A generalization of Mumford’s example (joint work with H. Nasu)

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Let \( \text{Hilb}^{sc} V \) be the Hilbert scheme parametrizing smooth curves in a smooth projective variety \( V \). In [3], Mumford showed that \( \text{Hilb}^{sc} \mathbb{P}^3 \) has a generically non-reduced component. More precisely the following is proved:

**Example A** Let \( S \) be a smooth cubic surface in \( \mathbb{P}^3 \), \( E \) a \((-1)\)-\( \mathbb{P}^1 \) in \( S \) and \( C \subset S \) a smooth member of the linear system \( |4h + 2E| \cong \mathbb{P}^{37} \) on \( S \). (\( C \) is of degree 14 and genus 24.) Such curves \( C \) are parametrized by \( W^{56} \subset \text{Hilb}^{sc} \mathbb{P}^3 \), an open subset of a \( \mathbb{P}^{37} \)-bundle over \( |3H| \cong \mathbb{P}^{19} \). \( H \) is a plane in \( \mathbb{P}^3 \) and \( h \) is its restriction to \( S \). Then \( W^{56} \) is an irreducible component of \((\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}} \) and \( \text{Hilb}^{sc} \mathbb{P}^3 \) is nowhere reduced along \( W^{56} \).

It is well known that every infinitesimal (embedded) deformation of \( C \subset V \) is unobstructed if \( H^1(N_C/V) = 0 \). Conversely we find a sufficient condition for a first order infinitesimal deformation of a curve \( C \) in a 3-fold \( V \) to be obstructed, abstracting an essence from the arguments in [1] and [4]. As application we construct generically non-reduced components of the Hilbert schemes of uniruled 3-folds \( V \) including Examples A and B as special cases:

**Example B** \([2]\) Let \( V_3 \) be a smooth cubic 3-fold in \( \mathbb{P}^4 \), \( S \) its general hyperplane section, \( E \) a \((-1)\)-\( \mathbb{P}^1 \) in \( S \) and \( C \subset S \) a smooth member of \( |2h + 2E| \cong \mathbb{P}^{12} \). (\( C \) is of degree 8 and genus 5.) Such curves \( C \) in \( V_3 \) are parametrized by \( W^{16} \subset \text{Hilb}^{sc} V_3 \), an open subset of a \( \mathbb{P}^{12} \)-bundle over the dual projective space \( \mathbb{P}^4, \vee \). Then \( W^{16} \) is an irreducible component of \((\text{Hilb}^{sc} V_3)_{\text{red}} \) and \( \text{Hilb}^{sc} V_3 \) is nowhere reduced along \( W^{16} \).

The curves \( C \) of genus 24 in Example A are not (moduli-theoretically) general but the curves \( C \) of genus 5 in Example B are general. Hence, with the help of Sylvester’s pentahedral theorem \([5]\), Example B gives a counterexample to the following problem:

**Problem 1** Is every component of the Hom scheme \( \text{Hom}(X, V') \) generically smooth for a smooth curve \( X \) with general modulus and for a general member \( V' \) in the Kuranishi family of \( V \)?

Let \( \text{Hom}_8(X_5, V_3) \) be the Hom scheme of morphisms of degree 8 from a curve \( X_5 \) of genus 5 with general modulus to a smooth cubic 3-fold \( V_3 \subset \mathbb{P}^4 \).

**Theorem** \([2]\) If \( V_3 \) is also moduli-theoretically general, then \( \text{Hom}_8(X_5, V_3) \) has a generically non-reduced component of expected dimension \((= 4)\).

The following seems still open:

**Problem 2** Let \( G/P \) be a projective homogeneous space, e.g., a Grassmann variety and \( X \) a curve with general modulus. Is every component of \( \text{Hom}(X, G/P) \) generically smooth?

The answer is affirmative for the projective space \( \mathbb{P}^n \) by virtue of Gieseker’s theorem (= Petri’s conjecture).
REFERENCES