colored Jones polynomial

\[ U_q(sl_2) \cong \mathbb{C}(q)-\text{alg.} \text{ with generators } E, F, K, K^\pm \]

relations
\[ KE = q^2 KE, \quad KF = q^{-2} FK, \quad KK^{-1} = \text{id} = K^{-1} K \]

\[ EF - FE = \frac{q - q^{-1}}{q^2 - q^{-2}} \]

**\textbf{Vv}** : \( \mathbb{C}(q) \)-vector space

with basis \( \{ v_0, \ldots, v_n \} \)

This has a structure of a \( U_q(sl_2) \)-repr.

\( \otimes \) : \( V_i \otimes V_i \to \mathbb{C}(q) \)

\[ \otimes (v_0 \otimes v_0) = 1 \]

\[ \otimes (v_i \otimes v_j) = (-q)^{-1} \quad \otimes (v_i \otimes v_i) = \otimes (v_0 \otimes v_0) = 0 \]

\( \otimes \text{id} : V_i \otimes V_i \to V_i^{(n+2)} \)

\( U : \mathbb{C}(q) \to V_i \otimes \quad U(q) = V_i \otimes v_0 - q^{-1} v_0 \otimes V_i \)

\( U_{\text{id}} : V_i \otimes \to V_i^{(n+2)} \)

\( \otimes : V_i \otimes V_i \to V_i \otimes \quad -q^2 \text{id} - q \otimes \)

\( \otimes_{\text{id}} : V_i \otimes \to V_i \otimes \)

\( \otimes = -q^2 \text{id} - q \otimes \quad \otimes_{\text{id}} \)

To each tangle diagram \( D \) we assign an intertwiner

\[ \Phi(D) : V_i \otimes \to V_i \otimes \quad \text{by } l \otimes l \to \otimes \text{id} \]

\( \text{etc.} \)
Let $D$ be an oriented diagram.

\[ \gamma(D) = \# \text{ of crossings of type } \nearrow \text{ or } \swarrow - \# \text{ of crossings of type } \nearrow \text{ or } \swarrow \]

$T$: oriented (r,s) tangle

$D_1, D_2$ two planar projections

\[ \delta^{3D(D_1)} \psi(D_1) = \delta^{3D(D_2)} \psi(D_2) : V_1^{\otimes r} \to V_1^{\otimes s} \]

In the special case of a (0,0) tangle
we get the Jones polynomial

\[ \mathcal{J}_n : V_n \to V_1^{\otimes n} \]

\[ \mathcal{J}_n(V_2) = \sum_{d^2} c_1(d^2) \otimes \ldots \otimes \otimes \]

\[ (d_1, \ldots, d_n) \]

\[ c_1 : \{0,1\}^n \to \mathbb{C}(\mathbb{R}) \]

$T\mathcal{J}_n : V_1^{\otimes n} \to V_n$

\[ T\mathcal{J}_n (\otimes \ldots \otimes \otimes \otimes) = c_2(d^2) \otimes \ldots \otimes \otimes \]

\[ c_2(d^2) \in \mathbb{C}(\mathbb{R}) \]

Now consider an oriented (r,s) tangle with each of the
strands colored by various finite dim. irr reps

This induces a coloring on the end points

(r,s)-tangle $\mapsto (d_1, \ldots, d_r), (e_1, \ldots, e_s)$ -tangles

Want a map $V_{d_1} \otimes \ldots \otimes V_{d_r} \to V_{e_1} \otimes \ldots \otimes V_{e_s}$
to each colored oriented diagram $D$ we associate its cabled diagram $cab(D)$

\[ D \quad \text{cable} \quad cab(D) \]

To an $((d_1, \ldots, d_r), (e_1, \ldots, e_s))$-tangle diagram $D$

We associate a map $C_{\text{col}}(D) : V_{d_1} \otimes \cdots \otimes V_{d_r} \otimes V_{e_1} \otimes \cdots \otimes V_{e_s}$

\[ = \bigotimes_{e_s}^{d_r} C_{\text{col}}(cab(D)) \otimes (\bigotimes_{e_r}^{d_1} \cdots) \]

**Theorem**: If $T$ be an oriented, framed $((d_1, \ldots, d_r), (e_1, \ldots, e_s))$-tangle $D_1, D_2$ two of its diagrams

\[ g \cdot 3\Delta(cab(D_1)) \cdot C_{\text{col}}(D_1) = g \cdot 3\Delta(cab(D_2)) \cdot C_{\text{col}}(D_2) \]

\[ : V_{d_1} \otimes \cdots \otimes V_{d_r} \otimes V_{e_1} \otimes \cdots \otimes V_{e_s} \]

In the case of $(0,0)$-tangle, this is the colored Jones poly.

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categorification of Jones polynomial

\[ C_x(q,h) = C_{\lambda}(q,h) \]

\[ \lambda = e_1 + \cdots + e_s - p \]

This can be viewed as a category of graded modules

**Theorem (BFKS)**

1) \[ C(g) \bigotimes_{Z(h, \Phi')} [ \bigoplus \bigotimes_{i=0}^{\infty} C_{\lambda}(q,h) ] \cong V_1 \bigotimes_{\text{functorial}} \]

2) projective functors $E, F, K, K'$ which satisfy $C(g(x))$-relation
$O_i^j(q^h) = \text{full subcategory of } O_i(q^h)$
of modules loc. finite with respect to $j$: \( \downarrow \)

$E_j : O_i^j(q^h) \to O_i(q^h)$ inclusion functor
$Z_j : O_i(q^h) \to O_i^j(q^h)$

It takes $M$ to its maximal locally finitequot. w.r.t. $E_j$

$Z_j$ is right exact.

We consider the left adjoint functor
$LZ_j : D^b(C_i(q^h)) \to D^b(C_i^j(q^h))$

\[ \mathbb{P}(\Phi_{Bk}) \]
\[ \cong \text{equiv. of categories } F_1, F_2 \text{ s.t.} \]

1) \[ \cong_{j,n} \overset{\text{def.}}{=} F_1 \circ LZ_j \] \[ \cong_{j,n} : D^b(\oplus O_i(q^h)) \to D^b(\oplus O_i(q^{h-2})) \]

2) \[ \cong_{j,n} \overset{\text{def.}}{=} E_j[-1] \circ F_2 \] \[ \cong_{j,n} : D^b(\oplus O_i(q^h)) \to D^b(\oplus O_i(q^{h+2})) \]

There exist natural transformations
\[ \| \text{id}[-2] \overset{\alpha}{\to} E_j \circ LZ_j[-1] \]
\[ \overset{\gamma}{\|} \text{id}(1) \overset{\beta}{\to} \]
\[ \overset{\gamma}{\|} \text{Cone } \alpha \]
\[ \overset{\gamma}{\|} \text{Cone } \beta \]
\[ \tilde{\times}_{j,m} = \text{cone } \alpha \quad \breve{\times}_{j,m} = \text{cone } \beta \]

\[ (\text{rs}) \]

To each tangle diagram \( D \), we have a functor
\[ \tilde{\Phi}(D) : D^b(\oplus C_i(x_i \gamma_i)) \to D^b(\oplus C_i(x_i \gamma_{i}')) \]

Then (Strapiel)

\[ T : \text{oriented (r.s) tangle} \]
\[ D_1, D_2 : \text{two of its diagram} \]
\[ \implies \tilde{\Phi}(D_1) \langle 3 \text{rs}(D_1) \rangle = \tilde{\Phi}(D_2) \langle 3 \text{rs}(D_2) \rangle \]

\[ [\tilde{\Phi}(T)] = \tilde{\Phi}(T) : V_1^{\otimes r} \to V_1^{\otimes s} \]

Let \( \mathcal{H}(\mathfrak{g}(n)) \) be the Harish-Chandra category of \( (\mathfrak{g}(\mathfrak{k}), \mathfrak{g}(\mathfrak{n})) \) bimodules

1) finitely generated
2) objects have finite length
3) \( \mathcal{H}(\mathfrak{g}(n)) \) is locally finite \( \text{w.r.t. the adjoint of } \mathfrak{g}(\mathfrak{n}) \)

Let \( \lambda \otimes \mathfrak{H}^{(1)}_{\mu} (\mathfrak{g}(k)) \) be the subset of \( \mathcal{H}(\mathfrak{g}(\mathfrak{n})) \)

Left action of \( \mathfrak{g}(\mathfrak{n}) \) has generalized central character (corresponding to int. dual with \( \lambda \))

Right action has central character to \( \mu \).

\[ d^\alpha = (d_1, \ldots, d_r) \quad |d^\alpha| = n \]

\[ \otimes^{(1)} \lambda = \otimes^{(1)} \mu \quad \text{where the stab. of } \lambda = S_i \times S_{n-i} \]

\[ \otimes^{(1)} \mu = S_{d_1} \times \cdots \times S_{d_r} \]
The functors \( E, \mathcal{F}, K, K^{-1} \) satisfy
the functional \( \mathcal{U}(\mathfrak{g}, k) \)-rel.

\[ \text{Rem} \quad \mathfrak{h}^{(0)}(\mathfrak{g}_{\mu}) \cong \mathfrak{g}_{\mu} \]

Let \( \tilde{\Lambda}_\mu : \mathfrak{h}^{\mu}(\mathfrak{gl}_n) \to \mathfrak{g}_{\mu}(\mathfrak{gl}_n) \)

\[ \tilde{\Lambda}_\mu M = M \otimes_{\mathfrak{h}_{\mu}} M(\mu) \quad \text{dominant Verma module} \]

\[ \tilde{\Pi}_\mu : \mathfrak{g}_{\mu}(\mathfrak{gl}_n) \to \mathfrak{h}^{\mu}(\mathfrak{gl}_n) \]

\[ M \mapsto \text{Hom}_\mathfrak{g}(M(\mu), M)^{\mathfrak{h}_{\mu}} \]

- In the case that \( \mu \) is regular (trivial stabilizer)
these functors are inverse equivalence of categories.

- The image of \( \tilde{\Lambda}_\mu \) in \( \mathcal{O}_{\mu}(\mathfrak{gl}_n) \) is some subcategory \( \mathcal{O}_{\mu}(\mathfrak{gl}_n) \)

\[ \text{This is the subcat of modules with proj. presentations } \quad \text{where} \]
\[ \text{the projectives allowed} \quad \text{depend on the data } \lambda = \mu. \]
\[ \bigoplus_i \tilde{\pi}_d^i = \tilde{\pi}_d : \bigoplus_i \omega_d^i (g_{kn}) \to \bigoplus_i \omega_d^i (g_{kn}) \]

\[ \bigoplus_i \tilde{\pi}_d^i = \tilde{\pi}_d : \bigoplus_i \omega_d^i (g_{kn}) \to \bigoplus_i \omega_d^i (g_{kn}) \]

\( \tilde{\pi}_d \) is not exact, its left derived functor

\[ \tilde{\mathcal{L}} : \mathcal{G} \left( \bigoplus_i \omega_d^i \right) \to \mathcal{G} \left( \bigoplus_i \omega_d^i \right) \]

**Prop. 1)** \[ [\tilde{\mathcal{L}} d^2] = L_{d_1} \odot \cdots \odot L_{d_r} : V_{d_1} \odot \cdots \odot V_{d_r} \to V_1 \odot m \]

**Prop. 2)** \[ [\tilde{\pi}_d^i] = \pi_{d_1} \odot \cdots \odot \pi_{d_r} : V_1 \odot m \to V_{d_1} \odot \cdots \odot V_{d_r} \]

Let \( \tilde{\mathcal{X}}_{d_j, d_{j-1}} \) be the functor associated with

\[ \begin{array}{c|c|c}
\vdots & \vdots & \vdots \\
\hline
d_1 & d_j & d_{j-1} \\
\hline
\end{array} \]

**Prop.** \[ D^\mathcal{G} (d_1 \cdots d_r) \omega_d^i (g_{kn}) \to D^\mathcal{G} (d_1 \cdots d_j \odot d_{j+1} \cdots d_r) \omega_d^i (g_{kn}) \]

\[ \tilde{\mathcal{X}}_{d_j, d_{j-1}} \]

To an oriented \((d_1 \cdots d_r), (e_1 \cdots e_s)\) tangle diagram \( D \), we define

\[ \mathcal{E}_{col} (D) : \bigoplus_{i=0}^{d_1} \mathcal{G} (i \odot \omega_d^i (g_{kn})) \to \bigoplus_{j=0}^{d_1} \mathcal{G} (j \odot \omega_d^i (g_{kn})) \]

\[ \mathcal{E}_{col} (D) = \tilde{\pi}_{e \to} \circ \tilde{\mathcal{E}} (cabD) \circ \tilde{\mathcal{L}}_{\tilde{d}} \]
In T: framed oriented (3, 2) tangle

Let $D_1, D_2$ be two of its diagram.

$$\Rightarrow \tilde{E}_{cot}(D_1)<3\delta(cab D_1)> \equiv \tilde{E}_{cot}(D_2)<3\delta(cab D_2)>$$