Birational Geometry of Algebraic Varieties

Open Problems

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Problems on characterization of the complex projective space

by

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A compact complex manifold \( X \) is a Fano manifold if its 1st Chern class \( c_1(X) \in H^1(X,\mathbb{Z}) \) is positive, or equivalently, the anticanonical class \( -K_X \) is ample. The projective space \( \mathbb{P}^n \) is the most typical example. In this note, I pose some problems on characterization of \( \mathbb{P}^n \) which was conceived during my study on Fano manifolds of coindex 3 [Mu].

1. Characterization by index

For a Fano manifold \( X \), the largest integer \( r \) which divides \( c_1(X) \) in \( H^2(X,\mathbb{Z}) \) is called the index of \( X \). The index of \( \mathbb{P}^n \) is equal to \( n+1 \).

Theorem 1. ([K-O]). Let \( X \) be a Fano manifold. Then index \( X \leq \dim X + 1 \). Moreover, the equality holds if and only if \( X \cong \mathbb{P}^n \).

If \( X \) is a Fano manifold of index \( r \), then the vector bundle \( \mathcal{O}_X(-K_X/r)^\otimes r \) is ample and its first Chern class is equal to \( c_1(X) \). So we consider ample vector bundles \( E \) on \( X \) with \( c_1(E) = c_1(X) \). How big can the rank \( r(E) \) of \( E \) be? By [Mo], there exists a rational curve \( C \) on \( X \) with \( (C \cdot c_1(X)) \leq \dim X + 1 \). Since every vector bundle on \( \mathbb{P}^1 \) is a direct sum of line bundles, we have \( r(E) = r(E|_C) \leq \dim X + 1 \).

Conjecture 1. Let \( X \) be a compact complex manifold and \( E \) an ample vector bundle on it with \( c_1(E) = c_1(X) \). If \( r(E) = \dim X + 1 \), then \( (X,E) \cong (\mathbb{P}^n, \mathcal{O}(1)^\otimes (n+1)) \).
2. Characterization by the tangent bundle

The following was conjectured by [Ha].

Theorem 2. ([Mo]). A compact complex manifold $X$ with ample tangent bundle $T_X$ is isomorphic to $\mathbb{P}^n$.

The tangent bundle $T_X$ is a vector bundle on $X$ with $r(T_X) = \dim X$ and $c_1(T_X) = c_1(X)$. The vector bundles $\mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$ over $\mathbb{P}^n$ and $\mathcal{O}(1)^{\oplus n}$ over a hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ also satisfy these conditions.

Conjecture 2. Let $E$ be an ample vector bundle on $X$ with $\text{rk } E = \dim X$ and $c_1(E) = c_1(X)$. Then the pair $(X, E)$ is isomorphic to $(\mathbb{P}^n, T_p), (\mathbb{P}^n, \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2))$ or $(Q^n, \mathcal{O}(1)^{\oplus n})$.

3. The logarithmic version of Hartshorne conjecture

The "log analogue" of the tangent bundle $T_X$ is the sheaf of vector fields with logarithmic zeroes along $D$, which is denoted by $T_X(-\log D)$. $T_X(-\log D)$ is characterized by the natural exact sequence

$$0 \rightarrow T_X(-\log D) \rightarrow T_X \rightarrow N_{D/X} \rightarrow 0,$$

where $N_{D/X}$ is the normal bundle $\mathcal{O}_D(D)$ of $D$ and we regard it as a sheaf on $X$ with support on $D$. If $X = \mathbb{P}^n$ and $D$ is a hyperplane, then $T_X(-\log D)$ is isomorphic to $\mathcal{O}_p(1)^{\oplus n}$.

Conjecture 3. (*) Let $X$ be a compact complex manifold and $D$ a nonzero reduced effective divisor on it. If the logarithmic tangent bundle $T_X(-\log D)$ is ample, then $(X, D) \approx (\mathbb{P}^n, \text{hyperplane})$.

(* ) In the problem session, Mori said that this would be proved by essentially the same argument as in [Mo].
The tangent bundle $T_X$ is ample if the bisectional curvature is positive.

Problem. Find a sufficient condition on the curvature for $T_X(-\log D)$ to be ample, that is, formulate a logarithmic version of the Frankel conjecture which characterizes $\mathbb{C}^n$.

4. Relation with the classification of Fano manifolds

Let $E$ be a rank $r$ vector bundle on $X$ with $c_1(E) = c_1(X)$ and put $Y = \mathbb{P}(E)$. Then $c_1(Y)$ is $r$ times the tautological line bundle $\mathcal{O}_Y(1)$. Hence if $E$ is ample then $Y$ is a Fano manifold of index $r$. If $r = n+1$, $n = \dim X$, then $Y$ is a Fano $2n$-fold of index $n+1$. We note $\rho(Y) = \rho(X)+1 \geq 2$, where $\rho$ denotes the Picard number. The following is a refinement of Theorem 1.

Conjecture 4. If $Y$ is a Fano manifold with Picard number $\rho$, then $\dim Y \leq \dim Y/\rho + 1$. Moreover, the equality holds iff $Y \cong (\mathcal{O}_{\text{index } Y-1})^\rho$.

For a Fano manifold $Y$, we define the coindex by $\dim Y - \text{index } Y + 1$, which is nonnegative by Theorem 1. Conjecture 4 implies

Conjecture 4'. If $Y$ is a Fano manifold with Picard number $\geq 2$, then $\dim Y \leq 2 \cdot \text{coindex } Y$. Moreover, the equality holds iff $Y \cong \mathcal{O}_{\text{coindex } Y} \times \mathcal{O}_{\text{coindex } Y}$.

This conjecture implies Conjecture 1. In the case $\text{coindex } Y \leq 3$, Conjecture 4' is easily obtained from the following;

Proposition. Let $Y$ be a Fano manifold of coindex $c \leq 3$ and $R$ an extremal ray of $Y$. Let $f : Y \to Z$ be the contraction morphism of $R$. Then we have either $\dim Z = \dim Y$ or $\dim Z \leq c$. 

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In the former case, \( f' \) is birational and contracts a divisor to a point or to a curve.

(This proposition is also observed in [Fuj].)

Proof of Conjecture 4' in the case coindex 3:

In the case \( \dim Y \geq 4 \), \( Y \) has a nef extremal ray \( R_1 \). Since \( \rho(Y) \geq 2 \), \( Y \) has another extremal ray \( R_2 \). Let \( F_2 \) be a fiber of maximal dimension of \( \text{cont}_{R_2} \). By the proposition, \( \dim F_2 \geq \dim Y - 3 \). Since the restriction of \( \text{cont}_{R_1} \) to \( F_1 \) is finite, we have \( \dim Y - 3 \leq 3 \). Moreover, if the equality holds, then both \( \text{cont}_{R_1} \) and \( \text{cont}_{R_2} \) are \( \mathbb{P}^3 \)-bundles over 3-folds. Hence we have \( Y \simeq \mathbb{P}^3 \times \mathbb{P}^3 \).

References


