Fano 3-folds

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Abstract: In the beginning of this century, G. Fano initiated the study of 3-dimensional projective varieties $X_{2g-2} \subset \mathbb{P}^{g+1}$ with canonical curve sections in connection with the Lüroth problem.¹ After a quick review of a modern treatment of Fano’s approach (§1), we discuss a new approach to Fano 3-folds via vector bundles, which has revealed their relation to certain homogeneous spaces (§§2 and 3) and varieties of sums of powers (§§5 and 6). We also give a new proof of the genus bound of prime Fano 3-folds (§4). In the maximum genus ($g = 12$) case, Fano 3-folds $X_{22} \subset \mathbb{P}^{13}$ yield a 4-dimensional family of compactifications of $\mathbb{C}^3$ (§8).

A compact complex manifold $X$ is Fano if its first Chern class $c_1(X)$ is positive, or equivalently, its anticanonical line bundle $\mathcal{O}_X(-K_X)$ is ample. If $\mathcal{O}_X(-K_X)$ is generated by global sections and $\Phi_{|−K_X|}$ is birational, then its image is called the anticanonical model of $X$. In the case $\dim X = 3$, every smooth curve section $C = X \cap H_1 \cap H_2 \subset \mathbb{P}^{g-1}$ of the anticanonical model $X \subset \mathbb{P}^{g+1}$ is canonical, that is, embedded by the canonical linear system $|K_C|$. Conversely, every projective 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ with a canonical curve section is obtained in this way. The integer $1/2(-K_X)^3 + 1$ is called the genus of a Fano 3-fold $X$ since it is equal to the genus $g$ of a curve section of the anticanonical model.

A projective 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ with a canonical curve section is a complete intersection of hypersurfaces if $g \leq 5$. In particular, the Picard group of $X$ is generated by $\mathcal{O}_X(-K_X)$. We call such a Fano 3-fold prime. If a Fano 3-fold $X$ is not prime, then either $-K_X$ is divisible by an integer $\geq 2$ or the Picard number $\rho$ of $X$ is greater than one. See [15], [7] and [9] for the classification in the former case and [24] and [25] in the latter case.

§1 Double projection The anticanonical line bundle $\mathcal{O}_X(-K_X)$ is very ample if $X$ is a prime Fano 3-fold of genus $\geq 5$ (cf. [15] and [41]). To classify prime Fano 3-folds $X_{2g-2} \subset \mathbb{P}^{g+1}$ of genus $g \geq 6$, Fano investigated the double projection from a line $\ell$ on $X_{2g-2}$, that is, the rational map associated to the linear system $|H − 2\ell|$ of hyperplane sections singular along $\ell$.

Example 1 Let $X_{16} \subset \mathbb{P}^{10}$ be a prime Fano 3-fold of genus 9. Then the double projection $\pi_{2\ell}$ from a line $\ell \subset X_{16}$ is a birational map onto $\mathbb{P}^3$. The union $D$ of conics which intersects $\ell$ is a divisor of $X$ and contracted to a space curve $C \subset \mathbb{P}^3$ of genus 3 and degree 7. The inverse rational map $\mathbb{P}^3 \dashrightarrow X_{16} \subset \mathbb{P}^{10}$ is given by the linear system $|7H − 2C|$ of surfaces of degree 7 which are singular along $C$.

The key for the analysis of $\pi_{2\ell}$ is the notion of flop. Let $X^−$ be the blow-up of $X$ along $\ell$. Since other lines intersect $\ell$, $X^−$ is not Fano. But $X^−$ is almost Fano in the sense that $|−K_{X^−}|$ is free and gives a birational morphism contracting no divisors. The anticanonical model $\hat{X}$ of $X^−$ is the image of the projection $X^− \rightarrow \mathbb{P}^8$ from $\ell$. The strict transform $D^− \subset X^−$ of $D$ is relatively negative over $\hat{X}$. By the theory of flops ([33], [19]), there exists another almost Fano 3-fold $X^+$ which has the same anticanonical model as $X^−$ and such that the strict transform $D^+ \subset X^+$ of $D^−$ is relatively ample over $\hat{X}$. $X^+$ is called the $D^−$-flop $\delta^3$ of $X^−$.

¹ A surface dominated by a rational variety is rational by Castelnuovo’s criterion. But this does not hold any more for 3-folds. See [5], [44] and [18].

² The existence of a line is proved by Shokurov [42].

³ The smoothness of $X^+$ follows from [19, 2.4] or from the classification [6, Theorem 15] of the singularity of $X$. 
Theorem ([23], [17]) Let $X$, $\ell$ and $D$ be as in Example 1. Then the $D$-flop $X^+$ of the blow-up $X^-$ of $X$ along $\ell$ is isomorphic to the blow-up of $\mathbb{P}^3$ along a space curve of genus 3 and degree 7.

For the proof, the theory of extremal rays ([22]) is applied to the almost Fano 3-fold $X^+$. If $X$ is a prime Fano 3-fold of genus 10, then $X^+$ is isomorphic to the blow-up of a smooth 3-dimensional hyperquadric $Q^3 \subset \mathbb{P}^4$ along a curve of genus 2 and degree 7. In the case genus 12, $X^+$ is the blow-up of a quintic del Pezzo 3-fold $V_5 \subset \mathbb{P}^6$ along a quintic normal rational curve.

§2 Bundle method A line on $X_{2g-2} \subset \mathbb{P}^{g+1}$ can move in a 1-dimensional family. Hence the double projection method does not give a canonical biregular description of $X_{2g-2} \subset \mathbb{P}^{g+1}$. In the case $g = 9$, e.g., there are infinitely many different space curves $C \subset \mathbb{P}^3$ which give the same Fano 3-fold $X_{16} \subset \mathbb{P}^{10}$. By the same reason, the double projection method does not classify $X_{2g-2} \subset \mathbb{P}^{g+1}$ over fields which are not algebraically closed. Even when a Fano 3-fold $X$ is defined over $k \subset \mathbb{C}$, it may not have a line defined over $k$. Our new classification makes up these defects. It is originated to solve the following:

Problem A: Classify all projective varieties $X_{2g-2}^{g+n-2} \subset \mathbb{P}^{g+n-2}$ of dimension $n \geq 3$ with a canonical curve section.

We restrict ourselves to the case that every divisor on $X$ is cut out by a hypersurface. In contrast with the case $g \leq 5$, the dimension $n$ cannot be arbitrarily large in the case $g \geq 6$. In each case $7 \leq g \leq 10$, the maximum dimension $n(g)$ is attained by a homogeneous space $\Sigma_{2g-2}$.

<table>
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<th>$g$</th>
<th>$n(g)$</th>
<th>$\Sigma_{2g-2} \subset \mathbb{P}^{g+n(g)-2}$</th>
<th>$r(E)$</th>
<th>$\chi(E)$</th>
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<td>10</td>
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<td>$G_2/P \subset \mathbb{P}^{14}$</td>
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We claim that every variety $X \subset \mathbb{P}$ with canonical curve section of genus $g \geq 6$ is a linear section of the above $\Sigma_{2g-2} \subset \mathbb{P}^{g+n(g)-2}$. Since each $\Sigma_{2g-2}$ has a natural morphism to a Grassmann variety, vector bundles play a crucial role in our classification. Instead of a line, we show the existence of a good vector bundle $E$ on $X$. Instead of the double projection, we embed $X$ into a Grassmann variety by the linear system $|E|$ and describe its image. The vector bundle is first constructed over a general (K3) surface section $S$ of $X$ and then extended to $X$ applying a Lefschetz type theorem (cf. [8]).

4 A smooth projective variety $V_d \subset \mathbb{P}^{d+n-2}$ with a normal elliptic curve section is called del Pezzo. The anticanonical class $-K_V$ is linearly equivalent to $(n-1)$ times hyperplane section. All quintic del Pezzo 3-folds are isomorphic to each other (see [15] and [9]).

5 The isomorphism classes of curves $C$ are uniquely determined by the Torelli theorem since the intermediate Jacobian variety of $X$ is isomorphic to the Jacobian variety of $C$.

6 Roth [36] [37] studied this problem by generalizing the double projection method.

7 The anticanonical class of $X_{2g-2}^2$ is $(n-2)$-times hyperplane section. In the case $n = 2$, $X_{2g-2}^2$ is a (polarized) K3 surface. The integer $g$ is called the genus of $X$.

8 $G(s,n)$ denotes the Grassmann variety of $s$-dimensional subspaces of a fixed $n$-dimensional vector space.

9 By our assumption on $X$ and [21], there exists a surface section with Picard number one. Hence every member of $|O_S(-K_X)|$ is irreducible. We use this property to analyze $\Phi_{|E|}$.
are as in the above table.\(^\text{10}\) All higher cohomology groups of \(E\) vanish and \(E\) is generated by its global sections. The morphism\(^\text{11}\) \(\Phi_{(E)} : X \rightarrow G(H^0(E), r(E))\) is an embedding if \(g \geq 7\). The first Chern class \(c_1(E)\) is equal to \(2c_1(X)\) if \(g = 7\) and equal to \(c_1(X)\) otherwise. \(E\) is characterized by the following two properties:

1) \(r(E), c_1(E)\) and \(c_2(E)\) are as above, and
2) the restriction\(^\text{12}\) of \(E\) to a general surface section is stable.

In the case \(g = 9, |E|\) embeds \(X\) into the 9-dimensional Grassmann variety \(G(V, 3)\), where \(V = H^0(X, E)\). Consider the natural map

\[\lambda_2 : \bigwedge^2 H^0(X, E) \rightarrow H^0(X, \bigwedge^2 E).\]

The kernel is generated by a nondegenerate bivector \(\sigma\) on \(V\). Hence the image of \(X\) is contained in the zero locus \(G(V, 3, \sigma)\) of the global section of \(\bigwedge^2 E\) corresponding to \(\sigma\), where \(E\) is the universal quotient bundle on \(G(V, 3)\). \(G(V, 3, \sigma)\) is a 6-dimensional homogeneous space of \(Sp(V, \sigma)\) and a projective variety \(\Sigma_{16} \subset \mathbb{P}^{13}\) with a canonical curve section of genus 9. In the case \(\dim X = 3\), we have

**Theorem 2** A prime Fano 3-fold \(X_{16} \subset \mathbb{P}^{10}\) of genus 9 is isomorphic to the intersection of \(\Sigma_{16}\) and a linear subspace \(\mathbb{P}^{10}\) in \(\mathbb{P}^{13}\).

By the above characterization, \(E\) is defined over \(k \subset \mathbb{C}\) if \(X\) is so. Hence the theorem holds true for every Fano 3-fold \(X_{16} \subset \mathbb{P}^{10}\) over \(k \subset \mathbb{C}\) such that \(X \otimes \mathbb{C}\) is prime.

The results are similar for \(g = 7, 8\) and 10. In the case \(g = 7\) and 10, the natural mappings \(\sigma_2 : S^2 H^0(X, E) \rightarrow H^0(X, S^2 E)\) and \(\lambda_4 : \bigwedge^4 H^0(X, E) \rightarrow H^0(X, \bigwedge^4 E)\) are considered instead of \(\lambda_2\). In the case \(g = 6\), \(X\) is a double cover of a linear section of \(G(2, 5) \subset \mathbb{P}^9\) if the linear subspace \(P\) passes through the vertex of the Grassmann cone. Otherwise, \(X\) is isomorphic to the complete intersection of a 6-dimensional hyperquadric \(Q \subset P\) and \(G(2, 5) \subset \mathbb{P}^9\).

§3 Fano 3-fold of genus 12 A prime Fano 3-fold\(^\text{13}\) \(X\) of genus 12 cannot be an ample divisor of a 4-fold. But the vector bundle \(E\) gives a canonical description of \(X\) in the 12-dimensional Grassmann variety \(G(V, 3), V = H^0(X, E)\). Consider the natural map \(\lambda_2 : \bigwedge^2 H^0(X, E) \rightarrow H^0(X, \bigwedge^2 E)\) as in the case \(g = 9\). Its kernel \(N\) is of dimension 3. Let \(\{\sigma_1, \sigma_2, \sigma_3\}\) be a basis of \(N\).

**Theorem 3** A prime Fano 3-fold \(X_{22} \subset \mathbb{P}^{12}\) of genus 12 is isomorphic to the common zero locus \(G(V, 3, N)\) of the three global sections of \(\bigwedge^2 E\) corresponding to \(\sigma_1, \sigma_2\) and \(\sigma_3\), where \(E\) is the universal quotient bundle on \(G(V, 3)\).

The third Chern number deg \(c_3(E)\) is equal to 2. Hence every general global section of \(E\) vanishes at two points. Conversely, since \(V\) is of dimension 7, there exists a nonzero global section \(s_{x,y}\) vanishing at \(x\) and \(y\) for every pair of distinct points \(x\) and \(y\). If \(x\) and \(y\) are general, then \(s_{x,y}\) is unique up to constant multiplications. The correspondence \((x, y) \mapsto [s_{x,y}]\) gives the birational mappings \(\Pi : S^2X_\ast \rightarrow \mathbb{P}_\ast(V) \simeq \mathbb{P}^6\) and \(\Pi_x : X_\ast \rightarrow \mathbb{P}_\ast(V_x) \simeq \mathbb{P}^3\) for general \(x\), where \(V_x \subset V\) is the space of global sections of \(E\) which vanish at \(x\). In particular, \(X\) is rational. The birational mapping \(\Pi_x\) is the same as the triple projection of \(X_{22} \subset \mathbb{P}^{13}\) from \(x\).

The bundle method gives another canonical description of prime Fano 3-folds of genus 12 in the variety of twisted cubics \((\text{[29, } \S 3]\)). This description is useful to analyze the double projection of \(X_{22} \subset \mathbb{P}^{13}\) from a line.

\(^{10}\) The bundle method works for other values of \(g\), e.g., 18 and 20 and gives a description of polarized K3 surfaces (see [30]).

\(^{11}\) For a vector space \(V\), \(G(V, r)\) denotes the Grassmann variety of \(r\)-dimensional quotient space of \(V\).

\(^{12}\) The restriction of \(E\) is rigid and characterized by its numerical invariants and stability \((\text{[27, } \S 3]\))

\(^{13}\) Prime Fano 3-folds of genus 12 were omitted in [38, Chap. V, §7] and first constructed by Iskovskih [16].
Remark 4 The third Betti number of a prime Fano 3-fold of genus $g \geq 7$ is equal $2(n(g) - 3)$. In particular, prime Fano 3-folds of genus 12 have the same homology group as $\mathbb{P}^3$.

§4 Genus bound The descriptions given in §§2 and 3 complete the classification of prime Fano 3-folds by virtue of Iskovskih’s genus bound:

Theorem 5 The genus $g$ of a prime Fano 3-fold satisfies $g \leq 10$ or $g = 12$.

This is proved in the course of the classification by the double projection method. Here we sketch a simple proof using a correspondence between the moduli spaces of K3 surfaces and curves. Let $\mathcal{F}_g$ be the moduli space of polarized K3 surfaces $(S, h)$ of degree $2g - 2$. A smooth member of $|h|$ is a curve of genus $g$. Hence we obtain the rational map $\phi_g$ from the $\mathbb{P}^g$-bundle $\mathcal{P}_g := \bigsqcup_{(S, h) \in \mathcal{F}_g} |h|$ over $\mathcal{F}_g$ to the moduli space $\mathcal{M}_g$ of stable curves of genus $g$. The key observation is this.

Proposition 6 If a prime Fano 3-fold of genus $g$ exists, then the rational map $\phi_g : \mathcal{P}_g \to \mathcal{M}_g$ is not generically finite.

By a simple deformation argument, we have that the generic hyperplane section $(S, h)$ of the generic prime Fano 3-fold is generic in $\mathcal{F}_g$. Take a generic pencil $P$ of hyperplane sections of $X_{2g - 2} \subset \mathbb{P}^{g+1}$. The isomorphism classes of the members of $P$ vary since the pencil $P$ contains a singular member. But every member of $P$ contains the base locus of $P$, which is a curve of genus $g$. This shows the proposition.

Since $\dim \mathcal{P}_g = g + 19$ and $\dim \mathcal{M}_g = 3g - 3$, $\dim \mathcal{P}_g \leq \dim \mathcal{M}_g$ holds if and only if $g \geq 11$. We recall the proof of the generic finiteness of $\phi_{11}$ in [25]. Let $C \subset \mathbb{P}^5$ be a sextic normal elliptic curve and $S$ a smooth complete intersection of three hyperquadrics containing $C$. Let $H$ be a general hyperplane section of $S$ and put $\Gamma = H \cup C$. The $S$ is a K3 surface and $\Gamma$ is a stable curve of genus 11.

Theorem([25, (1.2)]) For every embedding $i : \Gamma \to S'$ of $\Gamma$ in to a K3 surface $S'$, there exists an isomorphism $I : S \to S'$ whose restriction to $\Gamma$ coincides with $i$.

This implies that the point $\xi \in \mathcal{P}_{11}$ corresponding to $(S, \Gamma)$ is isolated in $\phi_{11}^{-1}(\phi_{11}(\xi))$. Hence $\phi_{11}$ is generically finite and a prime Fano 3-fold of genus 11 does not exist. The non-existence of prime Fano 3-folds of genus $\geq 13$ is proved in a similar way. Note that the elliptic curve $C$ induces an elliptic fibration of $S$, which we denote by $\pi : S \to \mathbb{P}^1$. We consider the case in which $\pi$ has two singular fibers of the following types:

i) $E_1 \cup E_2 \cup E_3$ with $(E_2, E_3) = (E_3, E_1) = (E_1, E_2) = 1$, and

ii) $E_2' \cup E_4$ with $(E_2', E_4) = 2$,

where $E_\nu$ is isomorphic to $\mathbb{P}^1$ and satisfies $(E_\nu, H) = \nu$ for every $1 \leq \nu \leq 4$. It is easy to construct a stable curve $\Gamma_g$ of genus $\geq 13$ on $S$ from $\Gamma$ by adding fibers of $\pi$. For example, $\Gamma \cup E_3$, $\Gamma \cup E_4$ and $\Gamma \cup E_2 \cup E_3$ are of genus 13, 14 and 15, respectively. Note that to add one general fibre of $\pi$ increases the genus by 6. By the above theorem, it is easy to show that every embedding of $\Gamma_g$ into a K3 surface $S'$ is extended to an isomorphism from $S'$ onto $S'$. Hence we have

Theorem 7 The rational map $\phi_g : \mathcal{P}_g \to \mathcal{M}_g$ is generically finite if and only if $g = 11$ or $g \geq 13$.

This completes the proof of Theorem 5.

Remark 8 The map $\phi_g$ is generically of maximal rank except for $g = 10, 12$. In the case of $g = 10$, the image of $\phi_{10}$ is is a divisor of $\mathcal{M}_{10}$ (See [28]).
§5 Theory of polars  Prime Fano 3-folds of genus 12 are related to the classical problem on sums of powers, which is a polynomial version of the Waring problem. Let \( F_d \) be a homogeneous polynomial of degree \( d \) in \( n \) variables.

1) Are there \( N \) linear forms \( f_1, \ldots, f_N \) such that \( F_d = \sum_1^N f_i^d \)?

2) If so, then how many?

In the following cases, every general \( F_d \) is a sum of \( d \)-th powers of \( N \) linear forms and the expression is unique:

1. \( n = 2 \) and \( d = 2N \) (Sylvester[43]),
2. \( n = 4, d = 3 \) and \( N = 5 \) (Sylvester’s pentahedral theorem [34] [39]), and
3. \( n = 3, d = 5 \) and \( N = 7 \) (Hilbert [14, p. 153], Richmond [34] and Palatini [32]).

We consider the case \( n = 3 \). Let \( C \) and \( \Gamma \) be the plane curves defined by \( F_d \) and \( \prod_1^N f_i \), respectively. \( \Gamma \) is called a polar \( N \)-side of \( C \) if \( F_d = \sum_1^N f_i^d \). The name comes from the following:

Example 9  Let \( C \) be a smooth conic and \( \ell_1, \ell_2 \) and \( \ell_3 \) three distinct lines. Then the following are equivalent:

1. \( \Delta = \ell_1 + \ell_2 + \ell_3 \) is a polar 3-side of \( C \) in the above sense, and
2. the triangle \( \Delta \) is self polar with respect to \( C \), that is, each side is the polar of its opposite vertex.

§6 Variety of sums of powers  We regard the set of polar \( N \)-sides of \( C : F_d(x, y, z) = 0 \) as a subvariety of the projective space of plane curves of degree \( N \). We denote its closure\(^{14} \) by \( VSP(C, N) \) or \( VSP(F_d, N) \). The homogeneous forms of degree \( N \) form a vector space of dimension \( \frac{1}{2}(d+1)(d+2) \). The \( N \)-ples of linear forms form a vector space of dimension \( 3N \). Hence the dimension of \( VSP(C, N) \) is expected to be \( 3N - \frac{1}{2}(d+1)(d+2) \) for general \( C \).

In the case \( (d, N) = (2, 3) \), this is true.

Proposition 10  If \( C \) is a smooth conic, then \( VSP(C, 3) \) is a smooth quintic del Pezzo 3-fold.

Let \( V_2 \) be the vector space of quadratic forms. If \( \Delta : f_1 f_2 f_3 = 0 \) is a polar 3-side of \( C \), then the defining equation \( F_2 \) of \( C \) is contained in the subspace \( \langle f_1^2, f_2^2, f_3^2 \rangle \) of \( V_2 \). Therefore, \( \Delta \) determines a 2-dimensional subspace \( W \) of \( V^* := V_2/C F_2 \). Hence we have the morphism from \( VSP(C, 3) \) to the 6-dimensional Grassmann variety \( G(2, V^*) \subset P^9 \). Let \( q : V_2 \rightarrow C \) be the linear map associated to the dual conic of \( C \). For a pair of quadratic forms \( f \) and \( g \), consider the three minors \( J_i(f, g), i = 1, 2, 3 \), of the Jacobian matrix

\[
\begin{pmatrix}
  f_x & f_y & f_z \\
  g_x & g_y & g_z
\end{pmatrix}
\]

and put \( \sigma_i(f, g) = q(J_i(f, g)) \). Then \( \sigma_i \) are skew-symmetric forms on \( V_2 \) and \( F_2 \) is their common radical. Therefore, each \( \sigma_i, i = 1, 2, 3 \), determine three hyperplanes \( H_i \) of \( P^9 = P_*(A^2 V^*) \). \( VSP(C, 3) \) is isomorphic to the quintic del Pezzo 3-fold \( G(2, V^*) \cap H_1 \cap H_2 \cap H_3 \).

Now we consider plane quartic curves \( C : F_4(x, y, z) = 0 \). The dimension count

\[
3N - 15 \leq \dim VSP(C, N)
\]

does not hold for \( N = 5 \):

Let \( \{ \partial_1 = \partial^2/\partial x^2, \ldots, \partial_6 = \partial^2/\partial z^2 \} \) be a basis of the space of homogeneous second order partial differential operators.

\(^{14}\) The closure is taken in the symmetric product \( \text{Sym}^N P^2 \). But this is a temporary definition. In practice, we choose a suitable model of \( \text{Sym}^N P^2 \) to define \( VSP(C, N) \).
Theorem (Clebsch [4]) If a plane quartic curve \( C : F_4(x, y, z) = 0 \) has a polar 5-side, then
\[
\Omega(F) := \det(\partial_i \partial_j F)_{1 \leq i, j \leq 6} = 0.
\]
In particular, general plane quartic curves have no polar 5-sides.

In other words, polar 5-sides are not equally distributed to quartic curves. Once a quartic curve has a polar 5-side, it has a 1-dimensional family of polar 5-sides. (The same happens for polar 2-sides of conics.)

Polar 6-sides of plane quartics was studied by Rosanes [35] and Scorza [40]. The dimension count is correct for \( N = 6 \) and we obtain 3-folds.

**Theorem 11** (1) If a quartic curve \( C \) has no polar 5-sides or no complete quadrangles as its polar 6-sides, then the variety \( VSP(C, 6) \) of polar 6-sides of \( C \) is a prime Fano 3-fold of genus 12.

(2) Conversely every prime Fano 3-fold \( X \) of genus 12 is obtained in this way. The isomorphism class of \( C \) is uniquely determined by that of \( X \).

By virtue of Theorem 3, it suffice to show that \( G(V, 3, N) \) is isomorphic to \( VSP(C, 6) \). Let \( V_3 \) be the vector space of cubic forms. If \( \Gamma : f_1 f_2 \cdots f_6 = 0 \) is a polar 6-side of \( C \), then the partial derivatives \( F_x, F_y \) and \( F_z \) of the defining equation \( F_4 \) are contained in \( -\langle f_1^3, f_2^3, \cdots, f_6^3 \rangle \). Hence \( \Gamma \) determines a 3-dimensional subspace of \( V^* := V_3/\langle F_x, F_y, F_z \rangle \) and we obtain a morphism \( \phi \) from \( VSP(C, 6) \) to \( G(3, V^*) \). Three skew-symmetric forms \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) on \( V^* \) are defined as in the case of \( VSP(F_2, 3) \) and the image of \( \phi \) is contained in \( G(3, V^*, \sigma_1, \sigma_2, \sigma_3) \).

Conversely, let \( V \) and \( N \subset \Lambda^2 V \) be as in Theorem 3. The multiplication in the exterior algebra \( \Lambda^* V \) induces the map \( \sigma_3 : S^3 N \to \Lambda^6 V \). This is surjective and its kernel is of dimension 3.

**Lemma 12** There exists a quartic polynomial \( F(x, y, z) \in S^4 N \) whose partial derivatives \( F_x, F_y \) and \( F_z \) form a basis of the kernel of \( \sigma_3 \), where \( \{x, y, z\} \) is a basis of \( N \) and \( \{X, Y, Z\} \) is its dual.

The conics on (the anticanonical model of) \( G(V, 3, N) \) is parametrized by the projective plane\(^{15} \mathbb{P}_*(N) \) For every point \( x \) of \( G(V, 3, N) \), there exist exactly six conics \( \{Z_{\lambda_i}\}_{1 \leq i \leq 6} \), \( \lambda_i \in \mathbb{P}_*(N) \), passing through \( x \), counted with their multiplicities. Let \( \Lambda_i, 1 \leq i \leq 6 \), be the lines on \( \mathbb{P}(N) \) with coordinates \( \lambda_i \). Then \( \Gamma = \sum_1^6 \Lambda_i \) is a polar 6-side of the plane curve \( C \) on \( \mathbb{P}(N) \) defined by the quartic form \( F(x, y, z) \) in the lemma. This correspondence \( x \mapsto \Gamma \) gives the inverse of the above morphism \( \phi \).

**Remark 13** (1) Assume that a plane quartic \( C' \) has a polar 5-side and that the 5lines are in general position.

When \( C \) in Theorem 11 deforms to \( C' \), the variety \( VSP(C, 6) \) deforms to a Fano 3-fold \( X' \) with an ordinary double point. \( X' \) is isomorphic to the anti canonical model of \( \mathbb{P}(E) \), where \( E \) is a stable vector bundle on \( \mathbb{P}^2 \) with \( c_1 = 0 \) and \( c_2 = 4 \) (cf. [2]).

(2) If \( C \) is a general plane septic curve, then the variety \( VSP(C, 10) \) is a polarized K3 surface of genus 20.

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\(^{15}\) For a vector space \( V \), \( \mathbb{P}_*(V) \) is the projective space of 1-dimensional subspaces of \( V \). \( \mathbb{P}(V) \), or \( \mathbb{P}^*(V) \), is its dual.
§7 Almost homogeneous Fano 3-fold  Varieties of sums of powers give two examples of almost homogeneous spaces of $SO(3, \mathbb{C})$ and their compactifications. Apply Theorem 11 to a double conic, say $2C_0 : (XZ + Y^2)^2 = 0$.

The variety $VSP(2C_0, 6)$ is a Fano 3-fold and has an action of $SO(3, \mathbb{C})$. It is easy to check
\[
30(XZ + Y^2)^2 = 25Y^4 + \sum_{i=0}^{4}(\zeta^i X + Y + \zeta^{-i}Z)^4,
\]
where $\zeta$ is a fifth root of unity. The polar 6-side
\[
\Gamma : Y \prod_{i=0}^{4}(\zeta^i X + Y + \zeta^{-i}Z) = 0
\]
intersects the 2-sphere $C_0$ at the 12 vertices of a regular icosahedron. The stabilizer group at $\Gamma$ of $SO(3, \mathbb{C})$ is the icosahedral group $\cong A_5$. Hence we have

**Theorem 14** The variety $VSP(2C_0, 6)$ is a smooth equivariant compactification of $SO(3, \mathbb{C})/Icosa$.

Similarly the quintic del Pezzo 3-fold $VSP(C_0, 3)$ is a smooth equivariant compactification of the quotient of $SO(3, \mathbb{C})$ by an octahedral group $\cong S_4$ by Proposition 10. These two compactifications are described in [31, §§3 and 6] by another method. We remark that $Q^3$ and $P^4$ are also almost homogeneous spaces of $SO(3, \mathbb{C})$. The stabilizer groups are tetrahedral group $\cong A_4$ and a dihedral group of order 6 $\cong S_3$, respectively.

§8 Compactification of $\mathbb{C}^3$  There are four types of Fano 3-folds with the same homology groups as $P^3$: $P^3$ itself, $Q^3 \subset P^4$, $V_5 \subset P^6$ and the 6-dimensional family of prime Fano 3-folds $X_{22} \subset P^{13}$ of genus 12 (see Remark 4). These Fano 3-folds are related to not only $SO(3, \mathbb{C})$ but also $\mathbb{C}^3$, the affine 3-space. It is well-known that $P^3$ and $Q^3$ are smooth compactifications of $\mathbb{C}^3$ with irreducible boundary divisors. The quintic del Pezzo 3-fold $V_5 \subset P^6$ is a compactifications of $\mathbb{C}^3$ in two ways (see [10] and [13]).

Furushima has found that the almost homogeneous Fano 3-fold $U_{22} := VSP(2C_0, 6)$ also is a Compactification of $\mathbb{C}^3$. This fact is proved in three ways using

i) the defining equation ([31] p.506) of $U_{22} \subset P^{12}$ (see [11]),

ii) the double projection of $U_{22} \subset P^{13}$ from a line (see [12]), and

iii) the action of a torus $\mathbb{C}^* \subset SO(3, \mathbb{C})$ on $U_{22}$ (see [1] and [20]).

In the last case, $U_{22}$ is decomposed into a disjoint union of affine spaces by virtue of [3]. The four compactifications by $P^3$, $Q^3$ and $V_5$ are rigid but that by $U_{22}$ is not. In fact, by a careful analysis of the double projection of $VSP(C, 6) \subset P^{13}$ from a line, we have

**Theorem 15** The variety $VSP(C, 6)$ in Theorem 11 is a compactification of $\mathbb{C}^3$ if $C$ has a non-ordinary singular point.

The variety $VSP(C, 6)$ has a line $\ell$ (on its anticanonical model) with normal bundle $O(1) \oplus O(-2)$ corresponding to a non-ordinary singular point of $C$. Let $D$ be the union of conics which intersect $\ell$ as in Example 1. Then the complement of $D$ is isomorphic to $\mathbb{C}^3$. 


References


[38] Roth, L.: Algebraic threefold with special regard to problems of rationality, Springer Verlag, 1955.


