Enriques surfaces and root lattices
— Enriques surfaces of type $E_7$ —

Shigeru Mukai

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Enriques surface $S = X/\varepsilon = (\text{K3 surface})/ (\text{free involution})$

$\mathbb{Z}^\omega := (\mathbb{Z} \times X)/(-1, \varepsilon) \to S$ nontrivial local system on $S$

$H := H_S := H^2(S, \mathbb{Z}^\omega) \simeq \mathbb{Z}^{12}$: Hodge structure of weight 2, that is, $H_S \otimes \mathbb{Z} \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. Hodge $\# = (1, 10, 1)$

$H$ carries an integral symmetric bilinear form induced by $\mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}$. As a lattice, $H \simeq I_{2,10}$, odd unimodular lattice with signature $(2, 10)$. $H_S$ is a polarized Hodge structure.

**Twisted Picard group** $\text{Pic}^\omega S := H^2(S, \mathbb{Z}^\omega) \cap H^{1,1}$

is a negative definite lattice which does not contain $(-1)$ elements.

**Relation with traditional formulation**

$H^2(S, \mathbb{Z}^\omega) = \text{Ker}[H^2(X, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{Z})(\simeq \mathbb{Z}^{12} \oplus \mathbb{Z}/2)]$

$\text{Pic}^\omega S = \text{Ker}[\text{Pic} X \longrightarrow \text{Pic} S](1/2)$, twisted Picard $\# = \rho(X) - 10$. 
**Torelli type theorem** (in new formulation)

$S, S'$: two Enriques surfaces.

$H^2(S, \mathbb{Z}^\omega) \simeq H^2(S', \mathbb{Z}^\omega)$ as polarized Hodge structures $\Rightarrow S \simeq S'$.

In other words,

\[
\{\text{Enriques surface}\} / \text{isom.} \rightarrow D^{10}/O(I_{2,10}), \quad S \mapsto H_S,
\]

is injective (and almost surjective).

**Inverse Problem:** (Re)construct $S$ from its period $H_S$, or $\text{KS}(H_S)$, the Kuga-Satake abelian variety of dimension $2^{10}$.

Two parts: a) Construct the K3-cover $X = \tilde{S}$.

b) Construct the free involution $\varepsilon$.

Today I answer in the case of type $E_7$. In this case $\text{KS}(H_S)$ is isogeneous to the self product $A^{28}$ for an abelian surface $A$. Still both a) and b) are non-trivial.
Definition Let $L$ be a negative definite lattice which does not contain a $(-1)$ element. An Enriques surface $S$ is of (lattice) type $L$ if there exists a primitive embedding $L \rightarrow \text{Pic}^\omega S$.

Assume

(*) the primitive embeddings of $L$ into $I_{2,10}$ is unique

and let $M := L^\perp$ be the orthogonal complement.

By Torelli, the period map

\[ \{\text{Enriques surface of type } L\}/L\text{-isom.} \rightarrow D^{m-2}/O(M)', \]

$S \mapsto$ Hodge structure on $M$,

is injective, where $m$ is the rank of $M$,

\[ D^{m-2} = \{ z \in M \otimes \mathbb{C} | (z,z) = 0, (z, \overline{z}) > 0 \} \]

is the $(m - 2)$-dimensional bounded symmetric domain of type IV, on which the orthogonal group $O(M)$ acts, and $O(M)'$ is the image of $O(I_{2,10}, L) \rightarrow O(M)$. 
I consider the case where $L = E_7$. $E_7$ satisfies (*) and the orthogonal complement is $< 1, 1, -1, -1, -2 >$, that is, $\mathbb{Z}^5$ with inner product $\text{diag}[1, 1, -1, -1, -2]$. Moreover, $O(I_{2,10}, E_7) \rightarrow O(< 1, 1, -1, -1, -2 >)$ is surjective. Hence the period map

\{Enriques of type $E_7$\}/isom. $\rightarrow D^3/O(< 1, 1, -1, -1, -2 >)$

is injective.

**Lemma** (1) $D^3$ is the Siegel upper half space $H_2$ of degree 2, and $D^3/O(< 1, 1, -1, -1, -2 >)$ is the quotient of $H_2/\Gamma_0(2)$ by the Fricke (or Atkin-Lehner) involution.

(2) $H_2/\Gamma_0(2)$ is the moduli space of pairs $(A, G)$ of a principally polarized abelian surface $A$ and a Göpel subgroup $G \subset A_{(2)}$.

(3) The Fricke involution maps $(A, G)$ to $(A', G') := (A/G, A_{(2)}/G)$.
Inverse Problem for $E_7$

Construct an Enriques surface $S$ of type $E_7$ from $(A, G)$ corresponding to the period of $S$.

**Enriques surfaces of type $E_7$, $E_8$ and $(E_7 + A_1)^+$**

Consider the quartic surfaces in $\mathbb{P}^3_{x:y:z:t}$

$$\{a(xt + yz) + b(yt + xz) + c(zt + xy)\}^2 - xyzt = 0.$$  

for nonzero constants $a, b, c \in \mathbb{C}$. This is a K3 surface with 4 rational double points of type $D_4$. Let $X$ be the minimal resolution. Standard Cremona transformation

$$(x : y : z : t) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right)$$

induces an involution $\varepsilon$ of $X$, which is free if $\pm a \pm b \pm c \neq 1/2$. 
Projecting from (0001), we get a double plane expression

\[ X : \tau^2 = p\left(\frac{x}{z} + \frac{z}{x}\right) + q\left(\frac{y}{z} + \frac{z}{y}\right) + r\left(\frac{x}{y} + \frac{y}{x}\right) + s \]

for constants \( p, q, r, s \in \mathbb{C} \) with \( pqr \neq 0 \).

**Theorem** (1) The quotient of a quartic \( \{a(xt + yz) + b(yt + xz) + c(zt + xy)\}^2 - xyzt = 0 \) by Cremona is an Enriques of type \( E_7 \) (if \( \pm a \pm b \pm c \neq 1/2 \)).

(2) Every Enriques \( S \) of type \( E_7 \) is obtained in this way.

(3) The coefficient \( (p : q : r : s) \) is explicitly determined from the period \( H_S \) explicitly.
Remark on \( \{a(xt + yz) + b(yt + xz) + c(zt + xy)\}^2 - xyzt = 0 \)

(1) In characteristic 2, this is the equation of the Jacobian Kummer surface \( \text{Km}(\text{Jac}(C)) \) (Laszlo-Pauly).

(2) This K3 is not a Jacobian Kummer but \textit{isogeneous} to it.

(3) This deforms to the double covering of the quadric
\( Q : a(xt + yz) + b(yt + xz) + c(zt + xy) = 0 \) with branch
\( Q \cap \{xyzt = 0\} \). This is a product Kummer surface \( \text{Km}(E_1 \times E_2) \)
and its quotient by Cremona is an Enriques surface of type
\( (E_7 + A_1)^+ \).

(4) An Enriques surfrace becomes of type \( E_8 \) if one of \( p, q, r \) becomes 0 in the double plane expression
\[
\tau^2 = p\left(\frac{x}{z} + \frac{z}{x}\right) + q\left(\frac{y}{z} + \frac{z}{y}\right) + r\left(\frac{x}{y} + \frac{y}{x}\right) + s.
\]
How to recover $S$ from $H_S$

$(A, G)$ the pair corresponding to $H_S$

$A$ is a p.p.a.s. and $G = \{0, a, b, c\} \subset A_{(2)}$ is a Göpel.

I consider the case where $A$ is the Jacobian of a curve $C$ of genus 2. (When $A$ is product, then $S$ is $E_8$-type.)

$C$ is a double cover of $\mathbb{P}^1$ with 6 points $P_1, \ldots, P_6$. A 2-torsion point $a \neq 0 \in A_{(2)}$ is $\tilde{P}_{i(a)} + \tilde{P}_{j(a)} - K_C$ for different $i(a), j(a)$.

Regard $\mathbb{P}^1$ as a conic $Q$ in $\mathbb{P}^2_{u:v:w}$. Let $l_{i(a)} + l_{j(a)} : q_a(u, v, w) = 0$ be the sum of two tangent lines of $Q$ at $P_{i(a)}, P_{j(a)}$.

Then $\tilde{S}$ is the double plane

$$\tau^2 = \det(xq_a + yq_b + zq_c)/xyz$$

and $S$ is its quotient by $(x, y, \tau) \mapsto (1/x, 1/y, -\tau)$.

Remark (1) $G$ is Göpel $\Leftrightarrow \{i(a), j(a), \ldots, j(c)\} = \{1, \ldots, 6\}$. 
Remark (2) \((A, G)\) and \((A/G, A_{(2)}/G)\) give the same \(S\) by Richelot’s theorem. 6 points \(Q_1, \ldots, Q_6\) corresponding to \((A/G, A_{(2)}/G)\) is given by the diagram.

\(Q_1, \ldots, Q_6\) are not equivalent to \(P_1, \ldots, P_6\) in general. But their nets of conics have the same invariant.
Proof of Theorem

\( \text{Km}(A/G') \) is the double \( \mathbb{P}^2_{u:v:w} \) with equation \( \tau^2 = q_a q_b q_c \).

\( \text{Km}(A) \) is the \((2, 2, 2)\)-covering of \( \mathbb{P}^2_{u:v:w} \) with equation

\[
\tau_1^2 = q_a, \quad \tau_2^2 = q_b, \quad \tau_3^2 = q_c.
\]

This is a complete intersection of three quadrics in \( \mathbb{P}^5 \).

Double plane

\[
\tau^2 = \det(xq_a + yq_b + zq_c)/xyz
\]

is the moduli space of 2-bundles on \( \text{Km}(A) \) with Mukai vector

\((2, h, 2)\). Its period is \( v^\perp/\mathbb{Z} \cdot v \). By computation this is the same as the period of \( \tilde{S} \). By Torelli, the double plane is isomorphic to \( \tilde{S} \).

Remark Fixed point condition \( \pm a \pm b \pm c \neq 1/2 \iff A \) has a real \( \sqrt{2} \)-multiplication \( \varphi \in \text{Aut} A, \varphi^2 = 2, \) and \( G = \text{Ker} \varphi \).
Enriques surfaces of type \((D_6 + A_1)^+\)

\(D_6 + A_1\) has two type of primitive embeddings into \(I_{2,10}\). One has odd orthogonal complement and the other even one. The latter embedding is denoted by \((D_6 + A_1)^+\).

Let \(q(u, v, w) = 0\) be a smooth plane conic and consider the quartic surface \(q(xt + yz, yt + xz, zt + xy) + xyzt = 0\).

This is Kummer’s quartic surface \(Km (Jac C)\) with 16 nodes. 4 nodes at the coordinate points form a Göpel subgroup \(G\) of the Jacobian. Standard Cremona transformation induces a free involution and we obtain an Enriques surface \((Km Jac C)/\varepsilon\).

**Theorem** (1) \((Km Jac C)/\varepsilon\) is an Enriques surface of type \((D_6 + A_1)^+\).

(2) Every Enriques surface \(S\) of type \((D_6 + A_1)^+\) is obtained in this way or of type \((E_7 + A_1)^+\) or \((D_8)^+\). (In the latter two cases, the K3-cover \(\tilde{S}\) is \(Km (E_1 \times E_2)\).)