Manolescu

Grid diagram

\[ \begin{array}{c}
\text{Lemma} \quad \text{Every knot has a grid diagram}
\end{array} \]

\[ \begin{array}{c}
\text{Versions of Knot Floer Homology}
\end{array} \]

\[ G: \text{grid diagram} \quad F=\mathbb{Z}/2 \]

\[ S(G) = \{ x=(x_{i0}, \ldots, x_{iz_{\text{max}}}) : \sigma \in \mathbb{S}_n \} \quad x_{ij} = (i, j) \]

\[ C^+(G) = (F \langle S(G) \rangle [T_1, \ldots, T_n]) \]

\[ \delta : C^+(G) \to \delta^- x = \sum_{g \in S(G)} \sum_{v \in \text{Rad}(g, \sigma)} \sum_{x_i(r) = 0} u_1^{(r)} \cdots u_n^{(r)} \]

\[ H_*(C^+(G), \delta) = HF^*(K) \]

\[ \text{module} \quad U = U_0 \cdot \Lambda_c \]

\[ \hat{C}(G) = \{ F \langle u_1, \ldots, u_n \rangle \subset S(G) \} \quad \text{set} \quad u_i = 0 \]

\[ H_*(\hat{C}(G), \hat{\delta}) = \hat{HF}(K) \]

\[ \text{finite dim.} \]

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\[ 0 \to C(G) \xrightarrow{u_1} C(G) \to \hat{C}(G) \to 0 \]

Long exact sequence \[ \to HF^k \xrightarrow{u} HF^k \to \hat{HF} \to \cdots \]

Set \( u_2 = 0 \)
\[ \hat{C}(G) = \{ F[u_3, \ldots, u_n] \mid SCG \} \]
\[ \hat{HF} \]
\[ \hat{HF} \xrightarrow{0} \hat{HF} \to \hat{HF} \]
\[ \hat{HF} = \hat{HF} \otimes V \]
Result \( V = 2 \)

3. \( \tilde{C}(G) \) Set \( u_1 = u_2 = \cdots = u_n = 0 \)
\[ \tilde{C}(G) = F[SCG] \]
\[ \tilde{x_i} = \sum \sum \right] \]

just count completely empty rectangles

Result \[ \hat{HF} = \hat{HF} \otimes V^{n-1} \]

not quite a knot invariant

4. Most complete theory

count all \( r \in \text{Rect}_i \)
\[ \exists \text{ int}(r) = \emptyset \]
allow \( x_i(r), \alpha(r) = 0 \)

\[ C(G) = C(G) = \{ F[u_1, \ldots, u_n] \mid SCG \} \]
\[ \tilde{x_i} = \sum \sum \right] \]

It turns out \[ H_0(CCG, \partial) = \{ F[2] \} \]
for any knot
Recall Alexander, homological gradings on $S(\mathcal{G})$

If $r \in \text{Real}(\mathcal{A})$

$$A(\mathcal{A}) - A(\mathcal{G}) = \sum x_i(r) - 0_i(r)$$

$$M(\mathcal{A}) - M(\mathcal{G}) = 1 + 2G(r) - 2\sum 0_i(r)$$

multiply by $\mathcal{C}_0$ changes $A$ by $-1$, $M$ by $-2$

$\mathcal{A}$ lowers $M$ by $1$

$$r \in \text{Real}(\mathcal{A}) \Rightarrow A(\mathcal{A}) - A(U_{1}^{m_1}, \ldots, U_{n}^{m_n})$$

$$= A(\mathcal{A}) - A(\mathcal{G}) + \sum 0_i(r) = \sum x_i(r) \leq 0$$

$\mathcal{A}$ never increase $A$

$\Rightarrow$ Alexander filtration on $(C(\mathcal{G}), \mathcal{A})$

[Full HFK]: filtered chain homotopy type of $(C(\mathcal{G}), \mathcal{A})$

e.g., associated graded

$$\text{CFK}^- \Rightarrow \text{HFK}^-$$

Example unknot $n=2$

$$S(\mathcal{G}) = \{x, y\}$$

$$A(x) - A(y) = 1 \quad x \text{ has bigrading } (0,0)$$

$$M(x) - M(y) = 1 \quad y \quad (-1,1)$$

1. $C^-$

$$\langle \alpha, \beta \rangle < x, y \rangle$$

$$\partial x = 0$$

$$\partial y = (y_1 - y_2)x$$

$$\text{HFK}^-(\text{unknot}) = \mathbb{F}[\mathcal{U}] | \mathcal{U}_5 = \mathcal{U}_6$$
2. \( \hat{C} \) set \( u_1 = 0 \) even \( \mathbb{F}(\Sigma) \)
\[
\hat{HFK} = \mathbb{F} \text{ supported in degree } (0, 0)
\]
\[
\chi(\hat{HFK}) = 1 = \Delta k(y)
\]
\( \text{genus } (k) = 0 \)
\( \hat{r}k \hat{HFK} |_{A=0} = 1 \)

3. \( \tilde{C} = \mathbb{F}(x, y) \) \( \partial = 0 \)
\[
\hat{HFK} = \tilde{C} = V \quad \hat{r}k V = 2
\]
\[
= \hat{HFK} \partial V
\]

4. \( C(G) = \mathbb{F}(x, y) \langle \tilde{x}, \tilde{y} \rangle \)
\[
\partial \tilde{y} = (u - u_2) \tilde{x}
\]
\[
\partial \tilde{x} = \tilde{y} + \tilde{y} = 0
\]
\( H_+ (C(G), \partial) = \mathbb{F}(u) \)

**NB.** If we keep track of \( \tilde{x} \)
\[
\partial \tilde{x} = (x_1 - x_2) \tilde{y}
\]
\[
\partial \tilde{y} = (c_1 - c_2) \tilde{y}
\]
\( \Rightarrow \partial \neq 0 \)

More about the Alexander gradings

\[
A(\tilde{y}) - A(\tilde{x}) = \sum x_i (v_i) - \sum c_i (v_i)
\]
\( = \Delta k (\partial \tilde{y}, k) \)

linking #: 

\[
\sum w(k, x_i) - w(k, y_i)
\]

windings #: 

\[ A(x) = \sum_i w(K, x_i) + \text{some constant depending only on the grid diagram} \]

\[ \chi(\hat{HF}_K) = \sum_{M,A} (-1)^M g^A \text{rank } \hat{HF}_M(K, A) = \Delta_k(g) \text{ Alexander polynomial} \]

\[ \chi(\hat{HF}_K) = \chi(\hat{HF}_K \otimes V^{n-1}) = (1 - \frac{1}{g})^{n-1} \Delta_k(g) \]

\[ = \sum_{g \in S(g)} \pm g A(g) \]

\[ = \sum_{g \in S(g)} g \cdot \text{w}(K, x_0, \ldots, x_{n-1}) \approx \sum_{g \in S(g)} g \cdot \text{w}(K, x_0, \ldots, x_{n-1}) \]

\[ \approx \text{det}(g \cdot \text{w}(K, x_0, \ldots, x_{n-1})) \]

\[ \approx \Delta_k(g) (1 - \frac{1}{g})^{n-1} \]

\[ \text{new formula for the Alexander polynomial} \]

Thus, the filtered chain homotopy type of \((\mathcal{CG}, \mathfrak{g})\) is a knot invariant. (Hence \(\hat{HF}^-, \hat{HF}: \text{knot inv.}\))
(Proof) Any two grid diagrams for the same knot $K$ are related by a sequence of Crowell-Dynnikov moves.

1. Cyclic permutation of rows & columns
   - By def. CGG does not change because we work on torus

2. Commutation of columns (or row)

3. Stabilization

NB: 1. Change the crossing number