TRUNCATED COUNTING FUNCTION AND ITS DIOPHANTINE ANALOGUE (A PROGRESS REPORT)

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Abstract. We introduce a Diophantine analogue of the truncated counting function and formulate a Diophantine analogue of the Lemma on Logarithmic Derivative.

§0. Introduction.

Motivation.

$\exists$ Analogy between Diophantine Approximation and Nevanlinna Theory.

Typical Example : Roth's Theorem v.s. Nevanlinna's Second Main Theorem.

Fundamental Question.

$\exists$? a "geometry" unifying Diophantine approximation and Nevanlinna theory.

Comparison.

Diophantine approximation is the theory on the ring of integers.

Nevanlinna theory is based on the calculus of functions on $\mathbb{C}$.

$\Rightarrow$

$\nexists$ intrinsic definition of derivative in Diophantine approximation.

$\exists$ intrinsic definition of derivative in Nevanlinna theory.

$\Rightarrow$

$\nexists$ ramification term in Roth(-Schmidt)'s Theorem.

$\exists$ ramification term (counting zeros of the Wronskian) in Nevanlinna(-Cartan)'s Second Main Theorem.

Expectation.

Constructing a geometric framework which recovers the **lost** "ramification counting function" in **Roth's theorem** occupies an essential part toward that of a unifying geometry.

Aim of this Talk.

We introduce a Diophantine analogue of the "truncated counting function" and formulate a Diophantine analogue of the Lemma on Logarithmic Derivative.

Basic Strategy.

Step 1. Both Diophantine approximation and Nevanlinna theory are theories on "approximation". Therefore we start with distinguishing two basic states :
"**exact state**" and "**approximate state**"

Step 2. Characterizing two basic states in terms of the "Wronskian". Here "derivative" appears. Therefore the main part of our approach is to establish a **Diophantine analogue of the Wronskian**.

Step 3. Establishing the "Lemma on Logarithmic Derivative" as a tool of measurement : how points of $\mathbb{P}_n(k)$ (k being a number field) (resp. holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$) deviates from "being in the exact state" w.r.to to a particular set of hyperplanes.

Step 4. Geometry. This part combines "Lemma on Logarithmic Derivative" to the canonical class of the target space under consideration.

Geometry involved in Step 4 will occupy the main part in the attempt toward constructing "unifying geometry". In this talk, however, we will focus on the problem arising from the absence / presence of "ramification term".

§1. Exact State v.s. Approximate State.

The Nevanlinna-Cartan Second Main Theorem (strengthened by Vojta [V3]) is the following:

Theorem 1. *Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then there exists a finite union Z_D of proper linear subspaces depending only on D such that the following statement holds: Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve such that $f(\mathbb{C}) \not\subset D$. Assume that the image of f is not contained in Z_D. Then*

$$\sum_{i=1}^{q} m_{f,D_i}(r) + T_{f,K_{\mathbb{P}^n(\mathbb{C})}}(r) \leq S_f(r) /\!/ \ . \tag{1}$$

Let $W(f) = f^{(1)} \wedge f^{(2)} \wedge \cdots \wedge f^{(n)}$ be the Wronskian of f defined in terms of affine coordinates of $\mathbb{P}^n(\mathbb{C})$.

Under the stronger assumption that $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is linearly non-degenerate (i.e., $f(\mathbb{C})$ not contained in a proper linear subspace), we have a stronger conclusion:

$$\sum_{i=1}^{q} m_{f,D_i}(r) + T_{f,K_{\mathbb{P}^n(\mathbb{C})}}(r) + N_{W(f),0}(r) \leq S_f(r) /\!/ \ .$$

Question. *Does there exist any stronger inequality of this type, i.e., with* **ramification term** *$N_{W(f),0}(r)$ (or its any "modification") in the left hand side, which holds for $\forall f$ s.t. $f(\mathbb{C}) \not\subset Z_D$ (Z_D being a finite union of proper linear subspaces depending only on D) ?*

Remark. *To answer this question and its Diophantine analogue (formulated in this talk) is the next step toward constructing "unifying geometry" (cf. [V2,3]).*

We interpret Theorem 1 by comparing two states: the exact and the approximate states.

Given holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we say that f is in the **exact** state if

$$f(\mathbb{C}) \subset D$$

holds.

Given holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we say that f is in the **approximate** state, if

$$f(\mathbb{C}) \not\subset D$$

holds.

Nevanlinna-Cartan's proof of Theorem 1 consists of three steps.

Step 1. The characterization of the exact state. Given holomorphic curve f being in the exact state w.r.to any linear divisor if and only if

$$W(f) \equiv 0 \ . \tag{2}$$

In this stage we cannot distinguish the linear divisor D under under question.

Step 2. The measurement : how given f in the approximate state deviates from being in the exact state w.r.to the particular linear divisor D (a linear divisor means a finite collection of hyperplanes in general position). This is done by Nevanlinna's Lemma on Logarithmic Derivative:

$$
\begin{aligned}
&m_{f,D}(r) \leq m_{W(f),0}(r) + S_f(r)/\!/ \,, \\
&m_{W(f),\infty}(r) \leq S_f(r)/\!/ \,.
\end{aligned}
\tag{3}
$$

The inequality (3) is the "approximate counterpart" of the characterization (1) of the exact state.

Step 3. Combine the geometry of the space where the Wronskian $W(f)$ lives (indeed, $W(f)$ takes values in the total space of the anticanonical bundle $K^{-1}_{\mathbb{P}^n(\mathbb{C})} \to \mathbb{P}^n(\mathbb{C})$) to the inequality (3).

We then end up with the inequality in Theorem 1.

The Diophantine analogue of Theorem 1 is Roth-Schmidt's SST (Sub-Space Theorem).

Let k be a number field and S a fixed finite set of places of k including all Archimedean ones. The following is Vojta's refinement [V2] of SST.

Theorem 2. *Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position with algebraic coefficients in $\mathbb{P}^n(k)$ and ε any positive number. Then there exists an "effectively computable" finite set of proper linear subspaces Z_D such that the set of solutions outside of Z for the Diophantine inequality*

$$
m_S(x, D) + \mathrm{ht}_{K_{\mathbb{P}^n}}(x) \;>\; \varepsilon\, \mathrm{ht}_{\mathcal{O}(1)}(x)
\tag{4}
$$

for points of $\mathbb{P}^n(k) - D$ is finite.

Remark. *(1) "effectively computable" means : (a) Z is expressed using only elements of $D_1, \ldots, D_q$ plus their translates under* $\mathrm{Gal}(\overline{k}/k)$ *and the operations $\cap$ and $\langle \cdot, \cdot \rangle$. (b) The complexity of the expression is bounded by a function in n and q.*

(2) The set of solutions of (4) in $\mathbb{P}^n(k) - D - Z$ is a finite set depending on k, S, ε and $D_1, \ldots, D_q$. To get an effective bound for the height of this set is an unsolved problem.

In Theorem 2 :

Exact state $= \{$rational points in $D\}$

Approximate state $= \{$rational points outside of $D\}$.

Vojta's idea ([V1, Theorem 6.4.3]).

Vojta proposed a Diophantine analogue

$$
x \mapsto x'
$$

(this is defined for $x \in \mathcal{O}_{k,S}^{n+1}$) of the derivative of a holomorphic curve (lifted to $\mathbb{C}^{n+1} - \{0\}$) using the **adèle version of Minkowski's Convex Body Theorem**.

Convex Body Theorem requires a **Length Function**. Vojta's choice is the following :

Length Function $=$ a function modelled after

Ahlfors' variant of **LLD** .

This means that Vojta interpreted the Diophantine analogue of Ahlfors' variant of the Lemma on Logarithmic Derivative

$$\begin{aligned} &m_{f,D}(r) \le m_{f^{(1)},D^{(1)}}(r) + S_f(r)/\!/ \,, \\ &m_{f^{(1)},\infty}(r) \le S_f(r)/\!/ \end{aligned}$$

as the **defining equation** of the association $x \mapsto x'$.

Also :

The Diophantine analogue of the Lemma on Logarithmic Derivative for higher derivatives

$$\begin{aligned} &m_{f,D}(r) \le m_{f^{(k)},D^{(k)}}(r) + S_f(r)/\!/ \,, \\ &m_{f^{(k)},\infty}(r) \le S_f(r)/\!/ \end{aligned}$$

is the associated system of **successive minima**, which is interpreted as successive differentiation $x', x'', \ldots, x^{(n)}$ (n being the dimension of the projective space under consideration).

Point :

The association $x \mapsto x', x'', \ldots, x^{(n)}$ is a **relative** notion which makes no sense without the approximation target D.

Need : defining equation arising from LLD.

Theorem is that **these defining equations have solutions (adèle version of Minkowski's Cobvex Body Theorem)**.

In [V1, Chapt. 6], Vojta incorporated the role of the Diophantine analogue of the derivative into Schmidt's proof of Theorem 2.

However, the 3-steps structure characterizing Nevanlinna-Cartan theory is not so recognizable. It is then natural to try to identify the 3-steps structure in the proof of Theorem 2.

We note that

the ramification term $N_{W(f),0}(r)$ exists in Theorem 1 under the linear non-degeneracy condition

but

no Diophantine counterpart exists in Theorem 2.

Question. *How to recover "ramification term" in Roth-Schmidt's SST from the definition of the association $x \mapsto x', x'', \ldots, x^{(n)}$?*

§2. Truncated Counting Function.

– An attempt toward recovering the Diophantine "ramification counting function".

2.1. Ramification term and truncated counting function in Nevanlinna Theory.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve s.t. $f(\mathbb{C}) \not\subset D$ (i.e. f is in approximate state).

"Counting zeros" of f'

$\Leftrightarrow$

Counting zeros of $W(f)$ with multiplicity.

Remark. *(1) $\exists$ intrinsic meaning in counting zeros of f' for $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. This has nothing to do with the target D.*
(2) $\exists$ "relative" Diophantine notion of "derivative" of the distribution of $\{f(x)\}_{x\in X(k)}$ for a rational function $f : X \to \mathbb{P}^1(k)$. This is "absolute" in the sense that it does not need a divisor of $\mathbb{P}^1(k)$.

Remark. *$\nexists$ "intrinsic" Diophantine analogue of counting zeros of f'. However, $\exists$ Diophantine analogue of counting zeros of f'* **only at** *$z \in \mathbb{C}$ s.t. $f(z) \in D$.*

Indeed, this is just to associate to each x the set of finite places on which the Zariski closures (over the ring of intergers $\mathcal{O}_k$) of x and D intersect.

Definition. *Let $N^n_{f,D}(r)$ be defined by replacing $\deg_z(f^*D)$ in the usual counting function by $\max\{\deg_z(f^*D) - n, 0\}$.*

Note. *The difference $N_{f,D}(r) - N^n_{f,D}(r)$ is the usual notion of the level n truncated counting function.*

Since D is linear and $W(f)$ is defined w.r.to affine coordinates, the inequality

$$\deg_z(f^*D) \geq n \;\Rightarrow\; \deg_z(f^*D) - n \leq \deg_z W(f)$$

holds ("generically" equality halds). This imples

$$N^n_{f,D}(r) \leq N_{W(f),0}(r) \ .$$

2.2 Ramification term and truncated counting function in Diophantine setting.

We can now define the Diophantine analogue of $N^n_{f,D}(r)$ (via Vojta's dictionary). This consists of

Definition. *(1) The association $x \mapsto S^n_x$ is defined by setting*

$$S^n_x := \{\text{all non-Archemedean places } v \,|\, \text{Zariski closure of } x \text{ and } D \text{ intersect over } v \text{ with multiplicity } \geq n\} \ .$$

(2) The "counting function" $N^n(x, D)$ counts the intersection of the Zariski closures of x and D over the places v in S^n_x with the multiplicity $\deg_v(x^*D)$ (in the usual counting function) replaced by $\max\{\deg_v(x^*D) - n, 0\}$ ($\forall v \in S^n_x$).

In contrast to working with a fixed set S of places (as in Theorem 2), we cannot fix the set of finite places if we try to define the the Diophantine analogue of the truncated counting function.

The conclusion of this lecture is the following analogue of the Lemma on Logarithmic Derivative in the setting of varying $S = S(x)$ where

$$S(x) = S_\infty \cup S_x^n \ .$$

Here, S_∞ is the set of all Archimedean places.

Theorem 3. *Let $F_0, \dots, F_q$ be a set of linear forms in k^{n+1} in general position and ε any positive number. Then there exists a finite set $\mathcal{S}$ of proper linear subspaces of k^{n+1} with the following property. If $x \in k^{n+1}$ is not a vector in the union of the linear subspaces in $\mathcal{S}$, then we can inductively construct a sequence $x^{(1)}, \dots, x^{(n)} \in \mathcal{O}_k^{n+1}$ of vectors with the following properties:*
(i) $x, x^{(1)}, \dots, x^{(n)}$ are linearly independent:

$$x \wedge x^{(1)} \wedge \cdots \wedge x^{(n)} \neq 0 \ .$$

(ii) $\mathrm{ord}_v(x^{(t)} \cdot F_i)$ decreases 1 as t increases 1, i.e., if $\mathrm{ord}_v(x \cdot F_i) \geq n$, we have

$$\mathrm{ord}_v(x^{(t)} \cdot F_i) \geq \mathrm{ord}_v(x \cdot F_i) - t$$

for $\forall t = 1, 2, \dots, n$.
*(iii) [***Diophantine analogue of LLD***] If we set $x^{\leq p-1} := x \wedge x^{(1)} \wedge \cdots \wedge x^{(p-1)}$ and $F_{i,p} = F_i \wedge F_{n-p+2} \wedge \cdots \wedge F_n$ for $p = 1, \dots, n$, we have the following inequality: after suitably re-ordering the F's for each $v \in S(x)$ (according to the v-adic approximation of x to D), we have*

$$\sum_{v \in S(x)} \log \frac{||(x^{\leq p-1} \wedge (x^{\leq p-2} \wedge x^{(p)})) \cdot F_{i,p}||_v}{||x^{\leq p-1}||_v ||x^{\leq p-1} \cdot F_{i,p}||_v^{\mathrm{modified}}} < \varepsilon \, \mathrm{ht}(x) \tag{5}$$

for $\forall i = 0, 1, \dots, q$ and $\forall x$ such that $x^{\leq p-1} \cdot F_{i,p} \neq 0$. If $x^{\leq p-1} \cdot F_{i,p} = 0$ then $(x^{\leq p-2} \wedge x^{(p)}) \cdot F_{i,p} = 0$. Here, $||x^{\leq p-1} \cdot F_{i,p}||_v^{\mathrm{modified}}$ means that if $v \in S_x^n$ we replace $\mathrm{ord}_v(x^{\leq p-1} \cdot F_{i,p})$ in the original definition by $\max\{\mathrm{ord}_v(x^{\leq p-1} \cdot F_{i,p}) - 1, 0\}$, and if $v \in S_\infty$, we need no modification.

To prove Theorem 3, we must incorporate the association $x \mapsto S_x^n$ into Vojta's interpretation [V, Chapt. 6] of Schmidt's proof of SST.

The point of the proof in this interpretation is the choice of the length function (which reflects on the left hand side of (5)) used in the adèle version of Minkowski Convex Body Theorem (i.e., successive minima with estimates) with varying $S(x)$.

We use the **proof by contradiction**, just as in the traditional proof of SST. Therefore the final stage of the proof is (a variant of) **Roth's Lemma**.

The inequality (5) implies that we may replace the inequality in the condition $\mathrm{ord}_v(x^{(t)} \cdot F_i) \geq \mathrm{ord}_v(x \cdot F_i) - t$ by the equality.

We can put the inequality (5) in more geometric form.

Theorem 4. *Let D be a linear divisor of $\mathbb{P}^n(k)$ in general position. Let $D^{(p)}$ denote the union of the p-th jet space of all irreducible components of D. Then there exists a finite union $\mathcal{S}$ of proper linear subspaces of $\mathbb{P}^n(k)$ such that, if $x \notin \mathcal{S}$, then there exist $\overline{x}^{(1)}, \dots, \overline{x}^{(n)} \in T_{[x]}\mathbb{P}^n(\mathcal{O}_k)$ which satisfy the inequalities*

$$m_{S_\infty}(x, D) \leq m_{S_\infty}(\overline{x}^{(p)}, D^{(p)}) + \varepsilon \, \mathrm{ht}(x)$$
$$m_{S_\infty}(\overline{x}^{(p)}, \infty) \leq \varepsilon \, \mathrm{ht}(x)$$

and the condition

$$\mathrm{ord}_v(x^{(t)} \cdot F_i) = \mathrm{ord}_v(x \cdot F_i) - t \quad \forall v \in S_x^n$$

for $\forall p = 1, 2, \ldots, n$ *(up to uniform error). Here,* $S(x)$ *is the finite set of places of* k *defined by* $S(x) = S_\infty \cup S_x^n$ *where* S_x^n *is the set of non-Archimedean places of* k *over which the section* $x : \mathrm{Spec}(\mathcal{O}_k) \to \mathbb{P}^n(\mathcal{O}_k)$ *and the linear divisor* D *in* $\mathbb{P}^n(\mathcal{O}_k)$ *intersect with multiplicity* $m \geq n$.

Problem. *What is the effect of this separation into inequalities over* S_∞ *and those over* S_x^n *to the effectivity problem ?*

We have an equivalence Theorem 3 $\Leftrightarrow$ Theorem 4. The following is the direct consequence of Theorem 4:

Corollary 5. *We have*

$$\begin{aligned} N^n(x, D) \leq N_{S_\infty}(x, D)&(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) \\ &- N_{S(x)}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + \varepsilon\, \mathrm{ht}(x) \end{aligned}$$

outside a finite union $\mathcal{S}$ *of proper linear subspaces of* $\mathbb{P}^n(k)$*, where the counting functions measure the* v*-adic approximation of* $x^{(1)} \wedge \cdots \wedge x^{(n)}$ *to* 0 *for appropriate finite places (*N_S *measures the* v*-adic approximation for* v *outside of* S*) in the total space of the anticanonical bundle of* $\mathbb{P}^n(k)$.

I propose that the "**exact state**" in the Diophantine setting is characterized by

$$x^{(1)} \wedge \cdots \wedge x^{(n)} = 0 \qquad \text{in } K^{-1}_{\mathbb{P}^n(k)} \ .$$

This is reasonable, because if we perform the successive minima restricted to a hyperplane, the sequence of **linearly independent** vectors $x, x^{(1)}, \ldots, x^{(t)}$ ends at $t = n-1$ and therefore $x^{(1)} \wedge \cdots \wedge x^{(n-1)} \wedge x^{(n)} = 0$.

It follows from Theorem 4 and Corollary 5 that the corresponding "**approximate state**" is characterized by the inequality

$$\begin{aligned} &m_{S_\infty}(x, D) + N^n(x, D) \\ \leq\ & m_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + N_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) \\ &- N_{S(x)}(x^{(1)} \wedge \cdots \wedge x^{(n)}, 0) + \varepsilon\, \mathrm{ht}(x) \ . \end{aligned} \tag{6}$$

The right hand side of (6) is bounded above by

$$-\mathrm{ht}_{K_{\mathbb{P}^n}}(x) + m_{S_\infty}(x^{(1)} \wedge \cdots \wedge x^{(n)}, \infty) + \varepsilon\, \mathrm{ht}(x) \ .$$

Using Theorem 4 again, we conclude that this is bounded above by

$$-\mathrm{ht}_{K_{\mathbb{P}^n}}(x) + \varepsilon\, \mathrm{ht}(x) \ .$$

Therefore Theorem 4 impplies Schmidt's Subspace Theorem with truncated counting functions:

Corollary 6. *Suppose that Conjecture 3 is true. Then the following improvement of Schmidt's SST is true: Let $D = \{D_1, \ldots, D_q\}$ be a collection of hyperplanes in general position in $\mathbb{P}^n(k)$ and ε any positive number. Then there exists a finite number of proper linear subspaces Z such that the set of the solutions outside of Z for the Diophantine inequality*

$$m_S(x, D) + N^n(x, D) + \mathrm{ht}_{K_{\mathbb{P}^n}}(x) \; > \; \varepsilon \, \mathrm{ht}_{\mathcal{O}(1)}(x)$$

for points of $\mathbb{P}^n(k) - D$ is finite.

Remark. *This is an unsatisfactory result, because the there no effective estimates for the height of Z (especially zero-dimensional components).*

Conclusion. *I introduced a natural geometric framework to recover the lost "ramification term" in Roth-Schmidt's SST. Effectivity in this framework is the next problem to study.*

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