TOPICS IN ABSOLUTE ANABELIAN GEOMETRY II: DECOMPOSITION GROUPS AND ENDOMORPHISMS

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Abstract. The present paper, which forms the second part of a three-part series in which we study **absolute anabelian geometry** from an **algorithmic** point of view, focuses on the study of the closely related notions of **decomposition groups** and **endomorphisms** in this anabelian context. We begin by studying an *abstract combinatorial analogue* of the algebro-geometric notion of a stable polycurve [i.e., a "successive extension of families of stable curves"] and showing that the "geometry of log divisors on stable polycurves" may be extended, in a purely group-theoretic fashion, to this abstract combinatorial analogue; this leads to various **anabelian** results concerning **configuration spaces**. We then turn to the study of the *absolute pro-Σ anabelian geometry* of hyperbolic curves over mixed-characteristic local fields, for Σ a set of primes of cardinality ≥ 2 that contains the residue characteristic of the base field. In particular, we prove a certain *"pro-p resolution of nonsingularities"* type result, which implies a "conditional" anabelian result to the effect that the condition, on an isomorphism of arithmetic fundamental groups, of preservation of decomposition groups of "most" closed points implies that the isomorphism arises from an isomorphism of schemes — i.e., in a word, **"point-theoreticity implies geometricity"**; a "non-conditional" version of this result is then obtained for *"pro-curves"* obtained by removing from a proper curve some set of closed points which is *"p-adically dense in a Galois-compatible fashion"*. Finally, we study, from an algorithmic point of view, the theory of **Belyi** and **elliptic cuspidalizations**, i.e., group-theoretic reconstruction algorithms for the arithmetic fundamental group of an *open subscheme* of a hyperbolic curve that arise from consideration of certain *endomorphisms* determined by *Belyi maps* and *endomorphisms of elliptic curves*.

Contents:

Introduction

In the present paper, which forms the second part of a three-part series, we continue our discussion of various topics in **absolute anabelian geometry** from

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a **"group-theoretic algorithmic"** point of view, as discussed in the Introduction to [Mzk15]. The topics presented in the present paper center around the following *two themes*:

(A) [the subgroups of arithmetic fundamental groups constituted by] **decomposition groups** of subvarieties of a given variety [such as closed points, divisors] as a crucial tool that leads to absolute anabelian results;

(B) **"hidden endomorphisms"** — which may be thought of as **"hidden symmetries"** — of hyperbolic curves that give rise to various absolute anabelian results.

In fact, "decomposition groups" and "endomorphisms" are, in a certain sense, related notions — that is to say, the monoid of "endomorphisms" of a variety may be thought of as a sort of "decomposition group of the generic point"!

With regard to the theme (B), we recall that the endomorphisms of an abelian variety play a fundamental role in the theory of *abelian varieties* [e.g., *elliptic curves*!]. Unlike abelian varieties, hyperbolic curves [say, in characteristic zero] do not have sufficient "endomorphisms" in the *literal, scheme-theoretic sense* to form the basis for an interesting theory. This difference between abelian varieties and hyperbolic curves may be thought of, at a certain level, as reflecting the difference between *linear Euclidean geometries* and *non-linear hyperbolic geometries*. From this point of view, it is natural to search for *"hidden endomorphisms"* that are, in some way, related to the *intrinsic non-linear hyperbolic geometry* of a hyperbolic curve. Examples [that appear in previous papers of the author] of such "hidden endomorphisms" — which exhibit a remarkable tendency to be related [for instance, via some induced action on the *arithmetic fundamental group*] to some sort of *"anabelian result"* — are the following:

(i) the interpretation of the **automorphism group** $PSL_2(\mathbb{R})$ of the universal covering of a hyperbolic Riemann surface as an object that gives rise to a certain *"Grothendieck Conjecture-type result"* in the "geometry of categories" [cf. [Mzk11], Theorem 1.12];

(ii) the interpretation of the theory of **Teichmüller mappings** [a sort of endomorphism — cf. (iii) below] between hyperbolic Riemann surfaces as a *"Grothendieck Conjecture-type result"* in the "geometry of categories" [cf. [Mzk11], Theorem 2.3];

(iii) the use of the *endomorphisms* constituted by **Frobenius liftings** — in the form of *p-adic Teichmüller theory* — to obtain the *absolute anabelian result* constituted by [Mzk6], Corollary 3.8;

(iv) the use of the *endomorphism* rings of **Lubin-Tate groups** to obtain the *absolute anabelian result* constituted by [Mzk15], Corollaries 3.8, 3.9.

The *main results* of the present paper — in which both themes (A) and (B) play a central role — are the following:

(1) In §1, we develop a *purely combinatorial approach* to the algebro-geometric notion of a *stable polycurve* [cf. [Mzk2], Definition 4.5]. This approach may

be thought of as being motivated by the purely combinatorial approach to the notion of a stable curve given in [Mzk13]. Moreover, in §1, we apply the theory of [Mzk13] to give, in effect, **"group-theoretic algorithms"** for reconstructing the *"abstract combinatorial analogue"* of the **geometry** of the various **divisors** — in the form of *inertia* and *decomposition* groups — associated to the canonical log structure of a stable polycurve [cf. Theorem 1.7]. These techniques, together with the theory of [MT], give rise to

> *various* **relative** *and* **absolute anabelian** *results concerning* **configuration spaces** *associated to hyperbolic curves*

[cf. Corollaries 1.10, 1.11]. Relative to the discussion above of "hidden endomorphisms", we observe that such configuration spaces may be thought of as representing a sort of *"tautological endomorphism/correspondence"* of the hyperbolic curve in question.

(2) In §2, we study the *absolute pro-Σ anabelian geometry* of hyperbolic curves over mixed-characteristic local fields, for Σ a set of primes of cardinality ≥ 2 that contains the residue characteristic of the base field. In particular, we show that the *condition*, on an isomorphism of arithmetic fundamental groups, *of preservation of decomposition groups of "most" closed points implies that the isomorphism arises from an isomorphism of schemes* — i.e., in a word,

*"***point-theoreticity** *implies* **geometricity***"*

[cf. Corollary 2.9]. This condition may be removed if one works with *"procurves"* obtained by removing from a proper curve some set of closed points which is *"p-adically dense in a Galois-compatible fashion"* [cf. Corollary 2.10]. The key technical result that underlies these anabelian results is a certain **"pro-*p* resolution of nonsingularities"** type result [cf. Lemma 2.6; Remark 2.6.1; Corollary 2.11] — i.e., a result reminiscent of the main [profinite] results of [Tama2]. This technical result allows one to apply the theory of **uniformly toral neighborhoods** developed in [Mzk15], §3. Relative to the discussion above of "hidden endomorphisms", this technical result is interesting [cf., e.g., (iii) above] in that one central step of the proof of the technical result is quite similar to the well-known classical argument that implies the **nonexistence of a Frobenius lifting** for stable curves over the ring of Witt vectors of a finite field [cf. Remark 2.6.2].

(3) In §3, we re-examine the theory of [Mzk8], §2, for reconstructing the decomposition groups of closed points from the point of view of the present series of developing *"group-theoretic algorithms"*. In particular, we observe that these group-theoretic algorithms allow one to use

> **Belyi maps** *and* **endomorphisms of elliptic curves** *to construct [not only decomposition groups of closed points, but also]* **"cuspidalizations"**

[i.e., the full arithmetic fundamental groups of the open subschemes obtained by removing various closed points — cf. the theory of [Mzk14]] associated to various types of closed points [cf. Corollaries 3.3, 3.4, 3.7, 3.8]. Relative to the discussion above of "hidden endomorphisms", the theory of *Belyi* and *elliptic cuspidalizations* given in §3 illustrates quite explicitly how *endomorphisms* [arising from Belyi maps or endomorphisms of elliptic curves] can give rise to group-theoretic reconstruction algorithms.

Finally, we *remark* that although the "algorithmic approach" to stating anabelian results is not carried out very explicitly in §1, §2 [by comparison to §3 or [Mzk15]], the translation into "algorithmic language" of the more traditional "Grothendieck Conjecture-type" statements of the main results of §1, §2 is quite routine. [Here, it should be noted that the results of §1 that depend on *"Uchida's theorem"* — i.e., Theorem 1.8, (ii); Corollary 1.11, (iv) — constitute a *notatable exception* to this "remark", an exception that will be discussed in more detail in [Mzk16] — cf., e.g., [Mzk16], Remark 1.9.5.] That is to say, this translation was not carried out *explicitly* by the author solely because of the *complexity* of the algorithms implicit in §1, §2, i.e., not as a result of any substantive mathematical obstacles.

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Section 0: Notations and Conventions

We shall continue to use the "Notations and Conventions" of [Mzk15], §0. In addition, we shall use the following notation and conventions:

Topological Groups:

Let G be a *topologically finitely generated, slim profinite group*. Thus, G admits a *basis* of *characteristic open subgroups*. Any such basis determines a *profinite topology* on the groups $\mathrm{Aut}(G)$, $\mathrm{Out}(G)$. If $\rho : H \to \mathrm{Out}(G)$ is any *continuous homomorphism* of profinite groups, then we denote by

$$G \overset{\mathrm{out}}{\rtimes} H$$

the *profinite group* obtained by pulling back the *natural exact sequence* of profinite groups $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$ via ρ. Thus, we have a *natural exact sequence* of profinite groups $1 \to G \to G \overset{\mathrm{out}}{\rtimes} H \to H \to 1$.

Semi-graphs:

Let Γ be a *connected semi-graph* [cf., e.g., [Mzk9], §1, for a review of the theory of semi-graphs]. We shall refer to the [possibly infinite] dimension over $\mathbb{Q}$ of the singular homology module $H_1(\Gamma, \mathbb{Q})$ as the *loop-rank* $\text{lp-rk}(\Gamma)$ of Γ. We shall say that Γ is *loop-ample* if, for any edge e of Γ, the semi-graph obtained from Γ by removing e remains connected. We shall say that Γ is *untangled* if every closed edge of Γ abuts to two *distinct* vertices [cf. [Mzk9], §1]. We shall say that Γ is *edge-paired* (respectively, *edge-even*) if Γ is untangled, and, moreover, for any two [not necessarily distinct!] vertices v, v' of Γ, the set of edges e of Γ such that e abuts to a vertex w of Γ if and only if $w \in \{v, v'\}$ is either empty or of cardinality ≥ 2 (respectively, empty or of even cardinality). [Thus, one verifies immediately that if Γ is *edge-paired* (respectively, *edge-even*), then it is *loop-ample* (respectively, *edge-paired*).] We shall refer to as a *simple path* in Γ any connected subgraph $\gamma \subseteq \Gamma$ such that the following conditions are satisfied: (a) γ is a *finite tree* that has at least one edge; (b) given any vertex v of γ, there exist at most two branches of edges of γ that abut to v. Thus, [one verifies easily that] a simple path γ has precisely two vertices v such that there exists precisely one branch of an edge of γ that abuts to v; we shall refer to these two vertices as the *terminal vertices* of the simple path γ. If γ, γ' are simple paths in Γ such that the terminal vertices of γ, γ' coincide, then we shall say that γ, γ' are *co-terminal*.

Log Schemes:

We shall often regard a *scheme* as a log scheme equipped the trivial log structure. Any *fiber product* of fs [i.e., fine, saturated] log schemes is to be taken in the *category of fs log schemes*. In particular, the underlying scheme of such a product is *finite* over, but *not necessarily isomorphic* to, the fiber product of the underlying schemes.

Curves:

We shall refer to a hyperbolic orbicurve X as *semi-elliptic* [i.e., "of type $(1,1)_\pm$" in the terminology of [Mzk12], §0] if there exists a finite étale double covering $Y \to X$, where Y is a once-punctured elliptic curve, and the covering is given by the stack-theoretic quotient of Y by the "action of ± 1" [i.e., relative to the group operation on the elliptic curve given by the canonical compactification of Y].

For $i = 1, 2$, let X_i be a hyperbolic orbicurve over a field k_i. Then we shall say that X_1, X_2 are *isogenous* [cf. [Mzk14], §0] if there exists a hyperbolic orbicurve X over a field k, together with finite étale morphisms $X \to X_i$, for $i = 1, 2$. Note that in this situation, the morphisms $X \to X_i$ induce *finite separable inclusions of fields* $k_i \hookrightarrow k$. [Indeed, this follows immediately from the easily verified fact that every subgroup $G \subseteq \Gamma(X, \mathcal{O}_X^\times)$ such that $G \bigcup \{0\}$ determines a *field* is necessarily contained in $k^\times$.]

We shall use the term *stable log curve* as it was defined in [Mzk9], §0. Let

$$X^{\log} \to S^{\log}$$

be a *stable log curve* over an fs log scheme $S^{\log}$, where $S = \mathrm{Spec}(k)$ for some field k; $\overline{k}$ a *separable closure* of k. Then we shall refer to as the *loop-rank* $\text{lp-rk}(X^{\log})$ [or $\text{lp-rk}(X)$] of $X^{\log}$ [or X] the loop-rank of the dual graph of $X^{\log} \times_k \overline{k}$ [or $X \times_k \overline{k}$]. We shall say that $X^{\log}$ [or X] is *loop-ample* (respectively, *untangled*; *edge-paired*; *edge-even*) if the *dual semi-graph with compact structure* [cf. [Mzk5], Appendix] of $X^{\log} \times_k \overline{k}$ is loop-ample (respectively, untangled; edge-paired; edge-even) [as a connected semi-graph]. We shall say that $X^{\log}$ [or X] is *sturdy* if the normalization of every irreducible component of X is of genus ≥ 2 [cf. [Mzk13], Remark 1.1.5].

Observe that for any prime number l invertible on S, there exist an fs log scheme $T^{\log}$ over $S^{\log}$, where $T = \mathrm{Spec}(k')$, for some finite separable extension k' of k, and a *connected Galois log admissible covering* $Y^{\log} \to X^{\log} \times_{S^{\log}} T^{\log}$ [cf. [Mzk1], §3] of degree a power of l such that $Y^{\log}$ is *sturdy* and *edge-paired* [hence, in particular, *untangled* and *loop-ample*]; if, moreover, $l = 2$, then one may also take $Y^{\log}$ to be *edge-even*. [Indeed, to verify this observation, we may assume that $k = \overline{k}$. Then note that any hyperbolic curve U over k admits a connected finite étale Galois covering $V \to U$ of degree a power of l such that V is of genus ≥ 2 and ramified with ramification index l^2 at each of the cusps of V — cf. the discussion at the end of the present §0. Thus, by gluing together such coverings at the nodes of X, one concludes that there exists a connected Galois log admissible covering $Z_1^{\log} \to X^{\log} \times_{S^{\log}} T^{\log}$ of degree a power of l which is totally ramified over every node of X with ramification index l^2 such that every irreducible component of Z_1 is of genus ≥ 2 — i.e., $Z_1^{\log}$ is *sturdy*. Next, observe that there exists a connected Galois log admissible covering $Z_2^{\log} \to Z_1^{\log}$ of degree a power of l that arises from a covering of the dual graph of Z_1 such that $Z_2^{\log}$ is *untangled* [and still sturdy]. Finally, observe that there exists a connected Galois log admissible covering $Z_3^{\log} \to Z_2^{\log}$ of degree a positive power of l which restricts to a connected finite étale covering over every irreducible component of Z_2 [hence is *unramified* at the nodes] such that $Z_3^{\log}$ is *edge-paired* [and still sturdy and untangled] for arbitrary l and *edge-even* when $l = 2$. Thus, we may take $Y^{\log} \stackrel{\text{def}}{=} Z_3^{\log}$.]

Observe that if X is *loop-ample*, then for every point $x \in X(k)$ which is *not a unique cusp* of X [i.e., either x is not a cusp or if x is a cusp, then it is not the unique cusp of X], the *evaluation map*

$$H^0(X, \omega_{X^{\log}/S^{\log}}) \to \omega_{X^{\log}/S^{\log}}|_x$$

is *surjective*. Indeed, by considering the long exact sequence associated to the short exact sequence $0 \to \omega_{X^{\log}/S^{\log}} \otimes_{\mathcal{O}_X} \mathcal{I}_x \to \omega_{X^{\log}/S^{\log}} \to \omega_{X^{\log}/S^{\log}}|_x \to 0$, where $\mathcal{I}_x \subseteq \mathcal{O}_X$ is the sheaf of ideals corresponding to x, one verifies immediately that it suffices to show that the surjection

$$\mathcal{H}_x \stackrel{\text{def}}{=} H^1(X, \omega_{X^{\log}/S^{\log}} \otimes_{\mathcal{O}_X} \mathcal{I}_x) \twoheadrightarrow \mathcal{H} \stackrel{\text{def}}{=} H^1(X, \omega_{X^{\log}/S^{\log}})$$

is *injective*. If x is a *node*, then the fact that X is *loop-ample* implies [by computing via *Serre duality*] that either $\dim_k(\mathcal{H}_x) = \dim_k(\mathcal{H}) = 1$ [if X has no cusps] or $\dim_k(\mathcal{H}_x) = \dim_k(\mathcal{H}) = 0$ [if X has cusps]. Thus, we may assume that x is *not a node*, so the surjection $\mathcal{H}_x \twoheadrightarrow \mathcal{H}$ is dual to the injection

$$\mathcal{M} \stackrel{\text{def}}{=} H^0(X, \mathcal{O}_X(-D)) \hookrightarrow \mathcal{M}_x \stackrel{\text{def}}{=} H^0(X, \mathcal{O}_X(x - D)) \quad (\subseteq \mathcal{N}_x \stackrel{\text{def}}{=} H^0(X, \mathcal{O}_X(x)))$$

— where we write $D \subseteq X$ for the *divisor of cusps* of X. If x is a *cusp*, then it follows that D is of degree ≥ 2, and one computes easily that $\dim_k(\mathcal{M}) = \dim_k(\mathcal{M}_x) = 0$. Thus, we may assume that x is *not a cusp*. Write C for the irreducible component of X containing x. Now suppose that the injection $\mathcal{M} \hookrightarrow \mathcal{M}_x$ is *not surjective*. Then it follows that $\dim_k(\mathcal{N}_x) = 2$, and that $\mathcal{N}_x$ determines a *basepoint-free linear system* on X. In particular, $\mathcal{N}_x$ determines a morphism $\phi : X \to \mathbb{P}^1_k$ that is *of degree* 1 on C — i.e., determines an *isomorphism* $C \xrightarrow{\sim} \mathbb{P}^1_k$ — and *constant* on the other irreducible components of X. Since X is *loop-ample*, it follows that the dual graph Γ of X either has *no edges* or admits a *loop* containing the vertex determined by C. On the other hand, the existence of such a loop contradicts the fact that ϕ determines an isomorphism $C \xrightarrow{\sim} \mathbb{P}^1_k$. Thus, we conclude that $X = C \cong \mathbb{P}^1_k$, so D is of degree ≥ 3. But this implies that $\mathcal{M}_x = 0$, a *contradiction*.

Finally, let U be a *hyperbolic curve* over an algebraically closed field k and l a prime number invertible in k. Suppose that the cardinality r of the set of cusps of U is ≥ 2, and, moreover, that, if $l = 2$, then r is *even*. Then *observe* that it follows immediately from the well-known structure of the maximal pro-l quotient of the abelianization of the étale fundamental group of U that

> for every power l^n of l, where n is a positive integer, there exists a *cyclic covering* $V \to U$ of degree l^n that is *totally ramified* over the cusps of U.

Indeed, this observation is an immediate consequence of the elementary fact that, in light of our assumptions on r, there always exist $r - 1$ integers prime to l whose sum is also prime to l. We shall often make use of the assumption that a stable log curve is *edge-paired* — or, when $l = 2$, *edge-even* — by applying the above *observation* to the various connected components of the complement of the cusps and nodes of the stable log curve.

Section 1: A Combinatorial Analogue of Stable Polycurves

In the present §1, we apply the theory of [Mzk13] to study a sort of *purely group-theoretic, combinatorial* analogue [cf. Definition 1.5 below] of the notion of a *stable polycurve* introduced in [Mzk2], Definition 4.5. This allows one to reconstruct the *"abstract combinatorial analogue"* of the *"geometry of log divisors"* [i.e., divisors associated to the log structure of a stable polycurve] of such a combinatorial object via group theory [cf. Theorem 1.7]. Finally, we apply the theory of [MT] to obtain various consequences of the theory of the present §1 [cf. Corollaries 1.10, 1.11] concerning the *absolute anabelian geometry of configuration spaces.*

We begin by recalling the discussion of [Mzk13], Example 2.5.

Example 1.1. Stable Log Curves over a Logarithmic Point (Revisited).

(i) Let k be a *separably closed field*; Σ a *nonempty set of prime numbers* invertible in k; $\mathbb{M} \subseteq \mathbb{Q}$ the monoid of positive rational numbers with denominators invertible in k; $S^{\log}$ (respectively, $T^{\log}$) the log scheme obtained by equipping $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$ (respectively, $T \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$) with the log structure determined by the chart $\mathbb{N} \ni 1 \mapsto 0 \in k$ (respectively, $\mathbb{M} \ni 1 \mapsto 0 \in k$); $T^{\log} \to S^{\log}$ the morphism determined by the natural inclusion $\mathbb{N} \hookrightarrow \mathbb{M}$;

$$X^{\log} \to S^{\log}$$

a *stable log curve* over $S^{\log}$. Thus, the profinite group $I_{S^{\log}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(T^{\log}/S^{\log})$ admits a natural isomorphism $I_{S^{\log}} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, k^{\times})$ and fits into an *natural exact sequence*

$$1 \to \Delta_{X^{\log}} \stackrel{\mathrm{def}}{=} \pi_1(X^{\log} \times_{S^{\log}} T^{\log}) \to \Pi_{X^{\log}} \stackrel{\mathrm{def}}{=} \pi_1(X^{\log}) \to I_{S^{\log}} \to 1$$

— where we write "$\pi_1(-)$" for the *"log fundamental group"* of the log scheme in parentheses [which amounts, in this case, to the fundamental group arising from the *admissible coverings* of $X^{\log}$], relative to an appropriate choice of basepoint [cf. [Ill] for a survey of the theory of log fundamental groups]. In particular, if we write $I^{\Sigma}_{S^{\log}}$ for the *maximal pro-*Σ *quotient* of $I_{S^{\log}}$, then as abstract profinite groups, $I^{\Sigma}_{S^{\log}} \cong \widehat{\mathbb{Z}}^{\Sigma}$, where we write $\widehat{\mathbb{Z}}^{\Sigma}$ for the *maximal pro-*Σ *quotient* of $\widehat{\mathbb{Z}}$.

(ii) On the other hand, $X^{\log}$ determines a *semi-graph of anabelioids* [cf. [Mzk9], Definition 2.1] *of pro-*Σ *PSC-type* [cf. [Mzk13], Definition 1.1, (i)], whose underlying semi-graph we denote by $\mathbb{G}$. Thus, for each vertex v [corresponding to an irreducible component of $X^{\log}$] (respectively, edge e [corresponding to a node or cusp of $X^{\log}$]) of $\mathbb{G}$, we have a connected anabelioid [i.e., a Galois category] $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$), and for each branch b of an edge e abutting to a vertex v, we are given a morphism of anabelioids $\mathcal{G}_e \to \mathcal{G}_v$. Then the *maximal pro-*Σ *completion* of $\Delta_{X^{\log}}$ may be *identified* with the *"PSC-fundamental group"* $\Pi_{\mathcal{G}}$ associated to $\mathcal{G}$. Also, we recall that $\Pi_{\mathcal{G}}$ is *slim* [cf., e.g., [Mzk13], Remark 1.1.3], and that the groups $\mathrm{Aut}(\mathcal{G})$, $\mathrm{Out}(\Pi_{\mathcal{G}})$ may be equipped with *profinite topologies* in such a way that the *natural morphism*

$$\mathrm{Aut}(\mathcal{G}) \to \mathrm{Out}(\Pi_{\mathcal{G}})$$

is a *continuous injection* [cf. the discussion at the beginning of [Mzk13], §2], which we shall use to *identify* $\mathrm{Aut}(\mathcal{G})$ with its image in $\mathrm{Out}(\Pi_{\mathcal{G}})$. In particular, we obtain a *natural continuous homomorphism* $I_{S^{\log}} \to \mathrm{Aut}(\mathcal{G})$. Moreover, it follows immediately from the well-known structure of *admissible coverings at nodes* [cf., e.g., [Mzk1], §3.23] that this homomorphism *factors* through $I^{\Sigma}_{S^{\log}}$, hence determines a *natural continuous homomorphism* $\rho_I : I^{\Sigma}_{S^{\log}} \to \mathrm{Aut}(\mathcal{G})$. Also, we recall that each *vertex* v (respectively, *edge* e) of $\mathbb{G}$ determines a(n) *verticial subgroup* $\Pi_v \subseteq \Pi_{\mathcal{G}}$ (respectively, *edge-like subgroup* $\Pi_e \subseteq \Pi_{\mathcal{G}}$), which is well-defined up to conjugation — cf. [Mzk13], Definition 1.1, (ii). Here, the edge-like subgroups Π_e may be either *nodal* or *cuspidal*, depending on whether e corresponds to a node or to a cusp. If an edge e corresponds to a node (respectively, cusp), then we shall simply say that e "is" a *node* (respectively, *cusp*).

(iii) Let e be a *node* of X. Write M_e for the *stalk of the characteristic sheaf* of the log scheme $X^{\log}$ at e; M_S for the stalk of the characteristic sheaf of the log scheme $S^{\log}$ at the tautological S-valued point of S. Thus, $M_S \cong \mathbb{N}$; we have a natural inclusion $M_S \hookrightarrow M_e$, with respect to which we shall often [by abuse of notation] identify M_S with its image in M_e. Write $\sigma \in M_e$ for the *unique generator* of [the image of] M_S. Then there exist elements $\xi, \eta \in M_e$ satisfying the relation

$$\xi + \eta = i_e \cdot \sigma$$

for some positive integer i_e, which we shall refer to as the *index* of the node e, such that M_e is generated by ξ, η, σ. Also, we shall write i^{Σ}_e for the largest positive integer j such that i_e/j is a product of primes $\notin \Sigma$ and refer to i^{Σ}_e as the Σ-*index* of the node e. One verifies easily that the set of elements $\{\xi, \eta\}$ of M_e may be *characterized intrisically* as the set of elements $\theta \in M_e \backslash M_S$ such that any relation $\theta = n \cdot \theta' + \theta''$ for n a positive integer, $\theta' \in M_e \backslash M_S$, $\theta'' \in M_S$ implies that $n = 1$, $\theta'' = 0$. In particular, i_e, i^{Σ}_e are *well-defined* and *depend only on the isomorphism class* of the pair consisting of the monoid M_e and the submonoid $\subseteq M_e$ given by the image of M_S.

Remark 1.1.1. Of course, in Example 1.1, it is *not necessary* to assume that k is *separably closed* [cf. [Mzk13], Example 2.5]. If k is not separably closed, then one must also contend with the action of the *absolute Galois group* of k. More generally, for the theory of the present §1, it is *not even necessary* to assume that an "additional profinite group" acting on $\mathcal{G}$ *arises "from scheme theory"*. It is this point of view that formed the motivation for Definition 1.2 below.

Definition 1.2. In the notation of Example 1.1:

(i) Let $\rho_H : H \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$) be a *continuous homomorphism* of profinite groups; suppose that $X^{\log}$ is *nonsingular* [i.e., has no nodes]. Then we shall refer to as a *[pro-Σ] PSC-extension* [i.e., "pointed stable curve extension"] any extension of profinite groups that is *isomorphic* — via an isomorphism which we shall refer to as the *"structure of [pro-Σ] PSC-extension"* — to an extension of the form

$$1 \to \Pi_{\mathcal{G}} \to \Pi_H \stackrel{\mathrm{def}}{=} (\Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} H) \to H \to 1$$

[cf. §0 for more on the notation "$\stackrel{\text{out}}{\rtimes}$"], which we shall refer to as the *PSC-extension associated to the construction data* $(X^{\log} \to S^{\log}, \Sigma, \mathcal{G}, \rho_H)$. In this situation, each [necessarily *cuspidal*] edge e of $\mathbb{G}$ determines [up to conjugation in $\Pi_{\mathcal{G}}$] a subgroup $\Pi_e \subseteq \Pi_{\mathcal{G}}$, whose *normalizer* $D_e \stackrel{\text{def}}{=} N_{\Pi_H}(\Pi_e)$ in Π_H we shall refer to as the *decomposition group* associated to the cusp e; we shall refer to $I_e \stackrel{\text{def}}{=} \Pi_e = D_e \bigcap \Pi_{\mathcal{G}} \subseteq D_e$ [cf. [Mzk13], Proposition 1.2, (ii)] as the *inertia group* associated to the cusp e. Finally, we shall apply the terminology applied to objects associated to $1 \to \Pi_{\mathcal{G}} \to \Pi_H \to H \to 1$ to the objects associated to an arbitrary PSC-extension via its "structure of PSC-extension" isomorphism.

(ii) Let $\rho_H : H \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$) be a *continuous homomorphism* of profinite groups; $\iota : I^{\Sigma}_{S^{\log}} \hookrightarrow H$ a *continuous injection* of profinite groups with *normal image* such that $\rho_H \circ \iota = \rho_I$. Suppose that $X^{\log}$ is *arbitrary* [i.e., X may be *singular* or *nonsingular*]. Then we shall refer to as a *[pro-Σ] DPSC-extension* [i.e., "degenerating pointed stable curve extension"] any extension of profinite groups that is *isomorphic* — via an isomorphism which we shall refer to as the *"structure of [pro-Σ] DPSC-extension"* — to an extension of the form

$$1 \to \Pi_{\mathcal{G}} \to \Pi_H \stackrel{\text{def}}{=} (\Pi_{\mathcal{G}} \stackrel{\text{out}}{\rtimes} H) \to H \to 1$$

— which we shall refer to as the *DPSC-extension associated to the construction data* $(X^{\log} \to S^{\log}, \Sigma, \mathcal{G}, \rho_H, \iota)$. In this situation, we shall refer to the image $I \subseteq H$ of ι as the *inertia subgroup* of H and to the extension

$$1 \to \Pi_{\mathcal{G}} \to \Pi_I \stackrel{\text{def}}{=} (\Pi_{\mathcal{G}} \stackrel{\text{out}}{\rtimes} I) \to I \to 1$$

[so $\Pi_I = \Pi_H \times_H I \subseteq \Pi_H$] as the *[pro-Σ] IPSC-extension* [i.e., "inertial pointed stable curve extension"] associated to the *construction data* $(X^{\log} \to S^{\log}, \Sigma, \mathcal{G}, \rho_H, \iota)$; each vertex v (respectively, edge e) of $\mathbb{G}$ determines [up to conjugation in $\Pi_{\mathcal{G}}$] a subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}}$ (respectively, $\Pi_e \subseteq \Pi_{\mathcal{G}}$), whose *normalizer*

$$D_v \stackrel{\text{def}}{=} N_{\Pi_H}(\Pi_v) \text{ (respectively, } D_e \stackrel{\text{def}}{=} N_{\Pi_H}(\Pi_e))$$

in Π_H we shall refer to as the *decomposition group* associated to v (respectively, e); for v *arbitrary* (respectively, e a *node*), we shall refer to the *centralizer*

$$I_v \stackrel{\text{def}}{=} Z_{\Pi_I}(\Pi_v) \subseteq D_v \text{ (respectively, } I_e \stackrel{\text{def}}{=} Z_{\Pi_I}(\Pi_e) \subseteq D_e)$$

as the *inertia group* associated to v (respectively, e). If e is a *cusp* of $\mathbb{G}$, then we shall refer to $I_e \stackrel{\text{def}}{=} \Pi_e = D_e \bigcap \Pi_{\mathcal{G}} \subseteq D_e$ [cf. [Mzk13], Proposition 1.2, (ii)] as the *inertia group* associated to the cusp e. Finally, we shall apply the terminology applied to objects associated to $1 \to \Pi_{\mathcal{G}} \to \Pi_H \to H \to 1$ to the objects associated to an arbitrary DPSC-extension via its "structure of DPSC-extension" isomorphism.

Remark 1.2.1. Note that in the situation of Definition 1.2, (i) (respectively, (ii)), any *open subgroup* of Π_H [equipped with the induced extension structure] admits a structure of [pro-Σ] *PSC-extension* (respectively, *DPSC-extension*) for appropriate construction data that may be derived from the original construction data. On the other hand, in the situation of Definition 1.2, (ii), given an *open subgroup* of Π_I [equipped with the induced extension structure], in order to endow this open subgroup with a structure of IPSC-extension, it may be necessary — even if, for instance, this open subgroup of Π_I *surjects* onto I — to *replace* the inertia subgroup I of H by some open subgroup of I [i.e., in effect, to replace the given open subgroup of Π_I with the intersection of this given open subgroup with the inverse image in Π_I of some open subgroup of I]. Such replacements may be regarded as a sort of abstract group-theoretic analogue of the operation of passing to a finite extension of a discretely valued field in order to achieve a situation in which a given hyperbolic curve over the original field has *stable reduction*.

Remark 1.2.2. Note that in the situation of Definition 1.2, (ii), the inertia subgroup $I \subseteq H$ is *not intrinsically determined* in the sense that any open subgroup of I may also serve as the inertia subgroup of H — cf. the *replacement operation* discussed in Remark 1.2.1.

Remark 1.2.3. Recall that for $l \in \Sigma$, one may construct directly from $\mathcal{G}$ a *pro-l cyclotomic character* $\chi_l : \mathrm{Aut}(\mathcal{G}) \to \mathbb{Z}_l^\times$ [cf. [Mzk13], Lemma 2.1]. In particular, any ρ_H as in Definition 1.2, (i), (ii), determines a *pro-l cyclotomic character* $\chi_l|_H : H \to \mathbb{Z}_l^\times$. The action ρ_H is called *l-cyclotomically full* [cf. [Mzk13], Definition 2.3, (ii)] if the image of $\chi_l|_H$ is *open*. We shall also apply this terminology "l-cyclotomically full" to the corresponding *PSC-, DPSC-extensions*. In fact, it follows immediately from the first portion of [Mzk13], Proposition 2.4, (iv), that the issue of whether or not ρ_H is l-cyclotomically full depends only on the *outer representation* $H \to \mathrm{Out}(\Pi_{\mathcal{G}})$ of H on $\Pi_{\mathcal{G}}$ determined by ρ_H.

Proposition 1.3. **(Basic Properties of Inertia and Decomposition Groups)** *In the notation of Definition 1.2, (ii):*

(i) If e is a **cusp** *of $\mathcal{G}$, then as abstract profinite groups, $I_e \cong \widehat{\mathbb{Z}}^\Sigma$.*

(ii) If e is a **node** *of $\mathcal{G}$, then we have a* **natural exact sequence** $1 \to \Pi_e \to I_e \to I \to 1$*; as abstract profinite groups, $I_e \cong \widehat{\mathbb{Z}}^\Sigma \times \widehat{\mathbb{Z}}^\Sigma$. If e abuts to vertices v, v', then [for appropriate choices of conjugates of the various inertia groups involved] we have inclusions $I_v, I_{v'} \subseteq I_e$, and the natural morphism $I_v \times I_{v'} \to I_e$ is an* **open injective homomorphism***, with image of index equal to i_e^Σ.*

(iii) If v is a **vertex** *of $\mathcal{G}$, then we have a* **natural isomorphism** $I_v \xrightarrow{\sim} I$*; $D_v \bigcap \Pi_I = I_v \times \Pi_v$; as abstract profinite groups, $I_v \cong \widehat{\mathbb{Z}}^\Sigma$. If e is a* **cusp** *that abuts to v, then [for appropriate choices of conjugates of the various inertia and decomposition groups involved] we have inclusions $I_e, I_v \subseteq D_e \bigcap \Pi_I$, and the natural morphism $I_e \times I_v \to D_e \bigcap \Pi_I$ is an* **isomorphism***; in particular, as abstract profinite groups, $D_e \bigcap \Pi_I \cong \widehat{\mathbb{Z}}^\Sigma \times \widehat{\mathbb{Z}}^\Sigma$, and we have a* **natural exact sequence** $1 \to I_e \to D_e \bigcap \Pi_I \to I \to 1$.

(iv) Let v, v' be **vertices** *of $\mathcal{G}$. If $D_v \bigcap D_{v'} \bigcap \Pi_I \neq \{1\}$, then one of the following three [mutually exclusive] properties holds: (1) $v = v'$; (2) v and v' are*

distinct, *but* **adjacent** *[i.e., there exists a node e that abuts to v, v']; (3) v and v' are distinct and non-adjacent, but there exists a vertex $v'' \neq v, v'$ of $\mathcal{G}$ such that v'' is* **adjacent** *to v and v'. Moreover, in the situation of (2), we have $I_v \bigcap I_{v'} = \{1\}$, [for appropriate choices of conjugates of the various inertia and decomposition groups involved] $D_v \bigcap D_{v'} \bigcap \Pi_I = I_e$; in the situation of (3), we have $\Pi_v \bigcap \Pi_{v'} = I_v \bigcap I_{v'} = I_v \bigcap I_{v''} = I_{v'} \bigcap I_{v''} = \{1\}$, [for appropriate choices of conjugates of the various inertia and decomposition groups involved] $D_v \bigcap D_{v'} \bigcap \Pi_I = I_{v''}$. In particular, $I_v \bigcap I_{v'} \neq \{1\}$ implies that $v = v'$.*

(v) Let v be a **vertex** *of $\mathcal{G}$. Then $D_v = C_{\Pi_H}(I_v) = N_{\Pi_H}(I_v)$ is* **commensurably terminal** *in Π_H; $D_v \bigcap \Pi_I = C_{\Pi_I}(I_v) = N_{\Pi_I}(I_v) = Z_{\Pi_I}(I_v)$ is* **commensurably terminal** *in Π_I; $D_v \bigcap \Pi_{\mathcal{G}} = \Pi_v$ is* **commensurably terminal** *in $\Pi_{\mathcal{G}}$.*

(vi) Let v be a **vertex** *of $\mathcal{G}$. Then the image of D_v in H is* **open***; on the other hand, if $\mathbb{G}$ has* **more than one vertex** *[i.e., the curve X is* **singular***], then D_v is* **not open** *in Π_H.*

(vii) Let e be an **edge** *of $\mathcal{G}$. Then $D_e = C_{\Pi_H}(\Pi_e) = N_{\Pi_H}(\Pi_e)$ is* **commensurably terminal** *in Π_H. If e is a* **node***, then $I_e = D_e \bigcap \Pi_I$.*

(viii) Let e, e' be **edges** *of $\mathcal{G}$. If $D_e \bigcap D_{e'} \bigcap \Pi_I \neq \{1\}$, then one of the following two [mutually exclusive] properties holds: (1) $e = e'$; (2) e and e' are* **distinct***, but abut to the* **same vertex** *v, and $D_e \bigcap D_{e'} \bigcap \Pi_{\mathcal{G}} = \{1\}$. Moreover, in the situation of (2), [for appropriate choices of conjugates of the various inertia and decomposition groups involved] we have $I_v = D_e \bigcap D_{e'} \bigcap \Pi_I$.*

(ix) Let e be an **edge** *of $\mathcal{G}$. Then the image of D_e in H is* **open***, but D_e is* **not open** *in Π_H.*

(x) Let $\tau_I : I \to \Pi_I$ be the [outer] homomorphism that arises [by functoriality!] from a **"log point"** *$\tau_S \in X^{\log}(S^{\log})$. Let us call τ_I* **non-verticial** *(respectively,* **non-edge-like***) if $\tau_I(I)$ is not contained in I_v (respectively, I_e) for any vertex v (respectively, edge e) of $\mathcal{G}$. Then if τ_I is non-verticial and non-edge-like, then the image of τ_S is the* **unique cusp** *e_τ of X such that [for an appropriate choice of conjugate of D_{e_τ}] $\tau_I(I) \subseteq D_{e_\tau}$. Now suppose that the image of τ_S is* **not** *a cusp. Then τ_I satisfies the condition $\tau_I(I) = I_{v_\tau}$ for some vertex v_τ of $\mathcal{G}$ [and an appropriate choice of conjugate of I_{v_τ}] if and only if the image of τ_S is a* **non-nodal point** *of the irreducible component of X corresponding to v_τ; τ_I is non-verticial and satisfies the condition $\tau_I(I) \subseteq I_{e_\tau}$ for some node e_τ of $\mathcal{G}$ [and an appropriate choice of conjugate of I_{e_τ}] if and only if the image of τ_S is the* **node** *of X corresponding to e_τ.*

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). Write ν ($\cong S$) for the closed subscheme of X determined by the node of X corresponding to e; $\nu^{\log}$ for the result of equipping ν with the log structure pulled back from $X^{\log}$. Thus, we obtain a natural [outer] homomorphism $\Pi_\nu \stackrel{\text{def}}{=} \pi_1(\nu^{\log}) \to \pi_1(X^{\log}) = \Pi_{X^{\log}} \twoheadrightarrow \Pi_I$. Now [in the notation of Example 1.1, (iii)] one computes easily [by considering the Galois groups of the various Kummer log étale coverings of $\nu^{\log}$] that we have *natural isomorphisms* $\Pi_\nu \xrightarrow{\sim} \mathrm{Hom}(M_e^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times)$,

$I_{S^{\log}} \xrightarrow{\sim} \mathrm{Hom}(M_S^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times)$ [where "gp" denotes the *groupification* of a monoid]. Moreover, if we write $\Pi_\nu \twoheadrightarrow \Pi_\nu^\Sigma$ for the *maximal pro-*Σ *quotient* of Π_ν, then one verifies immediately that the isomorphisms induced on maximal pro-Σ quotients by these natural isomorphisms are *compatible*, relative to the *surjection*

$$(\Pi_\nu^\Sigma \xrightarrow{\sim}\ ) \ \mathrm{Hom}(M_e^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times) \otimes \widehat{\mathbb{Z}}^\Sigma \twoheadrightarrow \mathrm{Hom}(M_S^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times) \otimes \widehat{\mathbb{Z}}^\Sigma \ (\ \xrightarrow{\sim} I_{S^{\log}}^\Sigma)$$

induced by the inclusion $M_S \hookrightarrow M_e$, with the morphism $\Pi_\nu^\Sigma \to I_{S^{\log}}^\Sigma$ induced by the composite morphism $\Pi_\nu \to \Pi_I \twoheadrightarrow I \xrightarrow{\sim} I_{S^{\log}}^\Sigma \cong \mathrm{Hom}(M_S^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times) \otimes \widehat{\mathbb{Z}}^\Sigma$. The kernel of this surjection $\Pi_\nu^\Sigma \twoheadrightarrow I_{S^{\log}}^\Sigma$ may be identified with the profinite group $\mathrm{Hom}(M_e^{\mathrm{gp}}/M_S^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times) \otimes \widehat{\mathbb{Z}}^\Sigma$, and one verifies immediately [from the definition of $\mathcal{G}$] that this kernel maps *isomorphically* onto $\Pi_e \subseteq \Pi_I$. In particular, it follows that we obtain an *injection* $\Pi_\nu^\Sigma \hookrightarrow \Pi_I$ whose image *contains* Π_e and *surjects onto* I. Since Π_ν^Σ is *abelian*, it follows that the image $\mathrm{Im}(\Pi_\nu^\Sigma)$ of this injection is contained in I_e; since $I_e \bigcap \Pi_{\mathcal{G}} = \Pi_e$ [cf. [Mzk13], Proposition 1.2, (ii)], we thus conclude that $\mathrm{Im}(\Pi_\nu^\Sigma) = I_e$. Now it follows immediately from the definitions that $I_v, I_{v'} \subseteq I_e$; moreover, one *computes* immediately that [in the notation of Example 1.1, (iii)] the subgroups $I_v, I_{v'} \subseteq I_e$ correspond to the subgroups of $\mathrm{Hom}(M_e^{\mathrm{gp}} \otimes \mathbb{Q}/\mathbb{Z}, k^\times) \otimes \widehat{\mathbb{Z}}^\Sigma$ consisting of homomorphisms that *vanish* on ξ, η, respectively. Now the various assertions contained in the statement of assertion (ii) follow immediately. This completes the proof of assertion (ii).

Next, we consider assertion (iii). Since Π_v is *slim* [cf., e.g., [Mzk13], Remark 1.1.3] and *commensurably terminal* in $\Pi_{\mathcal{G}}$ [cf. [Mzk13], Proposition 1.2, (ii)], it follows that $D_v \bigcap \Pi_{\mathcal{G}} = \Pi_v$ and $I_v \bigcap \Pi_{\mathcal{G}} = \{1\}$, so we obtain a natural injection $I_v \hookrightarrow I$. The fact that this injection is, in fact, *surjective* is immediate from the definitions when X is *smooth* over k and follows from the computation of "I_v" performed in the proof of assertion (ii) when X is *singular*. Next, let us observe that since I_v *commutes* [by definition!] with Π_v, we obtain a natural morphism $I_v \times \Pi_v \to D_v \bigcap \Pi_I$, which is both *injective* [since $I_v \bigcap \Pi_v = \{1\}$] and *surjective* [cf. the isomorphism $I_v \xrightarrow{\sim} I$; the fact that $D_v \bigcap \Pi_{\mathcal{G}} = \Pi_v$]. Now suppose that e is a *cusp* that abuts to v. Then [for appropriate choices of conjugates] it follows immediately from the definitions that we have inclusions $I_e, I_v \subseteq D_e \bigcap \Pi_I$, and that I_e *commutes* with I_v. Note, moreover, that $D_e \bigcap \Pi_{\mathcal{G}} = I_e$ [cf. [Mzk13], Proposition 1.2, (ii)]. Thus, the fact that the natural projection $I_v \to I$ is an *isomorphism* implies that we have a *natural exact sequence* $1 \to I_e \to D_e \bigcap \Pi_I \to I \to 1$, and that the natural morphism $I_e \times I_v \to D_e \bigcap \Pi_I$ is an *isomorphism*. This completes the proof of assertion (iii).

Next, we consider assertion (vii). Since $D_e \bigcap \Pi_{\mathcal{G}} = \Pi_e$ [cf. [Mzk13], Proposition 1.2, (ii)], it follows that $D_e \subseteq C_{\Pi_H}(D_e) \subseteq C_{\Pi_H}(\Pi_e)$; on the other hand, by [Mzk13], Proposition 1.2, (i), it follows that $C_{\Pi_H}(\Pi_e) = N_{\Pi_H}(\Pi_e)$ $(= D_e)$; thus, $D_e = C_{\Pi_H}(D_e) = C_{\Pi_H}(\Pi_e) = N_{\Pi_H}(\Pi_e)$, as desired. Now it remains only to consider the case where e is a *node*. In this case, since I_e is *abelian* [cf. assertion (ii)], it follows that $I_e \subseteq Z_{\Pi_I}(I_e) \subseteq C_{\Pi_I}(I_e) \subseteq C_{\Pi_I}(I_e \bigcap \Pi_{\mathcal{G}}) = C_{\Pi_I}(\Pi_e)$; thus, the fact that $D_e \bigcap \Pi_I = C_{\Pi_I}(\Pi_e) = C_{\Pi_I}(I_e) = I_e$ follows from the fact that I_e *surjects onto* I [cf. assertion (ii)], together with the *commensurable terminality* of Π_e in $\Pi_{\mathcal{G}}$ [cf. [Mzk13], Proposition 1.2, (ii)]. This completes the proof of assertion (vii).

Next, we consider assertion (iv). Suppose that (2) holds. Then it follows from assertions (ii), (iii) [and the definitions] that $I_v \bigcap I_{v'} = \{1\}$, $I_e = \Pi_e \cdot I_v =$

$\Pi_e \cdot I_{v'} \subseteq D_v \bigcap D_{v'} \bigcap \Pi_I \subseteq Z_{\Pi_I}(I_v \times I_{v'}) \subseteq C_{\Pi_I}(I_e)$. On the other hand, by assertion (vii), we have $C_{\Pi_I}(I_e) = I_e$. But this implies that $D_v \bigcap D_{v'} \bigcap \Pi_I = I_e$. Next, suppose that (3) holds, and that $\Pi_v \bigcap \Pi_{v'} = \{1\}$. Then it follows immediately from our discussion of the situation in which (2) holds [cf. also assertion (iii)] that $(\widehat{\mathbb{Z}}^\Sigma \cong)\ I_{v''} \subseteq D_v \bigcap D_{v'} \bigcap \Pi_I \hookrightarrow I$ [so $I_{v''} = D_v \bigcap D_{v'} \bigcap \Pi_I$], $I_v \bigcap I_{v'} \subseteq I_v \bigcap I_{v''} = I_{v'} \bigcap I_{v''} = \{1\}$. Thus, to complete the proof of assertion (iv), it suffices to verify — under the assumption that (1) and (2) are *false*! — that we have an equality $(D_v \bigcap D_{v'} \bigcap \Pi_I \supseteq)\ \Pi_v \bigcap \Pi_{v'} = \{1\}$, and that (3) *holds*.

Write C_v, $C_{v'}$ for the irreducible components of X corresponding to v, v'. Suppose that both (1) and (2) are *false*. Recall that $\Pi_{\mathcal{G}}$ [cf. [MT], Remark 1.2.2] and I $(\cong \widehat{\mathbb{Z}}^\Sigma)$, hence also Π_I, are *torsion-free*. Thus, $I_v \times \Pi_v = D_v \bigcap \Pi_I$ [cf. assertion (iii)] is *torsion-free*, so by replacing Π_H by an open subgroup of Π_H [cf. Remark 1.2.1], we may assume without loss of generality that $\mathcal{G}$ [i.e., $X^{\log}$] is *sturdy* [cf. §0], and that $\mathbb{G}$ is *edge-paired* [cf. §0]. Also, by projecting to the *maximal pro-l quotients*, for some $l \in \Sigma$, of suitable open subgroups [cf. Remark 1.2.1] of the various pro-Σ groups involved, one verifies immediately that we may assume without loss of generality [for the remainder of the proof of assertion (iv)] that $\Sigma = \{l\}$. When $l = 2$, we may also assume without loss of generality [by replacing Π_H by an open subgroup of Π_H] that $\mathbb{G}$ is *edge-even* [cf. §0].

Now I *claim* that $D_v \bigcap D_{v'} \bigcap \Pi_{\mathcal{G}} = \Pi_v \bigcap \Pi_{v'} = \{1\}$. Indeed, suppose that $\Pi_v \bigcap \Pi_{v'} \neq \{1\}$. Then one verifies immediately that there exist *log admissible coverings* [cf. [Mzk1], §3] $Y^{\log} \to X^{\log} \times_{S^{\log}} T^{\log}$, corresponding to open subgroups $J \subseteq \Pi_{\mathcal{G}}$, which are *split* over $C_{v'}$ [so $\Pi_{v'} \subseteq J$, $\Pi_v \bigcap \Pi_{v'} \subseteq \Pi_v \bigcap J$], but determine *arbitrarily small neighborhoods* $\Pi_v \bigcap J$ of the identity element in Π_v. [Here, we note that the existence of such coverings follows immediately from the fact that $X^{\log}$ is *edge-paired* for arbitrary l and *edge-even* when $l = 2$. That is to say, one starts by constructing the covering over C_v in such a way that the *ramification indices* at the nodes and cusps of C_v are *all equal*; one then extends the covering over the irreducible components of X *adjacent* to v [by applying the fact that $X^{\log}$ is *edge-paired* for arbitrary l and *edge-even* when $l = 2$ — cf. the discussion of §0] in such a way that the covering is *unramified* over the nodes of these irreducible components that do *not* abut to C_v; finally, one extends the covering to a *split* covering over the remaining portion of X [which includes $C_{v'}$!].] But the existence of such J implies that $\Pi_v \bigcap \Pi_{v'} = \{1\}$, a *contradiction*. This completes the proof of the *claim*. Thus, the natural projection $D_v \bigcap D_{v'} \bigcap \Pi_I \to I$ has *nontrivial open image* [since $\Sigma = \{l\}$], which we denote by $I_{v,v'} \subseteq I$. Moreover, to complete the proof of assertion (iv), it suffices to derive a *contradiction* under the assumption that (1), (2), and (3) are *false*. Thus, for the remainder of the proof of assertion (iv), we assume that (1), (2), and (3) are *false*.

Write $C_v^+ \subseteq X$ for the *union* of C_v and the irreducible components of X that are *adjacent* to C_v. We shall refer to a vertex of $\mathcal{G}$ as a C_v^+*-vertex* if it corresponds to an irreducible component of C_v^+. We shall say that a node e is a *bridge node* if it abuts both to a C_v^+-vertex and to a non-C_v^+-vertex. Thus, *no bridge node abuts to* v. Now let us write i_v for the *least common multiple* of the indices i_e of the bridge nodes e; i_v^Σ for the largest nonnegative power of l dividing i_v. Let $d \stackrel{\text{def}}{=} l \cdot i_v \cdot [I : I_{v,v'}]$; $d^\Sigma \stackrel{\text{def}}{=} l \cdot i_v^\Sigma \cdot [I : I_{v,v'}]$ [so d^Σ is the largest positive power of l dividing d]. Here, we observe that for any *open subgroup* $J_0 \subseteq \Pi_I$ such that

$D_v \bigcap J_0$ *surjects* onto I [cf. assertion (iii)], and $D_{v'} \bigcap \Pi_I \subseteq J_0$, it holds that

$$[I : I_{v,v'}] = [(D_v \bigcap J_0) : (\Pi_v \bigcap J_0) \cdot (D_v \bigcap D_{v'} \bigcap \Pi_I)]$$

[where we note that $D_v \bigcap D_{v'} \bigcap \Pi_I \subseteq D_v \bigcap J_0$]. Thus, it suffices to construct *open subgroups* $J \subseteq J_0 \subseteq \Pi_I$ such that $D_v \bigcap J_0$ *surjects* onto I, $\Pi_v \bigcap J_0 \subseteq J$, and $D_{v'} \bigcap \Pi_I \subseteq J$ [which implies that $(\Pi_v \bigcap J_0) \cdot (D_v \bigcap D_{v'} \bigcap \Pi_I) \subseteq D_v \bigcap J$], but $[D_v \bigcap J_0 : D_v \bigcap J] > [I : I_{v,v'}]$.

To this end, let us first observe that the *characteristic sheaf* of the log scheme $X^{\log}$ admits a *section* ζ over X satisfying the following properties:

(a) ζ *vanishes* on the open subscheme of X given by the complement of C_v^+ [hence, in particular, on $C_{v'}$];

(b) ζ *coincides* with $i_v \cdot \sigma \in M_S$ [cf. the notation of Example 1.1, (iii)] at the generic points of C_v^+;

(c) ζ *coincides* with either $(i_v/i_e) \cdot \xi \in M_e$ or $(i_v/i_e) \cdot \eta \in M_e$ [cf. the notation of Example 1.1, (iii)] at each bridge node e.

[Indeed, the existence of such a section ζ follows immediately from the discussion of Example 1.1, (iii), together with our definition of i_v.] Thus, by taking the inverse image of ζ in the monoid that defines the log structure of $X^{\log}$, we obtain a line bundle $\mathcal{L}$ on X. Let $Y \to X$ be a *finite étale cyclic covering* of order a positive power of l such that $\mathcal{L}|_Y$ has *degree divisible by* d^Σ on every irreducible component of Y, and $Y \to X$ restricts to a *connected* covering over every irreducible component of X that is $\neq C_{v'}$ [e.g., C_v], but *splits* over $C_{v'}$; $Y^{\log} \stackrel{\mathrm{def}}{=} X^{\log} \times_X Y$; $C_w \stackrel{\mathrm{def}}{=} C_v \times_X Y$; $C_w^+ \stackrel{\mathrm{def}}{=} C_v^+ \times_X Y$. [Note that the fact that such a covering exists follows immediately from our assumption that $\mathcal{G}$ is *sturdy*.] Now let

$$Z^{\log} \to Y^{\log}$$

be a *log étale cyclic covering of degree* d^Σ satisfying the following properties:

(d) $Z^{\log} \to Y^{\log}$ restricts to an *étale* covering of schemes over the complement of C_w^+ and *splits* over the irreducible components of Y that lie over $C_{v'}$ [cf. (a); the fact that (1), (2), and (3) are assumed to be *false*!];

(e) $Z^{\log} \to Y^{\log}$ is *ramified*, with ramification index d^Σ / i_v^Σ, over the generic points of C_w^+, but induces the *trivial extension* of the function field of C_w [cf. (b)];

(f) for each node f of Y that lies over a bridge node e of X, the restriction of $Z^{\log} \to Y^{\log}$ to the branch of f that does not abut to an irreducible component of C_w^+ is *ramified*, with ramification index $d^\Sigma \cdot i_e^\Sigma / i_v^\Sigma$ [cf. (c)].

Indeed, to construct such a covering $Z^{\log} \to Y^{\log}$, it suffices to construct a covering satisfying (d), (f) over the complement of C_w^+ [which is always possible, by the conditions imposed on Y, together with the fact that (1), (2), and (3) are assumed to be *false*], and then to *glue* this covering to a *suitable* [i.e., such that (e) is satisfied!] *Kummer log étale covering* of $(C_w^+)^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_Y C_w^+$ [by an fs log scheme!] obtained by *extracting a* d^Σ*-th root of* $\mathcal{L}|_{C_w^+}$ [cf. the *divisibility* condition on the degrees of $\mathcal{L}$ over the irreducible components of C_w^+]. [Here, we regard the $\mathbb{G}_\mathrm{m}$-torsor determined by $\mathcal{L}|_{C_w^+}$ as a *subsheaf of the monoid defining the log structure of* $(C_w^+)^{\log}$.] Now if we write $J_Z \subseteq J_Y \subseteq \Pi_I$ for the open subgroups defined by the coverings $Z^{\log} \to Y^{\log} \to X^{\log}$, then $D_v \bigcap J_Y$ *surjects* onto I; $D_{v'} \bigcap \Pi_I \subseteq J_Y$. On the other hand, $\Pi_v \bigcap J_Y \subseteq J_Z$ [cf. (e)] and $D_{v'} \bigcap \Pi_I \subseteq J_Z$ [cf. (d)], while [cf. (e)]

$$[D_v \bigcap J_Y : D_v \bigcap J_Z] = d^\Sigma / i_v^\Sigma > [I : I_{v,v'}]$$

[since $d^\Sigma = l \cdot i_v^\Sigma \cdot [I : I_{v,v'}]$]. Thus, it suffices to take $J_0 \stackrel{\mathrm{def}}{=} J_Y$, $J \stackrel{\mathrm{def}}{=} J_Z$. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that it follows from assertion (iv) [i.e., by applying assertion (iv) to various *open subgroups* of Π_H, Π_I — cf. also Remark 1.2.1] that if, for $\gamma \in \Pi_H$, $I_v \bigcap (\gamma \cdot I_v \cdot \gamma^{-1}) \neq \{1\}$, then $\Pi_v = \gamma \cdot \Pi_v \cdot \gamma^{-1}$. Thus, we conclude that $N_{\Pi_H}(I_v) \subseteq C_{\Pi_H}(I_v) \subseteq N_{\Pi_H}(\Pi_v) = D_v$. On the other hand, since [by definition] $I_v = Z_{\Pi_I}(\Pi_v)$, and I is *normal* in H, it follows that $D_v = N_{\Pi_H}(\Pi_v) \subseteq N_{\Pi_H}(I_v)$, so $N_{\Pi_H}(I_v) = C_{\Pi_H}(I_v) = D_v$, as desired. In particular, $D_v \bigcap \Pi_I = N_{\Pi_I}(I_v) = C_{\Pi_I}(I_v)$. Next, let us observe that $D_v \bigcap \Pi_\mathcal{G} = \Pi_v$ [cf. [Mzk13], Proposition 1.2, (ii)]. Thus, $D_v \subseteq C_{\Pi_H}(D_v) \subseteq C_{\Pi_H}(\Pi_v)$. Moreover, by [Mzk13], Proposition 1.2, (i), it follows that $C_{\Pi_H}(\Pi_v) = N_{\Pi_H}(\Pi_v) = D_v$; thus, we conclude that D_v (respectively, $D_v \bigcap \Pi_I$; $D_v \bigcap \Pi_\mathcal{G}$) is *commensurably terminal* in Π_H (respectively, Π_I; $\Pi_\mathcal{G}$). Finally, by assertion (iii), we have $D_v \bigcap \Pi_I = I_v \times \Pi_v \subseteq Z_{\Pi_I}(I_v) \subseteq N_{\Pi_I}(I_v) = D_v \bigcap \Pi_I$, so $D_v \bigcap \Pi_I = Z_{\Pi_I}(I_v)$, as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). The fact that the image of D_v in H is *open* follows immediately from the fact that since the semi-graph $\mathbb{G}$ is *finite*, some open subgroup of H necessarily *fixes* v. On the other hand, if $\mathbb{G}$ admits a vertex $v' \neq v$, then $\Pi_{v'} \bigcap \Pi_v$ is *not open* in $\Pi_{v'}$ [cf. [Mzk13], Proposition 1.2, (i)]; since $D_v \bigcap \Pi_\mathcal{G} = \Pi_v$ [cf. [Mzk13], Proposition 1.2, (ii)], this implies that D_v is *not open* in Π_H. This completes the proof of assertion (vi).

Next, we consider assertion (viii). First, we observe that if property (2) holds, then by assertions (ii), (iii), [for appropriate choices of conjugates] $(\widehat{\mathbb{Z}}^\Sigma \cong)\ I_v \subseteq D_e \bigcap D_{e'} \bigcap \Pi_I \hookrightarrow I$, so $I_v = D_e \bigcap D_{e'} \bigcap \Pi_I$. Thus, it suffices to verify that either (1) or (2) holds. Next, let us observe that, since, as observed above, Π_I, hence also $D_e \bigcap D_{e'} \bigcap \Pi_I$, is *torsion-free*, by projecting to the *maximal pro-l quotients*, for some $l \in \Sigma$, of suitable open subgroups [cf. Remark 1.2.1] of the various pro-Σ groups involved, one verifies immediately we may assume without loss of generality [for the remainder of the proof of assertion (viii)] that $\Sigma = \{l\}$. Now if $D_e \bigcap D_{e'} \bigcap \Pi_\mathcal{G} \neq \{1\}$, then [since $\Sigma = \{l\}$] $D_e \bigcap D_{e'} \bigcap \Pi_\mathcal{G}$ is *open* in $D_e \bigcap \Pi_\mathcal{G}$ $(\cong \widehat{\mathbb{Z}}^\Sigma)$, $D_{e'} \bigcap \Pi_\mathcal{G}$ $(\cong \widehat{\mathbb{Z}}^\Sigma)$ [cf. assertions (ii), (iii), (vii)], so we conclude from [Mzk13], Proposition 1.2, (i), that $e = e'$. Thus, to complete the proof of assertion (viii), it suffices to derive a *contradiction* under the further assumption that $D_e \bigcap D_{e'} \bigcap \Pi_\mathcal{G} = \{1\}$, and e and e' do *not* abut to a common vertex. Moreover, by replacing Π_H by an open subgroup of Π_H [cf. Remark 1.2.1], we may assume without loss of generality that $\mathcal{G}$ [i.e., $X^{\log}$] is *sturdy* [cf. §0], and that $\mathbb{G}$ is *edge-paired* [cf. §0] for arbitrary l and *edge-even* [cf. §0] when $l = 2$.

Now if, say, e is a *cusp* that abuts to a vertex v, then one verifies immediately that there exist *log étale cyclic coverings* $Y^{\log} \to X^{\log}$ of degree an *arbitrarily large power of* l which are *totally ramified* over e, but *unramified* over the *nodes* of X, as well as over the *cusps* of X that abut to vertices $\neq v$. [Indeed, the existence of such coverings follows immediately from the fact that $X^{\log}$ is *edge-paired* for arbitrary l and *edge-even* when $l = 2$ — cf. the discussion of §0.] In particular, such coverings are *unramified* over e', as well as over the generic point of the irreducible component of X corresponding to v, hence correspond to *open subgroups* $J \subseteq \Pi_I$ such that $D_{e'} \bigcap \Pi_I \subseteq J$ [so $D_e \bigcap D_{e'} \bigcap \Pi_I \subseteq J \bigcap D_e \bigcap \Pi_I$], and, moreover, J may be chosen so that the subgroup $J \bigcap D_e \bigcap \Pi_I \subseteq D_e \bigcap \Pi_I = I_e \times I_v$ [cf. assertion (iii)] forms an *arbitrarily small neighborhood* of I_v. Thus, we conclude that $D_e \bigcap D_{e'} \bigcap \Pi_I \subseteq I_v$. On the other hand, if e' is also a *cusp* that abuts to a vertex v', then [by symmetry] we conclude that $D_e \bigcap D_{e'} \bigcap \Pi_I \subseteq I_{v'}$, hence that $I_v \bigcap I_{v'} \neq \{1\}$. But, by assertion (iv), this implies that $v = v'$, a *contradiction.* Thus, we may assume that, say, e is a *node*, so $I_e = D_e \bigcap \Pi_I$ [cf. assertion (vii)].

Write v_1, v_2 for the two distinct vertices to which e abuts; C_1, C_2 for the irreducible components of X corresponding to v_1, v_2; $C \stackrel{\mathrm{def}}{=} C_1 \bigcup C_2 \subseteq X$; $U_C \subseteq C$ for the open subscheme obtained by removing the nodes that abut to vertices $\neq v_1, v_2$. Let us refer to the nodes and cusps of U_C as *inner*, to the nodes of X that were removed from C to obtain U_C as *bridge nodes*, and to the nodes and cusps of X which are neither inner nodes/cusps nor bridge nodes as *external.* [Thus, e is *inner*; e' is *external.*] Observe that the natural projection to I yields an inclusion $D_e \bigcap D_{e'} \bigcap \Pi_I \hookrightarrow I$ with *open image* [since $\Sigma = \{l\}$]; denote the image of this inclusion by I_C. Write i_C for the *least common multiple* of the indices i_f of the *bridge nodes* f; i_C^{Σ} for the largest nonnegative power of l dividing i_C. Let $d \stackrel{\mathrm{def}}{=} l \cdot i_C \cdot [I : I_C]$; $d^{\Sigma} \stackrel{\mathrm{def}}{=} l \cdot i_C^{\Sigma} \cdot [I : I_C]$ [so d^{Σ} is the largest positive power of l dividing d]. Here, we observe that

$$[I : I_C] = [I_e : \Pi_e \cdot (D_e \bigcap D_{e'} \bigcap \Pi_I)]$$

[cf. assertion (ii)]. Then it suffices to construct an *open subgroup* $J \subseteq \Pi_I$ such that $\Pi_e \subseteq J$ and $D_{e'} \bigcap \Pi_I \subseteq J$ [which implies that $\Pi_e \cdot (D_e \bigcap D_{e'} \bigcap \Pi_I) \subseteq I_e \bigcap J$], but $[I_e : I_e \bigcap J] > [I : I_C]$.

To this end, let us first observe that the *characteristic sheaf* of the log scheme $X^{\log}$ admits a *section* ζ over X satisfying the following properties:

(a) ζ *vanishes* on the open subscheme of X given by the complement of C;

(b) ζ *coincides* with $i_C \cdot \sigma \in M_S$ [cf. the notation of Example 1.1, (iii)] at the generic points of $C \stackrel{\mathrm{def}}{=} C_1 \bigcup C_2$;

(c) ζ *coincides* with either $(i_C/i_f) \cdot \xi \in M_f$ or $(i_C/i_f) \cdot \eta \in M_f$ [cf. the notation of Example 1.1, (iii), where we take "e" to be f] at each *bridge node* f.

[Indeed, the existence of such a section ζ follows immediately from the discussion of Example 1.1, (iii), together with our definition of i_C.] Thus, by taking the inverse image of ζ in the monoid that defines the log structure of $X^{\log}$, we obtain a line bundle $\mathcal{L}$ on X. Let $Y \to X$ be a *finite étale Galois covering* of order a positive power of l such that $\mathcal{L}|_Y$ has *degree divisible by* d^Σ on every irreducible component of Y, and $Y \to X$ restricts to a *connected* covering over every irreducible component of X; $Y^{\log} \stackrel{\mathrm{def}}{=} X^{\log} \times_X Y$. [Note that the fact that such a covering exists follows immediately from our assumption that $\mathcal{G}$ is *sturdy*.] Write C_1^Y, C_2^Y for the irreducible components of Y lying over C_1, C_2, respectively; we shall apply the terms "internal", "external", and "bridge" to nodes/cusps of Y that lie over such nodes/cusps of X. Now let

$$Z^{\log} \to Y^{\log}$$

be a *log étale cyclic covering of degree* d^Σ satisfying the following properties:

(d) $Z^{\log} \to Y^{\log}$ restricts to an *étale* covering of schemes over the complement of $C^Y \stackrel{\mathrm{def}}{=} C_1^Y \bigcup C_2^Y$ [cf. (a)], hence, in particular, over the *external nodes/cusps* of Y;

(e) $Z^{\log} \to Y^{\log}$ is *ramified*, with ramification index d^Σ/i_C^Σ, over the generic points of C_1^Y, C_2^Y, and, at each *internal node* of Y lying over e, determines a covering corresponding to an open subgroup of I_e that contains Π_e [cf. (b)];

(f) for each *bridge node* f of Y, the restriction of $Z^{\log} \to Y^{\log}$ to the branch of f that does not abut to C^Y is *ramified*, with ramification index $d^\Sigma \cdot i_f^\Sigma / i_C^\Sigma$ [cf. (c)].

Indeed, to construct such a covering $Z^{\log} \to Y^{\log}$, it suffices to construct a covering satisfying (d), (f) over the complement of C^Y [which is always possible, by the conditions imposed on Y], and then to *glue* this covering to a *suitable* [i.e., such that (e) is satisfied!] *Kummer log étale covering* of $(C^Y)^{\log} \stackrel{\mathrm{def}}{=} Y^{\log} \times_Y C^Y$ [by an fs log scheme!] obtained by *extracting a* d^Σ*-th root of* $\mathcal{L}|_{C^Y}$ [cf. the *divisibility* condition on the degrees of $\mathcal{L}|_{C_1^Y}$, $\mathcal{L}|_{C_2^Y}$]. [Here, we regard the $\mathbb{G}_\mathrm{m}$-torsor determined by $\mathcal{L}|_{C^Y}$ as a *subsheaf of the monoid defining the log structure of* $(C^Y)^{\log}$.] Now if we write $J_Z \subseteq J_Y \subseteq \Pi_I$ for the open subgroups defined by the coverings $Z^{\log} \to Y^{\log} \to X^{\log}$, then $I_e, D_{e'} \bigcap \Pi_I \subseteq J_Y$. On the other hand, $\Pi_e \subseteq J_Z$ [cf. (e)] and $D_{e'} \bigcap \Pi_I \subseteq J_Z$ [cf. (d)], while [cf. (e)]

$$[I_e : I_e \bigcap J_Z] = d^\Sigma / i_C^\Sigma > [I : I_C]$$

[since $d^\Sigma = l \cdot i_C^\Sigma \cdot [I : I_C]$]. Thus, it suffices to take $J \stackrel{\mathrm{def}}{=} J_Z$. This completes the proof of assertion (viii).

Next, we consider assertion (ix). The fact that the image of D_e in H is *open* follows immediately from the fact that since the semi-graph $\mathbb{G}$ is *finite*, some open subgroup of H necessarily *fixes* e. On the other hand, since $D_e \bigcap \Pi_{\mathcal{G}} = N_{\Pi_{\mathcal{G}}}(\Pi_e) = \Pi_e$ [cf. [Mzk13], Proposition 1.2, (ii)] is *abelian*, hence *not open* in the *slim*, *nontrivial* profinite group $\Pi_{\mathcal{G}}$, it follows that D_e is *not open* in Π_H. This completes the proof of assertion (ix).

Finally, we consider assertion (x). First, let us observe that an easy computation reveals that if the image of τ_S is a *non-nodal, non-cuspidal point* of the irreducible component of X corresponding to a vertex v_τ of $\mathcal{G}$, then $\tau_I(I) = I_{v_\tau}$. Next, let us suppose that the image of τ_S is the *node* of X corresponding to some node e_τ of $\mathcal{G}$. Then an easy computation [cf. the computations performed in the proof of assertion (ii)] reveals that $\tau_I(I) \subseteq I_{e_\tau}$, but that $\tau_I(I)$ is *not* contained in $I_{v'}$ for any vertex v' to which e_τ abuts. If, moreover, $\tau_I(I) \subseteq I_v$ for some vertex v to which e_τ does *not* abut, then [since the very existence of the node e_τ implies that X is *singular*] there exists a node $e \neq e_\tau$ that abuts to v, so $\tau_I(I) \subseteq I_v \subseteq I_e$ [cf. assertion (ii)]; but this implies that $\tau_I(I) \subseteq I_e \bigcap I_{e_\tau}$, so, by assertion (viii), it follows that $\tau_I(I) \subseteq I_{v'}$ for some vertex to which both e and e_τ abut — a contradiction. Thus, in summary, we conclude that in this case, τ_I is *non-verticial*.

Now suppose that τ_I is *non-verticial* and *non-edge-like*. Then the observations of the preceding paragraph imply that the image of τ_S is a *cusp* of X. Write e_τ for the corresponding cusp of $\mathcal{G}$. Thus, one verifies immediately that $\tau_I(I) \subseteq D_{e_\tau}$. The *uniqueness* of e_τ then follows from assertion (viii) [and the fact that τ_I is *non-verticial*]. Thus, for the remainder of the proof of assertion (x), we may assume that the image of τ_S is *not* a cusp. Now the remainder of assertion (x) follows formally, in light of what of we have done so far, from assertions (iv), (viii). This completes the proof of assertion (x). ◯

Corollary 1.4. (Graphicity of Isomorphisms of (D)PSC-Extensions) *Let l be a prime number. For $i = 1, 2$, let $1 \to \Pi_{\mathcal{G}_i} \to \Pi_{H_i} \to H_i \to 1$ be an* ***l*-cyclotomically full** *[cf. Remark 1.2.3]* **DPSC-extension** *(respectively,* **PSC-extension***), associated to* **construction data** *$(X_i^{\log} \to S_i^{\log}, \Sigma_i, \mathcal{G}_i, \rho_{H_i}, \iota_i)$ (respectively, $(X_i^{\log} \to S_i^{\log}, \Sigma_i, \mathcal{G}_i, \rho_{H_i})$) such that $l \in \Sigma_i$; in the non-resp'd case, write $I_i \subseteq H_i$ for the* **inertia subgroup***. Let*

$$\phi_H : H_1 \xrightarrow{\sim} H_2; \quad \phi_\Pi : \Pi_{\mathcal{G}_1} \xrightarrow{\sim} \Pi_{\mathcal{G}_2}$$

be **compatible** *[i.e., with the respective outer actions of H_i on $\Pi_{\mathcal{G}_i}$]* **isomorphisms of profinite groups***; in the non-resp'd case, suppose further that $\phi_H(I_1) = I_2$. Then $\Sigma_1 = \Sigma_2$; ϕ_Π is* **graphic** *[cf. [Mzk13], Definition 1.4, (i)], i.e., arises from an* **isomorphism of semi-graphs of anabelioids** *$\mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$.*

Proof. This follows immediately from [Mzk13], Corollary 2.7, (i), (iii). Here, as in the proof of [Mzk13], Corollary 2.8, we first apply [Mzk13], Corollary 2.7, (i) [which suffices to complete the proof of Corollary 1.4 in the resp'd case and allows one to reduce to the *noncuspidal* case in the non-resp'd case], then apply [Mzk13], Corollary 2.7, (iii), to the *compactifications* of corresponding sturdy finite étale coverings of the $\mathcal{G}_i$. ◯

We are now ready to define a *purely group-theoretic, combinatorial* analogue of the notion of a *stable polycurve* given in [Mzk2], Definition 4.5.

Definition 1.5. We shall refer to an extension of profinite groups as a *PPSC-extension* [i.e., "poly-PSC-extension"] if, for some positive integer n and some nonempty set of primes Σ, it admits a "structure of pro-Σ PPSC-extension of dimension n". Here, for n a positive integer, Σ a nonempty set of primes, and

$$1 \to \Delta \to \Pi \to H \to 1$$

an extension of profinite groups, we define the notion of a *structure of pro-*Σ *PPSC-extension of dimension* n as follows [*inductively* on n]:

(i) A *structure of pro-*Σ *PPSC-extension of dimension* 1 on the extension $1 \to \Delta \to \Pi \to H \to 1$ is defined to be a structure of pro-Σ PSC-extension. Suppose that the extension $1 \to \Delta \to \Pi \to H \to 1$ is equipped with a structure of pro-Σ PPSC-extension of dimension 1. Thus, we have an associated *semi-graph of anabelioids* $\mathcal{G}$, together with a *continuous action* of H on $\mathcal{G}$, and a compatible isomorphism $\Delta \xrightarrow{\sim} \Pi_{\mathcal{G}}$. We define the *[horizontal] divisors* of this PPSC-extension to be the *cusps* of the PSC-extension $1 \to \Delta \to \Pi \to H \to 1$. Thus, each divisor c of the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ has associated *inertia* and *decomposition groups* $I_c \subseteq D_c \subseteq \Pi$ [cf. Definition 1.2, (i)]. Moreover, by [Mzk13], Proposition 1.2, (i), a divisor is *completely determined* by [the conjugacy class of] its *inertia group*, as well as by [the conjugacy class of] its *decomposition group*. Finally, we shall refer to the extension $1 \to \Delta \to \Pi \to H \to 1$ [itself] as the *fiber extension* associated to the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ of dimension 1.

(ii) A *structure of pro-*Σ *PPSC-extension of dimension* $n + 1$ on the extension $1 \to \Delta \to \Pi \to H \to 1$ is defined to be a collection of data as follows:

(a) a *quotient* $\Pi \twoheadrightarrow \Pi^*$ such that $\Delta^\dagger \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi \twoheadrightarrow \Pi^*) \subseteq \Delta$; thus, the image $\Delta^* \subseteq \Pi^*$ of Δ in Π^* determines an extension

$$1 \to \Delta^* \to \Pi^* \to H \to 1$$

— which we shall refer to as the *associated base extension*; the subgroup $\Delta^\dagger \subseteq \Pi$ determines an extension

$$1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1$$

— which we shall refer to as the *associated fiber extension*;

(b) a *structure of pro-*Σ *PPSC-extension of dimension* n on the *base extension* $1 \to \Delta^* \to \Pi^* \to H \to 1$;

(c) a *structure of pro-*Σ *PPSC-extension of dimension* 1 on the *fiber extension* $1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1$;

(d) for each *base divisor* [i.e., divisor of the base extension] c^*, a structure of *DPSC-extension* on the extension

$$1 \to \Delta^\dagger \to \Pi_{c^*} \stackrel{\mathrm{def}}{=} \Pi \times_{\Pi^*} D_{c^*} \to D_{c^*} \to 1$$

— which we shall refer to as the *extension at* c^* — which is *compatible* with the PSC-extension structure on the fiber extension [cf. (c)], in the sense that both structures yield the *same cuspidal inertia subgroups* $\subseteq \Delta^\dagger$; also, we require that the *inertia subgroup* of D_{c^*} [i.e., that arises from this structure of DPSC-extension] be equal to I_{c^*}.

In this situation, we shall refer to Σ as the *fiber prime set* of the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$; we shall refer to as a *divisor* of the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ any element of the union of the set of *cusps* — which we shall refer to as *horizontal* divisors — of the PSC-extension $1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1$ and, for each base divisor c^*, the set of *vertices* of the DPSC-extension $1 \to \Delta^\dagger \to \Pi_{c^*} \to D_{c^*} \to 1$ — which we shall refer to as *vertical divisors [lying over c^*]*. Thus, each divisor c of the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ has associated *inertia* and *decomposition groups* $I_c \subseteq D_c \subseteq \Pi$. In particular, whenever c is *vertical* and lies over a base divisor c^*, we have $I_c \subseteq D_c \subseteq \Pi_{c^*}$.

Remark 1.5.1. Thus, [the collection of *fiber extensions* arising from] any structure of *PPSC-extension of dimension n* on an extension $1 \to \Delta \to \Pi \to H \to 1$ determine two *compatible sequences of surjections*

$$\Delta_n \stackrel{\mathrm{def}}{=} \Delta \twoheadrightarrow \Delta_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow \Delta_1 \twoheadrightarrow \Delta_0 \stackrel{\mathrm{def}}{=} \{1\}$$
$$\Pi_n \stackrel{\mathrm{def}}{=} \Pi \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow \Pi_1 \twoheadrightarrow \Pi_0 \stackrel{\mathrm{def}}{=} H$$

such that each [extension determined by a] surjection $\Pi_m \twoheadrightarrow \Pi_{m-1}$, for $m = 1, \ldots, n$, is a *fiber extension* [hence equipped with a structure of *PSC-extension*]; $\Delta_m = \mathrm{Ker}(\Pi_m \twoheadrightarrow H)$. If $c = c_n$ is a *divisor* of [the extension determined by] $\Pi = \Pi_n$, then [cf. Definition 1.5, (ii)] there exists a uniquely determined *sequence of divisors*

$$c_n \mapsto c_{n-1} \mapsto \ldots \mapsto c_{n_c-1} \mapsto c_{n_c}$$

— where $n_c \leq n$ is a positive integer; for $m = n_c, \ldots, n$, c_m is a divisor of Π_m; c_{n_c} is a *horizontal* divisor; the notation "$\mapsto$" denotes the relation of *"lying over"* [so c_{m+1} is a *vertical* divisor that lies over c_m, for $n_c \leq m < n$] — together with *sequences of [conjugacies classes of] inertia and decomposition groups*

$$I_{c_n} \to I_{c_{n-1}} \to \ldots \to I_{c_{n_c-1}} \to I_{c_{n_c}}$$
$$D_{c_n} \to D_{c_{n-1}} \to \ldots \to D_{c_{n_c-1}} \to D_{c_{n_c}}$$

[i.e., for $n_c \leq m < n$, $I_{c_{m+1}} \subseteq \Pi_{m+1}$ maps into $I_{c_m} \subseteq \Pi_m$, and $D_{c_{m+1}} \subseteq \Pi_{m+1}$ maps into $D_{c_m} \subseteq \Pi_m$].

Remark 1.5.2. Let $1 \to \Delta \to \Pi \to H \to 1$ be a *PPSC-extension* of dimension n [where n is a positive integer]. Then one verifies immediately that if $\Pi_\bullet \subseteq \Pi$ is any *open subgroup* of Π, then there exists an *open subgroup* $\Pi_{\bullet\bullet}$ of $\Pi_\bullet$ that [when equipped with the induced extension structure] admits a structure of *PPSC-extension* of dimension n — cf. Remark 1.2.1. Here, we note that one must, in general, pass to *"some open subgroup"* $\Pi_{\bullet\bullet}$ of $\Pi_\bullet$ in order to achieve a situation in which all of the fiber [PSC-]extensions have *"stable reduction"* [cf. Remark 1.2.1; Definition 1.5, (ii), (d)].

Remark 1.5.3. For l a prime number, we shall say that a PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ of dimension n is *l-cyclotomically full* if each of its n associated fiber extensions [cf. Remark 1.5.1] is *l-cyclotomically full* as a PSC-extension [cf. Remark 1.2.3]. Thus, it follows immediately from the final portion of Remark 1.2.3 that the issue of whether or not the PPSC-extension $1 \to \Delta \to \Pi \to H \to 1$ is l-cyclotomically full depends only on the sequence of surjections of profinite groups $\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow \Pi_1 \twoheadrightarrow \Pi_0$ [cf. Remark 1.5.1].

Remark 1.5.4. Let k be a *field*; $\overline{k}$ a separable closure of k; $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$; $S \stackrel{\text{def}}{=} \text{Spec}(k)$;

$$Z^{\log} \to S$$

the log scheme determined by a *stable polycurve* over S — i.e., $Z^{\log}$ admits a *successive fibration* by generically smooth *stable log curves* [cf. [Mzk2], Definition 4.5, for more details]; $U_Z \subseteq Z$ the *interior* of $Z^{\log}$; $D_Z \stackrel{\text{def}}{=} Z \backslash U_Z$ [with the reduced induced structure]; n the dimension of the scheme Z;

$$1 \to \Delta_Z \stackrel{\text{def}}{=} \pi_1(U_Z \times_k \overline{k}) \to \Pi_Z \stackrel{\text{def}}{=} \pi_1(U_Z) \to G_k \to 1$$

the exact sequence of *étale fundamental groups* [well-defined up to inner automorphism] associated to the structure morphism $U_Z \to S$. Then by repeated application of the discussion of Example 1.1 to the *fibers* of the successive fibration [mentioned above] of $Z^{\log}$ by stable log curves, one verifies immediately that:

(i) If k is of *characteristic zero*, then the structure of *stable polycurve* on $Z^{\log}$ determines a structure of *profinite PPSC-extension* of dimension n on the extension $1 \to \Delta_Z \to \Pi_Z \to G_k \to 1$.

Moreover, one verifies immediately that:

(ii) In the situation of (i), the *divisors of* Π_Z [in the sense of Definition 1.5] are in *natural bijective correspondence* with the *irreducible divisors* of D_Z in a fashion that is *compatible* with the *inertia* and *decomposition groups* of divisors of Π_Z [in the sense of Definition 1.5] and of irreducible divisors of D_Z [in the usual sense].

Finally, whether or not k is of characteristic zero, depending on the structure of $Z^{\log}$ [cf., e.g., Corollary 1.10 below], various *quotients* of the extension $1 \to \Delta_Z \to \Pi_Z \to G_k \to 1$ may be equipped with a structure of *pro-Σ PPSC-extension* [induced by the structure of stable polycurve on Z], for various nonempty sets of prime numbers Σ that are *not* equal to the set of all prime numbers; a similar observation to (ii) concerning a *natural bijective correspondence of "divisors"* then applies to such quotients. When considering such quotients $1 \to \Delta \to \Pi \to H \to 1$ of the extension $1 \to \Delta_Z \to \Pi_Z \to G_k \to 1$, it is useful to observe that the *slimness* of Δ [cf. Proposition 1.6, (i), below] implies that such a quotient $\Pi_Z \twoheadrightarrow \Pi$ is *completely determined* by the induced quotients $\Delta_Z \twoheadrightarrow \Delta$, $G_k \twoheadrightarrow H$ [cf. the discussion of the notation "$\stackrel{\text{out}}{\rtimes}$" in §0]; we shall refer to such a quotient $1 \to \Delta \to \Pi \to H \to 1$ as a *PPSC-extension arising from* $Z^{\log}$, $\widetilde{k}$ — where we write $\widetilde{k} \subseteq \overline{k}$ for the subfield fixed by $\mathrm{Ker}(G_k \twoheadrightarrow H)$.

Proposition 1.6. (Basic Properties of PPSC-Extensions) *Let*

$$1 \to \Delta \to \Pi \to H \to 1$$

be a pro-Σ **PPSC-extension** *of dimension* n *[where* n *is a positive integer];* $1 \to \Delta^\dagger \to \Pi \to \Pi^* \to 1$ *the associated* **fiber extension***;* c, c' **divisors** *of* Π. *Then:*

(i) Δ *is* **slim**. *In particular, if* H *is* **slim***, then so is* Π.

(ii) D_c *is* **commensurably terminal** *in* Π.

(iii) We have: $C_\Pi(I_c) = D_c$. *As abstract profinite groups,* $I_c \cong \widehat{\mathbb{Z}}^\Sigma$.

(iv) D_c *is* **not open** *in* Π. *The divisor* c *is* **horizontal** *if and only if* D_c *projects to an* **open** *subgroup of* Π^*. *If* c *is* **vertical** *and lies over a* **base divisor** c^**, then* D_c *projects onto an* **open** *subgroup of* D_{c^*}.

(v) If $D_c \bigcap D_{c'}$ *is* **open** *in* D_c, $D_{c'}$*, then* $c = c'$. *In particular, a divisor of* Π *is* **completely determined** *by its associated* **decomposition group***.*

(vi) If $I_c \bigcap I_{c'}$ *is* **open** *in* I_c, $I_{c'}$*, then* $c = c'$. *In particular, a divisor of* Π *is* **completely determined** *by its associated* **inertia group***.*

Proof. Assertion (i) follows immediately from the *"slimness of* $\Pi_\mathcal{G}$*"* discussed in Example 1.1, (ii) [cf. Definition 1.5, (i); Definition 1.5, (ii), (c)]. Next, we consider assertion (ii). We apply induction on n. If c is *horizontal*, then assertion (ii) follows from [the argument applied in] Proposition 1.3, (vii) [cf. also Definition 1.5, (ii), (c)]. If c is *vertical*, then c lies over some *base divisor* c^*, and we are in the situation of Definition 1.5, (ii), (d). By Proposition 1.3, (vi), it follows that D_c *surjects* onto some *open* subgroup of D_{c^*}, hence that $C_\Pi(D_c)$ maps into $C_{\Pi^*}(D_{c^*})$; by the induction hypothesis, $C_{\Pi^*}(D_{c^*}) = D_{c^*}$, so $C_\Pi(D_c) \subseteq \Pi_{c^*}$. Thus, the fact that $C_\Pi(D_c) = D_c$ follows from Proposition 1.3, (v). This completes the proof of assertion (ii).

Next, we consider assertion (iii). Again we apply induction on n. If c is *horizontal*, then assertion (iii) follows from [the argument applied in] Proposition 1.3, (i), (vii). If c is *vertical*, then c lies over some *base divisor* c^*, and we are in the situation of Definition 1.5, (ii), (d). By Proposition 1.3, (iii) [cf. Definition 1.5, (ii), (d)], we have isomorphisms $I_c \xrightarrow{\sim} I_{c^*} \cong \widehat{\mathbb{Z}}^\Sigma$. In particular, $C_\Pi(I_c)$ maps into $C_{\Pi^*}(I_{c^*})$; by the induction hypothesis, $C_{\Pi^*}(I_{c^*}) = D_{c^*}$. Thus, $C_\Pi(I_c) \subseteq \Pi_{c^*}$, so the fact that $C_\Pi(I_c) = D_c$ follows from Proposition 1.3, (v). This completes the proof of assertion (iii).

Next, we consider assertion (iv). Again we apply induction on n. If c is *horizontal*, then by [the argument applied in] Proposition 1.3, (ix), D_c is *not open* in Π, but D_c projects to an *open* subgroup of Π^*. If c is *vertical*, then c lies over some *base divisor* c^*, and we are in the situation of Definition 1.5, (ii), (d); $D_c \subseteq \Pi_{c^*}$. By Proposition 1.3, (vi), D_c projects onto an *open* subgroup of D_{c^*}. By the induction hypothesis, D_{c^*} is *not open* in Π^*, so Π_{c^*} is *not open* in Π; thus, D_c is *not open* in Π, and its image in Π^* is *not open* in Π^*. This completes the proof of assertion (iv).

Next, we consider assertion (v). Again we apply induction on n. By assertion (iv), c is *horizontal* if and only if c' is. If c, c' are *horizontal*, then the fact that $c = c'$ follows from [Mzk13], Proposition 1.2, (i), (ii). Thus, we may suppose that c, c' are *vertical* and lie over respective *base divisors* c^*, $(c')^*$. By assertion (iv), it follows that $D_{c^*} \bigcap D_{(c')^*}$ is *open* in D_{c^*}, $D_{(c')^*}$; by the induction hypothesis, this implies that $c^* = (c')^*$. Thus, by intersecting with "$\Pi_{\mathcal{G}}$" [cf. Proposition 1.3, (v)] and applying [Mzk13], Proposition 1.2, (i), we conclude that $c = c'$. This completes the proof of assertion (v). Finally, we observe that assertion (vi) is an immediate consequence of assertions (iii), (v). ◯

We are now ready to state and prove the *main result* of the present §1.

Theorem 1.7. (Graphicity of Isomorphisms of PPSC-Extensions) *Let l be a prime number; n a positive integer. For $\Box = \alpha, \beta$, let $\Sigma^\Box$ be a nonempty set of primes; $1 \to \Delta^\Box \to \Pi^\Box \to H^\Box \to 1$ an* ***l*-cyclotomically full** *[cf. Remark 1.5.3]* **pro-$\Sigma^\Box$ PPSC-extension** *of dimension n;*

$$\Pi_n^\Box \stackrel{\mathrm{def}}{=} \Pi^\Box \twoheadrightarrow \Pi_{n-1}^\Box \twoheadrightarrow \ldots \twoheadrightarrow \Pi_1^\Box \twoheadrightarrow \Pi_0^\Box \stackrel{\mathrm{def}}{=} H^\Box$$

the sequence of **successive fiber extensions** *associated to $\Pi^\Box$ [cf. Remark 1.5.1]. Let*

$$\phi : \Pi^\alpha \xrightarrow{\sim} \Pi^\beta$$

be an **isomorphism of profinite groups** *that induces isomorphisms $\phi_m : \Pi_m^\alpha \xrightarrow{\sim} \Pi_m^\beta$, for $m = 0, 1, \ldots, n$ [so $\phi = \phi_n$]. Then:*

(i) We have $\Sigma^\alpha = \Sigma^\beta$.

(ii) For $m \in \{1, \ldots, n\}$, ϕ_m induces a **bijection** *between the set of* **divisors** *of Π_m^α and the set of divisors of Π_m^β.*

(iii) For $m \in \{1, \ldots, n\}$, suppose that c^α, c^β are divisors of Π_m^α, Π_m^β, respectively, that correspond via the bijection of (ii). Then $\phi_m(I_{c^\alpha}) = I_{c^\beta}$, $\phi_m(D_{c^\alpha}) = D_{c^\beta}$. That is to say, ϕ_m is **compatible** *with the* **inertia** *and* **decomposition groups** *of divisors.*

(iv) For $m \in \{0, \ldots, n-1\}$, the isomorphism

$$\mathrm{Ker}(\Pi_{m+1}^\alpha \twoheadrightarrow \Pi_m^\alpha) \xrightarrow{\sim} \mathrm{Ker}(\Pi_{m+1}^\beta \twoheadrightarrow \Pi_m^\beta)$$

induced by ϕ_{m+1} is **graphic** *[i.e., compatible with the semi-graphs of anabelioids that appear in the respective collections of construction data of the PSC-extensions $\Pi_{m+1}^\Box \twoheadrightarrow \Pi_m^\Box$, for $\Box = \alpha, \beta$].*

(v) For $m \in \{1, \ldots, n-1\}$, c^α, c^β corresponding divisors of Π_m^α, Π_m^α, the isomorphism

$$\mathrm{Ker}((\Pi_{m+1}^\alpha)_{c^\alpha} \twoheadrightarrow D_{c^\alpha}) \xrightarrow{\sim} \mathrm{Ker}((\Pi_{m+1}^\beta)_{c^\beta} \twoheadrightarrow D_{c^\beta})$$

induced by ϕ_{m+1} is **graphic** *[i.e., compatible with the semi-graphs of anabelioids that appear in the respective collections of construction data of the DPSC-extensions $(\Pi^{\Box}_{m+1})_{c^{\Box}} \twoheadrightarrow D_{c^{\Box}}$, for $\Box = \alpha, \beta$].*

Proof. All of the assertions of Theorem 1.7 follow immediately from [the various definitions involved, together with] *repeated application* of Corollary 1.4 to the *PSC-extensions* $\Pi^{\Box}_{m+1} \twoheadrightarrow \Pi^{\Box}_m$ [cf. Definition 1.5, (ii), (c)] and the *DPSC-extensions* $(\Pi^{\Box}_{m+1})_{c^{\Box}} \twoheadrightarrow D_{c^{\Box}}$ [cf. Definition 1.5, (ii), (d)], for $\Box = \alpha, \beta$. ◯

Remark 1.7.1. In Theorem 1.7, instead of phrasing the result as an assertion concerning the *preservation of structures via some isomorphism* between *two* PPSC-extensions, one may instead phrase the result as an assertion concerning the existence of an *explicit "group-theoretic algorithm"* for reconstructing, from a *single* given PPSC-extension, the various structures corresponding to *graphicity*, *divisors*, and *inertia* and *decomposition groups* of divisors — i.e., in the fashion of [Mzk15], Lemma 4.5, for *cuspidal decomposition groups*; a similar remark may be made concerning Corollary 1.4. [We leave the routine details to the interested reader.] Indeed, both Corollary 1.4 and Theorem 1.7 are, in essence, formal consequences of the *"graphicity theory"* of [Mzk13], which [just as in the case of [Mzk15], Lemma 4.5] consists precisely of such *explicit "group-theoretic algorithms"* for reconstructing the various structures corresponding to graphicity in the case of semi-graphs of anabelioids of PSC-type.

Before proceeding, we observe the following result, which is, in essence, *independent* of the theory of the present §1.

Theorem 1.8. (PPSC-Extensions over Galois Groups of Arithmetic Fields) *For $\Box = \alpha, \beta$, let $k_{\Box}$ be a field of* **characteristic zero***; $\widetilde{k}_{\Box}$ a* **solvably closed** *[cf. [Mzk15], Definition 1.4] Galois extension of $k_{\Box}$; $H^{\Box} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{k}_{\Box}/k_{\Box})$; $(Z_{\Box})^{\log}$ the log scheme determined by a* **stable polycurve** *over $k_{\Box}$; $\Sigma^{\Box}$ a nonempty set of primes; n a positive integer; $1 \to \Delta^{\Box} \to \Pi^{\Box} \twoheadrightarrow H^{\Box} \to 1$ a* **pro-$\Sigma^{\Box}$ PPSC-extension** *of dimension n associated to $(Z_{\Box})^{\log}$, $\widetilde{k}_{\Box}$ [cf. Remark 1.5.4];*

$$\Pi^{\Box}_n \stackrel{\mathrm{def}}{=} \Pi^{\Box} \twoheadrightarrow \Pi^{\Box}_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow \Pi^{\Box}_1 \twoheadrightarrow \Pi^{\Box}_0 \stackrel{\mathrm{def}}{=} H^{\Box}$$

the sequence of **successive fiber extensions** *associated to $\Pi^{\Box}$ [cf. Remark 1.5.1]. Let*

$$\phi : \Pi^{\alpha} \xrightarrow{\sim} \Pi^{\beta}$$

be an **isomorphism of profinite groups** *that induces isomorphisms $\phi_m : \Pi^{\alpha}_m \xrightarrow{\sim} \Pi^{\beta}_m$, for $m = 0, 1, \ldots, n$ [so $\phi = \phi_n$]. Then:*

(i) **(Relative Version of the Grothendieck Conjecture for Stable Polycurves over Generalized Sub-p-adic Fields)** *Suppose that for $\Box = \alpha, \beta$, $k_{\Box}$ is* **generalized sub-p-adic** *[cf. [Mzk4], Definition 4.11] for some prime number $p \in \Sigma^{\alpha} \bigcap \Sigma^{\beta}$, and that the isomorphism of Galois groups $\phi_0 : H^{\alpha} \xrightarrow{\sim} H^{\beta}$ arises from a pair of* **isomorphisms of fields** *$\widetilde{k}_{\alpha} \xrightarrow{\sim} \widetilde{k}_{\beta}$, $k_{\alpha} \xrightarrow{\sim} k_{\beta}$. Then $\Sigma^{\alpha} = \Sigma^{\beta}$; there*

exists a **unique isomorphism of log schemes** $(Z_\alpha)^{\log} \xrightarrow{\sim} (Z_\beta)^{\log}$ *that gives rise to* ϕ*.*

(ii) **(Absolute Version of the Grothendieck Conjecture for Stable Polycurves over Number Fields)** *Suppose that for* $\Box = \alpha, \beta$*,* $k_\Box$ *is a* **number field**. *Then* $\Sigma^\alpha = \Sigma^\beta$*; there exists a* **unique isomorphism of log schemes** $(Z_\alpha)^{\log} \xrightarrow{\sim} (Z_\beta)^{\log}$ *that gives rise to* ϕ*.*

Proof. Assertions (i), (ii) follow immediately from *repeated application* of [Mzk4], Theorem 4.12 [cf. also [Mzk2], Corollary 7.4], together with [in the case of assertion (ii)] *"Uchida's theorem"* [cf., e.g., [Mzk10], Theorem 3.1]. ◯

Finally, we study the consequences of the theory of the present §1 in the case of *configuration spaces*. We refer to [MT] for more details on the theory of configuration spaces.

Definition 1.9. Let l be a prime number; Σ a set of primes which is either *of cardinality one* or equal to the *set of all primes*; X a *hyperbolic curve* of type (g, r) over a *field* k of characteristic $\notin \Sigma$; $\overline{k}$ a *separable closure* of k; $n \geq 1$ an integer; X_n the *n-th configuration space* associated to X [cf. [MT], Definition 2.1, (i)]; E the *index set* [i.e., the set of factors — cf. [MT], Definition 2.1, (i)] of X_n;

$$\pi_1(X_n \times_k \overline{k}) \twoheadrightarrow \Theta$$

the *maximal pro-*Σ *quotient* of $\pi_1(X_n \times_k \overline{k})$; $\Delta \subseteq \Theta$ a *product-theoretic open subgroup* [cf. [MT], Definition 2.3, (ii)]; $1 \to \Delta \to \Pi \to H \to 1$ an *extension of profinite groups*.

(i) We shall refer to as a *labeling* on E a bijection $\Lambda : \{1, 2, \ldots, n\} \xrightarrow{\sim} E$. Thus, for each labeling Λ on E, we obtain a *structure of hyperbolic polycurve* [i.e., a collection of data exhibiting X_n as a *hyperbolic polycurve* — cf. [Mzk2], Definition 4.6] on X_n, arising from the various *natural projection morphisms* associated to X_n [cf. [MT], Definition 2.1, (ii)], by projecting in the order specified by Λ. In particular, for each labeling Λ on E, we obtain a *structure of PPSC-extension* on [the extension $1 \to \Delta_\Lambda \to \Delta_\Lambda \to \{1\} \to 1$ associated to] some open subgroup $\Delta_\Lambda \subseteq \Delta$ [which may be taken to be *arbitrarily small* — cf. Remark 1.5.2].

(ii) Let Λ be a *labeling* on E. Then we shall refer to a *structure of PPSC-extension* on [the extension $1 \to \Delta_\Lambda \to \Pi_\Lambda \to H_\Lambda \to 1$ arising from $1 \to \Delta \to \Pi \to H \to 1$ by intersecting with] an open subgroup $\Pi_\Lambda \subseteq \Pi$ as Λ*-admissible* if it induces the structure of PPSC-extension on Δ_Λ discussed in (i).

(iii) We shall refer to as a *structure of [pro-*Σ*] CPSC-extension [of type* (g, r) *and dimension* n*, with index set* E*]* [i.e., "configuration (space) pointed stable curve extension"] on the extension $1 \to \Delta \to \Pi \to H \to 1$ any collection of data as follows: for each labeling Λ on E, a Λ*-admissible structure of PPSC-extension* on some open subgroup $\Pi_\Lambda \subseteq \Pi$ [which may be taken to be *arbitrarily small* — cf. Remark 1.5.2]. We shall refer to a structure of CPSC-extension on Π as *l-cyclotomically full* if, for each labeling Λ on E, the Λ-admissible structure of PPSC-extension that constitutes the given structure of CPSC-extension is l-cyclotomically full. We shall refer to a structure of CPSC-extension on Π as *strict* if one may take Δ to be Θ, and, for each labeling Λ on E, one may take Π_Λ to be Π. We shall refer to $1 \to \Delta \to \Pi \to H \to 1$ as a *[pro-Σ] CPSC-extension [of type (g, r) and dimension n, with index set E]* if it admits a structure of CPSC-extension; if this structure of CPSC-extension may be taken to be l-cyclotomically full (respectively, strict), then we shall refer to the CPSC-extension itself as *l-cyclotomically full* (respectively, *strict*). If $1 \to \Delta \to \Pi \to H \to 1$ is a *CPSC-extension*, then we shall refer to (Σ, X, k, Θ) as *construction data* for this CPSC-extension.

(iv) Let $\widetilde{k} \subseteq \overline{k}$ be a *solvably closed* [cf. [Mzk15], Definition 1.4] Galois extension of k; suppose that

$$Z \to X_n$$

is a *finite étale covering* such that $Z \times_k \overline{k} \to X_n \times_k \overline{k}$ is the [connected] covering determined by the open subgroup $\Delta \subseteq \Theta$ [so we have a natural surjection $\pi_1(Z \times_k \overline{k}) \twoheadrightarrow \Delta$]. Then [cf. the discussion of Remark 1.5.4] we shall refer to a [structure of] CPSC-extension [on] $1 \to \Delta \to \Pi \to H \to 1$ as *arising from Z, $\widetilde{k}/k$* if there exist a surjection $\pi_1(Z) \twoheadrightarrow \Pi$ and an isomorphism $\mathrm{Gal}(\widetilde{k}/k) \xrightarrow{\sim} H$ that are *compatible* with one another as well as with the natural surjections $\pi_1(Z \times_k \overline{k}) \twoheadrightarrow \Delta$, $\pi_1(Z) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k) \twoheadrightarrow \mathrm{Gal}(\widetilde{k}/k)$ and, moreover, satisfy the property that the *structure of CPSC-extension* on $1 \to \Delta \to \Pi \to H \to 1$ is induced by the various structures of hyperbolic polycurve on Z, X_n, associated to a suitable *labeling* of E [cf. (i)].

Corollary 1.10. **(Cominatorial Configuration Spaces)** *Let l be a* **prime number**. *For $\square = \alpha, \beta$, let*

$$1 \to \Delta^\square \to \Pi^\square \to H^\square \to 1$$

be an **extension of profinite groups** *equipped with some [***fixed***!]* ***l*-cyclotomically full structure of CPSC-extension** *of type $(g_\square, r_\square) \notin \{(0,3),(1,1)\}$ and dimension $n_\square$, with index set $E_\square$. If this fixed structure of CPSC-extension is* **not strict** *for* **either** *$\square = \alpha$* **or** *$\square = \beta$, then we assume that* **both** *g_α, g_β are ≥ 2. Let*

$$\phi : \Pi^\alpha \xrightarrow{\sim} \Pi^\beta$$

be an **isomorphism of profinite groups** *such that $\phi(\Delta^\alpha) = \Delta^\beta$. Then:*

(i) The isomorphism ϕ determines a **bijection** *$E_\alpha \xrightarrow{\sim} E_\beta$ of* **index sets**. *In particular, $n_\alpha = n_\beta$, so we write $n \stackrel{\mathrm{def}}{=} n_\alpha = n_\beta$.*

(ii) For each pair of **compatible** *[i.e., relative to the bijection of (i)]* **labelings** *$\Lambda = (\Lambda_\alpha, \Lambda_\beta)$ of E_α, E_β, there exist* **open subgroups** *$\Pi^\square_\Lambda \subseteq \Pi^\square$ [for $\square = \alpha, \beta$] such that the following properties hold: (a) $\phi(\Pi^\alpha_\Lambda) = \Pi^\beta_\Lambda$; (b) for $\square = \alpha, \beta$, the open subgroup $\Pi^\square_\Lambda$ admits an* **$\Lambda_\square$-admissible structure of PPSC-extension**; *(c) if we write*

$$(\Pi^\square_\Lambda)_n \stackrel{\mathrm{def}}{=} (\Pi^\square_\Lambda) \twoheadrightarrow (\Pi^\square_\Lambda)_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow (\Pi^\square_\Lambda)_1 \twoheadrightarrow (\Pi^\square_\Lambda)_0 \stackrel{\mathrm{def}}{=} H^\square_\Lambda$$

for the sequence of **successive fiber extensions** *associated to the structures of PPSC-extension of (b) [cf. Remark 1.5.1], then* ϕ *induces* **isomorphisms**

$$(\Pi^\alpha_\Lambda)_m \xrightarrow{\sim} (\Pi^\beta_\Lambda)_m$$

[for $m = 0, \dots, n$*]. In particular,* ϕ *satisfies the hypotheses of Theorem 1.7.*

Proof. By [MT], Corollaries 4.8, 6.3 [cf. our hypotheses on $(g_\Box, r_\Box)$], ϕ induces a *bijection* $E_\alpha \xrightarrow{\sim} E_\beta$ between the respective *index sets*, together with compatible *isomorphisms* between the various *fiber subgroups* of Δ^α, Δ^β. [Note that even though these results of [MT] are stated only in the case where the field appearing in the construction data is of *characteristic zero*, the results generalize immediately to the case where this field is of *characteristic invertible in* $\Sigma^\Box$, since any hyperbolic curve in positive characteristic may be lifted to a hyperbolic curve in characteristic zero in a fashion that is compatible with the maximal pro-$\Sigma^\Box$ quotients of the étale fundamental groups of the associated configuration spaces — cf., e.g., [MT], Proposition 2.2, (v).] To obtain open subgroups $\Pi^\Box_\Lambda \subseteq \Pi^\Box$ satisfying the desired properties, it suffices to argue by *induction on* n, by applying Remark 1.5.2. ○

Remark 1.10.1. A similar remark to Remark 1.7.1 may be made for Corollary 1.10.

Corollary 1.11. **(Configuration Spaces over Arithmetic Fields)** *For* $\Box = \alpha, \beta$*, let* $k_\Box$ *be a* **perfect field***;* $\widetilde{k}_\Box$ *a* **solvably closed** *[cf. [Mzk15], Definition 1.4] Galois extension of* $k_\Box$*;* $X_\Box$ *a* **hyperbolic curve** *of type* $(g_\Box, r_\Box) \notin \{(0,3),(1,1)\}$ *over* $k_\Box$*;* $n_\Box$ *a positive integer;*

$$Z_\Box \to (X_\Box)_{n_\Box}$$

a geometrically connected [over $k_\Box$*]* **finite étale covering** *of the* $n_\Box$*-th* **configuration space** $(X_\Box)_{n_\Box}$ *of* $X_\Box$*;* $\Sigma^\Box$ *a nonempty set of primes;*

$$1 \to \Delta^\Box \to \Pi^\Box \to H^\Box \to 1$$

an **extension of profinite groups** *equipped with some [***fixed!***]* **structure of pro-**$\Sigma^\Box$ **CPSC-extension** *arising from* $Z_\Box$, $\widetilde{k}_\Box$ *[cf. Definition 1.9, (iv)]. If this fixed structure of CPSC-extension is* **not strict** *for* **either** $\Box = \alpha$ **or** $\Box = \beta$*, then we assume that* **both** g_α, g_β *are* ≥ 2*. Let*

$$\phi : \Pi^\alpha \xrightarrow{\sim} \Pi^\beta$$

be an **isomorphism of profinite groups***. Then:*

(i) **(Relative Version of the Grothendieck Conjecture for Configuration Spaces over Generalized Sub-*p*-adic Fields)** *Suppose, for* $\Box = \alpha, \beta$*, that* $k_\Box$ *is* **generalized sub-*p*-adic** *[cf. [Mzk4], Definition 4.11] for some prime number* $p \in \Sigma^\alpha \bigcap \Sigma^\beta$*, and that* ϕ *lies over an isomorphism of Galois groups* $\phi_0 : H^\alpha \xrightarrow{\sim} H^\beta$ *that arises from a pair of* **isomorphisms of fields** $\widetilde{k}_\alpha \xrightarrow{\sim} \widetilde{k}_\beta$*,* $k_\alpha \xrightarrow{\sim} k_\beta$*. Then*

$\Sigma^\alpha = \Sigma^\beta$; there exists a **unique isomorphism of schemes** *$Z_\alpha \xrightarrow{\sim} Z_\beta$ that gives rise to ϕ.*

(ii) **(Strict Semi-absoluteness)** *Suppose, for $\Box = \alpha, \beta$, that $k_\Box$ is either an* **FF***, an* **MLF***, or an* **NF** *[cf. [Mzk15], §0]. Then $\phi(\Delta^\alpha) = \Delta^\beta$ [i.e., ϕ is* **"strictly semi-absolute"***].*

(iii) **(Absolute Version of the Grothendieck Conjecture for Configuration Spaces over MLF's)** *Suppose, for $\Box = \alpha, \beta$, that $k_\Box$ is an* **MLF***, that $n_\Box \geq 2$, that $n_\Box \geq 3$ if $X_\Box$ is* **proper***, and that $\Sigma^\Box$ is the set of* **all primes***. Then there exists a* **unique isomorphism of schemes** *$Z_\alpha \xrightarrow{\sim} Z_\beta$ that gives rise to ϕ.*

(iv) **(Absolute Version of the Grothendieck Conjecture for Configuration Spaces over NF's)** *Suppose, for $\Box = \alpha, \beta$, that $k_\Box$ is an* **NF***. Then $\Sigma^\alpha = \Sigma^\beta$; there exists a* **unique isomorphism of schemes** *$Z_\alpha \xrightarrow{\sim} Z_\beta$ that gives rise to ϕ.*

Proof. Assertion (i) (respectively, (iv)) follows immediately from Corollary 1.10, (ii), and Theorem 1.8, (i) (respectively, Theorem 1.8, (ii)) [applied to the coverings of Z_α, Z_β determined by the *open subgroups* of Corollary 1.10, (ii)]. Assertion (ii) follows immediately from [Mzk15], Corollary 2.8, (ii). Note that *in the situation of assertion (ii)*, assertion (ii) implies that $\Sigma^\alpha = \Sigma^\beta$ [since $\Sigma^\Box$ may be characterized as the unique minimal set of primes Σ' such that $\Delta^\Box$ is a *pro-Σ' group*]; moreover, in light of our assumptions on $k_\Box$, it follows immediately that $\Pi^\Box$ is *l-cyclotomically full* for any $l \in \Sigma^\alpha \bigcap \Sigma^\beta = \Sigma^\alpha = \Sigma^\beta$.

Finally, we consider assertion (iii). First, let us observe that by Corollary 1.10, (ii), and Theorem 1.8, (i) [applied to the coverings of Z_α, Z_β determined by the *open subgroups* of Corollary 1.10, (ii)], it suffices to verify that the isomorphism $\phi_H : H^\alpha \xrightarrow{\sim} H^\beta$ induced by ϕ [cf. assertion (ii)] arises from an *isomorphism of fields* $k_\alpha \xrightarrow{\sim} k_\beta$. To this end, let us observe that by Corollary 1.10, (ii), we may apply Theorem 1.7 to the present situation. Also, by Corollary 1.10, (i), $n = n_\alpha = n_\beta$ is *always* ≥ 2; moreover, if either of the $X_\Box$ is *proper*, then $n \geq 3$. Next, let us observe that if $X_\Box$ is *proper* (respectively, *affine*), then the *stable log curve* that appears in the logarithmic compactification of the fibration $(X_\Box)_3 \to (X_\Box)_2$ (respectively, $(X_\Box)_2 \to (X_\Box)_1$) over the generic point of the diagonal divisor of $(X_\Box)_2$ (respectively, over any cusp of $X_\Box$) contains an irreducible component whose interior is a *tripod* [i.e., a copy of the projective line minus three marked points]. In particular, if we apply Theorem 1.7, (iii), to the *vertical divisor* determined by such an irreducible component, then we may conclude that ϕ induces an *isomorphism* between the *decomposition groups* of these vertical divisors. In particular, [after possibly replacing the given k_α, k_β by corresponding finite extensions of k_α, k_β] we obtain, for $\Box = \alpha, \beta$, a *hyperbolic curve* $C_\Box$ over $k_\Box$, together with an *isomorphism of profinite groups*

$$\phi_C : \pi_1(C_\alpha \times_{k_\alpha} \widetilde{k}_\alpha) \xrightarrow{\sim} \pi_1(C_\beta \times_{k_\beta} \widetilde{k}_\beta)$$

induced by ϕ [so the $\pi_1(C_\Box \times_{k_\Box} \widetilde{k}_\Box)$ correspond to the respective "Π_v's" of the *vertical divisors* under consideration] that is *compatible* with the *outer action* of

$H^{\Box}$ on $\pi_1(C_{\Box} \times_{k_{\Box}} \widetilde{k}_{\Box})$ and the isomorphism ϕ_H; moreover, here we may assume that, say, C_{α} is a *finite étale covering* of a *tripod*. [In particular, we observe that the existence of the natural outer action of $H^{\Box}$ on $\pi_1(C_{\Box} \times_{k_{\Box}} \widetilde{k}_{\Box})$ implies — cf. the argument given in the proof of [Mzk10], Proposition 3.3 — that $\widetilde{k}_{\Box}$ is necessarily an *algebraic closure* of $k_{\Box}$.] On the other hand, since the *"absolute p-adic version of the Grothendieck Conjecture"* is known to hold in this situation [cf. [Mzk14], Corollary 2.3], we thus conclude that ϕ_H does indeed arise from an *isomorphism of fields* $k_{\alpha} \xrightarrow{\sim} k_{\beta}$, as desired. This completes the proof of assertion (iii). ◯

Remark 1.11.1. At the time of writing Corollary 1.11, (iii), constitutes the *only absolute isomorphism version* of the Grothendieck Conjecture *over MLF's* [to the knowledge of the author] that may be applied to *arbitrary hyperbolic curves*.

Section 2: Geometric Uniformly Toral Neighborhoods

In the present §2, we prove a certain *"resolution of nonsingularities"* type result [cf. Lemma 2.6; Remark 2.6.1; Corollary 2.11] — i.e., a result reminiscent of the main results of [Tama2] [cf. also the techniques applied in the verification of "observation (iv)" given in the proof of [Mzk9], Corollary 3.11] — that allows us to apply the theory of *uniformly toral neighborhoods* developed in [Mzk15], §3, to prove a certain *"conditional absolute p-adic version of the Grothendieck Conjecture"* — namely, that *"point-theoreticity implies geometricity"* [cf. Corollary 2.9]. This condition of point-theoreticity may be removed if, instead of starting with a hyperbolic curve, one starts with a *"pro-curve"* obtained by removing from a proper curve some [necessarily infinite] set of closed points which is *"p-adically dense in a Galois-compatible fashion"* [cf. Corollary 2.10].

First, we recall the following *"positive slope version of Hensel's lemma"* [cf. [Serre], Chapter II, §2.2, Theorem 1, for a discussion of a similar result].

Lemma 2.1. **(Positive Slope Version of Hensel's Lemma)** *Let k be a* **complete discretely valued field***; $\mathcal{O}_k \subseteq k$ the ring of integers of k [equipped with the topology determined by the valuation]; $\mathfrak{m}_k$ the maximal ideal of $\mathcal{O}_k$; $\pi \in \mathfrak{m}_k$ a uniformizer of $\mathcal{O}_k$; $A \stackrel{\mathrm{def}}{=} \mathcal{O}_k[[X_1, \ldots, X_m]]$; $B \stackrel{\mathrm{def}}{=} \mathcal{O}_k[[Y_1, \ldots, Y_n]]$. Let us suppose that A (respectively, B) is equipped with the topology determined by its maximal ideal; write $\mathcal{X} \stackrel{\mathrm{def}}{=} \mathrm{Spf}(A)$ (respectively, $\mathcal{Y} \stackrel{\mathrm{def}}{=} \mathrm{Spf}(B)$), K_A (respectively, K_B) for the quotient field of A (respectively, B), and Ω_A (respectively, Ω_B) for the module of continuous differentials of A (respectively, B) over $\mathcal{O}_k$ [so Ω_A (respectively, Ω_B) is a free A- (respectively, B-) module of rank m (respectively, n)]. Let $\phi : B \to A$ be the continuous $\mathcal{O}_k$-algebra homomorphism induced by an assignment*

$$B \ni Y_j \mapsto f_j(X_1, \ldots, X_m) \in A$$

[where $j = 1, \ldots, n$]; let us suppose that the induced morphism $d\phi : \Omega_B \otimes_B A \to \Omega_A$ satisfies the property that the **image of** *$d\phi \otimes_A K_A$ is a K_A-subspace of rank n in $\Omega_A \otimes_A K_A$ [so $n \leq m$]. Then there exists a point $\beta_0 \in \mathcal{Y}(\mathcal{O}_k)$ and a positive integer r satisfying the following property: Let k' be a finite extension of k, with ring of integers $\mathcal{O}_{k'}$; write $B(\beta_0, k', r)$ for the "ball" of points $\beta' \in \mathcal{Y}(\mathcal{O}_{k'})$ such that β', β_0 map to the same point of $\mathcal{Y}(\mathcal{O}_{k'}/(\pi^r))$. Then the* **image** *of the map*

$$\mathcal{X}(\mathcal{O}_{k'}) \to \mathcal{Y}(\mathcal{O}_{k'})$$

induced by ϕ **contains the "ball"** *$B(\beta_0, k', r)$.*

Proof. First, let us observe that by Lemma 2.2 below, after possibly re-ordering the X_i's, we may assume that the differentials $dX_i \in \Omega_A$, where $i = n+1, \ldots, m$, together with the differentials $df_j \in \Omega_A$, where $j = 1, \ldots, n$, form a K_A-*basis* of $\Omega_A \otimes_A K_A$. Thus, by adding indeterminates $Y_{n+1}, \ldots, Y_m$ to B and extending ϕ by sending $Y_i \mapsto X_i$ for $i = n+1, \ldots, m$, we may assume without loss of generality that $n = m$, $A = B$, $\mathcal{X} = \mathcal{Y}$, i.e., that the morphism $\mathrm{Spf}(\phi) : \mathcal{X} \to \mathcal{Y} = \mathcal{X}$ is *"generically formally étale"*.

Write M for the n by n matrix with coefficients $\in A$ given by $\{df_i/dX_j\}_{i,j=1,\ldots,n}$; $g \in A$ for the *determinant* of M. Thus, by elementary linear algebra, it follows that there exists an n by n matrix N with coefficients $\in A$ such that $M \cdot N = N \cdot M = g \cdot I$ [where we write I for the n by n identity matrix]. By our assumption concerning the image of $d\phi \otimes_A K_A$, it follows that $g \neq 0$, hence, by Lemma 2.3 below, that there exist elements $x_i \in \mathfrak{m}_k$, where $i = 1, \ldots, n$, such that $g_0 \stackrel{\text{def}}{=} g(x_1, \ldots, x_n) \in \mathfrak{m}_k$ is *nonzero*. By applying appropriate "affine translations" to the domain and codomain of ϕ, we may assume without loss of generality that, for $i = 1, \ldots, n$, we have $x_i = f_i(0, \ldots, 0) = 0 \in \mathcal{O}_k$. Write $M_0 \stackrel{\text{def}}{=} M(0, \ldots, 0)$, $N_0 \stackrel{\text{def}}{=} N(0, \ldots, 0)$ [so M_0, N_0 are n by n matrices with coefficients $\in \mathcal{O}_k$].

Next, suppose that $g_0 \in \mathfrak{m}_k^s \backslash \mathfrak{m}_k^{s+1}$. In the remainder of the present proof, all "vectors" are to be understood as *column vectors with coefficients* $\in \mathcal{O}_{k'}$, where k' is as in the statement of Lemma 2.1. Then I *claim* that for every vector $\vec{y} = (y_1, \ldots, y_n) \equiv 0 \pmod{\pi^{3s}}$, there exists a vector $\vec{x} = (x_1, \ldots, x_n) \equiv 0 \pmod{\pi^{2s}}$ such that $\vec{f}(\vec{x}) \stackrel{\text{def}}{=} (f_1(\vec{x}), \ldots, f_n(\vec{x})) = \vec{y}$. Indeed, since $\mathcal{O}_{k'}$ is *complete* and $x_i = f_i(0, \ldots, 0) = 0$, it suffices [by taking $\vec{x}\{2\}$ to be $(0, \ldots, 0)$] to show, for each integer $l \geq 2$, that the existence of a vector $\vec{x}\{l\} \equiv 0 \pmod{\pi^{2s}}$ such that $\vec{f}(\vec{x}\{l\}) \equiv \vec{y} \pmod{\pi^{(l+1)s}}$ implies the existence of a vector $\vec{x}\{l+1\}$ such that $\vec{x}\{l+1\} \equiv \vec{x}\{l\} \pmod{\pi^{ls}}$, $\vec{f}(\vec{x}\{l+1\}) \equiv \vec{y} \pmod{\pi^{(l+2)s}}$. To this end, we compute: Set $\vec{\epsilon} \stackrel{\text{def}}{=} \vec{y} - \vec{f}(\vec{x}\{l\})$, $\vec{\eta} \stackrel{\text{def}}{=} g_0^{-1} \cdot \vec{\epsilon}$, $\vec{\delta} \stackrel{\text{def}}{=} N_0 \cdot \vec{\eta}$, $\vec{x}\{l+1\} \stackrel{\text{def}}{=} \vec{x}\{l\} + \vec{\delta}$. Thus, $\vec{\epsilon} \equiv 0 \pmod{\pi^{(l+1)s}}$, $\vec{\eta} \equiv 0 \pmod{\pi^{ls}}$, $\vec{\delta} \equiv 0 \pmod{\pi^{ls}}$, $\vec{x}\{l+1\} \equiv \vec{x}\{l\} \pmod{\pi^{ls}}$. In particular, the squares of the elements of $\vec{\delta}$ all belong to $\pi^{(l+2)s} \cdot \mathcal{O}_{k'}$ [since $l \geq 2$], so we obtain that

$$\begin{aligned}\vec{f}(\vec{x}\{l+1\}) \equiv \vec{f}(\vec{x}\{l\} + \vec{\delta}) \equiv \vec{f}(\vec{x}\{l\}) + M_0 \cdot \vec{\delta} \quad &\pmod{\pi^{(l+2)s}} \\ \equiv \vec{y} - \vec{\epsilon} + M_0 \cdot N_0 \cdot \vec{\eta} \equiv \vec{y} - \vec{\epsilon} + g_0 \cdot \vec{\eta} \equiv \vec{y} \quad &\pmod{\pi^{(l+2)s}}\end{aligned}$$

— as desired. This completes the proof of the *claim*. On the other hand, one verifies immediately that the content of this claim is sufficient to complete the proof of Lemma 2.1. ○

Remark 2.1.1. Thus, the usual *"(slope zero version of) Hensel's lemma"* corresponds, in the notation of Lemma 2.1, to the case where the image of the morphism $d\phi$ is a *direct summand* of Ω_A. In this case, we may take $r = 1$.

Remark 2.1.2. In fact, according to oral communication to the author by *F. Oort*, it appears that the sort of "positive slope version" of "Hensel's lemma" given in Lemma 2.1 [i.e., where the derivative is only *generically* invertible] *preceded* the "slope zero version" that is typically referred to "Hensel's lemma" in modern treatments of the subject.

Lemma 2.2. **(Subspaces and Bases of a Vector Space)** *Let k be a field; V a finite-dimensional k-vector space with basis $\{e_i\}_{i \in I}$; $W \subseteq V$ a k-subspace. Then there exists a subset $J \subseteq I$ such that if we write $V_J \subseteq V$ for the k-subspace generated by the e_j, for $j \in J$, then the natural inclusions $V_J \hookrightarrow V$, $W \hookrightarrow V$ determine an isomorphism $V_J \oplus W \xrightarrow{\sim} V$ of k-vector spaces.*

Proof. This result is a matter of elementary linear algebra. ○

Lemma 2.3. (Nonzero Values of Functions Defined by Power Series) *Let k, $\mathcal{O}_k$, A be as in Lemma 2.1; $f = f(X_1, \dots, X_m) \in A$ a* **nonzero** *element. Then there exist elements $x_i \in \mathfrak{m}_k$, where $i = 1, \dots, m$, such that $f(x_1, \dots, x_m) \in \mathfrak{m}_k$ is* **nonzero**.

Proof. First, I *claim* that by induction on m, it suffices to verify Lemma 2.3 when $m = 1$. Indeed, for arbitrary $m \geq 2$, one may write

$$f = \sum_{i=0}^{\infty} c_i \ X_m^i$$

— where $c_i = c_i(X_1, \dots, X_{m-1}) \in \mathcal{O}_k[[X_1, \dots, X_{m-1}]]$. Since $f \neq 0$, it follows that there exists at least one *nonzero* c_j. Thus, by the induction hypothesis, it follows that there exist $x_i \in \mathfrak{m}_k$, where $i = 1, \dots, m-1$, such that $\mathfrak{m}_k \ni c_j(x_1, \dots, x_{m-1}) \neq 0$. Thus, $f(x_1, \dots, x_{m-1}, X_m) \in \mathcal{O}_k[[X_m]]$ is *nonzero*, so, again by the induction hypothesis, there exists an $x_m \in \mathfrak{m}_k$ such that $\mathfrak{m}_k \ni f(x_1, \dots, x_{m-1}, x_m) \neq 0$. This completes the proof of the *claim*.

Thus, for the remainder of the proof, we assume that $m = 1$ and write $X \stackrel{\text{def}}{=} X_1$,

$$f = \sum_{i=0}^{\infty} c_i \ X^i$$

— where $c_i \in \mathcal{O}_k$. Suppose that $c_j \neq 0$, but that $c_i = 0$ for $i < j$. Then there exists a positive integer s such that $c_j \notin \mathfrak{m}_k^s$. Let $x \in \mathfrak{m}_k^s$ be any nonzero element. Then $c_j x^j \notin x^j \cdot \mathfrak{m}_k^s$, while $c_i x^i \in x^j \cdot x \cdot \mathcal{O}_k \subseteq x^j \cdot \mathfrak{m}_k^s$ for any $i > j$. But this implies that $\mathfrak{m}_k \ni f(x) \neq 0$, as desired. ○

In the following, we shall often work with [two-dimensional] *log regular* log schemes. For various basic facts on *log regular* log schemes, we refer to [Kato]; [Mzk2], §1. If $X^{\log}$ is a *log regular* log scheme, then for integers $j \geq 0$, we shall write

$$U_X^{[j]} \subseteq X$$

for the *j-interior* of $X^{\log}$, i.e., the open subscheme of points at which the fiber of the groupification of the characteristic sheaf of $X^{\log}$ is of rank $\leq j$ [cf. [MT], Definition 5.1, (i); [MT], Proposition 5.2, (i)]. Thus, the complement of $U_X^{[j]}$ in X is a closed subset of *codimension* $> j$ [cf. [MT], Proposition 5.2, (ii)]; $U_X \stackrel{\text{def}}{=} U_X^{[0]}$ is the *interior* of $X^{\log}$ [i.e., the maximal open subscheme over which the log structure is trivial]. Also, we shall write

$$D_X \subseteq X$$

for the closed subscheme $X \backslash U_X$ with the reduced induced structure. Finally, we remind the reader that in the following, all *fiber products* of fs log schemes are to be taken in the *category of fs log schemes* [cf. §0].

Next, let k be a *complete discretely valued field* with *perfect residue field* $\underline{k}$; $\mathcal{O}_k \subseteq k$ the ring of integers of k; $\overline{k}$ an algebraic closure of k; $\overline{\underline{k}}$ the resulting algebraic closure of $\underline{k}$; $\mathfrak{m}_k$ the maximal ideal of $\mathcal{O}_k$; $\pi \in \mathfrak{m}_k$ a uniformizer of $\mathcal{O}_k$;

$$X \to S \stackrel{\text{def}}{=} \operatorname{Spec}(\mathcal{O}_k)$$

a *stable curve* over S whose *generic fiber* $X_\eta \stackrel{\text{def}}{=} X \times_S \eta$, where we write $\eta \stackrel{\text{def}}{=} \operatorname{Spec}(k)$, is *smooth*. Thus, X_η is a *proper hyperbolic curve* over k, whose *genus* we denote by g_X; the open subschemes $\eta \subseteq S$, $X_\eta \subseteq X$ determine *log regular* log structures on S, X, respectively. We denote the resulting *morphism of log schemes* by $X^{\log} \to S^{\log}$.

Definition 2.4.

(i) We shall refer to a morphism of log schemes

$$\phi^{\log} : V^{\log} \to X^{\log}$$

[or to the log scheme $V^{\log}$] as a *log-modification* if $\phi^{\log}$ admits a factorization

$$V^{\log} \to X^{\log} \times_{S^{\log}} S_V^{\log} \to X^{\log}$$

— where $S_V \stackrel{\text{def}}{=} \operatorname{Spec}(\mathcal{O}_{k_V})$, $\mathcal{O}_{k_V}$ is the ring of integers of a finite separable extension k_V of k; $S_V^{\log}$ is the log regular log scheme determined by the open immersion $\eta_V \stackrel{\text{def}}{=} \operatorname{Spec}(k_V) \hookrightarrow S_V$; the morphism $X^{\log} \times_{S^{\log}} S_V^{\log} \to X^{\log}$ is the projection morphism [where we observe that the underlying morphism of schemes $X \times_S S_V \to S_V$ is a stable curve over S_V]; the morphism $V^{\log} \to X^{\log} \times_{S^{\log}} S_V^{\log}$ is a *log étale* morphism whose underlying morphism of schemes is *proper* and *birational*; we shall refer to k_V as the *base-field* of the log-modification $\phi^{\log}$.

(ii) For $i = 1, 2$, let $\phi_i^{\log} : V_i^{\log} \to X^{\log}$ be a *log-modification* that admits a factorization $V_i^{\log} \to X^{\log} \times_{S^{\log}} S_i^{\log} \to X^{\log}$ as in (i); $\psi^{\log} : V_2^{\log} \to V_1^{\log}$ an $X^{\log}$-morphism. Then let us observe that the log scheme $V_i^{\log}$ is always *log regular of dimension* 2 [cf. Proposition 2.5, (iv), below]. We shall refer to the log-modification $\phi_i^{\log}$ as *regular* if the log structure of $V_i^{\log}$ is defined by a divisor with normal crossings [which implies that V_i is a regular scheme]. We shall refer to the log-modification $\phi_i^{\log}$ as *unramified* if $U_{V_i}^{[1]}$ is a *smooth* scheme over S_i. We shall refer to the morphism $\psi^{\log}$ as a *base-field-isomorphism* [or *base-field-isomorphic*] if the morphism $S_2 \to S_1$ induced by $\psi^{\log}$ is an *isomorphism*. We shall refer to the points of V_1 over which the underlying morphism ψ of $\psi^{\log}$ fails to be finite as the *critical* points of $\psi^{\log}$ [or ψ]. We shall refer to the reduced divisor in V_2 determined by the closed set of points of V_2 at which ψ fails to be quasi-finite as the *exceptional divisor* of $\psi^{\log}$ [or ψ]. We shall refer to the log scheme- (respectively, scheme-) theoretic fiber of $V_i^{\log}$ (respectively, V_i) over the unique closed point of S_i as the *log special fiber* (respectively, *special fiber*) of $V_i^{\log}$ (respectively, V_i); we shall use the notation

$$\underline{V}_i^{\log} \text{ (respectively, } \underline{V}_i)$$

to denote the *log special fiber* (respectively, *special fiber*) of $V_i^{\log}$ (respectively, V_i). Suppose that C is an irreducible component of $\underline{V}_i$. Then we shall say that C is *stable* if it maps finitely to X via ϕ_i; we shall say that the log-modification $\phi_i^{\log}$ as *unramified at* C if $U_{V_i}^{[1]}$ is smooth over S_i at the generic point of C.

Remark 2.4.1. Recall that there exists a *base-field isomorphic log-modification*

$$V^{\log} \to X^{\log}$$

which is *uniquely* determined up to unique isomorphism [over X] by the following two properties: (a) V is *regular*; (b) $V \to S$ is a *semi-stable curve*. In fact, it was this example that served as the *primary motivating example* for the author in developing the notion of a "log-modification". Note, moreover, that unlike property (a), however, the principal condition that defines a [base-field-isomorphic] "log-modification" — i.e., the condition that the morphism $V \to X$ be a *proper*, *birational* morphism that extends to a *log étale* morphism $V^{\log} \to X^{\log}$ of log schemes — is a condition on the morphism $V \to X$ that has the virtue of being *manifestly stable under base-change* [i.e., via morphisms that satisfy suitable, relatively mild conditions]. This property of stability under base-change will be applied repeatedly in the remainder of the present §2.

Proposition 2.5. **(First Properties of Log-modifications)** *For* $i = 1, 2$, *let*

$$\phi_i^{\log} : V_i^{\log} \to X^{\log}$$

be a **log-modification** *that admits a factorization* $V_i^{\log} \to X^{\log} \times_{S^{\log}} S_i^{\log} \to X^{\log}$ *as in Definition 2.4, (i);* $\psi^{\log} : V_2^{\log} \to V_1^{\log}$ *an* $X^{\log}$*-morphism. Write* $S_i = \mathrm{Spec}(\mathcal{O}_{k_i})$, *where, for simplicity, we assume that the extension* k_i *of* k *is a subfield of* $\overline{k}$; $\underline{k}_i$ *for the residue field of* k_i; $U_\psi^{\mathrm{noncr}} \subseteq V_1$ *for the open subscheme given by the* **complement** *of the* **critical points** *of* $\psi^{\log}$. *Let* $\bullet \in \{1, 2\}$. *Then:*

(i) **(The Noncriticality of the 1-Interior)** *We have:* $U_{V_1}^{[1]} \subseteq U_\psi^{\mathrm{noncr}}$.

(ii) **(Isomorphism over the Noncritical Locus)** *The morphism* $V_2^{\log} \to V_1^{\log} \times_{S_1^{\log}} S_2^{\log}$ *determined by* $\psi^{\log}$ *is an* **isomorphism** *over* U_ψ^{noncr}.

(iii) **(Log Smoothness and Unramified Log-modifications)** $V_\bullet^{\log}$ *is* **log smooth** *over* $S_\bullet^{\log}$. *In particular, the sheaf of relative logarithmic differentials of the morphism* $V_\bullet^{\log} \to S_\bullet^{\log}$ *is a* **line bundle**, *which we shall denote* $\omega_{V_\bullet^{\log}/S_\bullet^{\log}}$; *we have a natural isomorphism* $\psi^* \omega_{V_1^{\log}/S_1^{\log}} \cong \omega_{V_2^{\log}/S_2^{\log}}$. *Finally, there exists a finite extension* $k_\circ$ *of* $k_\bullet$ *such that, if we write* $S_\circ \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_{k_\circ})$, *then the morphism* $V_\circ^{\log} \stackrel{\mathrm{def}}{=} V_\bullet^{\log} \times_{S_\bullet^{\log}} S_\circ^{\log} \to X^{\log}$ *determined by* $\phi_\bullet^{\log}$, *is an* **unramified log-modification**.

(iv) **(Regularity and Log Regularity)** $V_\bullet^{\log}$ *is* **log regular**; $U_{V_\bullet}^{[1]}$ *is* **regular**. *Moreover, there exists a* **regular log-modification** $V_\circ^{\log} \to X^{\log}$ *that admits*

a **base-field-isomorphic** $X^{\log}$*-morphism* $V_\circ^{\log} \to V_\bullet^{\log}$ *such that every irreducible component* C *of the special fiber* $\underline{V}_\circ$ *is* **smooth** *over the residue field* $\underline{k}_\circ$ *of the base-field* $k_\circ$ *of* $V_\circ^{\log}$*. Finally, if the log-modification* $V_\circ^{\log} \to X^{\log}$ *is* **unramified** *at such an irreducible component* C*, and we write* $D_C \subseteq C$ *for the reduced divisor determined by the complement of* $C \bigcap U_{V_\circ}^{[1]}$ *in* C*, then we have a* **natural isomorphism**

$$\omega_{C/\underline{k}_\circ}(D_C) \xrightarrow{\sim} \omega_{V_\circ^{\log}/S_\circ^{\log}}|_C$$

of line bundles on C*.*

(v) **(Chains of Projective Lines)** *Suppose that* $\phi_2^{\log}$ *is a* **regular log-modification** *[so* D_{V_2} *is a divisor with normal crossings]. Then, after possibly replacing* k_2 *by a finite unramified extension of the discretely valued field* k_2*, every irreducible component* C *of* D_{V_2} *that lies in the* **exceptional divisor** *of* $\psi^{\log}$ *is isomorphic to the* **projective line** *over the residue field* $\underline{k}(c)$ *of the point* c *of* $V_1^{\log} \times_{S^{\log}} S_2^{\log}$ *to which* C *maps. Moreover,* C *meets the other irreducible components of* D_{V_2} *at* **precisely two** $\underline{k}(c)$*-valued points of* C*. That is to say, every connected component of the exceptional divisor of* $\psi^{\log}$ *is a* **"chain of $\mathbb{P}^1$'s"***.*

(vi) **(Dual Graphs of Special Fibers)** *The spectrum* $\underline{V}_x$ *of the local ring obtained by completing the geometric special fiber* $\underline{V}_\bullet \times_{\underline{k}_\bullet} \overline{\underline{k}}$ *at any point* x *which* **does not** *(respectively,* **does***) belong to the* 1*-interior has* **precisely two** *(respectively,* **precisely one***) irreducible component(s). In particular, the special fiber* $\underline{V}_\bullet$ *determines a* **dual graph** $\Gamma_{\underline{V}_\bullet}$*, whose* **vertices** *correspond bijectively to the irreducible components of* $\underline{V}_\bullet \times_{\underline{k}_\bullet} \overline{\underline{k}}$*, and whose* **edges** *correspond bijectively to the points of* $(V_\bullet \backslash U_{V_\bullet}^{[1]}) \times_{\underline{k}_\bullet} \overline{\underline{k}}$ *[so each edge abuts to the vertices corresponding to the irreducible components in which the point corresponding to the edge lies]. In discussions of* $\Gamma_{\underline{V}_\bullet}$*, we shall frequently* **identify** *the vertices and edges of* $\Gamma_{\underline{V}_\bullet}$ *with the corresponding irreducible components and points of* $\underline{V}_\bullet \times_{\underline{k}_\bullet} \overline{\underline{k}}$*. If the natural Galois action of* $\mathrm{Gal}(\overline{\underline{k}}/\underline{k}_\bullet)$ *on* $\Gamma_{\underline{V}_\bullet}$ *is* **trivial***, then we shall say that* $V_\bullet^{\log}$ *is* **split***. Finally, the* **loop-rank** $\text{lp-rk}(\underline{V}_\bullet) \stackrel{\text{def}}{=} \text{lp-rk}(\Gamma_{\underline{V}_\bullet})$ *[cf. §0] is equal to the loop-rank* $\text{lp-rk}(\underline{X})$*.*

(vii) **(Filtered Projective Systems)** *Given any log-modification* $V_\circ^{\log} \to X^{\log}$*, there exists a log-modification* $V_{\bullet\circ}^{\log} \to X^{\log}$ *that admits* $X^{\log}$*-morphisms* $V_{\bullet\circ}^{\log} \to V_\bullet^{\log}$*,* $V_{\bullet\circ}^{\log} \to V_\circ^{\log}$*. That is to say, the log-modifications over* $X^{\log}$ *form a* **filtered projective system***.*

(viii) **(Functoriality)** *Let* $Y \to S$ *be a* **stable curve***, with* **smooth** *generic fiber* $Y_\eta \stackrel{\text{def}}{=} Y \times_S \eta$*;* $Y^{\log}$ *the log regular log scheme determined by the open subscheme* $Y_\eta \subseteq Y$*. Then every finite morphism* $Y_\eta \to X_\eta$ *extends to a* **commutative diagram**

$$\begin{array}{ccc} W_\bullet^{\log} & \longrightarrow & V_\bullet^{\log} \\ \big\downarrow & & \big\downarrow \\ Y^{\log} & \longrightarrow & X^{\log} \end{array}$$

— where $W_\bullet^{\log} \to Y^{\log}$ *is a* **log-modification***. If* $\text{lp-rk}(\underline{Y}) = \text{lp-rk}(\underline{X})$*, then we shall say that the morphisms* $Y_\eta \to X_\eta$*,* $Y^{\log} \to X^{\log}$*,* $W_\bullet^{\log} \to V_\bullet^{\log}$ *are* **loop-preserving***; if* $\text{lp-rk}(\underline{Y}) > \text{lp-rk}(\underline{X})$ *[or, equivalently,* $\text{lp-rk}(\underline{Y}) \neq \text{lp-rk}(\underline{X})$*], then*

we shall say that the morphisms $Y_\eta \to X_\eta$, $Y^{\log} \to X^{\log}$, $W_\bullet^{\log} \to V_\bullet^{\log}$ are **loop-ifying**. *Let C be an irreducible component of $\underline{W}_\bullet$. Then we shall refer to C as* **base-stable** *[relative to $Y_\eta \to X_\eta$] (respectively,* **base-semi-stable** *[relative to $W_\bullet^{\log} \to V_\bullet^{\log}$]) if it maps* **finitely** *to a(n) stable (respectively, arbitrary) irreducible component of $\underline{V}_\bullet$. If there exist log-modifications $W_\circ^{\log} \to Y^{\log}$, $V_\circ^{\log} \to X^{\log}$ that fit into a commutative diagram*

$$\begin{array}{ccc} W_\circ^{\log} & \longrightarrow & V_\circ^{\log} \\ \downarrow & & \downarrow \\ W_\bullet^{\log} & \longrightarrow & V_\bullet^{\log} \end{array}$$

[where the left-hand vertical arrow is a $Y^{\log}$-morphism; the right-hand vertical arrow is an $X^{\log}$-morphism] such that C is the image of a base-semi-stable [relative to $W_\circ^{\log} \to V_\circ^{\log}$] irreducible component of $\underline{W}_\circ$, then we shall say that C is **potentially base-semi-stable** *[relative to $Y_\eta \to X_\eta$].*

(ix) **(Centers in the 1-Interior)** *Let $k_\circ$ be a finite separable extension of k; $K_\circ$ a* **discretely valued field** *containing $k_\circ$ which induces an inclusion $\mathcal{O}_{k_\circ} \subseteq \mathcal{O}_{K_\circ}$ between the respective rings of integers and a* **bijection** *$k_\circ/\mathcal{O}_{k_\circ}^\times \xrightarrow{\sim} K_\circ/\mathcal{O}_{K_\circ}^\times$ between the respective* **value groups**; *$x_\circ \in X(K_\circ)$ a $K_\circ$-valued point. Then there exists an* **unramified log-modification** *$V_\circ^{\log} \to X^{\log}$ with base-field $k_\circ$ such that the morphism $V_\circ \to S_\circ \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_{k_\circ})$ is a* **semi-stable curve**, *and $x_\circ$ extends to a point $\in U_{V_\circ}^{[1]}(\mathcal{O}_{K_\circ})$.*

(x) **(Maps to the Jacobian)** *Suppose [for simplicity] that $\phi_\bullet^{\log}$ is a* **base-field-isomorphism**. *Let $x_\bullet \in U_{V_\bullet}^{[1]}(\mathcal{O}_k)$; C the [unique, by (vi)] irreducible component of $\underline{V}_\bullet$ that meets [the image of] $x_\bullet$; $F_{x_\bullet} \stackrel{\mathrm{def}}{=} \underline{V}_\bullet \backslash (C \bigcap U_{V_\bullet}^{[1]}) \subseteq \underline{V}_\bullet$ [regarded as a closed subset]; $U_{x_\bullet} \stackrel{\mathrm{def}}{=} V_\bullet \backslash F_{x_\bullet} \subseteq V_\bullet$ [so the image of $x_\bullet$ lies in $U_{x_\bullet}$]. Write $J_\eta \to \eta$ for the* **Jacobian** *of X_η; $J \to S$ for the* **uniquely determined semi-abelian scheme** *over S that extends J_η; $\iota_\eta : X_\eta \to J_\eta$ for the morphism that sends a T-valued point ξ [where T is a k-scheme] of X_η, regarded as a divisor on $X_\eta \times_k T$, to the point of J_η determined by the degree zero divisor $\xi - (x_\bullet|_T)$. Then ι_η* **extends uniquely** *to a morphism $U_{x_\bullet} \to J$. If, moreover, $\underline{X}$ is* **loop-ample** *[cf. §0], then this morphism $U_{x_\bullet} \to J$ is* **unramified**.

(xi) **(Lifting Simple Paths)** *In the situation of (viii), suppose further that the following conditions hold:*

(a) the log-modifications $W_\bullet^{\log} \to Y^{\log}$, $V_\bullet^{\log} \to X^{\log}$ are **base-field-isomorphic** *and* **split** *[cf. (vi)];*

(b) the morphism $W_\bullet^{\log} \to V_\bullet^{\log}$ is **finite**.

Let γ_V be a **simple path** *[cf. §0] in the* **dual graph** *$\Gamma_{\underline{V}_\bullet}$ of $\underline{V}_\bullet$ [cf. (vi)]. Then there exists a simple path γ_W in the dual graph $\Gamma_{\underline{W}_\bullet}$ of $\underline{W}_\bullet$ that* **lifts** *γ_V in the sense that the morphism $\underline{W}_\bullet \to \underline{V}_\bullet$ induces an isomorphism of graphs $\gamma_W \xrightarrow{\sim} \gamma_V$. Suppose further that the following condition holds:*

(c) the morphism $W_\bullet^{\log} \to V_\bullet^{\log}$ is **loop-preserving**.

Then γ_W is **unique** *in the sense that if γ'_W is any simple path in $\Gamma_{\underline{W}_\bullet}$ that lifts γ_V and is* **co-terminal** *[cf. §0] with γ_W, then $\gamma_W = \gamma'_W$.*

(xii) **(Loop-preservation and Wild Ramification)** *In the situation of (xi), suppose that, in addition to the conditions (a), (b), (c) of (xi), the following conditions hold:*

- *(d) there exists a prime number p such that $\underline{k}$ is of* **characteristic** $\boldsymbol{p}$, *and the morphism $Y_\eta \to X_\eta$ is* **finite étale Galois** *and of* **degree** $\boldsymbol{p}$*;*
- *(e) the morphism $Y_\eta \to X_\eta$ is* **wildly ramified** *over the* **terminal vertices** *[cf. §0] of the simple path γ_V.*

Let w_{exc} be a vertex of $\Gamma_{\underline{W}_\bullet}$ that corresponds to an irreducible component of the **exceptional divisor** *of $W_\bullet^{\log} \to Y^{\log}$ [i.e., a* **non-stable** *irreducible component of $\underline{W}_\bullet$] and, moreover, maps to a vertex v_{exc} of $\Gamma_{\underline{V}_\bullet}$ lying in γ_V. Then the morphism $Y_\eta \to X_\eta$ is* **wildly ramified** *at w_{exc}.*

Proof. First, we consider assertion (i). We may assume without loss of generality that ψ is a *base-field-isomorphism.* Then it follows from the simple structure of the monoid $\mathbb{N}$ that any *log étale birational* morphism over $U^{[1]}_{V_1}$ is, in fact, *étale.* This completes the proof of assertion (i). Next, we consider assertion (iii). Since the morphism $X^{\log} \to S^{\log}$ is *log smooth*, and the morphism $V_\bullet^{\log} \to X^{\log} \times_{S^{\log}} S_\bullet^{\log}$ is *log étale*, we conclude that $V_\bullet^{\log}$ is *log smooth* over $S_\bullet^{\log}$; the portion of assertion (iii) concerning $\omega_{V_\bullet^{\log}/S_\bullet^{\log}}$ then follows immediately. To verify the portion of assertion (iii) concerning unramified log-modifications, it suffices to observe that, in light of the well-known local structure of the nodes of the stable log curve $X^{\log} \times_{S^{\log}} S_\bullet^{\log} \to S_\bullet^{\log}$, there exists a finite extension $k_\circ$ of $k_\bullet$ such that, if we write $S_\circ \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_{k_\circ})$, then the morphism $V_\circ^{\log} \stackrel{\mathrm{def}}{=} V_\bullet^{\log} \times_{S_\bullet^{\log}} S_\circ^{\log} \to S_\circ^{\log}$ admits *sections that intersect with every irreducible component of* $\underline{V}_\circ \bigcap U^{[1]}_{V_\circ}$; thus, the fact that $V_\circ^{\log} \to X^{\log}$ is an *unramified log-modification* follows immediately from the *log smoothness* of $U^{[1]}_{V_\bullet}$, in light of the simple structure of the monoid $\mathbb{N}$. This completes the proof of assertion (iii).

Next, we consider assertion (iv). The fact that $V_\bullet^{\log}$ is *log regular* follows immediately from the log smoothness of $V_\bullet^{\log}$ over $S_\bullet^{\log}$ [cf. assertion (iii)]; the fact that $U^{[1]}_{V_\bullet}$ is *regular* then follows from the *log regularity* of $U^{[1]}_{V_\bullet}$, in light of the simple structure of the monoid $\mathbb{N}$. To construct a *regular log-modification* $V_\circ^{\log} \to X^{\log}$ that admits a base-field-isomorphic $X^{\log}$-morphism $V_\circ^{\log} \to V_\bullet^{\log}$, it suffices to "resolve the singularities" at the *finitely many points* of $V_\bullet \backslash U^{[1]}_{V_\bullet}$. To give a "resolution of singularities" of the sort desired, it suffices to construct, for each such v, a *"fan"* arising from a "locally finite nonsingular subdivision of the strongly convex rational polyhedral cone associated to the stalk of the characteristic sheaf of $V_\bullet^{\log}$ at v that is equivariant with respect to the Galois action on the stalk" [cf., e.g., the discussion at the beginning of [Mzk2], §2]. Since this is always possible [cf., e.g., the references quoted in the discussion of *loc. cit.*], we thus obtain a *regular log-modification* $V_\circ^{\log} \to X^{\log}$ that admits a base-field-isomorphic $X^{\log}$-morphism $V_\circ^{\log} \to V_\bullet^{\log}$; moreover, by replacing $V_\circ^{\log}$ with the result of blowing up once more at various points of $V_\circ \backslash U^{[1]}_{V_\circ}$, we may assume that each irreducible component C of $\underline{V}_\circ$

is *smooth* over $\underline{k}_\circ$, as desired. Finally, the construction of the natural isomorphism $\omega_{C/\underline{k}_\circ}(D_C) \xrightarrow{\sim} \omega_{V_\circ^{\log}/S_\circ^{\log}}|_C$ is immediate over $C \bigcap U_{V_\circ}^{[1]}$ [cf. our assumption that the log-modification $V_\circ^{\log} \to X^{\log}$ is *unramified* at C!]; one may then extend this natural isomorphism to C by means of an easy local calculation at the points of D_C. This completes the proof of assertion (iv). By assertions (iii) and (iv), the underlying schemes of the domain and codomain of the morphism $V_2^{\log} \to V_1^{\log} \times_{S_1^{\log}} S_2^{\log}$ of assertion (ii) are *normal*. Thus, assertion (ii) follows immediately from *Zariski's main theorem*.

Next, we consider assertion (v). We may assume without loss of generality that the $\phi_i^{\log}$ are *base-field-isomorphic* [so $\psi^{\log}$ is *log étale*]. Also, by blowing up once more at various points of $V_2 \backslash U_{V_2}^{[1]}$ [cf. the proof of assertion (iv)], one verifies immediately that we may assume without loss of generality that C is *smooth* over $\underline{k}_2$. Let us write $C^{\log}$ for the log scheme obtained by equipping C with the log structure determined by the points of C that meet the other irreducible components of D_{V_2}. Thus, after possibly replacing k_2 by a suitable finite unramified extension of k_2, we may assume that the interior $U_C \subseteq C$ of $C^{\log}$ is the open subscheme of a smooth proper curve of genus g over $\underline{k}(c)$ obtained by removing a divisor $D_C \subseteq C$ of degree $r > 0$ over $\underline{k}(c)$. On the other hand, it follows immediately from the definition of a *log-modification* [cf. Definition 2.4, (i)] and the well-known general theory of *toric varieties* [cf. the discussion of *"fans"* in the proof of assertion (iv)] that the pair (C, D_C) determines a *toric variety of dimension one*, and hence that U_C is isomorphic to a copy of $\mathbb{G}_{\mathrm{m}}$ over $\underline{k}(c)$, i.e., that $g = 0$, $r = 2$, as desired. Now the remaining portions of assertion (v) follow immediately. This completes the proof of assertion (v).

Next, we consider assertion (vi). First, we observe that if x *belongs* to the 1-interior, then it follows immediately from the simple structure of the monoid $\mathbb{N}$ [and the definition of the 1-interior] that V_x is *irreducible*. Thus, it suffices to consider the case where x *does not belong* to the 1-interior. Let $V_\circ^{\log} \to X^{\log}$, $V_\circ^{\log} \to V_\bullet^{\log}$ be as in assertion (iv). We may assume without loss of generality that the log-modifications $V_\bullet^{\log} \to X^{\log}$, $V_\circ^{\log} \to X^{\log}$ are *base-field-isomorphisms*. Also, by replacing k be a finite unramified extension of k, we may assume that every irreducible component of $\underline{V}_\bullet$ is geometrically irreducible over $\underline{k}$, and that every point of $V_\bullet \backslash U_{V_\bullet}^{[1]}$ is defined over $\underline{k}$. Then, since $V_\circ^{\log} \to V_\bullet^{\log}$ is *birational*, and $V_\bullet^{\log}$ is *log regular*, hence, in particular, *normal* [cf. assertion (iv)], it follows from *Zariski's main theorem* that the points of $V_\bullet \backslash U_{V_\bullet}^{[1]}$ correspond precisely [via $V_\circ^{\log} \to V_\bullet^{\log}$] to the *connected components* of the inverse image of $V_\bullet \backslash U_{V_\bullet}^{[1]}$ via $V_\circ^{\log} \to V_\bullet^{\log}$. Thus, the remainder of assertion (vi) follows immediately from assertion (v), applied to $V_\circ^{\log} \to V_\bullet^{\log}$, $V_\circ^{\log} \to X^{\log}$. This completes the proof of assertion (vi).

Next, we consider assertion (vii). We may assume without loss of generality that the log-modifications $V_\bullet^{\log} \to X^{\log}$, $V_\circ^{\log} \to X^{\log}$ are *base-field-isomorphic*. Then to verify assertion (vii), it suffices to observe that one may take $V_{\bullet\circ}^{\log} \stackrel{\mathrm{def}}{=} V_\bullet^{\log} \times_{X^{\log}} V_\circ^{\log}$. This completes the proof of assertion (vii). In a similar vein, assertion (viii) follows by observing that the fact that $Y_\eta \to X_\eta$ extends to a morphism $Y^{\log} \to X^{\log}$ follows, for instance, from [Mzk2], Theorem A, (1); thus, one may take $W_\bullet^{\log} \stackrel{\mathrm{def}}{=} V_\bullet^{\log} \times_{X^{\log}} Y^{\log}$. This completes the proof of assertion (viii).

Next, we consider assertion (ix). We may assume without loss of generality that $k = k_\circ$. Let $V_\circ^{\log} \to X^{\log}$ be the *base-field-isomorphic unramified log-modification* determined by the *regular semi-stable model* of X over S [cf. Remark 2.4.1]. Since $V_\circ$ is *proper* over S, it follows that $x_\circ$ extends to a point $\in V_\circ(\mathcal{O}_{K_\circ})$; if this point *fails* to lie in $U^{[1]}_{V_\circ}$, then it follows that it meets one of the *nodes* of $V_\circ$. On the other hand, since [after possibly replacing $k_\circ$ by a finite unramified extension of the discretely valued field $k_\circ$] the completion of the regular scheme $V_\circ$ at such a node is necessarily isomorphic over $\mathcal{O}_{k_\circ}$ to a complete local ring of the form

$$\mathcal{O}_{k_\circ}[[s,t]]/(s \cdot t - \pi_\circ)$$

[where s, t are indeterminates; $\pi_\circ$ is a uniformizer of $\mathcal{O}_{k_\circ}$], this contradicts our assumption that $\mathcal{O}_{k_\circ} \subseteq \mathcal{O}_{K_\circ}$ induces a *bijection* $k_\circ/\mathcal{O}^\times_{k_\circ} \xrightarrow{\sim} K_\circ/\mathcal{O}^\times_{K_\circ}$ [i.e., by considering the images via pull-back by $x_\circ$ of s, t in these value groups, in light of the relation "$s \cdot t - \pi_\circ$"]. This completes the proof of assertion (ix).

Next, we consider assertion (x). Write $N \to S$ for the *Néron model* of J_η over S. Thus, J may be regarded as an open subscheme of N. Note that the existence of the rational point $x_\bullet$ implies that $U_{x_\bullet}$ is *smooth* over S [cf. the proof of assertion (iii)]. Thus, it follows from the *universal property* of the Néron model [which is typically used to define the Néron model] that ι_η extends to a morphism $U_{x_\bullet} \to N$. Since $C \bigcap U_{x_\bullet}$ is *connected* [cf. the definition of $U_{x_\bullet}$], the fact that the image of this morphism lies in $J \subseteq N$ follows immediately from the fact that [by definition] it maps $x_\bullet$ to the identity element of $J(\mathcal{O}_k)$. Thus, we obtain a morphism $U_{x_\bullet} \to J$. To verify that this morphism is *unramified*, it suffices [by considering appropriate *translation* automorphisms of J] to show that it induces a *surjection* on Zariski cotangent spaces at $x_\bullet$; but the induced map on Zariski cotangent spaces at $x_\bullet$ is easily computed [by considering the long exact sequence on cohomology associated to the short exact sequence $0 \to \mathcal{O}_{V_\bullet} \to \mathcal{O}_{V_\bullet}(x_\bullet) \to \mathcal{O}_{V_\bullet}(x_\bullet)|_{x_\bullet} \to 0$ on $V_\bullet$, then taking duals] to be the map

$$H^0(X^{\log}, \omega_{X^{\log}/S^{\log}}) \xrightarrow{\sim} H^0(V^{\log}, \omega_{V_\bullet^{\log}/S^{\log}}) \to \omega_{V_\bullet^{\log}/S^{\log}}|_{x_\bullet}$$

[where we recall the natural isomorphism $\omega_{X^{\log}/S^{\log}}|_{V_\bullet^{\log}} \xrightarrow{\sim} \omega_{V_\bullet^{\log}/S^{\log}}$, arising from the fact that $\phi_\bullet^{\log} : V_\bullet^{\log} \to X^{\log}$ is *log étale* — cf. assertion (iii)] given by *evaluating at* $x_\bullet$, hence is *surjective* so long as $\underline{X}$ is *loop-ample* [cf. §0]. This completes the proof of assertion (x).

Next, we consider assertion (xi). First, let us observe that it follows immediately from the surjectivity of $\underline{W}_\bullet \to \underline{V}_\bullet$ that every *vertex* of $\Gamma_{\underline{V}_\bullet}$ may be *lifted* to a vertex of $\Gamma_{\underline{W}_\bullet}$. Next, I *claim* that every *edge* of $\Gamma_{\underline{V}_\bullet}$ may be *lifted* to an edge of $\Gamma_{\underline{W}_\bullet}$. Indeed, let $y \in W_\bullet(\underline{k})$, $x \in V_\bullet(\underline{k})$ be such that $y \mapsto x$; write W_y, V_x for the respective spectra of the local rings obtained by *completing* $W_\bullet$, $V_\bullet$ at y, x. Then the morphism $W^{\log} \to V^{\log}$ induces a *finite, dominant*, hence *surjective*, morphism $W_y \to V_x$. In particular, this morphism $W_y \to V_x$ induces a *surjection* from the set I_y of irreducible components of $\underline{W}_\bullet$ that pass through y to the set I_x of irreducible components of $\underline{V}_\bullet$ that pass through x. Thus, if x corresponds to an edge of the dual graph $\Gamma_{\underline{V}_\bullet}$, then this set I_x is of cardinality 2 [cf. assertion (vi)]; since I_y is of cardinality ≤ 2 [cf. assertion (vi)], the existence of the surjection $I_y \twoheadrightarrow I_x$ thus implies that this surjection is, in fact, a bijection $I_y \xrightarrow{\sim} I_x$, hence that y corresponds to an edge of $\Gamma_{\underline{W}_\bullet}$ [cf. assertion (vi)]. This completes the proof of the *claim*. Thus, by starting at one of the terminal vertices of γ_V, and proceeding along γ_V from vertex to edge to vertex, etc., one concludes immediately the existence of a simple path γ_W *lifting* γ_V. To verify *uniqueness* when condition (c) holds, write v_1, v_2 for the terminal vertices of γ_V; e' for the edge of γ_V that is nearest to v_1 among those edges of γ_V that lift to *different* edges in γ_W, γ'_W; v'_1 (respectively, v'_2) for the vertex to which e' abuts that lies in the same connected component of the complement of e' in γ_V as v_1 (respectively, v_2); v_1^+ for the vertex of γ_V that is nearest to v_1 among those vertices of γ_V lying between v'_2 and v_2 which lift to the *same* vertex in γ_W, γ'_W. Then by traveling along γ_W from the vertex w_1 of γ_W lifting v_1 to the vertex w_1^+ of γ_W lifting v_1^+, then traveling back along γ'_W from w_1^+ [which, by definition, also belongs to γ'_W] to w_1 [which, by definition, also belongs to γ'_W], one obtains a *"nontrivial loop"* in $\Gamma_{\underline{W}_\bullet}$ [i.e., a nonzero element of $H_1(\Gamma_{\underline{W}_\bullet}, \mathbb{Q})$] that maps to a *"trivial loop"* in $\Gamma_{\underline{V}_\bullet}$ [i.e., the zero element of $H_1(\Gamma_{\underline{V}_\bullet}, \mathbb{Q})$]. But this contradicts the assumption that the morphism $W_\bullet^{\log} \to V_\bullet^{\log}$ is *loop-preserving* [cf. condition (c)]. This completes the proof of assertion (xi).

Finally, we consider assertion (xii). First, we observe that the hypotheses of assertion (xii) are *stable* with respect to base-change in S. In particular, we may always replace $S = \mathrm{Spec}(\mathcal{O}_k)$ by the normalization of S in some finite separable extension of k. Next, by assertion (iv), we may assume that there exists a *base-field-isomorphic, regular, split log-modification* $V_\circ^{\log} \to X^{\log}$, together with an $X^{\log}$-morphism $V_\circ^{\log} \to V_\bullet^{\log}$. Moreover, if we take $W_\circ^{\log} \stackrel{\mathrm{def}}{=} V_\circ^{\log} \times_{V_\bullet^{\log}} W_\bullet^{\log}$, then the composite morphism $W_\circ^{\log} \to W_\bullet^{\log} \to Y^{\log}$ forms a *base-field-isomorphic log-modification* such that the projection $W_\circ^{\log} \to V_\circ^{\log}$ is *finite* [by condition (b); cf. also the *finiteness* mentioned in the discussion entitled "Log Schemes" given in §0]. In particular, by replacing $W_\bullet^{\log} \to V_\bullet^{\log}$ by $W_\circ^{\log} \to V_\circ^{\log}$ [cf. also assertion (v), concerning the effect on the simple path γ_V; assertion (vi), concerning the effect on the *loop-rank*], we may assume that the log-modification $V_\bullet^{\log} \to X^{\log}$ is *regular*. Here, let us note that since $W_\bullet^{\log} \to V_\bullet^{\log}$ is *finite*, and $W_\bullet^{\log}$ is *log regular* [cf. assertion (iv)], which implies, in particular, that $W_\bullet$ is *normal* [so $W_\bullet$ is the *normalization* of $V_\bullet$ in Y_η], it follows that $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(Y_\eta/X_\eta)$ ($\cong \mathbb{Z}/p\mathbb{Z}$) acts on $W_\bullet^{\log}$. Also, by assertion (iv), we may assume that there exists a *base-field-isomorphic, regular, split log-modification* $W_\triangle^{\log} \to Y^{\log}$, together with a $Y^{\log}$-morphism $W_\triangle^{\log} \to W_\bullet^{\log}$; moreover, it follows immediately from the proof of assertion (iv) that we may choose $W_\triangle^{\log}$ so that the action of G *extends* to $W_\triangle^{\log}$. Finally, we observe that it follows from assertion (v) that every irreducible component of the exceptional divisors of $\underline{W}_\triangle$, $\underline{V}_\bullet$ [relative to the morphisms $W_\triangle^{\log} \to Y^{\log}$, $V_\bullet^{\log} \to X^{\log}$] is isomorphic to $\mathbb{P}^1_{\underline{k}}$.

Let E_W be the irreducible component of $\underline{W}_\bullet$ corresponding to w_{exc}. Thus, there exists a unique irreducible component F_W of $\underline{W}_\triangle$ that maps finitely to E_W; moreover, E_W maps finitely to an irreducible component E_V of $\underline{V}_\bullet$ [corresponding to v_{exc}] that lies in the exceptional divisor of $V_\bullet^{\log} \to X^{\log}$. Thus, we have *finite morphisms*

$$F_W \to E_W \to E_V$$

— where F_W, E_V are isomorphic to $\mathbb{P}^1_{\underline{k}}$; the first morphism $F_W \to E_W$ is a morphism between irreducible schemes that induces an *isomorphism* between the respective function fields. Now to complete the proof of assertion (xii), it suffices to assume that

the composite morphism $F_W \to E_V$ is *generically étale*

and derive a contradiction. Let us refer to the two k-valued points of F_W (respectively, E_V) [cf. assertion (v)] that lie outside $U^{[1]}_{W_\triangle}$ (respectively, $U^{[1]}_{V_\bullet}$) as the *critical points* of F_W (respectively, E_V). Then since the divisor on $W^{\log}_\bullet$ (respectively, $V^{\log}_\bullet$) at which the morphism $W^{\log}_\bullet \to V^{\log}_\bullet$ is [necessarily *wildly*] *ramified* does *not*, by our assumption, contain E_W (respectively, E_V), it follows that the divisor on F_W (respectively, E_V) at which $F_W \to E_V$ is ramified is supported in the divisor defined by the two *critical points* of F_W (respectively, E_V).

Next, let us write v_1, v_2 for the *terminal vertices* of γ_V. Note that by conditions (d), (e), it follows that v_1, v_2 lift, respectively, to *unique* vertices w_1, w_2 of $\Gamma_{\underline{W}_\bullet}$. In particular, it follows that *any* two simple paths in $\Gamma_{\underline{W}_\bullet}$ lifting γ_V are *co-terminal*. Now I *claim* that the morphism $F_W \to E_V$ is of *degree* p. Indeed, if this morphism is of degree 1, then it follows that there exists a G-conjugate w'_{exc} of w_{exc} such that $w'_{\mathrm{exc}} \neq w_{\mathrm{exc}}$. Thus, considering the *G-conjugates* of any simple path in $\Gamma_{\underline{W}_\bullet}$ lifting γ_V, it follows that we obtain *two distinct* [necessarily *co-terminal*!] simple paths in $\Gamma_{\underline{W}_\bullet}$ lifting γ_V. But this contradicts the *uniqueness* portion of assertion (xi). This completes the proof of the *claim*. Note that this claim implies that we have *G-equivariant* morphisms $F_W \to E_W \to E_V$, where G acts *trivially* on E_V.

Next, I *claim* that G *fixes* each of the critical points of F_W. Indeed, it follows immediately from the definitions that G preserves the divisor of critical points of F_W. Thus, if G *fails* to fix each of the critical points of F_W, then it follows that G permutes the two critical points of F_W, hence that $p = 2$. But since $F_W \to E_V$ is *of degree* p and *unramified* outside the divisor of critical points of F_W, this implies that $\mathbb{P}^1_{\underline{k}} \cong F_W \to E_V \cong \mathbb{P}^1_{\underline{k}}$ is *finite étale*, hence [since, as is well-known, the étale fundamental group of $\mathbb{P}^1_{\underline{k}}$ is *trivial*!] that $F_W \to E_V$ is an *isomorphism* — in contradiction to the fact that $F_W \to E_V$ is *of degree* $p > 1$. This completes the proof of the *claim*. Note that this claim implies that the morphism $F_W \to E_V$ is *ramified* at the critical points of F_W, and that the set of two critical points of F_W maps *bijectively* to the set of two critical points of E_V. In particular, it follows that $F_W \to E_V$ determines a *finite étale covering* $(\mathbb{G}_{\mathrm{m}})_{\underline{k}} \to (\mathbb{G}_{\mathrm{m}})_{\underline{k}}$ *of degree* p. On the other hand, any morphism $(\mathbb{G}_{\mathrm{m}})_{\underline{k}} \to (\mathbb{G}_{\mathrm{m}})_{\underline{k}}$ is determined by a *unit* on $(\mathbb{G}_{\mathrm{m}})_{\underline{k}}$, i.e., by a $\underline{k}^\times$-multiple of U^n, where U is the standard coordinate on $(\mathbb{G}_{\mathrm{m}})_{\underline{k}}$, and n is the *degree* of the morphism. Since the morphism determined by a $\underline{k}^\times$-multiple of U^p clearly *fails to be generically étale*, we thus obtain a *contradiction*. This completes the proof of assertion (xii). ○

In the following, we shall write

$$\text{“}\pi_1(-)\text{”}$$

for the *“log fundamental group”* of the log scheme in parentheses, relative to an appropriate choice of basepoint [cf. [Ill] for a survey of the theory of log fundamental groups]. Also, from now on, we shall assume, until further notice, that:

The discrete valuation ring $\mathcal{O}_k$ is *of mixed characteristic*, with residue field $\underline{k}$ *perfect of characteristic* p and of *countable cardinality.*

Recall that by *"Krasner's lemma"* [cf. [Kobl], pp. 69-70, as well as the proof given above of Lemma 2.1], given a splitting field k' over k of a monic polynomial $f(T)$ [where T is an indeterminate] of degree n with coefficients in k, every monic polynomial $h(T)$ of degree n with coefficients in k that are *sufficiently close* [in the topology of k] to the coefficients of $f(T)$ *also splits* in k'. Thus, it follows from our assumption that $\underline{k}$ is of countable cardinality that $\overline{k}$ admits a *countable* collection $\mathcal{F}$ of subfields which are finite and Galois over k such that *every* finite Galois extension of k contained in $\overline{k}$ is contained in a subfield that belongs to the collection $\mathcal{F}$.

In some sense, the *main technical result* of the present §2 is the following lemma.

Lemma 2.6. **(Prime-power Cyclic Coverings and Log-modifications)** *Suppose that* $\underline{X}$ *is* **loop-ample** *[cf. §0]. Then:*

(i) **(Existence of Wild Ramification)** *Let*

$$X_\eta^+ \to X_\eta$$

be a **finite étale Galois covering** *of hyperbolic curves over* η *with stable reduction over* S *such that* $\mathrm{Gal}(X_\eta^+/X_\eta)$ *is isomorphic to a product of* $2g_X$ *copies of* $\mathbb{Z}/p\mathbb{Z}$ *[so such a covering always exists after possibly replacing* k *by a finite extension of* k*];* $V^{\log} \to X^{\log}$ *a* **split, base-field-isomorphic log-modification**. *Then* $X_\eta^+ \to X_\eta$ *is* **wildly ramified** *over* **every** *irreducible component* C *of* $\underline{V}$.

(ii) **(Loopification vs. Component Crushing)** *After possibly replacing* k *by a finite extension of* k, *there exist data as follows: a* **stable curve** $Y \to S$ *with* **smooth** *generic fiber* $Y_\eta \stackrel{\mathrm{def}}{=} Y \times_S \eta$ *and associated log scheme* $Y^{\log}$*; a* **cyclic finite étale covering** $Y_\eta \to X_\eta$ *of* **degree** *a* **positive power of** $\boldsymbol{p}$ *— which determines a morphism*

$$Y^{\log} \to X^{\log}$$

— such that **at least one** *of the following two conditions is satisfied:*

(a) $Y_\eta \to X_\eta$ *is* **loopifying** *and* **wildly ramified** *at some [necessarily]* **stable** *irreducible component* C *of* $\underline{Y}$ *which is* **potentially base-semi-stable** *relative to* $Y_\eta \to X_\eta$*;*

(b) *there exists a [necessarily]* **stable** *irreducible component* C *of* $\underline{Y}$ *which is* **not potentially base-semi-stable** *relative to* $Y_\eta \to X_\eta$.

(iii) **(Components Crushed to the 1-Interior)** *In the situation of (ii), there exists a* **commutative diagram**

$$\begin{array}{ccc} W^{\log} & \longrightarrow & Q^{\log} \\ \downarrow & & \downarrow \\ Y^{\log} & \longrightarrow & X^{\log} \end{array}$$

— where the vertical morphisms are split, base-field-isomorphic **log-modifications***; the horizontal morphism in the bottom line is the morphism already referred to; the*

natural action of $\mathrm{Gal}(Y_\eta/X_\eta)$ *on* $Y^{\log}$ *extends to* $W^{\log}$ *— such that the following property is satisfied: If condition (a) (respectively, (b)) of (ii) is satisfied, then the unique irreducible component* C_W *of* $\underline{W}$ *that maps finitely to the irreducible component* C *of condition (a) (respectively, (b))* **maps finitely to** Q *(respectively,* **maps to a closed point** *of* $U_Q^{[1]}$*).*

(iv) **(Group-theoretic Characterization of Crushing)** *In the situation of (ii), let* C *be a [necessarily]* **stable** *irreducible component of* $\underline{Y}$*;* l *a prime* $\neq p$*. Write* $\mathrm{Gal}(\overline{k}/k) \twoheadrightarrow G_{\underline{k}^{\log}}$ *for the* **maximal tamely ramified quotient***;* $\Delta_{\underline{Y}^{\log}}$ *(respectively,* $\Delta_{\underline{X}^{\log}}$*) for the* **maximal pro-*l* quotient** *of the kernel of the natural [outer] surjection* $\pi_1(\underline{Y}^{\log}) \twoheadrightarrow G_{\underline{k}^{\log}}$ *(respectively,* $\pi_1(\underline{X}^{\log}) \twoheadrightarrow G_{\underline{k}^{\log}}$*);* $\Delta_C \subseteq \Delta_{\underline{Y}^{\log}}$ *for the* **decomposition group** *of* C *[well-defined up to conjugation by an element of* $\pi_1(\underline{Y}^{\log})$*]. [Thus,* $\Delta_{\underline{Y}^{\log}}$ *(respectively,* $\Delta_{\underline{X}^{\log}}$*) may be identified with the* **maximal pro-*l* quotient** *of* $\mathrm{Ker}(\pi_1(Y_\eta) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k))$ *(respectively,* $\mathrm{Ker}(\pi_1(X_\eta) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k))$*) — cf., e.g., [MT], Proposition 2.2, (v).] Then the following two conditions are* **equivalent***:*

(a) the image of Δ_C *in* $\Delta_{\underline{X}^{\log}}$ *is* **trivial***;*

(b) there exists a commutative diagram

$$\begin{array}{ccc} W^{\log} & \longrightarrow & Q^{\log} \\ \big\downarrow & & \big\downarrow \\ Y^{\log} & \longrightarrow & X^{\log} \end{array}$$

— where the vertical morphisms are split, base-field-isomorphic **log-modifications***; the horizontal morphism in the bottom line is the morphism already referred to — such that the unique irreducible component* C_W *of* $\underline{W}$ *that maps finitely to* C **maps to a closed point** *of* $U_Q^{[1]}$*.*

(v) **(Group-theoretic Characterization of Wild Ramification)** *In the situation of (iv), the morphism* $Y_\eta \to X_\eta$ *is* **wildly ramified** *at* C *if and only if* $\mathrm{Gal}(Y_\eta/X_\eta)$ **stabilizes** *[the conjugacy class of] and induces the* **identity** *[outer automorphism] on the [***normally terminal** *— cf. [Mzk13], Proposition 1.2, (ii); [Mzk17], Lemma 2.12] subgroup* Δ_C *of* $\Delta_{\underline{Y}^{\log}}$*.*

Proof. Let us write

$$T_X \stackrel{\mathrm{def}}{=} \pi_1(X_\eta \times_k \overline{k})^{\mathrm{ab}} \otimes \mathbb{Z}_p$$

for the *maximal pro-p abelian quotient* of the geometric fundamental group of X_η. Thus, T_X is a *free* $\mathbb{Z}_p$*-module of rank* $2g_X$.

Next, we consider assertion (i). Upon base-change to $\overline{k}$, the covering $X_\eta^+ \to X_\eta$ corresponds to the open subgroup $p \cdot T_X \subseteq T_X$. Let us write $J \to S$ for the *uniquely determined semi-abelian scheme* that extends the Jacobian $J_\eta \to \eta$ of X_η. After possibly replacing k by a finite extension of k, there exists a rational point $x \in U_V^{[1]}(\mathcal{O}_k)$ that meets C. Thus, for some Zariski open neighborhood U_x of the image of x in V, we obtain a morphism $\iota : U_x \to J$, as in Proposition 2.5, (x). Moreover, since we have assumed that X is *loop-ample*, it follows that this morphism ι is *unramified*. Now if the morphism $X_\eta^+ \to X_\eta$ is *tamely ramified* over

C, then it follows from our assumptions on the covering $X_\eta^+ \to X_\eta$, together with the interpretation of J in terms of *Néron models* [cf. the proof of Proposition 2.5, (x)], that [after possibly replacing k by a finite extension of k] there exists some finite *separable* extension L of the function field $\underline{k}(C)$ of C such that there exists a *commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(L) & \xrightarrow{\iota_L} & J \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle [p]} \\ \mathrm{Spec}(\underline{k}(C)) & \xrightarrow{\iota_C} & J \end{array}$$

— where ϵ is the [étale] morphism determined by the given inclusion $\underline{k}(C) \hookrightarrow L$; $[p]$ is the morphism given by *multiplication by* p on the group scheme J; ι_C is the restriction of ι to $\mathrm{Spec}(\underline{k}(C))$. On the other hand, since the restriction of $[p]$ to the special fiber $\underline{J}$ of J factors through the *Frobenius morphism* on $\underline{J}$, it follows that $[p] \circ \iota_L$ *fails to be unramified*. Thus, since $\iota_C \circ \epsilon$ *is unramified*, we obtain a *contradiction* to the commutativity of the diagram. This completes the proof of assertion (i).

Next, we consider assertion (ii). If $V^{\log} \to X^{\log}$ is a *split, base-field-isomorphic log-modification*, and C is an irreducible component of $\underline{V}$, then let us write

$$D_C \subseteq T_X$$

for the *decomposition group* associated to C and

$$I_C \subseteq D_C$$

for the *wild inertia group* associated to C. Note that since T_X is *abelian*, and $V^{\log}$ is *split*, it follows that the subgroups D_C, I_C are *well-defined* [and completely determined by C]. Moreover, by assertion (i), it follows that I_C has *nontrivial image* in $T_X \otimes \mathbb{Z}/p\mathbb{Z}$.

Next, I *claim* that if $V_\circ^{\log} \to X^{\log}$ is a split, base-field-isomorphic log-modification, then $V_\circ^{\log} \to X^{\log}$ is *completely determined*, as a log scheme over $X^{\log}$, up to *countably many possibilities*, by $X^{\log}$. Indeed, the morphism $V_\circ^{\log} \to X^{\log}$ is an *isomorphism* over the 1-*interior* of $X^{\log}$ [cf. Proposition 2.5, (i), (ii)]. Moreover, at each of the *finitely many* points x of $X^{\log}$ lying in the complement of the 1-interior, $V_\circ^{\log} \to X^{\log}$ is determined by *countably many choices* of certain combinatorial data involving the groupification of the stalk of the characteristic sheaf of $X^{\log}$ at x [cf. the proof of Proposition 2.5, (iv)]. This completes the proof of the *claim*. In particular, since $\underline{k}$ is assumed to be of *countable cardinality* [cf. the discussion preceding the present Lemma 2.6], it follows that:

> There exists a *countable cofinal collection* $\mathcal{M}$ *of split log-modifications of* $X^{\log}$.

In particular, it follows that if we write $\mathcal{C}$ for the set of all irreducible components of the special fibers of log-modifications belonging to $\mathcal{M}$, then the collection of [nontrivial] subgroups of T_X of the form "I_C", where $C \in \mathcal{C}$, is *of countable cardinality*. Thus, we may, for instance, *enumerate* the elements of $\mathcal{C}$ via the natural numbers so as to obtain a sequence $C_1, C_2, \ldots$ [i.e., which includes all elements of $\mathcal{C}$]. Since $\mathbb{Z}_p$, on the other hand, is *of uncountable cardinality*, we thus conclude that there exists a surjection

$$\Lambda : T_X \twoheadrightarrow \Lambda_X$$

— where $\Lambda_X \cong \mathbb{Z}_p$ — such that the following properties are satisfied:

(1) For every subgroup I_C, where $C \in \mathcal{C}$, we have $\Lambda_C \stackrel{\text{def}}{=} \Lambda(I_C) \neq \{0\}$.

(2) There exists a stable component C_0 of X such that $\Lambda_{C_0} = \Lambda_X$.

[For instance, by applying the fact that each I_{C_n} has *nontrivial image* in $T_X \otimes \mathbb{Z}/p\mathbb{Z}$, one may construct Λ by constructing inductively on n [a natural number] an increasing sequence of natural numbers $m_1 < m_2 < \ldots$ such that I_{C_n} maps to a nonzero subgroup of $\Lambda_X \otimes \mathbb{Z}/p^{m_n}\mathbb{Z}$.]

Let us refer to a connected finite étale Galois covering $X'_\eta \to X_\eta$ [which may only be defined after possibly replacing k by a finite extension of k] of hyperbolic curves over η with stable reduction over S as a Λ-*covering* if the covering $X'_\eta \times_k \overline{k} \to X_\eta \times_k \overline{k}$ arises from an open subgroup of Λ_X; thus, $\mathrm{Gal}(X'_\eta/X_\eta)$ may be thought of as a *finite quotient* of Λ_X by an open subgroup $\Lambda_{X'} \subseteq \Lambda_X$. Note that since every Λ_C, where $C \in \mathcal{C}$, is *isomorphic to* $\mathbb{Z}_p$, it follows that the following property also holds:

(3) For any pair $X''_\eta \to X'_\eta \to X_\eta$ of Λ-coverings of X_η such that Λ_C has *nontrivial image* in $\mathrm{Gal}(X'_\eta/X_\eta) \cong \Lambda_X/\Lambda_{X'}$, it follows that $\Lambda_C \bigcap \Lambda_{X'}$ *surjects* onto $\Lambda_{X'}/\Lambda_{X''}$ — i.e., that the covering $X''_\eta \to X'_\eta$ is *totally wildly ramified* over any valuation of the function field of X'_η whose center on X is equal to the generic point of C.

Now to complete the proof of assertion (ii), it suffices to *derive a contradiction* upon making the following two further assumptions:

(4) Every Λ-covering is *loop-preserving*.

(5) For every Λ-covering $X'_\eta \to X_\eta$ [which extends to a morphism $(X')^{\log} \to X^{\log}$ of log stable curves], there exists [after possibly replacing k by a finite extension of k] a *split, base-field-isomorphic log-modification* $V^{\log} \to X^{\log}$ such that the morphism $X'_\eta \to X_\eta \cong V_\eta \subseteq V$ *extends* to a *quasi-finite morphism* from some Zariski neighborhood in X' of the generic points of $\underline{X}'$ to V.

[Indeed, if assumption (4) is *false*, then it follows immediately that condition (a) of assertion (ii) holds [cf. property (2)]; if assumption (5) is *false*, then it follows immediately that condition (b) of assertion (ii) holds.] Note, moreover, that assumption (5) implies the following property:

(6) For every Λ-covering $X'_\eta \to X_\eta$ [which extends to a morphism $(X')^{\log} \to X^{\log}$ of log stable curves], there exist [after possibly replacing k by a finite extension of k], *split, base-field-isomorphic log-modifications* $(V')^{\log} \to$

$(X')^{\log}$, $V^{\log} \to X^{\log}$, together with a *finite* morphism $(V')^{\log} \to V^{\log}$ lying over $(X')^{\log} \to X^{\log}$.

[Indeed, if, in the notation of property (5), one takes $(V')^{\log} \stackrel{\text{def}}{=} V^{\log} \times_{X^{\log}} (X')^{\log}$, then the natural projection morphism $(V')^{\log} \to (X')^{\log}$ is a split, base-field-isomorphic log-modification. Moreover, every irreducible component of $\underline{V}'$ maps *finitely* either to some irreducible component of $\underline{V}$ or to some irreducible component of $\underline{X}'$. Thus, by property (5), we conclude that every irreducible component of $\underline{V}'$ maps *finitely* to some irreducible component of $\underline{V}$, hence [by *Zariski's main theorem*] that the natural projection morphism $(V')^{\log} \to V^{\log}$ is *finite*, as desired.]

Next, let us observe that properties (4), (6) imply the following properties:

(7) There exists a Λ-covering $X'_\eta \to X_\eta$ [after possibly replacing k by a finite extension of k] such that *every* Λ_C, where $C \in \mathcal{C}$, has *nontrivial image* in $\Lambda_X/\Lambda_{X'}$.

(8) There exist Λ-coverings $X''_\eta \to X'_\eta \to X_\eta$ [after possibly replacing k by a finite extension of k] such that $\Lambda_{X'}/\Lambda_{X''} \cong \mathbb{Z}/p\mathbb{Z}$, and, moreover, for *every* Λ_C, where $C \in \mathcal{C}$, the intersection $\Lambda_C \bigcap \Lambda_{X'}$ *surjects* onto $\Lambda_{X'}/\Lambda_{X''}$.

Indeed, by property (3), it follows that property (8) follows immediately from property (7). To verify property (7), we reason as follows: First, let

$$X^\dagger_\eta \to X^*_\eta \to X_\eta$$

be Λ-*coverings* [which exist after possibly replacing k by a finite extension of k] such that $\Lambda_{X^*}/\Lambda_{X^\dagger}$ is of order p, and Λ_C has *nontrivial image* in Λ_X/Λ_{X^*} [which implies that $\Lambda_C \bigcap \Lambda_{X^*}$ *surjects* onto $\Lambda_{X^*}/\Lambda_{X^\dagger}$ — cf. property (3)] for every *stable* irreducible component C of $\underline{X}$; write $X^{\dagger,\log} \to X^{*,\log} \to X^{\log}$ for the resulting morphisms of log stable curves. [Note that such Λ-coverings exist, precisely because there are only *finitely* many such stable C.] Let $V^{\dagger,\log} \to X^{\dagger,\log}$, $V^{*,\log} \to X^{*,\log}$, $V^{\log} \to X^{\log}$ be *split, base-field-isomorphic log-modifications* such that there exist *finite, loop-preserving* morphisms

$$V^{\dagger,\log} \to V^{*,\log} \to V^{\log}$$

lying over $X^{\dagger,\log} \to X^{*,\log} \to X^{\log}$ [cf. properties (4), (6)]. [Thus, $V^{\dagger,\log}$, $V^{*,\log}$ are *completely determined* by $V^{\log}$ — i.e., by taking the *normalization* of V in $X^\dagger_\eta$, X^*_η.] Next, let us observe that every node ν of $\underline{X}$ determines a *simple path* γ^ν_V in the dual graph $\Gamma_{\underline{V}}$ [i.e., by taking the inverse image of ν in $\underline{V}$ — cf. Proposition 2.5, (v)], whose *terminal vertices* are *stable* irreducible components of $\underline{V}$ [but whose non-terminal vertices are *non-stable* irreducible components of $\underline{V}$]. Thus, by Proposition 2.5, (xi) [which is applicable, in light of properties (4), (6)], it follows that γ^ν_V *lifts* [*uniquely* — i.e., once one fixes liftings of the terminal vertices] to simple paths $\gamma^\nu_{V^*}$ in $\Gamma_{\underline{V}^*}$, $\gamma^\nu_{V^\dagger}$ in $\Gamma_{\underline{V}^\dagger}$. Since, moreover, $X^\dagger_\eta \to X^*_\eta$ is *totally wildly ramified* over the *terminal vertices* of $\gamma^\nu_{V^*}$, it thus follows that we may apply Proposition 2.5, (xii), to conclude that $X^\dagger_\eta \to X^*_\eta$ is *wildly ramified* at every *non-stable* vertex of $\gamma^\nu_{V^\dagger}$. Write $\mathcal{B}$ for the set of irreducible components of $\underline{V}$ [which we think of as valuations of the function field of $X_\eta \times_k \overline{k}$] that are the images of *stable* vertices of $\gamma^\nu_{V^\dagger}$, for nodes ν

of $\underline{X}$. Observe that if we keep the coverings $X_\eta^\dagger \to X_\eta^* \to X_\eta$ *fixed*, but *vary* the log-modification $V^{\log} \to X^{\log}$ [among, say, elements of $\mathcal{M}$], then the set $\mathcal{B}$ *remains unchanged* [if we think of $\mathcal{B}$ as a set of valuations of the function field of $X_\eta \times_k \overline{k}$] and *of finite cardinality* [bounded by the cardinality of the set of stable irreducible components of $\underline{X}^\dagger$]. Thus, in summary, if we think of $\mathcal{B}$ as a *subset* of $\mathcal{C}$, then we may conclude the following:

$$\Lambda_C \bigcap \Lambda_{X^*} \ \textit{surjects} \text{ onto } \Lambda_{X^*}/\Lambda_{X^\dagger}, \text{ for all } C \in \mathcal{C}\backslash\mathcal{B}.$$

Since $\mathcal{B}$ is *finite*, it thus follows that there exists a Λ-*covering* $X'_\eta \to X_\eta^\dagger \to X_\eta$ such that $\Lambda_C \bigcap \Lambda_{X^\dagger}$ has *nontrivial image* in $\Lambda_{X^\dagger}/\Lambda_{X'}$, for all $C \in \mathcal{C}$ [cf. property (3)]. This completes the proof of property (7).

Next, let us consider Λ-*coverings*

$$X''_\eta \to X'_\eta \to X$$

[which exist after possibly replacing k by a finite extension of k] such that $\Lambda_{X'}/\Lambda_{X''}$ is of order p, and Λ_C has *nontrivial image* in $\Lambda_X/\Lambda_{X'}$ [which implies that $\Lambda_C \bigcap \Lambda_{X'}$ *surjects* onto $\Lambda_{X'}/\Lambda_{X''}$ — cf. property (3)] for all $C \in \mathcal{C}$ [cf. property (7)]; write $(X'')^{\log} \to (X')^{\log} \to X^{\log}$ for the resulting morphisms of log stable curves. Let $(V'')^{\log} \to (X'')^{\log}$, $(V')^{\log} \to (X')^{\log}$ be *split, base-field-isomorphic, unramified log-modifications* such that there exists a *finite, loop-preserving* morphism

$$(V'')^{\log} \to (V')^{\log}$$

lying over $(X'')^{\log} \to (X')^{\log}$ [which exist after possibly replacing k by a finite extension of k — cf. properties (4), (6); Proposition 2.5, (iii)].

Next, let us consider the *logarithmic derivative*

$$\delta : \omega_{(V')^{\log}/S^{\log}}|_{V''} \to \omega_{(V'')^{\log}/S^{\log}}$$

of the morphism $(V'')^{\log} \to (V')^{\log}$. Since this morphism is *finite étale* over η, it follows that δ is an *isomorphism* over η. On the other hand, since $X''_\eta \to X'_\eta$ is *totally wildly ramified* over every irreducible component C of $\underline{V}'$ [i.e., induces a *purely inseparable* extension of degree p of the function field of C], it follows that δ *vanishes* on the special fiber $\underline{V}''$. Write $\delta^* \stackrel{\text{def}}{=} \pi^{-n} \cdot \delta$, for the maximal integer n such that $\pi^{-n} \cdot \delta$ remains *integral*. Thus, we obtain a morphism

$$\delta^* : \omega_{(V')^{\log}/S^{\log}}|_{V''} \to \omega_{(V'')^{\log}/S^{\log}}$$

of line bundles on V'' which is *not identically zero* on $\underline{V}''$. Let C'' be an irreducible component of $\underline{V}''$ such that $\delta^*|_{C''} \not\equiv 0$. Note that since $\underline{V}''$ has *at least one* stable irreducible component, it follows that we may choose C'' such that *either* C'' is stable *or* C'' meets an irreducible component C^* of $\underline{V}''$ such that $\delta^*|_{C^*} \equiv 0$. Thus, if C'' is *not* stable, then it follows that $\delta^*|_{C''}$ has *at least one zero* [i.e., is not an isomorphism of line bundles]. Write C' for the irreducible component of $\underline{V}'$ which is the image of C''; $E'' \to C''$, $E' \to C'$ for the respective *normalizations*; $g_{E'}$, $g_{E''}$ for the respective *genera* of E', E''. Also, let us refer to the points of C'', E'', C',

E' that do not map to the respective 1-interiors of $(V'')^{\log}$, $(V')^{\log}$ as *critical points*. Write $D_{E'} \subseteq E'$, $D_{E''} \subseteq E''$ for the respective *divisors of critical points*; $r_{E'}$, $r_{E''}$ for the respective *degrees* of $D_{E'}$, $D_{E''}$.

Next, let us consider the morphism $C'' \to C'$. Since this morphism $C'' \to C'$ induces a *purely inseparable extension of degree* p on the respective function fields, it follows that we have an *isomorphism of* $\underline{k}$*-schemes* $E'' \times_{\underline{k}} \underline{k}' \xrightarrow{\sim} E'$ [so $g_{E'} = g_{E''}$], where we write $\underline{k} \hookrightarrow \underline{k}' \cong \underline{k}$ for the [degree one!] field extension determined by the *Frobenius morphism* on $\underline{k}$. Next, I *claim* that the critical points of C'' map to critical points of C'. Indeed, if a critical point of C'' maps to a *non-critical point* c of C', then let us write C''_c, C'_c for the spectra of the respective completions of the local rings of C'', C' at [the fiber over] c. Then observe that since V' is *regular* [of dimension two] at c [cf. Proposition 2.5, (iv)], while V'' is the *normalization* of V' in X''_η, it follows from elementary commutative algebra that V'' is *finite* and *flat* over V' of *degree* p at c. Thus, if we write η_c for the spectrum of the residue field of the *unique* generic point of the *irreducible* scheme C'_c, then $C''_c \times_{C'_c} \eta_c \to \eta_c$ is *finite, flat of degree* $\leq p$; [since $X''_\eta \to X'_\eta$ is *totally wildly ramified* over every irreducible component of $\underline{V}'$, it follows that] the degree of *each* of the ≥ 2 [cf. Proposition 2.5, (vi)] connected components of $C''_c \times_{C'_c} \eta_c$ over η_c is equal to p — in *contradiction* to the fact that the degree of $C''_c \times_{C'_c} \eta_c$ over η_c is $\leq p$. This completes the proof of the *claim*. In particular, it follows that $r_{E''} \leq r_{E'}$.

Now recall that we have *natural isomorphisms*

$$\omega_{E'/\underline{k}}(D_{E'}) \xrightarrow{\sim} \omega_{(V')^{\log}/S^{\log}}|_{E'}; \quad \omega_{E''/\underline{k}}(D_{E''}) \xrightarrow{\sim} \omega_{(V'')^{\log}/S^{\log}}|_{E''}$$

[cf. Proposition 2.5, (i), (ii), (iii), (iv), (v), (vi); the fact that the log-modifications $(V'')^{\log} \to (X'')^{\log}$, $(V')^{\log} \to (X')^{\log}$ are *unramified*]. Moreover, it follows immediately from the definitions that $\deg(\omega_{E'/\underline{k}}(D_{E'})) = 2g_{E'} + r_{E'}$, $\deg(\omega_{E''/\underline{k}}(D_{E''})) = 2g_{E''} + r_{E''}$. We thus conclude that $\deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) \leq \deg(\omega_{(V')^{\log}/S^{\log}}|_{C'})$. On the other hand, the existence of the *generically nonzero* morphism of line bundles $\delta^*|_{C''}$ implies that

$$\begin{aligned} \deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) &\geq \deg(\omega_{(V')^{\log}/S^{\log}}|_{C''}) \\ &= p \cdot \deg(\omega_{(V')^{\log}/S^{\log}}|_{C'}) \geq p \cdot \deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) \end{aligned}$$

— which implies that $\deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) \leq 0$. Now if C'' is *stable*, then we have $\deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) = 2g_{E''} + r_{E''} > 0$. We thus conclude that C'' is *non-stable*. But this implies that $\delta^*|_{C''}$ has *at least one zero*, so [cf. the above display of inequalities] we obtain that $\deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) < 0$, in *contradiction* to the equality $\deg(\omega_{(V'')^{\log}/S^{\log}}|_{C''}) = 0$ if C'' is *non-stable* [cf. the proof of Proposition 2.5, (v)]. This completes the proof of assertion (ii). In light of assertion (ii), assertion (iii) follows immediately from Proposition 2.5, (viii). This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, let us observe that by Proposition 2.5, (ix), it follows that we may assume that split, base-field-isomorphic log-modifications $W^{\log} \to Y^{\log}$, $Q^{\log} \to X^{\log}$, together with a morphism $W^{\log} \to Q^{\log}$ over $Y^{\log} \to X^{\log}$, have been chosen so that the generic point of the unique irreducible component C_W of $\underline{W}$ that maps finitely to C maps into $U_Q^{[1]}$. Then observe that there are *precisely two mutually exclusive possibilities*:

(c) some nonempty open subscheme of C maps *quasi-finitely* to $U_Q^{[1]}$;

(d) C maps to a closed point c of $U_Q^{[1]}$.

Moreover, by Proposition 2.5, (i), (ii), (vii), it follows immediately that (b) [as in the statement of assertion (iv)] $\iff$ (d). Thus, it suffices to show that (a) [as in the statement of assertion (iv)] $\iff$ (d). Note that it is immediate that (d) implies (a): Indeed, if we write $c^{\log}$ for the log scheme obtained by equipping c with the restriction to c of the log structure of $Q^{\log}$, then we obtain an *open injection* $\pi_1(c^{\log}) \hookrightarrow G_{\underline{k}^{\log}}$; but this implies that the natural homomorphism $\Delta_C \to \Delta_{\underline{X}^{\log}}$ *factors through* $\{1\} = \mathrm{Ker}(\pi_1(c^{\log}) \hookrightarrow G_{\underline{k}^{\log}})$, hence that condition (a) is satisfied. Thus, it remains to show that (a) implies (d), or, equivalently, that condition (c) implies that condition (a) *fails to hold*. But this follows immediately from the observation that condition (c) implies that Δ_C surjects onto an *open subgroup* of the *decomposition group* Δ_E in $\Delta_{\underline{X}^{\log}}$ of some irreducible component E of $\underline{Q}$. Here, we recall that the following well-known facts: if E is *stable*, then Δ_E may be identified with the *maximal pro-l quotient* of $\pi_1(U_E \times_{\underline{k}} \overline{k})$, where we write $U_E \stackrel{\mathrm{def}}{=} E \bigcap U_Q^{[1]}$, which is *infinite*; if E is *not stable*, then $\Delta_E \cong \mathbb{Z}_l(1)$ [cf. Proposition 2.5, (v)], hence *infinite*. This completes the proof of assertion (iv).

Finally, we consider assertion (v). First, we observe that Δ_C may be identified with the *maximal pro-l quotient* of $\pi_1(U_C \times_{\underline{k}} \overline{k})$, where we write $U_C \stackrel{\mathrm{def}}{=} C \bigcap U_Y^{[1]}$ [and assume, for simplicity, that $Y^{\log}$ is *split*]. In particular, an automorphism of U_C is equal to the identity if and only if it induces the *identity outer automorphism* of Δ_C [cf., e.g., [MT], Proposition 1.4, and its proof]. Note, moreover, that an automorphism of $Y^{\log}$ *stabilizes* C if and only if it stabilizes the conjugacy class of Δ_C [cf., e.g., [Mzk13], Proposition 1.2, (i)]. Thus, assertion (v) reduces to the [easily verified] assertion that the morphism $Y_\eta \to X_\eta$ is *wildly ramified* at C if and only if $\mathrm{Gal}(Y_\eta/X_\eta)$ *stabilizes* and induces the *identity* on C. This completes the proof of assertion (v). ◯

Remark 2.6.1. Note that the content of Lemma 2.6, (ii), (iii), is reminiscent of the main results of [Tama2] [cf. also Corollary 2.11 below]. By comparison to Tamagawa's *"resolution of nonsingularities"*, however, Lemma 2.6, (ii), (iii), assert a somewhat *weaker conclusion*, albeit for *pro-p geometric fundamental groups*, as opposed to profinite geometric fundamental groups.

Remark 2.6.2. The argument applied in the final portion of the proof of Lemma 2.6, (ii), is reminiscent of the well-known classical argument that implies the *nonexistence of a Frobenius lifting* for stable curves over the ring of Witt vectors of a finite field. That is to say, if k is absolutely unramified, and

$$\Phi : X \to X$$

is an S-morphism that induces the Frobenius morphism between the respective special fibers, then one obtains a contradiction as follows: Since Φ induces a morphism $X_\eta \to X_\eta$, it follows immediately that Φ extends to a morphism of log stable curves $\Phi^{\log} : X^{\log} \to X^{\log}$. Although the derivative

$$d\Phi^{\log} : \Phi^*(\omega_{X^{\log}/S^{\log}}) \to \omega_{X^{\log}/S^{\log}}$$

is $\equiv 0 \pmod{p}$, one verifies immediately [by an easy local calculation] that $\frac{1}{p}d\Phi^{\log}$ is necessarily $\not\equiv 0 \pmod{p}$ generically on each irreducible component of $\underline{X}$. Since $\omega_{X^{\log}/S^{\log}}$ is a line bundle of degree $2g_X - 2$, and Φ reduces to the *Frobenius morphism* between the special fibers, the existence of $d\Phi^{\log}$ thus implies [by taking degrees] that $p \cdot (2g_X - 2) \leq 2g_X - 2$, i.e., that $(p-1)(2g_X - 2) \leq 0$, in contradiction to the fact that $g_X \geq 2$.

Remark 2.6.3. Note that it follows immediately from *either* of the conditions (a), (b) of Lemma 2.6, (ii), that $\underline{Y}$ is *not* $\underline{k}$*-smooth* [i.e., *"singular"*].

Corollary 2.7. (Uniformly Toral Neighborhoods via Cyclic Coverings) *Suppose that we are* **either** *in the situation of Lemma 2.6, (ii), (a) — which we shall refer to in the following as* **case (a)** — **or** *in the situation of Lemma 2.6, (ii), (b) — which we shall refer to in the following as* **case (b)***; suppose further, in case (b), that* $\underline{X}$ *is* **not smooth** *over* $\underline{k}$*. Also, we suppose that we have been given a commutative diagram as in Lemma 2.6, (iii). Thus, in either case, we have an* **irreducible component** C_W *of* $\underline{W}$ *[lying over an irreducible component* C *of* $\underline{Y}$*] satisfying certain special properties, as in Lemma 2.6, (iii). Let* $y \in U_W^{[1]}(\mathcal{O}_k)$ $(\subseteq W_\eta(k) = Y_\eta(k))$ *be a* **point** *such that the image of* y **meets** C_W*, and, moreover,* y *maps to a point* $x \in U_Q^{[1]}(\mathcal{O}_k)$ $(\subseteq Q_\eta(k) = X_\eta(k))$*;* C_Q *the irreducible component of* $\underline{Q}$ *that meets the image of* x*;*

$$F_y \stackrel{\mathrm{def}}{=} \underline{W} \backslash (C_W \bigcap U_W^{[1]}) \subseteq \underline{W}; \quad F_x \stackrel{\mathrm{def}}{=} \underline{Q} \backslash (C_Q \bigcap U_Q^{[1]}) \subseteq \underline{Q}$$

[regarded as closed subsets of $\underline{W}$*,* $\underline{Q}$*];*

$$U_y \stackrel{\mathrm{def}}{=} W \backslash F_y \subseteq W; \quad U_x \stackrel{\mathrm{def}}{=} Q \backslash F_x \subseteq Q$$

[so the image of y *lies in* U_y*; the image of* x *lies in* U_x*]. Write* g_Y *for the genus of* Y_η*;* $J_\eta^Y \to \eta$ *(respectively,* $J_\eta^X \to \eta$*) for the* **Jacobian** *of* Y_η *(respectively,* X_η*);* $J^Y \to S$ *(respectively,* $J^X \to S$*) for the* **uniquely determined semi-abelian scheme** *over* S *that extends* J_η^Y *(respectively,* J_η^X*);* $\iota_\eta^Y : Y_\eta \to J_\eta^Y$ *for the morphism that sends a* T*-valued point* ζ *[where* T *is a* k*-scheme] of* Y_η*, regarded as a divisor on* $Y_\eta \times_k T$*, to the point of* J_η^Y *determined by the degree zero divisor* $\zeta - (y|_T)$*. In case (a), let* $\sigma \in \mathrm{Gal}(Y_\eta/X_\eta)$ *be a* **generator** *of* $\mathrm{Gal}(Y_\eta/X_\eta)$*;* $J_\eta \subseteq J_\eta^Y$ *the* **image abelian scheme** *of the restriction to* η *of the endomorphism* $(1-\sigma) : J^Y \to J^Y$*;* $J \to S$ *the uniquely determined semi-abelian scheme over* S *that extends* J_η *[which exists, for instance, by [BLR], §7.4, Lemma 2];*

$$\kappa : J^Y \to J$$

the [dominant] morphism induced by $(1-\sigma)$*. In case (b), let* $J \to S$ *be the semi-abelian scheme* $J^X \to S$*;* $\kappa : J^Y \to J$ *the [dominant] morphism induced by the covering* $Y_\eta \to X_\eta$*. Write*

$$\beta_\eta : Y_\eta \times_k \ldots \times_k Y_\eta \to J_\eta^Y \to J_\eta$$

[where the product is of g_Y *copies of* Y_η*] for the composite of the morphism given by adding* g_Y *copies of* ι^Y_η *with the morphism* $\kappa_\eta \stackrel{\mathrm{def}}{=} \kappa|_\eta$.

(i) Write $\widehat{J}$ *(respectively,* $\widehat{\mathbb{G}}_{\mathrm{m}}$*) for the* **formal group** *over* S *given by completing* J *(respectively, the multiplicative group* $(\mathbb{G}_{\mathrm{m}})_S$ *over* S*) at the origin. Then there exists an* **exact sequence**

$$0 \to \widehat{J}' \to \widehat{J} \to \widehat{J}'' \to 0$$

of [formally smooth] formal groups over S*, together with an* **isomorphism** $\widehat{J}' \xrightarrow{\sim} \widehat{\mathbb{G}}_{\mathrm{m}}$ *of formal groups over* S*. In the following, let us* **fix** *such an isomorphism* $\widehat{J}' \xrightarrow{\sim} \widehat{\mathbb{G}}_{\mathrm{m}}$ *and* **identify** $\widehat{J}'$ *with its image in* $\widehat{J}$.

(ii) The morphisms ι^Y_η, β_η **extend uniquely** *to morphisms*

$$\iota^Y : U_y \to J^Y; \quad \beta : U_y \times_k \ldots \times_k U_y \to J$$

[where the product is of g_Y *copies of* U_y*], respectively; the morphism* $W \to Q$ *restricts to a morphism* $U_y \to U_x$.

(iii) Suppose that k *is an* **MLF** *[or, equivalently, that* $\underline{k}$ *is* **finite***]. Then there exists a positive integer* M *— which, in fact, may be taken to be* 1 *in case (b) — such that the following condition holds: Let* $k_\bullet \subseteq \overline{k}$ *be a finite extension of* k *with ring of integers* $\mathcal{O}_{k_\bullet}$*, maximal ideal* $\mathfrak{m}_{k_\bullet} \subseteq \mathcal{O}_{k_\bullet}$*. Write* $I_{k_\bullet}$ *for the image in* $J(\mathcal{O}_{k_\bullet})$ *via* β *[cf. (ii)] of the product of* g_Y *copies of* $U_y(\mathcal{O}_{k_\bullet})$*. Then* $M \cdot I_{k_\bullet}$ *lies in the subgroup* $\widehat{J}(\mathcal{O}_{k_\bullet}) \subseteq J(\mathcal{O}_{k_\bullet})$*. Write* $\widehat{I}_{k_\bullet} \subseteq \widehat{J}(\mathcal{O}_{k_\bullet})$ *for the subgroup determined by* $M \cdot I_{k_\bullet}$*;*

$$N_{k_\bullet} \subseteq (\mathcal{O}^\times_{k_\bullet}) \otimes \mathbb{Q}_p \xrightarrow{\sim} \widehat{\mathbb{G}}_{\mathrm{m}}(\mathcal{O}_{k_\bullet}) \otimes \mathbb{Q}_p \xrightarrow{\sim} \widehat{J}'(\mathcal{O}_{k_\bullet}) \otimes \mathbb{Q}_p$$

for the image of the **intersection**

$$\widehat{I}_{k_\bullet} \bigcap \widehat{J}'(\mathcal{O}_{k_\bullet}) \quad (\subseteq \widehat{J}(\mathcal{O}_{k_\bullet}))$$

in $\widehat{J}'(\mathcal{O}_{k_\bullet}) \otimes \mathbb{Q}_p$*. Then as* $k_\bullet \subseteq \overline{k}$ *varies over the finite extensions of* k*, the subgroups* $N_{k_\bullet}$ *determine a* **uniformly toral neighborhood** *of* $\mathrm{Gal}(\overline{k}/k)$ *[cf. [Mzk15], Definition 3.6, (i), (ii)].*

Proof. First, we consider assertion (i). Recall from the well-known theory of *Néron models of Jacobians* [cf., e.g., [BLR], §9.2, Example 8] that the torus portion of the special fiber of J^Y (respectively, J^X) is [in the notation of Proposition 2.5, (vi)] of rank lp-rk($\underline{Y}$) (respectively, lp-rk($\underline{X}$)). In particular, the torus portion of the special fiber of J is of rank lp-rk($\underline{Y}$) $-$ lp-rk($\underline{X}$) in case (a), and of rank lp-rk($\underline{X}$) in case (b). Thus, in case (a), the fact that the morphism $Y_\eta \to X_\eta$ is *loopifying* implies that the torus portion of the special fiber of J is of *positive* rank; in case (b), since $\underline{X}$ is *not* $\underline{k}$*-smooth*, it follows from the *loop-ampleness* assumption of Lemma 2.6 that lp-rk($\underline{X}$) > 0, hence that the torus portion of the special fiber of J is of *positive* rank. Now the existence of an *exact sequence* as in assertion (i) follows from the well-known theory of *degeneration of abelian varieties* [cf., e.g., [FC], Chapter III, Corollary 7.3]. This completes the proof of assertion (i).

Next, we consider assertion (ii). The existence of the unique extension of ι_η^Y follows immediately from Proposition 2.5, (x); the existence of the unique extension of β_η follows immediately from the existence of this unique extension of ι_η^Y [together with the existence of the homomorphism of semi-abelian schemes $\kappa : J^Y \to J$]. In case (b), the existence of the morphism $U_y \to U_x$ follows immediately from the definitions. In case (a), if the morphism $U_y \to U_x$ *fails* to exist, then there exists a closed point $w \in W$ that maps to a closed point $q \in Q$ such that $w \in U_y \subseteq U_W^{[1]}$, but $q \notin U_Q^{[1]}$. On the other hand, since there exists an *irreducible* Zariski neighborhood of w in $\underline{W}$ [cf. the simple structure of the monoid $\mathbb{N}$], it follows from the fact that C_W maps *finitely* to C_Q [in case (a)], that $W \to Q$ is *quasi-finite* in a Zariski neighborhood of w. Thus, if we write R_w, R_q for the respective *strict henselizations* of W, Q at [some choice of $\overline{\underline{k}}$-valued points lifting] w, q, then R_w, R_q are *normal*, and the natural inclusion $R_q \hookrightarrow R_w$ is *finite* [cf. *Zariski's main theorem*]. In particular, if we write K_q for the quotient field of R_q, then we have $R_w^\times \bigcap K_q = R_q^\times$ [where "$\times$" denotes the subgroup of units], so the morphism $W^{\log} \to Q^{\log}$ induces an *injection* on the groupifications of the stalks of the characteristic sheaves at [some choice of $\overline{\underline{k}}$-valued points lifting] w, q — in contradiction to the fact that $w \in U_W^{[1]}$, but $q \notin U_Q^{[1]}$. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, we define the number M as follows: In case (a), the endomorphism $(1-\sigma) : J^Y \to J^Y$ admits a factorization $\theta \circ \kappa$, where $\theta : J \to J^Y$ is a *"closed immersion up to isogeny"* [cf., e.g., the situation discussed in [BLR], §7.5, Proposition 3, (b)] — i.e., there exists a morphism $\theta' : J^Y \to J$ such that $\theta' \circ \theta : J \to J$ is multiplication by some positive integer M_θ on J; then we take $M \stackrel{\text{def}}{=} M_\theta$. In case (b), we take $M \stackrel{\text{def}}{=} 1$. In the following, if G is a *group scheme* or *formal group* over $\mathcal{O}_k$, and $r \geq 1$ is an integer, then let us write

$$G_{\mathfrak{m}^r}(\mathcal{O}_{k_\bullet}) \subseteq G(\mathcal{O}_{k_\bullet})$$

for the subgroup of elements that are *congruent to the identity modulo* $\mathfrak{m}_k^r \cdot \mathcal{O}_{k_\bullet}$.

Next, let us make the following *observation*:

(1) We have $M \cdot I_{k_\bullet} \subseteq J_{\mathfrak{m}}(\mathcal{O}_{k_\bullet}) \subseteq J(\mathcal{O}_{k_\bullet})$.

Indeed, in case (a), we reason as follows: It suffices to show that $M \cdot \kappa(\iota^Y(U_y(\mathcal{O}_{k_\bullet}))) \subseteq J_{\mathfrak{m}}(\mathcal{O}_{k_\bullet})$. Since, moreover, the endomorphism $(1-\sigma) : J^Y \to J^Y$ admits a factorization $\theta \circ \kappa$, where, for some morphism $\theta' : J^Y \to J$, $\theta' \circ \theta$ is equal to multiplication by M, it suffices to show that

$$(1-\sigma)(\iota^Y(U_y(\mathcal{O}_{k_\bullet}))) \subseteq J^Y_{\mathfrak{m}}(\mathcal{O}_{k_\bullet})$$

[since applying θ' to this inclusion yields the desired inclusion $M \cdot \kappa(\iota^Y(U_y(\mathcal{O}_{k_\bullet}))) \subseteq J_{\mathfrak{m}}(\mathcal{O}_{k_\bullet})$]. On the other hand, since $Y_\eta \to X_\eta$ is *wildly ramified* at C_W, it follows that σ *acts as the identity* on C_W, hence that the composite morphism $(1-\sigma) \circ \iota^Y : U_y \to J^Y$ induces a morphism on special fibers $U_y \times_{\mathcal{O}_k} \underline{k} \to J^Y \times_{\mathcal{O}_k} \underline{k}$ that is *constant* [with image lying in the image of the identity section of $J^Y \times_{\mathcal{O}_k} \underline{k}$]. But this implies that $(1-\sigma)(\iota^Y(U_y(\mathcal{O}_{k_\bullet}))) \subseteq J^Y_{\mathfrak{m}}(\mathcal{O}_{k_\bullet})$. This completes the proof of *observation (1)* in case (a). In a similar [but slightly simpler] vein, in case (b), it suffices to observe that the morphism $\kappa \circ \iota^Y : U_y \to J$ admits a factorization $U_y \to U_x \to J^X$, where $U_y \to U_x$ is the morphism of assertion (ii), and $U_x \to J^X$ is the "analogue of ι^Y" for the point x of $X_\eta(k)$ [cf. Proposition 2.5, (x)]. That is to say, the fact that C_W maps to $x \in U_Q^{[1]}(\mathcal{O}_k)$ implies [by applying this factorization] that the morphism $\kappa \circ \iota^Y : U_y \to J$ induces a morphism on special fibers $U_y \times_{\mathcal{O}_k} \underline{k} \to J \times_{\mathcal{O}_k} \underline{k}$ that is *constant* [with image lying in the image of the identity section of $J \times_{\mathcal{O}_k} \underline{k}$]. This completes the proof of *observation (1)* in case (b).

Next, let us make the following *observation*:

(2) There exists a positive integer r which is *independent* of $k_\bullet$ such that $M \cdot I_{k_\bullet} \supseteq J_{\mathfrak{m}^r}(\mathcal{O}_{k_\bullet})$.

Indeed, since κ is clearly *dominant*, it follows immediately that the composite of the morphism $\beta : U_y \times_k \ldots \times_k U_y \to J$ with the morphism $J \to J$ given by multiplication by M is *dominant*, hence, in particular, *generically smooth* [since k is *of characteristic zero*]. Thus, [since $M \cdot I_{k_\bullet}$ is a *group*!] *observation (2)* follows immediately from the *"positive slope version of Hensel's lemma"* given in Lemma 2.1. Now since $\widehat{J}'_{\mathfrak{m}}(\mathcal{O}_{k_\bullet}) = J_{\mathfrak{m}}(\mathcal{O}_{k_\bullet}) \bigcap \widehat{J}'(\mathcal{O}_{k_\bullet})$ [cf. assertion (i)], we conclude that

$$\widehat{J}'_{\mathfrak{m}^r}(\mathcal{O}_{k_\bullet}) \subseteq \widehat{I}_{k_\bullet} \bigcap \widehat{J}'(\mathcal{O}_{k_\bullet}) \subseteq \widehat{J}'_{\mathfrak{m}}(\mathcal{O}_{k_\bullet})$$

[cf. the inclusions of *observations (1), (2)*], so assertion (iii) follows essentially formally [cf. [Mzk15], Definition 3.6, (i), (ii)]. This completes the proof of assertion (iii). ○

Remark 2.7.1. Note that in the situation of case (b), if f is a *rational function* on X whose value at x lies in $\mathcal{O}_k^\times$, then the *values* $\in \mathcal{O}_{k_\bullet}^\times$ *of* f *at points of* $U_y(\mathcal{O}_{k_\bullet})$ [cf. the notation of Corollary 2.7, (iii)] determine a *uniformly toral neighborhood.* It was precisely this observation that motivated the author to develop the theory of the present §2.

Definition 2.8. Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k.

(i) Suppose that $\overline{k}$ is equipped with a *topology.* Let X be a smooth, geometrically connected curve over k. Then we shall say that a subset $\Xi \subseteq X(\overline{k})$ is *Galois-dense* if, for every finite extension field $k' \subseteq \overline{k}$ of k, $\Xi \bigcap X(k')$ is *dense* in $X(k')$ [i.e., relative to the topology induced on $X(k')$ by $\overline{k}$].

(ii) We shall refer to as a *pro-curve* U over k [cf. the terminology of [Mzk3]] any k-scheme U that may be written as a projective limit of smooth, geometrically connected curves over k in which the transition morphisms are *birational.* Let U be a *pro-curve* over k. Then it makes sense to speak of the *function field* $k(U)$ of U. Write X for the smooth, proper, geometrically connected curve over k determined by the function field $k(U)$. Then one verifies immediately that U is *completely determined up to unique isomorphism* by $k(U)$, together with some $\mathrm{Gal}(\overline{k}/k)$-stable subset $\Xi \subseteq X(\overline{k})$ — i.e., roughly speaking, "U is obtained by removing Ξ from X".

If $\overline{k}$ is equipped with a *topology*, then we shall say that U is *co-Galois-dense* if the corresponding $\mathrm{Gal}(\overline{k}/k)$-stable subset $\Xi \subseteq X(\overline{k})$ is Galois-dense.

Remark 2.8.1. Suppose, in the notation of Definition 2.8, that k is an *MLF* [and that $\overline{k}$ is equipped with the *p-adic topology*]. Let X be a smooth, proper, geometrically connected curve over k, with function field $k(X)$. Then $\mathrm{Spec}(k(X))$ is a *co-Galois-dense pro-curve* over k. Suppose that $X = X_0 \times_{k_0} k$, where $k_0 \subseteq k$ is a *number field*, and X_0 is a smooth, proper, geometrically connected curve over k_0, with function field $k(X_0)$. Then $\mathrm{Spec}(k(X_0) \otimes_{k_0} k)$ [where we note that the ring $k(X_0) \otimes_{k_0} k$ is *not a field*!] also forms an example of a *co-Galois-dense pro-curve* over k.

Remark 2.8.2. Let k be a field of characteristic zero.

(i) Let us say that a pro-curve U over k is *of unit type* if there exists a connected finite étale covering of U that admits a *nonconstant unit*. Thus, any hyperbolic curve U over k for which there exists a connected finite étale covering $V \to U$ such that V admits a *dominant k-morphism* $V \to P$, where P is the projective line minus three points over k, is *of unit type*. That is to say, the hyperbolic curves considered in [Mzk15], Remark 3.8.1 — i.e., the sort of hyperbolic curves that motivated the author to prove [Mzk15], Corollary 3.8, (g) — are *necessarily of unit type*.

(ii) Suppose that k is an *MLF* of residue characteristic p, whose ring of integers we denote by $\mathcal{O}_k$. Let $n \geq 1$ be an integer; $\eta \in \mathcal{O}_k/(p^n)$. Then *observe* that the set E of elements of $\mathcal{O}_k$ that are $\equiv \eta \pmod{p^n}$ is of *uncountable* cardinality. In particular, it follows that the subfield of k generated over $\mathbb{Q}$ by E is of *uncountable* — hence, in particular, *infinite* — *transcendence degree* over $\mathbb{Q}$.

(iii) Let k be as in (ii); X_0 a proper hyperbolic curve over k_0, where $k_0 \subseteq k$ is a finitely generated extension of $\mathbb{Q}$; $k_1 \subseteq k$ a finitely generated extension of k_0; $r \geq 1$ an integer. Then recall from [MT], Corollary 5.7, that any curve U_1 obtained by removing from $X_1 \stackrel{\mathrm{def}}{=} X_0 \times_{k_0} k_1$ a set of r *"generic points"* $\in X_1(k_1) = X_0(k_1)$ — i.e., r points which determine a *dominant morphism* from $\mathrm{Spec}(k_1)$ to the product of r copies of X_0 over k_0 — is *not of unit-type*. In particular, it follows immediately from (ii) that:

There exist *co-Galois-dense pro-curves* U over k which are *not of unit type*.

For more on the significance of this fact, we refer to Remark 2.10.1 below.

Remark 2.8.3. Suppose that we are in the situation of Definition 2.8, (i). Let $Y \to X$ be a connected finite étale covering. Then one verifies immediately, by applying *"Krasner's Lemma"* [cf. [Kobl], pp. 69-70], that the inverse image $\Xi|_Y \subseteq Y(\overline{k})$ of a Galois-dense subset $\Xi \subseteq X(\overline{k})$ is itself *Galois-dense*.

Corollary 2.9. **(Point-theoreticity Implies Geometricity)** *For $i = 1, 2$, let k_i be an* **MLF** *of residue characteristic p_i; $\overline{k}_i$ an algebraic closure of k_i; Σ_i a set of primes of* **cardinality** ≥ 2 *such that $p_i \in \Sigma_i$; X_i a* **hyperbolic curve** *over k_i; $\overline{X}_i$ the smooth, proper, geometrically connected curve over k_i determined by the function field of X_i; $\Xi_i \subseteq \overline{X}_i(\overline{k}_i)$ a* **Galois-dense** *subset. Write "$\pi_1(-)$" for the étale fundamental group of a connected scheme, relative to an appropriate choice of basepoint; Δ_{X_i} for the* **maximal pro-Σ_i quotient** *of $\pi_1(X_i \times_{k_i} \overline{k}_i)$; Π_{X_i} for the quotient of $\pi_1(X_i)$ by the kernel of the natural surjection $\pi_1(X_i \times_{k_i} \overline{k}_i) \twoheadrightarrow \Delta_{X_i}$. Let*

$$\alpha : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$$

be an **isomorphism of profinite groups** *such that a closed subgroup of Π_{X_1} is a decomposition group of a point $\in \Xi_1$ if and only if it corresponds, relative to α, to a decomposition group in Π_{X_2} of a point $\in \Xi_2$. Then $p_1 = p_2$, $\Sigma_1 = \Sigma_2$, and α is* **geometric***, i.e., arises from a unique isomorphism of schemes $X_1 \xrightarrow{\sim} X_2$.*

Proof. First, we observe that by [Mzk15], Theorem 2.14, (i), α induces isomorphisms $\alpha_\Delta : \Delta_{X_1} \xrightarrow{\sim} \Delta_{X_2}$, $\alpha_G : G_1 \xrightarrow{\sim} G_2$ [where, for $i = 1, 2$, we write $G_i \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_i/k_i)$]; $p_1 = p_2$ [so we shall write $p \stackrel{\mathrm{def}}{=} p_1 = p_2$]; $\Sigma_1 = \Sigma_2$ [so we shall write $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_1 = \Sigma_2$]. Also, by [the portion concerning *semi-graphs* of] [Mzk15], Theorem 2.14, (i), it follows that α preserves the *decomposition groups of cusps*. Thus, by passing to corresponding open subgroups of Π_{X_i} [cf. Remark 2.8.3] and forming the *quotient* by the decomposition groups of cusps in Δ_{X_i}, we may assume, without loss of generality, that the X_i are *proper*. Next, let $l \in \Sigma$ be a prime $\neq p$. Then let us recall that by the well-known *stable reduction criterion* [cf., e.g., [BLR], §7.4, Theorem 6], X_i has stable reduction over $\mathcal{O}_{k_i}$ if and only if, for some $\mathbb{Z}_l$-submodule $M \subseteq \Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Z}_l$ of the maximal pro-l abelian quotient $\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Z}_l$ of Δ_{X_i}, the inertia subgroup of G_i acts *trivially* on M, $\Delta_{X_i}^{\mathrm{ab}} \otimes \mathbb{Z}_l/M$. Thus, we may assume, without loss of generality, that, for $i = 1, 2$, X_i admits a *log stable model* $\mathcal{X}_i^{\log}$ over $\mathrm{Spec}(\mathcal{O}_{k_i})^{\log}$ [where the last log structure is the log structure determined by the closed point]. Since, by [the portion concerning *semi-graphs* of] [Mzk15], Theorem 2.14, (i), it follows that α induces an isomorphism between the *dual graphs* of the special fibers $\underline{\mathcal{X}}_i$ of the $\mathcal{X}_i$, hence that $\underline{\mathcal{X}}_1$ is *loop-ample* (respectively, *singular*) if and only if $\underline{\mathcal{X}}_2$ is. Thus, by replacing X_i by a finite étale covering of X_i arising from an open subgroup of Π_{X_i}, we may assume that $\underline{\mathcal{X}}_i$ is *loop-ample* [cf. §0] and *singular* [cf. Remark 2.6.3]. Now, to complete the proof of Corollary 2.9, it follows from [Mzk15], Corollary 3.8, (e), that it suffices to show that α_G is *uniformly toral*.

Next, let us suppose that, for $i = 1, 2$, we are given a finite étale covering $Y_i \to X_i$ of hyperbolic curves over k_i with stable reduction over $\mathcal{O}_{k_i}$ arising from open subgroups of Π_{X_i} that correspond via α and are such that $\mathrm{Gal}(Y_i/X_i)$ is *cyclic of order a positive power of* p [cf. Lemma 2.6, (ii)]. Let us write $\mathcal{Y}_i^{\log}$ for the *log stable model* of Y_i over $(\mathcal{O}_{k_i})^{\log}$, $\underline{\mathcal{Y}}_i$ for the *special fiber* of $\mathcal{Y}_i$. By possibly replacing the k_i by corresponding [relative to α] finite extensions of k_i, we may assume that the $\mathcal{Y}_i$ are *split* [cf. Proposition 2.5, (vi)]. Note that by [the portion concerning *semi-graphs* of] [Mzk15], Theorem 2.14, (i), it follows that $Y_1 \to X_1$ is *loopifying* if and only if $Y_2 \to X_2$ is. Thus, by Lemma 2.6, (iv), (v) [cf. also Proposition 2.5, (i), (ii)] (respectively, Lemma 2.6, (iv)), it follows that $Y_1 \to X_1$ satisfies condition (a) (respectively, (b)) of Lemma 2.6, (ii), if and if $Y_2 \to X_2$ does. For $i = 1, 2$, let C_i be an irreducible component of $\underline{\mathcal{Y}}_i$ as in Corollary 2.7 [i.e., "C"]. By Lemma 2.6, (iv) [cf. also the portion concerning *semi-graphs* of [Mzk15], Theorem 2.14, (i)], we may assume that the C_i are *compatible* with α. Thus, to complete the proof of Corollary 2.9, it suffices to construct *uniformly toral neighborhoods* [cf. Corollary 2.7, (iii)] that are *compatible* with α.

Write $\Pi_{Y_i} \subseteq \Pi_{X_i}$, $\Delta_{Y_i} \subseteq \Delta_{X_i}$ for the open subgroups determined by Y_i; T_i^Y, T_i^X for the *maximal pro-p abelian quotients* of Δ_{Y_i}, Δ_{X_i}. If we are in *case (a)* [cf. Corollary 2.7], then we choose generators $\sigma_i \in \mathrm{Gal}(Y_i/X_i) \cong \Delta_{X_i}/\Delta_{Y_i}$ that correspond via α and write T_i for the intersection with T_i^Y of the image of the endomorphism $(1 - \sigma_i)$ of $T_i^Y \otimes \mathbb{Q}_p$, and

$$\kappa_{T_i} : T_i^Y \to T_i$$

for the morphism induced by $(1 - \sigma_i)$. If we are in *case (b)* [cf. Corollary 2.7], then we set $T_i \stackrel{\mathrm{def}}{=} T_i^X$; write $\kappa_{T_i} : T_i^Y \to T_i$ for the morphism induced by $Y_i \to X_i$ [i.e., by the inclusion $\Delta_{Y_i} \hookrightarrow \Delta_{X_i}$]. Thus, the formal group "J'" of Corollary 2.7, (i), corresponds to a G_i-submodule $T_i' \subseteq T_i$ such that $T_i' \cong \mathbb{Z}_p(1)$; by [Tate], Theorem 4 [cf. also [Mzk5], Proposition 1.2.1, (vi)], we may assume that these submodules T_i' are *compatible* with α.

Next, let us write $\Delta_{Y_i} \twoheadrightarrow \Delta_{Y_i}^{(l)}$ for the *maximal pro-l quotient* of Δ_{Y_i}; $\Delta_{C_i}^{(l)} \subseteq \Delta_{Y_i}^{(l)}$ for the *decomposition group of* C_i in $\Delta_{Y_i}^{(l)}$ [well-defined up to conjugation]. Thus, $\Delta_{C_i}^{(l)}$ may be identified with the *maximal pro-l quotient* of $\pi_1(U_{C_i} \times_{\underline{k}_i} \overline{\underline{k}}_i)$, where $U_{C_i} \stackrel{\mathrm{def}}{=} C_i \bigcap U_{\mathcal{Y}_i}^{[1]}$ [cf. the proof of Lemma 2.6, (iv)], $\underline{k}_i$ is the residue field of k_i, and $\overline{\underline{k}}_i$ is the algebraic closure of $\underline{k}_i$ induced by $\overline{k}_i$. Since $\Delta_{C_i}^{(l)}$ is *slim* [cf., e.g., [MT], Proposition 1.4], and the outer action of G_i on $\Delta_{C_i}^{(l)}$ clearly *factors* through the quotient $G_i \twoheadrightarrow \mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i)$, the resulting outer action of $\mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i)$ on $\Delta_{C_i}^{(l)}$ determines, in a fashion that is *compatible* with α, an *extension of profinite groups* $1 \to \Delta_{C_i}^{(l)} \to \Pi_{C_i}^{(l)} \to \mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i) \to 1$. In a similar vein, the outer action of G_i on $\Delta_{Y_i}^{(l)}$ *factors* through the *maximal tamely ramified quotient* $G_i \twoheadrightarrow G_{\underline{k}_i^{\log}}$, hence [since $\Delta_{Y_i}^{(l)}$ is *slim* — cf., e.g., [MT], Proposition 1.4] determines, in a fashion that is *compatible* with α, a *morphism of extensions of profinite groups*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{C_i}^{(l)} & \longrightarrow & \Pi_{C_i}^{(l)} \times_{\mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i)} G_{\underline{k}_i^{\log}} & \longrightarrow & G_{\underline{k}_i^{\log}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{Y_i}^{(l)} & \longrightarrow & \Pi_{Y_i}^{(l)} & \longrightarrow & G_{\underline{k}_i^{\log}} & \longrightarrow & 1
\end{array}$$

— in which the vertical morphisms are *inclusions*, and the vertical morphism on the right is the *identity morphism*; moreover, the images of the first two vertical morphisms are equal to the respective *decomposition groups* of C_i [well-defined up to conjugation].

Next, let us observe that, by our assumption concerning *decomposition groups of points* $\in \Xi_i$ in the statement of Corollary 2.9, it follows that α determines a *bijection* $Y_1(k_1, \Xi_1) \stackrel{\sim}{\to} Y_2(k_2, \Xi_2)$, where we write $Y_i(k_i, \Xi_i) \subseteq Y_i(k_i)$ for the subset of points lying over points $\in \Xi_i$. [Here, we recall that a point $\in Y_i(k_i)$ is *uniquely determined* by the conjugacy class of its decomposition group in Π_{Y_i} — cf., e.g., [Mzk3], Theorem C.] Now let us choose corresponding [i.e., via this bijection] points $y_i \in Y_i(k_i, \Xi_i)$ as our points "y" in the construction of the *uniformly toral neighborhoods* of Corollary 2.7, (iii). Here, we observe that [by our *Galois-density* assumption] we may assume that y_i is *compatible* with C_i, in the sense that the image in the quotient $\Pi_{Y_i} \twoheadrightarrow \Pi^{(l)}_{Y_i}$ of the decomposition group of y_i in Π_{Y_i} determines a subgroup of $\Pi^{(l)}_{C_i} \times_{\mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i)} G_{\underline{k}_i^{\log}}$ which contains the *kernel* of the surjection $\Pi^{(l)}_{C_i} \times_{\mathrm{Gal}(\overline{\underline{k}}_i/\underline{k}_i)} G_{\underline{k}_i^{\log}} \twoheadrightarrow \Pi^{(l)}_{C_i}$. Note that this condition that y_i be "compatible with C_i" is manifestly *"group-theoretic"*, i.e., *compatible with* α [cf. the portion concerning *semi-graphs* of [Mzk15], Theorem 2.14, (i); [Mzk5], Proposition 1.2.1, (ii)]. Moreover, let us recall from the theory of §1 that this condition that y_i be "compatible with C_i" is *equivalent* to the condition that *the closure in* $\mathcal{Y}_i$ *of* y_i *intersect* U_{C_i} [cf. Proposition 1.3, (x)].

Thus, by choosing *any* corresponding [i.e., via the bijection induced by α] points $y'_i \in Y_i(k_i, \Xi_i)$ that are *compatible* with the C_i, we may compute [directly from the decomposition groups of the y_i, y'_i in Π_{Y_i}] the *"difference"* of y_i, y'_i in $H^1(G_{k_i}, T_i^Y)$, as well as the *image*

$$\delta_{y_i, y'_i} \in H^1(G_{k_i}, T_i)$$

of this difference *via* κ_{T_i}. On the other hand, let us recall the *Kummer isomorphisms*

$$H^1(G_{k_i}, T'_i) \cong \mathcal{O}^\times_{k_i} \otimes \mathbb{Z}_p; \quad H^1(G_{k_i}, T_i) \cong J_i(k_i) \otimes \mathbb{Z}_p$$

[where J_i is the "J" of Corollary 2.7, (iii) — cf., e.g., the "well-known general nonsense" reviewed in the proof of [Mzk14], Proposition 2.2, (i), for more details]. By applying these isomorphisms, we conclude that the subset of

$$H^1(G_{k_i}, T'_i \otimes \mathbb{Q}_p) \cong \mathcal{O}^\times_{k_i} \otimes \mathbb{Q}_p$$

obtained by taking the image of the intersection in $H^1(G_{k_i}, T_i)$ with the image of $H^1(G_{k_i}, T'_i)$ of the *closure* [cf. our *Galois-density* assumption, together with the evident *p-adic continuity* of the assignment $y'_i \mapsto \delta_{y_i, y'_i}$] of the set obtained by adding g_{Y_i} [where g_{Y_i} is the genus of Y_i] elements of the form $M_i \cdot \delta_{y_i, y'_i}$ [where M_i is the "M" of Corollary 2.7, (iii)] yields — from the point of view of Corollary 2.7, (iii) — a subset that *coincides* with the subset "$N_{k_\bullet}$" [when "$k_\bullet$" is taken to be k_i] constructed in Corollary 2.7, (iii). Thus, by allowing the "k_i" to *vary* over arbitrary corresponding finite extensions $\subseteq \overline{k}_i$ of k_i, we obtain *uniform toral neighborhoods* of the G_i that are *compatible* with α. But this implies that α_G is *uniformly toral*, hence completes the proof of Corollary 2.9. ○

Remark 2.9.1. Corollary 2.9 may be regarded as a generalization of the [MLF portion of] [Mzk14], Corollary 2.2, to the case of *pro-*Σ [where Σ is of cardinality ≥ 2 and contains the residue characteristic — that is to say, Σ is *not necessarily the set of all primes*] geometric fundamental groups of *not necessarily affine* hyperbolic curves. From this point of view, it is interesting to note that in the theory of the present §2, Lemma 2.6, which, as is discussed in Remark 2.6.2, is reminiscent of a classical argument on the *"nonexistence of Frobenius liftings"*, takes the place of Lemma 4.7 of [Tama1], which is applied in [Mzk14], Corollary 2.1, to reconstruct the *additive structure* of the fields involved. In this context, we observe that the appearance of *"Frobenius endomorphisms"* in Remark 2.6.2 is interesting in light of the discussion of *"hidden endomorphisms"* in the Introduction, in which "Frobenius endomorphisms" also appear.

Remark 2.9.2. One way to think of Corollary 2.9 is as the statement that:

> The *"Section Conjecture"* over MLF's implies the *"absolute isomorphism version of the Grothendieck Conjecture"* over MLF's.

Here, we recall that in the notation of Corollary 2.9, the "Section Conjecture" over MLF's amounts to the assertion that *every* closed subgroup of Π_{X_i} that maps isomorphically to an open subgroup of $\mathrm{Gal}(\overline{k}_i/k_i)$ is the decomposition group associated to a closed point of X_i. In fact, in order to apply Corollary 2.9, a *"relatively weak version of the Section Conjecture"* is sufficient — cf. the point of view of [Mzk8].

Remark 2.9.3. The issue of verifying the *"point-theoreticity hypothesis"* of Corollary 2.9 [i.e., the hypothesis concerning the preservation of decomposition groups of closed points] may be thought of as consisting of *two steps*, as follows:

(a) First, one must show the *J-geometricity* [cf. [Mzk3], Definition 4.3] of the image via α of a decomposition group $D_\xi \subseteq \Pi_{X_1}$ of a closed point $\xi \in X_1(k_1)$. Once one shows this J-geometricity for all finite étale coverings of X_2 arising from open subgroups of Π_{X_2}, one concludes [cf. the arguments of [Mzk3], §7, §8] that there *exist* rational points of a certain tower of coverings of X_2 determined by $\alpha(D_\xi) \subseteq \Pi_{X_2}$ over *tame extensions* of k_2.

(b) Finally, one must show that these rational points over tame extensions of k_2 necessarily *converge* — an issue that the author typically refers to by the term *"tame convergence"*.

At the time of writing, it is not clear to the author how to complete either of these two steps. On the other hand, in the *"birational"* — or, more generally, the *"co-Galois-dense"* — case, one has Corollary 2.10 [given below].

Remark 2.9.4.

(i) By contrast to the *quite substantial difficulty* [discussed in Remark 2.9.3] of verifying "point-theoreticity" for hyperbolic curves over MLF's, in the case of *hyperbolic curves over finite fields*, there is a [relatively simple] *"group-theoretic" algorithm for reconstructing the decomposition groups of closed points*, which follows essentially from the theory of [Tama1] [cf. [Tama1], Corollary 2.10, Proposition 3.8]. Such an algorithm is discussed in [Mzk14], Remark 10, although the argument given there is somewhat sketchy and a bit misleading. A more detailed presentation may be found in [SdTm], Corollary 1.25.

(ii) A more concise version of this argument, along the lines of [Mzk14], Remark 10, may be given as follows: Let X be a *proper* [for simplicity] *hyperbolic curve* over a finite field k, with algebraic closure $\overline{k}$; Σ a set of prime numbers that contains a prime that is invertible in k; $\pi_1(X \times_k \overline{k}) \twoheadrightarrow \Delta_X$ the *maximal pro-*Σ *quotient* of the étale fundamental group $\pi_1(X \times_k \overline{k})$ of $X \times_k \overline{k}$; $\pi_1(X) \twoheadrightarrow \Pi_X$ the corresponding quotient of the étale fundamental group $\pi_1(X)$ of X; $\Pi_X \twoheadrightarrow G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ the natural quotient. Then it suffices to give a *"group-theoretic" characterization* of the *quasi-sections* $D \subseteq \Pi_X$ [i.e., closed subgroups that map isomorphically onto an open subgroup of G_k] which are decomposition groups of closed points of X. Write

$$\widetilde{X} \to X$$

for the *pro-finite étale covering* corresponding to Π_X. If $E \subseteq \Pi_X$ is a closed subgroup whose image in G_k is *open*, then let us write k_E for the finite extension field of k determined by this image. If $J \subseteq \Pi_X$ is an open subgroup, then let us write $X_J \to X$ for the covering determined by J and $J_\Delta \stackrel{\mathrm{def}}{=} J \bigcap \Delta_X$. If $J \subseteq \Pi_X$ is an open subgroup such that J_Δ is a characteristic subgroup of Δ_X, then we shall say that J is *geometrically characteristic*. Now let $J \subseteq \Pi_X$ be a *geometrically characteristic open subgroup*. Let us refer to as a *descent-group for* J any open subgroup $H \subseteq \Pi_X$ such that $J \subseteq H$, $J_\Delta = H_\Delta$. Thus, a descent-group H for J may be thought of as an intermediate covering $X_J \to X_H \to X$ such that $X_H \times_{k_H} k_J \cong X_J$. Write

$$X_J(k_J)^{\text{fld-def}} \subseteq X_J(k_J)$$

for the subset of k_J-valued points of X_J that do *not* arise from points $\in X_H(k_H)$ for any descent-group $H \neq J$ for J — i.e., the k_J-valued points whose *field of definition* is k_J with respect to *all possible "descended forms" of* X_J. [That is to say, this definition of "fld-def" differs slightly from the definition of "fld-def" in [Mzk14], Remark 10.] Thus, if $\widetilde{x}$ is a closed point of $\widetilde{X}$ that maps to $x \in X_J(k_J)$, and we write $D_{\widetilde{x}} \subseteq \Pi_X$ for the *stabilizer in* Π_X [i.e., "decomposition group"] of $\widetilde{x}$, then it is a *tautology* that x maps to a point $\in X_{H_x}(k_{H_x})$ for $H_x \stackrel{\mathrm{def}}{=} D_{\widetilde{x}} \cdot J_\Delta$ ($\supseteq J$) [so H_x forms a *descent-group for* J]; in particular, it follows immediately that:

$$x \in X_J(k_J)^{\text{fld-def}} \iff D_{\widetilde{x}} \subseteq J \iff H_x = J.$$

Now it follows immediately from this characterization of "fld-def" that if $J_1 \subseteq J_2 \subseteq \Pi_X$ are geometrically characteristic open subgroups such that $k_{J_1} = k_{J_2}$, then the natural map $X_{J_1}(k_{J_1}) \to X_{J_2}(k_{J_2})$ induces a map $X_{J_1}(k_{J_1})^{\text{fld-def}} \to X_{J_2}(k_{J_2})^{\text{fld-def}}$. Moreover, these considerations allow one to conclude [cf. the theory of [Tama1]] that:

> A quasi-section $D \subseteq \Pi_X$ is a *decomposition group* of a closed point of X if and and only if, for every geometrically characteristic open subgroup $J \subseteq \Pi_X$ such that $D \cdot J_\Delta = J$, it holds that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$.

Thus, to render this characterization of decomposition groups *"group-theoretic"*, it suffices to give a "group-theoretic" criterion for the condition that $X_J(k_J)^{\text{fld-def}} \neq \emptyset$. In [Tama1], the *Lefschetz trace formula* is applied to compute the cardinality of

$X_J(k_J)$. On the other hand, if we use the notation "$|-|$" to denote the cardinality of a finite set, then one verifies immediately that

$$|X_J(k_J)| = \sum_H |X_H(k_H)^{\text{fld-def}}|$$

— where $H \supseteq J$ ranges over the *descent-groups* for J. In particular, by applying *induction* on $[\Pi_X : J]$, one concludes immediately from the above formula that $|X_J(k_J)^{\text{fld-def}}|$ may be computed from the $|X_H(k_H)|$, as H ranges over the *descent-groups* for J [while $|X_H(k_H)|$ may be computed, as in [Tama1], from the *Lefschetz trace formula*]. This yields the desired "group-theoretic" characterization of the decomposition groups of Π_X.

Corollary 2.10. **(Geometricity of Absolute Isomorphisms for Co-Galois-dense Pro-curves)** *For $i = 1, 2$, let k_i be an* **MLF** *of residue characteristic p_i; $\overline{k}_i$ an algebraic closure of k_i; Σ_i a set of primes of* **cardinality** ≥ 2 *such that $p_i \in \Sigma_i$; U_i a* **co-Galois-dense pro-curve** *over k_i. Write "$\pi_1(-)$" for the étale fundamental group of a connected scheme, relative to an appropriate choice of basepoint; Δ_{U_i} for the* **maximal pro-Σ_i quotient** *of $\pi_1(U_i \times_{k_i} \overline{k}_i)$; Π_{U_i} for the quotient of $\pi_1(U_i)$ by the kernel of the natural surjection $\pi_1(U_i \times_{k_i} \overline{k}_i) \twoheadrightarrow \Delta_{U_i}$. Let*

$$\alpha : \Pi_{U_1} \xrightarrow{\sim} \Pi_{U_2}$$

be an **isomorphism of profinite groups**. *Then Δ_{U_i}, Π_{U_i} are* **slim**; *$p_1 = p_2$; $\Sigma_1 = \Sigma_2$; α is* **geometric**, *i.e., arises from a unique isomorphism of schemes $U_1 \xrightarrow{\sim} U_2$.*

Proof. First, we observe that Σ_i may be characterized as the set of primes l such that Π_{U_i} has *l-cohomological dimension* > 2. Thus, $\Sigma_1 = \Sigma_2$. Let us write $\Sigma \stackrel{\text{def}}{=} \Sigma_1 = \Sigma_2$; X_i for the smooth, proper, geometrically connected curve over k_i determined by U_i; Δ_{X_i} for the *maximal pro-Σ quotient* of $\pi_1(X_i \times_{k_i} \overline{k}_i)$; Π_{X_i} for the quotient of $\pi_1(X_i)$ by $\text{Ker}(\pi_1(X_i \times_{k_i} \overline{k}_i) \twoheadrightarrow \Delta_{X_i})$. Thus, U_i determines some *Galois-dense* subset $\Xi_i \subseteq X_i(\overline{k}_i)$. Since Δ_{U_i}, Π_{U_i} may be written as inverse limits of surjections of *slim* profinite groups [cf., e.g., [Mzk15], Proposition 2.3], it follows that Δ_{U_i}, Π_{U_i} are *slim*. Since the *kernel* of the natural surjection $\Pi_{U_i} \twoheadrightarrow \Pi_{X_i}$ is topologically generated by the *inertia groups* of points $\in \Xi_i$, and these inertia groups are isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}(1)$ [where the "(1)" denotes a Tate twist; we write $\widehat{\mathbb{Z}}^{\Sigma}$ for the *maximal pro-Σ quotient* of $\widehat{\mathbb{Z}}$] in a fashion that is compatible with the conjugation action of some open subgroup of $G_i \stackrel{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i)$, it follows that we obtain an *isomorphism*

$$\Pi_{U_i}^{\text{ab-t}} \xrightarrow{\sim} \Pi_{X_i}^{\text{ab-t}}$$

on *torsion-free abelianizations*. In particular, it follows [in light of our assumptions on Σ_i] that, in the notation of [Mzk15], Theorem 2.6,

$$\begin{aligned}\sup_{p',p''\in\Sigma} \{\delta^1_{p'}(\Pi_{U_i}) - \delta^1_{p''}(\Pi_{U_i})\} &= \sup_{p',p''\in\Sigma} \{\delta^1_{p'}(\Pi_{X_i}) - \delta^1_{p''}(\Pi_{X_i})\} \\ &= \sup_{p',p''\in\Sigma} \{\delta^1_{p'}(G_i) - \delta^1_{p''}(G_i)\} = [k_i : \mathbb{Q}_{p_i}]\end{aligned}$$

— cf. [Mzk15], Theorem 2.6, (ii). In particular, by applying this chain of equalities to arbitrary open subgroups of Π_{U_i}, we conclude that α induces an isomorphism $\alpha_G : G_1 \xrightarrow{\sim} G_2$. Moreover, by [Mzk5], Proposition 1.2.1, (i), (vi), the existence of α_G implies that $p_1 = p_2$ [so we set $p \stackrel{\mathrm{def}}{=} p_1 = p_2$], and α_G is compatible with the respective *cyclotomic characters*.

Let M be a *profinite abelian group* equipped with a continuous H-action, for $H \subseteq G_i$ [where $i \in \{1, 2\}$] an open subgroup. Then let us write $M \twoheadrightarrow \mathcal{Q}'(M)$ for the quotient of M by the closed subgroup generated by the *quasi-toral subgroups* of M [i.e., closed subgroups isomorphic as J-modules, for $J \subseteq H$ an open subgroup, to $\mathbb{Z}_l(1)$ for some prime l]; $M \twoheadrightarrow \mathcal{Q}'(M) \twoheadrightarrow \mathcal{Q}(M)$ for the *maximal torsion-free quotient* of $\mathcal{Q}'(M)$. Also, if M is topologically finitely generated, then let us write $M \twoheadrightarrow \mathcal{T}(M)$ for the *maximal torsion-free quasi-trivial quotient* [i.e., maximal torsion-free quotient on which H acts through a *finite quotient*]. Then one verifies immediately that the assignments $M \mapsto \mathcal{Q}(M)$, $M \mapsto \mathcal{T}(M)$ are *functorial*. Moreover, it follows from the observations of the preceding paragraph that the natural surjection $\Delta_{U_i} \twoheadrightarrow \Delta_{X_i}$ determines a surjection on *torsion-free abelianizations* $\Delta_{U_i}^{\text{ab-t}} \twoheadrightarrow \Delta_{X_i}^{\text{ab-t}}$ that induces an *isomorphism* $\mathcal{Q}(\Delta_{U_i}^{\text{ab-t}}) \xrightarrow{\sim} \mathcal{Q}(\Delta_{X_i}^{\text{ab-t}})$. Thus, it follows from *"Poincaré duality"* [i.e., the isomorphism $\Delta_{X_i}^{\text{ab-t}} \xrightarrow{\sim} \mathrm{Hom}(\Delta_{X_i}^{\text{ab-t}}, \widehat{\mathbb{Z}}(1))$ determined by the cup-product on the étale cohomology of X] that

$$\begin{aligned} 2g_{X_i} &= \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{X_i}^{\text{ab-t}}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\Delta_{X_i}^{\text{ab-t}}) \otimes \mathbb{Q}_l) \\ &= \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{X_i}^{\text{ab-t}}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\mathcal{Q}(\Delta_{X_i}^{\text{ab-t}})) \otimes \mathbb{Q}_l) \\ &= \dim_{\mathbb{Q}_l}(\mathcal{Q}(\Delta_{U_i}^{\text{ab-t}}) \otimes \mathbb{Q}_l) + \dim_{\mathbb{Q}_l}(\mathcal{T}(\mathcal{Q}(\Delta_{U_i}^{\text{ab-t}})) \otimes \mathbb{Q}_l) \end{aligned}$$

— where g_{X_i} is the *genus* of X_i, and $l \in \Sigma$. Thus, we conclude that $g_{X_1} = g_{X_2}$. In particular, by passing to corresponding [i.e., via α] open subgroups of the Π_{U_i}, we may assume that $g_{X_1} = g_{X_2} \geq 2$.

Next, by applying this equality "$g_{X_1} = g_{X_2}$" to corresponding [i.e., via α] open subgroups of the Π_{U_i}, it follows from the *Hurwitz formula* that the condition on a pair of open subgroups $J_i \subseteq H_i \subseteq \Delta_{U_i}$ that "the covering between J_i and H_i be *cyclic* of order a power of a prime number and *totally ramified* at precisely one closed point but unramified elsewhere" is *preserved by* α. Thus, it follows formally [cf., e.g., the latter portion of the proof of [Mzk5], Lemma 1.3.9] that α *preserves the inertia groups* of points $\in \Xi_i$. Moreover, by considering the *conjugation action* of Π_{U_i} on these inertia groups, we conclude that α *preserves the decomposition groups* $\subseteq \Pi_{U_i}$ *of points* $\in \Xi_i$. Thus, in summary, α induces an isomorphism $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ that *preserves the decomposition groups* $\subseteq \Pi_{X_i}$ *of points* $\in \Xi_i$; in particular, by applying Corollary 2.9 to this isomorphism $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$, we obtain an isomorphism of schemes $U_1 \xrightarrow{\sim} U_2$, as desired. This completes the proof of Corollary 2.10. ○

Remark 2.10.1. Thus, by contrast to the results of [Mzk14], Corollary 2.3, or [Mzk15], Corollary 3.8, (g) [cf. [Mzk15], Remark 3.8.1] — or, indeed, Corollary 1.11, (iii) of the present paper — Corollary 2.10 constitutes the first *"absolute isomorphism version of the Grothendieck Conjecture over MLF's"* known to the author that *does not rely on the use of Belyi maps*. One aspect of this independence of the theory of Belyi maps may be seen in the fact that Corollary 2.10 may be applied to pro-curves which are *not of unit type* [cf. Remark 2.8.2, (i), (iii)]. Another aspect of this independence of the theory of Belyi maps may be seen in the fact that Corollary 2.10 involves *geometrically pro-*Σ arithmetic fundamental groups for Σ which are not necessarily equal to the set of all prime numbers.

Finally, we observe that the techniques developed in the present §2 allow one to give a more *pedestrian treatment* of the [somewhat sketchy] treatment given in [Mzk9] [cf. the verification of "observation (iv)" given in the proof of [Mzk9], Corollary 3.11, as well as Remark 2.11.1 below] of the fact that *"cusps always appear as images of nodes"*.

Corollary 2.11. **(Cusps as Images of Nodes)** *Let k be a* **complete discretely valued field** *of characteristic zero, with* **perfect residue field** *$\underline{k}$ of characteristic $p > 0$ and ring of integers $\mathcal{O}_K$; $\eta \stackrel{\text{def}}{=} \text{Spec}(k)$; $S^{\log}$ the log scheme obtained by equipping $S \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}_K)$ with the log structure determined by the closed point $\underline{S} \stackrel{\text{def}}{=} \text{Spec}(\underline{k})$ of S; $X^{\log} \to S^{\log}$ a* **stable log curve** *over $S^{\log}$ such that the underlying scheme of the generic fiber $X^{\log} \times_S \eta$ is* **smooth***; $\xi \in X(S)$ a* **cusp** *of the stable log curve $X^{\log}$; $\underline{\xi} \in \underline{X}(\underline{S})$ the restriction of ξ to the special fiber $\underline{X}$ of X. In the following, we shall denote restrictions to η by means of a subscript η; also we shall often identify $\underline{\xi}$ with its image in $\underline{X}$. Then, after possibly replacing k by a finite extension of k, there* **exists** *a morphism of stable log curves over $S^{\log}$*

$$\phi^{\log} : Y^{\log} \to X^{\log}$$

such that the following properties are satisfied:

(a) the restriction $\phi^{\log}_\eta : Y^{\log}_\eta \to X^{\log}_\eta$ is a **finite log étale Galois** *covering;*

(b) $\underline{\xi}$ is the **image of a node** *of the special fiber $\underline{Y}$ of Y;*

(c) $\underline{\xi}$ is the **image of an irreducible component** *of $\underline{Y}$.*

If, moreover, $\underline{X}$ is **sturdy**, **loop-ample**, *and* **singular**, *then $\phi_\eta : Y_\eta \to X_\eta$ may be taken to be* **finite étale of degree p**.

Proof. By replacing k by an appropriate subfield of k, one verifies immediately that we may assume that $\underline{k}$ is of *countable cardinality*, hence that k satisfies the hypotheses of the discussion preceding Lemma 2.6. After possibly replacing k by a finite extension of k and $X^{\log}_\eta$ by a finite log étale Galois covering of $X^{\log}_\eta$ [which, in fact, may be taken to be of degree a power of $p \cdot l$, where l is a prime $\neq p$], we may assume that $\underline{X}$ is *sturdy* [cf. §0], *loop-ample* [cf. §0], *singular* [cf. Remark 2.6.3], and *split*. Next, let us recall from the well-known theory of pointed stable curves [cf. [Knud]] that if we write $V^{\log} \to S^{\log}$ for the *stable log curve* obtained by *forgetting the cusps* of $X^{\log}$ [so $V_\eta = X_\eta$], then it follows immediately from the fact that $\underline{X}$ is sturdy that $V = X$. Thus, by Lemma 2.6, (i) [cf. also the way in which Lemma 2.6, (i), is applied in the proof of Lemma 2.6, (ii)], it follows that, after possibly replacing k by a finite extension of k, there exists a morphism of stable log curves over $S^{\log}$

$$\phi^{\log} : Y^{\log} \to X^{\log}$$

such that if we write ϕ for the morphism of schemes underlying $\phi^{\log}$, then $\phi_\eta : Y_\eta \to X_\eta$ is a *finite étale Galois covering of degree* p that is *wildly ramified* over the *irreducible component* C of $\underline{X}$ containing $\underline{\xi}$.

Next, let us suppose that the property (c) is satisfied. Thus, there exists an irreducible component E of $\underline{Y}$ that maps to $\underline{\xi}$. Next, let us observe that there exists an irreducible component D of $\underline{Y}$ that maps *finitely* to C and meets the *connected component* F of the fiber $\phi^{-1}(\underline{\xi})$ that contains E [so D is *not contained* in F]. In particular, it follows that there exists a *chain of irreducible components*

$$E_1 = E, E_2, \ldots, E_n$$

[where $n \geq 1$ is an integer] of $\underline{Y}$ joining E to D such that each $E_j \subseteq F$ [for $j = 1, \ldots, n$]. Thus, E_n meets D at some node of $\underline{Y}$ that maps to $\underline{\xi}$. That is to say, property (b) is satisfied. Thus, to complete the proof of Corollary 2.11, it suffices to verify property (c).

Now suppose that property (c) *fails to hold*. Then ϕ is *finite* over some neighborhood of $\underline{\xi}$. Since ϕ_η is *wildly ramified* over C, it follows that there exists a *nontrivial* element $\sigma \in \mathrm{Gal}(Y_\eta^{\log}/X_\eta^{\log})$ that *fixes* and *acts as the identity* on some irreducible component D of $\underline{Y}$ that maps *finitely* to C. After possibly replacing k by a finite extension of k, it follows from the *finiteness* of ϕ over some neighborhood of $\underline{\xi}$ that we may assume that there exists a *cusp* $\zeta \in Y(S)$ of $Y^{\log}$ lying over ξ such that the restriction $\underline{\zeta}$ of ζ to $\underline{Y}$ lies in D. But then the *distinct* [since σ is *nontrivial*, and ϕ_η is *étale*] cusps ζ, ζ^σ of $Y^{\log}$ have *identical restrictions* $\underline{\zeta}$, $\underline{\zeta}^\sigma$ to $\underline{Y}$ — in contradiction to the definition of a *"stable log curve"* [i.e., of a "pointed stable curve"]. This completes the proof of property (c) and hence of Corollary 2.11. ◯

Remark 2.11.1.

(i) The statement of [Mzk9], Corollary 3.11, concerns smooth log curves over an MLF, but in fact, the same proof as the proof given in [Mzk9] for [Mzk9], Corollary 3.11, may be applied to smooth log curves over an *arbitrary mixed characteristic complete discretely valued field*. Here, we note that by passing to an appropriate extension, this discretely valued field may be assumed to have a *perfect residue field*, as in Corollary 2.11. In particular, Corollary 2.11 may be applied to [the portion corresponding to "observation (iv)" in *loc. cit.* of] the proof of such a generalization of [Mzk9], Corollary 3.11, for more general fields.

(ii) In the discussion of the *"pro-Σ version" of [Mzk9], Corollary 3.11,* in [Mzk9], Remark 3.11.1,

one should *assume* that $p_\alpha, p_\beta \in \Sigma$.

In fact, this assumption is, in some sense, *implicit* in the phraseology that appears in the first two lines of [Mzk9], Remark 3.11.1, but it should have been stated *explicitly*.

Section 3: Elliptic and Belyi Cuspidalizations

The sort of *preservation of decomposition groups of closed points* that is required in the hypothesis of Corollary 2.9 is shown [for certain types of hyperbolic curves] in the case of *profinite geometric fundamental groups* in [Mzk8], Corollary 3.2. On the other hand, at the time of writing, the author does not know of any such results in the case of *pro-*Σ *geometric fundamental groups*, when Σ is *not* equal to the set of all primes. Nevertheless, in the present §3, we observe that the techniques of [Mzk8], §2, concerning the preservation of *decomposition groups of torsion points of elliptic curves* do indeed hold for fairly general *pro-*Σ *geometric fundamental groups* [cf. Corollaries 3.3, 3.4]. Moreover, we observe that these techniques — which may be applied not only to [hyperbolic orbicurves related to] elliptic curves, but also, in the *profinite* case, to [hyperbolic orbicurves related to] *tripods* [i.e., hyperbolic curves of type $(0, 3)$ — cf. [Mzk15], §0], via the use of *Belyi maps* — allow one to recover not only the decomposition groups of [certain] closed points, but also the resulting *"cuspidalizations"* [i.e., the arithmetic fundamental groups of open subschemes obtained by removing such closed points] — cf. Corollaries 3.7, 3.8.

Let X be a *hyperbolic orbicurve* over a field k of *characteristic zero*; $\overline{k}$ an *algebraic closure* of k. We shall denote the base-change operation "$\times_k \overline{k}$" by means of a subscript $\overline{k}$. Thus, we have an exact sequence of fundamental groups $1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \to 1$.

Definition 3.1. Let $\pi_1(X) \twoheadrightarrow \Pi$ be a *quotient* of profinite groups. Write $\Delta \subseteq \Pi$ for the image of $\pi_1(X_{\overline{k}})$ in Π. Then we shall say that X is Π*-elliptically admissible* if the following conditions hold:

(a) X admits a *k-core* [in the sense of [Mzk6], Remark 2.1.1] $X \to C$;

(b) C is *semi-elliptic* [cf. §0], hence admits a double covering $D \to C$ by a once-punctured elliptic curve D;

(c) X admits a finite étale covering $Y \to X$ by a hyperbolic curve Y over a finite extension k_Y of k that arises from a *normal open subgroup* $\Pi_Y \subseteq \Pi$ such that the resulting finite étale covering $Y \to C$ *factors* as the composite of a covering $Y \to D$ with the covering $D \to C$ and, moreover, is such that, for every set of primes Σ such that some open subgroup of Δ is pro-Σ, it holds that Δ is pro-Σ, and, moreover, the degree of the covering $Y \to C \times_k k_Y$ is a *product of primes* [perhaps with multiplicities] $\in \Sigma$.

When $\Pi = \pi_1(X)$, we shall simply say that X is *elliptically admissible.*

Remark 3.1.1. In the notation of Definition 3.1, one verifies immediately that $D_{\overline{k}} \to C_{\overline{k}}$ may be characterized as the *unique* [up to isomorphism over $C_{\overline{k}}$] *finite étale double covering* of $C_{\overline{k}}$ by a *hyperbolic curve* [i.e., as opposed to an arbitrary *hyperbolic orbicurve*].

Example 3.2. Scheme-theoretic Elliptic Cuspidalizations.

(i) Let N be a *positive integer*; D a *once-punctured elliptic curve* over a *finite Galois extension* k' of k such that all of the N*-torsion points* of the underlying *elliptic curve* E of D are *defined* over k'; $D \to C$ a *semi-elliptic* k'*-core* of D [such that $D \to C$ is the double covering appearing in the definition of *"semi-elliptic"*]. Then the morphism $[N]_E : E \to E$ given by *multiplication by* N determines a *finite étale covering* $[N]_D : U \to D$ [of degree N^2], together with an *open embedding* $U \hookrightarrow D$ [which we use to identify U with its image in D], i.e., we have a diagram as follows:

$$\begin{array}{ccc} U & \hookrightarrow & D \\ \big\downarrow{\scriptstyle [N]_D} & & \\ D & & \end{array}$$

Suppose that the Galois group $\mathrm{Gal}(\overline{k}/k)$ is *slim*. Then, in the language of [Mzk15], §4, this situation may be described as follows [cf. [Mzk15], Definition 4.2, (i), where we take the extension "$1 \to \Delta \to \Pi \to G \to 1$" to be the extension $1 \to \pi_1(D \times_{k'} \overline{k}) \to \pi_1(D) \to \mathrm{Gal}(\overline{k}/k') \to 1$]: The above diagram yields a *chain*

$$D \rightsquigarrow U\ (\to D) \rightsquigarrow (U \hookrightarrow)\ U_n \rightsquigarrow (U_n \hookrightarrow)\ U_{n-1} \rightsquigarrow \dots$$
$$\rightsquigarrow (U_3 \hookrightarrow)\ U_2 \rightsquigarrow (U_2 \hookrightarrow)\ U_1 \stackrel{\mathrm{def}}{=} D$$

[where $n \stackrel{\mathrm{def}}{=} N^2 - 1$] whose associated *type-chain* is

$$\curlywedge, \bullet, \dots, \bullet$$

[i.e., a *finite étale covering*, followed by n *de-cuspidalizations*], together with a *terminal isomorphism*

$$U_1 \xrightarrow{\sim} D$$

[which, in our notation, amounts to the *identity morphism*] from the U_1 at the end of the above chain to the unique D of the trivial chain [of length 0]. In particular:

> The above chain may thought of as a construction of a *"cuspidalization"* [i.e., result of passing to an open subscheme by removing various closed points] $U \hookrightarrow D$ of D.

The remainder of the portion of the theory of the present §3 concerning *elliptic cuspidalizations* consists, in essence, of the unraveling of various consequences of this *"chain-theoretic formulation"* of the diagram that appears at the beginning of the present item (i).

(ii) A variant of the discussion of (i) may be obtained as follows. In the notation of (i), suppose further that X is an *elliptically admissible* hyperbolic orbicurve over k, and that we have been given *finite étale coverings* $V \to X$, $V \to D$, where V is a hyperbolic curve over k'. Also, [for simplicity] we suppose that $V \to X$ is a *Galois* covering such that $\mathrm{Gal}(V/X)$ *preserves* the open subscheme $U_V \stackrel{\mathrm{def}}{=} V \times_D U \subseteq V$ [i.e., the inverse image of $U \subseteq D$ via $V \to D$]. Thus, $U_V \subseteq V$ *descends* to an open subscheme $U_X \subseteq X$. Then by appending to the chain of (i) the "finite étale covering" $V \to X$, followed by the "finite étale quotient" $V \to D$ on the *left*, and the "finite étale covering" $V \to D$, followed by the "finite étale quotient" $V \to X$ on the *right*, we obtain a *chain*

$$X \rightsquigarrow V\ (\to X) \rightsquigarrow (V \to)\ D \rightsquigarrow U\ (\to D) \rightsquigarrow (U \hookrightarrow)\ U_n \rightsquigarrow (U_n \hookrightarrow)\ U_{n-1} \rightsquigarrow \dots$$
$$\rightsquigarrow (U_3 \hookrightarrow)\ U_2 \rightsquigarrow (U_2 \hookrightarrow)\ U_1 \stackrel{\mathrm{def}}{=} D \rightsquigarrow V\ (\to D) \rightsquigarrow (V \to)\ X_* \stackrel{\mathrm{def}}{=} X$$

whose associated *type-chain* is

$$\curlywedge, \curlyvee, \curlywedge, \bullet, \dots, \bullet, \curlywedge, \curlyvee$$

[where the "..." are all "$\bullet$'s"], together with a *terminal isomorphism* $X_* \xrightarrow{\sim} X$ [i.e., the *identity morphism*]. In particular, the above chain may thought of as a construction of a *"cuspidalization"* $U_X \hookrightarrow X$ of X via the construction of a "cuspidalization" $U_V \hookrightarrow V$ of V, equipped with *descent data* [i.e., a suitable collection of automorphisms] with respect to the finite étale Galois covering $V \to X$.

Now by translating the *scheme-theoretic* discussion of Example 3.2 into the language of *profinite groups* via the theory of [Mzk15], §4, we obtain the following result.

Corollary 3.3. **(Pro-Σ Elliptic Cuspidalization I: Algorithms)** *Let $\mathbb{D}$ be a* **chain-full** *set of collections of* **partial construction data** *[cf. [Mzk15], Definition 4.6, (i)] such that the* **rel-isom-$\mathbb{D}$GC** *holds [i.e., the "relative isomorphism version of the Grothendieck Conjecture for $\mathbb{D}$ holds" — cf. [Mzk15], Definition 4.6, (ii)]; G a* **slim** *profinite group;*

$$1 \to \Delta \to \Pi \to G \to 1$$

an **extension of GSAFG-type** *that admits* **partial construction data** (k, X, Σ), *where k is of* **characteristic zero**, *and X is a* **Π-elliptically admissible** *[cf. Definition 3.1]* **hyperbolic orbicurve**, *such that $([X], [k], \Sigma) \in \mathbb{D}$; $\alpha : \pi_1(X) \twoheadrightarrow \Pi$ the corresponding* **scheme-theoretic envelope** *[cf. [Mzk15], Definition 2.1, (iii)]; $\widetilde{X} \to X$ the* **pro-finite étale covering** *of X determined by α [so $\Pi \xrightarrow{\sim} \mathrm{Gal}(\widetilde{X}/X)$]; $\widetilde{k}$ the resulting field extension of k [so $G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{k}/k)$]. Suppose further that, for some $l \in \Sigma$, the* **cyclotomic character** *$G \to \mathbb{Z}_l^\times$ has* **open image**. *Thus, by the theory of [Mzk15], §4, we have associated categories*

$$\mathrm{Chain}(\Pi); \quad \mathrm{Chain}^{\text{iso-trm}}(\Pi); \quad \text{ÉtLoc}(\Pi)$$

which may be constructed via **purely "group-theoretic" operations** *from the extension of profinite groups $1 \to \Delta \to \Pi \to G \to 1$ [cf. [Mzk15], Definition 4.2, (iii), (iv), (v); [Mzk15], Lemma 4.5, (v); the proof of [Mzk15], Theorem 4.7, (ii)]. Then:*

(i) Let $G' \subseteq G$ be a normal open subgroup, corresponding to some finite extension $k' \subseteq \widetilde{k}$ of k; $\Pi' \stackrel{\mathrm{def}}{=} \Pi \times_G G'$; C a **k'-core** *of $X_{k'} \stackrel{\mathrm{def}}{=} X \times_k k'$. Then the finite étale covering $X_{k'} \to C$ determines a chain $X_{k'} \rightsquigarrow C$ of the category $\mathrm{Chain}(\widetilde{X}/X_{k'})$ [cf. [Mzk15], Definition 4.2, (i), (ii)] whose image $\Pi' \rightsquigarrow \Pi_C$ in $\mathrm{Chain}(\Pi')$ [via the natural functor of [Mzk15], Remark 4.2.1] may be characterized*

"group-theoretically", *up to isomorphism in* $\mathrm{Chain}(\Pi')$, *as the unique chain of length* 1 *in* $\mathrm{Chain}(\Pi')$, *with associated* **type-chain** $\curlyvee$, *such that the resulting object of* $\acute{\mathrm{E}}\mathrm{tLoc}(\Pi')$ *forms a* **terminal object** *of* $\acute{\mathrm{E}}\mathrm{tLoc}(\Pi')$.

(ii) The collection of open subgroups $\Pi_D \subseteq \Pi_C$ *that arise from* **finite étale double coverings** $D \to C$ *that exhibit* C *as* **semi-elliptic** *[cf. Remark 3.1.1] may be characterized* **"group-theoretically"** *as the collection of open subgroups* $J \subseteq \Pi_C$ *of index* 2 *such that* $J \bigcap \Delta_C$ *[where* $\Delta_C \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_C \twoheadrightarrow G')$*] is* **torsion-free** *[i.e., the covering determined by* J *is a* **scheme** *— cf. [Mzk15], Lemma 4.1, (iv)].*

(iii) Write $\Delta_D \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_D \twoheadrightarrow G')$. *Let* N *be a* **positive integer** *which is a product of primes [perhaps with multipliticites] of* Σ*;* $U \subseteq D$ *the open subscheme obtained by removing the* **N-torsion points** *of the elliptic curve underlying* D*;* $V \to X$, $V \to D$ **finite étale coverings**, *where* V *is a hyperbolic curve over* k'. *Suppose further that* $V \to X$ *arises from a normal open subgroup* $\Pi_V \subseteq \Pi$ *such that* $\mathrm{Gal}(V/X) \cong \Pi/\Pi_V$ **preserves** *the open subscheme* $U_V \stackrel{\mathrm{def}}{=} V \times_D U \subseteq V$ *[i.e., the inverse image of* $U \subseteq D$ *via* $V \to D$*], while* $V \to D$ *arises from an open immersion* $\Pi_V \hookrightarrow \Pi_D$. *Thus,* $U_V \subseteq V$ **descends** *to an open subscheme* $U_X \subseteq X$, *and* $U \subseteq D$, $U_V \subseteq V$, $U_X \subseteq X$ *determine* **extensions of GSAFG-type**

$$1 \to \Delta_U \to \Pi_U \to G' \to 1; \quad 1 \to \Delta_{U_V} \to \Pi_{U_V} \to G' \to 1$$

$$1 \to \Delta_{U_X} \to \Pi_{U_X} \to G \to 1$$

[i.e., by considering the finite étale Galois coverings of degree a product of primes [perhaps with multipliticites] $\in \Sigma$ *over coverings of* U, U_V, U_X *arising from* Π*], together with natural surjections* $\Pi_U \twoheadrightarrow \Pi_D$, $\Pi_{U_V} \twoheadrightarrow \Pi_V$, $\Pi_{U_X} \twoheadrightarrow \Pi$ *and open immersions* $\Pi_{U_V} \hookrightarrow \Pi_U$, $\Pi_{U_V} \hookrightarrow \Pi_{U_X}$. *[In particular,* Δ_U, Δ_{U_V}, *and* Δ_{U_X} *are* **pro-Σ groups***.] Then, for any* $G' \subseteq G$ *that is* **sufficiently small**, *where "sufficiently" depends only on* N, *the natural surjection*

$$\Pi_{U_X} \twoheadrightarrow \Pi$$

— i.e., **"cuspidalization"** *of* Π *— may be constructed via* **"group-theoretic"** *operations as follows:*

(a) There exists a [not necessarily unique] **Π-chain**, *which admits an* **entirely "group-theoretic" description**, *with associated* **type-chain**

$$\curlywedge, \curlyvee, \curlywedge, \bullet, \ldots, \bullet, \curlywedge, \curlyvee$$

— cf. Example 3.2, (ii) — that admits a **terminal isomorphism** *with the trivial* Π*-chain [of length* 0*], and whose final three groups consist of* $\Pi_D \rightsquigarrow \Pi_V$ $(\hookrightarrow \Pi_D) \rightsquigarrow (\Pi_V \hookrightarrow)$ Π *such that the* **natural surjection** $\Pi_U \twoheadrightarrow \Pi_D$ *may be recovered from the chain of "$\bullet$'s" terminating at the third to last group of the above-mentioned* Π*-chain; the* **natural surjection** $\Pi_{U_V} \twoheadrightarrow \Pi_V$ *may then be recovered from* $\Pi_U \twoheadrightarrow \Pi_D$ *by forming the fiber product with the inclusion* $\Pi_V \hookrightarrow \Pi_D$.

(b) The **natural surjection** $\Pi_{U_X} \twoheadrightarrow \Pi$ *may be recovered from* $\Pi_{U_V} \twoheadrightarrow \Pi_V$ *[where we note that* $\Pi_{U_V} \twoheadrightarrow \Pi_V$ *may be identified with the fiber product of*

$\Pi_{U_X} \twoheadrightarrow \Pi$ *with the inclusion* $\Pi_V \hookrightarrow \Pi$*] by forming the* "$\overset{\text{out}}{\rtimes}$" *[cf. §0] with respect to the* **unique** *lifting [relative to* $\Pi_{U_V} \twoheadrightarrow \Pi_V$*] of the outer action of the finite group* Π/Π_V *on* Π_V *to a group of outer automorphisms of* Π_{U_V}*.*

(c) The **decomposition groups of the closed points** *of* X *lying in the complement of* U_X *may be obtained as the images via* $\Pi_{U_X} \twoheadrightarrow \Pi$ *of the* **cuspidal decomposition groups** *of* Π_{U_X} *[cf. [Mzk15], Lemma 4.5, (v)].*

Proof. The assertions of Corollary 3.3 follow immediately from the definitions, together with the various references quoted in the course of the "group-theoretic" reconstruction algorithm described in the statement of Corollary 3.3, and the equivalences of [Mzk15], Theorem 4.7, (i). ○

Remark 3.3.1. Let p be a *prime number*. Then if one takes $\mathbb{F}$ to be set of isomorphism classes of *generalized sub-p-adic fields*, $\mathbb{S}$ the set of sets of prime numbers containing p, and $\mathbb{V}$ to be the set of isomorphism classes of *hyperbolic orbicurves* over fields whose isomorphism class $\in \mathbb{F}$, then $\mathbb{D} \stackrel{\text{def}}{=} \mathbb{V} \times \mathbb{F} \times \mathbb{S}$ satisfies the hypothesis of Corollary 3.3 concerning "$\mathbb{D}$" [cf. [Mzk15], Example 4.8, (i)].

Remark 3.3.2. Recall that when k is an *MLF* or an *NF*, the subgroup $\Delta \subseteq \Pi$ admits a purely *"group-theoretic"* characterization [cf. [Mzk15], Theorem 2.6, (v), (vi)]. Thus, when k is an *MLF* or an *NF*, the various "group-theoretic" reconstruction algorithms described in the statement of Corollary 3.3 may be thought of as being applied not to the extension $1 \to \Delta \to \Pi \to G \to 1$, but rather to the *single profinite group* Π.

Remark 3.3.3. One verifies immediately that Corollary 3.3 admits a *"tempered version"*, when the base field k is an *MLF* [cf. [Mzk15], Theorem 4.12, (i)]. We leave the routine details to the reader.

Remark 3.3.4. By applying the *tempered version* of Corollary 3.3 discussed in Remark 3.3.3, one may obtain *"explicit reconstruction algorithm versions"* of certain results of [Mzk12] [cf. [Mzk12], Theorem 1.6; [Mzk12], Remark 1.6.1] concerning the *étale theta function.* We leave the routine details to the reader.

The "group-theoretic" *algorithm* of Corollary 3.3 has the following immediate *"Grothendieck Conjecture-style"* consequence.

Corollary 3.4. (Pro-Σ Elliptic Cuspidalization II: Comparison) *Let* $\mathbb{D}$ *be a* **chain-full** *set of collections of* **partial construction data** *[cf. [Mzk15], Definition 4.6, (i)] such that the* **rel-isom-$\mathbb{D}$GC** *holds [cf. [Mzk15], Definition 4.6, (ii)]. For* $i = 1, 2$*, let* G_i *be a* **slim** *profinite group;*

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

an **extension of GSAFG-type** *that admits* **partial construction data** (k_i, X_i, Σ_i), *where* k_i *is of* **characteristic zero**, *and* X_i *is a* Π_i**-elliptically admissible** *[cf. Definition 3.1]* **hyperbolic orbicurve**, *such that* $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$; $\alpha_i : \pi_1(X_i) \twoheadrightarrow \Pi_i$ *the corresponding* **scheme-theoretic envelope** *[cf. [Mzk15], Definition 2.1, (iii)];* $\widetilde{X}_i \to X_i$ *the* **pro-finite étale covering** *of* X *determined by* α_i *[so* $\Pi_i \xrightarrow{\sim} \mathrm{Gal}(\widetilde{X}_i/X_i)$*];* $\widetilde{k}_i$ *the resulting field extension of* k_i *[so* $G_i \xrightarrow{\sim} \mathrm{Gal}(\widetilde{k}_i/k_i)$*];* C_i *a* k_i**-core** *of* X_i*;* $D_i \to C_i$ *a* **finite étale double covering** *that exhibits* C_i *as* **semi-elliptic** *[cf. Remark 3.1.1];* $\Pi_i \subseteq \Pi_{C_i}$, $\Pi_{D_i} \subseteq \Pi_{C_i}$ *the open subgroups determined by* $X_i \to C_i$, $D_i \to C_i$*;* N *a* **positive integer** *which is a product of primes [perhaps with multipliticites]* $\in \Sigma_1 \bigcap \Sigma_2$*;* $U_i \subseteq D_i$ *the open subscheme obtained by removing the* $\boldsymbol{N}$**-torsion points** *of the elliptic curve underlying* D_i*;* $V_i \to X_i$, $V_i \to D_i$ **finite étale coverings** *that arise from a* **normal** *open subgroup* $\Pi_{V_i} \subseteq \Pi_i$ *and an open immersion* $\Pi_{V_i} \hookrightarrow \Pi_{D_i}$ *such that* $\mathrm{Gal}(V_i/X_i) \cong \Pi_i/\Pi_{V_i}$ **preserves** *the open subscheme* $U_{V_i} \stackrel{\mathrm{def}}{=} V_i \times_{D_i} U_i \subseteq V_i$ *[i.e., the inverse image of* $U_i \subseteq D_i$ *via* $V_i \to D_i$*];* $U_{X_i} \subseteq X_i$ *the resulting open subscheme [obtained by* **descending** $U_{V_i} \subseteq V_i$*];*

$$1 \to \Delta_{U_{X_i}} \to \Pi_{U_{X_i}} \to G_i \to 1$$

the **extension of GSAFG-type** *obtained [via* α_i*] by considering the finite étale Galois coverings of degree a product of primes [perhaps with multipliticites]* $\in \Sigma_i$ *over coverings of* U_{X_i} *arising from* Π_i*;* $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$ *the* **natural surjection** *[relative to* α_i*]. Suppose further that, for some* $l \in \Sigma_1 \bigcap \Sigma_2$*, the* **cyclotomic characters** $G_i \to \mathbb{Z}_l^\times$ *have* **open image** *for* $i = 1, 2$*. Let*

$$\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$$

be an **isomorphism of profinite groups** *such that* $\phi(\Delta_1) = \Delta_2$*. Then there exists an* **isomorphism of profinite groups**

$$\phi_U : \Pi_{U_{X_1}} \xrightarrow{\sim} \Pi_{U_{X_2}}$$

that is **compatible** *with* ϕ*, relative to the natural surjections* $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$*. Moreover, such an isomorphism is* **unique** *up to composition with an inner automorphism arising from an element of the kernel of* $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$*.*

Proof. The construction of ϕ_U follows immediately from Corollary 3.3; the asserted *uniqueness* then follows immediately from our assumption that the *rel-isom-*$\mathbb{D}$*GC* holds. ◯

Remark 3.4.1. Just as in the case of Corollary 3.3 [cf. Remark 3.3.3], Corollary 3.4 admits a *"tempered version"*, when the base fields k_i invoved are *MLF's*. We leave the routine details to the reader.

Remark 3.4.2. By applying Corollary 3.4 [cf. also Remarks 3.3.4, 3.4.1], one may obtain *"pro-*Σ *tempered" versions* of certain results of [Mzk12] [cf. [Mzk12], Theorem 1.6; [Mzk12], Remark 1.6.1] concerning the *étale theta function*. We leave the routine details to the reader.

Now we return to the notation introduced at the beginning of the present §3: Let X be a *hyperbolic orbicurve* over a field k of *characteristic zero*; $\overline{k}$ an *algebraic closure* of k. Thus, we have an exact sequence of fundamental groups $1 \to \pi_1(X \times_k \overline{k}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \to 1$.

Definition 3.5. We shall say that X is *of strictly Belyi type* if it is defined over a number field and *isogenous* [cf. §0] to a hyperbolic curve of genus zero. [Thus, this definition generalizes the definition of [Mzk14], Definition 2.3, (i).]

Example 3.6. Scheme-theoretic Belyi Cuspidalizations.

(i) Let P be a copy of the *projective line minus three points* over a *finite Galois extension* k' of k; V an *arbitrary hyperbolic curve* over k'; $U \subseteq V$ a *nonempty open subscheme* [hence, in particular, a *hyperbolic curve* over k']. Suppose that U [hence also V] is *defined over a number field*. Then it follows from the *existence of Belyi maps* [cf. [Belyi]; [Mzk7]] that, for some *nonempty open subscheme* $W \subseteq U$, there exists a diagram as follows:

$$\begin{array}{ccccc} W & \hookrightarrow & U & \hookrightarrow & V \\ \downarrow \beta & & & & \\ P & & & & \end{array}$$

[where the "$\hookrightarrow$'s" are the natural open immersions; the "Belyi map" β is *finite étale*]. By replacing k' by some finite extension of k', let us suppose further [for simplicity] that the *cusps* of W are all defined over k'. Also, let us suppose that the Galois group $\mathrm{Gal}(\overline{k}/k)$ is *slim*. Then, in the language of [Mzk15], §4, this situation may be described as follows [cf. [Mzk15], Definition 4.2, (i), where we take the extension "$1 \to \Delta \to \Pi \to G \to 1$" to be the extension $1 \to \pi_1(P \times_{k'} \overline{k}) \to \pi_1(P) \to \mathrm{Gal}(\overline{k}/k') \to 1$]: For some nonnegative integers n, m, the above diagram yields a *chain*

$$\begin{aligned} P \rightsquigarrow W\ (\to P) \rightsquigarrow (W \hookrightarrow)\ W_n \rightsquigarrow (W_n \hookrightarrow)\ W_{n-1} \rightsquigarrow \dots \\ \rightsquigarrow (W_2 \hookrightarrow)\ W_1 \stackrel{\mathrm{def}}{=} U \rightsquigarrow (U \hookrightarrow)\ U_m \rightsquigarrow (U_m \hookrightarrow)\ U_{m-1} \rightsquigarrow \dots \\ \rightsquigarrow (U_2 \hookrightarrow)\ U_1 \stackrel{\mathrm{def}}{=} V \end{aligned}$$

whose associated *type-chain* is

$$\curlywedge, \bullet, \dots, \bullet$$

[i.e., a *finite étale covering*, followed by $n + m$ *de-cuspidalizations*]. In particular:

> The above chain may thought of as a construction of a *"cuspidalization"* [i.e., result of passing to an open subscheme by removing various closed points] $U \hookrightarrow V$ of V.

The remainder of the portion of the theory of the present §3 concerning *Belyi cuspidalizations* consists, in essence, of the unraveling of various consequences of this *"chain-theoretic formulation"* of the diagram that appears at the beginning of the present item (i).

(ii) A variant of the discussion of (i) may be obtained as follows. In the notation of (i), suppose further that X is a *hyperbolic orbicurve of strictly Belyi type* over k, and that we have been given *finite étale coverings* $V \to X$, $V \to Q$, together with an *open immersion* $Q \hookrightarrow P$ [so Q is a *hyperbolic curve* of genus zero over k']. Also, [for simplicity] we suppose that $V \to X$ is *Galois*, that $U \subseteq V$ *descends* to an open subscheme $U_X \subseteq X$, and [by possibly replacing k' by a finite extension of k'] that the *cusps* of Q are defined over k'. Then by appending to the chain of (i) the "finite étale covering" $V \to X$, followed by the "finite étale quotient" $V \to Q$, followed by the *de-cuspidalizations* $Q \hookrightarrow Q_l \hookrightarrow \ldots \hookrightarrow Q_1 \stackrel{\mathrm{def}}{=} P$ [for some nonnegative integer l], on the *left*, and the "finite étale quotient" $V \to X$ on the *right*, we obtain a *chain*

$$X \rightsquigarrow V\ (\to X) \rightsquigarrow (V \to)\ Q \rightsquigarrow (Q \hookrightarrow)\ Q_l \rightsquigarrow \ldots \rightsquigarrow (Q_2 \hookrightarrow)\ Q_1 \stackrel{\mathrm{def}}{=} P$$
$$\rightsquigarrow W\ (\to P) \rightsquigarrow (W \hookrightarrow)\ W_n \rightsquigarrow \ldots \rightsquigarrow (W_2 \hookrightarrow)\ W_1 \stackrel{\mathrm{def}}{=} U \rightsquigarrow (U \hookrightarrow)\ U_m \rightsquigarrow \ldots$$
$$\rightsquigarrow (U_2 \hookrightarrow)\ U_1 \stackrel{\mathrm{def}}{=} V \rightsquigarrow (V \to)\ X_* \stackrel{\mathrm{def}}{=} X$$

whose associated *type-chain* is

$$\curlywedge, \curlyvee, \bullet, \ldots, \bullet, \curlywedge, \bullet, \ldots, \bullet, \curlyvee$$

[where the "$\ldots$" are all "$\bullet$'s"], together with a *terminal isomorphism* $X_* \stackrel{\sim}{\to} X$ [i.e., the *identity morphism*]. In particular, the above chain may thought of as a construction of a *"cuspidalization"* $U_X \hookrightarrow X$ of X via the construction of a "cuspidalization" $U \hookrightarrow V$ of V, equipped with *descent data* [i.e., a suitable collection of automorphisms] with respect to the finite étale Galois covering $V \to X$.

Now by translating the *scheme-theoretic* discussion of Example 3.6 into the language of *profinite groups* via the theory of [Mzk15], §4, we obtain the following result.

Corollary 3.7. (Profinite Belyi Cuspidalization I: Algorithms) *Let $\mathbb{D}$ be a* **chain-full** *set of collections of* **partial construction data** *[cf. [Mzk15], Definition 4.6, (i)] such that the* **rel-isom-$\mathbb{D}$GC** *holds [i.e., the "relative isomorphism version of the Grothendieck Conjecture for $\mathbb{D}$ holds" — cf. [Mzk15], Definition 4.6, (ii)]; G a* **slim** *profinite group;*

$$1 \to \Delta \to \Pi \to G \to 1$$

an **extension of GSAFG-type** *that admits* **partial construction data** (k, X, Σ), *where k is of* **characteristic zero**, *X is a* **hyperbolic orbicurve of strictly Belyi type** *[cf. Definition 3.5], and Σ is the* **set of all primes**, *such that $([X], [k], \Sigma) \in \mathbb{D}$; $\alpha : \pi_1(X) \stackrel{\sim}{\to} \Pi$ the corresponding* **scheme-theoretic envelope** *[cf. [Mzk15], Definition 2.1, (iii)], which is an isomorphism of profinite groups; $\widetilde{X} \to X$ the* **pro-finite étale covering** *of X determined by α [so $\Pi \stackrel{\sim}{\to} \mathrm{Gal}(\widetilde{X}/X)$]; $\widetilde{k}$ the resulting algebraic closure of k [so $G \stackrel{\sim}{\to} \mathrm{Gal}(\widetilde{k}/k)$]. Suppose further that, for some $l \in \Sigma$, the* **cyclotomic character** $G \to \mathbb{Z}_l^\times$ *has* **open image**. *Thus, by the theory of [Mzk15], §4, we have associated categories*

$$\mathrm{Chain}(\Pi); \quad \mathrm{Chain}^{\text{iso-trm}}(\Pi); \quad \text{ÉtLoc}(\Pi)$$

which may be constructed via **purely "group-theoretic" operations** *from the extension of profinite groups* $1 \to \Delta \to \Pi \to G \to 1$ *[cf. [Mzk15], Definition 4.2, (iii), (iv), (v); [Mzk15], Lemma 4.5, (v); the proof of [Mzk15], Theorem 4.7, (ii)]. Then for every* **nonempty open subscheme**

$$U_X \subseteq X$$

defined over a number field, *the natural surjection*

$$\Pi_{U_X} \stackrel{\text{def}}{=} \pi_1(U_X) \twoheadrightarrow \pi_1(X) \stackrel{\sim}{\to} \Pi$$

[where the final " $\stackrel{\sim}{\to}$ *" is given by the inverse of* α*] — i.e.,* **"cuspidalization"** *of* Π *— may be constructed via* **"group-theoretic"** *operations as follows:*

(a) *For some normal open subgroup* $\Pi_V \subseteq \Pi$*, which corresponds to a finite covering* $V \to X$ *of hyperbolic orbicurves, there exists a [not necessarily unique]* **Π-chain***, which admits an* **entirely "group-theoretic" description***, with associated* **type-chain**

$$\curlywedge, \curlyvee, \bullet, \ldots, \bullet, \curlywedge, \bullet, \ldots, \bullet, \curlyvee$$

— *cf. Example 3.6, (ii) — that admits a* **terminal isomorphism** *with the trivial* Π*-chain [of length* 0*] such that if we write* $U \stackrel{\text{def}}{=} V \times_X U_X$*,* $\Pi_U \stackrel{\text{def}}{=} \Pi_V \times_\Pi \Pi_{U_X}$*, then the* **natural surjection** $\Pi_U \twoheadrightarrow \Pi_V$ *may be recovered from the chain of "*$\bullet$*'s" terminating at the second to last group of the above-mentioned* Π*-chain.*

(b) *The* **natural surjection** $\Pi_{U_X} \twoheadrightarrow \Pi$ *may be recovered from* $\Pi_U \twoheadrightarrow \Pi_V$ *by forming the "* $\stackrel{\text{out}}{\rtimes}$ *" [cf. §0] with respect to the* **unique** *lifting [relative to* $\Pi_U \twoheadrightarrow \Pi_V$*] of the outer action of the finite group* Π/Π_V *on* Π_V *to a group of outer automorphisms of* Π_U*.*

(c) *The* **decomposition groups of the closed points** *of* X *lying in the complement of* U_X *may be obtained as the images via* $\Pi_{U_X} \twoheadrightarrow \Pi$ *of the* **cuspidal decomposition groups** *of* Π_{U_X} *[cf. [Mzk15], Lemma 4.5, (v)].*

Proof. The assertions of Corollary 3.7 follow immediately from the definitions, together with the various references quoted in the course of the "group-theoretic" reconstruction algorithm described in the statement of Corollary 3.7, and the equivalences of [Mzk15], Theorem 4.7, (i). ○

Remark 3.7.1. Similar remarks to Remarks 3.3.1, 3.3.2, 3.3.3 may be made for Corollary 3.7.

Remark 3.7.2. In the situation of Corollary 3.7, when the field k is an *MLF*, one then obtains an *algorithm* for constructing the *decomposition groups of arbitrary closed points* of X, by combining the algorithms of Corollary 3.7 — cf., especially, Corollary 3.7, (c), which allows one to construct the decomposition groups of those closed points of X which [like U_X!] are *defined over a number field* — with the *"p-adic approximation lemma"* of [Mzk8] [i.e., [Mzk8], Lemma 3.1]. A *"Grothendieck Conjecture-style"* version of this sort of reconstruction of decomposition groups of arbitrary closed points of X may be found in [Mzk8], Corollary 3.2.

The "group-theoretic" *algorithm* of Corollary 3.7 has the following immediate *"Grothendieck Conjecture-style"* consequence.

Corollary 3.8. **(Profinite Belyi Cuspidalization II: Comparison)** *Let $\mathbb{D}$ be a* **chain-full** *set of collections of* **partial construction data** *[cf. [Mzk15], Definition 4.6, (i)] such that the* **rel-isom-$\mathbb{D}$GC** *holds [cf. [Mzk15], Definition 4.6, (ii)]. For $i = 1, 2$, let G_i be a* **slim** *profinite group;*

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

an **extension of GSAFG-type** *that admits* **partial construction data** (k_i, X_i, Σ_i), *where k_i is of* **characteristic zero**, *X_i is a* **hyperbolic orbicurve of strictly Belyi type** *[cf. Definition 3.5], and Σ_i is the* **set of all primes**, *such that $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$; $\alpha_i : \pi_1(X_i) \xrightarrow{\sim} \Pi_i$ the corresponding* **scheme-theoretic envelope** *[cf. [Mzk15], Definition 2.1, (iii)], which is an isomorphism of profinite groups; $\widetilde{X}_i \to X_i$ the* **pro-finite étale covering** *of X_i determined by α_i [so $\Pi_i \xrightarrow{\sim} \mathrm{Gal}(\widetilde{X}_i/X_i)$]; $\widetilde{k}_i$ the resulting algebraic closure of k_i [so $G_i \xrightarrow{\sim} \mathrm{Gal}(\widetilde{k}_i/k_i)$]. If, for $i = 1, 2$, $U_{X_i} \subseteq X_i$ is a* **nonempty open subscheme** *which is* **defined over a number field**, *then write*

$$1 \to \Delta_{U_{X_i}} \to \Pi_{U_{X_i}} \to G_i \to 1$$

for the **extension of GSAFG-type** *determined by [α_i and] the natural surjection $\pi_1(U_{X_i}) \twoheadrightarrow \mathrm{Gal}(\widetilde{k}_i/k_i)$ ($\cong G_i$); $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$ for the* **natural surjection** *[relative to α_i]. Suppose further that, for some $l \in \Sigma_1 \bigcap \Sigma_2$, the* **cyclotomic characters** *$G_i \to \mathbb{Z}_l^\times$ have* **open image** *for $i = 1, 2$. Let*

$$\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$$

be an **isomorphism of profinite groups** *such that $\phi(\Delta_1) = \Delta_2$. Then, for each nonempty open subscheme $U_{X_1} \subseteq X_1$ defined over a number field, there exist a nonempty open subscheme $U_{X_2} \subseteq X_2$ defined over a number field and an* **isomorphism of profinite groups**

$$\phi_U : \Pi_{U_{X_1}} \xrightarrow{\sim} \Pi_{U_{X_2}}$$

that is **compatible** *with ϕ, relative to the natural surjections $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$. Moreover, such an isomorphism ϕ_U is* **unique** *up to composition with an inner automorphism arising from an element of the kernel of $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$.*

Proof. The construction of ϕ_U for a suitable nonempty open subscheme U_{X_2} follows immediately from Corollary 3.7; the asserted *uniqueness* then follows immediately from our assumption that the *rel-isom-$\mathbb{D}GC$* holds. ◯

Remark 3.8.1. A similar remark to Remark 3.4.1 may be made for Corollary 3.8.

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