Curves and Grassmannians

Dedicated to Prof. Hideyuki Matsumura on his 60th Birthday

Shigeru MUKAI

Let C be a compact Riemann surface, or more generally, a smooth complete algebraic curve. The graded ring $R_C = \bigoplus_{k=0}^{\infty} H^0(\omega_C^k)$ of pluri-canonical forms on C is called the *canonical ring* of C. There are two fundamental results on R_C (cf. [1] and [5]):

Theorem (Noether) *If C is not hyperelliptic, then R_C is generated by $H^0(\omega_C)$.*

R_C is a quotient of the polynomial ring $S = k[X_1, \cdots, X_g]$ of g variables by a homogeneous ideal I_C, where g is the genus of C.

Theorem (Petri) *I_C is generated in degree 2 if C is neither trigonal nor a plane quintic.*

If C is not hyperelliptic, then the canonical linear system $|K_C|$ is very ample. The image $C_{2g-2} \subset \mathbf{P}^{g-1}$ of the morphism $\Phi_{|K_C|}$ is called *the canonical model* of C. It is called a canonical curve when C is not specified. By Noether's theorem, R_C is the homogeneous coordinate ring of the canonical model. If C is trigonal or a smooth plane quintic, then the quadric hull of $C_{2g-2} \subset \mathbf{P}^{g-1}$ is a surface of degree $g-2$. Otherwise, $C_{2g-2} \subset \mathbf{P}^{g-1}$ is an intersection of quadrics (Enriques-Petri's theorem). For curves C of genus $g \leq 5$, it is easy to determine the structure of R_C by the geometry of $C_{2g-2} \subset \mathbf{P}^{g-1}$ and/or by a general structure theorem of Gorenstein ring ([3]). But it seems not for curves of higher genus. In [11], we have announced linear section theorems which enable us to describe R_C for all curves of genus $g \leq 9$. In this article, we treat the case $g = 8$.

Let $G(2,6) \subset \mathbf{P}^{14}$ be the 8-dimensional Grassmannian embedded in $\mathbf{P}^{14}$ by the Plücker coordinates. It is classically known that a transversal linear subspace P of dimension 7 cuts out a canonical curve C of genus 8. In [10], we have shown that the generic curve of genus 8 is obtained in this manner. The main purpose of this article is to show the following:

Main Theorem *A curve C of genus 8 is a transversal linear section of the 8-dimensional Grassmannian $G(2,6) \subset \mathbf{P}^{14}$ if and only if C has no g^2_7.*

Since the defining ideal of $G(2,6) \subset \mathbf{P}^{14}$ is generated by Pfaffians, so is the ideal I_C. More precisely, we have

Corollary *Let C be as above. Then there exists a skew-symmetric matrix $M(X)$ of size 6 whose components are linear forms of $X_1, \cdots, X_8$ and such that the ideal I_C is generated by the 15 Pfaffians of 4×4 principal minors of $M(X)$.*

By [7], the graded ring R_C has the following free resolution as an S-module:

$$\begin{array}{ccccccccc}
0 \longleftarrow & R_C & \longleftarrow & S & \longleftarrow & S(-2)\otimes U^{15} & \longleftarrow S(-3)\otimes U^{35} \longleftarrow & S(-4)\otimes U^{21} \\
 & & & & & & & \oplus \\
 & 0 & \longrightarrow & S(-9) & \longrightarrow & S(-7)\otimes V^{15} & \longrightarrow S(-6)\otimes V^{35} \longrightarrow & S(-5)\otimes V^{21}
\end{array}$$

where U^i denotes an i-dimensional representation of $GL(6)$ and V^i is its dual.

For the proof of the main theorem, the use of vector bundles is essential. Let E be an (algebraic) vector bundle of rank 2 on C generated by global sections. Then each fibre of E is a 2-dimensional quotient space of $H^0(E)$. Hence we obtain a Grassmannian morphism of C, which we denote by $\Phi_{|E|} : C \longrightarrow G(H^0(E), 2)$. The determinant line bundle $\bigwedge^2 E$ is also generated by global sections and we obtain the morphism $\Phi_{|\bigwedge E|}$ to a projective space. A pair of global sections s_1 and s_2 of E determines a global section $[s_1 \wedge s_2]$ of $\bigwedge^2 E$. This correspondence $H^0(E) \times H^0(E) \longrightarrow H^0(\bigwedge^2 E)$ is bilinear and skew-symmetric. Hence we obtain the linear map

$$\lambda : \bigwedge^2 H^0(E) \longrightarrow H^0(\bigwedge^2 E). \tag{0.11}$$

The two morphisms $\Phi_{|E|}$ and $\Phi_{|\bigwedge E|}$ are related by the rational map $\mathbf{P}^*(\lambda)$ associated to λ and we obtain the commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\Phi_{|E|}} & G(H^0(E), 2) \\
\downarrow & & \downarrow \quad \text{Plücker} \\
\mathbf{P}^*(H^0(\bigwedge^2 E)) & \xrightarrow{\mathbf{P}(\lambda)} & \mathbf{P}^*(\bigwedge^2 H^0(E)).
\end{array} \tag{0.11}$$

Hence our task is to find of a 2-bundle E with the following properties:

(0.3) E has canonical determinant, that is, $\bigwedge^2 E \simeq \omega_C$,

(0.4) $\dim H^0(E) = 6$ and E is generated by global sections,

(0.5) the map λ is surjective, and

(0.6) the diagram (0.2) is cartesian.

A stable 2-bundle E with canonical determinant which maximizes $\dim H^0(E)$ is the desired one:

Theorem A *Let C be a curve of genus* 8 *without g^2_7. When F runs over all stable* 2*-bundles with canonical determinant on C, the maximum of* $\dim H^0(F)$ *is equal to* 6*. Moreover, such vector bundles F_{max} on C with* $\dim H^0(F_{max}) = 6$ *are unique up to isomorphism and generated by global sections.*

We denote F_{max} by E and put $V = H^0(E)$. The commutative diagram (0.2) becomes

$$\begin{array}{cccc}
 & C & \xrightarrow{\Phi_{|E|}} & G(V, 2) \\
\text{canonical} & \downarrow & & \downarrow \quad \text{Plücker} \\
 & \mathbf{P}^*(H^0(\omega_C)) & \xrightarrow{\mathbf{P}(\lambda)} & \mathbf{P}^*(\bigwedge^2 V).
\end{array} \tag{0.15}$$

The hyperplanes of $\mathbf{P}^*(\bigwedge^2 V)$ are parametrized by $\mathbf{P}_*(\bigwedge^2 V)$ and those containing the image of C by $\mathbf{P}_*(\operatorname{Ker} \lambda)$. A hyperplane corresponds to a point in the *dual* Grassmannian $G(2, V) \subset \mathbf{P}_*(\bigwedge^2 V)$ if and only if it cuts out a Schubert subvariety.

Theorem B *There exists a bijection between the intersection* $\mathbf{P}_*(\operatorname{Ker}\lambda) \cap G(2,V)$ *and the set* $W^1_5(C)$ *of* g^1_5*'s of* C.

The finiteness of $W^1_5(C)$ will be proved in §4 using the geometry of space curves. The 'if' part of Main Theorem is a consequence of

Theorem C *Let* E *be a* 2*-bundle with canonical determinant on a non-trigonal curve* C *of genus* 8*. If* E *satisfies* (0.4) *and if the intersection* $\mathbf{P}_*(\operatorname{Ker}\lambda) \cap G(2,V)$ *is finite, then* λ *is surjective and the diagram* (0.7) *is cartesian.*

We prove Theorem A, B and C in §3 after a brief review of basic materials on Grassmannians in §1 and the proof of 'only if part' of Main Theorem in §2. Results similar to these theorems will be proved for curves of genus 6 in the final section.

If the ground field is the complex number field $\mathbf{C}$, then C is the quotient of the upper half plane $H = \{\Im z > 0\}$ by the (cocompact) discrete subgroup $\pi_1(C) \subset PSL(2,\mathbf{R})$. Let $\Gamma \subset SL(2,\mathbf{R})$ be the pull-back of $\pi_1(C)$. The canonical ring R_C of C is isomorphic to the ring $\bigoplus_{k=0}^{\infty} S_{2k}(\Gamma)$ of holomorphic automorphic forms

$$f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z),\ z \in H, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

of even weight. By virtue of a theorem of Narasimhan and Ramanan ([13] and [4]), there exists a bijection between

1) the set of isomorphism classes of stable 2-bundles E with canonical determinant, and

2) the set of conjugacy classes (with respect to $SU(2)$) of odd $SU(2)$-irreducible representations $\rho : \Gamma \longrightarrow SU(2)$ of Γ,

where a representation ρ of Γ is *odd* if $\rho(-1) = -1$. $H^0(E)$ is isomorphic to the space $S_1(\Gamma,\rho)$ of vector-valued holomorphic automorphic forms

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \begin{pmatrix} f(\frac{az+b}{cz+d}) \\ g(\frac{az+b}{cz+d}) \end{pmatrix} = (cz+d) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix},\ z \in H, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

of weight one with coefficient in ρ. If $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in S_1(\Gamma,\rho)$, then $f_1g_2 - f_2g_1$ belongs to $S_2(\Gamma)$. Hence we obtain the linear map $\bigwedge^2 S_1(\Gamma,\rho) \longrightarrow S_2(\Gamma)$ which is nothing but λ in (0.1). By Theorem A and C, we have

Theorem D *Let* C *be a curve of genus* 8 *without* g^2_7*. When* ρ *runs all odd irreducible* $SU(2)$*-representations of* Γ*, the maximum of* $\dim S_1(\Gamma,\rho)$ *is equal to* 6*. Moreover, such representations* ρ_{max} *with* $\dim S_1(\Gamma,\rho_{max}) = 6$ *are unique up to conjugacy and satisfy the following:*

(1) $\bigwedge^2 S_1(\Gamma,\rho_{max}) \longrightarrow S_2(\Gamma)$ *is surjective, and*

(2) *the matrix* $M(z) = \begin{pmatrix} f_1(z) & \cdots & f_6(z) \\ g_1(z) & \cdots & g_6(z) \end{pmatrix}$ *is of rank* 2 *for every* $z \in H$*, where the column vectors of* $M(z)$ *are base of* $S_1(\Gamma,\rho)$*.*

By the property (2), $M(z)$ gives a holomorphic map of H to the 8-dimensional Grassmannian $G(2,6)$. By the automorphicity of $M(z)$, this map factors through C and its image is a linear section of $G(2,6)$.

Let $G(8, \bigwedge^2 \mathbf{C}^6)$ be the Grassmannian of 7-dimensional linear subspaces P of $\mathbf{P}_*(\bigwedge^2 \mathbf{C}^6)$ and $G(8, \bigwedge^2 \mathbf{C}^6)^s$ its open subset consisting of all stable points with respect to the action of $SL(6)$. The algebraic group $PGL(6)$ acts on $G(8, \bigwedge^2 \mathbf{C}^6)$ effectively and the geometric quotient $G(8, \bigwedge^2 \mathbf{C}^6)^s/PGL(6)$ exists as a normal quasi-projective variety ([12]). By Theorem A and C, the linear subspaces P transversal to $G(2,6)$ form an open subset Ξ of $G(8, \bigwedge^2 \mathbf{C}^6)^s$ and $\Xi/PGL(6)$ is isomorphic to the moduli space $\mathcal{M}'_8$ of curves of genus 8 without g^2_7.

Remark (1) The non-existence of g^2_7 is equivalent to the triple point freeness of the theta divisor of the Jacaobian variety of C.

(2) The curves with g^2_7 form a closed irreducible subvariety of codimension one in the moduli space $\mathcal{M}_8$ of curves of genus 8. See [11] for the canonical model of such curves of genus 8.

Notation and conventions. By a g^r_d, we mean a line bundle L on a curve C of degree d and with $\dim H^0(L) \geq r+1$. The map associated to the complete linear system $|L|$ is denoted by $\Phi_{|L|}$. The line bundle $\omega_C L^{-1}$ is called *the Serre adjoint* of L. We fix an algebraically closed field k and consider all vector spaces, varieties and schemes over it. For a vector space V, its dual is denoted by $V^\vee$. We denote by $G(r,V)$ and $G(V,r)$ the Grassmannians of r-dimensional subspaces and quotient spaces of V, respectively. They are abbreviated to $G(r,n)$ and $G(n,r)$ when $V = k^n$. Two projective spaces $G(1,V)$ and $G(V,1)$ associated to V are denoted by $\mathbf{P}_*(V)$ and $\mathbf{P}^*(V)$. $\mathbf{P}^*$ is a contravariant functor.

1 Grassmannians

The Grassmannian $G(r,V)$ is defined to be the set of r-dimensional (linear) subspaces of a vector space V. We consider the case $r = 2$. A 2-dimensional subspace U of k^n is spanned by two rows of a $2 \times n$ matrix

$$R = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

of rank 2. Hence $G(2,n)$ is covered by $\binom{n}{2}$ affine spaces Z_{ij}, $1 \leq i < j \leq n$, of dimension $2(n-2)$, where Z_{12} is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & a_3 & \cdots & a_n \\ 0 & 1 & b_3 & \cdots & b_n \end{pmatrix}$$

and other Z_{ij}'s are obtained from Z_{12} by permutation of columns. It is easy to check that $G(2,n)$ is an algebraic variety with respect to this atlas. Furthermore, $G(2,n)$ is a projective algebraic variety. We set $p_{ij}(R) = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ for $1 \leq i,j \leq n$. The ratio

$p_{ij}(R) : p_{kl}(R)$ is uniquely determined by U and does not depend on the choice of R. Hence the point

$$(p_{12}(R) : \cdots : p_{ij}(R) : \cdots : p_{n-1,n}(R)) \in \mathbf{P}^{\binom{n}{2}-1}, \quad 1 \leq i < j \leq n,$$

depends only on U. We call this the *Plücker coordinate* of U and denote by $p(U)$.

Proposition 1.1 *The map $\pi : G(2,n) \longrightarrow \mathbf{P}^{\binom{n}{2}-1}$, $[U] \mapsto p(U)$ is an embedding.*

Proof. It is obvious that the restriction of π to each Z_{ij} is an embedding. Since $p(U)$ belongs to Z_{ij} if and only if $p_{ij}(U) \neq 0$, π is injective. □

The defining equation of $G(2,n) \subset \mathbf{P}^{\binom{n}{2}-1}$ is easy to find. For a $2 \times n$ matrix R, let M_R be the $n \times n$ matrix whose ijth component is $p_{ij}(R)$. This matrix is skew-symmetric. Let Alt_n be the space of all skew-symmetric matrices of size n. The ambient projective space of the Grassmannian $G(2,n)$ is canonically identified with the projectivization of Alt_n. A skew-symmetric matrix M is equal to M_R for some R if and only if $\operatorname{rank} M = 2$. Hence the Grassmannian $G(2,n) \subset \mathbf{P}_*(Alt_n)$ is set-theoretically the intersection of $\binom{n}{4}$ quadrics defined by Pfaffians of 4×4 principal minors. Writing down the Pfaffians in the affine coordinate of Z_{ij}, it is easy to check

Proposition 1.2 *The Grassmannian $G(2,n) \subset \mathbf{P}_*(Alt_n)$ is scheme-theoretically the intersection of $\binom{n}{4}$ quadrics defined by Pfaffians of principal minors of size* 4.

We make the Plücker embedding and this proposition free from coordinates. Let A be a vector space. If U is a 2-dimensional subspace of A, then $\bigwedge^2 U$ is a 1-dimensional subspace of $\bigwedge^2 A$. Hence the Grassmannian $G(2,A)$ is a subvariety of $\mathbf{P}_*(\bigwedge^2 A)$ by Proposition 1.1. Similarly $G(A,2)$ is a subvariety of $\mathbf{P}^*(\bigwedge^2 A)$. For a bivector

$$w = \sum_{1 \leq i < j \leq n} a_{ij} v_i \wedge v_j \in \bigwedge^2 A$$

we define its *reduced square* $w^{[2]} \in \bigwedge^4 A$ by

$$w^{[2]} = \sum_{1 \leq i<j<k<l \leq n} \operatorname{Pfaff} \begin{pmatrix} 0 & a_{ij} & a_{ik} & a_{il} \\ a_{ji} & 0 & a_{jk} & a_{jl} \\ a_{ki} & a_{kj} & 0 & a_{kl} \\ a_{li} & a_{lj} & a_{lk} & 0 \end{pmatrix} v_i \wedge v_j \wedge v_k \wedge v_l, \tag{1.2}$$

where we put $a_{ji} = -a_{ij}$, $a_{ki} = -a_{ik}$ and so on. Then $w \wedge w = 2w^{[2]}$ and $w^{[2]}$ does not depend on the choice of a basis $\{v_1, \cdots, v_n\}$ of A. Similarly the reduced power $w^{[p]} \in \bigwedge^{2p} A$ is defined for every positive integer p so that $w^{\wedge p} = p! w^{[p]}$ by using the Pfaffians of principal minors of size $2p$. The point $[w] \in \mathbf{P}_*(\bigwedge^2 A)$ belongs to the Grassmannian $G(2,A)$ if and only if $w^{[2]} = 0$. By Proposition 1.2, we have

Proposition 1.3 *The Grassmannian $G(2, A) \subset \mathbf{P}_*(\bigwedge^2 A)$ is scheme-theoretically the zero locus of the quadratic form*

$$sq_A : \bigwedge^2 A \longrightarrow \bigwedge^4 A, \quad w \mapsto w^{[2]}$$

with values in $\bigwedge^4 A$.

For a 4-dimensional quotient space W of A, we call the composite q_W of sq_A and $\bigwedge^4 A \longrightarrow \bigwedge^4 W \simeq k$ the *Plücker quadratic form* associated to W. q_W is of rank 6. By the proposition, we have the linear system $L \simeq \mathbf{P}^*(\bigwedge^4 A)$ of quadrics containing $G(2, A)$. The zero loci of Plücker quadratic forms, called *Plücker quadrics*, are parametrized by the Grassmannian $G(A, 4) \subset L$.

If $\dim A = 4$, then $G(2, A)$ is a smooth 4-dimensional quadric in $\mathbf{P}_*(\bigwedge^2 A) = \mathbf{P}^5$. If $\dim A = 5$, every $Q \in L$ is a Plücker quadric. In the case $\dim A = 6$, $\bigwedge^4 A$ is the dual of $\bigwedge^2 A$ by the pairing

$$\bigwedge^2 A \times \bigwedge^4 A \longrightarrow \bigwedge^6 A \simeq k,$$

and $G(A, 4)$ is isomorphic to $G(2, A)$. Under the natural action of $PGL(A)$, the linear system L is decomposed into three orbits $G(A, 4), \Delta - G(A, 4)$ and $L - \Delta$ according as the rank of bivectors, where Δ is the cubic hypersurface defined by the Pfaffian. According as the three orbits, there are three types of quadrics in L. Take a basis $\{v_1, \cdots, v_6\}$ of A and let $p_{ij}, 1 \le i < j \le 6$, be the Plücker coordinates. The Plücker quadrics associated to the 4-dimensional quotient spaces $A/ < v_1, v_2 >, A/ < v_3, v_4 >$ and $A/ < v_5, v_6 >$ are

$$\begin{cases} Q_1 & : \quad q_1 = p_{34}p_{56} - p_{35}p_{46} + p_{36}p_{45} = 0, \\ Q_3 & : \quad q_3 = p_{12}p_{56} - p_{15}p_{26} + p_{16}p_{25} = 0 \ \text{ and} \\ Q_5 & : \quad q_5 = p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \end{cases} \tag{1.3}$$

respectively. The sum $q_3 + q_5$ is equal to

$$p_{12}(p_{34} + p_{56}) - p_{13}p_{24} + p_{14}p_{23} - p_{15}p_{26} + p_{16}p_{25} \tag{1.3}$$

and of rank 10. The sum $q_1 + q_3 + q_5$ is of rank 15. So we have proved

Proposition 1.4 *Assume that $\dim A = 6$. Then the linear system L has exactly three orbits L_6, L_{10} and L_{15} of dimension $8, 13$ and 14 under the natural action of $PGL(A)$. Moreover,*

a) every $Q \in L_6$ is a Plücker quadric and of rank 6,

b) every $Q \in L_{10}$ is of rank 10 and defined by a linear combination of two Plücker quadratic forms, and

c) every $Q \in L_{15}$ is smooth.

Remark 1.5 (1) The set L_6 of Plücker quadrics is canonically isomorphic to the Grassmannian $G(2, A) \subset \mathbf{P}_*(\bigwedge^2 A)$. The direct isomorphism between them is given as follows: The hypersurface Δ defined by the Pfaffian

$$r : \bigwedge^2 A \longrightarrow \bigwedge^6 A \simeq k, w \mapsto w^{[3]}$$

is singular along $G(2, A)$. Hence the partial derivatives $\partial r/\partial w, w \in \bigwedge^2 A$ are quadratic forms which vanish on $G(2, A)$. The correspondence $w \mapsto \partial r/\partial w$ gives a $PGL(A)$-equivariant isomorphism $\mathbf{P}_*(\bigwedge^2 A) \simeq L$, which maps $G(2, A)$ onto L_6.

(2) The secant variety S of $G(2, 6) \subset \mathbf{P}^{14}$ is the Pfaffian cubic hypersurface Δ and satisfies $\dim S = \frac{3}{2} \dim X + 1$. $G(2, 6) \subset \mathbf{P}^{14}$ is one of the Severi varieties classified by Zak [14] (see also [8]).

We recall an elementary fact on the projective dual of a hyperquadric $Q \subset \mathbf{P}$. The projective dual $\check{Q} \subset \mathbf{P}^\vee$ of Q consists of the points $[H]$ of the dual projective space $\mathbf{P}^\vee$ such that $\operatorname{rank} H \cap Q \leq \operatorname{rank} Q - 2$. The following is easily verified.

Proposition 1.6 *The projective dual $\check{Q} \subset \mathbf{P}^\vee$ is a smooth hyperquadric in the linear span $< \check{Q} >$ of $\check{Q}$. The linear span $< \check{Q} >$ coincides with the complementary linear subspace of* $\operatorname{Sing} Q \subset \mathbf{P}$ *and consists of $[H]$ such that* $\operatorname{rank} H \cap Q \leq \operatorname{rank} Q - 1$. *In particular,* $\dim \check{Q}$ *is equal to* $\operatorname{rank} Q - 2$.

A linear subspace P contained in Q is called *Lagrangean* if it is maximal among such subspaces. We can choose a system of coordinates $(x_1 : x_2 : x_3 : \cdots)$ of $\mathbf{P}$ so that

$$\begin{cases} P: & x_1 = x_2 = \cdots = x_n = 0 \\ Q: & x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} = 0 \end{cases}$$

when $\operatorname{rank} Q$ is even and so that

$$\begin{cases} P: & x_1 = x_2 = \cdots = x_n = x_{2n+1} = 0 \\ Q: & x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} + x_{2n+1}^2 = 0 \end{cases}$$

when $\operatorname{rank} Q$ is odd. In both cases, hyperplanes $H : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$, containing P, belongs to the dual $\check{Q}$ of Q. Moreover, they form a Lagrangean subspace of $\check{Q}$. Hence the complement $P^\perp \subset \mathbf{P}^\vee$ of P contains a Lagrangean of $\check{Q}$. If P_0 is a linear subspace of P, then $P_0^\perp \supset P^\perp$. Therefore, we have

Proposition 1.7 *If a linear subspace P is contained in a hyperquadric $Q \subset \mathbf{P}$, then its complement $P^\perp$ contains a Lagrangean of $\check{Q} \subset \mathbf{P}^\vee$ and hence* $\dim(P^\perp \cap \check{Q}) \geq [\frac{1}{2} \operatorname{rank} Q] - 1$.

The following is a key of the proof of Theorem C.

Proposition 1.8 *Let A, L_6 and L_{10} be as in Proposition 1.7.*

(1) *If $Q \in L_6$, then the projective dual $\check{Q} \subset \mathbf{P}^*(\bigwedge^2 A)$ of Q is a 4-dimensional quadric contained in $G(A, 2)$,*

(2) *If $Q \in L_{10}$, then $\check{Q}$ is an 8-dimensional quadric and the intersection $\check{Q} \cap G(A, 2)$ is of dimension 5.*

Proof. Let $\{v_1^*, \cdots, v_6^*\}$ be the dual basis of $\{v_1, \cdots, v_6\}$ and q_1, q_3 and q_5 as in (1.5).

(1) We may assume that Q is $Q_1 : q_1 = 0$. Since $\operatorname{rank} q_1 = 6$ and q_1 is a polynomial of the 6 variables $p_{34}, p_{56}, p_{35}, p_{46}, p_{36}$ and p_{45}, $< \check{Q}_1 >$ is the 5-plane spanned by the 6 points $[v_3^* \wedge v_4^*], [v_5^* \wedge v_6^*], [v_3^* \wedge v_5^*], [v_4^* \wedge v_6^*], [v_3^* \wedge v_6^*]$ and $[v_4^* \wedge v_5^*]$. A hyperplane

$$a_{34} p_{34} + a_{56} p_{56} + a_{35} p_{35} + a_{46} p_{46} + a_{36} p_{36} + a_{45} p_{45} = 0$$

is tangent to Q_1 if and only if $a_{34}a_{56} - a_{35}a_{46} + a_{36}a_{45} = 0$. Hence $\check{Q}$ is contained in $G(A, 2)$.

(2) We may assume that Q is defined by (1.6), that is, $q_3 + q_5 = 0$. $< \check{Q} >$ is the 9-plane spanned by $[v_3^* \wedge v_4^* - v_5^* \wedge v_6^*], [v_1^* \wedge v_2^*], \cdots, [v_2^* \wedge v_5^*]$. A hyperplane

$$a(p_{34} + p_{56}) + a_{12}p_{12} + \cdots + a_{25}p_{25} = 0$$

is tangent to Q if and only if

$$aa_{12} - a_{13}a_{24} + a_{14}a_{23} - a_{15}a_{26} + a_{16}a_{25} = 0.$$

The bivector $w = a(v_3^* \wedge v_4^* - v_5^* \wedge v_6^*) + a_{12}v_1^* \wedge v_2^* + \cdots + a_{25}v_2^* \wedge v_5^*$ is of rank ≤ 2 if and only if $a = 0$ and

$$\operatorname{rank}\begin{pmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix} \leq 1.$$

Therefore, $\check{Q} \cap G(A, 2)$ coincides with $< \check{Q} > \cap G(A, 2)$ and is set-theoretically the cone over the Segre variety $\mathbf{P}^1 \times \mathbf{P}^3 \subset \mathbf{P}^7$ with the vertex $[v_1^* \wedge v_2^*]$. □

We compute the canonical class and degree of Grassmannians.

Proposition 1.9 *The anti-canonical class of the Grassmannian $G(r, n)$ is n times the hyperplane section class of the Plücker embedding $G(r, n) \subset \mathbf{P}^{\binom{n}{r}-1}$.*

Proof. Let A be an n-dimensional vector space. For every r-dimensional subspace U of A, the tangent space of $G(r, A)$ at the point $[U]$ is canonically isomorphic to $\mathrm{Hom}\,(U, A/U)$. Let

$$0 \longrightarrow \mathcal{F}^\vee \longrightarrow A \otimes_k \mathcal{O}_G \longrightarrow \mathcal{E} \longrightarrow 0$$

be the universal exact sequence on $G(r, A)$. $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of rank r and $n-r$, respectively. Their determinant are the restriction of the tautological line bundle. Since the tangent bundle of $G(r, A)$ is isomorphic to $\mathcal{H}om(\mathcal{F}^\vee, \mathcal{E}) \simeq \mathcal{F} \otimes \mathcal{E}$, the anti-canonical class of $G(r, A)$ is n times the hyperplane section class. □

The Grassmannian $G(r, n)$ is a homogeneous space of $PGL(n)$. Let $\alpha_i = e_i - e_{i+1}$, $1 \leq i < n$, be the standard root basis of the Lie algebra $\underline{g}$ of $PGL(n)$. The stabilizer group P belongs to the conjugacy class of maximal parabolic subgroups corresponding to the rth fundamental weight w_r. Let $\underline{p} \subset \underline{g}$ be the Lie algebra of P. The tangent space of $G(r, n)$ (at the base point) is isomorphic to $\underline{g}/\underline{p}$ and spanned by $r(n-r)$ roots $e_i - e_j$ with $1 \leq i \leq r < j \leq n$, which are called *the positive complementary roots*. Their sum, which corresponds to the anti-canonical class of $G(r, n)$, is equal to nw_r. This is another proof of the above proposition since the line bundle L which gives the Plücker embedding of $G(r, n)$ corresponds to w_r. By [2], the self-intersection number of L is equal to

$$N! \prod_\beta \frac{(\beta.w_r)}{(\beta.\rho)},$$

where β runs over all positive complementary roots, $N = \dim G(r, n) = r(n-r)$ and $\rho = w_1 + \cdots + w_{n-1}$. Therefore, we have deduced the following classical formula:

Proposition 1.10 *The degree of the Grassmannian $G(r,n) \subset \mathbf{P}^{\binom{n}{r}-1}$ is equal to*

$$(r(n-r))! \prod_{1 \le i \le r < j \le n} (j-i)^{-1}$$

Corollary 1.11 *The degree of $G(2,n) \subset \mathbf{P}^{n(n-3)/2}$ is equal to the Catalan number* $(2n-4)!/(n-1)!(n-2)!$.

2 Linear sections of a Grassmannian

Let U_1, U_2, U_3 and U_4 be four distinct 2-dimensional subspaces of a vector space A. For $I \subset \{1,2,3,4\}$, we denote by P_I the linear span of $[U_i] \in G(2,A)$ with $i \in I$ in $\mathbf{P}_*(\bigwedge^2 A)$. We study the intersection of P_I and $G(2,A)$ and prove the 'only if' part of Main theorem.

Lemma 2.1 *The intersection $P_{12} \cap G(2,A)$ consists of $[U_1]$ and $[U_2]$ if $U_1 \cap U_2 = 0$. The line P_{12} is contained in $G(2,A)$ otherwise.*

The proof is straightforward.

Lemma 2.2 *The intersection $P_{123} \cap G(2,A)$ consists of $[U_1]$, $[U_2]$ and $[U_3]$ if $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = 0$ and $\dim U_1 + U_2 + U_3 \ge 5$. $P_{123} \cap G(2,A)$ is of positive dimension otherwise.*

Proof. Since P_{123} is contained in $\mathbf{P}_*(\bigwedge^2(U_1+U_2+U_3))$ and since $P_{123} \cap G(2,A) = P_{123} \cap G(2, U_1+U_2+U_3)$, we may assume that $A = U_1+U_2+U_3$. By Lemma 2.1, it suffices to consider the case $U_1 \cap U_2 = U_2 \cap U_3 = U_3 \cap U_1 = 0$, which implies $\dim A \ge 4$.

Case $\dim A = 4$: Since $G(2,A) \subset \mathbf{P}_*(\bigwedge^2 A)$ is a hyperquadric, we have $\dim P_{123} \cap G(2,A) > 0$.

Case $\dim A = 5$: We choose a basis $\{v_1, v_2, v_3, v_4, v_5\}$ of A so that $U_1 =< v_1, v_4 >$, $U_2 =< v_2, v_5 >$ and $U_3 =< v_3, -v_4 - v_5 >$. A point in P_{123} is represented by a bivector $w = av_1 \wedge v_4 + bv_2 \wedge v_5 + cv_3 \wedge (-v_4 - v_5)$. The reduced square $w^{[2]}$ defined in (1.3) is equal to

$$-abv_1 \wedge v_2 \wedge v_4 \wedge v_5 + acv_1 \wedge v_3 \wedge v_4 \wedge v_5 - bcv_2 \wedge v_3 \wedge v_4 \wedge v_5.$$

It follows that $P_{123} \cap G(2,A)$ contains no other points than $[U_1]$, $[U_2]$ and $[U_3]$.

Case $\dim A = 6$: A is the direct sum of U_1, U_2 and U_3. We have $P_{123} \cap G(2,A) = \{[U_1],[U_2],[U_3]\}$ by the same argument as above. □

If $\dim A = 5$, then $G(2,A) \subset \mathbf{P}_*(\bigwedge^2 A)$ is of degree 5 by Corollary 1.14 and of codimension 3. Hence for general U_1, U_2, U_3 and U_4, the intersection $P_{1234} \cap G(2,A)$ consists of five points. Now we assume that $\dim A = 6$.

Lemma 2.3 *The intersection $P_{1234} \cap G(2,A)$ consists of $[U_1]$, $[U_2]$, $[U_3]$ and $[U_4]$ if U_1, U_2, U_3 and U_4 satisfy,*
i) $U_i \cap U_j = 0$ *for every* $1 \le i < j \le 4$,
ii) $\dim U_i + U_j + U_k \ge 5$ *for every* $1 \le i < j < k \le 4$, *and*
iii) $U_1 + U_2 + U_3 + U_4 = A$.

Proof. First we consider the case where $U_i+U_j+U_k=A$ for every $1\le i<j<k\le 4$. A is the direct sum of U_1, U_2 and U_3. U_4 is generated by two vectors $v_+=v_1+v_3+v_5$ and $v_-=v_2+v_4+v_6$ for $v_1,v_2\in U_1$, $v_3,v_4\in U_2$ and $v_5,v_6\in U_3$. Then $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ is a basis of A. A point in P_{1234} is represented by a bivector

$$\begin{aligned} w &= av_1\wedge v_2+bv_3\wedge v_4+cv_5\wedge v_6+d(v_1+v_3+v_5)\wedge(v_2+v_4+v_6)\\ &= a'v_1\wedge v_2+b'v_3\wedge v_4+c'v_5\wedge v_6\\ &\quad +d(v_1\wedge v_4+v_1\wedge v_6-v_2\wedge v_3-v_2\wedge v_5+v_3\wedge v_6-v_4\wedge v_5),\end{aligned}$$

for some $a,b,c,d\in k$, where we put $a'=a+d$, $b'=b+d$ and $c'=c+d$. A direct computation shows

$$\begin{aligned} w^{[2]} &= (a'b'-d^2)v_1\wedge v_2\wedge v_3\wedge v_4+(a'c'-d^2)v_1\wedge v_2\wedge v_5\wedge v_6\\ &\quad +(b'c'-d^2)v_3\wedge v_4\wedge v_5\wedge v_6+(a'c'-d^2)v_1\wedge v_2\wedge v_5\wedge v_6\\ &\quad +(a'd-d^2)v_1\wedge v_2\wedge(v_3\wedge v_6-v_4\wedge v_5)\\ &\quad +(b'd-d^2)v_3\wedge v_4\wedge(v_1\wedge v_6-v_2\wedge v_5)\\ &\quad +(c'd-d^2)v_5\wedge v_6\wedge(v_1\wedge v_4-v_2\wedge v_3).\end{aligned}$$

Hence $[w]$ belongs to $G(2,A)$ if and only if $ad=bd=cd=ab=bc=ac=0$. Therefore, the intersection $P_{1234}\cap G(2,A)$ consists of $[U_1]$, $[U_2]$, $[U_3]$ and $[U_4]$.

Next we assume that three subspaces, say U_1, U_2 and U_3, do not generate A. By our assumption, we can take a basis $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ of A so that $U_1=<v_1,v_4>$, $U_2=<v_2,v_5>$, $U_3=<v_3,-v_4-v_5>$ and $v_6\in U_4$. U_4 is generated by v_6 and a nonzero vector v in $U_4\cap(U_1+U_2+U_3)$. A point in P_{1234} is represented by a bivector

$$w=av_1\wedge v_4+bv_2\wedge v_5+cv_3\wedge(-v_4-v_5)+dv\wedge v_6$$

for some $a,b,c,d\in k$ and we have

$$\begin{aligned} w^{[2]} &= -abv_1\wedge v_2\wedge v_4\wedge v_5+acv_1\wedge v_3\wedge v_4\wedge v_5-bcv_2\wedge v_3\wedge v_4\wedge v_5\\ &\quad +adv_1\wedge v_4\wedge v\wedge v_6+bdv_2\wedge v_5\wedge v\wedge v_6+cdv_3\wedge(-v_4-v_5)\wedge v\wedge v_6\end{aligned}$$

Assume that $[w]$ belongs to $G(2,A)$. Then $ab=ac=bc=0$ and two of a, b and c are zero. If $b=c=0$ for example, then $w=av_1\wedge v_4+dv\wedge v_6$. Since $v\notin U_1=<v_1,v_4>$ by our assumption i), either a or d is equal to zero. Therefore, $P_{1234}\cap G(2,A)$ consists of $[U_1]$, $[U_2]$, $[U_3]$ and $[U_4]$. □

Remark 2.4 As is seen from the proof, the intersection $P_{1234}\cap G(2,A)$ is the 0-dimensional reduced scheme consisting of $[U_1]$, $[U_2]$, $[U_3]$ and $[U_4]$ under the assumption i), ii) and iii).

By these lemmas, we have

Proposition 2.5 (1) *For every line ℓ in $\mathbf{P}_*(\bigwedge^2 A)$, the cardinality of the intersection $\ell\cap G(2,A)$ is either less than three or infinite.*

(2) *For every plane P in $\mathbf{P}_*(\bigwedge^2 A)$, the cardinality of the intersection $P\cap G(2,A)$ is either less than four or infinite.*

(3) *Assume that* $\dim A=6$ *and let R be a 3-plane in $\mathbf{P}_*(\bigwedge^2 A)$. If the cardinality of the intersection $R\cap G(2,A)$ is finite and greater than four, then there exists a 5-dimensional subspace A' of A such that $R\subset\mathbf{P}_*(\bigwedge^2 A')$.*

Let $\mathbf{P}$ be a linear subspace of $\mathbf{P}_*(\bigwedge^2 A)$ such that the intersection $C = \mathbf{P} \cap G(2, A)$ is of dimension one.

Corollary 2.6 (1) *$C \subset \mathbf{P}^7$ has no trisecant lines or 4-secant planes.*

(2) *Assume that $\dim A = 6$. If R is a 5-secant 3-plane of $C \subset \mathbf{P}^7$, then there exists a 5-dimensional subspace A' of A such that $R \cap C \subset G(2, A')$.*

Assume that $\dim A = 6$ and let $C \subset \mathbf{P}^7$ be a transversal intersection of $G(2, A) \subset \mathbf{P}_*(\bigwedge^2 A)$ and seven hyperplanes $H_1, \cdots, H_7$. The canonical class of C is linearly equivalent to a hyperplane section by Proposition 1.12 and the adjunction formula. By the lemma of Enriques-Severi-Zariski ([6], p. 244), C is connected and the linear map

$$(\bigwedge^2 A^\vee)/ < f_1, \cdots, f_7 > \longrightarrow H^0(C, \omega_C)$$

is injective, where f_i is a linear form defining the hyperplane H_i for $1 \le i \le 7$. Since $G(2, 6) \subset \mathbf{P}^{14}$ is of degree 14 by Corollary 1.14, C is of genus 8 and the above map is surjective. Hence we have

Proposition 2.7 *A transversal linear section $C \subset \mathbf{P}^7$ of $G(2, 6) \subset \mathbf{P}^{14}$ is a canonical curve of genus 8.*

For an effective divisor $D = p_1 + \cdots + p_d$ on a curve C of genus g, the Riemann-Roch theorem is written as

$$\dim |K_C| - \dim |K_C - p_1 - \cdots - p_d| - 1 = d - \dim H^0(\mathcal{O}_X(D)). \qquad (2.7)$$

The left hand side is the dimension of the linear span of the d points $p_1, \cdots, p_d \in C \subset \mathbf{P}^{g-1}$ on the canonical model. Hence the d points are linearly dependent if and only if $\dim |D| > 0$.

Lemma 2.8 *A transversal linear section C of $G(2, 6) \subset \mathbf{P}^{14}$ has no g^1_4. If an effective divisor D is a g^1_5 of C, then there exists a 5-dimensional subspace A' of A such that $D \subset C \cap G(2, A')$.*

Proof. Let ξ be a g^1_d and $\{D_t = p_{1,t} + \cdots + p_{d,t} | t \in \mathbf{P}^1\}$ the linear system associated to it. By (2.7), $p_{1,t}, \cdots, p_{d,t}$ are linearly dependent for every $t \in \mathbf{P}^1$. Hence C has no g^1_4 by Corollary 2.6 and Bertini's theorem. If $d = 5$ and if D_t is reduced, then there exists a 5-dimensional subspace A_t of A such that $D_t \subset C \cap G(2, A_t)$. Since $G(5, A) \simeq \mathbf{P}^5$ is complete, this holds true for every $t \in \mathbf{P}^1$. □

Assume that $C = G(2, 6) \cap \mathbf{P}^7$ has a g^2_7, which we denote by α. By the genus formula of a plane curve, $|\alpha|$ contains $D = D_1 \cup D_2$ such that both D_1 and D_2 are g^1_5's and $\deg D_1 \cap D_2 = 3$, where $D_1 \cup D_2$ is the smallest divisor dominating both D_1 and D_2, and $D_1 \cap D_2$ the largest one dominated by both. By the lemma, D_1 and D_2 are contained in $G(2, A_1)$ and $G(2, A_2)$ for 5-dimensional subspaces A_1 and A_2 of A. Hence $D_1 \cap D_2$ is contained in the 4-dimensional Grassmannian $G(2, A_1 \cap A_2)$, which is a contradiction. Thus we have proved the 'only if' part of the Main Theorem.

3 2-bundles with canonical determinant

Let C be a curve and E a vector bundle of rank 2 on C with $\bigwedge^2 E \simeq \omega_C$. The following is a variant of the base-point-free pencil trick and very useful for our study of bundles on a curve.

Proposition 3.1 *If a line bundle ζ on C is generated by global sections, then*

$$\dim \operatorname{Hom}(\zeta, E) \geq h^0(E) - \deg \zeta.$$

Proof. ζ is generated by two global sections and we have the exact sequence

$$0 \longrightarrow \zeta^{-1} \longrightarrow \mathcal{O}_C^{\oplus 2} \longrightarrow \zeta \longrightarrow 0.$$

Tensoring E and taking H^0, we have

$$h^0(\zeta^{-1}E) + h^0(\zeta E) \geq 2h^0(E).$$

By the Riemann-Roch theorem, we have

$$h^0(\zeta^{-1}E) - h^0(\omega_C \zeta E^\vee) = \deg(\zeta^{-1}E) + 2(1-g) = -2\deg \zeta.$$

Since $\zeta E \simeq \omega_C \zeta E^\vee$, the arithmetic mean of these two inequalities is the desired one. □

Let ξ be a line bundle and η its Serre adjoint. Then $\xi \oplus \eta$ is a 2-bundle with canonical determinant. Applying the proposition to this vector bundle, we have

Corollary 3.2 *If ζ is generated by global sections and if $\deg \zeta < h^0(\xi) + h^0(\eta)$, then there exists a nonzero homomorphism of ζ to ξ or to η.*

We recall the general existence theorem of special divisors (Chap. 7, [1]):

Theorem 3.3 *Let C be a curve of genus g, and d and r non-negative integers. If $(r+1)(r-d+g) \leq g$ holds, then C has a g^r_d.*

Let C be a curve of genus 8 and assume that C has no g^1_4. By the theorem, C has a g^1_5, which we denote by ξ. ξ is free by our assumption.

Lemma 3.4 *C has no g^2_6.*

Proof. We show the existence of a g^1_4 assuming that of a g^2_6. There exists a morphism $C \longrightarrow \mathbf{P}^2$ of degree ≤ 6, whose image $\bar{C}$ is not contained in a line. If $\bar{C}$ is a conic, C has a g^1_3. If $\bar{C}$ is a cubic, C has a g^1_4 since $\bar{C}$ has a g^1_2. If $\deg \bar{C} \geq 4$, then $C \longrightarrow \bar{C}$ is birational and $\bar{C}$ is singular by the genus formula. The projection from a singular point gives rise to a g^1_4. □

The Serre adjoint η of ξ is a g^3_9.

Lemma 3.5 *$|\eta|$ is free, $\dim |\eta| = 3$ and $\Phi_{|\eta|} : C \longrightarrow \mathbf{P}^3$ is birational onto its image.*

Proof. By Lemma 3.4, C has no g^3_8. Hence $\dim|\eta(-p)| \leq 2$ for every point $p \in C$ which shows the first two assertions. By our assumption, C is not trigonal, from which the last assertion follows. □

We consider extensions $0 \longrightarrow \xi \longrightarrow E \longrightarrow \eta \longrightarrow 0$ of ξ by η. Let $e \in \mathrm{Ext}\,(\eta, \xi)$ be the extension class and $\delta_e : H^0(\eta) \longrightarrow H^1(\xi)$ the coboundary map. Since $h^0(\xi) + h^0(\eta) = 6$, $h^0(E) = 6$ is equivalent to $\delta_e = 0$, that is, e lies in the kernel of the linear map

$$\Delta : \mathrm{Ext}\,(\eta, \xi) \longrightarrow H^0(\eta)^\vee \otimes H^1(\xi), \quad e \mapsto \delta_e.$$

Lemma 3.6 $\dim \mathrm{Ker}\,\Delta = 1$.

Proof. The group $\mathrm{Ext}\,(\eta, \xi)$ is isomorphic to the first cohomology group $H^1(\eta^{-1}\xi)$, which is the dual of $H^0(\eta^2)$ by the Serre duality. Hence the linear map Δ is the dual of the multiplication map

$$m : H^0(\eta) \otimes H^0(\eta) \longrightarrow H^0(\eta^2).$$

Since C has no g^1_4, no quadric surface contains the image $C_9 \subset \mathbf{P}^3$ of $\Phi_{|\eta|}$, that is, the linear map $S^2H^0(\eta) \longrightarrow H^0(\eta^2)$ induced by m is injective. Since $\dim H^0(\eta^2) = 11$ by the Riemann-Roch theorem, the cokernel of multiplication map m is of dimension one. □

By the lemma, there exists a unique non-trivial extension of η by ξ with linearly independent six global sections, which we denote by E_ξ . E_ξ is semi-stable by Lemma 3.4 and the following:

Lemma 3.7 $\dim H^0(\zeta) \geq 3$ *for every quotient line bundle* ζ *of* E_ξ.

Proof. Let f be the composite of the natural inclusion $\xi \hookrightarrow E_\xi$ and surjection $E_\xi \longrightarrow \zeta$. If $f = 0$, then $\zeta = \eta$ and $h^0(\zeta) = 4$. So we assume that $f \neq 0$. There exist a nonzero effective divisor D such that $\zeta \simeq \xi(D)$ and an exact sequence $0 \longrightarrow \eta(-D) \longrightarrow E \longrightarrow \xi(D) \longrightarrow 0$. Since $|\eta|$ is free by Lemma 3.5, we have $h^0(\xi(D)) \geq h^0(E) - h^0(\eta(-D)) \geq 3$. □

Proof of Theorem A: Let C be a curve of genus 8 and assume that C has no g^2_7.

Lemma 3.8 *C has no* g^1_4.

Proof. We show the existence of a g^2_7 assuming that of a g^1_4. Let ξ be a g^1_4 of C. We may assume that C has no g^2_6, which implies that C has no g^3_8 or g^1_3. In particular, $|\xi|$ is free and the Serre adjoint η of ξ is very ample. The image of $\Phi_{|\eta|}$ is a curve $C_{10} \subset \mathbf{P}^4$ of degree 10. Hence a g^2_7 is obtained by projecting off a trisecant line. The existence of a trisecant line follows from the Berzolari formula

$$\Theta(C) = (n-2)(n-3)(n-4)/6 - g(n-4)$$

([9]), where $n = \deg C$ and g is the genus. In fact, the number of trisecant lines $\Theta(C_{10})$ of $C_{10} \subset \mathbf{P}^4$ is equal to 8 in our case. □

Let ξ be a g^1_5 on C. E_ξ is stable by Lemma 3.7 and by our assumption. Let E be a stable bundle with canonical determinant and with $h^0(E) \geq 6$. Then there is a nonzero homomorphism $f : \xi \longrightarrow E$ by Proposition 3.1. $f(\xi)$ is a line subbundle by the lemma below. Therefore, we have $h^0(E) \leq h^0(\xi) + h^0(\omega_C \xi^{-1}) = 6$. The uniqueness of E follows from Lemma 3.6. Since η and ξ are generated by global sections, so is E. This completes the proof of Theorem A.

Lemma 3.9 *For every line subbundle L of E, $h^0(L) \leq 2$. Moreover, if $h^0(L) = 2$, then L is a g^1_5.*

Proof. Let L be a line subbundle of E with $h^0(L) \geq 2$. Then we have $\deg L < 7$ by the stability of E and $h^0(L) = 2$ since C has no g^2_6. Since $h^0(\omega_C L^{-1}) \geq h^0(E) - h^0(L) \geq 4$, we have $\deg L = h^0(L) - h^0(\omega_C L^{-1}) + 7 \leq 5$ by the Riemann-Roch theorem. Therefore, L is a g^1_5 by Lemma 3.8. □

Lemma 3.10 *For every g^1_5 ξ of C,* $\dim \mathrm{Hom}\,(\xi, E) \leq 1$.

Proof. Let f_1 and f_2 be two homomorphisms of ξ to E. We have two exact sequences

$$0 \longrightarrow f_1(\xi) \longrightarrow E \longrightarrow \eta \longrightarrow 0$$

and

$$0 \longrightarrow f_2(\xi) \longrightarrow E \longrightarrow \eta \longrightarrow 0,$$

where η is the Serre adjoint of ξ. By Lemma 3.6, there exists an isomorphism of E which maps $f_1(\xi)$ onto $f_2(\xi)$. Since E is simple, f_1 is a constant multiple of f_2. □

Proof of Theorem B: Let U be a 2-dimensional subspace of $H^0(E)$ such that $\lambda(\bigwedge^2 U) = 0$. Then the evaluation map $U \otimes \mathcal{O}_C \longrightarrow E$ is not generically surjective. Its image is a line subbundle and a g^1_5 by Lemma 3.9. Hence we obtain a map from the intersection $\mathbf{P}_*(\mathrm{Ker}\,\lambda) \cap G(2, V)$ to $W^1_5(C)$. This map is injective by Lemma 3.10 and surjective by Proposition 3.1.

Proof of Theorem C: The map $\lambda : \bigwedge^2 V \longrightarrow H^0(\omega_C)$ is surjective since $\dim G(2, V) = 8 = \dim H^0(\omega_C)$. Hence $\mathbf{P}^*(\lambda)$ is an embedding. Since $C \subset \mathbf{P}^7$ is an intersection of quadrics by the Enriques-Petri theorem, it suffices to show

Claim : The restriction map $I_{G(V,2),2} \longrightarrow I_{C,2}$ is surjective, where $I_{G(V,2),2}$ is the vector space $\simeq \bigwedge^4 V$ generated by the Plücker quadratic forms and $I_{C,2}$ is the vector space of quadratic forms which vanish on C.

Since $S^2 H^0(\omega_C) \longrightarrow H^0(\omega_C^2)$ is surjective by Noether's theorem, $I_{C,2}$ is of dimension 15. $I_{G(V,2),2}$ is also of dimension 15. So we show the injectivity of the restriction map, instead. By Proposition 1.7, $q \in I_{G(V,2),2}$ is of rank 6, 10 or 15. If $\mathrm{rank}\, q = 15$, $Q : q = 0$ is a smooth 13-dimensional quadric and contains no 7-plane. Hence q is not identically zero on the image $P \simeq \mathbf{P}^7$ of $\mathbf{P}^*(\lambda)$. If $\mathrm{rank}\, q = 6$, then the projective dual $\check{Q}$ of Q is contained in $G(2, V)$ by Proposition 1.11. Hence the intersection $\mathbf{P}_*(\mathrm{Ker}\,\lambda) \cap \check{Q}$ is finite by our assumption and Q does not contain P by Proposition 1.10. If $\mathrm{rank}\, q = 10$, $\check{Q}$ is an

8-dimensional quadric in the 9-plane $<\check{Q}>\subset \mathbf{P}_*(\bigwedge^2 V)$. By Proposition 1.11, the intersection $M = \check{Q} \cap G(2, V)$ is of dimension 5 and hence numerically equivalent to a positive multiple of the cubic power of a hyperplane section of $\check{Q}$. Hence every 4-dimensional subvariety of $\check{Q}$ intersects M in a positive dimensional set. Hence $\dim(\mathbf{P}_*(\mathrm{Ker}\,\lambda) \cap \check{Q}) \le 3$ by our assumption. Therefore, Q does not contain the image P by Proposition 1.10, which completes the proof of Theorem C.

4 4-secant lines of $C_9 \subset \mathbf{P}^3$

Let C be a curve of genus 8. If ξ is a g^1_5, then its Serre adjoint η is a g^3_9. In this section, investigating the image of $\Phi_{|\eta|}$, we prove the following

Theorem 4.1 *If C has no g^1_4, then C has only finitely many g^1_5's.*

Let $C \subset \mathbf{P}^3$ be a smooth space curve of genus 8 and degree 9.

Proposition 4.2 *The following two conditions are equivalent to each other.*
(1) $C \subset \mathbf{P}^3$ *has a 5-secant line.*
(2) $C \subset \mathbf{P}^3$ *is contained in a cubic surface.*
Moreover, if these equivalent conditions are satisfied, then $C \subset \mathbf{P}^3$ has only finitely many 4-secant lines.

Let $\ell \subset \mathbf{P}^3$ be a 5-secant line of $C \subset \mathbf{P}^3$ and put $I_\ell/I_C \simeq \mathcal{O}_C(-p_1 - \cdots - p_5) \subset \mathcal{O}_C$. Let $|3h - \ell|$ be the linear system of cubic surfaces containing ℓ and

$$|3h - \ell| \cdots \longrightarrow |3h_C - p_1 - \cdots - p_5|$$

the restriction (rational) map, where h_C is a hyperplane section class of C. Since $\dim |3h - \ell| = 15$ and $\dim |3h_C - p_1 - \cdots - p_5| = 14$, there exists a cubic surface containing C. This shows (1) $\Rightarrow$ (2).

Conversely assume that C is contained in a cubic surface S. Since C is not contained in a quadric surface by the genus formula, S is irreducible.

Lemma 4.3 *S has no triple points.*

Proof. Assume the contrary. Then S is a cone over a plane cubic. Since $\deg C = 9$, C does not pass the vertex of S and each generating line intersects C at three points. Since the blow-up of S at the vertex has Picard number 2, C is cut out by another cubic surface, which contradicts $g(C) = 8$. □

Lemma 4.4 *S has only isolated singularities.*

Proof. Assume the contrary. Then the singular locus is a line and the normalization $\tilde{S}$ of S is the blow-up of $\mathbf{P}^2$ at a point p. A plane section of $S \subset \mathbf{P}^3$ is transformed to a conic passing through the point p. Let $\bar{C} \subset \mathbf{P}^2$ be the transform of C. If $\bar{C}$ is of degree d and has multiplicity μ at p, then we have $(d-1)(d-2)/2 - m(m-1)/2 = 8$ and $2d - m = 9$, which has no integral solution. □

Lemma 4.5 *Let $S \subset \mathbf{P}^3$ be a cubic surface with only isolated double points as its singularity and C a smooth curve on S. Then there exists a birational morphism π from a minimal resolution $\tilde{S}$ of S onto $\mathbf{P}^2$ which satisfies*

(1) π is the blowing up of at six points $p_1, \cdots, p_6$, and

(2) the strict transform $\tilde{C} \subset \tilde{S}$ of C is linearly equivalent to $dL - a_1E_1 - \cdots - a_6E_6$ with $d \geq a_1 + a_2 + a_3$ and $a_1 \geq a_2 \geq \cdots \geq a_6 \geq 0$, where E_i is (the total transform of) the exceptional divisor over p_i for each $1 \leq i \leq 6$ and L is the pull-back of a line.

Proof. The existence of π satisfying (1) is well known in the case S is smooth. If S is singular, the projection off a singular point induces a morphism π satisfying (1). Relabeling $p_1, \cdots, p_6$, we may assume that $\tilde{C}$ is linearly equivalent to either

a) $dL - a_1E_1 - \cdots - a_6E_6$ with $a_1 \geq a_2 \geq \cdots \geq a_6 \geq 0$, or

b) E_3.

If $d < a_1 + a_2 + a_3$ in the former case or if $\tilde{C} \sim E_3$, we make the quadratic transformation with center p_1, p_2 and p_3. Then we have new expression

a) $\tilde{C} \sim d'L - a_1'E_1 - a_2'E_2 - a_3'E_3 - a_4E_4 - a_5E_5 - a_6E_6$, or

b) $\tilde{C} \sim 2L - E_1 - E_2$.

Since $d' = 2d - a_1 - a_2 - a_3 < d$, repeating this process, we have (2). $\square$

Applying the proposition to the space curve $C \subset S \subset \mathbf{P}^3$ of degree 9, we have that $\tilde{C}$ is linearly equivalent to $dL - a_1E_1 - \cdots - a_6E_6$ for integers $d, a_1, \cdots, a_6$ satisfying

$$\begin{cases} d \geq a_1 + a_2 + a_3, \ a_1 \geq a_2 \geq \cdots \geq a_6 \geq 0 \\ 3d - a_1 - a_2 - a_3 - a_4 - a_5 - a_6 = 9, \quad \text{and} \\ d(d-1) - a_1(a_1 - 1) - \cdots - a_6(a_6 - 1) = 16. \end{cases}$$

This has the unique integral solution

$$\tilde{C} \sim 7L - 3E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - E_6.$$

Let m be the strict transform by π of a conic passing through $p_2, \cdots, p_6$. Then m is a 5-secant line of $C \subset \mathbf{P}^3$ since $(m.\tilde{C}) = 5$ and $(-K_S.m) = 1$, which completes the proof of (2) $\Rightarrow$ (1). Every 4-secant line of C is contained in the cubic surface S, which contains only finitely many lines. Therefore, we have the second half of Theorem 4.2.

Proposition 4.6 *There exists a surface of degree ≤ 7 which is singular along $C \subset \mathbf{P}^3$.*

Proof. Since C is smooth, we have the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbf{P}}|_C \longrightarrow N_{C/\mathbf{P}} \longrightarrow 0.$$

Since $N_{C/\mathbf{P}}$ is of rank 2, we have

$$N_{C/\mathbf{P}}^\vee \simeq N_{C/\mathbf{P}} \otimes \det N_{C/\mathbf{P}}^{-1} \simeq N_{C/\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(-4) \otimes \omega_C^{-1}.$$

Since $T_{\mathbf{P}}$ is a quotient of $\mathcal{O}_{\mathbf{P}}(1)^{\oplus 4}$, $N_{C/\mathbf{P}}^\vee \otimes \mathcal{O}_{\mathbf{P}}(7)$ is a quotient of $(\mathcal{O}_{\mathbf{P}}(4) \otimes \omega_C^{-1})^{\oplus 4}$. Since $H^1(\mathcal{O}_{\mathbf{P}}(4) \otimes \omega_C^{-1})$ vanishes, $H^1(C, N_{C/\mathbf{P}}^\vee \otimes \mathcal{O}_{\mathbf{P}}(7))$ also vanishes and we have

$$\dim H^0(C, N_{C/\mathbf{P}}^\vee \otimes \mathcal{O}_{\mathbf{P}}(7)) = \deg(N_{C/\mathbf{P}}^\vee \otimes \mathcal{O}_{\mathbf{P}}(7)) + 2(1 - g(C)) = 62$$

by the Riemann-Roch theorem. Since $N^\vee_{C/\mathbf{P}} \simeq I_C/I_C^2$, we have

$$\begin{array}{rl} & \dim H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}}(7) \otimes I_C^2) \\ \geq & \dim H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}}(7)) - \dim H^0(C, \mathcal{O}_{\mathbf{P}}(7) \otimes \mathcal{O}_C) - \dim H^0(C, \mathcal{O}_{\mathbf{P}}(7) \otimes N^\vee_{C/\mathbf{P}}) \\ = & 120 - 56 - 62 = 2. \end{array}$$

□

The surface in the proposition contains all 4-secant lines of $C \subset \mathbf{P}^3$.

Lemma 4.7 *Assume that $C \subset \mathbf{P}^3$ is not contained in a cubic surface and that two 4-secant lines ℓ and m intersect at a point p. Then we have*
(1) *p lies on C, and*
(2) *the two lines ℓ, m and the tangent line of C at p are not contained in a plane.*

Proof. The idea of proof has already appeared in the proof of Proposition 4.2. Assume that $p \notin C$ and put $I_\ell/I_C \simeq \mathcal{O}_C(-p_1 - p_2 - p_3 - p_4)$ and $I_m/I_C \simeq \mathcal{O}_C(-q_1 - q_2 - q_3 - q_4)$. Let $|3h - \ell - m|$ be the linear system of cubic surfaces containing ℓ and m, and

$$|3h - \ell - m| \cdots \longrightarrow |3h_C - p_1 - \cdots - q_5|$$

the restriction map. Since $\dim |3h - \ell - m| = 12$ and $\dim |3h_C - p_1 - \cdots - q_5| = 11$, there exists a member of $|3h - \ell - m|$ which contains C. But this contradicts our assumption and shows (1). Next assume that the plane spanned by ℓ and m is tangent to C at p. Put $I_\ell/I_C \simeq \mathcal{O}_C(-p - p_1 - p_2 - p_3)$ and $I_m/I_C \simeq \mathcal{O}_C(-p - q_1 - q_2 - q_3)$. Every surface containing ℓ and m tangents C at p. Hence we have the restriction map

$$|3h - \ell - m| \cdots \longrightarrow |3h_C - 2p - p_1 - \cdots - q_3|.$$

The rest of the proof of (2) is same as (1). □

Proposition 4.8 *A smooth space curve $C \subset \mathbf{P}^3$ of genus 8 and degree 9 has only finitely many 4-secant lines.*

Proof. By Proposition 4.2, we may assume that $C \subset \mathbf{P}^3$ is not contained in a cubic surface and that $C \subset \mathbf{P}^3$ has no 5-secant lines. Assume that $C \subset \mathbf{P}^3$ has a 1-dimensional family of 4-secant lines and Let $\bar{S}$ be the surface swept out by them. The degree d of $\bar{S}$ is ≤ 7 by Proposition 4.6 and ≥ 4 by our assumption. Let X be the blow-up of $\mathbf{P}^3$ along C and S the strict transform of $\bar{S}$. By the Lemma 4.7, S has a $\mathbf{P}^1$-bundle structure $\pi : S \longrightarrow N$ over a curve N whose fibres are the strict transforms of 4-secant lines. Let D be the exceptional divisor of the blowing up $X \longrightarrow \mathbf{P}^3$ and H the pull-back of a plane. Then the canonical class K_X of X is linearly equivalent to $-4H + D$. Since S is a $\mathbf{P}^1$-bundle, we have $-2 = (K_S.f) = (K_X + S.f) = (S.f)$. Let μ be the multiplicity of $\bar{S}$ along C. Then S belongs to the linear system $|dH - \mu D|$ and we have $-2 = (dH - \mu D.f) = d - 4\mu$. Since $4 \leq d \leq 7$, we have $d = 6$ and $\mu = 2$. Since $(H^3) = 1$, $(H^2.D) = 0$, $(H.D^2) = -9$ and $(D^3) = -50$, we have

$$-8(p_a(N) - 1) = (K_S^2) = ((K_X + S)^2.S) = ((2H - D)^2.(6H - 2D)) = -2,$$

which is a contradiction. □

Proof of Theorem 4.1: Let C be a curve of genus 8 and assume that C has no g^1_4. Let ξ be a g^1_5 of C, which is generated by global sections by our assumption. By Corollary 3.2, we have

(*) $\mathrm{Hom}\,(\zeta, \eta) \neq 0$ for every g^1_5 ζ different from ξ,

where η is the Serre adjoint of ξ. The image $\bar{C}$ of $\Phi_{|\eta|}$ is a space curve of degree 9 by Lemma 3.5. If $\bar{C}$ is smooth, every g^1_5 different from ξ is induced from the projection off a 4-secant line of $\bar{C}$ by (*). Hence the number of g^1_5 is finite by Proposition 4.8. If $\bar{C}$ is singular, the projection off a singular point gives rise to a g^2_7, which we denote by α. $|\alpha|$ is free and $h^0(\alpha) = 3$ by Lemma 3.4. Hence the image of $\Phi_{|\alpha|}$ is a plane curve of degree 7. The same holds for the Serre adjoint β of α. Since $h^0(\alpha) + h^0(\beta) = 6$, either $\mathrm{Hom}\,(\zeta, \alpha) \neq 0$ or $\mathrm{Hom}\,(\zeta, \beta) \neq 0$ holds for every g^1_5 ζ of C, by Corollary 3.2. Hence every g^1_5 of C is induced from the projection off a double point of the plane curves $\Phi_{|\alpha|}(C)$ or $\Phi_{|\beta|}(C)$. Therefore, C has only finitely many g^1_5's.

5 Curves of genus 6

A 2-dimensional complete linear section $S_5 \subset \mathbf{P}^5$ of the 6-dimensional Grassmannian $G(2,5) \subset \mathbf{P}^9$ is a quintic del Pezzo surface. A hyperquadric section $C_{10} \subset \mathbf{P}^9$ of $S_5 \subset \mathbf{P}^5$ is a canonical curve of genus 6 by the adjunction formula and Proposition 1.12. Since $C_{10} \subset \mathbf{P}^9$ is an intersection of quadrics, C is neither trigonal nor a plane quintic.

Theorem 5.1 *Let C be a curve of genus 6 which are neither trigonal nor a plane quintic.*

(1) *When E runs over all stable 2-bundles with canonical determinant on C, the maximum of $\dim H^0(F)$ is equal to 5. Moreover, such vector bundles F_{max} on C with $\dim H^0(F_{max}) = 5$ are unique up to isomorphism and generated by global sections.*

(2) *There exists a bijection between the intersection $\mathbf{P}_*(\mathrm{Ker}\,\lambda) \cap G(2, H^0(F_{max}))$ and the set $W^1_4(C)$ of g^1_4's of C, where λ is the map (0.1) for F_{max}.*

Let ξ be a g^1_4 of C and η its Serre adjoint. Then every stable 2-bundle E with canonical determinant and with $h^0(E) \geq 5$ is an extension of η by ξ. The rest of the proof is quite similar to that of Theorem A and B. We omit it here.

Let E be a 2-bundle with canonical determinant on C and assume that $h^0(E) = 5$, E is generated by global sections and that the intersection $\mathbf{P}_*(\mathrm{Ker}\,\lambda_E) \cap G(2, H^0(E))$ is finite. Since $\dim G(2, H^0(E)) = 6 = \dim H^0(\omega_C)$, $\lambda_E : \bigwedge^2 H^0(E) \longrightarrow H^0(\omega_C)$ is surjective. Hence $\Phi_{|E|} : C \longrightarrow G(H^0(E), 2)$ is an embedding by the commutative diagram (0.7).

Claim : The restriction map $I_{G,2} \longrightarrow I_{C,2}$ is injective, where $I_{G,2}$ is the vector space of Plücker quadratic forms of $G(2, H^0(E))$ and $I_{C,2}$ is the vector space of quadratic forms which vanish on C.

For every $q \in I_{G,2}$, the projective dual $\check{Q}$ of $Q : q = 0$ is a 4-dimensional quadric contained in $G(2, H^0(E))$. Hence the intersection $\mathbf{P}_*(\mathrm{Ker}\,\lambda_E) \cap \check{Q}$ is finite by our assumption and Q does not contain the image $P \simeq \mathbf{P}^5$ of $\mathbf{P}_*(\lambda_E)$ by Proposition 1.10.

$I_{G,2}$ is of dimension 5 and $I_{C,2}$ is of dimension 6 by Noether's theorem. Hence there exists a hyperquadric Q such that $C = P \cap G(H^0(E), 2) \cap Q$ by Enriques-Petri's theorem. Hence, by Theorem 5.1, we have

Theorem 5.2 *If $W^1_4(C)$ is finite, then $\Phi_{|E|} : C \longrightarrow G(H^0(E), 2)$ is an embedding and its image is a complete intersection of $G(H^0(E), 2)$ and a 4-dimensional quadric in $\mathbf{P}^9 = \mathbf{P}^*(\bigwedge^2 H^0(E))$, where E is F_{max} in (1) of Theorem 5.1.*

Assume that C has no g^1_3 or g^2_5 and let ξ be a g^1_4 of C. Its Serre adjoint η is a g^2_6 by the Riemann-Roch and $h^0(\eta) = 3$ by Clifford's theorem. By our assumption, $|\eta|$ is free and the image $\bar{C}$ of $\Phi_{|\eta|} : C \longrightarrow \mathbf{P}^2$ is either a sextic or a smooth cubic. By Corollary 3.2, every g^1_4 different from ξ is obtained from the projection off a double point of $\Phi_{|\eta|}$, that is, a double points of the sextic $\bar{C}$ in the former case and any point of the cubic $\bar{C}$ in the latter case. Hence we have

Proposition 5.3 *For a curve C of genus 6, $W^1_4(C)$ is finite if and only if C is not bi-elliptic and has no g^1_3 or g^2_5.*

This is a special case of Mumford's refinement of Martens' theorem ([1], p. 193).

References

[1] Arbarello, E., Cornalba, M., Griffiths, P.A. and J. Harris: *Geometry of Algebraic Curves, I*, Springer-Verlag, 1985.

[2] Borel, A. and F. Hirzebruch: Characteristic classes and homogeneous spaces I, Amer. J. Math. **80** (1958), 458-538: II, Amer. J. Math., **81**(1959), 315-382.

[3] Buchsbaum, D.A. and D. Eisenbud: Algebra structure theorems for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. **99**(1977), 447-485.

[4] Donaldson, S.K.: A new proof of a theorem of Narasimhan and Seshadri, J. Diff. Geom. **18** (1983), 269-2277.

[5] Griffiths, P.A. and J. Harris: *Principles of Algebraic Geometry*, Wiley-Interscience, New-York, 1978.

[6] Hartshorne, R.: *Algebraic Geometry*, Springer-Verlag, 1977.

[7] Józefiak, T. and P. Paragacz: Ideals generated by Pfaffians, J. Algebra, **61**(1979), 189-198.

[8] Lazarsfeld, R. and A. Van de Ven.: Topics in the geometry of projective space "Recent result of F.L. Zak", DMV Seminar, Vol. 4, Birkhäuser, 1984.

[9] Le Barz, P.: Formules multi-sécants pour les courbes gauche quelconques, in '*Enumerative Geometry and Classical Algebraic Geometry*', P. Le Barz and Y. Hervier (eds.), pp. 165-197, Birkhäuser, Boston, 1982.

[10] Mukai, S.: Curves, K3 surfaces and Fano 3-folds of genus ≤ 10, in '*Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*', pp. 357-377, Kinokuniya, Tokyo, 1988.

[11] —— : Curves and symmetric spaces, Proc. Japan Acad. **68**(1992), 7-10.

[12] Mumford, D. and J. Fogarty: *Geometric Invariant Theory*, second enlarged edition, Springer-Verlag, 1982.

[13] Narasimhan, M.S. and C.S. Seshadri: Stable and unitary vector bundles on a compact Riemann surface, Ann. of Math. **82**(1965), 540-564.

[14] Zak, F.L.: Severi varieties, Math. USSR Sbornik 54 (1986), 113-127.

Department of Mathematics
School of Science
Nagoya University
464-01 *Furō-chō, Chikusa-ku*
Nagoya, Japan