Quiver varieties and Branching

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1. **Motivation.** Braverman-Finkelberg [1] recently proposed the geometric Satake correspondence for the affine Kac-Moody group G_{aff}. They conjecture that intersection cohomology sheaves on the Uhlenbeck compactification of the framed moduli space of G_{cpt}-instantons on $\mathbb{R}^4/\mathbb{Z}_r$ correspond to weight spaces of representations of the Langlands dual group $G_{\mathrm{aff}}^{\vee}$ at level r. When $G = \mathrm{SL}(l)$, the Uhlenbeck compactification is the quiver variety of type $\mathfrak{sl}(r)_{\mathrm{aff}}$, and their conjecture follows from the author's earlier result [5] and I. Frenkel's level-rank duality [4]. They further introduce a convolution diagram which conjecturally gives the tensor product multiplicity [2]. Since the tensor product multiplicy corresponds to the branching multiplicity under the level-rank duality, the author develop the theory for the branching in quiver varieties and check this conjecture for $G = \mathrm{SL}(l)$ in the paper [6].

2. **Quiver varieties.** Suppose that a finite graph is given. Let I be the set of vertices and E the set of edges. Suppose that there are no edge loops. Let $\mathbf{C}$ be the Cartan matrix. Let $\mathfrak{g}$ be the corresponding (symmetric) Kac-Moody Lie algebra. Let H be the set of oriented edges (hence $\#H = 2\#E$), and we choose an orientation Ω of the graph (I, E).

Suppose that I-graded vector spaces V, W are given. Then we consider the vector space

$$\mathbf{M}(V,W) = \bigoplus_{h\in H} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i),$$

where $o(h)$, $i(h)$ are the outgoing and incoming vertices of h. We denote the corresponding componets of the above decomposition by B_h, a_i, b_i. Let $G_V = \prod_{i\in I} \mathrm{GL}(V_i)$. It acts on $\mathbf{M}(V,W)$ by conjugation. The choice of the orientation gives us the symplectic form invariant under the G_V-action. Let $\mu\colon \mathbf{M}(V,W) \to (\mathrm{Lie}\, G_V)^*$ be the corresponding moment map vanishing at the origin. It is given by

$$\mu(B_h, a_i, b_i) = \sum_{h: i(h)=i} \varepsilon(h) B_h B_{\overline{h}} + a_i b_i$$

if we identify $(\mathrm{Lie}\, G_V)^*$ with $\mathrm{Lie}\, G_V$ by the trace. Here $\varepsilon(h)$ is 1 if $h \in \Omega$ and -1 otherwise, and $\overline{h}$ is the same edge with h but equipped with the opposite orientation.

We consider a quotient of $\mu^{-1}(0)$ by G_V in the sense of the geometric invariant theory. It depends on the choice, called the *stability parameter*. Let $\zeta = (\zeta_i) \in \mathbb{Z}^I$. We define the character χ_ζ of G_V given by $\chi_\zeta(g) = \prod_{i\in I} (\det g_i)^{-\zeta_i}$, and we consider the semi-invariants $A(\mu^{-1}(0))^{G,\chi_\zeta^n} = \{f \in A(\mu^{-1}(0) \mid f(gx) = \chi_\zeta(g)^n f(x)\}$. Then $\bigoplus_{n=0}^{\infty} A(\mu^{-1}(0))^{G,\chi_\zeta^n}$ is a graded ring, and we define the quiver variety by

$$\mathfrak{M}_\zeta(V,W) = \mathrm{Proj}(\bigoplus_{n=0}^{\infty} A(\mu^{-1}(0))^{G,\chi_\zeta^n}).$$

By a general result for the geometric invariant theory, $\mathfrak{M}_\zeta(V,W)$ is the set of ζ-semistable points modulo the so-called S-equivalences. (See [6] for the precise statement.) It contains the open subscheme $\mathfrak{M}^{\mathrm{s}}_\zeta(V,W)$ consisting of G_V-orbits of ζ-stable points. For example, if $\zeta = 0$, all points are ζ-semistable, and two points are S-equivalent if and only if their closure intersect. In this case, $\mathfrak{M}_0(V,W)$ is an affine algebraic variety given by $\operatorname{Spec}(A(\mu^{-1}(0))^{G_V})$.

The quiver variety depends on the choice of the stability parameter ζ, but its dependence is through the face F containing ζ. Here a face is given by the decomposition of the set $R_+(V)$ of positive roots with $\alpha = \sum m_i\alpha_i$ with $m_i \le \dim V_i$ into three parts $R_+(V) = R^+_+(V) \sqcup R^-_+(V) \sqcup R^0_+(V)$ as

$$F = \{\zeta \in \mathbb{Q}^I \mid \zeta\cdot\alpha > 0, < 0, = 0 \text{ for } \alpha \in R^+_+(V), \in R^-_+(V),\ R^0_+(V) \text{ respectively}\}.$$

We say a face F is a *chamber* if $R^0_+(V) = \emptyset$. For example, in [5] we use the parameter ζ^+ in the face given by $R^+_+(V) = R_+(V)$. If ζ is in a chamber, we have $\mathfrak{M}_\zeta(V,W) = \mathfrak{M}^{\mathrm{s}}_\zeta(V,W)$ and $\mathfrak{M}_\zeta(V,W)$ is nonsingular of dimension

$$\dim \mathfrak{M}_\zeta(V,W) = 2(\dim V, \dim W) - (\dim V, \mathbf{C}\dim V),$$

where $\dim V$, $\dim W$ are dimension vectors (in $\mathbb{Z}^I$) and $(\ ,\ )$ is the natural inner product on $\mathbb{Z}^I$.

If F' is in the closure of F, and if we take $\zeta' \in F'$, $\zeta \in F$, we have a projective morphism

$$\pi_{\zeta,\zeta'}\colon \mathfrak{M}_\zeta(V,W) \to \mathfrak{M}_{\zeta'}(V,W).$$

In particular, $\zeta' = 0$ is contained in the closure of any face, we always have $\mathfrak{M}_\zeta(V,W) \to \mathfrak{M}_0(V,W)$.

3. **Convolution algebra.** For the parameter $\zeta = 0$, we have a closed embedding $\mathfrak{M}_0(V,W) \subset \mathfrak{M}_0(V',W)$ for $V \subset V'$ by setting the data 0 on a subspace of V' complementary to V. We denote the direct limit by $\mathfrak{M}_0(W)$. If ζ is in a chamber, there is no obvious relation among different $\mathfrak{M}_\zeta(V,W)$'s, and we set $\mathfrak{M}_\zeta(W) = \bigsqcup_V \mathfrak{M}_\zeta(V,W)$ where V runs all isomorphism classes of I-graded vector spaces. For a general ζ, we have the closed embedding $\mathfrak{M}_\zeta(V,W) \subset \mathfrak{M}_\zeta(V',W)$ for $V \subset V'$, when the data $0 \in \mathbf{M}(V'/V, 0)$ is ζ-semitable. We denote the inductive limit by $\mathfrak{M}_\zeta(W)$. We consider the fiber product

$$Z_{\zeta,\zeta'}(W) = \mathfrak{M}_\zeta(W) \times_{\mathfrak{M}_{\zeta'}(W)} \mathfrak{M}_\zeta(W),$$

when the faces F', F containing ζ', ζ satisfy $F' \subset \overline{F}$ for any choice of V. This is a union $\mathfrak{M}_\zeta(V^1,W) \times_{\mathfrak{M}_{\zeta'}(V,W)} \mathfrak{M}_\zeta(V^2,W)$ of various V^1, V^2 and a big vector space V containing both V^1 and V^2. Any irreducible component has at most $\dim \mathfrak{M}_\zeta(V^1,W) \times \mathfrak{M}_\zeta(V^2,W)/2$.

We assume ζ is in a chamber and consider

$$H_{\mathrm{top}}(Z_{\zeta,\zeta'}(W)),$$

where top means the degree $\dim \mathfrak{M}_\zeta(V^1,W) \times \mathfrak{M}_\zeta(V^2,W)$ for each summand $\mathfrak{M}_\zeta(V^1,W) \times_{\mathfrak{M}_{\zeta'}(V,W)} \mathfrak{M}_\zeta(V^2,W)$. This has a structure of the algebra given by the convolution product

$$c * c' = p_{13*}(p_{12}^*(c) \cap p_{23}^*(c')),$$

where p_{ab} is the projection from the triple fiber product to the fiber product of a^{th} and b^{th} factors.

In [5] the author constructed an algebra homomorphism

$$\mathbf{U}(\mathfrak{g}) \to H_{\text{top}}(Z_{\zeta,0}(W)) \tag{1}$$

for $\zeta = \zeta^+$ as above. By the general theory of the convolution algebra (see [3]) the algebra $H_{\text{top}}(\mathfrak{M}_\zeta(W))$ is the endomorphism algebra

$$\operatorname{End}_{\operatorname{Perv}(\mathfrak{M}_0(W))}(\pi_{\zeta,0*}(\mathbb{C}_{\mathfrak{M}_\zeta(W)}[\dim \mathfrak{M}_\zeta(W)])),$$

where the shift $\dim \mathfrak{M}_\zeta(W)$ means that we shift $\dim \mathfrak{M}_\zeta(V, W)$ for each component $\mathfrak{M}_\zeta(V, W)$. One can show that $\pi_{\zeta,0*}(\mathbb{C}_{\mathfrak{M}_\zeta(W)}[\dim \mathfrak{M}_\zeta(W)])$ is canonically isomorphic to each other independent of the choice of the chamber (containing ζ) by using a one parameter deformation of $\mathfrak{M}_0(W)$ and its similtaneous resolution. So we have a homomorphism (1) for any ζ.

Theorem 1. (1) *Choose a subdiagram $I^\circ \subset I$. Take ζ' so that $\zeta'_i = 0$ for $i \in I^\circ$ and $\zeta'_i > 0$ for $i \notin I^\circ$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{g}_{I^\circ}) & \longrightarrow & H_{\text{top}}(Z_{\zeta,\zeta'}(W)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{U}(\mathfrak{g}) & \longrightarrow & H_{\text{top}}(Z_{\zeta,0}(W)), \end{array}$$

where $\mathfrak{g}_{I^\circ}$ is the Levi subalgebra of $\mathfrak{g}$ corresponding to I° and the bottom horizontal arrow is (1).

(2) *Suppose that the graph (I, E) is affine. We choose a subdiagram $I_0^\circ \subset I_0$ of the corresponding finite type graph $I_0 = I \setminus \{0\}$. Take ζ' so that $\zeta'_i = 0$ for $i \in I_0^\circ$ and $\zeta'_i > 0$ for $i \in I_0 \setminus I^\circ$ and $\zeta' \cdot \delta = 0$ for the imaginary root δ. And take ζ from a chamber containing ζ' in its closure. Then we have a commutative diagram as above replacing $\mathbf{U}(\mathfrak{g}_{I^\circ})$ by $\mathbf{U}(\widehat{\mathfrak{g}}_{I_0^\circ})$ the enveloping algebra of the affine Lie algebra of the Levi subalgebra $\mathfrak{g}_{I_0^\circ}$ of the finite dimensional Lie algebra $\mathfrak{g}_{I_0}$.*

References

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