Geometric realization of root systems and the Jacobians of del Pezzo surfaces

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In the conference talk, I reviewed the geometric part of my article [2] and announced the following result on the Jacobian of a del Pezzo surface, whose details will be published elsewhere.

A smooth complete algebraic surface $S$ (over an algebraically closed field) is del Pezzo if the anti-canonical class $-K_S$ is ample. The self-intersection number $(-K_S^2) =: d$ is called the degree, which ranges from 1 to 9. A del Pezzo surface $S$ is isomorphic to the projective plane $\mathbb{P}^2$ if $d = 9$, and to either a smooth quadric surface $Q$ or the blow-up of $\mathbb{P}^2$ at a point if $d = 8$. A del Pezzo surface $S$ of degree $d \leq 7$ is isomorphic to the blow-up of $\mathbb{P}^2$ at $(9 - d)$ points in a general position.

The anti-canonical system $|-K_S|$ is of dimension $d$ and its general member is a smooth elliptic curve. Let $C \subset S \times \mathbb{P}^d$ be the universal family of anti-canonical members $C \in |-K_S| = \mathbb{P}^d$ and $C_\eta$ be the generic fiber of $C \rightarrow \mathbb{P}^d$.

**Definition** A morphism $\varphi : \tilde{J} \rightarrow \mathbb{P}^d$ is a Jacobian fibration of $S$ if all fibers are of dimension one and its generic fiber is the Jacobian of the generic anti-canonical member $C_\eta$. A $(d + 1)$-dimensional variety $J$ with a smooth point $p$ is a reduced Jacobian of $S$ if the blow-up of $J$ at $p$ have a Jacobian fibration $\varphi$ such that the exceptional divisor over $p$ is the 0-section of $\varphi$.

In the case of degree $d = 1$, the anti-canonical system $|-K_S|$ is a pencil with a unique base point. Hence a del Pezzo surface $S$ itself is its reduced Jacobian.

**Theorem** For a del Pezzo surface $S$, there exists a reduced Jacobian $(J(S), p)$ which satisfies the following properties:

1. $J(S)$ is a $(d + 1)$-dimensional weak del Pezzo variety of degree one, that is, $-K_S = dH$, $H$ being a nef and big divisor with $(H^{d+1}) = 1$.

*Supported in part by the JSPS Grant-in-Aid for Scientific Research (B) 17340006.
(2) $p$ is the unique base point of the $d$-dimensional linear system $|H|$, 
(3) $J(S)$ is the blow-up of the projective space $\mathbb{P}^3$ at seven points in a general position if $d = 2$, 
(4) $J(S)$ is the blow-up of the product $\mathbb{P}^2 \times \mathbb{P}^2$ at five points in a general position if $d = 3$, 
(5) $J(S)$ is the blow-up of the 6-dimensional Grassmannian $G(2, 5)$ at four points $p_1, \ldots, p_4$ in a general position if $d = 5$, and 
(6) $J(S)$ is the blow-up of a singular hyperplane section $G(2, 5)'$ of $G(2, 5) \subset \mathbb{P}^9$ at four points $p_1, \ldots, p_4$ in a general position if $d = 4$.

In the case $d = 1, 2, 3$, the reduced Jacobian $J(S)$ belongs to the class of rational varieties studied in [2]. The augmented root system $\mathcal{N}(E_9-d, \text{adjoint})$ (cf. [3, §4] and [4, §4]) is realized in the second cohomology group $H^2(J(S), \mathbb{Z})$. The Weyl group $W(E_9-d)$ birationally acts on the universal family of $J(S)$ over the configuration space of $(9-d)$ points on $\mathbb{P}^2$. Similar properties hold true for a del Pezzo surface $S_d$ of degree $d = 5, 4$ with the following augmented root system.

\[
\begin{align*}
& e_1 - e_2 & e_2 - e_3 & e_3 - e_4 \quad & e_1 - e_2 & e_2 - e_3 & e_3 - e_4 & h_2 - e_1 - e_2 - e_3 \\
& h - 2e_1 - e_2 - e_3 - e_4 & e_4 & & h_1 - e_1 - e_2 & e_4 &
\end{align*}
\]

$d = 5, \quad (A_4, \text{adjoint}) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad d = 4, \quad (D_5, \text{adjoint})$

In these diagrams in $H^2(J(S_d), \mathbb{Z})$, $h$ denotes the pull back of a hyperplane section of the Plücker embedding $G(2, 5) \subset \mathbb{P}^9$, and $h_i$ denotes that of a divisor class of $G(2, 5)'$ with dim $|h_i| = i$. $e_1, \ldots, e_4$ are the classes of exceptional divisors over $p_1, \ldots, p_4$. The reflection with respect to the $(-2)$-class $h - 2e_1 - e_2 - e_3 - e_4 \in H^2(J(S_5), \mathbb{Z})$ is realized by the composite of two birational involutions of $J(S_5) = Bl_{p_1, \ldots, p_4} G(2, 5)$. One is the Geiser involution the blow-up of $G(2, 5)$ at $p_2, p_3$ and $p_4$, that is, the covering involution of the morphism

$$\Phi_{H - e_2 - e_3 - e_4} : Bl_{p_2, p_3, p_4} G(2, 5) \longrightarrow \mathbb{P}^6$$

of degree 2. The other is the Bertini involution, that is, the involution of $J(S_5)$ induced from the $(-1)$ of the elliptic fibration $\varphi : \tilde{J}(S_5) \longrightarrow \mathbb{P}^5$. (See [1, §7] for the Geiser and Bertini involutions of del Pezzo surfaces.)
References


