Instanton counting and wall-crossing in Donaldson invariants

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- X : oriented compact C^∞ 4-manifold g : Riemannian metric
- $P \to X$ principal $U(r)$-bundle

  A : a connection is anti-self-dual $\overset{\text{def.}}{\Longleftrightarrow}$ $* F_A = -F_A$

- $\mathcal{M}_g(P)$: moduli space of anti-self-dual connections on $P = \{A : \text{asd}\}/\text{gauge equiv.}$

— Donaldson invariants of X are originally defined via the intersection products on $\mathcal{M}_g(P)$.

— If X is a projective complex surface and g = a Hodge metric, $\mathcal{M}_g(P)$ = moduli space of stable holomorphic vector bundles.
Then an algebro-geometric approach is possible.

$\boxed{\text{HOPE}}$ ∘ Better understanding of Donaldson invariants
∘ A possible link to Donaldson-Thomas invariants

PLAN OF LECTURES

1.1. Quick review of Mochizuki theory

X : nonsingular complex projective surface, $\pi_1(X)=1$ for simplicity.
H : ample line bundle
$\chi(E(mH)) := \sum_{i=0}^{2} (-1)^i \dim H^i(X, E(mH))$: Hilbert polynomial
(polynomial in m)

A torsion free sheaf E is semistable

$$\overset{\text{def}}{\Longleftrightarrow} \frac{\chi(S(mH))}{\operatorname{rank} S} \leqq \frac{\chi(E(mH))}{\operatorname{rank} E} \quad (m \gg 0) \quad \text{for } 0 \neq {}^\forall S \subsetneq E \text{ subsheaf}$$

$M \equiv M_H(c)$: moduli space of semistable sheaves E with $\operatorname{ch} E = c$
$\cup$ open
$M^s_H(c)$: moduli space of stable sheaves

Deformation theory is controlled by $\operatorname{Ext}^i_0(E,E)$
trace-free part

$\operatorname{Ext}^0_0(E,E) = \operatorname{Hom}_0(E,E)$ ----- automorphism
$\operatorname{Ext}^1_0(E,E)$ ----- deformation
$\operatorname{Ext}^2_0(E,E)$ ----- obstruction

○ generic smoothness

Fact (Donaldson, Friedman, Qin, Gieseker-Li, O'Grady, ----)
Fix rank & c_1.
If $c_2 \gg 0 \Rightarrow \operatorname{Ext}^2_0(E,E) = 0$ except for $E \in$ lower dim. subvariety

Then M is of expected dimension.

Assume a universal family $\mathcal{E}$ over $X \times M$ exists for simplicity
We define the μ-map :

$$\mu_p : H_*(X) \longrightarrow H^*(M) ; \alpha \longmapsto (-1)^p \left[\operatorname{ch}(\mathcal{E}) e^{-c_1(\mathcal{E})/r}\right]_{p+1} / \alpha$$

Then we consider (generalised) Donaldson invariant

$$\boxed{\Phi_H^{r,c_1}\left(\exp \sum_{p=1}^{\infty} \alpha_p\right) := \sum_{\substack{c \\ (c_1, r \,:\, \text{fix})}} \Lambda^{\dim M_H(c)} \int_{M_H(c)} \exp\left(\sum_{p=1}^{\infty} \mu_p(\alpha_p)\right)}$$

(The definition when $M_H(c)$ is not of expectded dimension
$\longrightarrow$ discussed later.

<u>Remark.</u> In the differential geometric approach, we usually consider only $\mathcal{M}_p$ for $p \le r-1$, as $\mathcal{E}$ exists only as a vector b'dle over an open subset.

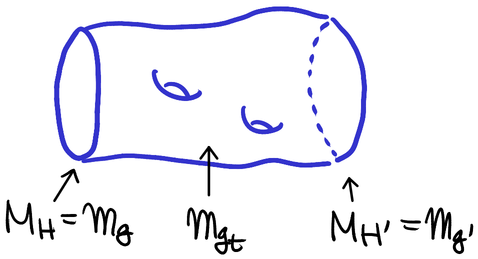
<u>Problem.</u> Is the invariant independent of the choice of the ample line bundle H?

We expect
- independence for $p_g > 0$
- wall-crossing for $p_g = 0$

  the difference depends only on homotopy type of X (Kotschick-Morgan conjecture)

from the differential geometric approach.

- $p_g > 0$ case: Cobordism



$\mathcal{M}_g$ = moduli space of antiself-dual connections

g_t : generic path of Riemannian metric connecting Kähler metrics for H & H'

But a generic path can be taken in the space of Riemannian metrics, not ample line bundles.

Also it seems difficult to study what happens for $p_g = 0$.
(Approaches by Feehan-Leness, Chen are not direct, and do not yield explicit wall-crossing formula.)

Takuro Mochizuki (ArXiv: 0210211) developed the theory of Donaldson invariants based on the perfect obstruction theory, and proved the wall-crossing formula via the virtual localization on the master space.

<u>Theorem</u> (Mochizuki) (rank = 2 for simplicity)
H_+, H_- : two ample line b'dles

$$B_+ := \left\{ \xi \in \mathrm{Pic}X \setminus \{0\} \;\middle|\; \begin{array}{l} \cdot\ \xi + a \text{ is divisible by } 2 \\ \cdot\ \langle \xi, H_+ \rangle > 0 > \langle \xi, H_- \rangle \end{array} \right\}$$

$$\Longrightarrow \boxed{\Phi^{c_1,2}_{H_+}(\exp \textstyle\sum \alpha_p) - \Phi^{c_1,2}_{H_-}(\exp \textstyle\sum \alpha_p) = \displaystyle\sum_{\xi \in B_+} \delta^X_\xi (\exp \textstyle\sum \alpha_p)}$$

where δ_ξ^X is the coeff. of t^{-1} of the following integral over Hilbert scheme of points :

$$\boxed{\delta_{\xi,t}^X := \sum_{\ell \geq 0} \Lambda^{4\ell-\xi^2-3} \int_{X_2^{[\ell]}} \frac{\exp\left(\sum (-1)^p \left[\mathrm{ch}(\mathcal{I}_1) e^{\frac{\xi-t}{2}} + \mathrm{ch}(\mathcal{I}_2) e^{-\frac{\xi-t}{2}}\right]_{p+1} / d_p\right)}{e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_2, \mathcal{I}_1(\xi)) e^t)\, e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_1(\xi), \mathcal{I}_2) e^{-t})}}$$

- $X_2 = X \amalg X$
- $X_2^{[\ell]}$ = Hilbert scheme of n points in X_2 $= \coprod_{m+n=\ell} X^{[m]} \times X^{[n]}$
- $\mathcal{I}_1, \mathcal{I}_2$: universal ideal sheaves on $X \times X^{[m]}$ & $X \times X^{[n]}$ pullbacked to $X \times X_2^{[\ell]} = \coprod X \times X^{[m]} \times X^{[n]}$
- $$\left.\begin{aligned} -\mathrm{Ext}_p^\bullet(\mathcal{I}_2, \mathcal{I}_1(\xi)) &:= -\sum_{i=0}^{2} (-1)^i \mathrm{Ext}_p^i(\mathcal{I}_2, \mathcal{I}_1(\xi)) \\ -\mathrm{Ext}_p^\bullet(\mathcal{I}_1, \mathcal{I}_2(-\xi)) &:= -\sum_{i=0}^{2} (-1)^i \mathrm{Ext}_p^i(\mathcal{I}_1, \mathcal{I}_2(-\xi)) \end{aligned}\right\} \in K(X_2^{[\ell]})$$
- e^t : trivial line b'dle with $\mathbb{C}^*$-action of the weight 1
- $e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_2, \mathcal{I}_1(\xi)) e^t)$: Euler class $\in H^*_{\mathbb{C}^*}(X_2^{[\ell]}) = H^*(X_2^{[\ell]}) \otimes \mathbb{C}[t]$
- $\dfrac{1}{e(-\mathrm{Ext}_p^\bullet(\mathcal{I}_2, \mathcal{I}_1(\xi)) e^t)} \in H^*(X_2^{[\ell]})[t, t^{-1}]]$ can be written by Segre class

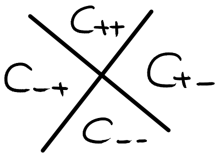
Comments on Higher rank case

1) $\exists$ similar formula in higher rank case, where $X_2^{[\ell]}$ is replaced by the union of the products of lower rank moduli spaces.

It corresponds to a semistable sheaf $E = E_1 \oplus \cdots \oplus E_k$ $(k \geq 2)$.

But the RHS of the formula still depends on the choice of the line b'dle H.

2) We consider the wall-crossing of the wall-crossing formula.



$$\{(\text{inv's for } C_{++}) - (\text{inv's for } C_{+-})\}$$
$$- \{(\text{inv's for } C_{-+}) - (\text{inv's for } C_{--})\}$$

If the RHS still depends on H, we should go further :

Wall-crossing of wall-crossing of wall-crossing of $\cdots\cdots$

If we repeat (rank -1) times, we arrive at an integral over r-copies of Hilbert schemes of points.

Very Brief review of Mochizuki's proof (of 217 pages)

- $Q \curvearrowleft G$ reductive group action on a projective scheme Q
- G-equivariant ample line bundles L_+, L_-

$\Longrightarrow M_\pm := Q /\!\!/_{L_\pm} G$; GIT quotient with respect to $L_\pm$
$\mathrm{Proj}(\bigoplus_{n\geq 0} H^0(L_\pm^{\otimes n})^G)$

We want to compare integrations over M_+ and M_-.

Construct the master space following Thaddeus

$\mathcal{M} := \mathbb{P}(L_+^{-1} \oplus L_-^{-1}) /\!\!/ G$ with respect to the line b'dle $\mathcal{O}_{\mathbb{P}}(1)$

$\mathbb{C}^*$ acts on $\mathcal{M}$ by $[z_+ : z_-] \mapsto [e^t z_+ : z_-]$

$$\mathcal{M}^{\mathbb{C}^*} = \underset{z_-=0}{M_+} \sqcup \underset{z_+=0}{M_-} \sqcup \text{exceptional fixed points}$$

Let $\alpha \in H_G^*(Q)$,
$T =$ trivial line b'dle with the nontrivial $\mathbb{C}^*$-action of weight 1.
Atiyah-Bott-Lefschetz Fixed point formula:

$$\int_{\mathcal{M}} \alpha \cup c_1(T) = \int_{M_+} \frac{\alpha \cup c_1(T)}{e(N_{M_+/\mathcal{M}})} + \int_{M_-} \frac{\alpha \cup c_1(T)}{e(N_{M_-/\mathcal{M}})} + \int_{\text{exceptional}} \frac{\alpha \cup c_1(T)}{e(\text{Normal Bundle})}$$

holds in $\mathbb{C}[t, t^{-1}]$ where $H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[t]$

Let $t=0$ (i.e. Take non-equivariant limit.)
$\mathrm{LHS} = 0$ since $c_1(T)\big|_{t=0} = 0$

$$\Longrightarrow \int_{M_+} \alpha - \int_{M_-} \alpha = \mathrm{Coeff}_{t^{-1}} \left[\int_{\substack{\text{exceptional}\\ \text{fixed points}}} \frac{\alpha}{e(\text{Normal Bundle})} \right]$$

Exceptional fixed point

$[z_+ : z_-] \bmod G \in \mathbb{P}(L_+ \oplus L_-)/\!/G$ is fixed by the $\mathbb{C}^*$-action

$\Rightarrow \quad [\lambda z_+ : z_-] = \underset{\underset{G}{\uparrow}}{\rho(\lambda)} \cdot [z_+ : z_-] \quad \Rightarrow$ The corresponding point in Q has nontrivial stabilizers

For moduli spaces of sheaves $M_H(c)$, $Q =$ the quot. scheme,
The exceptional fixed pts are direct sum of lower rank sheaves.

But this explanation is too much over-simplified as:

① We need to use the virtual fundamental classes.

② We cannot take the common space Q for M_{H_+} & M_{H_-}.
$\Longrightarrow$ Use moduli space of stable pairs
$=$ sheaves + sections

③ $\mathcal{M}$ is not a Deligne-Mumford stack, in general.
e.g. if a point having stabilizer $\supsetneq \mathbb{C}^*$ appears in $\mathcal{M}$.
$\sim$ like $E_1 \oplus E_2 \oplus E_3$
$\Longrightarrow$ consider further sheaves + flags in H^0

$\longrightarrow$ Read the original paper!

1.2. Intersection Products on Hilbert schemes

- $$\delta^X_{\xi,t} := \sum_{\ell \geq 0} \Lambda^{4\ell - \xi^2 - \xi} \int_{X_2^{[\ell]}} \frac{\exp\left(\sum (-1)^p \left[ch(\mathcal{J}_1) e^{\frac{\xi - t}{2}} + ch(\mathcal{J}_2) e^{-\frac{\xi - t}{2}} \right]_{p+1} / \alpha_p \right)}{e(-\mathrm{Ext}_p^{\bullet}(\mathcal{J}_2, \mathcal{J}_1(\xi)) e^{t})\, e(-\mathrm{Ext}_p^{\bullet}(\mathcal{J}_1(\xi), \mathcal{J}_2) e^{-t})}$$

$\delta^X_\xi =$ coeff. of t^{-1}

Theorem (Göttsche-Yoshioka-N) based on Ellingsrud-Göttsche-Lehn
$\exists$ universal polynomials $A_1, A_2, \cdots, A_8^p \in \mathbb{Q}((t^{-1}))[\![\Lambda]\!]$, independent of X

s.t. $$(-1)^{\chi(\mathcal{O}_X) + \xi(\xi - K_X)/2}\, t^{-\xi^2 - 2\chi(\mathcal{O}_X)}\, \Lambda^{\xi^2 + 3\chi(\mathcal{O}_X)}\, \delta^X_{\xi,t}$$

$$= \exp\Big[A_1 \cdot \int_X \xi^2 + A_2 \int_X c_1(X) \cdot \xi + A_3 \int_X c_1(X)^2 + A_4 \int_X c_2(X)$$
$$+ \sum_p A_5^p \int_X \xi \cdot \alpha_p + A_6^p \int_X c_1(X) \cdot \alpha_p + A_7^{p,p'} \int_X \alpha_p \alpha_{p'} + A_8^p \int_X \alpha_p \Big]$$

Sketch of the proof

Consider incidence variety $X_2^{[\ell,\ell+1]} \subset X_2^{[\ell]} \times X_2^{[\ell+1]}$

$$X_2^{[\ell,\ell+1]} = \{ (Z, Z') \mid Z \subset Z',\ Z' \setminus Z \text{ supported at a single point in } X_2 = X \sqcup X \}$$

$$X_2^{[\ell,\ell+1]} = X_{2,1}^{[\ell,\ell+1]} \sqcup X_{2,2}^{[\ell,\ell+1]} \qquad (Z' \setminus Z \text{ is in either } 1^{st} \text{ or } 2^{nd}.)$$

We have a natural diagram :

$$\begin{tikzcd} X_2 & X_2^{[\ell,\ell+1]} \arrow[l, "p"'] \arrow[r, "\psi"] \arrow[d, "\phi"] & X_2^{[\ell+1]} \\ & X_2^{[\ell]} & \end{tikzcd} \qquad p(Z, Z') = \text{support of } Z' \setminus Z$$

◎ $X_2^{[\ell,\ell+1]}$: nonsingular (well-known)

◎ $$\begin{tikzcd} X_2^{[\ell,\ell+1]} \arrow[r, "\psi"] \arrow[d, "\sigma = p \times \phi"', "\text{birational}"] & X_2^{[\ell+1]} \\ X_2 \times X_2^{[\ell]} & \end{tikzcd}$$ generically finite of degree $\ell+1$

$$\therefore \int_{X_2^{[\ell+1]}} f = \frac{1}{\ell+1} \int_{X_2^{[\ell,\ell+1]}} \psi^* f$$

Using this diagram, we rewrite $\int_{X_2^{[\ell+1]}}$ by $\int_{X_2 \times X_2^{[\ell]}}$, then $\int_{X_2^2 \times X_2^{[\ell-1]}}$, ... $\int_{X_2^{\ell+1}}$.

Let $\mathcal{L} := \operatorname{Ker}[H^0(\mathcal{O}_{Z'}) \to H^0(\mathcal{O}_Z)]$: line bundle over $X_2^{[\ell,\ell+1]}$

$= \mathcal{I}'_\alpha / \mathcal{I}_\alpha$ on $X_{2,\alpha}^{[\ell,\ell+1]}$

Then one can show

- $\psi^* \operatorname{ch}(\mathcal{I}'_\alpha)/c = \phi^* \operatorname{ch}(\mathcal{I}_\alpha)/c - \operatorname{ch}(\mathcal{L})\, p_\alpha^* c \quad (\alpha = 1, 2) \qquad p_\alpha : X_2^{[\ell,\ell+1]} \xrightarrow[p]{} X_2 \xrightarrow{\alpha^{th}\ \text{proj.}} X$
- $\psi^* \left(\sum (-1)^i \operatorname{Ext}^i_p(\mathcal{I}'_2, \mathcal{I}'_1(\zeta)) \right)$

  $= \phi^* \left(\sum (-1)^i \operatorname{Ext}^i_p(\mathcal{I}_2, \mathcal{I}_1(\zeta)) \right) - \sigma_\alpha^* \mathcal{I}_2^\vee \otimes p_\alpha^* \zeta \otimes \mathcal{L} - \sigma_\alpha^* \mathcal{I}_1 \otimes p_\alpha^*(\zeta \otimes \omega_X^\vee) \otimes \mathcal{L}^\vee$

  $X_{2,1}^{[\ell,\ell+1]}$ comp. $\qquad X_{2,2}^{[\ell,\ell+1]}$ component
- $\sigma_*(c_1(\mathcal{L})^k) = (-1)^k c_k(\mathcal{O}_Z)$

But this proof does not give us explicit expressions of A_i.

1.3 Hilbert scheme $\Longrightarrow$ instanton counting

From the universality of Theorem,
it is enough to compute $\delta^{X}_{\mathfrak{z},t}$ for X : toric surface.

Then we take an equivariant lift of $\delta^{X}_{\mathfrak{z},t}$, and compute it by the fixed point formula.

$T = \mathbb{C}^* \times \mathbb{C}^*$: torus $\curvearrowright X$
$\mathrm{Lie}\,T = \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2$
$X^T = \{p_1, \cdots, p_\chi\}$: torus fixed points
(x_i, y_i) : T-equivariant coordinates at p_i
$(w(x_i), w(y_i))$: T-weights of the tangent space $T_{p_i}X$

e.g. $\mathbb{P}^2 \ni [z_0 : z_1 : z_2] \longmapsto [z_0 : t_1 z_1 : t_2 z_2]$

fixed pts	$[1:0:0]$	$[0:1:0]$	$[0:0:1]$
(x_i, y_i)	$(z_1/z_0, z_2/z_0)$	$(z_0/z_1, z_2/z_1)$	$(z_0/z_2, z_1/z_2)$
wts	$\varepsilon_1, \varepsilon_2$	$-\varepsilon_1, \varepsilon_2 - \varepsilon_1$	$-\varepsilon_2, \varepsilon_1 - \varepsilon_2$

$X_2^{[\ell]} \curvearrowleft T$: induced torus action

$(X_2^{[\ell]})^T \ni Z$: a fixed point
$\quad\; \| $
$Z_1 \sqcup Z_2 \sqcup \cdots \sqcup Z_\chi \qquad Z_i$: supported at p_i

I_i : corresponding ideal sheaf $\longleftrightarrow$ monomial ideal in $\mathbb{C}[x_i, y_i]$

y_i^2		
y_i	$x_i y_i$	$x_i^2 y_i$
1	x_i	x_i^2

$I_i =$ span of monomials $\notin$ Young diagram

$\therefore (X_2^{[\ell]})^T$ is parametrised by

$\{ \vec{Y} = (Y_\alpha^i)\ (\alpha = 1, 2,\ i = 1, \cdots, \chi)$ 2χ-tuple of Young diagrams $\mid |\vec{Y}| = \sum |Y_\alpha^i| = \ell \}$

$$\therefore \tilde{\delta}^{X}_{\mathfrak{z},t} = \sum_{|\vec{Y}|} \Lambda^{|\vec{Y}| - \mathfrak{z}^2 - 3} \frac{1}{e(T_{\vec{Y}} X_2^{[\ell]})} \times \frac{\exp\left(\sum (-1)^p \left[ch(\mathcal{J}_1) e^{\frac{\mathfrak{z}-t}{2}} + ch(\mathcal{J}_2) e^{-\frac{\mathfrak{z}-t}{2}} \right]_{p+1} / \alpha_p \right)}{e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_2, \mathcal{J}_1(\mathfrak{z})) e^{t})\, e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_1(\mathfrak{z}), \mathcal{J}_2) e^{-t})} \Bigg|_{\vec{Y}}$$

$$= \Lambda^{-\xi^2-\xi} \frac{1}{e(-H^\bullet(\mathcal{O}(\xi))\, e(-H^\bullet(\mathcal{O}(-\xi)))}$$

$$\times \sum_{|\vec{Y}|} \Lambda^{|\vec{Y}|} \frac{\exp\left(\sum (-1)^p \left[ch(\mathcal{J}_1) e^{\frac{\xi-t}{2}} + ch(\mathcal{J}_2) e^{-\frac{\xi-t}{2}} \right]_{p+1} / \alpha_p \right)}{e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_1,\mathcal{J}_1))\, e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_2,\mathcal{J}_2))\, e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_2,\mathcal{J}_1) e^{t})\, e(-\mathrm{Ext}^\bullet_p(\mathcal{J}_1,\mathcal{J}_2) e^{-t})}$$

Since $\mathrm{Ext}^\bullet_p(\mathcal{J}_\alpha, \mathcal{J}_\beta)$ is the direct sum $\bigoplus \mathrm{Ext}^\bullet_p(\ ,\ )$ for p_i, and the Euler class and $\exp(ch(\cdot))$ are multiplicative,

the second part is the product of the local contribution of p_i !

It is enough to determine $\tilde{\delta}^X_{\xi,t}$ for $X = \mathbb{C}^2$.

It is nothing but the definition of Nekrasov's partition function.
(instanton counting)