Moduli of abelian surfaces, and regular polyhedral groups

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Abstract : Let $\mathcal{A}_d$ be the moduli space of polarized abelian surfaces of type $(1,d)$. For $d = 2,3$ and 4, the Satake compactification of $\mathcal{A}_d$ is isomorphic to the quotient of $\mathbf{P}^3$ by an action of $PSL(2,\mathbf{Z}/d) \times PSL(2,\mathbf{Z}/d)$. Let $PSL(2,\mathbf{Z}/5) =: G_5 \subset PGL(2)$ be the icosahedral group and $PGL(2) \subset \mathbf{P}^3$ the natural embedding into the projective space of 2×2 matrices. A small resolution of the Satake compactification of $\mathcal{A}_5$ (at the poit cusps) is isomorphic to the quotient of the blow-up $\tilde{\mathbf{P}}^3$ at the 60 points ($\simeq G_5$) by an action of $G_5 \times G_5$.

Let $X(d)$ be the moduli space of elliptic curves E with a full level d structure, *i.e.*, a symplectic isomorphism between the standard symplectic module

$$2[\mathbf{Z}/d)] := \left(\mathbf{Z}/d \oplus \mathbf{Z}/d, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

and the group E_d of d-torsion points with the Weil pairing. The modular curve $X(d)$ is rational if and only if $d \leq 5$. In particular, the finite group $PSL(2,\mathbf{Z}/d)$ is a regular polyhedral group $G_d \subset PGL(2)$ for $d = 2,3,4$ and 5. The compactified modular curve $\overline{X(d)}$ is identified with the circumscribing Riemann sphere of the regular polyhedron P_d with t vertices, where t is the number of cusps. The order of G_d is equal to dt.

d	2	3	4	5
P_d	triangle	tetrahedron	octahedron	icosahedron
t	3	4	6	12
G_d	S_3	A_4	S_4	A_5

Regular polyhedral groups are also closely related with Hilbert modular surfaces of small discriminant.

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Example Let $\mathcal{O}_{\sqrt{5}}$ be the ring of integers of the quadratic field $\mathbf{Q}(\sqrt{5})$ and put

$$\Gamma = SL(2, \mathcal{O}_{\sqrt{5}} : \sqrt{5}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1_2 \bmod \sqrt{5},\ a,b,c,d, \in \mathcal{O}_{\sqrt{5}},\ ad - bc = 1 \right\}.$$

Then Γ acts on the product $H \times H$ of two copies of the upper half planes. Let $\tilde{\Gamma}$ be the group generated by Γ and the switch involution of $H \times H$. Then the Hilbert modular surface $Y_\Gamma := \overline{\tilde{\Gamma} \backslash H \times H}$ added with 6 point cusps is isomorphic to the projective plane $\mathbf{P}^2$. This Y_Γ has an action of the icosdahedral group G_5. Moreover, $X_\Gamma := \overline{\Gamma \backslash H \times H}$ is the double cover of Y_Γ with branch a G_5-invariant plane curve of degree 10.

In the 3-dimensional case the wreath product $2 \| G_d$ plays the role of G_d. Let $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ be the Segre embedding. The ambient space is the projectivization of the space of 2 by 2 matrices and the quadric $\mathbf{P}^1 \times \mathbf{P}^1$ parametrizes the rank one marices (modulo constant multiplication). Hence the complement is naturally identified with $PGL(2)$. This $\mathbf{P}^3$ is an equivariant compactification of the algebraic group $PGL(2)$ and the polyhedral group G_d acts on it from both sides. Let τ be the involution of this $\mathbf{P}^3$ interchanging $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its cofactor matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The fixed locus is the union of the constant matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and the traceless ones $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. τ interchanges the two factors of $\mathbf{P}^1 \times \mathbf{P}^1$. So the bipolyhedral group $2 \| G_d$ acts on $\mathbf{P}^3$.

For a polarized abelian surface (X, L) of type $(1, d)$, a (symplectic) isomorphism α between the standard symplectic module $2[\mathbf{Z}/d]$ and the group

$$K(L) := \{x \in X \mid T_x^* L \simeq L\}$$

with the Weil pairing is called a *canonical level structure*. By a canonical *colevel* structure, we mean a (symplectic) isomorphism between $2[\mathbf{Z}/d]$ and the quotient group $X_d/K(L)$, which has a symplectic structure as the complement of $K(L)$ in X_d. We denote the moduli space of polarized abelian surfaces with canonical level and colevel structure by $\mathcal{A}(1, d)$ and $\mathcal{A}(d, 1)$, respectively. The forgetful morphisms

$$\mathcal{A}(1, d) \longrightarrow \mathcal{A}_d \quad \text{and} \quad \mathcal{A}(d, 1) \longrightarrow \mathcal{A}_d$$

are both Galois coverings with Galois group $PSL(2, \mathbf{Z}/d)$. The fibre product

$$\mathcal{A}_d^{wbl} := \mathcal{A}(1, d) \times_{\mathcal{A}} \mathcal{A}(d, 1)$$

is called the moduli space of abelian surfaces with a *weak bilevel structure*.

For a polarized abelian surface (X, L) of type $(1, d)$, its dual $\hat{X}$ has a natural polarization $\hat{L}$ of the same type such that $\phi_{\hat{L}} \phi_L = d_X$. The colevel

structure of (X, L) is equivalent to the level structure of its dual $(\hat{X}, \hat{L})$ and vice versa. Therefore, the moduli space $\mathcal{A}_d^{wbl}$ has an action of wreath product $2||PSL(2, \mathbf{Z}/d)$.

Remark 1 The moduli space $\mathcal{A}_d$ is the quotient of the Sigel upper half space of degree 2 by the full paramodular group

$$1_4 + \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & d\mathbf{Z} \\ d\mathbf{Z} & \mathbf{Z} & d\mathbf{Z} & d\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & d\mathbf{Z} \\ \mathbf{Z} & \frac{1}{d}\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{pmatrix} \cap Sp_4(\mathbf{Q}).$$

A pair (α, β) of isomorphisms $\alpha : 2[\mathbf{Z}/d] \xrightarrow{\sim} K(L)$ and $\beta : 2[\mathbf{Z}/d] \xrightarrow{\sim} K(\hat{L})$ is called a *canonical bilevel structure* of (X, L). The moduli space $\mathcal{A}_d^{bl}$ of polarized abelian surfaces of type $(1, d)$ with bilevel structure (X, L, α, β) is the quotient by the subgroup

$$1_4 + \begin{pmatrix} d\mathbf{Z} & & d\mathbf{Z} & d\mathbf{Z} \\ d\mathbf{Z} & d\mathbf{Z} & d\mathbf{Z} & d^2\mathbf{Z} \\ d\mathbf{Z} & & d\mathbf{Z} & d\mathbf{Z} \\ & & & d\mathbf{Z} \end{pmatrix} \cap Sp_4(\mathbf{Z}).$$

The moduli space $\mathcal{A}_d^{wbl}$ is the quotient of $\mathcal{A}_d^{bl}$ by the involution

$$(X, L, \alpha, \beta) \mapsto (X, L, \alpha, -\beta),$$

which corresponds to the element

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in Sp_4(\mathbf{Q}).$$

Theorem (1) *For $d = 2, 3$ and 4, the Satake compactification of $\mathcal{A}_d^{wbl}$ is $(2||G_d)$-equivariantly isomorphic to the projective 3-space $\mathbf{P}(M_{2\times 2}\ \mathbf{C})$.*

(2) *There exists a $(2||G_5)$-equivariant morphism*

$$\psi : \tilde{\mathbf{P}}^3 \longrightarrow \overline{\mathcal{A}}_d^{wbl}$$

onto the Satake compactificaiton and ψ contracts the strict transforms of the 72 special lines (see below) to the 72 point cusps, where $\tilde{\mathbf{P}}^3$ is the blow-up of $\mathbf{P}^3$ with center G_5. (The normal bundles of the strict transforms are isomorphic to $\mathcal{O}(-4) \oplus \mathcal{O}(-4)$.) ψ is an isomorphism elsewhere. Moreover, the exceptional divisors over the 60 points G_5 are the Hilbert modular surface Y_Γ in Example and parametrize the Comesatti surfaces, i.e., *abelian surfaces with real multiplication by $\mathcal{O}_{\sqrt{5}}$.*

(3) *In both cases (1) and (2),* $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}(M_{2\times 2}\ \mathbf{C})$ *parametrizes the products ow two elliptic curves (of degree* 1 *and* d*).*

Let $p_1, \ldots, p_t$ be the cusps of the elliptic modular curve $\overline{X(d)}$, $d = 2, 3, 4, 5$. Then by the theorem, the $2t$ lines $p_i \times \mathbf{P}^1$ and $\mathbf{P}^1 \times p_i$, $1 \leq i \leq t$, on $\mathbf{P}^1 \times \mathbf{P}^1$ are 1-dimensional (Satake) boundaries of $\mathcal{A}_d^{wbl}$. $\mathcal{A}_2^{wbl}$ and $\mathcal{A}_3^{wbl}$ are the complement of these $2t$ lines in $\mathbf{P}^3$. In order to describe $\mathcal{A}_4^{wbl}$ and $\mathcal{A}_5^{wbl}$, we need the following:

Definition A line in $\mathbf{P}^3$ joining two points $[g_1]$ and $[g_2]$ of $G_d \subset PGL(2)$ is *special* if $g_1 g_2^{-1} \in G_d$ is of order d.

The number of special lines is iqual to 9, 16, 18 and 72 for $d = 2, 3, 4$ and 5. In the case $d = 2, 3$, the special lines parametrize the polarized abeian surfaces (X, L) which have symplectic automorphism of order d.

Proposition (1) *The moduli space* $\mathcal{A}_4^{wbl}$ *is the complement of* 12 *lines* $p_i \times \mathbf{P}^1$, $\mathbf{P}^1 \times p_i$ *and the* 18 *special lines in* $\mathbf{P}^3$.

(2) *The moduli space* $\mathcal{A}_5^{wbl}$ *is the complement of the strict trransform of* 12 *lines* $p_i \times \mathbf{P}^1$, $\mathbf{P}^1 \times p_i$ *and the* 72 *special lines in the blow-up* $\tilde{\mathbf{P}}^3$.

Remark 2 Let K_4 be the klein's subgroup of the octahedral group G_4. The action of $K_4 \times K_4$ on $\mathbf{P}^3$ is the projectivization of the Schrödinger representation of the Heisenberg group. Each of the 15 involutions in $K_4 \times K_4$ has the union of two skew lines as fixed point locus. The 18 and 12 lines in (1) of the proposition coincide with these 30 fixed lines.

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