Stroppel

Kazhdan–Lusztig theory from rep. theory

Want: To categorify tensor products of finite dim modules $U_q(\mathfrak{g})$-modules

$\to$ RT-tangle inv.

$\to$ CK-invariants?

Recall of: a ss q,x Lie alg $\to O(\mathfrak{g})$

$\lambda \in \mathfrak{g}^*/W \to O(\mathfrak{g})_{\lambda} \subset O(\mathfrak{g})$ $W$ = Weyl group

From now on $\mathfrak{g} = sl_n$

categorify $\bigotimes_{i=1}^n V_i$ $V = \bigotimes_{i=1}^n U_q(\mathfrak{sl}_n)$

pick up for $\{1, \cdots, n\}$ pick $\lambda_i \in \mathfrak{g}^*$ integral dominant

sit. stabilizer of $\lambda_i$ is $S_i \times S_{n-i} \leq S_n = W$

e.g. $sl_3$

\begin{align*}
S_0 \times S_3 & \quad S_1 \times S_2 & \quad S_2 \times S_1 & \quad S_3 \times S_0
\end{align*}

orbit $\begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 2^3$ elements

Thus there is an isom. $K_0(\bigoplus_{i=0}^n O(\mathfrak{sl}_n)_{\lambda_i}) \cong V^{\otimes n}$ $\text{s.t.}$

\begin{align*}
\text{graded} & \quad \text{deep} \\
\text{Verma module} & \leftrightarrow \text{std basis} \\
\text{Simple module} & \leftrightarrow \text{dual canonical basis} \\
\text{twisted indec. proj. modules} & \leftrightarrow \text{canonical basis} \\
\text{(tilting)} &
\end{align*}
action of $E,F,K \leftrightarrow$ tensoring with natural, its rep $C^n$ dual grading shift

Remarks

- can be generalized to arbitrary tensor products of finite dimensional modules of $U_q(\mathfrak{sl}_2)$ using categories of Harish-Chandra modules (it with Frankel-Khovanov)
- generalization of $\Delta$ works well although the categories are not highest wt
- it work with Mazorchuk

Picture so far convenient for $U_q(\mathfrak{sl}_2)$-action now want $TL$-action

$g^\mathfrak{f} = \mathfrak{h}_n, \mathfrak{c} \subseteq \mathfrak{c} \subseteq \mathfrak{n}$

\[
\begin{bmatrix}
  \ast & \ast \\
  0 & \ast
\end{bmatrix}
\]

$g^\mathfrak{f}(\mathfrak{g}), C \circ \mathfrak{O}(\mathfrak{g})$

- all objects which are (cc. finite w.r.t. $\mathfrak{U}(\mathfrak{g})$)

- simple objects $L(x)$ $x \in S_i \times S_{n-i}$ shortest length coset representative

Koszul duality $\mathbb{Z}$-on picture ($\sim \delta \triangleright \delta$)
Th. 1) $K_0(\oplus \mathcal{O}(\mathcal{L}_e)^{\infty}) \cong \mathcal{V}^{\infty}$
parabolic Verma $\iff$ standard basis

2) Grothendieck group of proj. functors in the graded version

restriction of proj. functors $\mathcal{O}$

3) This extends to an invariant of tangles + cobordism as follows

\[ \begin{array}{c|c}
\text{tangles} & \text{Cat} \\
\hline
\{ \text{objects} \} & \mathcal{A}^n(\oplus \mathcal{O}(\mathcal{L}_e)^{\infty}) \\
\text{morphisms} & \text{functors} \\
\text{2-morphisms} & \text{natural trans.} / \text{scalars} \\
\end{array} \]

4) On the Grothendieck group $RT$

connection to $D^b(Coh \mathcal{L}_n)$?

Conj. $D^b(Coh \mathcal{L}_n) \cong D^b(B-\text{grad.}) \cong D^b(\oplus \mathcal{O}(\mathcal{L}_e)^{\infty})$
Koszul dual

NB. Temp. Lieb. is exact on functors in the $K_0(\oplus \mathcal{O}(\mathcal{L}_e)^{\infty})$ side
General fact

\[ G_i : (\mathfrak{m}_n)^n \cong \text{mod } A_n^i \text{ for some f.d. algebra } A_n^i \]

Badan gave description via generators & relations

(LHS \cong \text{per. sheave on Grassmann})

Not obvious that the algebra is graded

He conjectured \( H_n \) is a subquotient algebra of \( A_{2n}^n \)

\[ H_n \text{ is a natural subalgebra of } A_{2n}^n \text{, namely} \]

\[ H_n = \text{End}_{A_{2n}^n} (\oplus \text{ indec. proj. & injective}) \]

Some functors is trivial!

Restricting functors from \( H_n \) to LHS gives exactly Khovanov's picture.

\( A_n^i \) is \( H_n \) better than \( n \) is not Koszul. But the trade-off is not computable.

\( A_n^i \) via diagrams

Write a basis of \( V^\otimes n \) as "spin chains" using \( \wedge, \vee \)

\[ \text{e.g. } V^\otimes V \sim \wedge (\vee \wedge \vee) \wedge \]

\( i \text{th weight op. } = \# \text{ chains is } i \)

**Parabolic ** \( \leftrightarrow \text{ spin } \leftrightarrow \text{ chains} \)
Rule I) Given a spin chain $S$ connect all $\wedge \vee$ via a cup diagram (if they are not) 

E.g. $\wedge \vee \rightarrow$ do nothing $\dagger \dagger$

$\wedge \vee \rightarrow \cup$

Result cup diagram $C(S)$ with probably also $\dagger \dagger$

Lemma S a spin chain, $M(S)$ corresponding Verma $P(S)$ its proj. cover

$[P(S): M(x): j]> = \#$ orientations of $C(x)$ of degree $j$ and type $t$

E.g. $\uparrow \downarrow$ or $\downarrow \uparrow$

degree $= \# $of clockwise cups

E.g. $\wedge \vee$

$\dagger \dagger$

$\frac{P(\wedge \vee)}{M(\wedge \vee)} = M(\wedge \vee)$

$M(\wedge \vee) \langle \psi \rangle$

$\dagger \dagger$

only one orientation $P(\wedge \vee) = M(\wedge \vee)$

Rule II) Do the same thing with caps instead of cups gives $[M(s); L(s): j>]

interpretation of BGG-reciprocity $[P(s): M(x): j>] = (M(x): L(s): j>)]$

$\dim \text{Hom} (P(s), P(x)) = \sum_{y, e, j} [P(s), M(y): e> \rangle \langle \langle M(y), L(x): j>)]$
1) The algebra $A^n$ has a graded vector space basis given by all possible oriented cup-cap diagrams.

E.g. $A^2$

grading clockwise cap/cup shift by 1

0 1 1 2 0

2) Multiplication just as in Khovanov's

3) The cup-cap diagrams which give closed circles only form a subalgebra which is $H_n$ in the case $A^n$.

Rem.