ON THE CUSPIDALIZATION PROBLEM FOR HYPERBOLIC CURVES OVER FINITE FIELDS

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Abstract. In this paper, we study some group-theoretic constructions associated to arithmetic fundamental groups of hyperbolic curves over finite fields. One of the main results of this paper asserts that any Frobenius-preserving isomorphism between the geometrically pro-$l$ fundamental groups of hyperbolic curves with one given point removed induces an isomorphism between the geometrically pro-$l$ fundamental groups of the hyperbolic curves obtained by removing other points. Finally, we apply this result to obtain results concerning certain cuspidalization problems for fundamental groups of (not necessarily proper) hyperbolic curves over finite fields.

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Introduction

In the present paper, we consider the following problem:

Problem.

Suppose that we are given a hyperbolic curve over a finite field in which $l$ is invertible. Then, given the geometrically pro-$l$ fundamental group of the curve obtained by removing a specific point from this hyperbolic curve, is it possible to reconstruct the geometrically pro-$l$ fundamental groups of the curves obtained by removing other points which vary “continuously” in a suitable sense?
We shall formulate the above problem mathematically. Let $l$ be a prime number, $X$ a hyperbolic curve over a finite field $K$ in which $l$ is invertible. For $n$ a positive integer, we denote by $X_n$ the $n$-th configuration space associated to $X$ (hence, $X_1 = X$), and write $\Pi_{X_n}$ for the geometrically pro-$l$ fundamental group of $X$. Here, the fiber of $X_2 \to X$ over a $K$-rational point $x \in X$ may be naturally identified with $X \{ x \}$, so we may regard $X_2 \to X$ as a *continuous family of cuspidalizations* of $X$. Therefore, the above problem can be formulated as follows (where $Y$ denotes a hyperbolic curve over a finite field $L$ in which $l$ is also invertible, and we use similar notations for $Y$ to the notations used for $X$):

**Theorem A.**

Let

$$\alpha : \Pi_{X \setminus \{ x \}} \sim \longrightarrow \Pi_{Y \setminus \{ y \}}$$

be a Frobenius-preserving isomorphism [cf. Definition 3.5] which maps the decomposition group $D_x$ of $x$ (well-defined up to $\Pi_{X \setminus \{ x \}}$-conjugacy) onto the decomposition group $D_y$ of $y$ (well-defined up to $\Pi_{Y \setminus \{ y \}}$-conjugacy). Here, we shall denote by $\overline{\alpha} : \pi_X \sim \longrightarrow \pi_Y$ (resp., $\overline{D_x}$, $\overline{D_y}$) the isomorphism (resp., the decomposition group of $x$ in $\pi_X$, the decomposition group of $y$ in $\pi_Y$) obtained by passing to the quotients $\Pi_{X \setminus \{ x \}} \to \Pi_X$, $\Pi_{Y \setminus \{ y \}} \to \Pi_Y$.

Then there exists a unique isomorphism

$$\alpha_2 : \Pi_{X_2} \sim \longrightarrow \Pi_{Y_2}$$

which is compatible with the natural switching automorphisms up to an inner automorphism and fits into a commutative diagram

$$\begin{array}{ccc}
\Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\
\downarrow & & \downarrow \\
\Pi_X & \xrightarrow{\overline{\alpha}} & \Pi_Y
\end{array}$$

that induces $\alpha$ by restricting $\alpha_2$ to the inverse images (via the vertical arrows) of $\overline{D_x}$ and $\overline{D_y}$.

In particular, if $x'$ (resp., $y'$) is a $K$-rational point of $X$ (resp., an $L$-rational point of $Y$), and we assume that the decomposition groups of $x'$, $y'$ correspond...
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via \( \alpha \), then we have an isomorphism

\[
\alpha' : \Pi_{X \setminus \{x'\}} \xrightarrow{\sim} \Pi_{Y \setminus \{y'\}}
\]

such that \( \alpha \) and \( \alpha' \) induce the same isomorphism \( \Pi_X \xrightarrow{\sim} \Pi_Y \).

In Section 1, we recall the notion of the (log) configuration space associated to a hyperbolic curve and review group-theoretic properties of the various fundamental groups associated to such spaces. In particular, the splitting determined by the Frobenius action on the pro-\( l \) étale fundamental group \( \Delta_{X_n} \) of \( X_n \otimes_K \overline{K} \) gives rise to an explicit description of the graded Lie algebra obtained by considering the weight filtration on \( \Delta_{X_n} \) (cf. Definition 1.6). This explicit description will play an essential role in the proof of Theorem A.

In Section 2, we discuss a certain specific choice (among composites with inner automorphisms) of the morphism between geometrically pro-\( l \) fundamental groups obtained by switching the two ordered marked points parametrized by the second configuration space. This choice will play a key role in the proof of Theorem A.

Section 3 is devoted to proving Theorem A. Roughly speaking, starting from a given geometrically pro-\( l \) fundamental group \( \Pi_{X \setminus \{x\}} \), we reconstruct group-theoretically a suitable topological group, i.e., \( \Pi^{\text{Lie}}_{X_2} \) (cf. Definition 3.1), which contains the geometrically pro-\( l \) fundamental group of the second configuration space, by using the explicit description of graded Lie algebra studied in Section 1. Next, we reconstruct the automorphism on \( \Pi^{\text{Lie}}_{X_2} \) induced by the specific choice of the switching morphism studied in Section 2. Finally, we verify that \( \Pi_{X_2} \) can be generated, as a subgroup of \( \Pi^{\text{Lie}}_{X_2} \), by the given fundamental group \( \Pi_{X \setminus \{x\}} \) and the image of this fundamental group via the specific choice of the switching morphism studied in Section 2; this allows us to reconstruct \( \Pi_{X_2} \) as a subgroup of \( \Pi^{\text{Lie}}_{X_2} \).

In Section 4, as an application of (a slightly generalized version of) Theorem A, we give a group-theoretic construction of the cuspidalization of an affine hyperbolic curve \( X \) over a finite field at a point “infinitesimally close” to the cusp \( x \). That is to say, we give a construction, starting from the geometrically pro-\( l \) fundamental group \( \Pi_X \) of \( X \), of the geometrically pro-\( l \) fundamental group \( \Pi_{X x_{\log}} \) of the log scheme obtained by gluing \( X \) to a tripod (i.e., the projective line minus three points) at a cusp \( x \) of \( X \):

\[ \textbf{Theorem B.} \]

Let \( X \) (resp., \( Y \)) be an affine hyperbolic curve over a finite field \( K \) (resp., \( L \)), \( x \) a \( K \)-rational point of \( \overline{X} \setminus X \) (resp., \( y \) an \( L \)-rational point of \( \overline{Y} \setminus Y \)). Let

\[
\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y
\]

be a Frobenius-preserving isomorphism such that the decomposition groups of \( x \) and \( y \) (which are well-defined up to conjugacy) correspond via \( \alpha \). Then there
exists a unique isomorphism

$$\tilde{\alpha} : \Pi_{\log x} \xrightarrow{\sim} \Pi_{\log y}$$

well-defined up to composition with an inner automorphism which maps the
decomposition group (well-defined up to conjugacy) of $\tilde{x}$ in $\log x$ to that of $\tilde{y}$ in $\log y$, and induces $\alpha$ by passing to the quotients $\Pi_{\log x} \xrightarrow{\sim} \Pi_{\log y}$.

Finally, we consider the cuspidalization problem for (geometrically pro-$l$) fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields (cf. Theorem 4.4):

**Theorem C.**

Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let

$$\alpha_1 : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists a unique isomorphism

$$\alpha_n : \Pi_{X_n} \xrightarrow{\sim} \Pi_{Y_n}$$

well-defined up to composition with an inner automorphism, which is compatible with the natural respective outer actions of the symmetric group on $n$ letters and makes the diagram

$$\begin{array}{ccc}
\Pi_{X_{n+1}} & \xrightarrow{\alpha_{n+1}} & \Pi_{Y_{n+1}} \\
p_i & \downarrow & \downarrow p_i \\
\Pi_{X_n} & \xrightarrow{\alpha_n} & \Pi_{Y_n}
\end{array}$$

$(i = 1, \ldots, n + 1)$ commute.

This statement is already proved in [11] for the case where $n = 2$ and $X$ is proper, and in [4] for the case where $n \geq 3$ and $X$ is proper. On the other hand, by combining results obtained in this paper with the result obtained in [11], we obtain a shorter proof of the statement for $n \geq 3$ which includes, for the first time, the affine case.
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Notations and Conventions

Numbers:
We shall denote by \( \mathbb{Q} \) the field of rational numbers, by \( \mathbb{Z} \) the ring of rational integers, and by \( \mathbb{N} \subseteq \mathbb{Z} \) (resp., \( \mathbb{Z}_{\geq a} \subseteq \mathbb{Z} \)) the additive submonoid of integers \( n \geq 0 \) (resp., the subset of integers \( n \geq a \) for \( a \in \mathbb{Z} \)). If \( l \) is a prime number, then \( \mathbb{Z}_l \) (resp., \( \mathbb{Q}_l \)) denotes the \( l \)-adic completion of \( \mathbb{Z} \) (resp., \( \mathbb{Q} \)).

Topological Groups:
For an arbitrary Hausdorff topological group \( G \), the notation \( G^{\text{ab}} \) will be used to denote the abelianization of \( G \), i.e., the quotient of \( G \) by the closed subgroup of \( G \) topologically generated by the commutators of \( G \).

If \( G \) is a center-free, then we have a natural exact sequence
\[
1 \rightarrow G \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1
\]
where \( \text{Aut}(G) \) denotes the group of automorphisms of the topological group \( G \); the injective (since \( G \) is center-free) homomorphism \( G \rightarrow \text{Aut}(G) \) is obtained by letting \( G \) act on \( G \) by inner automorphisms; \( \text{Out}(G) \) is defined so as to render the sequence exact. If the profinite group \( G \) is topologically finitely generated, then the groups \( \text{Aut}(G) \), \( \text{Out}(G) \) are naturally endowed with a profinite topology, and the above sequence may be regarded as an exact sequence of profinite groups.

If \( J \rightarrow \text{Out}(G) \) is a homomorphism of groups, then we shall write
\[
G \rtimes^\text{out} J := \text{Aut}(G) \times_{\text{Out}(G)} J
\]
for the “outer semi-direct product of \( J \) with \( G \”). Thus, we have a natural exact sequence
\[
1 \rightarrow G \rightarrow G^{\text{out}} \rtimes J \rightarrow J \rightarrow 1.
\]
It is verified (cf. [4], Lemma 4.10) that if an automorphism \( \phi \) of \( G^{\text{out}} \rtimes J \) preserves the subgroup \( G \leq G^{\text{out}} \rtimes J \) and induces the identity morphism on \( G \) and the quotient \( J \), then \( \phi \) is the identity morphism of \( G^{\text{out}} \rtimes J \).

Log schemes:
Basic references for the notion of log scheme are [7] and [6]. In this paper, log structures are always considered on the étale sites of schemes. For a log
scheme $X^{\log}$, we shall denote by $X$ (resp., $\mathcal{M}_X$) the underlying scheme of $X^{\log}$ (resp., the sheaf of monoids defining the log structure of $X^{\log}$). Let $X^{\log}$ and $Y^{\log}$ be log schemes, and $f^{\log} : X^{\log} \to Y^{\log}$ a morphism of log schemes. Then we shall refer to the quotient of $\mathcal{M}_X$ by the image of the morphism $f^*\mathcal{M}_Y \to \mathcal{M}_X$ induced by $f^{\log}$ as the \textit{relative characteristic sheaf} of $f^{\log}$. Moreover, we shall refer to the relative characteristic sheaf of the morphism $X^{\log} \to X$ (where, by abuse of notation, we write $X$ for the log scheme obtained by equipping $X$ with the trivial log structure) induced by the natural inclusion $\mathcal{O}^* \hookrightarrow \mathcal{M}_X$ as the \textit{characteristic sheaf} of $X^{\log}$.

We shall say that a log scheme $X^{\log}$ is \textit{fs} if $\mathcal{M}_X$ is a sheaf of integral monoids, and locally for the \'{e}tale topology, has a chart modeled on a finitely generated and saturated monoid. If $X^{\log}$ is \textit{fs}, then, for $n$ a nonnegative integer, we shall refer to as the \textit{$n$-interior} of $X^{\log}$ the open subset of $X$ on which the associated sheaf of groupifications of characteristic sheaf of $X^{\log}$ is of rank $\leq n$. Thus, the $0$-interior of $X^{\log}$ is often referred to simply as the \textit{interior} of $X^{\log}$.

\textbf{Curves:}

Let $f : X \to S$ be a morphism of schemes. Then we shall say that $f$ is a \textit{family of curves of type (}$(g,r)$\textit{)} if it factors $X \hookrightarrow \overline{X} \to S$ as the composite of an open immersion $X \hookrightarrow \overline{X}$ whose image is the complement $\overline{X} \setminus D$ of a relative divisor $D \subseteq \overline{X}$ which is finite \'{e}tale over $S$ of relative degree $r$, and a morphism $\overline{X} \to S$ which is proper, smooth, and geometrically connected, and whose geometric fibers are one-dimensional of genus $g$. We shall refer to $\overline{X}$ as the \textit{compactification} of $X$.

We shall say that $f$ is a \textit{family of hyperbolic curves} (resp., \textit{tripod}) if $f$ is a family of curves of type $(g,r)$ such that $(g,r)$ satisfies $2g - 2 + r > 0$ (resp., $(g,r) = (0,3)$ and the relative divisor $D$ is split over $S$).

We shall denote by $\overline{M}_{g,[r]+s}$ the moduli stack of $r+s$-pointed stable curves of genus $g$ for which $s$ sections are equipped with an ordering. This moduli stack may be obtained as the quotient of the moduli stack of ordered $(r+s)$-pointed stable curves of genus $g$ (cf. [8] for an exposition of the theory of such curves) by a suitable symmetric group action on $r$ letters. We shall denote by $\overline{M}_{g,[r]+s}^{\log}$ the log stack obtained by equipping $\overline{M}_{g,[r]+s}$ with the log structure associated to the divisor with normal crossings which parametrizes singular curves.

\textbf{Fundamental Groups:}

A basic reference for the notion of \textit{Kummer \'{e}tale covering} is [6]. For a locally Noetherian, connected scheme $X$ (resp., a locally Noetherian, connected, fs log scheme $X^{\log}$) equipped with a geometric point $\overline{x} \to X$ (resp., log geometric point $\hat{x}^{\log} \to X^{\log}$), we shall denote by $\pi_1(X, \overline{x})$ (resp., $\pi_1(X^{\log}, \hat{x}^{\log})$) the \'{e}tale fundamental group of $X$ (resp., logarithmic fundamental group of $X^{\log}$). Since one knows that the \'{e}tale and logarithmic fundamental groups are determined...
up to inner automorphisms independently of the choice of basepoint, we shall omit the basepoint, and write \( \pi_1(X) \) (resp., \( \pi_1(X^{\log}) \)).

For a scheme \( X \) (resp., fs log scheme \( X^{\log} \)) which is geometrically connected and of finite type over a field \( K \) in which a prime number \( l \) is invertible, we shall refer to the quotient \( \Pi_X \) of \( \pi_1(X) \) (resp., \( \pi_1(X^{\log}) \)) by the closed normal subgroup obtained as the kernel of the natural projection from \( \pi_1(X \otimes_K \overline{K}) \) (resp., \( \pi_1(X^{\log} \otimes_K \overline{K}) \)) (where \( \overline{K} \) is a separable closure of \( K \)) to its maximal pro-\( l \) quotient \( \Delta_X \) (resp., \( \Delta_{X^{\log}} \)) as the geometrically pro-\( l \) étale fundamental group of \( X \) (resp., geometrically pro-\( l \) logarithmic fundamental group of \( X^{\log} \)). Thus, (if we write \( G_K \) for the Galois group of a separable closure of \( K \) over \( K \), then) we have a natural exact sequence

\[
1 \rightarrow \Delta_X \rightarrow \Pi_X \rightarrow G_K \rightarrow 1
\]

(resp., \( 1 \rightarrow \Delta_{X^{\log}} \rightarrow \Pi_{X^{\log}} \rightarrow G_K \rightarrow 1 \)).

Note that if the log structure of \( X^{\log} \) is trivial, then we have natural isomorphisms \( \Delta_X \cong \Delta_{X^{\log}}, \Pi_X \cong \Pi_{X^{\log}} \).

If \( K \) is finite, then write \( G^\dagger_K \subseteq G_K \) for the maximal pro-\( l \) subgroup of \( G_K \) (so \( G^\dagger_K \cong \mathbb{Z}_l \)). Also, we shall use the notation

\[
\Pi^\dagger := \Pi \times_{G_K} G^\dagger_K \subseteq \Pi
\]

— where \( \Pi \) denotes either the geometrically pro-\( l \) étale or logarithmic fundamental group of \( X \) — as the restricted pro-\( l \) étale or logarithmic fundamental group of \( X \).

### 1. Fundamental groups of (log) configuration spaces

The purpose of this section is to recall the notion of the (log) configuration space associated to a curve and review group-theoretic properties of the various fundamental groups associated to such spaces.

Let \( l \) be a prime number, \( K \) a field in which \( l \) is invertible, \( \overline{K} \) a separable closure of \( K \) — where we shall denote by \( G_K \) the Galois group of \( \overline{K} \) over \( K \) — and \( X \) a hyperbolic curve over \( \overline{K} \) of type \((g,r)\).

**Definition 1.1.**

(i) For \( n \in \mathbb{Z}_{\geq 1} \), Write \( X^{\times n} \) for the fiber product of \( n \) copies of \( X \) over \( K \). We shall denote by

\[
X_n(\subseteq X^{\times n})
\]
the \(n\)-th configuration space associated to \(X\), i.e., the scheme which represents the open subfunctor

\[ S \mapsto \{ (f_1, \cdots, f_n) \in X^{\times n}(S) \mid f_i \neq f_j \text{ if } i \neq j \} \]

of the functor represented by \(X^{\times n}\).

(ii) Let us denote by \(X_n^{\log}\) the \(n\)-th log configuration space associated to \(X\) (cf. [14]), i.e.,

\[ X_n^{\log} := \text{Spec} K \times_{\mathcal{M}^{\log}_{g,[r]+n}} \mathcal{M}^{\log}_{g,[r]+n} \]

— where the (1-)morphism \(\text{Spec} K \to \mathcal{M}^{\log}_{g,[r]}\) is the classifying morphism determined by the curve \(X \to \text{Spec} K\), and the (1-)morphism \(\mathcal{M}^{\log}_{g,[r]+n} \to \mathcal{M}^{\log}_{g,[r]}\) is obtained by forgetting the ordered \(n\) marked points of the tautological family of curves over \(\mathcal{M}^{\log}_{g,[r]+n}\). In the following, for simplicity, we shall write \(X_n^{\log}\) for \(X_1^{\log}\).

**Proposition 1.2.**

(i) The 0-interior (cf. §0) of the log scheme \(X_n^{\log}\) is naturally isomorphic to the \(n\)-th configuration space \(X_n\) associated to \(X\).

(ii) The log scheme \(X_n^{\log}\) is log regular and its underlying scheme is connected and regular.

(iii) The projection \(p_k^{\log} : X_n^{\log} \to X_{n-1}^{\log}\), induced from the (1-)morphism \(\mathcal{M}^{\log}_{g,[r]+n} \to \mathcal{M}^{\log}_{g,[r]+n-1}\) obtained by forgetting the \(k\)-th \((k = 1, \cdots, n)\) ordered points of the tautological family of curves over \(\mathcal{M}^{\log}_{g,[r]+n}\), is log smooth (cf. §0) and its underlying morphism of schemes is the natural projection \(p_k : X_n \to X_{n-1}\) obtained by forgetting the \(k\)-th factor, and hence, is flat, geometrically connected, and geometrically reduced.

**Proof.** See, for example, [4], Proposition 2.2. \(\square\)

**Definition 1.3.**

We shall denote (cf. §0) by

\[ \Pi_{X_n} \text{ (resp., } \Delta_{X_n} \text{)}\]

the geometrically pro-\(l\) étale fundamental group of \(X_n\) (resp., \(X_n \otimes \overline{K}\)), and

\[ \Pi_{X_n^{\log}} \text{ (resp., } \Pi_{X_n^{\log}}^{\times n} \text{)}\]

the geometrically pro-\(l\) log fundamental group of \(X_n^{\log}\) (resp., the fiber product \(X_n^{\log} \times \cdots \times n\) copies of \(X^{\log}\) over \(K\)). Moreover, we shall denote (cf. §0) by

\[ \Pi_{X_n}^l, \Delta_{X_n}^l \text{ (resp., } \Pi_{X_n^{\log}}^l, \Pi_{X_n^{\log}}^{\times n} \text{)}\]
respectively restricted pro-$l$ fundamental groups. If we write

$$i_k : \Delta_{X_{n/n-1}}^k \hookrightarrow \Delta_{X_n}$$

for the kernel of the surjection $p_k^\Delta : \Delta_{X_n} \to \Delta_{X_{n-1}}$, where $p_k^\Delta$ denotes the morphism induced by the projection $p_k : X_n \to X_{n-1}$ obtained by forgetting the $k$-th factor, then we have exact sequences

$$1 \to \Delta_{X_n} \to \Pi_{X_n}^{(-)} \to G_K^{(-)} \to 1$$

$$1 \to \Delta_{X_{n/n-1}}^k \to \Delta_{X_n} \xrightarrow{i_k} \Delta_{X_{n-1}} \to 1$$

$$1 \to \Delta_{X_{n/n-1}}^k \to \Pi_{X_n}^{(-)} \xrightarrow{i_k} \Pi_{X_{n-1}}^{(-)} \to 1$$

— where the symbol $(\cdot)$ denotes either the presence or absence of “†”, and when there is no fear of confusion, we shall write “$i_k$”, “$p_k$” (by abuse of notation) for the morphisms induced by $i_k$, $p_k$, respectively.

Also, we have a square diagram

$$\Pi_{X_{n-1}}^{(-)} \xleftarrow{p_k} \Pi_{X_n}^{(-)} \rightarrow \Pi_X^{(-)} \times G_{K_n}^{(-)} \cdots \times G_{K_2}^{(-)} \Pi_X^{(-)}$$

$$\Pi_{X_{n-1}}^{(-)} \log \xleftarrow{p_k^{log}} \Pi_{X_n}^{(-)} \log \rightarrow \Pi_{X_{n-1}}^{(-)} \log \times X_n$$

— which can be made commutative without conjugate-indeterminacy by choosing compatible base points — arising from a natural commutative diagram

$$X_{n-1} \xleftarrow{p_k} X_n \rightarrow X_{n\times}$$

$$X_n^{\log} \xleftarrow{p_k^{log}} X_n^{\log} \rightarrow X_{n\times}^{\log}.$$

Then, it follows from Proposition 1.2 (i), (ii) together with the log purity theorem (cf. [6], [9]) that the two vertical homomorphisms are isomorphisms. In the following, we shall identify $\Pi_{X_n}^{(-)}$ with $\Pi_{X_n}^{(-)}$, $\Pi_{X_{n\times}}^{(-)}$ with $\Pi_{X_n}^{(-)} \times G_{K_n}^{(-)} \cdots \times G_{K_2}^{(-)} \Pi_{X_n}^{(-)}$ and the surjection $p_k : \Pi_{X_n} \to \Pi_{X_{n-1}}$ with the surjection $p_k : \Pi_{X_n}^{(-)} \log \to \Pi_{X_{n-1}}^{(-)} \log$ by means of these specific isomorphisms.

**Proposition 1.4.**

(i) $\Delta_{X_{n/n-1}}^k$ may be naturally identified with the maximal pro-$l$ quotient of the étale fundamental group of a geometric fiber of the projection morphism $p_k : X_n \to X_{n-1}$.

(ii) The images of the $i_k : \Delta_{X_{n/n-1}}^k \to \Delta_{X_n}$, where $k = 1, \ldots, n$, generate $\Delta_{X_n}$.
(iii) The profinite groups \( \Delta_{X_n}, \Delta_{X_{n/n-1}}^k, \Pi_{X_n}^\dagger, \Pi_{X_{X^n}}^\dagger \) are slim (i.e., every open subgroup of each profinite group is center-free).

**Proof.** Assertion (i) follows from [14], Proposition 2.2, or [18], Proposition 2.3. Assertions (ii) and (iii) follow from induction on \( n \), together with the exact sequence

\[
1 \longrightarrow \Delta_{X_{n/n-1}}^n \xrightarrow{i_n} \Delta_{X_n} \xrightarrow{p_n} \Delta_{X_{n-1}} \longrightarrow 1
\]
displayed in Definition 1.3. Indeed, with regard to (ii), \( \Delta_{X_{n/n-1}}^k \) maps to \( \Delta_{X_{n-1}}^k \) via \( p_n : \Delta_{X_n} \to \Delta_{X_{n-1}} \), and it is verified that this map \( \Delta_{X_{n/n-1}}^k \to \Delta_{X_{n-1}}^k \) is surjective by regarding it as the morphism induced by an open immersion between the hyperbolic curves that arise as geometric fibers of the projection morphisms involved. With regard to (iii), the slimness of \( \Delta_X \) is well-known (cf., e.g., [10], Lemma 1.3.10); the slimness of \( \Pi_{X}^\dagger \) follows from the fact that the character of \( G^\dagger_{K} \) arising from the determinant of \( \Delta_{X}^{ab} \) coincides with some positive power of the cyclotomic character; the other statements follow from the fact that an extension of slim profinite groups is itself slim. \( \square \)

Next, we recall from [11], § 3, the theory of the weight filtration of fundamental groups and the associated graded Lie algebra.

**Definition 1.5.**

Let \( l \) be a prime number; \( G, H, A \) topologically finitely generated pro-\( l \) groups; \( \phi : H \to A \) a (continuous) surjective homomorphism. Suppose further that \( A \) is abelian, and that \( G \) is an \( l \)-adic Lie group.

(i) We shall refer to as the **central filtration** \( \{H(n)\}_{n \geq 1} \) on \( H \) with respect to the homomorphism \( \phi \) the filtration defined as follows:

\[
H(1) := H \\
H(2) := \text{Ker}(\phi) \\
H(m) := \langle [H(m_1), H(m_2)] \mid m_1 + m_2 = m \rangle \text{ for } m \geq 3
\]

where \( \langle N_i \mid i \in I \rangle \) is the group topologically generated by the \( N_i \)’s.

In the following, for \( a, b, n \in \mathbb{Z} \) such that \( 1 \leq a \leq b, n \geq 1 \), we shall write

\[
H(a/b) := H(a)/H(b) \\
\text{Gr}(H) := \bigoplus_{m \geq 1} H(m/m + 1) \\
\text{Gr}(H)(a/b) := \bigoplus_{b > m \geq a} H(m/m + 1) \\
\text{Gr}_{\mathbb{Q}_l}(H) := \text{Gr}(H) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \\
\text{Gr}_{\mathbb{Q}_l}(a/b) := \text{Gr}(H)(a/b) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l
\]
\[ H(a/\infty) := \lim_{b>a} H(a/b) . \]

(ii) We shall denote by \( \text{Lie}(G) \) the Lie algebra over \( \mathbb{Q}_l \) determined by the \( l \)-adic Lie group \( G \). We shall say that \( G \) is nilpotent if there exists a positive integer \( m \) such that if we denote by \( \{G(n)\} \) the central filtration with respect to the natural surjection \( G \twoheadrightarrow G^{ab} \) (cf. (i)), then \( G(m) = \{1\} \).

If \( G \) is nilpotent, then \( \text{Lie}(G) \) is a nilpotent Lie algebra over \( \mathbb{Q}_l \), hence determines a connected, unipotent linear algebraic group \( \text{Lin}(G) \), which we shall refer to as the linear algebraic group associated to \( G \). In this situation, there exists a natural (continuous) homomorphism (with open image)

\[ G \longrightarrow \text{Lin}(G)(\mathbb{Q}_l) \]

(from \( G \) to the \( l \)-adic Lie group determined by the \( \mathbb{Q}_l \)-valued points of \( \text{Lin}(G) \)) which is uniquely determined (since \( \text{Lin}(G) \) is connected and unipotent) by the condition that it induce the identity morphism on the associated Lie algebras.

In the situation of (i), if \( 1 \leq a \in \mathbb{Z} \), then we shall write

\[ \text{Lie}(H(a/\infty)) := \lim_{b>a} \text{Lie}(H(a/b)) \]

\[ \text{Lin}(H(a/\infty)) := \lim_{b>a} \text{Lin}(H(a/b)) \]

— where we note that each \( H(a/b) \) is a nilpotent \( l \)-adic Lie group.

**Definition 1.6.**

For \( n \in \mathbb{Z}_{\geq 1} \), we shall denote by

\[ \{\Delta_{X_n}(m)\} \]

the central filtration of \( \Delta_{X_n} \) with respect to the natural surjection \( \Delta_{X_n} \twoheadrightarrow \Delta^{ab}_{X \times n} \) (where \( X \)) denotes the smooth compactification of \( X \) (cf. §0), and refer to it as the weight filtration on \( \Delta_{X_n} \).

**Proposition 1.7.**

If we equip \( \Delta^k_{X_{n-1}} \) with the central filtration induced from the identification given by Proposition 1.4 (i) and its weight filtration, then the sequence of morphisms of graded Lie algebras

\[ 1 \longrightarrow \text{Gr}(\Delta^k_{X_{n-1}}) \xrightarrow{\iota_k} \text{Gr}(\Delta_{X_n}) \xrightarrow{p_k} \text{Gr}(\Delta_{X_{n-1}}) \longrightarrow 1 \]

induced by the second displayed exact sequence of Definition 1.3 is exact.

**Proof.** See [4], Proposition 4.1. \( \square \)
Next, let us fix a section $\sigma : G_K \rightarrow \Pi_{X_n}$ of the surjection $\Pi_{X_n} \twoheadrightarrow G_K$ induced by the structure morphism of $X_n$. This section $\sigma$ determines natural conjugate actions of $G_K$ on $\Delta_{X_n}$, hence also on

$$\text{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b)$$

$$\text{Lie}_{X_n}(a/b) := \text{Lie}(\Delta_{X_n}(a/b))$$

$$\text{Lin}_{X_n}(a/b) := \text{Lin}(\Delta_{X_n}(a/b))(\mathbb{Q}_l)$$

for $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$.

**Proposition 1.8.**

Let us assume that $K$ is a finite field whose cardinality we denote by $q_K$, and write $Fr \in G_K$ for the Frobenius element of $G_K$. Then:

(i) The eigenvalues of the action of $Fr$ on $\text{Lie}_{X_n}(a/a + 1)$ are algebraic numbers all of whose complex absolute values are equal to $q^{a/2}_K$ (i.e., weight $a$).

(ii) There is a unique $G_K$-equivariant isomorphism of Lie algebras

$$\text{Lie}_{X_n}(a/b) \sim \text{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(a/b)$$

which induces the identity isomorphism

$$\text{Lie}_{X_n}(c/c + 1) \sim \text{Gr}_{\mathbb{Q}_l}(\Delta_{X_n})(c/c + 1)$$

for all $c \in \mathbb{Z}_{\geq 1}$ such that $a \leq c < b$.

**Proof.** Assertion (i) follows from the “Riemann hypothesis for abelian varieties over finite fields” (cf., e.g., [15], p. 206). Assertion (ii) follows formally from assertion (i) by considering the eigenspaces with respect to the action of $Fr$. □

The following proposition is a special case of a result proven previously (cf. [17]). For simplicity, we discuss only the case used in the proofs of the present paper.

**Proposition 1.9.**

For $n = 1, 2$, the graded Lie algebra $\text{Gr}(\Delta_{X_n})$ has the following presentation.

(i) The case $n = 1$ (i.e., $X_n = X$):

- generators ($1 \leq j \leq r$, $1 \leq i \leq g$)
  
  - $\zeta_j \in \Delta_X(2/3)$
  - $\alpha_i, \beta_i \in \Delta_X(1/2)$

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\[ \sum_{j=1}^{r} \zeta_j + \sum_{i=1}^{g} \alpha_i \beta_i = 0 \]

where \( \zeta_j \) (\( j = 1, 2, \ldots, r \)) topologically generates the inertia subgroup in \( \Delta_X \) (well-defined up to conjugacy) associated to the \( j \)-th cusp [relative to some ordering of the cusps of \( X \times_K \overline{K} \)].

(ii) The case \( n = 2 \):

- generators (\( 1 \leq j \leq r, \ 1 \leq i \leq g, \ k = 1, 2 \))
  \[ \zeta \in \Delta_{X_2}(2/3) \]
  \[ \zeta_j^k \in \Delta_{X_2/1}(2/3) \]
  \[ \alpha_i^k, \beta_i^k \in \Delta_{X_2/1}(1/2) \]

- relations (\( 1 \leq j, j' \leq r, \ j \neq j', \ 1 \leq i, i' \leq g, \ i \neq i', \ \{k, k'\} = \{1, 2\} \))
  \[ \zeta + \sum_{j=1}^{r} \zeta_j^k + \sum_{i=1}^{g} \alpha_i^k \beta_i^k = 0 \]
  \[ [\alpha_i^k, \zeta_j^k'] = [\beta_i^k, \zeta_j^k] = [\zeta_j^k, \zeta_j^k'] = 0 \]
  \[ [\alpha_i^k, \alpha_i'^k] = [\beta_i^k, \beta_i'^k] = [\zeta_j^k, \beta_i^k] = 0 \]
  \[ [\alpha_i^k, \alpha_i'^k] = [\beta_i^k, \beta_i'^k] = [\zeta_j^k, \zeta_j^k'] = 0 \]
  \[ [\alpha_i^k, \beta_i'^k] = [\beta_i^k, \alpha_i'^k] = [\zeta_j^k, \beta_i^k] = 0 \]
  \[ [\alpha_i^k, \beta_i'^k] = [\beta_i^k, \alpha_i'^k] = [\zeta_j^k, \beta_i^k] = 0 \]

where \( \zeta \) topologically generates the image in \( \Delta_{X_2}(2/3) \) of the inertia subgroup in \( \Delta_{X_2} \) (well-defined up to conjugacy) associated to the diagonal divisor of \( X \times_K X \), and \( \zeta_j^k \) generates the image in \( \Delta_{X_2/1}(2/3) \) of the inertia subgroup in \( \Delta_{X_2/1} \) associated to the \( j \)-th cusp [relative to some ordering of the cusps of \( X \times_K \overline{K} \)] of the \( k \)-th factor of \( X_2 \).

2. Switching morphism on configuration spaces

We continue to use the notation of Section 1. In this section, we consider various automorphisms induced by the automorphism of \( \overline{X}_2^{log} \) determined by switching the two factors of \( X \). The group-theoretic uniqueness of such induced switching morphisms between fundamental groups (Proposition 2.5) plays a key role in the proof of Theorem A.
We denote by
\[ D_{\log} \]
the log scheme obtained by equipping the diagonal divisor \( \overline{X} \subseteq \overline{X}_2 \) (which is the restriction of the (1-)morphism \( \overline{M}_{g,\lfloor r \rfloor + 1} \to \overline{M}_{g,\lfloor r \rfloor + 2} \) obtained by gluing the tautological family of curves over \( \overline{M}_{g,\lfloor r \rfloor + 1}^{\log} \) to a trivial family of tripods along the final ordered marked section) with the log structure pulled back from \( \overline{X}_2^{\log} \).

Thus, if we write \( d : D_{\log} \to \overline{X}_2^{\log} \) for the natural diagonal embedding, then it follows immediately from the definitions that \( p_1 \circ d = p_2 \circ d : D_{\log} \to \overline{X}^{\log} \) is a morphism of type \( \mathbb{N} \) (cf. [2]), i.e., the underlying morphism of schemes is an isomorphism, and the relative characteristic sheaf (cf. § 0) is locally constant with stalk isomorphic to \( \mathbb{N} \).

Observe that the (1-)automorphism on \( \overline{M}_{g,\lfloor r \rfloor + 2}^{\log} \) over \( \overline{M}_{g,\lfloor r \rfloor}^{\log} \) given by switching the two ordered marked points of the tautological family of curves over \( \overline{M}_{g,\lfloor r \rfloor + 2}^{\log} \) induces automorphisms \( s \), \( s \), and \( s_D \), which fit into a commutative diagram as follows:

\[
\begin{array}{ccc}
D_{\log} & \xrightarrow{d} & \overline{X}_2^{\log} \\
\downarrow{s} & & \downarrow{s} \\
D_{\log} & \xrightarrow{d} & \overline{X}_2^{\log} \\
& & \downarrow{s} \\
& & \overline{X}_2^{\log} \times_K \overline{X}^{\log}
\end{array}
\]

\[ (*)^X \]

Lemma 2.1.

In the notation of the above situation,

(i) \( \overline{s} \) is the morphism determined by switching the two factors.

(ii) \( \overline{s} \) is the identity morphism on the underlying scheme; on the sheaf of monoids defining the log structure of \( D_{\log} \), for any \( \text{étale local section } \theta \) of \( \overline{M}_{\overline{D}} \) such that \( "\theta = 0" \) defines the diagonal divisor \( \overline{X} \subseteq \overline{X}_2 \),

\[ \overline{s}(\theta) = -\theta. \]

Proof. Recall that \( \overline{X}_2 \) is obtained by blowing-up \( \overline{X} \times_K \overline{X} \) along the intersection of the diagonal divisor and the pull-backs of the cusps via \( p_1, p_2 : \overline{X}_2 \to \overline{X} \). Thus, one verifies easily that assertions (i) and (ii) follow immediately from the fact that the ring homomorphism corresponding to \( \overline{s} \) in an affine neighborhood of any diagonal point may be expressed as

\[
A \otimes_K A \longrightarrow A \otimes_K A
\]

\[
\sum_j a_j \otimes a_j' \mapsto \sum_j a_j' \otimes a_j,
\]

hence maps \( \theta \) to \( -\theta \) for any local section \( \theta \) such that \( "\theta = 0" \) defines the diagonal divisor \( \overline{X} \subseteq \overline{X} \times_K \overline{X} \).
Remark 2.1.1.

Lemma 2.1 (ii) can be interpreted as the assertion that the automorphism induced by \( s \) on the sheaf of monoids \( M_\log \) defining the log structure of \( D_\log \) may be expressed, relative to the étale local splitting of \( M_\log \to M_\log / O_X^* \cong \mathbb{N} \) corresponding to \( \theta \), as

\[
\mathbb{N} \oplus O_X^* \xrightarrow{\sim} \mathbb{N} \oplus O_X^* \\
(m, v) \mapsto (m, (-1)^m v).
\]

The above diagram \((\ast)^X\) induces a diagram of profinite groups as follows:

\[
\Pi_{D_\log} \xrightarrow{[d_1]} \Pi_{X^2} \xrightarrow{[p_1]} \Pi_X \times_{G_K} \Pi_X \\
\Pi_{D_\log} \xrightarrow{[d_2]} \Pi_{X^2} \xrightarrow{[p_2]} \Pi_X \times_{G_K} \Pi_X.
\]

Note that the arrows in the diagram \((\ast)^\Pi\) are only defined (i.e., in the absence of appropriate choices of basepoints of respective log schemes) up to conjugacy.

Next, we observe that since the subgroups of the conjugacy class of subgroups determined by the image of \([d_\Pi]\) may be naturally regarded as decomposition groups associated to the diagonal divisor of \( X^2 \), any choice of a specific homomorphism \( d_\Pi : \Pi_{D_\log} \to \Pi_{X_\log^2} \) (i.e., among its various conjugates) determines a specific decomposition group

\[D_X \subseteq \Pi_{X_\log^2}\]

— where we write \( d_\Pi : D_X \hookrightarrow \Pi_{X_\log^2} \) for the natural inclusion — associated to the diagonal divisor (i.e., among its various \( \Pi_{X_\log^2}\)-conjugates), as well as a specific inertia subgroup

\[I_X \subseteq \Pi_{X_\log^2}\]

associated the diagonal divisor (i.e., among its various \( \Pi_{X_\log^2}\)-conjugates). Here, we recall that \( I_X \) is canonically isomorphic to \( \mathbb{Z}_l(1) \).

Definition 2.2.

Let \( x_\log \to X_\log \) be a strict morphism (cf. [6], 1.2) such that the underlying scheme of \( x_\log \) is equal to \( \text{Spec}(K) \). We shall write

\[
X_\log^x := X_\log^2 \times X_\log x_\log, \\
\tilde{X}_\log := \Pi_{\log} \times X_\log x_\log, \\
G_{K, \log} := \Pi_{x_\log}^{(-)}
\]

— where the morphism \( X_\log^x \to X_\log \) (resp., \( \tilde{X}_\log \to X_\log \)) in the fiber product defining \( X_\log^x \) (resp., \( \tilde{X}_\log \)) is \( p_1 \) (resp., \( p_1 \circ d = p_2 \circ d \)), and the symbol \( "((-)" \) denotes either the presence or absence of \( \dagger \) — and refer to \( X_\log^x \) (resp., \( \tilde{X}_\log \)).
as the cuspidalization of $X$ at $x$ (resp., diagonal cusp of $X^\log_x$). We note that both the log structure of $x^\log$ and the underlying scheme of $X^\log_x$ depend on the choice of $x \in \overline{X}$:

1. **The Case $x \in X$:**
   In this case, $x = x^\log$, i.e., the log structure of $x^\log$ is trivial. As we discussed in Section 1, the underlying scheme of $X^\log_x$ is naturally isomorphic to $\overline{X}$; this isomorphism maps $\tilde{x}$ to $x$ and the interior of $X^\log_x$ onto $X \setminus \{x\}$.

2. **The Case $x \in \overline{X} \setminus X$:**
   In this case, the log structure of $x^\log$ has a chart modeled on $\mathbb{N}$, which determines a local uniformizer of $X$ at $x$. The scheme $\overline{X}_x$ consists of precisely two irreducible components, one of which maps to the point $x \in X$ (resp., maps isomorphically to $\overline{X}$) via $\overline{X}^\log_x \xrightarrow{p_{\nu_x}} \overline{X}^\log$; denote this irreducible component by $\mathbb{P}_K$ (resp., $\overline{X}$, via a slight abuse of notation). Thus, $\overline{X}$, $\mathbb{P}_K$ are joined at a single *node* $\nu_x$. Let us refer to $\overline{X}$ (resp., $\mathbb{P}_K$, $\nu_x$) as the major cuspidal component (resp., the minor cuspidal component, the nexus) at $x$, and denote by $\overline{X}^\log_x$, $\mathbb{P}_K^\log$, $\nu_x^\log$ the log schemes obtained by equipping $\overline{X}$, $\mathbb{P}_K$, $\nu_x$ with the respective log structures pulled back from $X^\log$ (cf. [13], Definition 1.4). Note that the 1-interior of $\overline{X}^\log'$ (resp., $\mathbb{P}_K^\log$) is isomorphic to $X$ (resp., is a tripod).

\[
\begin{array}{c}
\text{X} \\
\text{Cuspidalization} \\
at x \in X(K) \\
\text{Case (1)} \\
\overline{X}^\log_x \\
x \\
\text{Case (2)} \\
\overline{X}^\log' \\
\mathbb{P}_K^\log \\
\nu_x^\log \\
\text{cusps}
\end{array}
\]

(the two thick arrows in the picture do not represent morphisms of log schemes)
In the following, we fix choices of specific [i.e., in the sense that they are not subject to conjugacy indeterminacy!] homomorphisms

\[ i_1^\Pi : \Pi_{X_2}^{\log} \to \Pi_{X_2}, \quad p_1^\Pi : \Pi_{X_2} \to \Pi_X \]

induced by the morphisms of log schemes \( i_1 : X_2^{\log} \to X_2^{\log} \), \( p_1 : X_2 \to X^{\log} \) and a choice of a specific decomposition group

\[ D_2 \subseteq \Pi_{X_2}^{\log} \]

associated to \( x^{\log} \) of \( X_2^{\log} \) among the various conjugates of this subgroup. Write

\[ \gamma : D_\tilde{x} \hookrightarrow \Pi_{X_2}^{\log} \]

for the natural inclusion. These choices determine a homomorphism of profinite groups

\[ f_X : \Pi_{X_2}^{\log} \to G_{K,\log}, \]

arising from the structure morphism \( X_2^{\log} \to x^{\log} \), by taking \( f_X := i_1^\Pi \circ p_1^\Pi : \Pi_{X_2}^{\log} \to G_{K,\log} \left( := \text{Im}(\Pi_{X_2}^{\log} \to \Pi_X) \right) \), as well as a profinite group

\[ I_\tilde{x} := \text{Ker}(D_\tilde{x} \xrightarrow{\gamma} \Pi_{X_2}^{\log} \xrightarrow{f} G_{K,\log}) \]

— where we note that \( I_\tilde{x} \) is naturally isomorphic to \( \mathbb{Z}_l(1) \) (i.e., a Tate twist of \( \mathbb{Z}_l \)).

**Lemma 2.3.**

(i) The subgroup \( D_\tilde{x} \xrightarrow{\gamma} \Pi_{X_2}^{\log} \) is the normalizer of \( I_\tilde{x} \) in \( \Pi_{X_2}^{\log} \).

(ii) For any choice of a specific decomposition group \( d^\Pi : D_X \hookrightarrow \Pi_{X_2} \) (i.e., among its various conjugates) associated to the diagonal divisor of \( X_2 \), the subgroup \( D_X \) of \( \Pi_{X_2} \) coincides with the normalizer of \( I_X \) in \( \Pi_{X_2} \).

**Proof.** Note that we have commutative diagrams

\[
\begin{array}{ccc}
I_\tilde{x} & \longrightarrow & D_\tilde{x} \\
\downarrow i_1^\Pi \circ \gamma & & \downarrow i_1^\Pi \circ \gamma \\
\Delta_{X_2/1}^{1} & \longrightarrow & \Pi_{X_2} \\
1 & \longrightarrow & I_X \\
\downarrow d^\Pi \circ i_X & & \downarrow d^\Pi \\
\Delta_{X_2/1}^{1} & \longrightarrow & \Pi_{X_2} \\
1 & \longrightarrow & 1
\end{array}
\]

— where the two horizontal sequences in the second diagram are exact, and both the first displayed diagram and the left-hand square in the second diagram are cartesian. Next, let us recall the well-known fact (cf., e.g., [16], (2.3.1)) that \( I_\tilde{x} \) and \( I_X \) coincide with their respective normalizers in \( \Delta_{X_2/1}^{1} \). Thus, assertion
(ii) follows immediately from the surjectivity of $p_1^\Pi \circ d^\Pi$. On the other hand, assertion (i) follows immediately from the observation that the images of $\tilde{D}_x$ and $\Pi_{X_2}^\log$ via $p_1^\Pi \circ i_1^\Pi$ coincide. This observation is a consequence of the geometry of the corresponding morphisms of log schemes, which implies that these images coincide with a decomposition group $\subseteq \Pi_X$ associated to the point $x$. □

**Lemma 2.4.**

Under the determination discussed preceding Lemma 2.3, we can uniquely take a pair of specific homomorphisms

$$d^\Pi : D_X \rightarrow \Pi_{X_2}, \quad p_2^\Pi : \Pi_{X_2} \rightarrow \Pi_X$$

among the various conjugates of these subgroups, obtained by morphisms between log schemes $d : D_{\log} \rightarrow X_{\log}$, $p_2 : X_{\log} \rightarrow X$, satisfying the following conditions:

1. The image of the inertia subgroup $I_X$ (via $d^\Pi$) coincides with the image of $I_{\tilde{x}}$ via $i_1^\Pi$.
2. $D_X$ maps (via $(p_1^\Pi, p_2^\Pi) \circ d^\Pi : D_X \rightarrow \Pi_X \times_{G_K} \Pi_X$) onto the image of the diagonal embedding $\Pi_X \hookrightarrow \Pi_X \times_{G_K} \Pi_X$.

**Proof.** Since $D_X$ is the normalizer of $I_X$ in $\Pi_{X_2}^\log$ by Lemma 2.3, it is enough to take $D_X$ as the normalizer of $I_{\tilde{x}}$ in $\Pi_{X_2}^\log$, and take $p_2^\Pi$ so as to $p_1^\Pi \circ d^\Pi = p_2^\Pi \circ d^\Pi$. Uniqueness follows from the required two conditions. □

Now, before we continue the discussion, we shall give comments for Proposition 2.5.

1. Recall that the natural surjection $D_{\tilde{x}} \rightarrow G_K$ (since $G_K$ is abelian, this map is uniquely determined without the discussion of base points) has a section. Indeed, fixing a choice of such a section is equivalent to extracting roots of any local uniformizers of the divisors $X_x \subseteq X_{\log}$ and $D \subseteq X_{\log}$ at $\tilde{x}$.

2. We shall consider the restriction map $H^1(G_K, Z_l(1)) \rightarrow H^1(G_K^\dagger, Z_l(1))$ of cohomology groups induced by the natural inclusion $G_K^\dagger \hookrightarrow G_K$. Since $G_K^\dagger$ is the maximal pro-$l$ subgroup of $G_K$ and $I_{\tilde{x}}$ is isomorphic to $Z_l(1)$ as $G_K$-module, $H^1(G_K, Z_l(1))$ is isomorphic to $H^1(G_K^\dagger, Z_l(1))$ and is isomorphic to the maximal pro-$l$ completion $(K^\times)^\wedge$ of the multiplicative group $K^\times$ of $K$. Therefore, if we denote by $Z^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) the set of (continuous) 1-cocycle maps of $G_K^\dagger$ (resp., $G_K$) with coefficients in $I_{\tilde{x}}$, then we can refer to any element of $Z^1(G_K^\dagger, I_{\tilde{x}})$ (resp., $Z^1(G_K, I_{\tilde{x}})$) belonging to the inverse image of $a \in (K^\times)^\wedge$.
Proposition 2.5.

Following the discussions and results in this section until now, we shall fix a choice of a section

$$
\sigma : G_K \rightarrow D_{\tilde{z}}
$$

(hence also a choice of an induced morphism $\sigma^1 : G^1_K \rightarrow D^1_{\tilde{z}}$) of the surjection $D_{\tilde{z}} \rightarrow G_K$.

Then, for any 1-cocycle map

$$
\delta : G^1_K \rightarrow I_{\tilde{z}}
$$

representing the Kummer class $-1 \in (K^\times)^\wedge$, there exists a unique triple $(\tilde{s}_\delta, s_\delta^1, \tilde{s}_\delta^1)$ of vertical automorphisms in the following diagram

$$
\begin{array}{ccc}
D^1_X & \xrightarrow{d^1} & \Pi^1_{X_2} \quad \xrightarrow{p^1} \quad \Pi^1_X \times_{G^1_K} \Pi^1_X \\
\downarrow s_\delta^1 & & \downarrow \tilde{s}_\delta^1 \\
D^1_X & \xrightarrow{\delta^1} & \Pi^1_{X_2} \quad \xrightarrow{\mu^1} \quad \Pi^1_X \times_{G^1_K} \Pi^1_X \\
\end{array}
$$

— where $p^1 : \Pi^1_{X_2} \rightarrow \Pi^1_X \times_{G^1_K} \Pi^1_X$ (resp., $d^1 : D^1_X \rightarrow \Pi^1_{X_2}$) denotes the morphism induced by $p^1, p^1_{II}$ (resp., $d^1_{II}$) determined in the preceding discussions — which makes $(\ast)^1$ commute and satisfies the following two conditions:

1. $\tilde{s}_\delta^1 : \Pi^1_X \times_{G^1_K} \Pi^1_X \xrightarrow{s_\delta^1 \circ \sigma^1 \circ (g \cdot (s_\delta^1 \circ \sigma^1 \circ g)^{-1})}$ is the morphism obtained by switching the two factors.

2. The continuous function from $G^1_K$ to $\Pi^1_X$ defined by

$$
g \mapsto (s_\delta^1 \circ \sigma^1 \circ (g \cdot (s_\delta^1 \circ \sigma^1 \circ g)^{-1})
$$

is valued in $I_{\tilde{z}} \subseteq \Pi^1_{X_2}$ and coincides with $\delta$.

Proof. We begin by proving the existence portion. By the surjectivity of $p^1$, we can take $s_\delta^1, \tilde{s}_\delta^1$ such that the right-hand square of the diagram $(\ast)^1$ commutes and the condition (1) is satisfied. If we take arbitrary $s_\delta^1 \in Aut(D^1_X)$ from the conjugacy class, commutativity of the rectangle in $(\ast)^1$ up to conjugacy implies that there exists $\lambda \in \Pi^1_X \times_{G^1_K} \Pi^1_X$ such that $\tilde{s}_\delta^1 \circ (p^1 \circ d^1) = \text{Inn}(\lambda) \circ (p^1 \circ d^1) \circ s_\delta^1$ (where $\text{Inn}(\lambda)$ denotes the inner automorphism obtained by conjugating by $\lambda$).

As obtained in Lemma 2.4, $p^1 \circ d^1$ maps $D^1_X$ onto the diagonal subgroup of $\Pi^1_X \times_{G^1_K} \Pi^1_X$, hence $\text{Inn}(\lambda)$ preserves the diagonal subgroup. Since $\Pi^1_X$ is center-free (by Lemma 1.4 (iii)), it is verified that $\lambda$ lies in the diagonal. By taking a lifting $\tilde{\lambda} \in D^1_X$ of $\lambda$ and replacing $\tilde{s}_\delta^1$ by $\text{Inn}(\tilde{\lambda}^{-1}) \circ s_\delta^1$, we can make the rectangle in $(\ast)^1$ commute in the strict sense. Next, we observe that $s_\delta^1 \circ d^1 = \text{Inn}(\mu) \circ d^1 \circ s_\delta^1$.
for some $\mu \in \Pi^1_{X_2}$. By the commutativity of the rectangle in $(\star)^\dagger$, $\mu$ projects via $p^\dagger$ into the center of $\Pi^1_k \times_{G^1_K} \Pi^1_k$ (hence, to the unit element). Therefore, replacing $s^\dagger_\delta$ by $\text{Inn}(\mu^{-1}) \circ s^\dagger_\delta$, we can make $(\star)^\dagger$ commute.

Thus, we obtain an automorphism of an exact sequence

$$1 \longrightarrow I_X \longrightarrow D^1_X \xrightarrow{p^1_{\text{odd}}-p^1_{\text{odd}}} \Pi^1_X \longrightarrow 1$$

which consists of the identity morphisms of $\Pi^1_X$ and $I_X$. Let $\mathbb{M} \subseteq \mathbb{Q}$ be the monoid of positive rational numbers with denominators of $l$-power, and $\mathcal{N}$ the global sections of the sheaf of monoids defining the log structure on a universal geometrically pro-$l$ két covering of $\mathbb{D}^{\text{log}} \times_{\mathbb{X}^{\text{log}}_l} \mathbb{Z}^{\text{log}}_l$. Then, $\mathcal{N}$ forms a direct product splitting $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus \widehat{\mathbb{K}}^\times$, where (cf. the discussion (1) preceding this proposition) the first (resp., second) factor of direct product is due to extracting roots of a local uniformizer of the divisor $\mathbb{X}_x \subseteq \mathbb{X}_2$ (resp., $\mathbb{D} \subseteq \mathbb{X}_2$) at $\tilde{x}$ inside of a choice of $\sigma$. Then the automorphism $s^1_\delta$ of $D^1_X$ may be characterized by the choice of a projective system $\{(-1)^{\overline{n}}\}_{n \in \mathbb{Z}_{>0}}$ of $l$-power roots of $-1$ (well-defined up to multiplications by projective systems of $l$-power roots of 1) in a way that the automorphism of $\mathcal{N} \cong \mathbb{M} \oplus \mathbb{M} \oplus \widehat{\mathbb{K}}^\times$ is expressed as

$$(\frac{a_1}{m_1}, \frac{a_2}{m_2}, k) \mapsto \left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, (-1)^{\overline{n}} \cdot k\right)$$

(cf. Lemma 2.1 (ii)). We observe that for any element $g$ of $G^1_K$, the automorphism of $\mathcal{N}$ corresponding to $\sigma^\dagger(g)$ (resp., $(s^\dagger \circ \sigma^\dagger_X)(g)$) in $\Pi^1_X$ is expressed as

$$(\frac{a_1}{m_1}, \frac{a_2}{m_2}, k) \mapsto \left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, g(k)\right)$$

(resp., $\left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, g((-1)^{\overline{n}} \cdot k))

Thus, $(s^\dagger \circ \sigma^\dagger_X)(g) \cdot \sigma^\dagger_X(g)^{-1}$ is valued in $I_X$ ($\cong I_K$ by Lemma 2.3) and the 1-cocycle $g \mapsto (s^\dagger \circ \sigma^\dagger_X)(g) \cdot \sigma^\dagger_X(g)^{-1}$ represents, by definition, the Kummer class $-1 \in (\mathbb{K}^\times)^{\wedge} \cong H^1(G_K^{\text{et}}, \mathbb{Z}_l(1))$. Therefore, after modifications of $s^\dagger_\delta$, $s^\dagger_\delta$ by some $I_X$-inner automorphisms, condition (2) is satisfied. This completes the existence assertion.

Next we prove the uniqueness portion. If we take two triples $(s^\dagger_1, s^\dagger_1, s^\dagger_1)$, $(s^\dagger_2, s^\dagger_2, s^\dagger_2)$ so as to satisfy conditions (1) and (2), then $s^\dagger_1 \circ (s^\dagger_2)^{-1} = \text{Inn}(\eta)\in \text{Aut}(\Pi^1_{X_2})$ for some $\eta \in \Pi^1_{X_2}$ and we see that $\text{Inn}(\eta)$ preserves the subgroup $D^1_X \subseteq \Pi^1_{X_2}$. Since $D^1_X$ is normally terminal in $\Pi^1_{X_2}$, it is verified that $\eta$ is in $\Pi^1_{D^1_X}$. Moreover, from condition (1) and the fact that $\Pi^1_X$ is center-free, $\eta$ lies in $\text{Ker}(D^1_X \to \Pi^1 \times_{G^1_K} \Pi^1)$, i.e., $\eta \in I_X$. On the other hand, as the section
σ acts on $I_X$ via the cyclotomic character, which is a faithful action, condition (2) implies that $η$ is the unit element, i.e., that $s_1^1 = s_2^1$, hence $(s_1^1, s_1^1, s_1^1) = (s_2^1, s_2^1, s_2^1)$.

\[ \square \]

**Remark 2.5.1.**

In the case $l \neq 2$, $-1$ coincides with the unit element 1 in $(K^\times)^\wedge$. Then, in the statement of Proposition 2.5, by taking a 1-cocycle map $δ$ to be trivial, we may obtain $s_δ^1$ satisfying $s_δ^1 \circ σ^1 = σ^1$.

### 3. The proof of Theorem A

We begin with a review of the notation and the setup. Let $l$ be a prime number, $K$ a finite field in which $l$ is invertible, and $\overline{K}$ a separable closure of $K$. We shall denote by $G_K$ the Galois group of $\overline{K}$ over $K$. Next, let $X$ be a hyperbolic curve over $K$ of type $(g, r)$, $x$ a strict $K$-rational log point of $X$ log := $X_2$, and write $X_2 := X_2 \times x$ log := $\mathbb{D}_{log} \times X_2$ log, $G_K log := \Pi_x log$. In addition, we assume that we have fixed choices of specific homomorphisms $\iota_1^{\Pi} : \Pi_{X log} x \rightarrow \Pi_X$, $p_1^{\Pi} : \Pi_X \rightarrow \Pi$ (they determine a structure morphism $f_X : \Pi_{X log} \rightarrow G_K log$ by taking $f_X = \iota_1^{\Pi} \circ p_1^{\Pi} : \Pi_{X log} \rightarrow G_K log$ := $\text{Im}(\Pi_{X log} i_1^{\Pi} \circ \gamma_x \rightarrow \Pi_X)$), within in the respective conjugacy classes determined by these homomorphisms, a choice of specific decomposition group $D_\hat{x} \subseteq \Pi_{X log}$ associated to $\hat{x}$, where we write $γ_\hat{x} : D_\hat{x} \rightarrow \Pi_{X log}$ for the natural inclusion, and a choice of a section

$σ_X : G_K \rightarrow D_\hat{x}$

of the composite surjection $D_\hat{x} \rightarrow \Pi_{X log} \rightarrow G_K log \rightarrow G_K$.

**Definition 3.1.**

- The section $σ_X$ determines, by composing with the morphisms $D_\hat{x} \rightarrow \Pi_{X log}$ (resp., $D_\hat{x} \rightarrow \Pi_X$, $D_\hat{x} \gamma_x \rightarrow \Pi_X \times^2$), a natural conjugate $G_K$ action on $\Delta_{X/2}^1 \cong \text{Ker}(\Pi_{X log} f_X \rightarrow G_K log)$ (resp., on $\Delta_X$, on $\Delta_X \times^2$), hence also on

$\text{Gr}_{\Delta_{X/2}^1} := \text{Gr}_{\Delta_1} \Delta_{X/2}^1$,

(resp., $\text{Gr}_X := \text{Gr}_{\Delta_X}$, $\text{Gr}_X \times^2 := \text{Gr}_{\Delta_X \times^2}$),
\[
\begin{align*}
\text{Lie}^1_{X_{2/1}} &= \text{Lie}(\Delta^1_{X_{2/1}}(1/\infty)), \\
\text{Lin}^1_{X_{2/1}} &= \text{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(Q_l), \\
\text{Lie}^1_{X_2} &= \text{Lie}(\Delta^1_{X_2}(1/\infty)), \\
\text{Lin}^1_{X_2} &= \text{Lin}(\Delta^1_{X_2}(1/\infty))(Q_l), \\
\text{Lie}^1_{X_{2/2}} &= \text{Lie}(\Delta^1_{X_{2/2}}(1/\infty)), \\
\text{Lin}^1_{X_{2/2}} &= \text{Lin}(\Delta^1_{X_{2/2}}(1/\infty))(Q_l).
\end{align*}
\]

In the following, we regard these objects as to be equipped with the \(G_K\)-actions in this sense. From the discussion in Definition 1.5 (ii), we have the following commutative diagram consisting of \(G_K\)-equivariant morphisms

\[
\begin{array}{ccc}
\Delta^1_{X_{2/1}} & \xrightarrow{i_1} & \Delta_{X_2} \\
\downarrow & & \downarrow \\
\text{Lin}^1_{X_{2/1}} & \xrightarrow{i_{\text{Lin}}} & \text{Lin}_{X_2} \\
\end{array}
\]

and topological groups equipped with the additional \(G_K\)-action structures

\[
\Delta^\text{Lie}_{X_2} := \Delta_{X_{2/2}} \times \text{Lin}_{X_{2/2}} \text{Lin}_{X_2}, \quad \Pi^\text{Lie}_{X_2} := \Delta^\text{Lie}_{X_2} \rtimes G_K
\]

as well as \(G_K\)-equivariant homomorphisms of topological groups

\[
\text{Int}^\Delta_{X_2} : \Delta_{X_2} \to \Delta^\text{Lie}_{X_2}, \quad \text{Int}^\Pi_{X_2} : \Pi_{X_2} \to \Pi^\text{Lie}_{X_2}.
\]

(ii) We shall fix a 1-cocycle map \(\delta_X : G_K \to I_{\tilde{\mathbb{Z}}} := \text{Ker}(D_{\tilde{\mathbb{Z}}} \to G_{K\text{-log}})\) representing the Kummer class \(-1\) (cf. the discussion preceding Proposition 2.5). Then, taking a product

\[
\sigma_{\delta_X} : G_K \to D_{\tilde{\mathbb{Z}}}; g \mapsto \delta_X(g) \cdot \sigma_X(g)
\]

— which is a homomorphism of topological groups — gives a new section of the surjective homomorphism \(D_{\tilde{\mathbb{Z}}} \to G_K\). In a similar way to (i), the section \(\sigma_{\delta_X}\) determines a natural conjugate \(G_K\)-action on

\[
\begin{align*}
\tilde{\text{Gr}}_{X_{2/1}} &= \text{Gr}_{Q_l}(\Delta^1_{X_{2/1}}), \\
\tilde{\text{Lie}}_{X_{2/1}} &= \text{Lie}(\Delta^1_{X_{2/1}}(1/\infty)), \\
\tilde{\text{Lin}}_{X_{2/1}} &= \text{Lin}(\Delta^1_{X_{2/1}}(1/\infty))(Q_l). \\
\end{align*}
\]

— where, in the following, we regard these objects as to be equipped with
the $G_K$-actions in this sense — as well as topological groups equipped with the additional $G_K$-action structures

$$\tilde{\Delta}_{X_2}^{\text{Lie}} := \Delta_{X_2} \times \text{Lin}_{X_2} \text{Lin}_{X_2}, \quad \tilde{\Pi}_{X_2}^{\text{Lie}} := \tilde{\Delta}_{X_2}^{\text{Lie}} \times G_K.$$ 

Next, by taking account of $\delta_X$ and the setup of $i_1^\Pi, p_1^\Pi, \gamma_x, \sigma_X$ fixed above, we may take an automorphism $s_{\Pi}^\delta_{X} : \Pi_{X_2} \rightarrow \Pi_{X_2}$ arising from the switching morphism $s_X : \log X_2 \rightarrow \log X_2$ inducing a uniquely determined morphism $s_X^{\dagger} : \Pi_{X_2} \rightarrow \Pi_{X_2}$ obtained in Section 2, and obtain $G_K$-equivariant homomorphisms of topological groups

$$s_{\delta_X}^{\Delta_{X_2}^{\text{Lie}}} : \Delta_{X_2}^{\text{Lie}} \xrightarrow{\sim} \tilde{\Delta}_{X_2}^{\text{Lie}}, \quad s_{\delta_X}^{\Pi_{X_2}^{\text{Lie}}} : \Pi_{X_2}^{\text{Lie}} \xrightarrow{\sim} \tilde{\Pi}_{X_2}^{\text{Lie}}$$

induced by $s_{\Pi}^\delta_{X}$.}

**Lemma 3.2.**

The $G_K$-action induced by $\sigma_{\delta_X}$ defined in Definition 3.1 (ii) on $\Delta_{X_2}$ (hence also on $\text{Gr}_{X_2}, \text{Lie}_{X_2}, \text{Lin}_{X_2}$ and $\Delta_{X_2}^{\text{Lie}}$) coincides with the action defined in a way that

$$G_K \longrightarrow \text{Aut}(\Delta_{X_2}); g \mapsto \text{Inn}(s_{\delta_X}^{\Pi} \circ i_1^\Pi \circ \gamma_x \circ \sigma_X(g)).$$

**Proof.** It follows immediately from the definition of $G_K$-action induced by $\sigma_{\delta_X}$. \qed

**Lemma 3.3.**

$\text{Int}_{\Delta_X}$ and $\text{Int}_{\Pi_X}^{\text{Lie}}$ are injective.

**Proof.** It is enough to verify that $\Delta_{X_2} \rightarrow \text{Lin}_{X_2}$ is injective. But it follows from the discussion in Definition 1.5 (ii) and the fact that $\bigcap_{m \geq 1} \Delta_X(m) = 1$ (cf. [17], Corollary 2.6). \qed

**Definition 3.4.**

(i) Let us fix a choice of each inertia subgroup $I_j \subseteq \Delta_{X_2}^{1} \cong \text{Ker}(\Pi_{X_2}^{\text{log}} \xrightarrow{f_X} G_{K^{\text{log}}})$ ($j = 1, 2, \ldots, r$) associated to the $j$-th cusp (relative to some ordering of the cusps of $X \times_K \bar{K}$) among the various $\Delta_{X_2}^{1}$-conjugates of these subgroups. Then, we have a canonical isomorphisms

$$\eta_j : I_{\bar{z}} \xrightarrow{\sim} I_j \quad (j = 1, 2, \ldots, r)$$

For $n = 1, 2$ we shall denote by $V^n$ the completion with respect to the filtration topology of the free Lie algebra generated by

$$V^n := I_{\bar{z}} \oplus \left( \bigoplus_{j=1}^r I_j \oplus \Delta_X^{ab} \right)^{\otimes n}.$$
equipped with a natural grading (hence also a filtration) by taking $I_x, I_j$ to be of weight 2, $\Delta^{ab}_X$ to be of weight 1.

(ii) If $X$ has genus $\geq 1$, then we shall write

$$M_X := \text{Hom}_{\mathbb{Z}_l}(H^2(\Delta_X, \mathbb{Z}_l), \mathbb{Z}_l).$$

Note that $M_X$ is canonically isomorphic to $I_\sim x$ as a $G_K$-module. The cup product on the group cohomology of $\Delta_X$

$$\bigwedge^2 H^1(\Delta_X, M_X) \longrightarrow H^2(\Delta_X, M_X \otimes_{\mathbb{Z}_l} M_X)$$

determines an isomorphism

$$\text{Hom}(\Delta^{ab}_X, M_X) \sim \longrightarrow \Delta^{ab}_X,$$

hence compositions of natural homomorphisms

$$\phi : I_\sim x \sim \longrightarrow M_X \longrightarrow \bigwedge^2 \Delta^{ab}_X, \quad \psi : \bigwedge^2 \Delta^{ab}_X \longrightarrow M_X \sim \longrightarrow I_\sim x$$

(iii) If $X$ has genus $\geq 1$ (resp., genus $= 0$), then we define $L^n_X$ to be the quotient of $V^n$ by the relations determined by the images of the following morphisms:

1. When $n = 1$,
   
   \[
   \bullet_1 I_\sim x \longrightarrow V^1(2/3) ; \quad m \mapsto (\text{id}_{I_\sim x} + \sum \eta_j + \phi)(m)
   \]
   (resp., \(\bullet_1 I_\sim x \longrightarrow V^1(2/3) ; \quad m \mapsto (\text{id}_{I_\sim x} + \sum \eta_j)(m)\)).

2. When $n = 2$ (1 $\leq i \leq g$, 1 $\leq j \leq r$, \(\{k, k'\} = \{1, 2\}\)),
   
   \[
   \bullet_1 I_\sim x \longrightarrow V^2(2/3) ; \quad m \mapsto m + i_k(\sum \eta_j + \phi)(m)
   \]
   \[
   \bullet_2 I_\sim x \otimes_{\mathbb{Z}_l} \Delta^{ab}_X \longrightarrow V^2(3/4) ; \quad m \otimes a \mapsto [i_k \circ \eta_j(m), i_{k'}(a)]
   \]
   \[
   \bullet_3 \bigwedge^2 \Delta^{ab}_X \longrightarrow V^2(2/3) ; \quad a \wedge a' \mapsto [i_k(a), i_{k'}(a')]
   \]
   \[
   \bullet_{4,5} \bigwedge^2 \Delta^{ab}_X \longrightarrow V^2(2/3) ; \quad a \wedge a' \mapsto i_k(a) \wedge i_{k'}(a') - \psi(a \wedge a')
   \]
   (resp., \(\bullet_1 I_\sim x \longrightarrow V^2(2/3) ; \quad m \mapsto m + i_k(\sum \eta_j)(m)\))

— where “[ , ]” denotes the Lie bracket, and for $k = 1, 2$, \(i_k : (\bigoplus I_j \oplus \bigwedge^2 \Delta^{ab}_X) \hookrightarrow (\bigoplus I_j \oplus \bigwedge^2 \Delta^{ab}_X)^{\otimes 2}\) denotes the inclusion into the $k$-th factor.

(iv) If we consider a $G_K$-action on $V^n$ as that taking the natural action on each direct summand in $V^n$, then the ideal generated by the relations defined in (iii) is invariant under this action. Hence, we may equip the graded Lie algebra

$$L^n_X \quad (\text{resp., } L^n_X)$$
with the additional $G_K$-action structure and have a $G_K$-equivariant homomorphism

$$i^L_1 : \mathcal{L}_X^1 \to \mathcal{L}_X^2$$

of graded Lie algebras given by

$$I_\pm \oplus \left( \bigoplus_{j=1}^r I_j \oplus \Delta^{ab}_X \right) \to I_\pm \oplus \left( \bigoplus_{j=1}^r I_j \oplus \Delta^{ab}_X \right)^{\otimes 2}$$

$$(a, b) \mapsto (a, i_1(b))$$

and a $G_K$-equivariant isomorphism

$$s_X^L : \mathcal{L}_X^2 \sim \to \mathcal{L}_X^2$$

of graded Lie algebras given by

$$I_\pm \oplus \left( \bigoplus_{j=1}^r I_j \oplus \Delta^{ab}_X \right)^{\otimes 2} \to I_\pm \oplus \left( \bigoplus_{j=1}^r I_j \oplus \Delta^{ab}_X \right)^{\otimes 2}$$

$$(a, b_1, b_2) \mapsto (a, b_2, b_1)$$

\begin{lemma}
Let $h_1^1 : \mathcal{L}_X^1 \to \text{Gr}^{1}_{X^2/1}$, $\tilde{h}_1^1 : \mathcal{L}_X^1 \to \text{Gr}^{1}_{X^2/1}$ be homomorphisms of graded Lie algebra given by the natural inclusions $\Delta^{ab}_X \hookrightarrow \text{Gr}_{Q_k}(\Delta^{1}_{X^2/1})(1/2)$, $I_\pm \hookrightarrow \text{Gr}_{Q_k}(\Delta^{1}_{X^2/1})(2/3)$ and $I_j \hookrightarrow \text{Gr}_{Q_k}(\Delta^{1}_{X^2/1})(2/3)$.

Then $h_1^1$ and $\tilde{h}_1^1$ are $G_K$-equivariant isomorphisms of graded Lie algebras.
\end{lemma}

\begin{proof}
It is enough to verify the assertion in the case where $x$ is cusp of $X$. But since $\Delta_{\mathcal{F}_X}$ is trivial, it follows that the natural inclusion $\Pi_{X^\log} \hookrightarrow \Pi_{x^\log}$ (well-defined up to inner automorphisms) induces a $G_K$-equivariant isomorphism $\Delta^{ab}_{X} \sim \to \Delta^{ab}_{X_x}$This completes the proof by Proposition 1.9 (i).
\end{proof}

\begin{lemma}
Let

$$i^\text{Lie}_1 : \text{Lie}^{1}_{X^2/1} \to \text{Lie}^{1}_{X^2}, \quad i^\text{Lie}_2 : \text{Lie}^{1}_{X^2/1} \to \text{Lie}^{1}_{X^2}, \quad s^\text{Lie}_1 : \text{Lie}_{X^2} \sim \to \tilde{\text{Lie}}_{X^2}$$

be the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i^\Pi_1 : \Pi_{X^\log} \to \Pi_{X^2}$, $i^\Pi_1 : \Pi_{\text{X}^\log} \to \Pi_{X^2}$ and $s^\Pi_1 : \Pi_{X^2} \sim \to \Pi_{X^2}$ respectively.

Then there exist $G_K$-equivariant isomorphisms of graded Lie algebras

$$h^1_X : \mathcal{L}_X^1 \sim \to \text{Lie}^{1}_{X^2/1}, \quad h^2_X : \mathcal{L}_X^1 \sim \to \text{Lie}^{1}_{X^2/1},$$

$$h^1_X : \mathcal{L}_X^2 \sim \to \text{Lie}^{1}_{X^2}, \quad h^2_X : \mathcal{L}_X^2 \sim \to \text{Lie}^{1}_{X^2}$$

\end{lemma}
which fit into the following commutative diagrams consisting of $G_K$-equivariant morphisms

\[
\begin{array}{ccccccc}
\mathcal{L}^1_X & \cong & \mathcal{L}^2_X & \cong & \mathcal{L}^1_X & \cong & \mathcal{L}^2_X \\
\downarrow^{h^1_X} & i & \downarrow^{h^2_X} & i & \downarrow^{h^1_X} & i & \downarrow^{h^2_X} \\
\text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} & \cong & \text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\mathcal{L}^1_X & \cong & \mathcal{L}^2_X & \cong & \mathcal{L}^1_X & \cong & \mathcal{L}^2_X \\
\downarrow^{h^1_X} & i & \downarrow^{h^2_X} & i & \downarrow^{h^1_X} & i & \downarrow^{h^2_X} \\
\text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} & \cong & \text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} \\
\end{array}
\]

**Proof.** Let $h^2 : \mathcal{L}^2_X \to \text{Gr}_{Q_1}(\Delta_{X_2})$ be the homomorphism given by

\[
I_e \oplus \bigoplus_{j=1}^r I_j \oplus \Delta^{ab}_X \cong \mathcal{L}^2_X \to \text{Gr}_{Q_1}(\Delta_{X_2})
\]

\[
(a, b_1, b_2) \mapsto i^1_{Gr} \circ h^1_{Gr}(a + b_1) + s^1_{Gr} \circ i^1_{Gr} \circ h^1_{Gr}(b_2).
\]

It is verified that $h^2$ is an isomorphism of graded Lie algebras by Proposition 1.9 (ii) and $G_K$-equivariant when we regard it as a map $h^2_{Gr} : \mathcal{L}^2_X \to \text{Gr}_{X_2}$ as well as $\tilde{h}^2_{Gr} : \mathcal{L}^2_X \to \tilde{\text{Gr}}_{X_2}$. If we denote $i^1_{Gr} : \text{Gr}_{X_{2/1}} \to \text{Gr}_{X_2}$, $\tilde{i}^1_{Gr} : \tilde{\text{Gr}}_{X_{2/1}} \to \tilde{\text{Gr}}_{X_2}$, $s^1_{Gr} : \text{Gr}_{X_2} \sim \tilde{\text{Gr}}_{X_2}$ be the $G_K$-equivariant homomorphisms of graded Lie algebras induced by $i^1_{Q_2} : \Pi_{X_{2/1}} \to \Pi_{X_2}$, $\tilde{i}^1_{Q_2} : \Pi_{\tilde{X}_{2/1}} \to \Pi_{\tilde{X}_2}$ and $s^1_{Q_2} : \Pi_{X_2} \sim \Pi_{\tilde{X}_2}$ respectively, then we have the following $G_K$-equivariant commutative diagram

\[
\begin{array}{ccccccc}
\mathcal{L}^1_X & \cong & \mathcal{L}^2_X & \cong & \mathcal{L}^1_X & \cong & \mathcal{L}^2_X \\
\downarrow^{h^1_{Gr}} & i & \downarrow^{h^2_{Gr}} & i & \downarrow^{h^1_{Gr}} & i & \downarrow^{h^2_{Gr}} \\
\text{Gr}_{X_{2/1}} & \cong & \text{Gr}_{X_2} & \cong & \text{Gr}_{X_{2/1}} & \cong & \text{Gr}_{X_2} \\
\end{array}
\]

On the other hand, by taking account of the compatibility of $G_K$-actions (cf. Lemma 3.2) together with Proposition 1.8, we have the following $G_K$-equivariant commutative diagram

\[
\begin{array}{ccccccc}
\text{Gr}_{X_{2/1}} & \cong & \text{Gr}_{X_2} & \cong & \text{Gr}_{X_{2/1}} & \cong & \text{Gr}_{X_2} \\
\downarrow & i & \downarrow & i & \downarrow & i & \downarrow \\
\text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} & \cong & \text{Lie}_{X_{2/1}} & \cong & \text{Lie}_{X_2} \\
\end{array}
\]

By composing the vertical arrows in these commutative diagrams, we obtain the required isomorphisms. \qed

Now, let $L$ be a finite field of cardinality prime to $l$, $Y$ a hyperbolic curve over $L$ of type $(g', r')$, $y^{\log}$ a strict $L$-rational log point of $\overline{Y}^{\log} := \overline{Y}_1^{\log}$; we shall use similar notation for objects obtained from $Y$ (e.g., $Y_2$, $\overline{Y}_2^{\log}$, $\Pi^{\log}_Y$, $G^{\log}$,
et al. to the notation used for objects obtained from $X$. Moreover, we shall fix a choice of homomorphisms

$$
i_1^\Pi : \Pi_{y_{\log}} \longrightarrow \Pi_Y, \quad \rho_1^\Pi : \Pi_Y \longrightarrow \Pi_Y$$

arising from the morphisms $Y_{y_{\log}} \xrightarrow{i_1} Y_2, Y_2 \xrightarrow{\rho_1} Y$ within in the respective conjugacy classes determined by these homomorphisms.

**Definition 3.7.**

Let $\beta$ be an isomorphism of profinite groups $\Pi_{X_{y_{\log}}} \sim \Pi_{Y_{y_{\log}}}$ or $\Pi_X \sim \Pi_Y$. Then the natural quotient $\Pi_{X_{y_{\log}}} \twoheadrightarrow G_K$ (resp., $\Pi_{Y_{y_{\log}}} \twoheadrightarrow G_L, \Pi_X \twoheadrightarrow G_K, \Pi_Y \twoheadrightarrow G_L$) arising from the structure morphism $\overline{X}_{x_{\log}} \twoheadrightarrow \text{Spec}(K)$ (resp., $\overline{Y}_{y_{\log}} \twoheadrightarrow \text{Spec}(L), X \twoheadrightarrow \text{Spec}(K), Y \twoheadrightarrow \text{Spec}(L)$) may be characterized group-theoretically (cf. [19], Proposition 3.3) as the (unique) maximal $($$\tilde{\mathbb{Z}}$$)$-free abelian quotient of $\Pi_{X_{y_{\log}}}$ (resp., $\Pi_{Y_{y_{\log}}}, \Pi_X, \Pi_Y$). Therefore $\beta$ induces an isomorphism $\beta_0 : G_K \sim G_L$.

Now we shall say that $\beta$ is Frobenius-preserving if the isomorphism $\beta_0 : G_K \sim G_L$ obtained as above preserves the Frobenius elements.

In the following, suppose further that we have given a Frobenius-preserving isomorphism $\alpha : \Pi_{X_{y_{\log}}} \sim \Pi_{Y_{y_{\log}}}$ that maps $D_{\overline{x}}(\overline{\pi}_{y_{\log}} \Pi_{X_{y_{\log}}})$ onto a specific decomposition group $D_{\overline{y}}$ of $\overline{y}$. Then we may take a section $\sigma_Y : G_L \rightarrow D_{\overline{y}}$ of the natural surjection $D_{\overline{y}} \rightarrow G_L$ and a 1-cocycle map $\delta_Y : G_L \rightarrow \Delta_{\overline{i}} := \text{Ker}(D_{\overline{y}} \rightarrow G_{L_{\log}})$ of $G_L$ with coefficients in $I_{\overline{y}}$ so as to be compatible with $\sigma_X, \delta_X$ via isomorphisms $\alpha$ and $\alpha_0 : G_K \sim G_L$ induces by $\alpha$. By applying the preceding discussion and results in Section 2 to $Y$, we obtain objects $\Pi_{Y_2}^{\text{Lie}}, \text{Int}_{Y_2}, s_{\beta Y}, \text{etc.}$

**Proposition 3.8.**

In the notation of the above situation, there exist a $G_K$-equivariant isomorphism $\alpha_2^{\text{Lie}} : \Delta_{X_2}^{\text{Lie}} \sim \Delta_{Y_2}^{\text{Lie}}$ such that it is also $G_L$-equivariant when we regard it as a map $\Delta_{X_2}^{\text{Lie}} \sim \Delta_{Y_2}^{\text{Lie}}$ and that if we denote by $\alpha_2^{\text{Lie,Lie}} : \Pi_{X_2}^{\text{Lie,Lie}} \sim \Pi_{Y_2}^{\text{Lie,Lie}}$, $\tilde{\alpha}_2^{\text{Lie,Lie}} : \tilde{\Pi}_{X_2}^{\text{Lie,Lie}} \sim \tilde{\Pi}_{Y_2}^{\text{Lie,Lie}}$ the semi-direct products of $\alpha_2^{\text{Lie,Lie}}$ in these two ways, then these morphisms make the following diagrams commute

$$
\begin{array}{ccc}
\Pi_{X_{y_{\log}}}^{\text{Lie}} & \xrightarrow{\text{Int}_{X_2}^{\text{Lie}} \alpha_{X_2}^{\text{Lie,Lie}}} & \Pi_{X_2}^{\text{Lie,Lie}} \\
\alpha_2^{\text{Lie,Lie}} & \downarrow \alpha_2^{\text{Lie}} & \downarrow \alpha_2^{\text{Lie,Lie}} \\
\Pi_{Y_{y_{\log}}}^{\text{Lie}} & \xrightarrow{\text{Int}_{Y_2}^{\text{Lie}} \alpha_{Y_2}^{\text{Lie,Lie}}} & \Pi_{Y_2}^{\text{Lie,Lie}}
\end{array}
$$

**Proof.** Since $\alpha$ is assumed to be Frobenius-preserving, it follows from [10], Lemma 1.3.9 and [12], Corollary 2.8 that $(g, r) = (g', r')$ and that $\alpha$ induces
an isomorphism $\alpha^\text{cpt} : \Delta^{ab}_X \xrightarrow{\sim} \Delta^{ab}_Y$ and a bijective correspondence between the respective sets of cusps of $X^\log_x, Y^\log_y$ as well as isomorphisms of the decompositions (inertia) groups of cusps corresponding via this bijection. By using these isomorphisms (together with constructions of $L^1_X, L^2_X, L^1_y, L^2_y$), Lemma 3.6 induces $G_K$-equivariant isomorphisms $\alpha^\text{Lie} : \text{Lie}_X^{1/2} \cong \text{Lie}_Y^{1/2}$, $\alpha^{\text{Lie}}_2 : \text{Lie}_X \cong \text{Lie}_Y$ and $\tilde{\alpha}^\text{Lie} : \text{Lie}_X \cong \text{Lie}_Y$ such that $\alpha^\text{Lie} = \tilde{\alpha}^\text{Lie}$ (by compatibility of $G_K$-actions induced by as a morphism of underlying graded Lie algebras and that these maps which make the following diagram commute in the sense of $G_K$-equivariant

$$
\begin{array}{cccccc}
\text{Lie}_X^{1/2} & \overset{\alpha^\text{Lie}}{\longrightarrow} & \text{Lie}_Y^{1/2} & \overset{s^\text{Lie}_X}{\longrightarrow} & \text{Lie}_Y^{1/2} & \overset{\tilde{\alpha}^\text{Lie}}{\longleftarrow} & \text{Lie}_X^{1/2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Lie}_X^{1/2} & \overset{\alpha^{\text{Lie}}_2}{\longrightarrow} & \text{Lie}_X & \overset{s^\text{Lie}_X}{\longrightarrow} & \text{Lie}_X & \overset{\tilde{\alpha}^{\text{Lie}}_2}{\longleftarrow} & \text{Lie}_X^{1/2} \\
\end{array}
$$

Since $\sigma_X, \sigma^{\text{Lie}}_X$ have been taken to be compatible with $\sigma_Y, \sigma^{\text{Lie}}_Y$ via $\alpha$ respectively, we have $\alpha^{\text{Lie}}_2 = \tilde{\alpha}^{\text{Lie}}_2, s^\text{Lie}_X = \tilde{s}^\text{Lie}_X$ as morphisms of underlying graded Lie algebras. On the other hand, $\text{Lie}_X$ (resp., $\text{Lie}_Y$) is, as a Lie algebra, generated by the images of $\text{Lie}_X^{1/2}$ (resp., $\text{Lie}_X^{1/2}$) and the composition $\text{Lie}_X^{1/2} \to \text{Lie}_X^{1/2} \to \text{Lie}_X^{1/2}$ $\text{Lie}_X = \text{Lie}_X$ (resp., $\text{Lie}_X^{1/2} \to \text{Lie}_X^{1/2} \to \text{Lie}_X^{1/2}$), hence we have $\alpha^{\text{Lie}}_2 = \tilde{\alpha}^{\text{Lie}}_2$. Therefore, it follows from the functoriality of $\text{Lin}(-)$ that this diagram induces the required diagram.

One of main results of this paper, i.e., Theorem A, is as follows:

**Theorem 3.9.**
Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$), $x$ a $K$-rational point of $X$ (resp., $y$ an $L$-rational point of $Y$), $X_2$ (resp., $Y_2$) the second configuration space associated to $X$ (resp., $Y$), $X^\log_x$ (resp., $Y^\log_y$) the cuspidalization of $X$ at $x$ (resp., of $Y$ at $y$) [cf. Definition 2.2], $D_x \xrightarrow{\gamma^x} \Pi_{X^\log_x}$ (resp., $D_y \xrightarrow{\gamma^y} \Pi_{Y^\log_y}$) the decomposition group of the diagonal cusp $x^\log$ (resp., $y^\log$).

Let

$$
\alpha : \Pi_{X^\log_x} \xrightarrow{\sim} \Pi_{Y^\log_y}
$$

be a Frobenius-preserving isomorphism [cf. Definition 3.5] which maps $D_x$ onto $D_y$. Here, we shall denote $\overline{\alpha} : \Pi_X \xrightarrow{\sim} \Pi_Y$ (resp., $D_x \xrightarrow{\overline{\alpha}} \Pi_X, D_y \xrightarrow{\overline{\alpha}} \Pi_Y$) by the isomorphism (resp., the decomposition group of $x$, the decomposition group of $y$) obtained by passing to the quotients $\Pi_{X^\log_x} \to \Pi_X, \Pi_{Y^\log_y} \to \Pi_Y$. 


Then there exists a unique isomorphism
\[ \alpha_2 : \Pi_{X_2} \overset{\sim}{\longrightarrow} \Pi_{Y_2} \]
which is compatible with the natural switching automorphisms [cf. the discussion following Remark 2.1.1] up to an inner automorphism and fits into the following commutative square
\[
\begin{array}{ccc}
\Pi_{X_2} & \xrightarrow{\alpha_2} & \Pi_{Y_2} \\
\downarrow & & \downarrow \\
\Pi_X & \xrightarrow{\pi} & \Pi_Y
\end{array}
\]
which induce \( \alpha \) by restricting \( \alpha_2 \) to the inverse images (via the vertical arrows) of \( D_x \) and \( D_y \).

Proof. Let us take \( s_{\Pi_X}^\Pi, s_{\Pi_Y}^\Pi, \alpha_2^{\Pi\text{Lie}}, \) and \( \tilde{\alpha}_2^{\Pi\text{Lie}} \) as we obtained up to now. If we identify \( (s_{\Pi_X}^\Pi \circ i_1^\Pi)(\Delta_{X_2/1}^1) \) with \( i_1^\Pi(\Pi_{X_2}^\text{log}) \) and \( (s_{\Pi_Y}^\Pi \circ i_1^\Pi)(\Delta_{Y_2/1}^1) \) generate \( \Pi_{X_2} \) (similarly, \( \Pi_{Y_2} \) is generated by \( i_1^\Pi(\Pi_{Y_2}^\text{log}) \) and \( (s_{\Pi_Y}^\Pi \circ i_1^\Pi)(\Delta_{Y_2/1}^1) \)). Therefore, since the diagram
\[
\begin{array}{ccc}
\Pi_{X_2}^\text{log} & \xrightarrow{i_1^\Pi} & \Pi_{X_2}^\text{Lie} \\
\downarrow^\alpha & & \downarrow^\alpha_2^\text{Lie} \\
\Pi_{Y_2}^\text{log} & \xrightarrow{i_1^\Pi} & \Pi_{Y_2}^\text{Lie}
\end{array}
\]
commute, \( \alpha_2^\Pi \) maps \( \Pi_{X_2} \) onto \( \Pi_{Y_2} \). Thus, the restriction \( \alpha_2 \) of \( \alpha_2^\Pi \) to \( \Pi_{X_2} \) makes the diagram (**) commutes.

Next we consider the uniqueness. Let us take two maps \( \beta, \beta' : \Pi_{X_2} \overset{\sim}{\longrightarrow} \Pi_{Y_2} \) both of which make the diagram (**) commutes. Then \( \beta^{-1} \circ \beta' \) induces an automorphism of the exact sequence
\[ 1 \longrightarrow \Delta_{X_2/1}^1 \longrightarrow \Pi_{X_2} \overset{\phi_1^\Pi}{\longrightarrow} \Pi_X \longrightarrow 1 \]
which consists of the identities of \( \Delta_{X_2/1}^1 \) and \( \Pi_X \). This implies that \( \beta^{-1} \circ \beta' \) is the identity morphism (cf. §0). \( \square \)

**Corollary 3.10.**

Let \( X \) (resp., \( Y \)) be a hyperbolic curve over a finite field \( K \) (resp., \( L \)), \( x, x' \) \( K \)-rational points of \( \overline{X} \) (resp., \( y, y' \) \( L \)-rational points of \( \overline{Y} \)). Let
\[ \alpha : \Pi_{X_2}^\text{log} \longrightarrow \Pi_{Y_2}^\text{log} \]
be a Frobenius-preserving isomorphism such that \( \alpha \) (resp., the isomorphism \( \overline{\alpha} : \Pi_X \overset{\sim}{\longrightarrow} \Pi_Y \) induced by passing to the quotients \( \Pi_{X_2}^\text{log} \rightarrow \Pi_X, \Pi_{Y_2}^\text{log} \rightarrow \Pi_Y \)) maps
the decomposition group of the diagonal cusp \( \tilde{x} \) (resp., \( x' \)) to the decomposition group of the diagonal cusp \( \tilde{y} \) (resp., \( y' \)) up to conjugation.

Then there exists a unique Frobenius-preserving isomorphism

\[
\alpha' : \Pi_{X_{x'}}^{\log} \longrightarrow \Pi_{Y_{y'}}^{\log}
\]

well-defined up to an inner automorphism which induces \( \pi \) by passing to the quotients and maps the decomposition group of the diagonal cusp \( \tilde{x} \) to the decomposition group of the diagonal cusp \( \tilde{y} \) up to conjugation.

Proof. The existence assertion follows from Theorem 3.8 and the fact that if \( D_{x'} \subseteq \Pi_X, D_{y'} \subseteq \Pi_Y \) denote the decomposition groups of \( x' \), \( y' \) respectively, then we have \( \Pi_{X_{x'}}^{\log} \cong D_{x'} \times_{\Pi_X} \Pi_{X_2}, \Pi_{Y_{y'}}^{\log} \cong D_{y'} \times_{\Pi_Y} \Pi_{Y_2}. \)

We consider the uniqueness assertion. Let us take two isomorphisms \( \hat{\alpha}', \bar{\alpha}' : \Pi_{X_{x'}}^{\log} \overset{\sim}{\longrightarrow} \Pi_{Y_{y'}}^{\log} \) which induce \( \pi : \Pi_X \overset{\sim}{\longrightarrow} \Pi_Y \) by passing to the quotients. and write \( \beta := (\hat{\alpha}')^{-1} \circ \bar{\alpha}' \in \text{Aut}(\Pi_{X_{x'}}^{\log}) \) which yields, from the existence assertion, \( \beta_2 \in \text{Aut}(\Pi_{X_2}) \) reducing to the identity morphism of \( \Pi_{X_{x'}}, \Pi_{Y_{y'}} \) by passing to the natural quotient \( \Pi_{X_2} \rightarrow \Pi_{X_{x'}} \). Then \( \beta_2 \) define an element \( [\beta_2] \) of \( \text{Out}^{FC}(\Delta_{X_2}) \) and \( [\beta_2] \) reduce to the unit element in \( \text{Out}(\Delta_X) \) by the definitions of \( \hat{\alpha}', \bar{\alpha}' \). But \( \text{Out}^{FC}(\Delta_{Y_2}) \rightarrow \text{Out}(\Delta_Y) \) is injective (cf. [5] for the definition and results concerning to “\( \text{Out}^{FC^n} \)”), so we have \( [\beta_2] = 1 \). This completes the proof. \( \square \)

Remark 3.10.1.

Any Frobenius-preserving isomorphism is quasi-point-theoretic (cf. [19], Corollary 2.10, Proposition 3.8 and [12], Corollary 2.8), i.e., induces a bijection between the set of decomposition groups of the points of \( \overline{X}, \overline{Y} \). Therefore, in the statement of Corollary 3.10, a closed point \( y' \) of \( Y \) which corresponds to \( x' \) via \( \pi \) necessarily exists (but this choice is not unique).

4. Cuspidalization Problems for hyperbolic curves

In this last section, we apply of Theorem 3.9 to obtain group-theoretical constructions of the cuspidalization of a hyperbolic curve at a point infinitesimally close to a cusp (cf. Theorem 4.3), as well as of arithmetic fundamental groups of configuration spaces (cf. Theorem 4.4).

We maintain the notation of Section 3; moreover, until the end of Theorem 4.3, we shall assume that both \( X \) and \( Y \) are affine (i.e., \( r, r' > 0 \)), and \( x, y \) are split cusps of \( X, Y \), respectively, i.e., \( x \in \overline{X}(K) \setminus X(K), y \in \overline{Y}(L) \setminus Y(L) \). As discussed in Definition 2.2, we obtain the major and minor cuspidal components.
\(X^\log, P_x^\log\) at \(x\), together with the nexus \(\nu_x^\log\) at \(x\) as strict log closed subschemes of \(X^\log\) (cf. [6], 1.2), which determine subgroups well-defined up to conjugacy,

\[
\Pi_{X^\log'}, \Pi_{P^\log}, \Pi_{\nu_x^\log} \subseteq \Pi_{X^\log}
\]

— which we shall refer to, respectively, as major vertical, minor vertical, and nexus subgroup (cf. [17], Definition 1.4).

**Lemma 4.1.**

The composition of morphisms \(X^\log' \to X^\log\) (resp., \(P^\log \to X^\log\)) and \(X^\log \to X^\log \times_K x^\log\) (resp., \(X^\log \to P^\log \times_K x^\log\)) induces an isomorphism

\[
\Pi_{X^\log'} \cong \Pi_{X \times_K G_{K_{L}}}
\]

(\(\text{resp., } \Pi_{P^\log} \cong \Pi_{P \times_K G_{K_{L}}}\)).

In particular, the major vertical subgroups may be thought of as defining sections of the projection \(\Pi_{X^\log} \to \Pi_{X \times_K G_{K_{L}}}\).

**Proof.** We shall only consider the non-resp’d portion due to a similar argument.

We recall that the category of \(\acute{e}t\) coverings has the \(\acute{e}t\)ale descent property and invariance for restriction from an henselian trait to its closed point (cf. [6]). Since \(X^\log \to X^\log \times_K x^\log\) is an isomorphism on \(X \setminus \{x\}\), it is enough to see isomorphicity of the morphism between the log inertia groups of \(\nu_x^\log\) and \(x^\log \times_K x^\log\), i.e., \(\text{Ker}(\Pi_{\nu_x^\log} \to G_K)\) and \(\text{Ker}(\Pi_{x^\log \times_K x^\log} \to G_K)\) (cf. [6], 4.7 for the terminology “log inertia subgroup”). If we fix a chart, modeled on \(N\), roots of a local uniformizer at \(x\) in \(X\), then we may give \(x^\log \times_K x^\log\) a chart of the form \(N \oplus N\), where the first factor of direct sum is that pulled back from the ground log scheme \(x^\log\) and the second factor is that pulled back from \(x^\log\) as an exact closed subscheme of \(X^\log\) (resp., via \(p_2 \circ i_1 : X^\log \to X^\log\)). By using these splittings, we may express a chart of the homomorphism of monoids induced by the morphism \(\nu_x^\log \to x^\log \times_K x^\log\) as

\[
N \oplus N \to N \oplus N
\]

\[(a, b) \mapsto (a, a + b)\]

Then, by applying the functor \(\text{Hom}((-)^{ep}, \mathbb{Z}_d(1))\) to this morphism of monoids, it is verified that the induced morphism of log inertia groups between \(\nu_x^\log\) and \(x^\log \times_K x^\log\) is an isomorphism. \(
\)

**Lemma 4.2.**

(i) Suppose that we fix a choice of a nexus subgroup \(\Pi_{\nu_x^\log} \subseteq \Pi_{X^\log}\) among its various \(\Pi_{X^\log}\)-conjugates. Then there exists a unique pair of inclusions

\[
\Pi_{X^\log'} \subseteq \Pi_{X^\log}, \quad \Pi_{P^\log} \subseteq \Pi_{X^\log}
\]
(among their various $\Pi_{X_{\log};x}$-conjugates) both of which contain $\Pi_{\nu_{x}^{\log}} \subseteq \Pi_{X_{\log}}$.

(ii) The compatible inclusions $\Pi_{\nu_{x}^{\log}} \subseteq \Pi_{X_{\log}';x} \subseteq \Pi_{X_{\log}x}$, $\Pi_{\nu_{X}^{\log}} \subseteq \Pi_{P_{\log}X} \subseteq \Pi_{X_{\log}}$ obtained in (i) make a commutative square

$$
\begin{array}{ccc}
\Pi_{\nu_{x}^{\log}} & \longrightarrow & \Pi_{\nu_{X}^{\log}} \\
\downarrow & & \downarrow \\
\Pi_{X_{\log}'} & \longrightarrow & \Pi_{X_{\log}}
\end{array}
$$

which is co-cartesian in the category of extensions of $G_{K_{\log}}$ by pro-$l$ groups.

Proof. We consider assertion (i). If we fix a universal (geometrically pro-$l$) kêt covering $U_{\log} \to X_{\log}'$, then the fixed inclusion $\Pi_{\nu_{x}^{\log}} \subseteq \Pi_{X_{\log}}$ corresponds, by definition, to a log geometric point $\tilde{\nu}_{x}^{\log}$ of $U_{\log}$ over $\nu_{x}^{\log}$. Therefore it is enough to take $\Pi_{X_{\log}'} \subseteq \Pi_{X_{\log}}$, $\Pi_{P_{\log}X} \subseteq \Pi_{X_{\log}}$ as those of corresponding to unique irreducible components of $U_{\log}$ over $\bar{X}_{\log}'$, $\bar{P}_{\log}$ which contain $\tilde{\nu}_{x}$. Assertion (ii) follows from the construction of colimit and the “van Kampen Theorem” in algebraic topology. □

Now we consider Theorem B. Once we shall consider, for simplicity of the proof, a slightly weaker statement with respect to the case where the types $(g, r)$ of the hyperbolic curves satisfy that $(g, r) = (0, 3)$ as stated below. But by giving directly a proof of Theorem 4.4, which contain the statement of Theorem B, we also can conclude the same statement as the case $(g, r) \neq (0, 3)$.

**Theorem 4.3.**

Let $X, Y$ be affine hyperbolic curves over a finite field $K, L$, respectively, of type $(g, r) \neq (0, 3)$ (resp., of type $(g, r) = (0, 3)$), $x$ a $K$-rational point of $X \setminus X$, $y$ an $L$-rational point of $Y \setminus Y$. Let

$$
\alpha : \Pi_{X} \xrightarrow{\sim} \Pi_{Y}
$$

(resp., $\alpha' : \Pi_{X \times_{K} K'} \xrightarrow{\sim} \Pi_{Y \times_{L} L'}$)

be a Frobenius-preserving isomorphism such that the decomposition groups of $x$ and $y$ (which are well-defined up to conjugacy) correspond via $\alpha$ (resp., where $K', L'$ are the unique extensions of $K, L$ whose degrees of extensions are two).

Then there exists a unique isomorphism

$$
\tilde{\alpha} : \Pi_{X_{\log}} \xrightarrow{\sim} \Pi_{Y_{\log}}
$$

(resp., $\tilde{\alpha}' : \Pi_{X_{\log}} \times_{G_{K}} G_{K'} \xrightarrow{\sim} \Pi_{Y_{\log}} \times_{G_{L}} G_{L'}$)

well-defined up to composition with an inner automorphism which maps the decomposition group (well-defined up to conjugacy) of $\tilde{x}$ in $X_{\log}$ to that of $\tilde{y}$.
in \( \overline{Y}_y \) and induces \( \alpha \) (resp., \( \alpha' \)) by passing to the quotients \( \Pi_{X \log} \rightarrow \Pi_X \), \( \Pi_{Y \log} \rightarrow \Pi_Y \) (resp., \( \Pi_{X \log} \times_{G_K} G_{K'} \rightarrow \Pi_{X \times K K'} \), \( \Pi_{Y \log} \times_{G_L} G_{L'} \rightarrow \Pi_{Y \times L L'} \)).

**Proof.** The proof of uniqueness is similar to that of Corollary 3.8.

Now we consider the existence assertion. In the case \( (g, r) = (0, 3) \), \( X \times K K' \) is isomorphic to a tripod. Let \( D'_x \hookrightarrow \Pi_{X \times K K'} \) be a natural inclusion of a specific decomposition group \( D'_x \) of \( x \) in \( \Pi_{X \times K K'} \) and \( \kappa : D'_x \times_{G_K} \Pi_{X \times K K'} \leftarrow \Pi_{X \log} \times_{G_K} G_{K'} \) (resp., \( \kappa : \Pi_{X \times K K'} \times_{G_K} D'_x \rightarrow \Pi_{X \log} \times_{G_K} G_{K'} \)) the injective morphism induced by the isomorphism, obtained in Lemma 4.1, with respect to the major (resp., minor) vertical subgroup. If we fix an identification \( D'_x \cong \mathbb{Z}_l(1) \times G_K \), then the automorphism \( \kappa \) of \( (\mathbb{Z}_l(1) \times \mathbb{Z}_l(1)) \times G_{K'} \) given by \( (s_1, s_2, t) \mapsto (s_1, s_2 - s_2, t) \) coincides with \( (\kappa)^{-1} \circ \kappa_X \) \( D'_x \times_{G_K} D'_x \rightarrow D'_x \times_{G_K} D'_x \). Hence the square

\[
\begin{array}{ccc}
D'_x \times_{G_K} D'_x & \longrightarrow & \Pi_{X \times K K'} \times_{G_K} D'_x \\
\downarrow & & \downarrow \\
D'_x \times_{G_K} \Pi_{X \times K K'} & \xrightarrow{\kappa_X} & \Pi_{X \log} \times_{G_K} G_{K'}
\end{array}
\]

— where the left hand vertical arrow is the natural inclusion and the upper horizontal arrow is the composition of \( \kappa \) and the natural inclusion — is co-cartesian in the category of extensions of \( D'_x \) by pro- \( l \) groups by Lemma 4.2. Therefore, by comparing this diagram to that of \( Y \), the proof is completed.

In the case \( (g, r) \neq (0, 3) \), by considering the number of topological generators of profinite groups, we take an open subgroup \( \Pi_{\tilde{Z}} \subseteq \Pi_X \) such that the corresponding étale covering \( \tilde{Z} \) of \( X \) satisfies the following condition: \( \tilde{Z} \) is also étale over \( x \) in their (smooth) compactification, and have at least two split points \( z, z' \) over \( x \) (where we shall choose \( z \) so that a suitable choice of decomposition group \( D_z \subseteq \Pi_{\tilde{Z}} \) of \( z \) coincides with \( D_z \)). Let \( Z \) be the partial compactification of \( Z \) at \( z' \), \( \overline{Z}_{z'} \) the cuspidalization of \( Z \) at \( z \). Moreover, let \( \overline{Z}_{z'} \), \( \overline{Z}_{z} \), \( \nu_{z} \) be the major, minor cuspidal component and the nexus of \( \overline{Z}_{z} \) at \( z \). Then we have a sequence of inclusions

\[
\Pi_{\nu_z} \longrightarrow \Pi_{\overline{Z}} \times_{G_K} D_z \longrightarrow \Pi_{\overline{Z}} \times_{G_K} D_z
\]

and a commutative diagram

\[
\begin{array}{ccc}
\Pi \times_{G_K} D_x & \leftarrow & \Pi \times_{G_K} D_z \\
\downarrow & & \downarrow \\
\Pi_{\overline{Z}} & \leftarrow & \Pi_{\overline{X}}
\end{array}
\]

— where the upper sequence induces a sequence of morphisms, by restricting to subgroups,

\[
\Pi \times_{G_K} D_x \leftarrow D_z \times_{G_K} D_z \longrightarrow D_z \times_{G_K} D_z \leftarrow \Pi_{\nu_z}
\]

\( \ast \ast \).
Hence, combining (⋆) with (⋆⋆) yields the lower horizontal sequence of the commutative diagram

\[
\begin{array}{ccc}
\Pi_{X}^{\text{log}} & \to & \Pi_{Y}^{\text{log}} & \to & \Pi_{F_{X}}^{\text{log}} \\
\downarrow{\iota} & & \downarrow{\iota} & & \downarrow{\iota} \\
\Pi_{X} \times_{G_{K}} D_{x} & \to & \Pi_{Y}^{\text{log}} & \to & \Pi_{F_{X}} \times_{G_{K}} D_{x}
\end{array}
\]

Therefore, the colimit of the following diagram

\[
\begin{array}{ccc}
\Pi_{X} \times_{G_{K}} D_{x} & \to & \Pi_{Y}^{\text{log}} & \to & \Pi_{F_{X}} \times_{G_{K}} D_{x} \\
& & \downarrow{\iota} & & \downarrow{\iota} \\
& & \Pi_{X} \times_{G_{K}} D_{x} & \to & \Pi_{Y}^{\text{log}} & \to & \Pi_{F_{X}} \times_{G_{K}} D_{x}
\end{array}
\]

in the category of extensions of $D_{x}$ by pro-$l$ groups coincides (by Lemma 4.2) with $\Pi_{X}^{\text{log}}$. Therefore, by comparing this diagram to that of $Y$, the proof is completed. □

Finally, we consider Theorem C, i.e., the cuspidalization problem for geometrically pro-$l$ fundamental groups of configuration spaces of (not necessarily proper) hyperbolic curves over finite fields:

**Theorem 4.4.** (cf. [11], Theorem 3.10, [4], Theorem 4.1)

Let $X$ (resp., $Y$) be a hyperbolic curve over a finite field $K$ (resp., $L$). Let

\[ \alpha_{1} : \Pi_{X} \sim \Pi_{Y} \]

be a Frobenius-preserving isomorphism. Then for any $n \in \mathbb{Z}_{\geq 0}$, there exists a unique isomorphism

\[ \alpha_{n} : \Pi_{X_{n}} \sim \Pi_{Y_{n}} \]

well-defined up to composition with an inner automorphism, which is compatible with the natural respective outer actions of the symmetric group on $n$ letters and makes the diagram

\[
\begin{array}{ccc}
\Pi_{X_{n+1}} & \xrightarrow{\alpha_{n+1}} & \Pi_{Y_{n+1}} \\
\downarrow{p_{i}^{n}} & & \downarrow{p_{i}^{n}} \\
\Pi_{X_{n}} & \xrightarrow{\alpha_{n}} & \Pi_{Y_{n}}
\end{array}
\]

$(i = 1, \ldots, n + 1)$ commute.

**Proof.** If $n = 2$ and $X$ is proper, it follows from [11], Theorem 3.1. We shall consider the case where $n = 2$ and $X$ is affine. It follows from Theorem 3.8 and Proposition 4.3 that $\alpha_{1}$ induces an isomorphism $\alpha_{0} : G_{K} \cong G_{L}$ and $\alpha_{2} : \Pi_{X_{2}} \times_{G_{K}} G_{K'} \cong \Pi_{Y_{2}} \times_{G_{L}} G_{L'}$ — where $G_{K'} \subseteq G_{K}$ (resp., $G_{L'} \subseteq G_{L}$) denotes an open subgroup corresponding to some finite extension $K'$ of $K$ (resp., $L'$ of $L$) — as well as an isomorphism $\alpha_{2}^{\Delta} : \Delta_{X_{2}} \cong \Delta_{Y_{2}}$. Now let us denote by $\tau_{X} : G_{K} \to \text{Out}(\Delta_{X_{2}})$ (resp., $\tau_{Y} : G_{L} \to \text{Out}(\Delta_{Y_{2}})$) the morphism obtained naturally by lifting elements of $G_{K}$ (resp., $G_{L}$) via the surjection $\Pi_{X_{2}} \to G_{K}$ (resp., $\Pi_{Y_{2}} \to G_{L}$). Then $\alpha_{2}^{\Delta} : \alpha_{0}$ construct two morphisms

\[ \tau_{Y} \circ \alpha_{0}, \ [\alpha_{2}^{\Delta}] \circ \tau_{X} : G_{K} \to \text{Out}(\Delta_{Y_{2}}) \]

— where \([\alpha^2\Delta]\) denotes the isomorphism \([\Delta_X] \xrightarrow{\sim} \Delta_Y\) that sends each element \(g \in \text{Aut}(\Delta_X)\) to \(\alpha^2\Delta \circ g \circ (\alpha^2\Delta)^{-1} \in \text{Aut}(\Delta_Y)\) — which coincide after composing with the natural homomorphism \([\Delta_Y] \rightarrow \text{Out}(\Delta_Y)\) by definitions of \(\alpha_0, \alpha^2\Delta\). On the other hand, the images of \(\tau_Y \circ \alpha_0\) and \([\alpha^2\Delta]\) lie in \(\text{Out}^\text{FC}(\Delta_Y)\) and \(\text{Out}^\text{FC}(\Delta_Y) \rightarrow \text{Out}(\Delta_Y)\) is injective (cf. [5] for the definition and results concerning to “\(\text{Out}^\text{FC}\)”), so \(\tau_Y \circ \alpha_0 = [\alpha^2\Delta] \circ \tau_X\). Therefore, since \(\Pi_X \cong \Delta_X \times G_K\) (resp., \(\Pi_Y \cong \Delta_Y \times G_L\)), we have an isomorphism \(\Pi_X \cong \Pi_Y\), which satisfies the required uniqueness and compatibility from the construction. This completes the assertion in the case where \(n = 2\) and \(X\) is affine.

The assertion of the case \(n \geq 3\) follows from a very similar argument as the above discussion. Indeed, we can apply an inductive argument on \(n\) together with the natural extension

\[
1 \longrightarrow \Delta_{(X \setminus \{x\})_{n-1}} \longrightarrow \Pi_{X_{n-1}} \longrightarrow \Pi_X \longrightarrow 1
\]

(hence \(\Pi_{X_n} \cong \Delta_{(X \setminus \{x\})_{n-1}} \times \Pi_X\)) for an arbitrary \(K\)-rational point \(x \in X\), and the fact that \(\text{Out}^\text{FC}(\Delta_{(X \setminus \{x\})_{n-1}}) \rightarrow \text{Out}^\text{FC}(\Delta_{(X \setminus \{x\})_{n-2}})\) is injective. \(\square\)

The following corollary follows immediately from Theorem 4.4, together with the fact that any Frobenius-preserving isomorphism between hyperbolic curves over finite fields preserves the set of decomposition groups of closed points (as stated Remark 3.10.1).

**Corollary 4.5.** (cf. [4], Corollary 4.1)

Let \(X\) (resp., \(Y\)) be a hyperbolic curve over a finite field \(K\) (resp., \(L\)), and \(n \in \mathbb{Z}_{\geq 0}\). Let

\[
\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y
\]

be a Frobenius-preserving isomorphism, and \(\{x_1, \cdots, x_n\}\) an ordered set of distinct \(K\)-rational points of \(X\). Then there exists a ordered set \(\{y_1, \cdots, y_n\}\) of distinct \(L\)-rational points of \(Y\) and uniquely exists an isomorphism

\[
\tilde{\alpha} : \Pi_{X \setminus \{x_1, \cdots, x_n\}} \xrightarrow{\sim} \Pi_{Y \setminus \{y_1, \cdots, y_n\}}
\]

well-defined up to composition with an inner automorphism, which induces \(\alpha\) by passing to quotients \(\Pi_{X \setminus \{x_1, \cdots, x_n\}} \xrightarrow{\sim} \Pi_X, \Pi_{Y \setminus \{y_1, \cdots, y_n\}} \xrightarrow{\sim} \Pi_Y\).

**References**


