equiv 0 \pmod{\pi^{2s}}$ such that $f(\vec{x}) = (f_1(\vec{x}), \ldots, f_n(\vec{x})) = \vec{y}$. Indeed, since $O_{k'}$ is complete and $x_i = f_i(0, \ldots, 0) = 0$, it suffices by taking $\vec{x}\{2\}$ to be $(0, \ldots, 0)$ to show, for each integer $l \geq 2$, that the existence of a vector $\vec{x}\{l\} \equiv 0 \pmod{\pi^{2s}}$ such that $f(\vec{x}\{l\}) = \vec{y} \pmod{\pi^{(l+1)s}}$ implies the existence of a vector $\vec{x}\{l+1\}$ such that $\vec{x}\{l+1\} \equiv \vec{x}\{l\} \pmod{\pi^{(l+2)s}}$. To this end, we compute: Set $\vec{\epsilon} \overset{\text{def}}{=} \vec{y} - f(\vec{x}\{l\}), \vec{\eta} \overset{\text{def}}{=} g_{l-1} \cdot \vec{\epsilon}, \vec{\delta} \overset{\text{def}}{=} N_0 \cdot \vec{\eta}, \vec{x}\{l+1\} \overset{\text{def}}{=} \vec{x}\{l\} + \vec{\delta}$. Thus, $\vec{\epsilon} \equiv 0 \pmod{\pi^{(l+1)s}}, \vec{\eta} \equiv 0 \pmod{\pi^{ls}}, \vec{\delta} \equiv 0 \pmod{\pi^{ls}}, \vec{x}\{l+1\} \equiv \vec{x}\{l\} \pmod{\pi^{(l+2)s}}$. In particular, the squares of the elements of $\vec{\delta}$ all belong to $\pi^{(l+2)s} \cdot O_{k'}$ [since $l \geq 2$], so we obtain that

\[
\vec{x}\{l+1\} \equiv f(\vec{x}\{l\}) + M_0 \cdot \vec{\delta} \pmod{\pi^{(l+2)s}}
\]

\[
\equiv \vec{y} - \vec{\epsilon} + M_0 \cdot N_0 \cdot \vec{\eta} \equiv \vec{y} - \vec{\epsilon} + g_0 \cdot \vec{\eta} \equiv \vec{y} \pmod{\pi^{(l+2)s}}
\]

— as desired. This completes the proof of the claim. On the other hand, one verifies immediately that the content of this claim is sufficient to complete the proof of Lemma 2.1. $\square$

**Remark 2.1.1.** Thus, the usual “(slope zero version of) Hensel’s lemma” corresponds, in the notation of Lemma 2.1, to the case where the image of the morphism $d\phi$ is a direct summand of $\Omega_A$. In this case, we may take $r = 1$.

**Remark 2.1.2.** In fact, according to oral communication to the author by F. Oort, it appears that the sort of “positive slope version” of “Hensel’s lemma” given in Lemma 2.1 [i.e., where the derivative is only generically invertible] preceded the “slope zero version” that is typically referred to “Hensel’s lemma” in modern treatments of the subject.

**Lemma 2.2.** (Subspaces and Bases of a Vector Space) Let $k$ be a field; $V$ a finite-dimensional $k$-vector space with basis $\{ e_i \}_{i \in I}$; $W \subseteq V$ a $k$-subspace. Then there exists a subset $J \subseteq I$ such that if we write $V_J \subseteq V$ for the $k$-subspace generated by the $e_j$ for $j \in J$, then the natural inclusions $V_J \hookrightarrow V$, $W \hookrightarrow V$ determine an isomorphism $V_J \oplus W \cong V$ of $k$-vector spaces.
Proof. This result is a matter of elementary linear algebra. $\Box$

Lemma 2.3. (Nonzero Values of Functions Defined by Power Series) Let $k, O_k, A$ be as in Lemma 2.1; $f = f(X_1, \ldots, X_m) \in A$ a nonzero element. Then there exist elements $x_i \in m_k$, where $i = 1, \ldots, m$, such that $f(x_1, \ldots, x_m) \in m_k$ is nonzero.

Proof. First, I claim that by induction on $m$, it suffices to verify Lemma 2.3 when $m = 1$. Indeed, for arbitrary $m \geq 2$, one may write

$$f = \sum_{i=0}^{\infty} c_i \ X_i^i$$

— where $c_i = c_i(X_1, \ldots, X_{m-1}) \in O_k[[X_1, \ldots, X_{m-1}]]$. Since $f \neq 0$, it follows that there exists at least one nonzero $c_j$. Thus, by the induction hypothesis, it follows that there exist $x_i \in m_k$, where $i = 1, \ldots, m-1$, such that $m_k \ni c_j(x_1, \ldots, x_{m-1}) \neq 0$. Thus, $f(x_1, \ldots, x_{m-1}, X_m) \in O_k[[X_m]]$ is nonzero, so, again by the induction hypothesis, there exists an $x_m \in m_k$ such that $m_k \ni f(x_1, \ldots, x_{m-1}, x_m) \neq 0$. This completes the proof of the claim.

Thus, for the remainder of the proof, we assume that $m = 1$ and write $X \overset{\text{def}}{=} X_1$,

$$f = \sum_{i=0}^{\infty} c_i \ X^i$$

— where $c_i \in O_k$. Suppose that $c_j \neq 0$, but that $c_i = 0$ for $i < j$. Then there exists a positive integer $s$ such that $c_j \not\in m_k^s$. Let $x \in m_k^s$ be any nonzero element. Then $c_j x^j \not\in x^j \cdot m_k^s$, while $c_i x^i \in x^j \cdot O_k \subseteq x^j \cdot m_k^s$ for any $i > j$. But this implies that $m_k \ni f(x) \neq 0$, as desired. $\Box$

In the following, we shall often work with two-dimensional log regular log schemes. For various basic facts on log regular log schemes, we refer to [Kato]; [Mzk2], §1. If $X^{\log}$ is a log regular log scheme, then for integers $j \geq 0$, we shall write

$$U^{[j]}_X \subseteq X$$

for the $j$-interior of $X^{\log}$, i.e., the open subscheme of points at which the fiber of the groupification of the characteristic sheaf of $X^{\log}$ is of rank $\leq j$ [cf. [MT], Definition 5.1, (i); [MT], Proposition 5.2, (i)]. Thus, the complement of $U^{[j]}_X$ in $X$ is a closed subset of codimension $> j$ [cf. [MT], Proposition 5.2, (ii)]; $U_X \overset{\text{def}}{=} U^{[0]}_X$ is the interior of $X^{\log}$ [i.e., the maximal open subscheme over which the log structure is trivial]. Also, we shall write

$$D_X \subseteq X$$

for the closed subscheme $X \setminus U_X$ with the reduced induced structure. Finally, we remind the reader that in the following, all fiber products of fs log schemes are to be taken in the category of fs log schemes [cf. §0].
Next, let $k$ be a complete discretely valued field with perfect residue field $k$; $\mathcal{O}_k \subseteq k$ the ring of integers of $k$; $\overline{k}$ an algebraic closure of $k$; $\overline{\mathcal{O}}_k$ the resulting algebraic closure of $\overline{k}$; $\mathfrak{m}_k$ the maximal ideal of $\mathcal{O}_k$; $\pi \in \mathfrak{m}_k$ a uniformizer of $\mathcal{O}_k$;

$$X \to S \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_k)$$

a stable curve over $S$ whose generic fiber $X_\eta \overset{\text{def}}{=} X \times_S \eta$, where we write $\eta \overset{\text{def}}{=} \text{Spec}(k)$, is smooth. Thus, $X_\eta$ is a proper hyperbolic curve over $k$, whose genus we denote by $g_X$; the open subschemes $\eta \subseteq S$, $X_\eta \subseteq X$ determine log regular log structures on $S$, $X$, respectively. We denote the resulting morphism of log schemes by $X^{\log} \to S^{\log}$.

**Definition 2.4.**

(i) We shall refer to a morphism of log schemes

$$\phi^{\log} : V^{\log} \to X^{\log}$$

[or to the log scheme $V^{\log}$] as a log-modification if $\phi^{\log}$ admits a factorization

$$V^{\log} \to X^{\log} \times_{S^{\log}} S^{\log}_V \to X^{\log}$$

where $S_V \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_{k_V})$, $\mathcal{O}_{k_V}$ is the ring of integers of a finite separable extension $k_V$ of $k$; $S^{\log}_V$ is the log regular log scheme determined by the open immersion $\eta_V \overset{\text{def}}{=} \text{Spec}(k_V) \hookrightarrow S_V$; the morphism $X^{\log} \times_{S^{\log}} S^{\log}_V \to X^{\log}$ is the projection morphism [where we observe that the underlying morphism of schemes $X \times_S S_V \to S_V$ is a stable curve over $S_V$]; the morphism $V^{\log} \to X^{\log} \times_{S^{\log}} S^{\log}_V$ is a log étale morphism whose underlying morphism of schemes is proper and birational; we shall refer to $k_V$ as the base-field of the log-modification $\phi^{\log}$.

(ii) For $i = 1, 2$, let $\phi_i^{\log} : V_i^{\log} \to X_i^{\log}$ be a log-modification that admits a factorization $V_i^{\log} \to X_i^{\log} \times_{S_i^{\log}} S_i^{\log} \to X_i^{\log}$ as in (i); $\psi^{\log} : V_2^{\log} \to V_1^{\log}$ an $X_1^{\log}$-morphism. Then let us observe that the log scheme $V_1^{\log}$ is always log regular of dimension 2 [cf. Proposition 2.5, (iv), below]. We shall refer to the log-modification $\phi_1^{\log}$ as regular if the log structure of $V_1^{\log}$ is defined by a divisor with normal crossings [which implies that $V_1$ is a regular scheme]. We shall refer to the log-modification $\phi_1^{\log}$ as unramified if $U_1^{[1]}_{V_1}$ is a smooth scheme over $S_1$. We shall refer to the morphism $\psi^{\log}$ as a base-field-isomorphism [or base-field-isomorphic] if the morphism $S_2 \to S_1$ induced by $\psi^{\log}$ is an isomorphism. We shall refer to the points of $V_1$ over which the underlying morphism $\psi$ of $\psi^{\log}$ fails to be finite as the critical points of $\psi^{\log}$ [or $\psi$]. We shall refer to the reduced divisor in $V_2$ determined by the closed set of points of $V_2$ at which $\psi$ fails to be quasi-finite as the exceptional divisor of $\psi^{\log}$ [or $\psi$]. We shall refer to the log scheme- (respectively, scheme-) theoretic fiber of $V_i^{\log}$ (respectively, $V_i$) over the unique closed point of $S_i$ as the log special fiber (respectively, special fiber) of $V_i^{\log}$ (respectively, $V_i$); we shall use the notation

$$V_i^{\log} \text{ (respectively, } V_i\text{)}$$
to denote the log special fiber (respectively, special fiber) of $V^\log_i$ (respectively, $V_i$).

Suppose that $C$ is an irreducible component of $V_i$. Then we shall say that $C$ is stable if it maps finitely to $X$ via $\phi_i$; we shall say that the log-modification $\phi^\log_i$ as unramified at $C$ if $U^\log_i/V_i$ is smooth over $S_i$ at the generic point of $C$.

**Remark 2.4.1.** Recall that there exists a base-field isomorphic log-modification

$$V^\log \to X^\log$$

which is uniquely determined up to unique isomorphism [over $X$] by the following two properties: (a) $V$ is regular; (b) $V \to S$ is a semi-stable curve. In fact, it was this example that served as the primary motivating example for the author in developing the notion of a “log-modification”. Note, moreover, that unlike property (a), however, the principal condition that defines a [base-field-isomorphic] “log-modification” — i.e., the condition that the morphism $V \to X$ be a proper, birational morphism that extends to a log étale morphism $V^\log \to X^\log$ of log schemes — is a condition on the morphism $V \to X$ that has the virtue of being manifestly stable under base-change [i.e., via morphisms that satisfy suitable, relatively mild conditions]. This property of stability under base-change will be applied repeatedly in the remainder of the present §2.

**Proposition 2.5. (First Properties of Log-modifications)** For $i = 1, 2$, let

$$\phi^\log_i : V^\log_i \to X^\log$$

be a log-modification that admits a factorization $V^\log_i \to X^\log \times S^\log_i S^\log_i \to X^\log$ as in Definition 2.4, (i); $\psi^\log : V^\log_2 \to V^\log_1$ an $X^\log$-morphism. Write $S_i = \text{Spec}(O_{k_i})$, where, for simplicity, we assume that the extension $k_i$ of $k$ is a subfield of $K$; $\bar{k}_i$ for the residue field of $k_i$; $U^\noncr_i \subseteq V_i$ for the open subscheme given by the complement of the critical points of $\psi^\log$. Let $i \in \{1, 2\}$. Then:

(i) (The Noncriticality of the 1-Interior) We have: $U^\log_1 \subseteq U^\noncr.$

(ii) (Isomorphism over the Noncritical Locus) The morphism $V^\log_2 \to V^\log_1 \times S^\log_1 S^\log_2$ determined by $\psi^\log$ is an isomorphism over $U^\noncr.

(iii) (Log Smoothness and Unramified Log-modifications) $V^\log_\circ$ is log smooth over $S^\log_\circ$. In particular, the sheaf of relative logarithmic differentials of the morphism $V^\log_\circ \to S^\log_\circ$ is a line bundle, which we shall denote $\omega_{V^\log_\circ/S^\log_\circ}$; we have a natural isomorphism $\psi^*\omega_{V^\log_1/S^\log_1} \cong \omega_{V^\log_2/S^\log_2}$. Finally, there exists a finite extension $k_\circ$ of $k_\circ$ such that, if we write $S_\circ = \text{Spec}(O_{k_\circ})$, then the morphism $V^\log_\circ \to S^\log_\circ \times S^\log_\circ S^\log_\circ \to X^\log$ determined by $\phi^\log_\circ$, is an unramified log-modification.

(iv) (Regularity and Log Regularity) $V^\log_\circ$ is log regular; $U^\log_1 \circ$ is regular. Moreover, there exists a regular log-modification $V^\log_\circ \to X^\log$ that admits
a base-field-isomorphic $X^\log$-morphism $V_{\log}^0 \to V_{\log}^*$ such that every irreducible component $C$ of the special fiber $V_{\log}^c$ is smooth over the residue field $k_{\log}$ of the base-field $k_{\log}$ of $V_{\log}^0$. Finally, if the log-modification $V_{\log}^0 \to X^\log$ is unramified at such an irreducible component $C$, and we write $D_C \subseteq C$ for the reduced divisor determined by the complement of $C \cap U_{V_{\log}^0}^{[1]}$ in $C$, then we have a natural isomorphism

$$\omega_{C/\mathbb{E}_{\log}}(D_C) \sim \omega_{V_{\log}^{S/E_{\log}}/C}$$

of line bundles on $C$.

(v) (Chains of Projective Lines) Suppose that $\phi_{\log}^1$ is a regular log-modification [so $D_{V_2}$ is a divisor with normal crossings]. Then, after possibly replacing $k_2$ by a finite unramified extension of the discretely valued field $k_2$, every irreducible component $C$ of $D_{V_2}$ that lies in the exceptional divisor of $\phi_{\log}$ is isomorphic to the projective line over the residue field $k(c)$ of the point $c$ of $V_{\log}^1 \times_{S_{\log}} S_{\log}$ to which $C$ maps. Moreover, $C$ meets the other irreducible components of $D_{V_2}$ at precisely two $k(c)$-valued points of $C$. That is to say, every connected component of the exceptional divisor of $\phi_{\log}$ is a “chain of $\mathbb{P}^1$’s”.

(vi) (Dual Graphs of Special Fibers) The spectrum $V_x$ of the local ring obtained by completing the geometric special fiber $V_{\log} \times_{k_{\log}} \overline{k}$ at any point $x$ which does not (respectively, does) belong to the 1-interior has precisely two (respectively, precisely one) irreducible component(s). In particular, the special fiber $V_{\log}^c$ determines a dual graph $\Gamma V_{\log}$, whose vertices correspond bijectively to the irreducible components of $V_{\log} \times_{k_{\log}} \overline{k}$, and whose edges correspond bijectively to the points of $(V_{\log} \setminus U_{V_{\log}}^{[1]}) \times_{k_{\log}} \overline{k}$ [so each edge abuts to the vertices corresponding to the irreducible components in which the point corresponding to the edge lies]. In discussions of $\Gamma V_{\log}$, we shall frequently identify the vertices and edges of $\Gamma V_{\log}$ with the corresponding irreducible components and points of $V_{\log} \times_{k_{\log}} \overline{k}$. If the natural Galois action of $\text{Gal}(\overline{k}/k_{\log})$ on $\Gamma V_{\log}$ is trivial, then we shall say that $V_{\log}^c$ is split. Finally, the loop-rank $\text{lp-rk}(V_{\log}^c) \equiv \text{lp-rk}(\Gamma V_{\log})$ [cf. §0] is equal to the loop-rank $\text{lp-rk}(X)$.

(vii) (Filtered Projective Systems) Given any log-modification $V_{\log}^0 \to X^\log$, there exists a log-modification $V_{\log}^* \to X^\log$ that admits $X^\log$-morphisms $V_{\log}^* \to V_{\log}^0$, $V_{\log}^* \to V_{\log}^0$. That is to say, the log-modifications over $X^\log$ form a filtered projective system.

(viii) (Functionality) Let $Y \to S$ be a stable curve, with smooth generic fiber $Y_\eta \equiv Y \times_S \eta$; $Y^\log$ the log regular log scheme determined by the open subscheme $Y_\eta \subseteq Y$. Then every finite morphism $Y_\eta \to X_\eta$ extends to a commutative diagram

$$
\begin{array}{ccc}
W_{\log}^* & \to & V_{\log}^* \\
\downarrow & & \downarrow \\
Y^\log & \to & X^\log
\end{array}
$$

where $W_{\log}^* \to Y^\log$ is a log-modification. If $\text{lp-rk}(Y) = \text{lp-rk}(X)$, then we shall say that the morphisms $Y_\eta \to X_\eta$, $Y^\log \to X^\log$, $W_{\log}^* \to V_{\log}^*$ are loop-preserving; if $\text{lp-rk}(Y) > \text{lp-rk}(X)$ [or, equivalently, $\text{lp-rk}(Y) \neq \text{lp-rk}(X)$], then
we shall say that the morphisms \( Y_\eta \to X_\eta \), \( Y^{\log} \to X^{\log} \), \( W^{\log}_\bullet \to V^{\log}_\bullet \) are **loopifying**. Let \( C \) be an irreducible component of \( W_\bullet \). Then we shall refer to \( C \) as **base-stable** [relative to \( Y_\eta \to X_\eta \)] (respectively, **base-semi-stable** [relative to \( W^{\log}_\bullet \to V^{\log}_\bullet \)]) if it maps **finitely** to a(n) stable (respectively, arbitrary) irreducible component of \( V_\bullet \). If there exist log-modifications \( W^{\log}_0 \to Y^{\log} \), \( V^{\log}_0 \to X^{\log} \) that fit into a commutative diagram

\[
\begin{array}{ccc}
W^{\log}_0 & \to & V^{\log}_0 \\
\downarrow & & \downarrow \\
W^{\log}_\bullet & \to & V^{\log}_\bullet
\end{array}
\]

[where the left-hand vertical arrow is a \( Y^{\log} \)-morphism; the right-hand vertical arrow is an \( X^{\log} \)-morphism] such that \( C \) is the image of a base-semi-stable [relative to \( W^{\log}_0 \to V^{\log}_0 \)] irreducible component of \( W_0 \), then we shall say that \( C \) is **potentially base-semi-stable** [relative to \( Y_\eta \to X_\eta \)].

(ix) **(Centers in the 1-Interior)** Let \( k_0 \) be a finite separable extension of \( k \); \( K_0 \) a discretely valued field containing \( k_0 \) which induces an inclusion \( \mathcal{O}_{k_0} \subseteq \mathcal{O}_{K_0} \) between the respective rings of integers and a bijection \( k_0/\mathcal{O}^\times_{k_0} \cong K_0/\mathcal{O}^\times_{K_0} \) between the respective value groups; \( x_0 \in X(K_0) \) a \( K_0 \)-valued point. Then there exists an unramified log-modification \( V^{\log}_0 \to X^{\log} \) with base-field \( k_0 \) such that the morphism \( V_0 \to S_0 \defeq \text{Spec}(\mathcal{O}_{k_0}) \) is a semi-stable curve, and \( x_0 \) extends to a point \( \in U^{[1]}_{V_0}(\mathcal{O}_{K_0}) \).

(x) **(Maps to the Jacobian)** Suppose [for simplicity] that \( \phi^\text{log}_\bullet \) is a base-field-isomorphism. Let \( x_\bullet \in U^{[1]}_{V_\bullet}(\mathcal{O}_k) \); \( C \) the [unique, by (vi)] irreducible component of \( V_\bullet \) that meets [the image of] \( x_\bullet \); \( F_\bullet \defeq V_\bullet \setminus (C \cap U^{[1]}_\bullet) \subseteq V_\bullet \) [regarded as a closed subset]; \( U_\bullet \defeq V_\bullet \setminus F_\bullet \subseteq V_\bullet \) [so the image of \( x_\bullet \) lies in \( U_\bullet \)]. Write \( J_\eta \to \eta \) for the Jacobian of \( X_\eta \); \( J \to S \) for the uniquely determined semi-abelian scheme over \( S \) that extends \( J_\eta \)-\( \eta \) for the morphism that sends a \( T \)-valued point \( \xi \) [where \( T \) is a \( k \)-scheme] of \( X_\eta \), regarded as a divisor on \( X_\eta \times_k T \), to the point of \( J_\eta \) determined by the degree zero divisor \( \xi - (x_\bullet|_T) \). Then \( \eta \) **extends uniquely** to a morphism \( U_\bullet \to J \). If, moreover, \( X \) is loop-ample [cf. §0], then this morphism \( U_\bullet \to J \) is unramified.

(xi) **(Lifting Simple Paths)** In the situation of (viii), suppose further that the following conditions hold:

(a) the log-modifications \( W^{\log}_\bullet \to Y^{\log} \), \( V^{\log}_\bullet \to X^{\log} \) are base-field-isomorphic and split [cf. (vi)];

(b) the morphism \( W^{\log}_\bullet \to V^{\log}_\bullet \) is finite.

Let \( \gamma_\nu \) be a simple path [cf. §0] in the dual graph \( \Gamma_{V_\bullet} \) of \( V_\bullet \) [cf. (vi)]. Then there exists a simple path \( \gamma_W \) in the dual graph \( \Gamma_{W_\bullet} \) of \( W_\bullet \) that lifts \( \gamma_\nu \) in the sense that the morphism \( W_\bullet \to V_\bullet \) induces an isomorphism of graphs \( \gamma_W \cong \gamma_\nu \). Suppose further that the following condition holds:

(c) the morphism \( W^{\log}_\bullet \to V^{\log}_\bullet \) is loop-preserving.
Then $\gamma_W$ is unique in the sense that if $\gamma'_W$ is any simple path in $\Gamma_{W^\bullet}$ that lifts $\gamma_V$ and is co-terminal [cf. §0] with $\gamma_W$, then $\gamma_W = \gamma'_W$.

(xi) (Loop-preservation and Wild Ramification) In the situation of (xi), suppose that, in addition to the conditions (a), (b), (c) of (xi), the following conditions hold:

(d) there exists a prime number $p$ such that $k$ is of characteristic $p$, and the morphism $Y \rightarrow X$ is finite étale Galois and of degree $p$;
(e) the morphism $Y \rightarrow X$ is wildly ramified over the terminal vertices [cf. §0] of the simple path $\gamma_V$.

Let $w_{exc}$ be a vertex of $\Gamma_{W^\bullet}$ that corresponds to an irreducible component of the exceptional divisor of $W_{\log} \rightarrow Y_{\log}$ [i.e., a non-stable irreducible component of $W_{\log}$] and, moreover, maps to a vertex $v_{exc}$ of $\Gamma_{V^\bullet}$ lying in $\gamma_V$. Then the morphism $Y \rightarrow X$ is wildly ramified at $w_{exc}$.

Proof. First, we consider assertion (i). We may assume without loss of generality that $\psi$ is a base-field-isomorphism. Then it follows from the simple structure of the monoid $N$ that any log étale birational morphism over $U_{V^\bullet}$ is, in fact, étale. This completes the proof of assertion (i). Next, we consider assertion (iii). Since the morphism $X_{\log} \rightarrow S_{\log}$ is log smooth, and the morphism $V_{\log} \rightarrow X_{\log} \times_{S_{\log}} S_{\log}$ is log étale, we conclude that $V_{\log}$ is log smooth over $S_{\log}$, the portion of assertion (iii) concerning $\omega_{V_{\log}/S_{\log}}$ then follows immediately. To verify the portion of assertion (iii) concerning unramified log-modifications, it suffices to observe that, in light of the well-known local structure of the nodes of the stable log curve $X_{\log} \times_{S_{\log}} S_{\log} \rightarrow S_{\log}$, there exists a finite extension $k_o$ of $k$ such that, if we write $S_o = \text{Spec}(O_{k_o})$, then the morphism $V_{\log} \equiv V_{\log} \times_{S_{\log}} S_o \rightarrow S_{\log}$ admits sections that intersect with every irreducible component of $V_o \cap U_{V^\bullet}$; thus, the fact that $V_{\log} \rightarrow X_{\log}$ is an unramified log-modification follows immediately from the log smoothness of $U_{V^\bullet}$, in light of the simple structure of the monoid $N$. This completes the proof of assertion (iii).

Next, we consider assertion (iv). The fact that $V_{\log}$ is log regular follows immediately from the log smoothness of $V_{\log}$ over $S_{\log}$ [cf. assertion (iii)]; the fact that $U_{V^\bullet}$ is regular then follows from the log regularity of $U_{V^\bullet}$, in light of the simple structure of the monoid $N$. To construct a regular log-modification $V_{\log} \rightarrow X_{\log}$ that admits a base-field-isomorphic $X_{\log}$-morphism $V_{\log} \rightarrow V_{\log}$, it suffices to “resolve the singularities” at the finitely many points of $V_o \setminus U_{V^\bullet}$. To give a “resolution of singularities” of the sort desired, it suffices to construct, for each such $v$, a “fan” arising from a “locally finite nonsingular subdivision of the strongly convex rational polyhedral cone associated to the stalk of the characteristic sheaf of $V_{\log}$ at $v$ that is equivariant with respect to the Galois action on the stalk” [cf., e.g., the discussion at the beginning of [Mzk2], §2]. Since this is always possible [cf., e.g., the references quoted in the discussion of loc. cit.], we thus obtain a regular log-modification $V_{\log} \rightarrow X_{\log}$ that admits a base-field-isomorphic $X_{\log}$-morphism $V_{\log} \rightarrow V_{\log}$; moreover, by replacing $V_{\log}$ with the result of blowing up once more at various points of $V_o \setminus U_{V^\bullet}$, we may assume that each irreducible component $C$ of $V_o$
is smooth over $k_2$, as desired. Finally, the construction of the natural isomorphism $\omega_{C/k_2}(D_C) \to \omega_{V_0^\log/S_0^\log}^\log$ is immediate over $C \cap U_V^{[1]}$ [cf. our assumption that the log-modification $V_0^\log \to X^\log$ is unramified at $C$!]; one may then extend this natural isomorphism to $C$ by means of an easy local calculation at the points of $D_C$. This completes the proof of assertion (iv). By assertions (iii) and (iv), the underlying schemes of the domain and codomain of the morphism $V_2^\log \to V_1^\log \times_S^\log V_2^\log$ of assertion (ii) are normal. Thus, assertion (ii) follows immediately from Zariski’s main theorem.

Next, we consider assertion (v). We may assume without loss of generality that the $\phi_i^\log$ are base-field-isomorphic [so $\psi^\log$ is log étale]. Also, by blowing up once more at various points of $V_2 \setminus U_V^{[1]}$ [cf. the proof of assertion (iv)], one verifies immediately that we may assume without loss of generality that $C$ is smooth over $k_2$. Let us write $C^\log$ for the log scheme obtained by equipping $C$ with the log structure determined by the points of $C$ that meet the other irreducible components of $D_{V_2}$. Thus, after possibly replacing $k_2$ by a suitable finite unramified extension of $k_2$, we may assume that the interior $U_C \subseteq C$ of $C^\log$ is the open subscheme of a smooth proper curve of genus $g$ over $k(c)$ obtained by removing a divisor $D_C \subseteq C$ of degree $r > 0$ over $k(c)$. On the other hand, it follows immediately from the definition of a log-modification [cf. Definition 2.4, (i)] and the well-known general theory of toric varieties [cf. the discussion of “fans” in the proof of assertion (iv)] that the pair $(C, D_C)$ determines a toric variety of dimension one, and hence that $U_C$ is isomorphic to a copy of $\mathbb{G}_m$ over $k(c)$, i.e., that $g = 0$, $r = 2$, as desired. Now the remaining portions of assertion (v) follow immediately. This completes the proof of assertion (v).

Next, we consider assertion (vi). First, we observe that if $x$ belongs to the 1-interior, then it follows immediately from the simple structure of the monoid $\mathbb{N}$ [and the definition of the 1-interior] that $V_x$ is irreducible. Thus, it suffices to consider the case where $x$ does not belong to the 1-interior. Let $V_0^\log \to X^\log$, $V_0^\log \to V_0^\log$ be as in assertion (iv). We may assume without loss of generality that the log-modifications $V_0^\log \to X^\log$, $V_0^\log \to X^\log$ are base-field-isomorphisms. Also, by replacing $k$ by a finite unramified extension of $k$, we may assume that every irreducible component of $V_0^\log$ is geometrically irreducible over $k$, and that every point of $V_0 \setminus U_V^{[1]}$ is defined over $k$. Then, since $V_0^\log \to V_0^\log$ is birational, and $V_0^\log$ is log regular, hence, in particular, normal [cf. assertion (iv)], it follows from Zariski’s main theorem that the points of $V_0 \setminus U_V^{[1]}$ correspond precisely [via $V_0^\log \to V_0^\log$] to the connected components of the inverse image of $V_0 \setminus U_V^{[1]}$ via $V_0^\log \to V_0^\log$. Thus, the remainder of assertion (vi) follows immediately from assertion (v), applied to $V_0^\log \to V_0^\log$, $V_0^\log \to X^\log$. This completes the proof of assertion (vi).

Next, we consider assertion (vii). We may assume without loss of generality that the log-modifications $V_0^\log \to X^\log$, $V_0^\log \to X^\log$ are base-field-isomorphic. Then to verify assertion (vii), it suffices to observe that one may take $V_0^\log \overset{\text{def}}{=} V_0^\log \times_{X^\log} V_0^\log$. This completes the proof of assertion (vii). In a similar vein, assertion (viii) follows by observing that the fact that $Y_\eta \to X_\eta$ extends to a morphism $Y^\log \to X^\log$ follows, for instance, from [Mzk2], Theorem A, (1); thus, one may take $W_0^\log \overset{\text{def}}{=} V_0^\log \times_{X^\log} Y^\log$. This completes the proof of assertion (viii).
Next, we consider assertion (ix). We may assume without loss of generality that \( k = k_o \). Let \( V_o^{\log} \to X^{\log} \) be the base-field-isomorphic unramified log-modification determined by the regular semi-stable model of \( X \) over \( S \) [cf. Remark 2.4.1]. Since \( V_o \) is proper over \( S \), it follows that \( x_o \) extends to a point in \( V_o(\mathcal{O}_{K_o}) \); if this point fails to lie in \( U^{[1]}_{V_o} \), then it follows that it meets one of the nodes of \( V_o \). On the other hand, since \( \pi \) is a finite unramified extension of the discretely valued field \( k_o \) the completion of the regular scheme \( V_o \) at such a node is necessarily isomorphic over \( \mathcal{O}_{k_o} \) to a complete local ring of the form

\[
\mathcal{O}_{k_o}[[s, t]]/(s \cdot t - \pi_o)
\]

[where \( s, t \) are indeterminates; \( \pi_o \) is a uniformizer of \( \mathcal{O}_{k_o} \)], this contradicts our assumption that \( \mathcal{O}_{k_o} \subseteq \mathcal{O}_{K_o} \) induces a bijection \( k_o/\mathcal{O}_{k_o} \to K_o/\mathcal{O}_{K_o} \) [i.e., by considering the images via pull-back by \( x_o \) of \( s, t \) in these value groups, in light of the relation “\( s \cdot t - \pi_o \)”]. This completes the proof of assertion (ix).

Next, we consider assertion (x). Write \( N \to S \) for the Néron model of \( J_\eta \) over \( S \). Thus, \( J \) may be regarded as an open subscheme of \( N \). Note that the existence of the rational point \( x_\bullet \) implies that \( U_{x_\bullet} \) is smooth over \( S \) [cf. the proof of assertion (iii)]. It follows from the universal property of the Néron model [which is typically used to define the Néron model] that \( \iota_\eta \) extends to a morphism \( U_{x_\bullet} \to N \).

Since \( C \cap U_{x_\bullet} \) is connected [cf. the definition of \( U_{x_\bullet} \)], the fact that the image of this morphism lies in \( J \subseteq N \) follows immediately from the fact that [by definition] it maps \( x_\bullet \) to the identity element of \( J(\mathcal{O}_k) \). Thus, we obtain a morphism \( U_{x_\bullet} \to J \).

To verify that this morphism is unramified, it suffices [by considering appropriate translation automorphisms of \( J \)] to show that it induces a surjection on Zariski cotangent spaces at \( x_\bullet \); but the induced map on Zariski cotangent spaces at \( x_\bullet \) is easily computed [by considering the long exact sequence on cohomology associated to the short exact sequence \( 0 \to \mathcal{O}_{V_\bullet} \to \mathcal{O}_{V_\bullet}(x_\bullet) \to \mathcal{O}_{V_\bullet}(x_\bullet)|_{x_\bullet} \to 0 \) on \( V_\bullet \), then taking duals] to be the map

\[
H^0(X^{\log}, \omega_{X^{\log}/S^{\log}}) \to H^0(V^{\log}, \omega_{V^{\log}/S^{\log}}) \to \omega_{V^{\log}/S^{\log}}|_{x_\bullet}
\]

[where we recall the natural isomorphism \( \omega_{X^{\log}/S^{\log}}|_{V^{\log}} \to \omega_{V^{\log}/S^{\log}} \), arising from the fact that \( \phi^{\log} : V^{\log} \to X^{\log} \) is log étale — cf. assertion (iii)] given by evaluating at \( x_\bullet \), hence is surjective so long as \( X \) is loop-ample [cf. §0]. This completes the proof of assertion (x).

Next, we consider assertion (xi). First, let us observe that it follows immediately from the surjectivity of \( W^\bullet \to V^\bullet \) that every vertex of \( \Gamma_{V^\bullet} \) may be lifted to a vertex of \( \Gamma_{W^\bullet} \). Next, I claim that every edge of \( \Gamma_{V^\bullet} \) may be lifted to an edge of \( \Gamma_{W^\bullet} \). Indeed, let \( y \in W^\bullet(\bar{k}) \), \( x \in V^\bullet(\bar{k}) \) be such that \( y \to x \); write \( W_y \), \( V_x \) for the respective spectra of the local rings obtained by completing \( W^\bullet \), \( V^\bullet \) at \( y \), \( x \). Then the morphism \( W^{\log} \to V^{\log} \) induces a finite, dominant, hence surjective, morphism \( W_y \to V_x \). In particular, this morphism \( W_y \to V_x \) induces a surjection from the set \( I_y \) of irreducible components of \( W^\bullet \) that pass through \( y \) to the set \( I_x \) of irreducible components of \( V^\bullet \) that pass through \( x \). Thus, if \( x \) corresponds to an edge of the dual graph \( \Gamma_{V^\bullet} \), then this set \( I_x \) is of cardinality 2 [cf. assertion (vi)]; since \( I_y \) is of cardinality \( \leq 2 \) [cf. assertion (vi)], the existence of the surjection \( I_y \to I_x \) thus
implies that this surjection is, in fact, a bijection $I_y \sim I_x$, hence that $y$ corresponds to an edge of $\Gamma_{W\ast}$ [cf. assertion (vi)]. This completes the proof of the claim. Thus, by starting at one of the terminal vertices of $\gamma_V$, and proceeding along $\gamma_V$ from vertex to edge to vertex, etc., one concludes immediately the existence of a simple path $\gamma_W$ lifting $\gamma_V$. To verify uniqueness when condition (c) holds, write $v_1, v_2$ for the terminal vertices of $\gamma_V$; $e'$ for the edge of $\gamma_V$ that is nearest to $v_1$ among those edges of $\gamma_V$ that lift to different edges in $\gamma_W$. $\gamma_W'$; $v_1'$ (respectively, $v_2'$) for the vertex to which $e'$ abuts that lies in the same connected component of the complement of $e'$ in $\gamma_V$ as $v_1$ (respectively, $v_2$); $v_1^+$ for the vertex of $\gamma_V$ that is nearest to $v_1$ among those vertices of $\gamma_V$ lying between $v_2'$ and $v_2$ which lift to the same vertex in $\gamma_W$, $\gamma_W'$. Then by traveling along $\gamma_W$ from the vertex $w_1$ of $\gamma_W$ lifting $v_1$ to the vertex $w_1^+$ of $\gamma_W$ lifting $v_1^+$, then traveling back along $\gamma_W'$ from $w_1^+$ [which, by definition, also belongs to $\gamma_W'$] to $w_1$ [which, by definition, also belongs to $\gamma_W'$], one obtains a “nontrivial loop” in $\Gamma_{W,\ast}$ [i.e., a nonzero element of $H_1(\Gamma_{W,\ast}, \mathbb{Q})$] that maps to a “trivial loop” in $\Gamma_{V,\ast}$ [i.e., the zero element of $H_1(\Gamma_{V,\ast}, \mathbb{Q})$]. But this contradicts the assumption that the morphism $W_{\log} \to V_{\log}^{\ast}$ is loop-preserving [cf. condition (c)]. This completes the proof of assertion (xii).

Finally, we consider assertion (xii). First, we observe that the hypotheses of assertion (xii) are stable with respect to base-change in $S$. In particular, we may always replace $S = \text{Spec} (\mathcal{O}_k)$ by the normalization of $S$ in some finite separable extension of $k$. Next, by assertion (iv), we may assume that there exists a base-field-isomorphic, regular, split log-modification $V_0^{\log} \to X^{\log}$, together with an $X^{\log}$-morphism $V_0^{\log} \to V_0^{\log}$. Moreover, if we take $W_0^{\log} = V_0^{\log} \times V_0^{\log} W_{\log}^{\ast}$, then the composite morphism $W_0^{\log} \to W_0^{\log} \to Y^{\log}$ forms a base-field-isomorphic log-modification such that the projection $W_0^{\log} \to V_0^{\log}$ is finite [by condition (b); cf. also the finiteness mentioned in the discussion entitled “Log Schemes” given in §0]. In particular, by replacing $W_0^{\log} \to V_0^{\log}$ by $W_0^{\log} \to V_0^{\log}$ [cf. also assertion (v), concerning the effect on the simple path $\gamma_V$; assertion (vi), concerning the effect on the loop-rank], we may assume that the log-modification $V_0^{\log} \to X^{\log}$ is regular. Here, let us note that since $W_0^{\log} \to V_0^{\log}$ is finite, and $W_0^{\log}$ is log regular [cf. assertion (iv)], which implies, in particular, that $W_0^{\ast}$ is normal [so $W_0^{\ast}$ is the normalization of $V_0^{\ast}$ in $Y_0^{\ast}$], it follows that $G = \text{Gal}(Y_0/X_0) (\cong \mathbb{Z}/p\mathbb{Z})$ acts on $W_0^{\log}$. Also, by assertion (iv), we may assume that there exists a base-field-isomorphic, regular, split log-modification $W_0^{\log} \to Y^{\log}$, together with a $Y^{\log}$-morphism $W_0^{\log} \to W_0^{\log}$; moreover, it follows immediately from the proof of assertion (iv) that we may choose $W_0^{\log}$ so that the action of $G$ extends to $W_0^{\log}$. Finally, we observe that it follows from assertion (v) that every irreducible component of the exceptional divisors of $W_0^{\ast}$, $V_0^{\ast}$ [relative to the morphisms $W_0^{\log} \to Y^{\log}$, $V_0^{\log} \to X^{\log}$] is isomorphic to $\text{Spec} \mathcal{O}_k$.

Let $E_W$ be the irreducible component of $W_0^{\ast}$ corresponding to $w_{\text{exc}}$. Thus, there exists a unique irreducible component $F_W$ of $W_0^{\ast}$ that maps finitely to $E_W$; moreover, $E_W$ maps finitely to an irreducible component $E_Y$ of $V_0^{\ast}$ [corresponding to $v_{\text{exc}}$] that lies in the exceptional divisor of $V_0^{\log} \to X^{\log}$. Thus, we have finite morphisms

$$F_W \to E_W \to E_Y.$$
where $F_W$, $E_V$ are isomorphic to $\mathbb{P}_k^1$, the first morphism $F_W \to E_W$ is a morphism between irreducible schemes that induces an isomorphism between the respective function fields. Now to complete the proof of assertion (xii), it suffices to assume that

the composite morphism $F_W \to E_V$ is generically étale

and derive a contradiction. Let us refer to the two $k$-valued points of $F_W$ (respectively, $E_V$) [cf. assertion (v)] that lie outside $U^{[1]}_{W\Delta}$ (respectively, $U^{[1]}_V$) as the critical points of $F_W$ (respectively, $E_V$). Then since the divisor on $W^{\log}_\bullet$ (respectively, $V^{\log}_\bullet$) at which the morphism $W^{\log}_\bullet \to V^{\log}_\bullet$ is [necessarily wildly] ramified does not, by our assumption, contain $E_W$ (respectively, $E_V$), it follows that the divisor on $F_W$ (respectively, $E_V$) at which $F_W \to E_V$ is ramified is supported in the divisor defined by the two critical points of $F_W$ (respectively, $E_V$).

Next, let us write $v_1, v_2$ for the terminal vertices of $\gamma_V$. Note that by conditions (d), (e), it follows that $v_1, v_2$ lift, respectively, to unique vertices $w_1, w_2$ of $\Gamma_{W, \bullet}$. In particular, it follows that any two simple paths in $\Gamma_{W, \bullet}$ lifting $\gamma_V$ are co-terminal. Now I claim that the morphism $F_W \to E_V$ is of degree $p$. Indeed, if this morphism is of degree 1, then it follows that there exists a $G$-conjugate $w'_\text{exc}$ of $w_{\text{exc}}$ such that $w'_\text{exc} \neq w_{\text{exc}}$. Thus, considering the $G$-conjugates of any simple path in $\Gamma_{W, \bullet}$ lifting $\gamma_V$, it follows that we obtain two distinct [necessarily co-terminal] simple paths in $\Gamma_{W, \bullet}$ lifting $\gamma_V$. But this contradicts the uniqueness portion of assertion (xi). This completes the proof of the claim. Note that this claim implies that we have $G$-equivariant morphisms $F_W \to E_W \to E_V$, where $G$ acts trivially on $E_V$.

Next, I claim that $G$ fixes each of the critical points of $F_W$. Indeed, it follows immediately from the definitions that $G$ preserves the divisor of critical points of $F_W$. Thus, if $G$ fails to fix each of the critical points of $F_W$, then it follows that $G$ permutes the two critical points of $F_W$, hence that $p = 2$. But since $F_W \to E_V$ is of degree $p$ and unramified outside the divisor of critical points of $F_W$, this implies that $\mathbb{P}_k^1 \cong F_W \to E_V \cong \mathbb{P}_k^1$ is finite étale, hence [since, as is well-known, the étale fundamental group of $\mathbb{P}_k^1$ is trivial!] that $F_W \to E_V$ is an isomorphism — in contradiction to the fact that $F_W \to E_V$ is of degree $p > 1$. This completes the proof of the claim. Note that this claim implies that the morphism $F_W \to E_V$ is ramified at the critical points of $F_W$, and that the set of two critical points of $F_W$ maps bijectively to the set of two critical points of $E_V$. In particular, it follows that $F_W \to E_V$ determines a finite étale covering $(\mathbb{G}_m)_k \to (\mathbb{G}_m)_k$ of degree $p$. On the other hand, any morphism $(\mathbb{G}_m)_k \to (\mathbb{G}_m)_k$ is determined by a unit on $(\mathbb{G}_m)_k$, i.e., by a $k^\times$-multiple of $U^n$, where $U$ is the standard coordinate on $(\mathbb{G}_m)_k$ and $n$ is the degree of the morphism. Since the morphism determined by a $k^\times$-multiple of $U^p$ clearly fails to be generically étale, we thus obtain a contradiction. This completes the proof of assertion (xii). ☐

In the following, we shall write

“$\pi_1(-)$”

for the “log fundamental group” of the log scheme in parentheses, relative to an appropriate choice of basepoint [cf. [III] for a survey of the theory of log fundamental groups]. Also, from now on, we shall assume, until further notice, that:
The discrete valuation ring \( \mathcal{O}_k \) is of mixed characteristic, with residue field \( k \) perfect of characteristic \( p \) and of countable cardinality.

Recall that by “Krasner’s lemma” [cf. [Kobl], pp. 69-70, as well as the proof given above of Lemma 2.1], given a splitting field \( k' \) over \( k \) of a monic polynomial \( f(T) \) [where \( T \) is an indeterminate] of degree \( n \) with coefficients in \( k \), every monic polynomial \( h(T) \) of degree \( n \) with coefficients in \( k \) that are sufficiently close [in the topology of \( k \)] to the coefficients of \( f(T) \) also splits in \( k' \). Thus, it follows from our assumption that \( k \) is of countable cardinality that \( k \) admits a countable collection \( \mathcal{F} \) of subfields which are finite and Galois over \( k \) such that every finite Galois extension of \( k \) contained in \( \bar{k} \) is contained in a subfield that belongs to the collection \( \mathcal{F} \).

In some sense, the main technical result of the present §2 is the following lemma.

**Lemma 2.6.** (Prime-power Cyclic Coverings and Log-modifications) Suppose that \( X \) is loop-ample [cf. §0]. Then:

(i) **(Existence of Wild Ramification)** Let

\[
X^+_\eta \rightarrow X_\eta
\]

be a finite étale Galois covering of hyperbolic curves over \( \eta \) with stable reduction over \( S \) such that \( \text{Gal}(X^+_\eta/X_\eta) \) is isomorphic to a product of \( 2g_X \) copies of \( \mathbb{Z}/p\mathbb{Z} \) [so such a covering always exists after possibly replacing \( k \) by a finite extension of \( k \)]; \( V^{\log} \rightarrow X^{\log} \) a split, base-field-isomorphic log-modification. Then \( X^+_\eta \rightarrow X_\eta \) is wildly ramified over every irreducible component \( C \) of \( V \).

(ii) **(Loopification vs. Component Crushing)** After possibly replacing \( k \) by a finite extension of \( k \), there exist data as follows: a stable curve \( Y \rightarrow S \) with smooth generic fiber \( Y_\eta \overset{\text{def}}{=} Y \times_S \eta \) and associated log scheme \( Y^{\log} \); a cyclic finite étale covering \( Y_\eta \rightarrow X_\eta \) of degree a positive power of \( p \) — which determines a morphism

\[
Y^{\log} \rightarrow X^{\log}
\]

such that at least one of the following two conditions is satisfied:

(a) \( Y_\eta \rightarrow X_\eta \) is loopifying and wildly ramified at some [necessarily] stable irreducible component \( C \) of \( Y \) which is potentially base-semistable relative to \( Y_\eta \rightarrow X_\eta \);

(b) there exists a [necessarily] stable irreducible component \( C \) of \( Y \) which is not potentially base-semistable relative to \( Y_\eta \rightarrow X_\eta \).

(iii) **(Components Crushed to the 1-Interior)** In the situation of (ii), there exists a commutative diagram

\[
\begin{array}{ccc}
W^{\log} & \rightarrow & Q^{\log} \\
\downarrow & & \downarrow \\
Y^{\log} & \rightarrow & X^{\log}
\end{array}
\]

where the vertical morphisms are split, base-field-isomorphic log-modifications; the horizontal morphism in the bottom line is the morphism already referred to; the
natural action of $\text{Gal}(Y_\eta/X_\eta)$ on $Y^{\text{log}}$ extends to $W^{\text{log}}$ — such that the following property is satisfied: If condition (a) (respectively, (b)) of (ii) is satisfied, then the unique irreducible component $C_W$ of $W$ that maps finitely to the irreducible component $C$ of condition (a) (respectively, (b)) maps finitely to $Q$ (respectively, maps to a closed point of $U_Q^{\lbrack 1 \rbrack}$).

(iv) (Group-theoretic Characterization of Crushing) In the situation of (ii), let $C$ be a [necessarily] stable irreducible component of $Y$; $l$ a prime $\neq p$. Write $\text{Gal}(\overline{k}/k) \to G_{k^{\text{log}}}$ for the maximal tamely ramified quotient; $\Delta_{Y^{\text{log}}}$ (respectively, $\Delta_{X^{\text{log}}}$) for the maximal pro-$l$ quotient of the kernel of the natural [outer] surjection $\pi_1(Y^{\text{log}}) \to G_{k^{\text{log}}}$ (respectively, $\pi_1(X^{\text{log}}) \to G_{k^{\text{log}}}$); $\Delta_C \subseteq \Delta_{Y^{\text{log}}}$ for the decomposition group of $C$ [well-defined up to conjugation by an element of $\pi_1(Y^{\text{log}})$]. [Thus, $\Delta_{Y^{\text{log}}}$ (respectively, $\Delta_{X^{\text{log}}}$) may be identified with the maximal pro-$l$ quotient of $\ker(\pi_1(Y_\eta) \to \text{Gal}(\overline{k}/k))$ (respectively, $\ker(\pi_1(X_\eta) \to \text{Gal}(\overline{k}/k))$) — cf., e.g., [MT], Proposition 2.2, (v).] Then the following two conditions are equivalent:

(a) the image of $\Delta_C$ in $\Delta_{X^{\text{log}}}$ is trivial;
(b) there exists a commutative diagram

$$
\begin{array}{ccc}
W^{\text{log}} & \to & Q^{\text{log}} \\
\downarrow & & \downarrow \\
Y^{\text{log}} & \to & X^{\text{log}}
\end{array}
$$

— where the vertical morphisms are split, base-field-isomorphic log-modifications; the horizontal morphism in the bottom line is the morphism already referred to — such that the unique irreducible component $C_W$ of $W$ that maps finitely to $C$ maps to a closed point of $U_Q^{\lbrack 1 \rbrack}$.

(v) (Group-theoretic Characterization of Wild Ramification) In the situation of (iv), the morphism $Y_\eta \to X_\eta$ is wildly ramified at $C$ if and only if $\text{Gal}(Y_\eta/X_\eta)$ stabilizes [the conjugacy class of] and induces the identity [outer automorphism] on the normally terminal — cf. [Mzk13], Proposition 1.2, (ii); [Mzk17], Lemma 2.12] subgroup $\Delta_C$ of $\Delta_{Y^{\text{log}}}$.

Proof. Let us write $T_X \overset{\text{def}}{=} \pi_1(X_\eta \times_k \overline{k})^{\text{ab}} \otimes \mathbb{Z}_p$ for the maximal pro-$p$ abelian quotient of the geometric fundamental group of $X_\eta$. Thus, $T_X$ is a free $\mathbb{Z}_p$-module of rank $2g_X$.

Next, we consider assertion (i). Upon base-change to $\overline{k}$, the covering $X^+_\eta \to X_\eta$ corresponds to the open subgroup $p \cdot T_X \subseteq T_X$. Let us write $J \to S$ for the uniquely determined semi-abelian scheme that extends the Jacobian $J_\eta \to \eta$ of $X_\eta$. After possibly replacing $k$ by a finite extension of $k$, there exists a rational point $x \in U_V^{\lbrack 1 \rbrack}(\mathcal{O}_k)$ that meets $C$. Thus, for some Zariski open neighborhood $U_x$ of the image of $x$ in $V$, we obtain a morphism $\iota : U_x \to J$, as in Proposition 2.5, (x). Moreover, since we have assumed that $X$ is loop-ample, it follows that this morphism $\iota$ is unramified. Now if the morphism $X^+_\eta \to X_\eta$ is tamely ramified over
$C$, then it follows from our assumptions on the covering $X^+_\eta \to X_\eta$, together with the interpretation of $J$ in terms of Néron models [cf. the proof of Proposition 2.5, (x)], that [after possibly replacing $k$ by a finite extension of $k$] there exists some finite separable extension $L$ of the function field $\kappa(C)$ of $C$ such that there exists a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(L) & \xrightarrow{\iota_L} & J \\
\downarrow \epsilon & & \downarrow [p] \\
\text{Spec}(\kappa(C)) & \xrightarrow{\iota_C} & J \\
\end{array}
$$

where $\epsilon$ is the [étale] morphism determined by the given inclusion $\kappa(C) \hookrightarrow L$; $[p]$ is the morphism given by multiplication by $p$ on the group scheme $J$; $\iota_C$ is the restriction of $\iota$ to Spec($\kappa(C)$). On the other hand, since the restriction of $[p]$ to the special fiber $\mathcal{J}$ of $J$ factors through the Frobenius morphism on $\mathcal{J}$, it follows that $[p] \circ \iota_L$ fails to be unramified. Thus, since $\iota_C \circ \epsilon$ is unramified, we obtain a contradiction to the commutativity of the diagram. This completes the proof of assertion (i).

Next, we consider assertion (ii). If $V^\log \to X^\log$ is a split, base-field-isomorphic log-modification, and $C$ is an irreducible component of $V$, then let us write $D_C \subseteq T_X$ for the decomposition group associated to $C$ and $I_C \subseteq D_C$ for the wild inertia group associated to $C$. Note that since $T_X$ is abelian, and $V^\log$ is split, it follows that the subgroups $D_C, I_C$ are well-defined [and completely determined by $C$]. Moreover, by assertion (i), it follows that $I_C$ has nontrivial image in $T_X \otimes \mathbb{Z}/p\mathbb{Z}$.

Next, I claim that if $V^\log_0 \to X^\log_0$ is a split, base-field-isomorphic log-modification, then $V^\log_0 \to X^\log_0$ is completely determined, as a log scheme over $X^\log$, up to countably many possibilities, by $X^\log$. Indeed, the morphism $V^\log_0 \to X^\log$ is an isomorphism over the 1-interior of $X^\log$ [cf. Proposition 2.5, (i), (ii)]. Moreover, at each of the finitely many points $x$ of $X^\log$ lying in the complement of the 1-interior, $V^\log_0 \to X^\log$ is determined by countably many choices of certain combinatorial data involving the groupification of the stalk of the characteristic sheaf of $X^\log$ at $x$ [cf. the proof of Proposition 2.5, (iv)]. This completes the proof of the claim. In particular, since $k$ is assumed to be of countable cardinality [cf. the discussion preceding the present Lemma 2.6], it follows that:

There exists a countable cofinal collection $\mathcal{M}$ of split log-modifications of $X^\log$.

In particular, it follows that if we write $\mathcal{C}$ for the set of all irreducible components of the special fibers of log-modifications belonging to $\mathcal{M}$, then the collection of [non-trivial] subgroups of $T_X$ of the form $\langle I_C \rangle$, where $C \in \mathcal{C}$, is of countable cardinality. Thus, we may, for instance, enumerate the elements of $\mathcal{C}$ via the natural numbers
so as to obtain a sequence $C_1, C_2, \ldots$ [i.e., which includes all elements of $C$]. Since $\mathbb{Z}_p$, on the other hand, is of uncountable cardinality, we thus conclude that there exists a surjection

$$\Lambda : T_X \twoheadrightarrow \Lambda_X$$

— where $\Lambda_X \cong \mathbb{Z}_p$ — such that the following properties are satisfied:

1. For every subgroup $I_C$, where $C \in \mathcal{C}$, we have $\Lambda_C \overset{\text{def}}{=} \Lambda(I_C) \neq \{0\}$.
2. There exists a stable component $C_0$ of $X$ such that $\Lambda_{C_0} = \Lambda_X$.

[For instance, by applying the fact that each $I_{C_n}$ has nontrivial image in $T_X \otimes \mathbb{Z}/p\mathbb{Z}$, one may construct $\Lambda$ by constructing inductively on $n$ [a natural number] an increasing sequence of natural numbers $m_1 < m_2 < \ldots$ such that $I_{C_n}$ maps to a nonzero subgroup of $\Lambda_X \otimes \mathbb{Z}/p^{m_n}\mathbb{Z}$.

Let us refer to a connected finite étale Galois covering $X'_\eta \to X_\eta$ [which may only be defined after possibly replacing $k$ by a finite extension of $k$] of hyperbolic curves over $\eta$ with stable reduction over $S$ as a $\Lambda$-covering if the covering $X'_\eta \times_k \overline{k} \to X_\eta \times_k \overline{k}$ arises from an open subgroup of $\Lambda_X$; thus, $\text{Gal}(X'_\eta/X_\eta)$ may be thought of as a finite quotient of $\Lambda_X$ by an open subgroup $\Lambda_X' \subseteq \Lambda_X$. Note that since every $\Lambda_C$, where $C \in \mathcal{C}$, is isomorphic to $\mathbb{Z}_p$, it follows that the following property also holds:

3. For any pair $X''_\eta \to X'_\eta \to X_\eta$ of $\Lambda$-coverings of $X_\eta$ such that $\Lambda_C$ has nontrivial image in $\text{Gal}(X'_\eta/X_\eta) \cong \Lambda_X/\Lambda_X'$, it follows that $\Lambda_C \cap \Lambda_X'$ surjects onto $\Lambda_X'/\Lambda_X''$ — i.e., that the covering $X''_\eta \to X'_\eta$ is totally wildly ramified over any valuation of the function field of $X'_\eta$ whose center on $X$ is equal to the generic point of $C$.

Now to complete the proof of assertion (ii), it suffices to derive a contradiction upon making the following two further assumptions:

4. Every $\Lambda$-covering is loop-preserving.
5. For every $\Lambda$-covering $X'_\eta \to X_\eta$ [which extends to a morphism $(X')^{\log} \to X^{\log}$ of log stable curves], there exists [after possibly replacing $k$ by a finite extension of $k$] a split, base-field-isomorphic log-modification $V^{\log} \to X^{\log}$ such that the morphism $X'_\eta \to X_\eta \cong V_\eta \subseteq V$ extends to a quasi-finite morphism from some Zariski neighborhood in $X'$ of the generic points of $X'_\eta$ to $V$.

[Indeed, if assumption (4) is false, then it follows immediately that condition (a) of assertion (ii) holds [cf. property (2)]; if assumption (5) is false, then it follows immediately that condition (b) of assertion (ii) holds.] Note, moreover, that assumption (5) implies the following property:

6. For every $\Lambda$-covering $X'_\eta \to X_\eta$ [which extends to a morphism $(X')^{\log} \to X^{\log}$ of log stable curves], there exist [after possibly replacing $k$ by a finite extension of $k$], split, base-field-isomorphic log-modifications $(V')^{\log} \to$
A covering $X'_\eta \to X_\eta$ [after possibly replacing $k$ by a finite extension of $k$] such that every $\Lambda_C$, where $C \in \mathcal{C}$, has nontrivial image in $\Lambda_X/\Lambda_{X'}$. Indeed, by property (3), it follows that property (8) follows immediately from property (4). To verify property (7), we reason as follows: First, let

$$X^+_\eta \to X^*_\eta \to X_\eta$$

be $\Lambda$-coverings [which exist after possibly replacing $k$ by a finite extension of $k$] such that $\Lambda_{X'}/\Lambda_{X^+}$ is of order $p$, and $\Lambda_C$ has nontrivial image in $\Lambda_X/\Lambda_{X'}$ [which implies that $\Lambda_C \cap \Lambda_{X^+}$ surjects onto $\Lambda_{X'}/\Lambda_{X^+}$ — cf. property (3)] for every stable irreducible component $C$ of $X'$. Write $X^+_{\eta,log} \to X^*_{\eta,log} \to X^*_{\eta}$ for the resulting morphisms of log stable curves. [Note that such $\Lambda$-coverings exist, precisely because there are only finitely many such stable $C$.] Let $V^+_{\eta,log} \to V^*_{\eta,log}$, $V^*_{\eta,log} \to X^*_{\eta,log}$, $V^*_{\eta,log} \to X^*_{\eta}$ be split, base-field-isomorphic log-modifications such that there exist finite, loop-preserving morphisms

$$V^+_{\eta,log} \to V^*_{\eta,log} \to V_{\eta,log}$$

lying over $X^+_\eta \to X^*_{\eta,log} \to X^*_{\eta,log}$ [cf. properties (4), (6)]. Thus, $V^+_{\eta,log}, V^*_{\eta,log}$ are completely determined by $V_{\eta,log}$ — i.e., by taking the normalization of $V$ in $X^+_{\eta}, X^*_{\eta}$] Next, let us observe that every node $\nu$ of $X$ determines a simple path $\gamma^\nu$ in the dual graph $\Gamma_V$ [i.e., by taking the inverse image of $\nu$ in $V$ — cf. Proposition 2.5, (v)], whose terminal vertices are stable irreducible components of $V$ [but whose non-terminal vertices are non-stable irreducible components of $V$]. Thus, by Proposition 2.5, (xi) [which is applicable, in light of properties (4), (6)], it follows that $\gamma^\nu$ lifts [uniquely — i.e., once one fixes liftings of the terminal vertices] to simple paths $\gamma^\nu_{\nu}$ in $\Gamma_{V^+}$, $\gamma^\nu_{V^*}$ in $\Gamma_{V^*}$. Since, moreover, $X^+_{\eta,log} \to X^*_{\eta,log}$ is totally wildly ramified over the terminal vertices of $\gamma^\nu_{\nu}$, it thus follows that we may apply Proposition 2.5, (xii), to conclude that $X^+_{\eta,log} \to X^*_{\eta,log}$ is wildly ramified at every non-stable vertex of $\gamma^\nu_{\nu}$. Write $\mathcal{B}$ for the set of irreducible components of $V$ [which we think of as valuations of the function field of $X_\eta \times_k \overline{k}$] that are the images of stable vertices of $\gamma^\nu_{\nu}$, for nodes $\nu$
of \(X\). Observe that if we keep the coverings \(X^\dagger_\eta \to X^*_\eta \to X_\eta\) fixed, but vary the log-modification \(V^{\log} \to X^{\log}\) [among, say, elements of \(M\)], then the set \(\mathcal{B}\) remains unchanged [if we think of \(\mathcal{B}\) as a set of valuations of the function field of \(X^\dagger_\eta \times_k \overline{k}\)] and of finite cardinality [bounded by the cardinality of the set of stable irreducible components of \(X^\dagger\)]. Thus, in summary, if we think of \(\mathcal{B}\) as a subset of \(\mathcal{C}\), then we may conclude the following:

\[
\Lambda_C \bigcap \Lambda_X \text{ surjects onto } \Lambda_{X'/\Lambda_{X^*}}, \text{ for all } C \in \mathcal{C}\setminus\mathcal{B}.
\]

Since \(\mathcal{B}\) is finite, it thus follows that there exists a \(\Lambda\)-covering \(X'\eta \to X^\dagger_\eta \to X_\eta\) such that \(\Lambda_C \bigcap \Lambda_X\) has nontrivial image in \(\Lambda_{X'/\Lambda_{X^*}}\), for all \(C \in \mathcal{C}\) [cf. property (3)]. This completes the proof of property (7).

Next, let us consider \(\Lambda\)-coverings

\[
X''_\eta \to X'_\eta \to X
\]

[which exist after possibly replacing \(k\) by a finite extension of \(k\)] such that \(\Lambda_{X'/\Lambda_{X''}}\) is of order \(p\), and \(\Lambda_C\) has nontrivial image in \(\Lambda_X/\Lambda_{X'}\) [which implies that \(\Lambda_C \bigcap \Lambda_X\) surjects onto \(\Lambda_{X'/\Lambda_{X''}}\) — cf. property (3)] for all \(C \in \mathcal{C}\) [cf. property (7)]; write \((X'')^{\log} \to (X')^{\log} \to X^{\log}\) for the resulting morphisms of log stable curves. Let \((V'')^{\log} \to (X'')^{\log}, (V')^{\log} \to (X')^{\log}\) be split, base-field-isomorphic, unramified log-modifications such that there exists a finite, loop-preserving morphism

\[
(V'')^{\log} \to (V')^{\log}
\]

lying over \((X'')^{\log} \to (X')^{\log}\) [which exist after possibly replacing \(k\) by a finite extension of \(k\) — cf. properties (4), (6); Proposition 2.5, (iii)].

Next, let us consider the logarithmic derivative

\[
\delta : \omega_{(V')^{\log}/S^{\log}}|_{V'} \to \omega_{(V'')^{\log}/S^{\log}}
\]

of the morphism \((V'')^{\log} \to (V')^{\log}\). Since this morphism is finite étale over \(\eta\), it follows that \(\delta\) is an isomorphism over \(\eta\). On the other hand, since \(X''_\eta \to X'_\eta\) is totally wildly ramified over every irreducible component \(C\) of \(V'\) [i.e., induces a purely inseparable extension of degree \(p\) of the function field of \(\overline{C}\)], it follows that \(\delta\) vanishes on the special fiber \(\overline{V''}\). Write \(\delta^* \overset{\text{def}}{=} \pi^{-n} \cdot \delta\), for the maximal integer \(n\) such that \(\pi^{-n} \cdot \delta\) remains integral. Thus, we obtain a morphism

\[
\delta^* : \omega_{(V')^{\log}/S^{\log}}|_{V'} \to \omega_{(V'')^{\log}/S^{\log}}
\]

of line bundles on \(V''\) which is not identically zero on \(\overline{V''}\). Let \(C''\) be an irreducible component of \(V''\) such that \(\delta^*|_{C''} \neq 0\). Note that since \(V''\) has at least one stable irreducible component, it follows that we may choose \(C''\) such that either \(C''\) is stable or \(C''\) meets an irreducible component \(C^*\) of \(V''\) such that \(\delta^*|_{C^*} \equiv 0\). Thus, if \(C''\) is not stable, then it follows that \(\delta^*|_{C''}\) has at least one zero [i.e., is not an isomorphism of line bundles]. Write \(C'\) for the irreducible component of \(V'\) which is the image of \(C''\); \(E'' \to C'', E' \to C''\) for the respective normalizations; \(g_{E'}, g_{E''}\) for the respective genera of \(E', E''\). Also, let us refer to the points of \(C'', E'', C'\),
$E'$ that do not map to the respective 1-interiors of $(V'')^{\log}$, $(V')^{\log}$ as critical points. Write $D_{E'} \subseteq E'$, $D_{E''} \subseteq E''$ for the respective divisors of critical points; $r_{E'}$, $r_{E''}$ for the respective degrees of $D_{E'}$, $D_{E''}$.

Next, let us consider the morphism $C'' \to C'$. Since this morphism $C'' \to C'$ induces a purely inseparable extension of degree $p$ on the respective function fields, it follows that we have an isomorphism of $k$-schemes $E'' \times_k k' \to E'$ [so $g_{E'} = g_{E''}$], where we write $k \to k' \cong k$ for the [degree one] field extension determined by the Frobenius morphism on $k$. Next, I claim that the critical points of $C''$ map to critical points of $C'$. Indeed, if a critical point of $C''$ maps to a non-critical point $c$ of $C'$, then let us write $C''_c$, $C'_c$ for the spectra of the respective completions of the local rings of $C''$, $C'$ at [the fiber over] $c$. Then observe that since $V'$ is regular [of dimension two] at $c$ [cf. Proposition 2.5, (iv)], while $V''$ is the normalization of $V'$ in $X''_p$, it follows from elementary commutative algebra that $V''$ is finite and flat over $V'$ of degree $p$ at $c$. Thus, if we write $\eta_c$ for the spectrum of the residue field of the unique generic point of the irreducible scheme $C'_c$, then $C''_c \times_{C'_c} \eta_c \to \eta_c$ is finite, flat of degree $\leq p$; [since $X''_p \to X''$ is totally wildly ramified over every irreducible component of $V'$, it follows that] the degree of each of the $\geq 2$ [cf. Proposition 2.5, (vi)] connected components of $C''_c \times_{C'_c} \eta_c$ over $\eta_c$ is equal to $p$ — in contradiction to the fact that the degree of $C''_c \times_{C'_c} \eta_c$ over $\eta_c$ is $\leq p$. This completes the proof of the claim. In particular, it follows that $r_{E''} \leq r_{E'}$.

Now recall that we have natural isomorphisms

$$\omega_{E'k}(D_{E'}) \cong \omega_{(V')^{\log}}(S_{E'}^{\log}|E'); \quad \omega_{E''k}(D_{E''}) \cong \omega_{(V'')^{\log}}(S_{E''}^{\log}|E'')$$

[cf. Proposition 2.5, (i), (ii), (iii), (iv), (v), (vi)]; the fact that the log-modifications $(V'')^{\log} \to (X'')^{\log}$, $(V')^{\log} \to (X')^{\log}$ are unramified. Moreover, it follows immediately from the definitions that $\deg(\omega_{E'k}(D_{E'})) = 2g_{E'} + r_{E'}$, $\deg(\omega_{E''k}(D_{E''})) = 2g_{E''} + r_{E''}$. We thus conclude that $\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) \leq \deg(\omega_{V'k}^{\log}(S_{E'}^{\log}|C'))$. On the other hand, the existence of the generically nonzero morphism of line bundles $\delta^*|_{C''}$ implies that

$$\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) \geq \deg(\omega_{V'k}^{\log}(S_{E'}^{\log}|C'))$$

— which implies that $\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) \leq 0$. Now if $C''$ is stable, then we have $\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) = 2g_{E''} + r_{E''} > 0$. We thus conclude that $C''$ is non-stable. But this implies that $\delta^*|_{C''}$ has at least one zero, so [cf. the above display of inequalities] we obtain that $\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) < 0$, in contradiction to the equality $\deg(\omega_{V''k}^{\log}(S_{E''}^{\log}|C'')) = 0$ if $C''$ is non-stable [cf. the proof of Proposition 2.5, (v)]. This completes the proof of assertion (ii). In light of assertion (ii), assertion (iii) follows immediately from Proposition 2.5, (viii). This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, let us observe that by Proposition 2.5, (ix), it follows that we may assume that split, base-field-isomorphic log-modifications $W^{\log} \to Y^{\log}$, $Q^{\log} \to X^{\log}$, together with a morphism $W^{\log} \to Q^{\log}$ over $Y^{\log} \to X^{\log}$, have been chosen so that the generic point of the unique irreducible component $C_W$ of $W$ that maps finitely to $C$ maps into $U_Q^{[1]}$. Then observe that there are precisely two mutually exclusive possibilities:
(c) some nonempty open subscheme of $C$ maps quasi-finitely to $U_Q^{[1]}$;
(d) $C$ maps to a closed point $c$ of $U_Q^{[1]}$.

Moreover, by Proposition 2.5, (i), (ii), (vii), it follows immediately that (b) [as in the statement of assertion (iv)] $\iff$ (d). Thus, it suffices to show that (a) [as in the statement of assertion (iv)] $\iff$ (d). Note that it is immediate that (d) implies (a): Indeed, if we write $c^{log}$ for the log scheme obtained by equipping $c$ with the restriction to $c$ of the log structure of $Q^{log}$, then we obtain an open injection $\pi_1(c^{log}) \to \Delta_{X^{log}}^{C}$ factors through $\{1\} = \text{Ker}(\pi_1(c^{log}) \to G_{E^{log}})$, hence that condition (a) is satisfied. Thus, it remains to show that (a) implies (d), or, equivalently, that condition (c) implies that condition (a) fails to hold. But this follows immediately from the observation that condition (c) implies that $\Delta_C$ surjects onto an open subgroup of the decomposition group $\Delta_E$ in $\Delta_{X^{log}}^{C}$ of some irreducible component $E$ of $Q$. Here, we recall that the following well-known facts: if $E$ is stable, then $\Delta_E$ may be identified with the maximal pro-$l$ quotient of $\pi_1(U_E \times_k \bar{E})$, where we write $U_E \overset{def}{=} E \cap U_Q^{[1]}$, which is infinite; if $E$ is not stable, then $\Delta_E \cong \mathbb{Z}_l(1)$ [cf. Proposition 2.5, (v)], hence infinite. This completes the proof of assertion (iv).

Finally, we consider assertion (v). First, we observe that $\Delta_C$ may be identified with the maximal pro-$l$ quotient of $\pi_1(U_C \times_k \bar{E})$, where we write $U_C \overset{def}{=} C \cap U_Y^{[1]}$ [and assume, for simplicity, that $Y^{log}$ is split]. In particular, an automorphism of $U_C$ is equal to the identity if and only if it induces the identity outer automorphism of $\Delta_C$ [cf., e.g., [MT], Proposition 1.4, and its proof]. Note, moreover, that an automorphism of $Y^{log}$ stabilizes $C$ if and only if it stabilizes the conjugacy class of $\Delta_C$ [cf., e.g., [Mzk13], Proposition 1.2, (i)]. Thus, assertion (v) reduces to the [easily verified] assertion that the morphism $Y_\eta \to X_\eta$ is wildly ramified at $C$ if and only if $\text{Gal}(Y_\eta/X_\eta)$ stabilizes and induces the identity on $C$. This completes the proof of assertion (v).

Remark 2.6.1. Note that the content of Lemma 2.6, (ii), (iii), is reminiscent of the main results of [Tama2] [cf. also Corollary 2.11 below]. By comparison to Tamagawa’s “resolution of nonsingularities”, however, Lemma 2.6, (ii), (iii), assert a somewhat weaker conclusion, albeit for pro-$p$ geometric fundamental groups, as opposed to profinite geometric fundamental groups.

Remark 2.6.2. The argument applied in the final portion of the proof of Lemma 2.6, (ii), is reminiscent of the well-known classical argument that implies the nonexistence of a Frobenius lifting for stable curves over the ring of Witt vectors of a finite field. That is to say, if $k$ is absolutely unramified, and

$$\Phi : X \to X$$

is an $S$-morphism that induces the Frobenius morphism between the respective special fibers, then one obtains a contradiction as follows: Since $\Phi$ induces a morphism $X_\eta \to X_\eta$, it follows immediately that $\Phi$ extends to a morphism of log stable curves $\Phi^{log} : X^{log} \to X^{log}$. Although the derivative

$$d\Phi^{log} : \Phi^*(\omega_{X^{log}/S^{log}}) \to \omega_{X^{log}/S^{log}}$$
is $\equiv 0 \pmod{p}$, one verifies immediately [by an easy local calculation] that $\frac{1}{p}d\Phi^\log$ is necessarily $\not\equiv 0 \pmod{p}$ generically on each irreducible component of $\overline{X}$. Since $\omega_{X^\log/S^\log}$ is a line bundle of degree $2g_X - 2$, and $\Phi$ reduces to the Frobenius morphism between the special fibers, the existence of $d\Phi^\log$ thus implies [by taking degrees] that $p \cdot (2g_X - 2) \leq 2g_X - 2$, i.e., that $(p - 1)(2g_X - 2) \leq 0$, in contradiction to the fact that $g_X \geq 2$.

**Remark 2.6.3.** Note that it follows immediately from either of the conditions (a), (b) of Lemma 2.6, (ii), that $Y$ is not $k$-smooth [i.e., “singular”].

**Corollary 2.7.** (Uniformly Toral Neighborhoods via Cyclic Coverings) Suppose that we are either in the situation of Lemma 2.6, (ii), (a) — which we shall refer to in the following as case (a) — or in the situation of Lemma 2.6, (ii), (b) — which we shall refer to in the following as case (b); suppose further, in case (b), that $\overline{X}$ is not smooth over $k$. Also, we suppose that we have been given a commutative diagram as in Lemma 2.6, (iii). Thus, in either case, we have an irreducible component $C_W$ of $\overline{W}$ [lying over an irreducible component $C$ of $\overline{Y}$] satisfying certain special properties, as in Lemma 2.6, (iii). Let $y \in U_{\overline{W}}(O_k) (\subseteq W_\eta(k) = Y_\eta(k))$ be a point such that the image of $y$ meets $C_W$, and, moreover, $y$ maps to a point $x \in U_Q^1(O_k) (\subseteq Q_\eta(k) = X_\eta(k)); C_Q$ the irreducible component of $\overline{Q}$ that meets the image of $x$;

$$F_y \overset{\text{def}}{=} \overline{W} \setminus (C_W \cap U_{\overline{W}}^1) \subseteq \overline{W}; \quad F_x \overset{\text{def}}{=} \overline{Q} \setminus (C_Q \cap U_Q^1) \subseteq \overline{Q}$$

[regarded as closed subsets of $\overline{W}$, $\overline{Q}$]: $U_y \overset{\text{def}}{=} \overline{W} \setminus F_y \subseteq \overline{W}; \quad U_x \overset{\text{def}}{=} \overline{Q} \setminus F_x \subseteq \overline{Q}$

[so the image of $y$ lies in $U_y$; the image of $x$ lies in $U_x$]. Write $g_y$ for the genus of $Y_\eta; J^Y_\eta \to \eta$ (respectively, $J^X_\eta \to \eta$) for the Jacobian of $Y_\eta$ (respectively, $X_\eta$); $J^Y \to S$ (respectively, $J^X \to S$) for the uniquely determined semi-abelian scheme over $S$ that extends $J^Y_\eta$ (respectively, $J^X_\eta$); $\iota_\eta : Y_\eta \to J^Y_\eta$ for the morphism that sends a $T$-valued point $\zeta$ [where $T$ is a $k$-scheme] of $Y_\eta$, regarded as a divisor on $Y_\eta \times_k T$, to the point of $J^Y_\eta$ determined by the degree zero divisor $\zeta - (y|_T)$. In case (a), let $\sigma \in \text{Gal}(Y_\eta/X_\eta)$ be a generator of $\text{Gal}(Y_\eta/X_\eta); J_\eta \subseteq J^Y_\eta$ the image of the endomorphism $(1 - \sigma) : J^Y \to J^Y$; $J \to S$ the uniquely determined semi-abelian scheme over $S$ that extends $J_\eta$ [which exists, for instance, by [BLR], §7.4, Lemma 2];

$$\kappa : J^Y \to J$$

the [dominant] morphism induced by $(1 - \sigma)$. In case (b), let $J \to S$ be the semi-abelian scheme $J^X \to S; \kappa : J^Y \to J$ the [dominant] morphism induced by the covering $Y_\eta \to X_\eta$. Write

$$\beta_\eta : Y_\eta \times_k \ldots \times_k Y_\eta \to J^Y_\eta \to J_\eta$$
[where the product is of $g_Y$ copies of $Y_{n|}$] for the composite of the morphism given by adding $g_Y$ copies of $t^Y_{n|}$ with the morphism $\kappa_{n|} \overset{\text{def}}{=} \kappa_{n|}$.

(i) Write $\hat{J}$ (respectively, $\hat{G}_m$) for the formal group over $S$ given by completing $J$ (respectively, the multiplicative group $(\mathbb{G}_m)_S$ over $S$) at the origin. Then there exists an exact sequence

$$0 \to \hat{J}' \to \hat{J} \to \hat{J}'' \to 0$$

of [formally smooth] formal groups over $S$, together with an isomorphism $\hat{J}' \sim \hat{G}_m$ of formal groups over $S$. In the following, let us fix such an isomorphism $\hat{J}' \sim \hat{G}_m$ and identify $\hat{J}'$ with its image in $\hat{J}$.

(ii) The morphisms $t^Y_{n|}, \beta_{n|}$ extend uniquely to morphisms

$$t^Y : U_y \to J^Y; \quad \beta : U_y \times_k \ldots \times_k U_y \to J$$

[where the product is of $g_Y$ copies of $U_y$], respectively; the morphism $W \to Q$ restricts to a morphism $U_y \to U_x$.

(iii) Suppose that $k$ is an MLF [or, equivalently, that $\bar{k}$ is finite]. Then there exists a positive integer $M$ — which, in fact, may be taken to be 1 in case (b) — such that the following condition holds: Let $k_* \subseteq \bar{k}$ be a finite extension of $k$ with ring of integers $\mathcal{O}_{k_*}$, maximal ideal $m_{k_*} \subseteq \mathcal{O}_{k_*}$. Write $I_{k_*}$ for the image in $J(\mathcal{O}_{k_*})$ via $\beta$ [cf. (ii)] of the product of $g_Y$ copies of $U_y(\mathcal{O}_{k_*})$. Then $M : I_{k_*}$ lies in the subgroup $\hat{J}(\mathcal{O}_{k_*}) \subseteq J(\mathcal{O}_{k_*})$. Write $\hat{I}_{k_*} \subseteq \hat{J}(\mathcal{O}_{k_*})$ for the subgroup determined by $M : I_{k_*}$:

$$N_{k_*} \subseteq (\mathcal{O}_{k_*}^\times) \otimes \mathbb{Q}_p \sim \hat{G}_m(\mathcal{O}_{k_*}) \otimes \mathbb{Q}_p \sim \hat{J}'(\mathcal{O}_{k_*}) \otimes \mathbb{Q}_p$$

for the image of the intersection

$$\hat{I}_{k_*} \cap \hat{J}'(\mathcal{O}_{k_*}) \quad (\subseteq \hat{J}(\mathcal{O}_{k_*}))$$

in $\hat{J}'(\mathcal{O}_{k_*}) \otimes \mathbb{Q}_p$. Then as $k_* \subseteq \bar{k}$ varies over the finite extensions of $k$, the subgroups $N_{k_*}$ determine a uniformly toral neighborhood of $\text{Gal}(\bar{k}/k)$ [cf. [Mzk15], Definition 3.6, (i), (ii)].

Proof. First, we consider assertion (i). Recall from the well-known theory of Néron models of Jacobians [cf., e.g., [BLR], §9.2, Example 8] that the torus portion of the special fiber of $J^Y$ (respectively, $J^X$) is [in the notation of Proposition 2.5, (vi)] of rank $\text{lp-rk}(Y)$ (respectively, $\text{lp-rk}(X)$). In particular, the torus portion of the special fiber of $J$ is of rank $\text{lp-rk}(Y) - \text{lp-rk}(X)$ in case (a), and of rank $\text{lp-rk}(X)$ in case (b). Thus, in case (a), the fact that the morphism $Y_{n|} \to X_{n|}$ is loopifying implies that the torus portion of the special fiber of $J$ is of positive rank; in case (b), since $X$ is not $k$-smooth, it follows from the loop-ampleness assumption of Lemma 2.6 that $\text{lp-rk}(X) > 0$, hence that the torus portion of the special fiber of $J$ is of positive rank. Now the existence of an exact sequence as in assertion (i) follows from the well-known theory of degeneration of abelian varieties [cf., e.g., [FC], Chapter III, Corollary 7.3]. This completes the proof of assertion (i).
Next, we consider assertion (ii). The existence of the unique extension of \( \iota_\eta^Y \) follows immediately from Proposition 2.5, (x); the existence of the unique extension of \( \beta_\eta \) follows immediately from the existence of this unique extension of \( \iota_\eta^Y \) [together with the existence of the homomorphism of semi-abelian schemes \( \kappa : J^Y \to J \)]. In case (b), the existence of the morphism \( U_y \to U_x \) follows immediately from the definitions. In case (a), if the morphism \( U_y \to U_x \) fails to exist, then there exists a closed point \( w \in W \) that maps to a closed point \( q \in Q \) such that \( w \in U_y \subseteq U_{[1]}^W \), but \( q \not\in U_{[1]}^Q \). On the other hand, since there exists an irreducible Zariski neighborhood of \( w \) in \( W \) [cf. the simple structure of the monoid \( \mathbb{N} \)], it follows from the fact that \( C_W \) maps finitely to \( C_Q \) [in case (a)], that \( W \to Q \) is quasi-finite in a Zariski neighborhood of \( w \). Thus, if we write \( R_w, R_q \) for the respective strict henselizations of \( W, Q \) at \( \{ \text{some choice of } \mathbb{F}_\mathfrak{k} \text{-valued points lifting} \} w, q \), then \( R_w, R_q \) are normal, and the natural inclusion \( R_q \subset R_w \) is finite [cf. Zariski’s main theorem]. In particular, if we write \( K_q \) for the quotient field of \( R_q \), then we have \( R_w \cap K_q = R_q^\times \) [where “\( \times \)” denotes the subgroup of units], so the morphism \( W^\log \to Q^\log \) induces an injection on the groupifications of the stalks of the characteristic sheaves at \( \{ \text{some choice of } \mathbb{F}_\mathfrak{k} \text{-valued points lifting} \} w, q \) in contradiction to the fact that \( w \in U_{[1]}^W \), but \( q \not\in U_{[1]}^Q \). This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, we define the number \( M \) as follows: In case (a), the endomorphism \((1 - \sigma) : J^Y \to J^Y \) admits a factorization \( \theta \circ \kappa \), where \( \theta : J \to J^Y \) is a “closed immersion up to isogeny” [cf., e.g., the situation discussed in [BLR], §7.5, Proposition 3, (b)] — i.e., there exists a morphism \( \theta' : J^Y \to J \) such that \( \theta' \circ \theta : J \to J \) is multiplication by some positive integer \( M_\theta \) on \( J \); then we take \( M \defeq M_\theta \). In case (b), we take \( M \defeq 1 \). In the following, if \( G \) is a group scheme or formal group over \( \mathcal{O}_k \), and \( r \geq 1 \) is an integer, then let us write

\[ G_{m^r}(\mathcal{O}_{k_*}) \subseteq G(\mathcal{O}_{k_*}) \]

for the subgroup of elements that are congruent to the identity modulo \( m^r \mathfrak{k} \cdot \mathcal{O}_{k_*} \).

Next, let us make the following observation:

\[ \text{(1) We have } M \cdot I_{k_*} \subseteq J_m(\mathcal{O}_{k_*}) \subseteq J(\mathcal{O}_{k_*}). \]

Indeed, in case (a), we reason as follows: It suffices to show that \( M \cdot \kappa(\iota^Y(\mathcal{U}_y(\mathcal{O}_{k_*}))) \subseteq J_m(\mathcal{O}_{k_*}) \). Since, moreover, the endomorphism \((1 - \sigma) : J^Y \to J^Y \) admits a factorization \( \theta \circ \kappa \), where, for some morphism \( \theta' : J^Y \to J \), \( \theta' \circ \theta \) is equal to multiplication by \( M \), it suffices to show that

\[ (1 - \sigma)(\iota^Y(\mathcal{U}_y(\mathcal{O}_{k_*}))) \subseteq J_m(\mathcal{O}_{k_*}) \]

[since applying \( \theta' \) to this inclusion yields the desired inclusion \( M \cdot \kappa(\iota^Y(\mathcal{U}_y(\mathcal{O}_{k_*}))) \subseteq J_m(\mathcal{O}_{k_*}) \)]. On the other hand, since \( Y_\eta \to X_\eta \) is wildly ramified at \( C_W \), it follows that \( \sigma \) acts as the identity on \( C_W \), hence that the composite morphism \((1 - \sigma) \circ \iota^Y : U_y \to J^Y \) induces a morphism on special fibers \( U_y \times_{\mathcal{O}_k} \mathbb{k} \to J^Y \times_{\mathcal{O}_k} \mathbb{k} \) that is constant [with image lying in the image of the identity section of \( J^Y \times_{\mathcal{O}_k} \mathbb{k} \)]. But this implies that \((1 - \sigma)(\iota^Y(\mathcal{U}_y(\mathcal{O}_{k_*}))) \subseteq J_m^Y(\mathcal{O}_{k_*}) \). This completes the proof of observation (1) in case (a). In a similar [but slightly simpler] vein, in case (b), it suffices to observe
that the morphism $\kappa \circ \iota^y : U_y \to J$ admits a factorization $U_y \to U_x \to J^X$, where $U_y \to U_x$ is the morphism of assertion (ii), and $U_x \to J^X$ is the “analogue of $\iota^y$” for the point $x$ of $X_\eta(k)$ [cf. Proposition 2.5, (x)]. That is to say, the fact that $C_W$ maps to $x \in U^1[1](O_k)$ implies [by applying this factorization] that the morphism $\kappa \circ \iota^y : U_y \to J$ induces a morphism on special fibers $U_y \times_{O_k} \hat{k} \to J \times_{O_k} \hat{k}$ that is constant [with image lying in the image of the identity section of $J \times_{O_k} \hat{k}$]. This completes the proof of observation (1) in case (b).

Next, let us make the following observation:

(2) There exists a positive integer $r$ which is independent of $k_*$ such that $M \cdot I_{k_*} \supset J_m'(O_{k_*})$.

Indeed, since $\kappa$ is clearly dominant, it follows immediately that the composite of the morphism $\beta : U_y \times_k \ldots \times_k U_y \to J$ with the morphism $J \to J$ given by multiplication by $M$ is dominant, hence, in particular, generically smooth [since $k$ is of characteristic zero]. Thus, [since $M \cdot I_{k_*}$ is a group] observation (2) follows immediately from the “positive slope version of Hensel’s lemma” given in Lemma 2.1. Now since $\hat{J}_m'(O_{k_*}) = J_m(O_{k_*}) \cap \hat{J}'(O_{k_*})$ [cf. assertion (i)], we conclude that

$$\hat{J}_m'(O_{k_*}) \subseteq \hat{I}_{k_*} \cap \hat{J}'(O_{k_*}) \subseteq \hat{J}_m'(O_{k_*})$$

[cf. the inclusions of observations (1), (2)], so assertion (iii) follows essentially formally [cf. [Mzk15], Definition 3.6, (i), (ii)]. This completes the proof of assertion (iii). $\Box$

**Remark 2.7.1.** Note that in the situation of case (b), if $f$ is a rational function on $X$ whose value at $x$ lies in $O_k^X$, then the values $\in O_{k_*}^X$ of $f$ at points of $U_y(O_{k_*})$ [cf. the notation of Corollary 2.7, (iii)] determine a uniformly toral neighborhood. It was precisely this observation that motivated the author to develop the theory of the present §2.

**Definition 2.8.** Let $k$ be a field of characteristic zero, $\overline{k}$ an algebraic closure of $k$.

(i) Suppose that $\overline{k}$ is equipped with a topology. Let $X$ be a smooth, geometrically connected curve over $k$. Then we shall say that a subset $\Xi \subseteq X(\overline{k})$ is Galois-dense if, for every finite extension field $k' \subseteq \overline{k}$ of $k$, $\Xi \cap X(k')$ is dense in $X(k')$ [i.e., relative to the topology induced on $X(k')$ by $\overline{k}$].

(ii) We shall refer to as a pro-curve over $k$ [cf. the terminology of [Mzk3]] any $k$-scheme $U$ that may be written as a projective limit of smooth, geometrically connected curves over $k$ in which the transition morphisms are birational. Let $U$ be a pro-curve over $k$. Then it makes sense to speak of the function field $k(U)$ of $U$. Write $X$ for the smooth, proper, geometrically connected curve over $k$ determined by the function field $k(U)$. Then one verifies immediately that $U$ is completely determined up to unique isomorphism by $k(U)$, together with some $\text{Gal}(\overline{k}/k)$-stable subset $\Xi \subseteq X(\overline{k})$ — i.e., roughly speaking, “$U$ is obtained by removing $\Xi$ from $X$.”
If \( \bar{k} \) is equipped with a topology, then we shall say that \( U \) is co-Galois-dense if the corresponding \( \text{Gal}(\bar{k}/k) \)-stable subset \( \Xi \subseteq X(\bar{k}) \) is Galois-dense.

**Remark 2.8.1.** Suppose, in the notation of Definition 2.8, that \( k \) is an MLF [and that \( \bar{k} \) is equipped with the \( p \)-adic topology]. Let \( X \) be a smooth, proper, geometrically connected curve over \( k \), with function field \( k(X) \). Then \( \text{Spec}(k(X)) \) is a co-Galois-dense pro-curve over \( k \). Suppose that \( k = k_0 \times_{k_0} k \), where \( k_0 \subseteq k \) is a number field, and \( k_0 \) is a smooth, proper, geometrically connected curve over \( k_0 \), with function field \( k_0(X_0) \). Then \( \text{Spec}(k_0(X_0) \otimes_{k_0} k) \) [where we note that the ring \( k_0(X_0) \otimes_{k_0} k \) is not a field!] also forms an example of a co-Galois-dense pro-curve over \( k \).

**Remark 2.8.2.** Let \( k \) be a field of characteristic zero.

(i) Let us say that a pro-curve \( U \) over \( k \) is of unit type if there exists a connected finite étale covering of \( U \) that admits a nonconstant unit. Thus, any hyperbolic curve \( U \) over \( k \) for which there exists a connected finite étale covering \( V \to U \) such that \( V \) admits a dominant \( k \)-morphism \( V \to P \), where \( P \) is the projective line minus three points over \( k \), is of unit type. That is to say, the hyperbolic curves considered in \([\text{Mzk15}], \text{Remark 3.8.1}\) — i.e., the sort of hyperbolic curves that motivated the author to prove \([\text{Mzk15}], \text{Corollary 3.8, (g)}\) — are necessarily of unit type.

(ii) Suppose that \( k \) is an MLF of residue characteristic \( p \), whose ring of integers we denote by \( \mathcal{O}_k \). Let \( n \geq 1 \) be an integer; \( \eta \in \mathcal{O}_k/(p^n) \). Then observe that the set \( E \) of elements of \( \mathcal{O}_k \) that are \( \equiv \eta \pmod{p^n} \) is of uncountable cardinality. In particular, it follows that the subfield of \( k \) generated over \( \mathbb{Q} \) by \( E \) is of uncountable — hence, in particular, infinite — transcendence degree over \( \mathbb{Q} \).

(iii) Let \( k \) be as in (ii); \( X_0 \) a proper hyperbolic curve over \( k_0 \), where \( k_0 \subseteq k \) is a finitely generated extension of \( \mathbb{Q} \); \( k_1 \subseteq k \) a finitely generated extension of \( k_0 \); \( r \geq 1 \) an integer. Then recall from \([\text{MT}], \text{Corollary 5.7}\), that any curve \( U_1 \) obtained by removing from \( X_1 = X_0 \times_{k_0} k_1 \) a set of \( r \) “generic points” \( \in X_1(k_1) = X_0(k_1) \) — i.e., \( r \) points which determine a dominant morphism from \( \text{Spec}(k_1) \) to the product of \( r \) copies of \( X_0 \) over \( k_0 \) — is not of unit-type. In particular, it follows immediately from (ii) that:

There exist co-Galois-dense pro-curves \( U \) over \( k \) which are not of unit type.

For more on the significance of this fact, we refer to Remark 2.10.1 below.

**Remark 2.8.3.** Suppose that we are in the situation of Definition 2.8, (i). Let \( Y \to X \) be a connected finite étale covering. Then one verifies immediately, by applying “Krasner’s Lemma” [cf. [Kobl], pp. 69-70], that the inverse image \( \Xi|_Y \subseteq Y(\bar{k}) \) of a Galois-dense subset \( \Xi \subseteq X(\bar{k}) \) is itself Galois-dense.
Corollary 2.9. (Point-theoreticity Implies Geometricity) For \( i = 1, 2 \), let \( k_i \) be an MLF of residue characteristic \( p_i \); \( \overline{k}_i \) an algebraic closure of \( k_i \); \( \Sigma_i \) a set of primes of cardinality \( \geq 2 \) such that \( p_i \in \Sigma_i \); \( X_i \) a hyperbolic curve over \( k_i \); \( \overline{X}_i \) the smooth, proper, geometrically connected curve over \( k_i \) determined by the function field of \( X_i \); \( \Xi_i \subseteq \overline{X}_i(\overline{k}_i) \) a Galois-dense subset. Write \( \pi_1(\cdot) \) for the \( \acute{e}tale fundamental group of a connected scheme, relative to an appropriate choice of basepoint; \( \Delta_{X_i} \) for the maximal pro-\( \Sigma_i \) quotient of \( \pi_1(X_i \times_{k_i} \overline{k}_i) \); \( \Pi_{X_i} \) for the quotient of \( \pi_1(X_i) \) by the kernel of the natural surjection \( \pi_1(X_i \times_{k_i} \overline{k}_i) \to \Delta_{X_i} \). Let

\[
\alpha : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}
\]

be an isomorphism of profinite groups such that a closed subgroup of \( \Pi_{X_1} \) is a decomposition group of a point in \( \Xi_1 \) if and only if it corresponds, relative to \( \alpha \), to a decomposition group in \( \Pi_{X_2} \) of a point in \( \Xi_2 \). Then \( p_1 = p_2, \Sigma_1 = \Sigma_2, \) and \( \alpha \) is geometric, i.e., arises from a unique isomorphism of schemes \( X_1 \xrightarrow{\sim} X_2 \).

Proof. First, we observe that by [Mzk15], Theorem 2.14, (i), \( \alpha \) induces isomorphisms \( \alpha_\Delta : \Delta_{X_1} \xrightarrow{\sim} \Delta_{X_2} \), \( \alpha_G : G_1 \xrightarrow{\sim} G_2 \) [where, for \( i = 1, 2 \), we write \( G_i \overset{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i) \); \( p_1 = p_2 \) [so we shall write \( p \overset{\text{def}}{=} p_1 = p_2 \)]; \( \Sigma_1 = \Sigma_2 \) [so we shall write \( \Sigma \overset{\text{def}}{=} \Sigma_1 = \Sigma_2 \)]. Also, by [the portion concerning semi-graphs of] [Mzk15], Theorem 2.14, (i), it follows that \( \alpha \) preserves the decomposition groups of cusps. Thus, by passing to corresponding open subgroups of \( \Pi_{X_i} \) [cf. Remark 2.8.3] and forming the quotient by the decomposition groups of cusps in \( \Delta_{X_i} \), we may assume, without loss of generality, that the \( X_i \) are proper. Next, let \( l \in \Sigma \) be a prime \( \neq p \). Then let us recall that by the well-known stable reduction criterion [cf., e.g., [BLR], §7.4, Theorem 6], \( X_i \) has stable reduction over \( \mathcal{O}_{k_i} \) if and only if, for some \( \mathbb{Z}_l \)-submodule \( M \subseteq \Delta_{X_i}^{ab} \otimes \mathbb{Z}_l \) of the maximal pro-\( l \) abelian quotient \( \Delta_{X_i}^{ab} \otimes \mathbb{Z}_l \) of \( \Delta_{X_i} \), the inertia subgroup of \( G_i \) acts trivially on \( M, \Delta_{X_i}^{ab} \otimes \mathbb{Z}_l/M \). Thus, we may assume, without loss of generality, that, for \( i = 1, 2 \), \( X_i \) admits a log stable model \( X_i^{\log} \) over \( \text{Spec}(\mathcal{O}_{k_i})^{\log} \) [where the last log structure is the log structure determined by the closed point]. Since, by [the portion concerning semi-graphs of] [Mzk15], Theorem 2.14, (i), it follows that \( \alpha \) induces an isomorphism between the dual graphs of the special fibers \( \mathcal{X}_i \) of the \( X_i \), hence that \( \mathcal{X}_1 \) is loop-ample (respectively, singular) if and only if \( \mathcal{X}_2 \) is. Thus, by replacing \( X_i \) by a finite étale covering of \( X_i \) arising from an open subgroup of \( \Pi_{X_i} \), we may assume that \( \mathcal{X}_i \) is loop-ample [cf. §0] and singular [cf. Remark 2.6.3]. Now, to complete the proof of Corollary 2.9, it follows from [Mzk15], Corollary 3.8, (e), that it suffices to show that \( \alpha_G \) is uniformly toral.

Next, let us suppose that, for \( i = 1, 2 \), we are given a finite étale covering \( Y_i \to X_i \) of hyperbolic curves over \( k_i \) with stable reduction over \( \mathcal{O}_{k_i} \) arising from open subgroups of \( \Pi_{X_i} \) that correspond via \( \alpha \) and are such that \( \text{Gal}(Y_i/X_i) \) is cyclic of order a positive power of \( p \) [cf. Lemma 2.6, (ii)]. Let us write \( Y_i^{\log} \) for the log stable model of \( Y_i \) over \( (\mathcal{O}_{k_i})^{\log} \), \( \mathcal{Y}_i \), for the special fiber of \( Y_i \). By possibly replacing the \( k_i \) by the corresponding [relative to \( \alpha \)] finite extensions of \( k_i \), we may assume that the \( \mathcal{Y}_i \) are split [cf. Proposition 2.5, (vi)]. Note that by [the portion concerning semi-graphs of] [Mzk15], Theorem 2.14, (i), it follows that \( Y_1 \to X_1 \) is loopifying if and only if \( Y_2 \to X_2 \) is. Thus, by Lemma 2.6, (iv), (v) [cf. also Proposition 2.5, (i), (ii)] (respectively, Lemma 2.6, (iv)), it follows that \( Y_1 \to X_1 \) satisfies condition (a).
(respectively, (b)) of Lemma 2.6, (ii), if and if $Y_2 \rightarrow X_2$ does. For $i = 1, 2$, let $C_i$ be an irreducible component of $Y_i$ as in Corollary 2.7 [i.e., “C’’]. By Lemma 2.6, (iv) [cf. also the portion concerning semi-graphs of [Mzk15], Theorem 2.14, (i)], we may assume that the $C_i$ are compatible with $\alpha$. Thus, to complete the proof of Corollary 2.9, it suffices to construct \textit{uniformly toral neighborhoods} [cf. Corollary 2.7, (iii)] that are compatible with $\alpha$.

Write $\Pi_{Y_i} \subseteq \Pi_{X_i}$, $\Delta_{Y_i} \subseteq \Delta_{X_i}$ for the open subgroups determined by $Y_i; T^Y_i,$ $T^X_i$ for the maximal pro-$p$ abelian quotients of $\Delta_{Y_i}$, $\Delta_{X_i}$. If we are in case (a) [cf. Corollary 2.7], then we choose generators $\sigma_i \in \text{Gal}(Y_i/X_i) \cong \Delta_{X_i}/\Delta_{Y_i}$ that correspond via $\alpha$ and write $T_i$ for the intersection with $T^Y_i$ of the image of the endomorphism $(1 - \sigma_i)$ of $T^Y_i \otimes \mathbb{Q}_p$, and

$$\kappa_{T_i} : T^Y_i \rightarrow T_i$$

for the morphism induced by $(1 - \sigma_i)$. If we are in case (b) [cf. Corollary 2.7], then we set $T_i \overset{\text{def}}{=} T^X_i$; write $\kappa_{T_i} : T^Y_i \rightarrow T_i$ for the morphism induced by $Y_i \rightarrow X_i$ [i.e., by the inclusion $\Delta_{Y_i} \hookrightarrow \Delta_{X_i}$]. Thus, the formal group “$J$” of Corollary 2.7, (i), corresponds to a $G_i$-submodule $T'_i \subseteq T_i$ such that $T'_i \cong \mathbb{Z}_p(1)$; by [Tate], Theorem 4 [cf. also [Mzk5], Proposition 1.2.1, (vi)], we may assume that these submodules $T'_i$ are compatible with $\alpha$.

Next, let us write $\Delta_{Y_i} \rightarrow \Delta^{(l)}_{Y_i}$ for the \textit{maximal pro-$l$ quotient} of $\Delta_{Y_i}$; $\Delta^{(l)}_{C_i} \subseteq \Delta^{(l)}_{Y_i}$ for the \textit{decomposition group of} $C_i$ in $\Delta^{(l)}_{Y_i}$ [well-defined up to conjugation]. Thus, $\Delta^{(l)}_{C_i}$ may be identified with the \textit{maximal pro-$l$ quotient} of $\pi_1(U_{C_i} \times \kbar, \ellbar)$, where $U_{C_i} \overset{\text{def}}{=} C_i \cap U_{[l]}$, [cf. the proof of Lemma 2.6, (iv)], $\kbar$ is the residue field of $k_i$, and $\ellbar$ is the algebraic closure of $\kbar$ induced by $\kbar$. Since $\Delta^{(l)}_{C_i}$ is \textit{slim} [cf., e.g., [MT], Proposition 1.4], and the outer action of $G_i$ on $\Delta^{(l)}_{C_i}$ clearly \textit{factors} through the quotient $G_i \rightarrow \text{Gal}(\ellbar/\kbar)$, the resulting outer action of $\text{Gal}(\ellbar/\kbar)$ on $\Delta^{(l)}_{C_i}$ determines, in a fashion that is compatible with $\alpha$, an \textit{extension of profinite groups} $1 \rightarrow \Delta^{(l)}_{C_i} \rightarrow \Pi_{C_i}^{(l)} \rightarrow \text{Gal}(\ellbar/\kbar) \rightarrow 1$. In a similar vein, the outer action of $G_i$ on $\Delta^{(l)}_{Y_i}$ factors through the \textit{maximal tamely ramified quotient} $G_i \rightarrow G_{\ellbar}^{\text{log}}$, hence [since $\Delta^{(l)}_{Y_i}$ is slim — cf., e.g., [MT], Proposition 1.4] determines, in a fashion that is \textit{compatible} with $\alpha$, a \textit{morphism of extensions of profinite groups}

$$1 \rightarrow \Delta^{(l)}_{C_i} \rightarrow \Pi_{C_i}^{(l)} \times \text{Gal}(\ellbar/\kbar) G_{\ellbar}^{\text{log}} \rightarrow G_{\ellbar}^{\text{log}} \rightarrow 1$$

— in which the vertical morphisms are \textit{inclusions}, and the vertical morphism on the right is the \textit{identity morphism}; moreover, the images of the first two vertical morphisms are equal to the respective \textit{decomposition groups} of $C_i$ [well-defined up to conjugation].

Next, let us observe that, by our assumption concerning \textit{decomposition groups of points} $\in \Xi_i$ in the statement of Corollary 2.9, it follows that $\alpha$ determines a \textit{bijection} $Y_i(k_1, \Xi_i) \xrightarrow{\sim} Y_2(k_2, \Xi_2)$, where we write $Y_i(k_i, \Xi_i) \subseteq Y_i(k_i)$ for the
subset of points lying over points ∈ Ξ. Here, we recall that a point ∈ Yi(ki) is uniquely determined by the conjugacy class of its decomposition group in ΠYi — cf., e.g., [Mzk3], Theorem C.] Now let us choose corresponding [i.e., via this bijection] points yi ∈ Yi(ki, Ξi) as our points “y” in the construction of the uniformly toral neighborhoods of Corollary 2.7, (iii). Here, we observe that [by our Galois-density assumption] we may assume that yi is compatible with Ci, in the sense that the image in the quotient ΠYi → ΠYi(l) of the decomposition group of yi in ΠYi determines a subgroup of ΠYi(l) × Gal(Ξi/k), G(Ξi) log which contains the kernel of the surjection ΠYi(l) × Gal(Ξi/k), G(Ξi) log → ΠYi. Note that this condition that yi be “compatible with Ci” is manifestly “group-theoretic”, i.e., compatible with α [cf. the portion concerning semi-graphs of [Mzk15], Theorem 2.14, (i); [Mzk5], Proposition 1.2.1, (ii)]. Moreover, let us recall from the theory of §1 that this condition that yi be “compatible with Ci” is equivalent to the condition that the closure in Yi of yi intersect UCi, [cf. Proposition 1.3, (x)].

Thus, by choosing any corresponding [i.e., via the bijection induced by α] points yi′ ∈ Yi(ki, Ξi) that are compatible with the Ci, we may compute [directly from the decomposition groups of the yi, yi′ in ΠYi] the “difference” of yi, yi′ in H1(Gki, Ti), as well as the image

δyiy′i ∈ H1(Gki, Ti)

of this difference via κTi. On the other hand, let us recall the Kummer isomorphisms

H1(Gki, Ti) ≅ Oki × Zp; H1(Gki, Ti) ≅ Ji(ki) ⊗ Zp

[where Ji is the “J” of Corollary 2.7, (iii) — cf., e.g., the “well-known general nonsense” reviewed in the proof of [Mzk14], Proposition 2.2, (i), for more details]. By applying these isomorphisms, we conclude that the subset of

H1(Gki, Ti′ ⊗ Qp) ≅ Oki × Qp

obtained by taking the image of the intersection in H1(Gki, Ti) with the image of H1(Gki, Ti′) of the closure [cf. our Galois-density assumption, together with the evident p-adic continuity of the assignment yi′l → δyi′,y′l] of the set obtained by adding gYi, [where gYi is the genus of Yi] elements of the form Mi · δyi′,y′l [where Mi is the “M” of Corollary 2.7, (iii)] yields — from the point of view of Corollary 2.7, (iii) — a subset that coincides with the subset “Nk•” [where “k•” is taken to be ki] constructed in Corollary 2.7, (iii). Thus, by allowing the “ki” to vary over arbitrary corresponding finite extensions ⊆ ki of ki, we obtain uniform toral neighborhoods of the Gi that are compatible with α. But this implies that αC is uniformly toral, hence completes the proof of Corollary 2.9.

Remark 2.9.1. Corollary 2.9 may be regarded as a generalization of the [MLF portion of] [Mzk14], Corollary 2.2, to the case of pro-Σ [where Σ is of cardinality ≥ 2 and contains the residue characteristic — that is to say, Σ is not necessarily the set of all primes] geometric fundamental groups of not necessarily affine hyperbolic curves. From this point of view, it is interesting to note that in the theory of the
present §2, Lemma 2.6, which, as is discussed in Remark 2.6.2, is reminiscent of a classical argument on the “nonexistence of Frobenius liftings”, takes the place of Lemma 4.7 of [Tama1], which is applied in [Mzk14], Corollary 2.1, to reconstruct the additive structure of the fields involved. In this context, we observe that the appearance of “Frobenius endomorphisms” in Remark 2.6.2 is interesting in light of the discussion of “hidden endomorphisms” in the Introduction, in which “Frobenius endomorphisms” also appear.

Remark 2.9.2. One way to think of Corollary 2.9 is as the statement that:

The “Section Conjecture” over MLF’s implies the “absolute isomorphism version of the Grothendieck Conjecture” over MLF’s.

Here, we recall that in the notation of Corollary 2.9, the “Section Conjecture” over MLF’s amounts to the assertion that every closed subgroup of $\Pi_X$ that maps isomorphically to an open subgroup of $\text{Gal}(\overline{k_i}/k_i)$ is the decomposition group associated to a closed point of $X_i$. In fact, in order to apply Corollary 2.9, a “relatively weak version of the Section Conjecture” is sufficient — cf. the point of view of [Mzk8].

Remark 2.9.3. The issue of verifying the “point-theoreticity hypothesis” of Corollary 2.9 [i.e., the hypothesis concerning the preservation of decomposition groups of closed points] may be thought of as consisting of two steps, as follows:

(a) First, one must show the $J$-geometricity [cf. [Mzk3], Definition 4.3] of the image via $\alpha$ of a decomposition group $D_\xi \subseteq \Pi_{X_1}$ of a closed point $\xi \in X_1(k_1)$. Once one shows this $J$-geometricity for all finite étale coverings of $X_2$ arising from open subgroups of $\Pi_{X_2}$, one concludes [cf. the arguments of [Mzk3], §7, §8] that there exist rational points of a certain tower of coverings of $X_2$ determined by $\alpha(D_\xi) \subseteq \Pi_{X_2}$ over tame extensions of $k_2$.

(b) Finally, one must show that these rational points over tame extensions of $k_2$ necessarily converge — an issue that the author typically refers to by the term “tame convergence”.

At the time of writing, it is not clear to the author how to complete either of these two steps. On the other hand, in the “birational” — or, more generally, the “co-Galois-dense” — case, one has Corollary 2.10 [given below].

Remark 2.9.4.

(i) By contrast to the quite substantial difficulty [discussed in Remark 2.9.3] of verifying “point-theoreticity” for hyperbolic curves over MLF’s, in the case of hyperbolic curves over finite fields, there is a [relatively simple] “group-theoretic” algorithm for reconstructing the decomposition groups of closed points, which follows essentially from the theory of [Tama1] [cf. [Tama1], Corollary 2.10, Proposition 3.8]. Such an algorithm is discussed in [Mzk14], Remark 10, although the argument given there is somewhat sketchy and a bit misleading. A more detailed presentation may be found in [SdTm], Corollary 1.25.
(ii) A more concise version of this argument, along the lines of [Mzk14], Remark 10, may be given as follows: Let \( X \) be a proper [for simplicity] hyperbolic curve over a finite field \( k \), with algebraic closure \( \bar{k} \); \( \Sigma \) a set of prime numbers that contains a prime that \( \equiv 3 \pmod{4} \); \( \pi_1(X \times_{\bar{k}} \bar{k}) \to \Delta_X \) the maximal pro-\( \Sigma \) quotient of the étale fundamental group \( \pi_1(X \times_{\bar{k}} \bar{k}) \) of \( X \times_{\bar{k}} \bar{k} \); \( \pi_1(X) \to \Pi_X \) the corresponding quotient of the étale fundamental group \( \pi_1(X) \) of \( X \); \( \Pi_X \to G_k \overset{\text{def}}{=} \text{Gal}(\bar{k}/k) \) the natural quotient. Then it suffices to give a “group-theoretic” characterization of the quasi-sections \( D \subseteq \Pi_X \) [i.e., closed subgroups that map isomorphically onto an open subgroup of \( G_k \)] which are decomposition groups of closed points of \( X \). Write

\[
\tilde{X} \to X
\]

for the pro-finite étale covering corresponding to \( \Pi_X \). If \( E \subseteq \Pi_X \) is a closed subgroup whose image in \( G_k \) is open, then let us write \( k_E \) for the finite extension field of \( k \) determined by this image. If \( J \subseteq \Pi_X \) is an open subgroup, then let us write \( X_J \to X \) for the covering determined by \( J \) and \( J_{\Delta} \overset{\text{def}}{=} J \cap \Delta_X \). If \( J \subseteq \Pi_X \) is an open subgroup such that \( J_{\Delta} \) is a characteristic subgroup of \( \Delta_X \), then we shall say that \( J \) is geometrically characteristic. Now let \( J \subseteq \Pi_X \) be a geometrically characteristic open subgroup. Let us refer to as a descent-group for \( J \) any open subgroup \( H \subseteq \Pi_X \) such that \( J \subseteq H \), \( J_{\Delta} = H_{\Delta} \). Thus, a descent-group \( H \) for \( J \) may be thought of as an intermediate covering \( X_J \to X_H \to X \) such that \( X_H \times_{k_H} k_J \cong X_J \). Write

\[
X_J(k_J)^{\text{fld-def}} \subseteq X_J(k_J)
\]

for the subset of \( k_J \)-valued points of \( X_J \) that do not arise from points \( \in X_H(k_H) \) for any descent-group \( H \neq J \) for \( J \) — i.e., the \( k_J \)-valued points whose field of definition is \( k_J \) with respect to all possible “descended forms” of \( X_J \). [That is to say, this definition of “fld-def” differs slightly from the definition of “fld-def” in [Mzk14], Remark 10.] Thus, if \( \tilde{x} \) is a closed point of \( \tilde{X} \) that maps to \( x \in X_J(k_J) \), and we write \( D_{\tilde{x}} \subseteq \Pi_X \) for the stabilizer in \( \Pi_X \) [i.e., “decomposition group”] of \( \tilde{x} \), then it is a tautology that \( x \) maps to a point \( \in X_{H_x}(k_{H_x}) \) for \( H_x \overset{\text{def}}{=} D_{\tilde{x}}^{-1} J_{\Delta} (\supset J) \) [so \( H_x \) forms a descent-group for \( J \)]; in particular, it follows immediately that:

\[
x \in X_J(k_J)^{\text{fld-def}} \iff D_{\tilde{x}} \subseteq J \iff H_x = J.
\]

Now it follows immediately from this characterization of “fld-def” that if \( J_1 \subseteq J_2 \subseteq \Pi_X \) are geometrically characteristic open subgroups such that \( k_{J_1} = k_{J_2} \), then the natural map \( X_{J_1}(k_{J_1}) \to X_{J_2}(k_{J_2}) \) induces a map \( X_{J_1}(k_{J_1})^{\text{fld-def}} \to X_{J_2}(k_{J_2})^{\text{fld-def}} \). Moreover, these considerations allow one to conclude [cf. the theory of [Tama1]] that:

A quasi-section \( D \subseteq \Pi_X \) is a decomposition group of a closed point of \( X \) if and only if, for every geometrically characteristic open subgroup \( J \subseteq \Pi_X \) such that \( D \cdot J_{\Delta} = J \), it holds that \( X_J(k_J)^{\text{fld-def}} \neq \emptyset \).

Thus, to render this characterization of decomposition groups “group-theoretic”, it suffices to give a “group-theoretic” criterion for the condition that \( X_J(k_J)^{\text{fld-def}} \neq \emptyset \). In [Tama1], the Lefschetz trace formula is applied to compute the cardinality of
$X_J(k_J)$. On the other hand, if we use the notation "$| - |$" to denote the cardinality of a finite set, then one verifies immediately that

$$|X_J(k_J)| = \sum_H |X_H(k_H)^{\text{fld-def}}|$$

— where $H \supseteq J$ ranges over the descent-groups for $J$. In particular, by applying induction on $[\Pi_X : J]$, one concludes immediately from the above formula that $|X_J(k_J)^{\text{fld-def}}|$ may be computed from the $|X_H(k_H)|$, as $H$ ranges over the descent-groups for $J$ [while $|X_H(k_H)|$ may be computed, as in [Tama1], from the Lefschetz trace formula]. This yields the desired "group-theoretic" characterization of the decomposition groups of $\Pi_X$.

**Corollary 2.10. (Geometricity of Absolute Isomorphisms for Co-Galois-dense Pro-curves)** For $i = 1, 2$, let $k_i$ be an MLF of residue characteristic $p_i$; $\overline{k}_i$ an algebraic closure of $k_i$; $\Sigma_i$ a set of primes of cardinality $\geq 2$ such that $p_i \in \Sigma_i$; $U_i$ a co-Galois-dense pro-curve over $k_i$. Write "$\pi_1( - )$" for the étale fundamental group of a connected scheme, relative to an appropriate choice of basepoint; $\Delta_U$, for the maximal pro-$\Sigma_i$ quotient of $\pi_1(U_i \times_k \overline{k}_i)$; $\Pi_U$, for the quotient of $\pi_1(U_i)$ by the kernel of the natural surjection $\pi_1(U_i \times_k \overline{k}_i) \twoheadrightarrow \Delta_U$. Let

$$\alpha : \Pi_{U_1} \cong \Pi_{U_2}$$

be an isomorphism of profinite groups. Then $\Delta_{U_i}$, $\Pi_{U_i}$ are slim; $p_1 = p_2$; $\Sigma_1 = \Sigma_2$; $\alpha$ is geometric, i.e., arises from a unique isomorphism of schemes $U_1 \cong U_2$.

**Proof.** First, we observe that $\Sigma_i$ may be characterized as the set of primes $l$ such that $\Pi_{U_i}$ has $l$-cohomological dimension $> 2$. Thus, $\Sigma_1 = \Sigma_2$. Let us write $\Sigma \overset{\text{def}}{=} \Sigma_1 = \Sigma_2$; $X_i$ for the smooth, proper, geometrically connected curve over $k_i$ determined by $U_i$; $\Delta_{X_i}$ for the maximal pro-$\Sigma$ quotient of $\pi_1(X_i \times_k \overline{k}_i)$; $\Pi_{X_i}$ for the quotient of $\pi_1(X_i)$ by $\text{Ker}(\pi_1(X_i \times_k \overline{k}_i) \twoheadrightarrow \Delta_{X_i})$. Thus, $U_i$ determines some Galois-dense subset $\Xi \subseteq X_i(\overline{k}_i)$. Since $\Delta_{U_i}$, $\Pi_{U_i}$ may be written as inverse limits of surjections of slim profinite groups [cf., e.g., [Mzk15], Proposition 2.3], it follows that $\Delta_{U_i}$, $\Pi_{U_i}$ are slim. Since the kernel of the natural surjection $\Pi_{U_i} \twoheadrightarrow \Pi_{X_i}$ is topologically generated by the inertia groups of points $\in \Xi_i$, and these inertia groups are isomorphic to $\hat{\mathbb{Z}}^\Sigma(1)$ [where the "$(1)$" denotes a Tate twist; we write $\hat{\mathbb{Z}}^\Sigma$ for the maximal pro-$\Sigma$ quotient of $\hat{\mathbb{Z}}$] in a fashion that is compatible with the conjugation action of some open subgroup of $G_i \overset{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i)$, it follows that we obtain an isomorphism

$$\Pi_{U_i}^{\text{ab-t}} \cong \Pi_{X_i}^{\text{ab-t}}$$

on torsion-free abelianizations. In particular, it follows [in light of our assumptions on $\Sigma_i$] that, in the notation of [Mzk15], Theorem 2.6,

$$\sup_{p', p'' \in \Sigma} \{\delta^{1}_{p'}(\Pi_{U_i}) - \delta^{1}_{p''}(\Pi_{U_i})\} = \sup_{p', p'' \in \Sigma} \{\delta^{1}_{p'}(\Pi_{X_i}) - \delta^{1}_{p''}(\Pi_{X_i})\} = \sup_{p', p'' \in \Sigma} \{\delta^{1}_{p'}(G_i) - \delta^{1}_{p''}(G_i)\} = [k_i : \mathbb{Q}_{p_i}]$$
from the observations of the preceding paragraph that the natural surjection \( \Delta \subseteq \Pi \) induces an isomorphism \( \alpha \colon G_1 \cong G_2 \). Moreover, by [Mzk5], Proposition 1.2.1, (i), (vi), the existence of \( \alpha \) implies that \( p_1 = p_2 \) [so we set \( p \equiv p_1 = p_2 \)], and \( \alpha \) is compatible with the respective cyclotomic characters.

Let \( M \) be a profinite abelian group equipped with a continuous \( H \)-action, for \( H \subseteq G_i \) [where \( i \in \{1, 2\} \)] an open subgroup. Then let us write \( M \to \mathcal{Q}'(M) \) for the quotient of \( M \) by the closed subgroup generated by the quasi-toral subgroups of \( M \) [i.e., closed subgroups isomorphic as \( J \)-modules, for \( J \subseteq H \) an open subgroup, to \( \mathbb{Z}_l(1) \) for some prime \( l \)]; \( M \to \mathcal{Q}'(M) \to \mathcal{Q}(M) \) for the maximal torsion-free quotient of \( \mathcal{Q}'(M) \). Also, if \( M \) is topologically finitely generated, then let us write \( M \to \mathcal{T}(M) \) for the maximal torsion-free quasi-trivial quotient [i.e., maximal torsion-free quotient on which \( H \) acts through a finite quotient]. Then one verifies immediately that the assignments \( M \mapsto \mathcal{Q}(M) \), \( M \mapsto \mathcal{T}(M) \) are functorial. Moreover, it follows from the observations of the preceding paragraph that the natural surjection \( \Delta_{U_i} \to \Delta_{X_i} \) determines a surjection on torsion-free abelianizations \( \Delta_{U_i}^{ab-t} \to \Delta_{X_i}^{ab-t} \) that induces an isomorphism \( \mathcal{Q}(\Delta_{U_i}^{ab-t}) \cong \mathcal{Q}(\Delta_{X_i}^{ab-t}) \). Thus, it follows from “Poincaré duality” [i.e., the isomorphism \( \Delta_{X_i}^{ab-t} \cong \text{Hom}(\Delta_{X_i}^{ab-t}, \hat{\mathbb{Z}}(1)) \) determined by the cup-product on the étale cohomology of \( X \)] that

\[
2g_{X_1} = \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{X_i}^{ab-t}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\Delta_{X_i}^{ab-t}) \otimes \mathbb{Q}_l)
= \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{X_i}^{ab-t}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\mathcal{Q}(\Delta_{X_i}^{ab-t})) \otimes \mathbb{Q}_l)
= \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{U_i}^{ab-t}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\mathcal{Q}(\Delta_{U_i}^{ab-t})) \otimes \mathbb{Q}_l)
\]

where \( g_{X_i} \) is the genus of \( X_i \), and \( l \in \Sigma \). Thus, we conclude that \( g_{X_1} = g_{X_2} \). In particular, by passing to corresponding [i.e., via \( \alpha \)] open subgroups of the \( \Pi_{U_i} \), we may assume that \( g_{X_1} = g_{X_2} \geq 2 \).

Next, by applying this equality “\( g_{X_1} = g_{X_2} \)” to corresponding [i.e., via \( \alpha \)] open subgroups of the \( \Pi_{U_i} \), it follows from the Hurwitz formula that the condition on a pair of open subgroups \( J_i \subseteq H_i \subseteq \Delta_{U_i} \) that “the covering between \( J_i \) and \( H_i \) be cyclic of order a power of a prime number and totally ramified at precisely one closed point but unramified elsewhere” is preserved by \( \alpha \). Thus, it follows formally [cf., e.g., the latter portion of the proof of [Mzk5], Lemma 1.3.9] that \( \alpha \) preserves the inertia groups of points in \( \Xi_i \). Moreover, by considering the conjugation action of \( \Pi_{U_i} \) on these inertia groups, we conclude that \( \alpha \) preserves the decomposition groups \( \subseteq \Pi_{U_i} \) of points \( \in \Xi_i \). Thus, in summary, \( \alpha \) induces an isomorphism \( \Pi_{X_1} \sim \Pi_{X_2} \) that preserves the decomposition groups \( \subseteq \Pi_{X_i} \) of points \( \in \Xi_i \); in particular, by applying Corollary 2.9 to this isomorphism \( \Pi_{X_1} \sim \Pi_{X_2} \), we obtain an isomorphism of schemes \( U_1 \sim U_2 \), as desired. This completes the proof of Corollary 2.10. \( \square \)

**Remark 2.10.1.** Thus, by contrast to the results of [Mzk14], Corollary 2.3, or [Mzk15], Corollary 3.8, (g) [cf. [Mzk15], Remark 3.8.1] — or, indeed, Corollary 1.11, (iii) of the present paper — Corollary 2.10 constitutes the first “absolute isomorphism version of the Grothendieck Conjecture over MLF’s” known to the author that does not rely on the use of Belyi maps. One aspect of this independence of the theory of Belyi maps may be seen in the fact that Corollary 2.10 may be
applied to pro-curves which are not of unit type [cf. Remark 2.8.2, (i), (iii)]. Another aspect of this independence of the theory of Belyi maps may be seen in the fact that Corollary 2.10 involves geometrically pro-$\Sigma$ arithmetic fundamental groups for $\Sigma$ which are not necessarily equal to the set of all prime numbers.

Finally, we observe that the techniques developed in the present §2 allow one to give a more pedestrian treatment of the [somewhat sketchy] treatment given in [Mzk9] [cf. the verification of “observation (iv)” given in the proof of [Mzk9], Corollary 3.11, as well as Remark 2.11.1 below] of the fact that “cusps always appear as images of nodes”.

**Corollary 2.11. (Cusps as Images of Nodes)** Let $k$ be a complete discretely valued field of characteristic zero, with perfect residue field $k$ of characteristic $p > 0$ and ring of integers $\mathcal{O}_K$; $\eta = \text{Spec}(k)$; $S^{\log}$ the log scheme obtained by equipping $S \overset{\text{def}}{=} \text{Spec}(\mathcal{O}_K)$ with the log structure determined by the closed point $S \overset{\text{def}}{=} \text{Spec}(k)$ of $S$; $X^{\log} \to S^{\log}$ a stable log curve over $S^{\log}$ such that the underlying scheme of the generic fiber $X^{\log} \times_S \eta$ is smooth; $\xi \in X(S)$ a cusp of the stable log curve $X^{\log}$; $\xi \in X(S)$ the restriction of $\xi$ to the special fiber $X$ of $X$. In the following, we shall denote restrictions to $\eta$ by means of a subscript $\eta$; also we shall often identify $\xi$ with its image in $X$. Then, after possibly replacing $k$ by a finite extension of $k$, there exists a morphism of stable log curves over $S^{\log}$

$$\phi^{\log} : Y^{\log} \to X^{\log}$$

such that the following properties are satisfied:

(a) the restriction $\phi^{\log}_\eta : Y^{\log}_\eta \to X^{\log}_\eta$ is a finite log étale Galois covering;

(b) $\xi$ is the image of a node of the special fiber $Y$ of $Y$;

(c) $\xi$ is the image of an irreducible component of $Y$.

If, moreover, $X$ is sturdy, loop-ample, and singular, then $\phi_\eta : Y_\eta \to X_\eta$ may be taken to be finite étale of degree $p$.

**Proof.** By replacing $k$ by an appropriate subfield of $k$, one verifies immediately that we may assume that $k$ is of countable cardinality, hence that $k$ satisfies the hypotheses of the discussion preceding Lemma 2.6. After possibly replacing $k$ by a finite extension of $k$ and $X^{\log}_\eta$ by a finite log étale Galois covering of $X^{\log}_\eta$ [which, in fact, may be taken to be of degree a power of $p \cdot l$, where $l$ is a prime $\neq p$], we may assume that $X$ is sturdy [cf. §0], loop-ample [cf. §0], singular [cf. Remark 2.6.3], and split. Next, let us recall from the well-known theory of pointed stable curves [cf. [Knud]] that if we write $V^{\log} \to S^{\log}$ for the stable log curve obtained by forgetting the cusps of $X^{\log}$ [so $V_\eta = X_\eta$], then it follows immediately from the fact that $X$ is sturdy that $V = X$. Thus, by Lemma 2.6, (i) [cf. also the way in which Lemma 2.6, (i), is applied in the proof of Lemma 2.6, (ii)], it follows that, after
possibly replacing $k$ by a finite extension of $k$, there exists a morphism of stable log curves over $S^{\log}$

$$
\phi^{\log} : Y^{\log} \to X^{\log}
$$
such that if we write $\phi$ for the morphism of schemes underlying $\phi^{\log}$, then $\phi_\eta : Y_\eta \to X_\eta$ is a finite étale Galois covering of degree $p$ that is wildly ramified over the irreducible component $C$ of $X$ containing $\xi$.

Next, let us suppose that the property (c) is satisfied. Thus, there exists an irreducible component $E$ of $Y$ that maps to $\xi$. Next, let us observe that there exists an irreducible component $D$ of $Y$ that maps finitely to $C$ and meets the connected component $F$ of the fiber $\phi^{-1}(\xi)$ that contains $E$ [so $D$ is not contained in $F$]. In particular, it follows that there exists a chain of irreducible components

$$E_1 = E, E_2, \ldots, E_n$$

[where $n \geq 1$ is an integer] of $Y$ joining $E$ to $D$ such that each $E_j \subseteq F$ [for $j = 1, \ldots, n$]. Thus, $E_n$ meets $D$ at some node of $Y$ that maps to $\xi$. That is to say, property (b) is satisfied. Thus, to complete the proof of Corollary 2.11, it suffices to verify property (c).

Now suppose that property (c) fails to hold. Then $\phi$ is finite over some neighborhood of $\xi$. Since $\phi_\eta$ is wildly ramified over $C$, it follows that there exists a nontrivial element $\sigma \in \text{Gal}(Y^{\log}_{\eta}/X^{\log}_{\eta})$ that fixes and acts as the identity on some irreducible component $D$ of $Y$ that maps finitely to $C$. After possibly replacing $k$ by a finite extension of $k$, it follows from the finiteness of $\phi$ over some neighborhood of $\xi$ that we may assume that there exists a cusp $\zeta \in Y(S)$ of $Y^{\log}$ lying over $\xi$ such that the restriction $\zeta'$ of $\zeta$ to $Y$ lies in $D$. But then the distinct [since $\sigma$ is nontrivial, and $\phi_\eta$ is étale] cusps $\zeta, \zeta'^{\sigma}$ of $Y^{\log}$ have identical restrictions $\zeta', \zeta'^{\sigma}$ to $Y$ — in contradiction to the definition of a “stable log curve” [i.e., of a “pointed stable curve”]. This completes the proof of property (c) and hence of Corollary 2.11.

\[\text{Remark 2.11.1.}\]

(i) The statement of [Mzk9], Corollary 3.11, concerns smooth log curves over an MLF, but in fact, the same proof as the proof given in [Mzk9] for [Mzk9], Corollary 3.11, may be applied to smooth log curves over an arbitrary mixed characteristic complete discretely valued field. Here, we note that by passing to an appropriate extension, this discretely valued field may be assumed to have a perfect residue field, as in Corollary 2.11. In particular, Corollary 2.11 may be applied to [the portion corresponding to “observation (iv)” in loc. cit. of] the proof of such a generalization of [Mzk9], Corollary 3.11, for more general fields.

(ii) In the discussion of the “pro-$\Sigma$ version” of [Mzk9], Corollary 3.11, in [Mzk9], Remark 3.11.1,

one should assume that $p_\alpha, p_\beta \in \Sigma$.

In fact, this assumption is, in some sense, implicit in the phraseology that appears in the first two lines of [Mzk9], Remark 3.11.1, but it should have been stated explicitly.
Section 3: Elliptic and Belyi Cuspidalizations

The sort of preservation of decomposition groups of closed points that is required in the hypothesis of Corollary 2.9 is shown [for certain types of hyperbolic curves] in the case of profinite geometric fundamental groups in [Mzk8], Corollary 3.2. On the other hand, at the time of writing, the author does not know of any such results in the case of pro-$\Sigma$ geometric fundamental groups, when $\Sigma$ is not equal to the set of all primes. Nevertheless, in the present §3, we observe that the techniques of [Mzk8], §2, concerning the preservation of decomposition groups of torsion points of elliptic curves do indeed hold for fairly general pro-$\Sigma$ geometric fundamental groups [cf. Corollaries 3.3, 3.4]. Moreover, we observe that these techniques — which may be applied not only to [hyperbolic orbicurves related to] elliptic curves, but also, in the profinite case, to [hyperbolic orbicurves related to] tripods [i.e., hyperbolic curves of type $(0,3)$ — cf. [Mzk15], §0], via the use of Belyi maps — allow one to recover not only the decomposition groups of [certain] closed points, but also the resulting “cuspidalizations” [i.e., the arithmetic fundamental groups of open subschemes obtained by removing such closed points] — cf. Corollaries 3.7, 3.8.

Let $X$ be a hyperbolic orbicurve over a field $k$ of characteristic zero; $\overline{k}$ an algebraic closure of $k$. We shall denote the base-change operation “$\times_k\overline{k}$” by means of a subscript $\overline{k}$. Thus, we have an exact sequence of fundamental groups $1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \text{Gal}(\overline{k}/k) \to 1$.

**Definition 3.1.** Let $\pi_1(X) \to \Pi$ be a quotient of profinite groups. Write $\Delta \subseteq \Pi$ for the image of $\pi_1(X_{\overline{k}})$ in $\Pi$. Then we shall say that $X$ is $\Pi$-elliptically admissible if the following conditions hold:

(a) $X$ admits a $k$-core [in the sense of [Mzk6], Remark 2.1.1] $X \to C$;

(b) $C$ is semi-elliptic [cf. §0], hence admits a double covering $D \to C$ by a once-punctured elliptic curve $D$;

(c) $X$ admits a finite étale covering $Y \to X$ by a hyperbolic curve $Y$ over a finite extension $k_Y$ of $k$ that arises from a normal open subgroup $\Pi_Y \subseteq \Pi$ such that the resulting finite étale covering $Y \to C$ factors as the composite of a covering $Y \to D$ with the covering $D \to C$ and, moreover, is such that, for every set of primes $\Sigma$ such that some open subgroup of $\Delta$ is pro-$\Sigma$, it holds that $\Delta$ is pro-$\Sigma$, and, moreover, the degree of the covering $Y \to C \times_k k_Y$ is a product of primes [perhaps with multiplicities] $\in \Sigma$.

When $\Pi = \pi_1(X)$, we shall simply say that $X$ is elliptically admissible.

**Remark 3.1.1.** In the notation of Definition 3.1, one verifies immediately that $D_{\overline{k}} \to C_{\overline{k}}$ may be characterized as the unique [up to isomorphism over $C_{\overline{k}}^{-1}$] finite étale double covering of $C_{\overline{k}}$ by a hyperbolic curve [i.e., as opposed to an arbitrary hyperbolic orbicurve].
Example 3.2. Scheme-theoretic Elliptic Cuspidalizations.

(i) Let $N$ be a positive integer; $D$ a once-punctured elliptic curve over a finite Galois extension $k'$ of $k$ such that all of the $N$-torsion points of the underlying elliptic curve $E$ of $D$ are defined over $k'$; $D 	o C$ a semi-elliptic $k'$-core of $D$ [such that $D 	o C$ is the double covering appearing in the definition of “semi-elliptic”]. Then the morphism $[N]_E : E \to E$ given by multiplication by $N$ determines a finite étale covering $[N]_D : U \to D$ [of degree $N^2$], together with an open embedding $U \hookrightarrow D$ [which we use to identify $U$ with its image in $D$], i.e., we have a diagram as follows:

$$
\begin{array}{c}
U \hookrightarrow D \\
\downarrow [N]_D \\
D
\end{array}
$$

Suppose that the Galois group $\text{Gal}(\overline{k}/k)$ is slim. Then, in the language of [Mzk15], §4, this situation may be described as follows [cf. [Mzk15], Definition 4.2, (i), where we take the extension “$1 \to \Delta \to \Pi \to G \to 1$” to be the extension $1 \to \pi_1(D \times_{k'} \overline{k}) \to \pi_1(D) \to \text{Gal}(\overline{k}/k') \to 1$]: The above diagram yields a chain

$$
D \leadsto U \hookrightarrow (U \hookrightarrow) U_n \leadsto (U_n \hookrightarrow) U_{n-1} \leadsto \ldots
$$

$$
\leadsto (U_3 \hookrightarrow) U_2 \leadsto (U_2 \hookrightarrow) U_1 \overset{\text{def}}{=} D
$$

[where $n \overset{\text{def}}{=} N^2 - 1$] whose associated type-chain is

$\lambda, \bullet, \ldots, \bullet$

[i.e., a finite étale covering, followed by $n$ de-cuspidalizations], together with a terminal isomorphism

$$
U_1 \cong D
$$

[which, in our notation, amounts to the identity morphism] from the $U_1$ at the end of the above chain to the unique $D$ of the trivial chain [of length 0]. In particular:

The above chain may thought of as a construction of a “cuspidalization” [i.e., result of passing to an open subscheme by removing various closed points] $U \hookrightarrow D$ of $D$.

The remainder of the portion of the theory of the present §3 concerning elliptic cuspidalizations consists, in essence, of the unraveling of various consequences of this “chain-theoretic formulation” of the diagram that appears at the beginning of the present item (i).

(ii) A variant of the discussion of (i) may be obtained as follows. In the notation of (i), suppose further that $X$ is an elliptically admissible hyperbolic orbicurve over $k$, and that we have been given finite étale coverings $V \to X$, $V \to D$, where $V$ is a hyperbolic curve over $k'$. Also, [for simplicity] we suppose that $V \to X$ is a Galois covering such that $\text{Gal}(V/X)$ preserves the open subscheme $U_V \overset{\text{def}}{=} V \times_D U \subseteq V$ [i.e., the inverse image of $U \subseteq D$ via $V \to D$]. Thus, $U_V \subseteq V$ descends to an open subscheme $U_X \subseteq X$. Then by appending to the chain of (i) the “finite étale
covering” \( V \to X \), followed by the “finite étale quotient” \( V \to D \) on the left, and the “finite étale covering” \( V \to D \), followed by the “finite étale quotient” \( V \to X \) on the right, we obtain a chain

\[
X \sim V \ (\to X) \sim (V \to) D \sim (U \to) (\to D) \sim (U \leftarrow) U_n \sim (U_n \leftarrow) U_{n-1} \sim \ldots
\]

\[
\sim (U_3 \leftarrow) U_2 \sim (U_2 \leftarrow) U_1 \overset{\text{def}}{=} D \sim (V \to) (\to D) \sim (V \rightarrow) X_\ast \overset{\text{def}}{=} X
\]

whose associated type-chain is

\[\lambda, \gamma, \lambda, \ast, \ldots, \ast, \lambda, \gamma\]

[where the “…” are all “\( \ast \)’s”], together with a terminal isomorphism \( X_\ast \xrightarrow{\sim} X \) [i.e., the identity morphism]. In particular, the above chain may thought of as a construction of a “cuspidalization” \( U_X \leftarrow X \) of \( X \) via the construction of a “cuspidalization” \( U_V \leftarrow V \) of \( V \), equipped with descent data [i.e., a suitable collection of automorphisms] with respect to the finite étale Galois covering \( V \to X \).

Now by translating the scheme-theoretic discussion of Example 3.2 into the language of profinite groups via the theory of [Mzk15], §4, we obtain the following result.

**Corollary 3.3.** (Pro-\( \Sigma \) Elliptic Cuspidalization I: Algorithms) Let \( \mathbb{D} \) be a chain-full set of collections of partial construction data [cf. [Mzk15], Definition 4.6, (i)] such that the rel-isom-\( \mathbb{D} \)-GC holds [i.e., the “relative isomorphism version of the Grothendieck Conjecture for \( \mathbb{D} \) holds”] — cf. [Mzk15], Definition 4.6, (ii)]; \( G \) a slim profinite group;

\[
1 \to \Delta \to \Pi \to G \to 1
\]

an extension of GSAFG-type that admits partial construction data \((k, X, \Sigma)\), where \( k \) is of characteristic zero, and \( X \) is a \( \Pi \)-elliptically admissible [cf. Definition 3.1] hyperbolic orbicurve, such that \([(X), [k], \Sigma)] \in \mathbb{D}; \alpha : \pi_1(X) \to \Pi\) the corresponding scheme-theoretic envelope [cf. [Mzk15], Definition 2.1, (iii)]; \( \tilde{X} \to X \) the pro-finite étale covering of \( X \) determined by \( \alpha \) [so \( \Pi \to \text{Gal}(\tilde{X}/X) \)]; \( k \) the resulting field extension of \( k \) [so \( G \to \text{Gal}(k/k) \)]. Suppose further that, for some \( l \in \Sigma \), the cyclotomic character \( G \to \mathbb{Z}_l^\times \) has open image. Thus, by the theory of [Mzk15], §4, we have associated categories

\[
\text{Chain}(\Pi); \quad \text{Chain}^{\text{iso-trm}}(\Pi); \quad \text{\acute{E}tLoc}(\Pi)
\]

which may be constructed via purely “group-theoretic” operations from the extension of profinite groups \( 1 \to \Delta \to \Pi \to G \to 1 \) [cf. [Mzk15], Definition 4.2, (iii), (iv), (v); [Mzk15], Lemma 4.5, (v); the proof of [Mzk15], Theorem 4.7, (ii)]. Then:

(i) Let \( G' \subseteq G \) be a normal open subgroup, corresponding to some finite extension \( k' \subseteq k \) of \( k \); \( \Pi' \overset{\text{def}}{=} \Pi \times_G G' \); \( C \) a \( k' \)-core of \( X_{k'} \overset{\text{def}}{=} X \times_k k' \). Then the finite étale covering \( X_{k'} \to C \) determines a chain \( X_{k'} \sim C \) of the category \( \text{Chain}(\tilde{X}/X_{k'}) \) [cf. [Mzk15], Definition 4.2, (i), (ii)] whose image \( \Pi' \sim \Pi'_{\text{def}} \) in \( \text{Chain}(\Pi') \) [via the natural functor of [Mzk15], Remark 4.2.1] may be characterized
“group-theoretically”, up to isomorphism in \(\text{Chain}(\Pi')\), as the unique chain of length 1 in \(\text{Chain}(\Pi')\), with associated type-chain \(\gamma\), such that the resulting object of \(\text{EtLoc}(\Pi')\) forms a terminal object of \(\text{EtLoc}(\Pi')\).

(ii) The collection of open subgroups \(\Pi_D \subseteq \Pi_C\) that arise from finite étale double coverings \(D \to C\) that exhibit \(C\) as semi-elliptic [cf. Remark 3.1.1] may be characterized “group-theoretically” as the collection of open subgroups \(J \subseteq \Pi_C\) of index 2 such that \(J \cap \Delta_C\) [where \(\Delta_C \overset{\text{def}}{=} \ker(\Pi_C \to G')\)] is torsion-free [i.e., the covering determined by \(J\) is a scheme — cf. [Mzk15], Lemma 4.1, (iv)].

(iii) Write \(\Delta_D \overset{\text{def}}{=} \ker(\Pi_D \to G')\). Let \(N\) be a positive integer which is a product of primes [perhaps with multiplicities] of \(\Sigma\); \(U \subseteq D\) the open subscheme obtained by removing the \(N\text{-torsion points}\) of the elliptic curve underlying \(D\); \(V \to X, V \to D\) finite étale coverings, where \(V\) is a hyperbolic curve over \(k'\). Suppose further that \(V \to X\) arises from a normal open subgroup \(\Pi_V \subseteq \Pi\) such that \(\text{Gal}(V/X) \cong \Pi/\Pi_V\) preserves the open subscheme \(U_V \overset{\text{def}}{=} V \times_D U \subseteq V\) [i.e., the inverse image of \(U \subseteq D\) via \(V \to D\)], while \(V \to D\) arises from an open immersion \(\Pi_V \hookrightarrow \Pi_D\). Thus, \(U_V \subseteq V\) descends to an open subscheme \(U_X \subseteq X\), and \(U \subseteq D, U_V \subseteq V, U_X \subseteq X\) determine extensions of GSAFG-type

\[
1 \to \Delta_U \to \Pi_U \to G' \to 1; \quad 1 \to \Delta_{U_V} \to \Pi_{U_V} \to G' \to 1
\]

[\(\text{i.e., by considering the finite étale Galois coverings of degree a product of primes [perhaps with multiplicities] of } \Sigma\; \text{over coverings of } U, U_V, U_X \text{ arising from } \Pi\), together with natural surjections \(\Pi_U \to \Pi_D, \Pi_{U_V} \to \Pi_V, \Pi_{U_X} \to \Pi\) and open immersions \(\Pi_{U_V} \hookrightarrow \Pi_U, \Pi_{U_V} \hookrightarrow \Pi_{U_X}\). [In particular, \(\Delta_U, \Delta_{U_V}\), and \(\Delta_{U_X}\) are pro-\(\Sigma\) groups.] Then, for any \(G' \subseteq G\) that is sufficiently small, where “sufficiently” depends only on \(N\), the natural surjection \(\Pi_{U_X} \to \Pi\)

— i.e., “cuspidalization” of \(\Pi\) — may be constructed via “group-theoretic” operations as follows:

(a) There exists a [not necessarily unique] \(\Pi\)-chain, which admits an entirely “group-theoretic” description, with associated type-chain \(\lambda, \gamma, \lambda, \bullet, \ldots, \bullet, \lambda, \gamma\)

— cf. Example 3.2, (ii) — that admits a terminal isomorphism with the trivial \(\Pi\)-chain [of length 0], and whose final three groups consist of \(\Pi_D \sim \Pi_V \sim (\Pi_V \hookrightarrow)\) \(\Pi\) such that the natural surjection \(\Pi_U \to \Pi_D\) may be recovered from the chain of “\(\bullet\)’s” terminating at the third to last group of the above-mentioned \(\Pi\)-chain; the natural surjection \(\Pi_{U_V} \to \Pi_V\) may then be recovered from \(\Pi_U \to \Pi_D\) by forming the fiber product with the inclusion \(\Pi_V \hookrightarrow \Pi_D\).

(b) The natural surjection \(\Pi_{U_X} \to \Pi\) may be recovered from \(\Pi_{U_V} \to \Pi_V\) [where we note that \(\Pi_{U_V} \to \Pi_V\) may be identified with the fiber product of
The decomposition groups of the closed points of $X$ lying in the complement of $U_X$ may be obtained as the images via $\Pi_{U_X} \twoheadrightarrow \Pi$ of the cuspidal decomposition groups of $\Pi_{U_X}$ [cf. [Mzk15], Lemma 4.5, (v)].

**Proof.** The assertions of Corollary 3.3 follow immediately from the definitions, together with the various references quoted in the course of the “group-theoretic” reconstruction algorithm described in the statement of Corollary 3.3, and the equivalences of [Mzk15], Theorem 4.7, (i). □

**Remark 3.3.1.** Let $p$ be a prime number. Then if one takes $F$ to be set of isomorphism classes of generalized sub-$p$-adic fields, $S$ the set of sets of prime numbers containing $p$, and $V$ to be the set of isomorphism classes of hyperbolic orbicurves over fields whose isomorphism class $\in F$, then $D \overset{\text{def}}{=} V \times F \times S$ satisfies the hypothesis of Corollary 3.3 concerning “$D$” [cf. [Mzk15], Example 4.8, (i)].

**Remark 3.3.2.** Recall that when $k$ is an MLF or an NF, the subgroup $\Delta \subseteq \Pi$ admits a purely “group-theoretic” characterization [cf. [Mzk15], Theorem 2.6, (v), (vi)]. Thus, when $k$ is an MLF or an NF, the various “group-theoretic” reconstruction algorithms described in the statement of Corollary 3.3 may be thought of as being applied not to the extension $1 \to \Delta \to \Pi \to G \to 1$, but rather to the single profinite group $\Pi$.

**Remark 3.3.3.** One verifies immediately that Corollary 3.3 admits a “tempered version”, when the base field $k$ is an MLF [cf. [Mzk15], Theorem 4.12, (i)]. We leave the routine details to the reader.

**Remark 3.3.4.** By applying the tempered version of Corollary 3.3 discussed in Remark 3.3.3, one may obtain “explicit reconstruction algorithm versions” of certain results of [Mzk12] [cf. [Mzk12], Theorem 1.6; [Mzk12], Remark 1.6.1] concerning the étale theta function. We leave the routine details to the reader.

The “group-theoretic” algorithm of Corollary 3.3 has the following immediate “Grothendieck Conjecture-style” consequence.

**Corollary 3.4.** (Pro-$\Sigma$ Elliptic Cuspidalization II: Comparison) Let $D$ be a chain-full set of collections of partial construction data [cf. [Mzk15], Definition 4.6, (i)] such that the rel-isom-$\mathcal{D}$GC holds [cf. [Mzk15], Definition 4.6, (ii)]. For $i = 1, 2$, let $G_i$ be a slim profinite group;

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$
an extension of GSAFG-type that admits partial construction data \((k_i, X_i, \Sigma_i)\), where \(k_i\) is of characteristic zero, and \(X_i\) is a \([\Pi_i]\)-elliptically admissible [cf. Definition 3.1] hyperbolic orbicurve, such that \(((X_i), (k_i), \Sigma_i) \in D; \alpha_i : \pi_1(X_i) \to \Pi_i\) the corresponding scheme-theoretic envelope [cf. [Mzk15], Definition 2.1, (iii)]; \(\tilde{X}_i \to X_i\) the pro-finite étale covering of \(X\) determined by \(\alpha_i\) [so \(\Pi_i \xrightarrow{\sim} \text{Gal}(\tilde{X}_i/X_i)\)]; \(\tilde{k}_i\) the resulting field extension of \(k_i\) [so \(G_i \xrightarrow{\sim} \text{Gal}(\tilde{k}_i/k_i)\)]; \(C_i\) a \(k_i\)-core of \(X_i\); \(D_i \to C_i\) a finite étale double covering that exhibits \(C_i\) as semi-elliptic [cf. Remark 3.1.1]; \(\Pi_i \subseteq \Pi_{C_i}\), \(\Pi_{D_i} \subseteq \Pi_{C_i}\) the open subgroups determined by \(X_i \to C_i\), \(D_i \to C_i\); \(N\) a positive integer which is a product of primes [perhaps with multiplicites] \(\in \Sigma_1 \cap \Sigma_2\); \(U_i \subseteq D_i\) the open subscheme obtained by removing the \(N\)-torsion points of the elliptic curve underlying \(D_i\); \(V_i \to X_i\), \(V_i \to D_i\) finite étale coverings that arise from a normal open subgroup \(\Pi_{V_i} \subseteq \Pi_i\) and an open immersion \(\Pi_{V_i} \hookrightarrow \Pi_{D_i}\) such that \(\text{Gal}(V_i/X_i) \cong \Pi_{V_i}/\Pi_{V_i}\) preserves the open subscheme \(U_{V_i} \mathrel{\overset{\text{def}}{=} V_i \times_{D_i} U_i} \subseteq V_i\) [i.e., the inverse image of \(U_i \subseteq D_i\) via \(V_i \to D_i\)]; \(U_{X_i} \subseteq X_i\) the resulting open subscheme [obtained by descending \(U_{V_i} \subseteq V_i\)];

\[
1 \to \Delta_{U_{X_1}} \to \Pi_{U_{X_1}} \to G_i \to 1
\]

the extension of GSAFG-type obtained [via \(\alpha_i\)] by considering the finite étale Galois coverings of degree a product of primes [perhaps with multiplicites] \(\in \Sigma_1\) over coverings of \(U_{X_i}\) arising from \(\Pi_i\); \(\Pi_{U_{X_i}} \to \Pi_i\) the natural surjection [relative to \(\alpha_i\)]. Suppose further that, for some \(l \in \Sigma_1 \cap \Sigma_2\), the cyclotomic characters \(G_i \to \mathbb{Z}_l^\times\) have open image for \(i = 1, 2\). Let

\[
\phi : \Pi_1 \xrightarrow{\sim} \Pi_2
\]

be an isomorphism of profinite groups such that \(\phi(\Delta_1) = \Delta_2\). Then there exists an isomorphism of profinite groups

\[
\phi_{U} : \Pi_{U_{X_1}} \xrightarrow{\sim} \Pi_{U_{X_2}}
\]

that is compatible with \(\phi\), relative to the natural surjections \(\Pi_{U_{X_i}} \to \Pi_i\). Moreover, such an isomorphism is unique up to composition with an inner automorphism arising from an element of the kernel of \(\Pi_{U_{X_i}} \to \Pi_i\).

**Proof.** The construction of \(\phi_U\) follows immediately from Corollary 3.3; the asserted uniqueness then follows immediately from our assumption that the rel-isom-\(\mathbb{D}GCC\) holds. \(\Box\)

**Remark 3.4.1.** Just as in the case of Corollary 3.3 [cf. Remark 3.3.3], Corollary 3.4 admits a “tempered version”, when the base fields \(k_i\) involved are MLF’s. We leave the routine details to the reader.

**Remark 3.4.2.** By applying Corollary 3.4 [cf. also Remarks 3.3.4, 3.4.1], one may obtain “pro-\(\Sigma\) tempered” versions of certain results of [Mzk12] [cf. [Mzk12], Theorem 1.6; [Mzk12], Remark 1.6.1] concerning the étale theta function. We leave the routine details to the reader.
Now we return to the notation introduced at the beginning of the present §3: Let $X$ be a hyperbolic orbicurve over a field $k$ of characteristic zero; $\overline{k}$ an algebraic closure of $k$. Thus, we have an exact sequence of fundamental groups $1 \to \pi_1(X \times_k \overline{k}) \to \pi_1(X) \to \text{Gal}(\overline{k}/k) \to 1$.

**Definition 3.5.** We shall say that $X$ is of strictly Belyi type if it is defined over a number field and isogenous [cf. §0] to a hyperbolic curve of genus zero. [Thus, this definition generalizes the definition of [Mzk14], Definition 2.3, (i).]

**Example 3.6.** Scheme-theoretic Belyi Cuspidalizations.

(i) Let $P$ be a copy of the projective line minus three points over a finite Galois extension $k'$ of $k$; $V$ an arbitrary hyperbolic curve over $k'$; $U \subseteq V$ a nonempty open subscheme [hence, in particular, a hyperbolic curve over $k'$]. Suppose that $U$ [hence also $V$] is defined over a number field. Then it follows from the existence of Belyi maps [cf. [Belyi]; [Mzk7]] that, for some nonempty open subscheme $W \subseteq U$, there exists a diagram as follows:

\[
\begin{array}{c}
W \leftrightarrow U \leftrightarrow V \\
\downarrow \beta \\
\end{array}
\]

[where the “$\leftrightarrow$’s” are the natural open immersions; the “Belyi map” $\beta$ is finite étale]. By replacing $k'$ by some finite extension of $k'$, let us suppose further [for simplicity] that the cusps of $W$ are all defined over $k'$. Also, let us suppose that the Galois group $\text{Gal}(\overline{k}/k')$ is slim. Then, in the language of [Mzk15], §4, this situation may be described as follows [cf. [Mzk15], Definition 4.2, (i), where we take the extension “$1 \to \Delta \to \Pi \to G \to 1$” to be the extension $1 \to \pi_1(P \times_{k'} \overline{k}) \to \pi_1(P) \to \text{Gal}(\overline{k}/k') \to 1$]: For some nonnegative integers $n$, $m$, the above diagram yields a chain

\[
P \rightsquigarrow W (\to P) \rightsquigarrow (W \leftrightarrow) W_n \rightsquigarrow (W_n \leftrightarrow) W_{n-1} \rightsquigarrow \ldots
\]

\[
\rightsquigarrow (W_2 \leftrightarrow) W_1 \overset{\text{def}}{=} U \rightsquigarrow (U \leftrightarrow) U_m \rightsquigarrow (U_m \leftrightarrow) U_{m-1} \rightsquigarrow \ldots
\]

\[
\rightsquigarrow (U_2 \leftrightarrow) U_1 \overset{\text{def}}{=} V
\]

whose associated type-chain is

\[
\lambda, \bullet, \ldots, \bullet
\]

[i.e., a finite étale covering, followed by $n + m$ de-cuspidalizations]. In particular:

The above chain may thought of as a construction of a “cuspidalization” [i.e., result of passing to an open subscheme by removing various closed points] $U \hookrightarrow V$ of $V$.

The remainder of the portion of the theory of the present §3 concerning Belyi cuspidalizations consists, in essence, of the unraveling of various consequences of
this “chain-theoretic formulation” of the diagram that appears at the beginning of the present item (i).

(ii) A variant of the discussion of (i) may be obtained as follows. In the notation of (i), suppose further that \( X \) is a hyperbolic orbicurve of strictly Belyi type over \( k \), and that we have been given finite \( \acute{e}tale \) coverings \( V \to X \), \( V \to Q \), together with an open immersion \( Q \hookrightarrow P \) [so \( Q \) is a hyperbolic curve of genus zero over \( k' \)]. Also, [for simplicity] we suppose that \( V \to X \) is Galois, that \( U \subseteq V \) descends to an open subscheme \( U_X \subseteq X \), and [by possibly replacing \( k' \) by a finite extension of \( k' \)] that the cusps of \( Q \) are defined over \( k' \). Then by appending to the chain of (i) the “finite \( \acute{e}tale \) covering” \( V \to X \), followed by the “finite \( \acute{e}tale \) quotient” \( V \to Q \), followed by the \( de-cuspidalizations \) \( Q \to Q_1 \to \ldots \to Q_1 \overset{\text{def}}{=} P \) [for some nonnegative integer \( l \)], on the left, and the “finite \( \acute{e}tale \) quotient” \( V \to X \) on the right, we obtain a chain

\[
X \rightsquigarrow V \ (\to X) \rightsquigarrow (V \to) \ Q \rightsquigarrow (Q \hookrightarrow) \ Q_1 \rightsquigarrow \ldots \rightsquigarrow (Q_2 \hookrightarrow) \ Q_1 \overset{\text{def}}{=} P \\
\rightsquigarrow W \ (\to P) \rightsquigarrow (W \hookrightarrow) \ W_1 \rightsquigarrow \ldots \rightsquigarrow (W_2 \hookrightarrow) \ W_1 \overset{\text{def}}{=} U \rightsquigarrow (U \hookrightarrow) \ U_m \rightsquigarrow \ldots \\
\rightsquigarrow (U_2 \hookrightarrow) \ U_1 \overset{\text{def}}{=} V \rightsquigarrow (V \to) \ X_\ast \overset{\text{def}}{=} X
\]

whose associated type-chain is

\[\lambda, \gamma, \bullet, \ldots, \bullet, \lambda, \gamma, \ldots, \lambda, \gamma\]

[where the “…” are all “\( \bullet \)’s”], together with a terminal isomorphism \( X_\ast \sim X \) [i.e., the identity morphism]. In particular, the above chain may thought of as a construction of a “cuspidalization” \( U_X \hookrightarrow X \) of \( X \) via the construction of a “cuspidalization” \( U \hookrightarrow V \) of \( V \), equipped with descent data [i.e., a suitable collection of automorphisms] with respect to the finite \( \acute{e}tale \) Galois covering \( V \to X \).

Now by translating the scheme-theoretic discussion of Example 3.6 into the language of profinite groups via the theory of [Mzk15], §4, we obtain the following result.

**Corollary 3.7.** (Profinite Belyi Cuspidalization I: Algorithms) Let \( \mathbb{D} \) be a chain-full set of collections of partial construction data [cf. [Mzk15], Definition 4.6, (i)] such that the rel-isom-\( \mathbb{D}GC \) holds [i.e., the “relative isomorphism version of the Grothendieck Conjecture for \( \mathbb{D} \) holds” — cf. [Mzk15], Definition 4.6, (ii)]; \( G \) a slim profinite group;

\[1 \to \Delta \to \Pi \to G \to 1\]

an extension of GSAFG-type that admits partial construction data \((k, X, \Sigma)\), where \( k \) is of characteristic zero, \( X \) is a hyperbolic orbicurve of strictly Belyi type [cf. Definition 3.5], and \( \Sigma \) is the set of all primes, such that \([X], [k], \Sigma] \in \mathbb{D}; \alpha : \pi_1(X) \to \Pi \) the corresponding scheme-theoretic envelope [cf. [Mzk15], Definition 2.1, (iii)], which is an isomorphism of profinite groups; \( \widetilde{X} \to X \) the pro-finite \( \acute{e}tale \) covering of \( X \) determined by \( \alpha \) [so \( \Pi \to Gal(\widetilde{X}/X) \)]; \( \tilde{k} \) the resulting algebraic closure of \( k \) [so \( G \to Gal(\tilde{k}/k) \)]. Suppose further that, for some \( \ell \in \Sigma \), the cyclotomic character \( G \to \mathbb{Z}_\ell^\times \) has open image. Thus, by the theory of [Mzk15], §4, we have associated categories

\[
\text{Chain}(\Pi); \quad \text{Chain}^{\text{iso-trm}}(\Pi); \quad \text{\acute{E}tLoc}(\Pi)
\]
which may be constructed via purely “group-theoretic” operations from the extension of profinite groups $1 \to \Delta \to \Pi \to G \to 1$ [cf. [Mzk15], Definition 4.2, (iii), (iv), (v); [Mzk15], Lemma 4.5, (v); the proof of [Mzk15], Theorem 4.7, (ii)]. Then for every nonempty open subscheme

$$U_X \subseteq X$$

defined over a number field, the natural surjection

$$\Pi_{U_X} \overset{\text{def}}{=} \pi_1(U_X) \to \pi_1(X) \overset{\sim}{\to} \Pi$$

[where the final “$\sim$” is given by the inverse of $\alpha$] — i.e., “cuspidalization” of $\Pi$ — may be constructed via “group-theoretic” operations as follows:

(a) For some normal open subgroup $\Pi_V \subseteq \Pi$, which corresponds to a finite covering $V \to X$ of hyperbolic orbicurves, there exists a [not necessarily unique] $\Pi$-chain, which admits an entirely “group-theoretic” description, with associated type-chain

$$\lambda, \gamma, \bullet, \ldots, \bullet, \lambda, \bullet, \ldots, \bullet, \gamma$$

— cf. Example 3.6, (ii) — that admits a terminal isomorphism with the trivial $\Pi$-chain [of length 0] such that if we write $U \overset{\text{def}}{=} V \times_X U_X$, $\Pi_U \overset{\text{def}}{=} \Pi_V \rtimes \Pi_{U_X}$, then the natural surjection $\Pi_U \to \Pi_V$ may be recovered from the chain of “$\bullet$’s” terminating at the second to last group of the above-mentioned $\Pi$-chain.

(b) The natural surjection $\Pi_{U_X} \to \Pi$ may be recovered from $\Pi_U \to \Pi_V$ by forming the “$\rtimes$” [cf. §0] with respect to the unique lifting [relative to $\Pi_U \to \Pi_V$] of the outer action of the finite group $\Pi/\Pi_V$ on $\Pi_V$ to a group of outer automorphisms of $\Pi_U$.

(c) The decomposition groups of the closed points of $X$ lying in the complement of $U_X$ may be obtained as the images via $\Pi_{U_X} \to \Pi$ of the cuspidal decomposition groups of $\Pi_{U_X}$ [cf. [Mzk15], Lemma 4.5, (v)].

Proof. The assertions of Corollary 3.7 follow immediately from the definitions, together with the various references quoted in the course of the “group-theoretic” reconstruction algorithm described in the statement of Corollary 3.7, and the equivalences of [Mzk15], Theorem 4.7, (i).

Remark 3.7.1. Similar remarks to Remarks 3.3.1, 3.3.2, 3.3.3 may be made for Corollary 3.7.

Remark 3.7.2. In the situation of Corollary 3.7, when the field $k$ is an MLF, one then obtains an algorithm for constructing the decomposition groups of arbitrary closed points of $X$, by combining the algorithms of Corollary 3.7 — cf., especially,
Corollary 3.7, (c), which allows one to construct the decomposition groups of those
closed points of $X$ which [like $U_X$!] are defined over a number field — with the "p-
adic approximation lemma" of [Mzk8] [i.e., [Mzk8], Lemma 3.1]. A "Grothendieck
Conjecture-style" version of this sort of reconstruction of decomposition groups of
arbitrary closed points of $X$ may be found in [Mzk8], Corollary 3.2.

The "group-theoretic" algorithm of Corollary 3.7 has the following immediate
"Grothendieck Conjecture-style" consequence.

Corollary 3.8. (Profinite Belyi Cuspidalization II: Comparison) Let
$\mathcal{D}$ be a chain-full set of collections of partial construction data [cf. [Mzk15],
Definition 4.6, (i)] such that the rel-isom-$\mathcal{D}GC$ holds [cf. [Mzk15], Definition 4.6,
(ii)]. For $i = 1, 2$, let $G_i$ be a slim profinite group;
$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$
an extension of GSAFG-type that admits partial construction data $(k_i, X_i, \Sigma_i)$,
where $k_i$ is of characteristic zero, $X_i$ is a hyperbolic orbicurve of strictly
Belyi type [cf. Definition 3.5], and $\Sigma_i$ is the set of all primes, such that
$([X_i], [k_i], \Sigma_i) \in \mathcal{D}$; $\alpha_i : \pi_1(X_i) \sim \to \Pi_i$ the corresponding scheme-theoretic en-
velope [cf. [Mzk15], Definition 2.1, (iii)], which is an isomorphism of profinite
groups; $\tilde{X}_i \to X_i$ the pro-finite étale covering of $X_i$ determined by $\alpha_i$ [so
$\Pi_i \sim \to \text{Gal}(\tilde{X}_i/X_i)$]; $\tilde{k}_i$ the resulting algebraic closure of $k_i$ [so $G_i \sim \to \text{Gal}(\tilde{k}_i/k_i)$].
If, for $i = 1, 2$, $U_{X_i} \subseteq X_i$ is a nonempty open subscheme which is defined
over a number field, then write
$$1 \to \Delta_{U_{X_i}} \to \Pi_{U_{X_i}} \to G_i \to 1$$
for the extension of GSAFG-type determined by $[\alpha_i]$ and $[\Pi_i]$ the natural surjection
$\pi_1(U_{X_i}) \to \text{Gal}(\tilde{k}_i/k_i) \cong G_i$; $\Pi_{U_{X_i}} \to \Pi_i$ for the natural surjection [relative
to $\alpha_i$]. Suppose further that, for some $l \in \Sigma_1 \cap \Sigma_2$, the cyclotomic characters
$G_i \to \mathbb{Z}_l^\times$ have open image for $i = 1, 2$. Let
$$\phi : \Pi_1 \sim \to \Pi_2$$
be an isomorphism of profinite groups such that $\phi(\Delta_1) = \Delta_2$. Then, for each
nonempty open subscheme $U_{X_1} \subseteq X_1$ defined over a number field, there exist a
nonempty open subscheme $U_{X_2} \subseteq X_2$ defined over a number field and an isomor-
phism of profinite groups
$$\phi_U : \Pi_{U_{X_1}} \sim \to \Pi_{U_{X_2}}$$
that is compatible with $\phi$, relative to the natural surjections $\Pi_{U_{X_i}} \to \Pi_i$. More-
over, such an isomorphism $\phi_U$ is unique up to composition with an inner automor-
phism arising from an element of the kernel of $\Pi_{U_{X_i}} \to \Pi_i$.

Proof. The construction of $\phi_U$ for a suitable nonempty open subscheme $U_{X_2}$
follows immediately from Corollary 3.7; the asserted uniqueness then follows im-
mEDIATELY from our assumption that the rel-isom-$\mathcal{D}GC$ holds. $\Box$

Remark 3.8.1. A similar remark to Remark 3.4.1 may be made for Corollary
3.8.
Bibliography


