Polarized K3 surfaces of genus 18 and 20

Dedicated to Professor Hisasi Morikawa on his 60th Birthday

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A surface, i.e., 2-dimensional compact complex manifold, S is of type K3 if its canonical line bundle $\mathcal{O}_S(K_S)$ is trivial and if $H^1(S, \mathcal{O}_S) = 0$. An ample line bundle L on a K3 surface S is a polarization of genus g if its self intersection number (L^2) is equal to $2g-2$, and called *primitive* if $L \simeq M^k$ implies $k = \pm 1$. The moduli space $\mathcal{F}_g$ of primitively polarized K3 surfaces (S, L) of genus g is a quasi-projective variety of dimension 19 for every $g \geq 2$ ([15]). In [12], we have studied the generic primitively polarized K3 surfaces (S, L) of genus $6 \leq g \leq 10$. In each case, the K3 surface S is a comlete intersection of divisors in a homogeneous space X and the polarization L is the restriction of the ample generator of the Picard group $\operatorname{Pic} X \simeq \mathbf{Z}$ of X.

In this article, we shall study the generic (polarized) K3 surfaces (S, L) of genus 18 and 20. (Polarization of genus 18 and 20 are always primitive.) The K3 surface S has a *canonical* embedding into a homogeneous space X such that L is the restriction of the ample generator of $\operatorname{Pic} X \simeq \mathbf{Z}$. S is not a comlete intersection of divisors any more but a complete intersection in X with respect to a homogeneous vector bundle $\mathcal{V}$ (Definition 1.1): S is the zero locus of a global section s of $\mathcal{V}$. Moreover, the global section s is uniquely determined by the isomorphism class of (S, L) up to the automorphisms of the pair $(X, \mathcal{V})$. As a corollary, we obtain a description of birational types of $\mathcal{F}_{18}$ and $\mathcal{F}_{20}$ as orbit spaces (Theorem 0.3 and Corolary 5.10).

In the case of genus 18, the ambient space X is the 12-dimensional variety of 2-planes in the smooth 7-dimensional hyperquadric $Q^7 \subset \mathbf{P}^8$. The complex special orthogonal group $G = SO(9, \mathbf{C})$ acts on X transitively. Let $\mathcal{F}$ be the homogeneous vector bundle corresponding to the fourth fundamental weight $w_4 = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4)/2$ of the root system

$$B_4 : \overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ} \Longrightarrow \overset{\alpha_4}{\circ} \tag{0.1}$$

of G. $\mathcal{F}$ is of rank 2 and the determinant line bundle $\bigwedge^2 \mathcal{F}$ of $\mathcal{F}$ generates the Picard group of X. The vector bundle $\mathcal{V}$ is the direct sum of five copies of $\mathcal{F}$.

Theorem 0.2 *Let $S \subset X$ be the common zero locus of five global sections of the homogeneousn vector bundle $\mathcal{F}$. If S is smooth and of dimension 2, then $(S, \bigwedge^2 \mathcal{F}|_S)$ is a (polarized) K3 surface of genus 18.*

Remark Consider the variety X of lines in the 5-dimensional hyperquadric $Q^5 \subset \mathbf{P}^6$ instead. This is a 7-dimensional homogeneous space of $SO(7, \mathbf{C})$ and has a homogeneous vector bundle of rank 2 on it. The zero locus Z of its general global section is a Fano 5-fold of index 3 and a homogeneous space of the exceptional group of type G_2. See [12] and [13] for other description of Z and its relation to K3 surfaces of genus 10.

The space $H^0(X, \mathcal{F})$ of global sections of $\mathcal{F}$ is the (16-dimensional) spin representation U^{16} of the universal covering group $\tilde{G} = Spin(9, \mathbf{C})$ (see [5]). Let $G(5, U^{16})$ be the

Grassmann variety of 5-dimensional subspaces of U^{16} and $G(5, U^{16})^s$ be its open subset consisting of stable points with respect to the action of $\tilde{G}$. The orthogonal group G acts on $G(5, U^{16})$ effectively and the geometric quotient $G(5, U^{16})^s/G$ exists as a normal quasi-projective variety ([14]).

Theorem 0.3 *The generic K3 surface of genus 18 is the common zero locus of five global sections of the rank 2 homogeneous vector bundle $\mathcal{F}$ on X. Moreover, the classification (rational) map $G(5, U^{16})^s/SO(9, \mathbf{C}) - \to \mathcal{F}_{18}$ is birational.*

Remark The spin represetation U^{16} is the restriction of the half spin representation H^{16} of $Spin(10, \mathbf{C})$ to $Spin(9, \mathbf{C})$. The quotient
$SO(10, \mathbf{C})/SO(9, \mathbf{C})$ is isomorphic to the complement of the 8-dimensional hyperquadric $Q^8 \subset \mathbf{P}^9$. Hence $\mathcal{F}_{18}$ is birationally equivalent to a $\mathbf{P}^9$-bundle over the 10-dimensional orbit space $G(5, H^{16})^s/SO(10, \mathbf{C})$.

Let $\mathcal{E}$ be the homogeneous vector bundle corresponding to the first fundamental weight $w_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and E_N its restriction to S_N. The genericity of S_N is a consequence of the simpleness of E_N (Proposition 4.1). The uniqueness of the expression follows from the rigidity of E_N and the following:

Proposition 0.4 *Let E be a stable vector bundle on a K3 surface S and assume that E is rigid, i.e., $\chi(sl(E)) = 0$. If a semi-stable vector bundle F has the same rank and Chern classes as E, then F is isomorphic to E.*

This is a consequence of the Riemann-Roch theorem

$$(0.4) \quad \begin{aligned} \dim \operatorname{Hom}(E, F) + \dim \operatorname{Hom}(F, E) &\geq \chi(E^\vee \otimes F) \\ &= \chi(E^\vee \otimes E) = \chi(sl(E)) + \chi(\mathcal{O}_S) = 2 \end{aligned}$$

on a K3 surface (*cf.* [11], Corollary 3.5).

In the case of genus 20, the ambient homogeneous space X is the (20-dimensional) Grassmsann variety $G(V, 4)$ of 4-dimensional quotient spaces of a 9-dimensional vector space V and the homogeneous vector bundle $\mathcal{V}$ is the direst sum of three copies of $\bigwedge^2 \mathcal{E}$, where $\mathcal{E}$ is the (rank 4) universal quotient bundle on X. The generic K3 surface of genus 20 is a complete intersection in $G(V, 4)$ with respect to $(\bigwedge^3 \mathcal{E})^{\oplus 3}$ in a unique way (Theorem 5.1 and Theorem 5.9). This description of K3 surfaces is a generalization of one of the three descriptions of Fano threefolds of genus 12 given in [13].

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Notation and Conventions. All varieties are considered over the complex number field $\mathbf{C}$. A vector bundle E on a variety X is a locally free $\mathcal{O}_X$-module. Its rank is denoted by $r(E)$. The determinant line bundle $\bigwedge^{r(E)} E$ is denoted by $\det E$. The dual vector bundle $\mathcal{H}om(E, \mathcal{O}_X)$ of E is deoted by $E^\vee$. The subbundle of $\mathcal{E}nd(E) \simeq E \otimes E^\vee$ consisting of trace zero endomorphisms of E is denoted by $sl(E)$.

1 Complete intersections with respect to vector bundles

We generalize Bertini's theorem for vector bundles. Let $s \in H^0(U, E)$ be a global section of a vector bundle E on a variety U. Let $\mathcal{O}_V \longrightarrow E$ be the multiplication by s and $\eta : E^\vee \longrightarrow \mathcal{O}_V$ its dual homomorphism. The subscheme $(s)_0$ of U defined by the ideal $I = \operatorname{Im} \eta \subset \mathcal{O}_U$ is called the scheme of zeroes of s.

Definition 1.1 *(1) Let $\{e_1, \cdots, e_r\}$ be a local frame of E at $x \in U$. A global section $s = \sum_{i=1}^r f_i e_i, f_i \in \mathcal{O}_X$, of E is nondegenerate at x if $s(x) = 0$ and $(f_1, \cdots, f_r)$ is a regular sequence. s is nondegenerate if it is so at every point x of $(s)_0$.*

(2) A subscheme Y of U is a complete intersection with respect to E if Y is the scheme of zeroes of a nondegenerate global section of E.

In the case U is Cohen-Macaulay, a global section s of E is nondegenerate if and only if the codimension of $Y = (s)_0$ is equal to the rank of E.

The wedge product by $s \in H^0(U, E)$ gives rise to a complex

$$\Lambda^\bullet : \quad \mathcal{O}_U \longrightarrow E \longrightarrow \bigwedge^2 E \longrightarrow \cdots \longrightarrow \bigwedge^{r-1} E \longrightarrow \bigwedge^r E \tag{1.2}$$

called the *Koszul complex* of s. The dual complex

$$K^\bullet : \quad \bigwedge^r E^\vee \longrightarrow \bigwedge^{r-1} E^\vee \longrightarrow \cdots \longrightarrow \bigwedge^2 E^\vee \longrightarrow E^\vee \longrightarrow \mathcal{O}_U \tag{1.3}$$

is called the Koszul complex of Y.

Proposition 1.4 *The Koszul complex $K^\bullet$ is a resolution of the structure sheaf $\mathcal{O}_Y$ of Y by vector bundles, that is, the sequence*

$$0 \longrightarrow K^\bullet \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

is exact.

In particular, the conormal bundle I/I^2 of Y in U is isomorphic to $E^\vee$ and we have the adjunction formula

$$K_Y \cong (K_U + \det E)|_Y. \tag{1.5}$$

Since the pairing

$$\bigwedge^i E \times \bigwedge^{r-i} E \longrightarrow \det E$$

is nondegenerate for every i, we have

Lemma 1.6 *The complex $K^\bullet$ is isomorphic to $\Lambda^\bullet \otimes (det E)^{-1}$.*

Let $\pi : \mathbf{P}(E) \longrightarrow X$ be the $\mathbf{P}^{r-1}$-bundle associated to E in the sense of Grothendieck and $\mathcal{O}_{\mathbf{P}}(1)$ the tautological line bundle on it. By the construction of $\mathbf{P}(E)$, we have the canonical isomorphisms

$$\pi_* \mathcal{O}_{\mathbf{P}}(1) \simeq E \quad \textit{and} \quad H^0(\mathbf{P}(E), \mathcal{O}(1)) \simeq H^0(U, E). \tag{1.7}$$

The two linear systems associated to E and $\mathcal{O}_{\mathbf{P}}(1)$ have several common properties.

Proposition 1.8 *A vector bundle E is generated by its global section if and only if the tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$ is so.*

Proposition 1.9 *Let σ be the global section of the tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$ corresponding to $s \in H^0(U, E)$ via (1.7). If U is smooth, then the following are equivalent:*
i) s is nondegenerate and $(s)_0$ is smooth, and
ii) the divisor $(\sigma)_0 \subset \mathbf{P}(E)$ is smooth.

Proof. Since the assertion is local, we may assume E is trivial, i.e., $E \simeq \mathcal{O}_U^{\oplus r}$. Let $f_1, \cdots, f_r$ be a set of generators of the ideal I defining $(s)_0$. A point $x \in (s)_0$ is singular if and only if $df_1, \cdots, df_r$ are linearly dependent at x, that is, there is a set of constants $(a_1, \cdots, a_r) \neq (0, \cdots, 0)$ such that $a_1 df_1 + \cdots + a_r df_r = 0$ at x. This condition is equivalent to the condition that the divisor

$$(\sigma)_0 : f_1 X_1 + \cdots + f_r X_r = 0$$

in $\mathbf{P}(E) \simeq U \times \mathbf{P}^{r-1}$ is singular at $x \times (a_1 : \cdots : a_r)$. Therefore, ii) implies i). Since $(\sigma)_0$ is smooth off $(s)_0 \times \mathbf{P}^{r-1}$, i) implies ii). *q.e.d.*

By these two propositions, Bertini's theorem (see [7, p. 137]) is generalized for vector bundles:

Theorem 1.10 *Let E be a vector bundle on a smooth variety. If E is generated by its global sections, then every general global section is nondegenerate and its scheme of zeroes is smooth.*

2 A homogeneous space of $SO(9, \mathbf{C})$

Let X be the subvariety of $Grass(\mathbf{P}^2 \subset \mathbf{P}^7)$ consisting of 2-planes in the smooth 7-dimensional hyperquadric

$$Q^7 : q(X) = X_1X_5 + X_2X_6 + X_3X_7 + X_4X_8 + X_9^2 = 0 \tag{2.1}$$

in $\mathbf{P}^8$. The special orthogonal group $G = SO(9, \mathbf{C})$ acts transitively on X. Let P be the stabilizer group at the 2-plane $X_4 = X_5 = \cdots = X_9 = 0$. X is isomorphic to G/P and the reductive part L of P consists of matrices

$$\begin{pmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & B \end{pmatrix}$$

with $A \in GL(3, \mathbf{C})$ and $B \in SO(3, \mathbf{C})$. We denote the diagonal matrix

$$[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}, 1]$$

by $< x_1, x_2, x_3, x_4 >$. All the invertible diagonal matrices $< x_1, x_2, x_3, x_4 >$ form a maximal torus H of G contained in L. Let $X(H) \simeq \mathbf{Z}^{\oplus 4}$ be the character group and $\{e_1, e_2, e_3, e_4\}$ its standard basis. For a character $\alpha = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, let $\underline{g}^\alpha$ be the α-eigenspace $\{Z : Ad < x_1, x_2, x_3, x_4 > \cdot Z = x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} Z\}$ of the adjoint action Ad on the Lie algebra $\underline{g}$ of G. Then we have the well-known decomposition

$$\underline{g} = \underline{h} \oplus \bigoplus_{0 \neq \alpha \in X(H)} \underline{g}^\alpha.$$

A character α is a *root* if $\underline{g}^\alpha \neq 0$. In our case, there are 16 positive roots

$$e_1, e_2, e_3, e_4 \text{ and } e_i \pm e_j (1 \leq i \leq j \leq 4)$$

ant their negatives. The basis of roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4 \text{ and } \alpha_4 = e_4.$$

Since e_1, e_2, e_3 and e_4 are orthonormal with respect to the Killing form, the Dynkin diagram of G is of type B_2 (see (0.1)). The fundamental weights are

(2.2)
$$\begin{aligned} w_1 &= e_1, w_2 = e_1 + e_2, w_3 = e_1 + e_2 + e_3 \textit{ and} \\ w_4 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \end{aligned}$$

(Cf. [4]). The positive roots of $L \simeq GL(3, \mathbf{C}) \times SO(3, \mathbf{C})$ with respect to H are

(2.3)
$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2 \textit{ and } \alpha_4.$$

The root basis of L is of type $A_2 \coprod A_1$ and the Weyl group W' is a dihedral group of order 12. There are 12 positive roots other than (2.3) and their sum is equal to $5(e_1 + e_2 + e_3) = 5w_3$. Hence by [2, §16], we have

Proposition 2.4 *X is a 12-dimensional Fano manifold of index 5.*

Let $\tilde{G}$ be the universal covering group of G and $\tilde{H}$ and $\tilde{L}$ the pull-back of H and L, respectively. The character group $X(\tilde{H})$ of $\tilde{H}$ is canonically isomorphic to the weight lattice. Let $\rho_i, 1 \leq i \leq 4$, be the irreducible representation of $\tilde{L}$ with the highest weight w_i. Since the W'-orbit of w_4 consists of two weights $p_+ = w_4$ and $p_- = w_4 - e_4$ and since p_- is the reflection of p_+ by e_4, the representation ρ_4 is of dimension 2. ρ_1 is induced from the vector representation of the $GL(3, \mathbf{C})$-factor of L. From the equality

$$e_1 + e_2 + e_3 = p_+ + p_- = w_3,$$

we obtain the isomorphism

(2.5)
$$\bigwedge^3 \rho_1 \simeq \bigwedge^2 \rho_4 \simeq \rho_3.$$

Let $\mathcal{E}$ (resp. $\mathcal{O}_X(1), \mathcal{F}$) be the homogeneous vector bundle over $X = G/P$ induced from the representation ρ_1 (resp. ρ_3, ρ_4). $\mathcal{O}_X(1)$ is the positive generator of $\operatorname{Pic} X$. $\mathcal{E}$ and $\mathcal{F}$ are of rank 2 and 3, respectively. By the above isomorphism, we have

(2.6)
$$\bigwedge^3 \mathcal{E} \simeq \bigwedge^2 \mathcal{F} \simeq \mathcal{O}_X(1).$$

We shall study the property of the zero locus of a glogal section of $\mathcal{F}^{\oplus 5}$ in Section 3. For its study we need vanishing of cohomology groups of homogeneous vector bundles on X and apply the theorem of Bott[3]. The sum δ of the four fundamental weights w_i in (2.2) is equal to $(7e_1 + 5e_2 + 3e_3 + e_4)/2$. The sum of all positive roots is equal to 2δ.

Theorem 2.7 *Let $\mathcal{E}(w)$ bethe homogeneous vector bundles on $X = \tilde{G}/\tilde{P}$ induced from the representation of $\tilde{L}$ with the heighest weight $w \in X(\tilde{H})$. Then we have*

1) $H^i(X, \mathcal{E}(w))$ vanishes for every i if there is a root α with $(\alpha.\delta + w) = 0$, and

2) Let i_0 be the number of positive roots α with $(\alpha.\delta + w) < 0$. Then $H^i(X, \mathcal{E}(w))$ vanishes for every i except i_0.

We apply the theorem to the following four cases:
1) $w = jw_3 + nw_4$ and $\mathcal{E}(w) \simeq S^n\mathcal{F}(j)$ for $n \le 6$,
2) $w = w_1 + jw_3 + nw_4$ and $\mathcal{E}(w) \simeq \mathcal{E} \otimes S^n\mathcal{F}(j)$ for $n \le 5$,
3) $w = 2w_1 + jw_3 + nw_4$ and $\mathcal{E}(w) \simeq S^2\mathcal{E} \otimes S^n\mathcal{F}(j)$ for $n \le 5$, and
4) $w = w_1 + w_2 + (j-1)w_3 + nw_4$ and $\mathcal{E}(w) \simeq sl(\mathcal{E}) \otimes S^n\mathcal{F}(j)$ for $n \le 5$.

Proposition 2.8 *(1) The cohomology group $H^i(X, S^n\mathcal{F}(j))$, vanishes for every (i,n,j) with $0 \le n \le 6$ except the following:*

i	0	3	9	12
n	n	6	6	n
j	≥ 0	-4	-7	$\le -n-5$

(2) The cohomology group $H^i(X, \mathcal{E} \otimes S^n\mathcal{F}(j))$, vanishes for every (i,n,j) with $0 \le n \le 5$ except the following:

i	0	2	2	11	11	11	11	12
n	n	4	5	2	3	4	5	n
j	≥ 0	-3	-3	-6	-7	-9	-9	$\le -n-6$

(3) The cohomology group $H^i(X, S^2\mathcal{E} \otimes S^n\mathcal{F}(j))$, vanishes for every (i,n,j) with $0 \le n \le 5$ except the following:

i	0	2	2	10	11	11	11	11	11	12
n	n	4	5	2	3	4	4	5	5	n
j	≥ 1	-3	-3	-6	-8	-8	-9	-9	-10	$\le -n-7$

(4) The cohomology group $H^i(X, sl(\mathcal{E}) \otimes S^n\mathcal{F}(j))$, vanishes for every (i,n,j) with $0 \le n \le 5$ except the following:

i	0	1	1	1	1	3	9	11	11	11	11	12
n	n	2	3	4	5	4	4	2	3	4	5	n
j	≥ 1	-1	-1	-1	-1	-3	-6	-6	-7	-8	-9	$\le -n-6$

3 K3 surfaces of genus 18

In this section, we prove Theorem 0.2 and prepare the proof of Theorem 0.3. Let X, $\mathcal{E}$ and $\mathcal{F}$ be as in Section 2. Let N be a 5-dimensional subspace of $H^0(X, \mathcal{F})$ and $\{s_1, \cdots, s_5\}$ a basis of N. The common zero locus $S_N \subset X$ of N coincides with the zero locus of the global section $s = (s_1, \cdots, s_5)$ of $\mathcal{F}^{\oplus 5}$. Let Ξ_{SCI} be the subset of $G(5, H^0(X, \mathcal{F}))$ consisting of $[N]$ such that S_N is smooth and of dimension 2. $\mathcal{F}$ is generated by its global section and $\dim X - r(\mathcal{F}^{\oplus 5}) = 2$. Hence by Theorem 1.10, we have

Proposition 3.1 *Ξ_{SCI} is a non-empty (Zariski) open subset of $G(5, H^0(X, \mathcal{F}))$.*

We compute the cohomology groups of vector bundles on S_N, $\dim S_N = 2$, using the Koszul complex

$$\Lambda^\bullet : \quad \mathcal{O}_X \longrightarrow \mathcal{F}^{\oplus 5} \longrightarrow \bigwedge^2(\mathcal{F}^{\oplus 5}) \longrightarrow \cdots \longrightarrow \bigwedge^9(\mathcal{F}^{\oplus 5}) \longrightarrow \bigwedge^{10}(\mathcal{F}^{\oplus 5}). \tag{3.2}$$

The terms $\Lambda^i = \bigwedge^i(\mathcal{F}^{\oplus 5})$ of this complex have the following symmetry:

Lemma 3.3 $\Lambda^i \simeq \Lambda^{10-i} \otimes \mathcal{O}_X(i-5)$.

Proof. By (1.6), Λ^i is isomorphic to $(\Lambda^{10-i})^\vee \otimes \mathcal{O}_X(5)$. Since $\mathcal{F}$ is of rank 2, $\mathcal{F}^\vee$ is isomorphic to $\mathcal{F} \otimes \mathcal{O}_X(-1)$, which shows the lemma. *q.e.d.*

We need the decomposition of Λ^i into irreducible homogeneous vector bundles. $\Lambda^i = \bigwedge^i(\mathcal{F}^{\oplus 5})$, $i \leq 5$, has the following vector bundles as its irreducible factors:

(3.4)

Λ^0	$\mathcal{O}$
Λ^1	$\mathcal{F}$

Λ^2	$S^2\mathcal{F}, \mathcal{O}(1)$
Λ^3	$S^3\mathcal{F}, \mathcal{F}(1)$

Λ^4	$S^4\mathcal{F}, S^2\mathcal{F}(1), \mathcal{O}(2)$
Λ^5	$S^5\mathcal{F}, S^3\mathcal{F}(1), \mathcal{F}(2)$

Proposition 3.5 *If* $[N] \in \Xi_{SCI}$, *then* S_N *is a K3 surface.*

Proof. By Proposition 2.4 and (2.6), the vector bundle $\mathcal{F}^{\oplus 5}$ and the tangent bundle T_X have the same determinant bundle. Hence S_N has a trivial canonical bundle. By Proposition 1.4 and Lemma 1.6, we have the exact sequence $0 \longrightarrow K^\bullet \longrightarrow \mathcal{O}_{S_N} \longrightarrow 0$ and the isomorphism $K^\bullet \simeq \Lambda^\bullet \otimes \mathcal{O}(-5)$. By (1) of Proposition 2.8 and the Serre duality $H^i(K^{10}) \simeq H^{12-i}(\mathcal{O}_X)^\vee$, we have

$$H^1(K^1) = H^2(K^2) = \cdots = H^{10}(K^{10}) = 0$$

and

$$H^1(K^0) = H^2(K^1) = \cdots = H^{10}(K^9) = H^{11}(K^{10}) = 0.$$

Therefore, the restriction map $H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_{S_N})$ is surjective and $H^1(\mathcal{O}_{S_N})$ vanishes. Hence S_N is connected and regular. *q.e.d.*

Let F_N be the restriction of $\mathcal{F}$ to S_N. The complex $K^\bullet \otimes \mathcal{F}$ gives a resolution of F_N. Since $S^n\mathcal{F} \otimes \mathcal{F} \simeq S^{n+1}\mathcal{F} \oplus S^{n-1}\mathcal{F}(1)$, we have the following 4 series of vanishings by (1) of Proposition 2.8:

(a) $H^1(\mathcal{F} \otimes K^2) = H^2(\mathcal{F} \otimes K^3) = \cdots = H^9(\mathcal{F} \otimes K^{10}) = 0$,

(b) $H^1(\mathcal{F} \otimes K^1) = H^2(\mathcal{F} \otimes K^2) = \cdots = H^9(\mathcal{F} \otimes K^9) = H^{10}(\mathcal{F} \otimes K^{10}) = 0$,

(c) $H^1(\mathcal{F}) = H^2(\mathcal{F} \otimes K^1) = \cdots = H^{10}(\mathcal{F} \otimes K^9) = H^{11}(\mathcal{F} \otimes K^{10}) = 0$

and

(d) $H^2(\mathcal{F}) = H^3(\mathcal{F} \otimes K^1) = \cdots = H^{11}(\mathcal{F} \otimes K^9) = H^{12}(\mathcal{F} \otimes K^{10}) = 0$.

By (c) and (d), both $H^1(F_N)$ and $H^2(F_N)$ vanish. By (a) and (b), the sequence

$$0 \longrightarrow H^0(\mathcal{F} \otimes K^1) \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(F_N) \longrightarrow 0$$

is exact. So we have proved

Proposition 3.6 *If* $\dim S_N = 2$, *then we have*

1) $H^1(S_N, F_N) = H^2(S_N, F_N) = 0$, *and*

2) the restriction map $H^0(X, \mathcal{F}) \longrightarrow H^0(S_N, F_N)$ *is surjective and its kernel coincides with* N.

Let E_N be the restriction of $\mathcal{E}$ to S_N. Arguing similarly for the complexes $K^\bullet \otimes \mathcal{E}$, $K^\bullet \otimes S^2\mathcal{E}$ and $K^\bullet \otimes sl(\mathcal{E})$, we have

Proposition 3.7 *If* $\dim S_N = 2$, *then we have*

(1) all the higher cohomology groups of E_N *and* S^2E_N *vanish,*

(2) the restriction maps $H^0(X, \mathcal{E}) \longrightarrow H^0(S_N, E_N)$ *and* $H^0(X, S^2\mathcal{E}) \longrightarrow H^0(S_N, S^2E_N)$ *are bijective, and*

(3) all the cohomology groups of $sl(E_N)$ *vanish.*

Corollary 3.8 *The natural map* $S^2H^0(S_N, E_N) \longrightarrow H^0(S_N, S^2E_N)$ *is surjective and its kernel is generated by a nondegenerate symmetric tensor.*

Proof. The assertion holds for the pair of X and $\mathcal{E}$ since $H^0(X, S^2\mathcal{E})$ is an irreducible representation of G. Hence it also holds for the pair S_N and E_N by (2) of the proposition. *q.e.d.*

Corollary 3.9 $\chi(E_N) = \dim H^0(X, \mathcal{E}) = 9$ *and* $\chi(sl(E_N)) = 0$.

Theorem 0.2 is a consequence of Proposition 3.5 and the following:

Proposition 3.10 *The self intersection number of* $c_1(E_N)$ *is equal to 34.*

Proof. By (3.9), and the Riemann-Roch theorem, we have

$$9 = \chi(E_N) = (c_1(E_N)^2)/2 - c_2(E_N) + 3 \cdot 2$$

and

$$0 = \chi(sl(E_N)) = -c_2(sl(E_N)) + 8 \cdot 2 = 2(c_1(E_N)^2) - 6c_2(E_N) + 16,$$

which imply $(c_1(E_N)^2) = 34$ and $c_2(E_N) = 14$. *q.e.d.*

4 Proof of Theorem 0.3

We need the following general fact on deformations of vector bundles on K3 surfaces, which is implicit in [10].

Proposition 4.1 *Let* E *be a simple vector bundle on a K3 surface* S *and* (S', L') *be a small deformation of* $(S, \det L)$. *Then there is a deformation* (S', E') *of the pair* (S, E) *such that* $\det E' \simeq L'$.

Proof. The obstruction $ob(E)$ for E to deform a vector bundle on S' is contained in $H^2(S, \mathcal{E}nd(E))$. Its trace is the obstruction for $\det E$ to deform a line bundle on S', which is zero by assumption. Since the trace map $H^2(S, \mathcal{E}nd(E)) \longrightarrow H^2(S, \mathcal{O}_S)$ is injective, the obstruction $ob(E)$ itself is zero. *q.e.d.*

We fix a 5-dimensional subspace N of $H^0(X, \mathcal{F}) \simeq U^{16}$ belonging to Ξ_{SCI} and consider deformations of the polarized K3 surface $(S_N, \mathcal{O}_S(1))$, where $\mathcal{O}_S(1)$ is the restriction of $\mathcal{O}_X(1)$ to S_N.

Proposition 4.2 *Let* (S, L) *be a sufficiently small deformation of* $(S_N, \mathcal{O}_S(1))$. *Then there exists a vector bundle* E *on* S *which satisfies the following:*

i) $\det E \simeq L$,

ii) The pair (S, E) *is a deformation of* (S_N, E_N),

iii) E *is generated by its global sections and* $H^1(S, E) = H^2(S, E) = 0$,

iv) the natural linear map

$$S^2H^0(S, E) \longrightarrow H^0(S, S^2E)$$

is surjective and its kernel is generated by a nondegenerate symmetric tensor, and

v) the morphism $\Phi_{|E|} : S \longrightarrow G(H^0(E), 3)$ *associated to* E *is an embedding.*

Proof. The existence of E which satisfies i) and ii) follows from Proposition 4.1. The pair (S_N, E_N) satifies iii) and iv) by Proposition 3.7and Corollary 3.8. Since (S, E) is a small deformation of (S_N, E_N), E satisfies iii) and v). Since $H^1(S_N, S^2E_N)$ vanishes, E satisfies iv), too. *q.e.d.*

By iii) of the proposition, we identify S with its image in $G(H^0(E), 3)$. By iv) of the proposition, (the image of) S lies in the 12-dimensional homogeneous space X of $SO(9, \mathbf{C})$. By Proposition 3.6, S is contained in the common zero locus of a 5-dimensional subspace N' of $H^0(X, \mathcal{F})$. Since S is a small deformation of S_N, S is also a complete intersection with respect to $\mathcal{F}^{\oplus 5}$. Therefore, we have shown

Proposition 4.3 *The image of the classification morphism* $\Xi_{SCI} \longrightarrow \mathcal{F}_{18}$, $[N] \mapsto (S_N, \mathcal{O}_S(1))$, *is open.*

The Picard group of a K3 surface S is isomorphic to $H^{1,1}(S, \mathbf{Z}) = H^2(S, \mathbf{Z}) \cap H^0(\Omega^2)^{\perp}$. Since the local Torelli type theorem holds for the period map of K3 surfaces ([1, §7, Chap. VIII]), the subset $\{(S, L) | \mathrm{Pic}\, S \neq \mathbf{Z} \cdot [L]\}$ of $\mathcal{F}_g$ is a countable union of subvarieties. Hence by the proposition and Baire's property, we have

Proposition 4.4 *There exists* $[N] \in \Xi_{SCI}$ *such that* $(S_N, \mathcal{O}_S(1))$ *is Picard general, i.e.,* $\mathrm{Pic}\, S_N$ *is generated by* $\mathcal{O}_S(1)$.

The stability of vector bundles is easy to check over a Picard general variety.

Proposition 4.5 *If* $(S_N, \mathcal{O}_S(1))$ *is Picard general, then* E_N *is* μ*-stable with respect to* $\mathcal{O}_S(1)$.

Proof. Let B be a locally free subsheef of E_N. By our assumption, $\det B$ is isomorphic to $\mathcal{O}_S(b)$ for an integer b. In the case B is a line bundle, we have $b \leq 0$ since $\dim H^0(B) \leq \dim H^0(E_N) = 9$. In the case B is of rank 2, $\bigwedge^2 B \simeq \mathcal{O}_S(b)$ is a subsheaf of $\bigwedge^2 E_N \simeq E_N^{\vee} \otimes \mathcal{O}_S(1)$. We have $H^0(S_N, E_N^{\vee}) = H^2(S_N, E_N)^{\vee} = 0$ by (1) of Proposition 3.7 and the Serre duality. Hence b is nonpositive. Therefore we have $b/r(B) < 1/3$ for every B with $r(B) < r(E_N) = 3$. If F is a subsheaf of E_N, then its double dual $F^{\vee\vee}$ is a locally free subsheaf of E_N. Hence $c_1(F)/r(F) < c_1(E_N)/r(E_N)$ for every subsheaf F of E_N with $0 < r(F) < r(E_N)$. *q.e.d.*

Let Ξ be the subset of Ξ_{SCI} consisting of $[N]$ such that E_N is stable with respect to $\mathcal{O}_S(1)$ in the sense of Gieseker [6]. Ξ is non-empty by the above two propositions.

Theorem 4.6 *Let* M *and* N *be 5-dimensional subspaces of* $H^0(X, \mathcal{F})$ *with* $[M], [N] \in \Xi$.

(1) If $(S_M, \mathcal{O}_S(1))$ *and* $(S_N, \mathcal{O}_S(1))$ *are isomorphic to each other, then* $[M]$ *and* $[N]$ *belong to the same* $SO(9, \mathbf{C})$*-orbit.*

(2) The automorphism group of $(S_N, \mathcal{O}_S(1))$ *is isomorphic to the stabilizer group of* $SO(9, \mathbf{C})$ *at* $[N] \in G(5, U^{16})$.

Proof. Let $\phi : S_M \longrightarrow S_N$ be an isomorphism such that $\phi^*\mathcal{O}_S(1) \simeq \mathcal{O}_S(1)$. Two vector bundles E_M and ϕ^*E_N have the same rank and Chern classes. Since E_N is rigid by (3) of Proposition 3.7 and since both are stable, there exists an isomorphism $f : E_M \longrightarrow \phi^*E_N$ by Proposition 0.4. By (2) of Proposition 3.7,

$$H^0(f) : H^0(S_M, E_M) \longrightarrow H^0(S_M, \phi^*E_N) \simeq H^0(S_N, E_N)$$

is an automorphism of $V = H^0(X, \mathcal{E})$. By Corollary 3.8, $H^0(f)$ preserves the 1-dimensional subspace $\mathbf{C}q$ of S^2V. Hence replacing f by cf for suitable constant c, we may assume that $H^0(f)$ belongs to the special orthogonal group $SO(V, q)$. Let L be a lift of $H^0(f)$ to $\tilde{G} = Spin(V, q)$. Since $\mathcal{F}$ is homogeneous, there exists an isomorphism $\ell : F_M \longrightarrow \phi^* F_N$ such that $H^0(\ell) = L$. By (2) of Proposition 3.6, L maps $M \subset H^0(X, \mathcal{F})$ onto $N \subset H^0(X, \mathcal{F})$, which shows (1). Putting $M = N$ in this argument, we have (2). *q.e.d.*

Let $\Phi : \Xi \longrightarrow \mathcal{F}_{18}, \quad [N] \mapsto (S_N, \mathcal{O}_S(1))$ be the classification morphism. By the theorem, every fibre of Φ is an orbit of $SO(9, \mathbf{C})$. By the openness of stability condition ([8]) and Proposition 4.3, the image of Φ is Zariski open in $\mathcal{F}_{18}$. Hence we have completed thr proof of Theorem 0.3.

5 K3 surfaces of genus 20

Let V be a vector space of dimension 9 and $\mathcal{E}$ the (rank 4) universal quotient bundle on the Grassmann variety $X = G(V, 4)$. The determinant bundle of $\mathcal{E}$ is the ample generator $\mathcal{O}_X(1)$ of $\operatorname{Pic} X \simeq \mathbf{Z}$. We denote the restrictions of $\mathcal{E}$ and $\mathcal{O}_X(1)$ to S_N by E_N and $\mathcal{O}_S(1)$, respectively.

Theorem 5.1 *Let N be a 3-dimensional subspace of $H^0(G(V, 4), \bigwedge^2 \mathcal{E}) \simeq \bigwedge^2 V$ and $S_N \subset G(V, 4)$ the common zero locus of N. If S_N is smooth and of dimension 2, then the pair $(S_N, \mathcal{O}_S(1))$ is a K3 surface of genus 20.*

The tangent bundle T_X of $G(V, 4)$ is isomorphic to $\mathcal{E} \otimes \mathcal{F}$, where $\mathcal{F}$ is the dual of the universal subbundle. Hence $X = G(V, 4)$ is a 20-dimensional Fano manifold of index 9. S_N is the scheme of zeroes of the section

$$s : \mathcal{O}_X \longrightarrow (\bigwedge^2 \mathcal{E}) \otimes_{\mathbf{C}} N^\vee \simeq (\bigwedge^2 \mathcal{E})^{\oplus 3}$$

induced by $\mathcal{O}_X \otimes_{\mathbf{C}} N \longrightarrow \bigwedge^2 \mathcal{E}$. The vector bundle $(\bigwedge^2 \mathcal{E})^{\oplus 3}$ is of rank $6 \cdot 3 = 18$, and has the same determinant as T_X. Hence, by Theorem 1.10 and Proposition 1.4, we have

Proposition 5.2 *Let Ξ_{SCI} be the subset of $G(3, \bigwedge^2 V)$ consisting of $[N]$ such that S_N is smooth and of dimension 2. Then Ξ_{SCI} is non-empty and S_N has trivial canonical bundle for every $[N] \in \Xi_{SCI}$.*

We show vanishing of cohomology groups of vector bundles on S_N using the Koszul complex
(5.3)

$$\Lambda^\bullet : \quad \mathcal{O}_X \longrightarrow (\bigwedge^2 \mathcal{E})^{\oplus 3} \longrightarrow \bigwedge^2 (\bigwedge^2 \mathcal{E})^{\oplus 3} \longrightarrow \cdots \longrightarrow \bigwedge^{17} (\bigwedge^2 \mathcal{E})^{\oplus 3} \longrightarrow \bigwedge^{18} (\bigwedge^2 \mathcal{E})^{\oplus 3}$$

of s and the Bott vanishing. $G(V, 4)$ is a homogeneous space of $GL(9, \mathbf{C})$. The stabilizer group P consists of the matrices of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with $A \in GL(4, \mathbf{C}), B \in M_{4,5}(\mathbf{C})$ and $D \in GL(5, \mathbf{C})$. The set H of invertible diagonal matrices is a maximal torus. The roots of $GL(9, \mathbf{C})$ are $e_i - e_j, i \neq j$, for the standard basis of the character group $X(H)$ of H. We take $\Delta = \{e_i - e_{i+1}\}_{1 \leq i \leq 8}$ as a root basis. The reductive part L of P is isomorphic to $GL(4, \mathbf{C}) \times GL(5, \mathbf{C})$ and its root basi is $\Delta \setminus \{e_4 - e_5\}$. Let $\rho(a_1, a_2, a_3, a_4)$ be the irreducible representation of $GL(4, \mathbf{C})$ (or L) with the heighest weight

$w = (a_1, a_2, a_3, a_4) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4, a_1 \geq a_2 \geq a_3 \geq a_4$. We denote by $\mathcal{E}(a_1, a_2, a_3, a_4)$ the homogeneus vector bundle on X induced from the representation $\rho(a_1, a_2, a_3, a_4)$. The universal quotient bundle $\mathcal{E}$ is $\mathcal{E}(1, 0, 0, 0)$ and its exterior products $\bigwedge^2 \mathcal{E}$, $\bigwedge^3 \mathcal{E}$ and $\bigwedge^4 \mathcal{E}$ are $\mathcal{E}(1, 1, 0, 0)$, $\mathcal{E}(1, 1, 1, 0)$ and $\mathcal{E}(1, 1, 1, 1)$, respectively.

We apply the Bott vanishing theorem ([3]). One half δ of the sum of all the positive roots is equal to $4e_1 + 3e_2 + 2e_3 + e_4 - e_6 - 2e_7 - 3e_8 - 4e_4$ and we have

$$\delta + w = (4 + a_1)e_1 + (3 + a_2)e_2 + (2 + a_3)e_3 + (1 + a_4)e_4 - e_6 - 2e_7 - 3e_8 - 4e_9.$$

All the cohomology groups of $\mathcal{E}(a_1, a_2, a_3, a_4)$ vanish if a number appears more than once among the coeficients. For the convenience of later use we state the vanishing theorem for $\mathcal{E}(a_1, a_2, a_3, a_4) \otimes \mathcal{O}_X(-9)$:

Proposition 5.4 *The cohomology group $H^i(X, \mathcal{E}(a_1, a_2, a_3, a_4) \otimes \mathcal{O}_X(-9))$ vanishes for every i if one of the following holds:*
i) $\lambda \leq a_\lambda \leq \lambda + 4$ for some $1 \leq \lambda \leq 4$, or
ii) $a_\mu - a_\nu = \mu - \nu$ for some pair $\mu \neq \nu$.

To apply this to the Koszul complex (5.3), we need the decomposition of $\Lambda^i = \bigwedge^i (\bigwedge^2 \mathcal{E})^{\oplus 3}$ into the sum of irreducible homogeneous vector bundles. Since $(\bigwedge^2 \mathcal{E})^\vee \simeq (\bigwedge^2 \mathcal{E})(-1)$, we have the following in the same manner as Lemma 3.3:

Lemma 5.5 $\Lambda^i \simeq \Lambda^{18-i} \otimes \mathcal{O}_X(i - 9)$.

Put $\rho_2 = \rho(1, 1, 0, 0)$. It is easy to check the following:

i	2	3	4	5	6
$\bigwedge^i \rho_2$	$\rho(2,1,1,0)$	$\rho(3,1,1,1)$ $\oplus\rho(2,2,2,0)$	$\rho(3,2,2,1)$	$\rho(3,3,2,2)$	$\rho(3,3,3,3)$

The representation $\bigwedge^i (\rho_2^{\oplus 3})$ is isomorphic to

$$\bigoplus_{p+q+r=i} (\bigwedge^p \rho_2) \otimes (\bigwedge^q \rho_2) \otimes (\bigwedge^r \rho_2).$$

By the computation using the Littlewood-Richardson rule ([9, Chap. I, §9]), we have

Proposition 5.6 *The set W_i of the highest weights of irreducible components of the representation $\bigwedge^i \rho_2^{\oplus 3}$ is as follows:*

i	W_i
1	$\{(1,1,0,0)\}$
2	$\{(2,2,0,0),(2,1,1,0),(1,1,1,1)\}$
3	$\{(3,3,0,0),(3,2,1,0),(3,1,1,1),(2,2,2,0),(2,2,1,1)\}$
4	$\{(4,3,1,0),(4,2,2,0),(4,2,1,1),(3,3,2,0),(3,3,1,1),(3,2,2,1),(2,2,2,2)\}$
5	$\{(5,3,2,0),(5,3,1,1),(5,2,2,1),(4,4,2,0),(4,3,3,0)\} \cup W_3 + (1,1,1,1)$
6	$\{(6,3,3,0),(6,3,2,1),(6,2,2,2),(5,4,3,0),(4,4,4,0)\} \cup W_4 + (1,1,1,1)$
7	$\{(7,3,3,1),(7,3,2,2),(6,4,4,0),(5,5,4,0)\} \cup W_5 + (1,1,1,1)$
8	$\{(8,3,3,2),(6,5,5,0)\} \cup W_6 + (1,1,1,1)$
9	$\{(9,3,3,3),(6,6,6,0)\} \cup W_7 + (1,1,1,1)$

The sets of the highest weights appearing in the decompositions of $\mathcal{E} \otimes \Lambda^\bullet$, $\bigwedge^2 \mathcal{E} \otimes \Lambda^\bullet$ and $\mathcal{E} \otimes \mathcal{E}^\vee \otimes \Lambda^\bullet$ are easily computed from Lemma 5.5 and the proposition using the following formula:

$$\rho(1,0,0,0)\otimes\rho(a_1,a_2,a_3,a_4)=\bigoplus_{\substack{\sum b_i=1+\sum a_i\\ a_i\le b_i\le a_i+1}}\rho(b_1,b_2,b_3,b_4),$$

$$\rho(1,1,0,0)\otimes\rho(a_1,a_2,a_3,a_4)=\bigoplus_{\substack{\sum b_i=2+\sum a_i\\ a_i\le b_i\le a_i+1}}\rho(b_1,b_2,b_3,b_4)$$

and

$$\rho(0,0,0,-1)\otimes\rho(a_1,a_2,a_3,a_4)=\bigoplus_{\substack{\sum b_i=-1+\sum a_i\\ a_i-1\le b_i\le a_i}}\rho(b_1,b_2,b_3,b_4).$$

Applying Proposition 5.4 to the exact sequence

$$0\longrightarrow\Lambda^\bullet\otimes\mathcal{O}_X(-9)\longrightarrow\mathcal{O}_{S_N}\longrightarrow 0,$$

$$0\longrightarrow\Lambda^\bullet\otimes\mathcal{E}\otimes\mathcal{O}_X(-9)\longrightarrow E_N\longrightarrow 0,$$

$$0\longrightarrow\Lambda^\bullet\otimes\bigwedge^2\mathcal{E}\otimes\mathcal{O}_X(-9)\longrightarrow\bigwedge^2 E_N\longrightarrow 0$$

and

$$0\longrightarrow\Lambda^\bullet\otimes sl(\mathcal{E})\otimes\mathcal{O}_X(-9)\longrightarrow sl(E_N)\longrightarrow 0,$$

we have the following in a similar way to Propositions 3.6 and 3.7:

Proposition 5.7 *If* $\dim S_N=2$*, then we have*

(1) the restriction map $H^0(\mathcal{O}_X)\longrightarrow H^0(\mathcal{O}_{S_N})$ *is surjective and* $H^1(\mathcal{O}_{S_N})$ *vanishes,*

(2) all higher cohomology groups of E_N *and* $\bigwedge^2 E_N$ *vanish,*

(4) the restriction map $H^0(X,\mathcal{E})\longrightarrow H^0(S_N,E_N)$ *is bijective,*

(4) the restriction map $H^0(X,\bigwedge^2\mathcal{E})\longrightarrow H^0(S_N,\bigwedge^2 E_N)$ *is surjective and its kernel coincides with* N*, and*

(5) all cohomology groups of $sl(E_N)$ *vanish.*

Corollary 5.8 $\chi(E_N)=\dim H^0(X,\mathcal{E})=9$ *and* $\chi(sl(E_N))=0$*.*

If $[N]$ belongs to Ξ_{SCI}, then S_N is a K3 surface by Proposition 5.2 and (1) of Proposition 5.7. We have $(c_1(E_N)^2)=38$ by the corollary in a similar manner to Proposition 3.10. This proves Theorem 5.1.

Theorem 5.9 *Let* Ξ *be the subset of* $G(3,\bigwedge^2 V)$*,* $\dim V=9$*, consisting of* $[N]$ *such that* S_N *is a K3 surface and* E_N *is stable with respect to* L_N*. Then we have*

(1) Ξ *is a non-empty Zariski open subset,*

(2) the image of the classification morphism $\Phi:\Xi\longrightarrow\mathcal{F}_{20}$ *is open,*

(3) every fibre of Φ *is an orbit of* $PGL(V)$*, and*

(4) the automorphism group of (S_N,L_N) *is isomorphic to the stabilizer group of* $PGL(V)$ *at* $[N]\in G(3,\bigwedge^2 V)$*.*

There exists a 3-dimensional subspace N of $\bigwedge^2 V$ such that the polarlized K3 surface $(S_N,\mathcal{O}_S(1))$ is Picard general. Let B be a locally free subsheaf of E_N. In the cases $r(B)=1$ and 3, we have $b\le 0$ in the same way as Proposition 4.4. In the case B is of rank 2, $\bigwedge^2 B\simeq\mathcal{O}_S(b)$ is a subsheaf of $\bigwedge^2 E_N$. Since $\bigwedge^2 E_N\simeq\bigwedge^2 E_N^\vee\otimes\mathcal{O}_S(1)$, we have

$$\mathrm{Hom}\,(\mathcal{O}(1),\bigwedge^2 E_N)\simeq H^0(S_N,(\bigwedge^2 E_N)(-1))\simeq H^2(S_N,\bigwedge^2 E_N)^\vee=0$$

by the Serre duality and 2) of Proposition 5.7. Hence b is nonpositve. Therefore, E_N is μ-stable if $(S_N, \mathcal{O}_S(1))$ is Picard general. This shows (1) of the theorem. The rest of the proof of Theorem 5.9 is the same as that of Theorem 0.3 in Section 4.

Let $G(3, \bigwedge^2 V)^s$ be the stable part of $G(3, \bigwedge^2 V)$ with respect to the action of $SL(V)$.

Corollary 5.10 *The moduli space $\mathcal{F}_{20}$ of K3 surfaces of genus 20 is birationally equivallent to the moduli space $G(3, \bigwedge^2 V)^s/PGL(V)$ of nets of bivectors on V.*

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