pole

$$\frac{\nabla}{ds} T_\alpha + \frac{1}{2} \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma} [T_\beta, T_\gamma] = 0$$

suppose $\quad T_\alpha = \frac{a_\alpha}{s} + \text{higher}$

$$\Longrightarrow \quad -\frac{a_\alpha}{s^2} + \frac{1}{2} \sum \varepsilon_{\alpha\beta\gamma} \left[\frac{a_\beta}{s}, \frac{a_\gamma}{s}\right] + \cdots = 0$$

$\therefore \quad a_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} [a_\beta, a_\gamma] \qquad a_1 = [a_2, a_3]$ etc

$\mathfrak{su}(2) \cong \mathfrak{sp}(1) \cong \mathrm{Im}\,\mathbb{H} \qquad \frac{1}{2} i = [\frac{1}{2} j, \frac{1}{2} k]$ etc

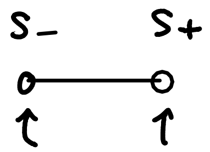
or $\frac{1}{2}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

$\therefore \quad i, j, k \mapsto 2a_1, 2a_2, 2a_3$ defines a Lie alg. hom.

complexification $\rightsquigarrow$ $\rho : \mathfrak{su}(2) \longrightarrow \mathfrak{u}(k)$

$\rho : \mathfrak{sl}(2) \longrightarrow \mathfrak{gl}(k)$

$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



can put $\rho_\pm$ at $s_\pm$

For Nahm's equations, corresponding to monopoles

$\Longrightarrow$ Both $\rho_\pm$ irreducible for $s_\pm$

But one can put any $\rho_\pm$

Classification of $\rho_\pm$

$\rho : su(2) \longrightarrow u(k)$ $\quad$ ($\rho : sl(2) \longrightarrow gl(k)$)

$\longleftrightarrow$ k-dim'l representation of $SU(2)$

$\longleftrightarrow$ partition of k $\quad (1^{n_1} 2^{n_2} \cdots\cdots)$ s.t. $\sum i n_i = k$

$$V = \bigoplus V(i)^{\oplus n_i} \leftarrow \text{multiplicity}$$

$\uparrow$ i-dim'l irr. rep.

$\longleftrightarrow$ nilpotent elements of $gl(k)$ up to conjugacy

Jordan normal form

This correspondence holds for arbitrary simple cpx Lie algebra $/\mathbb{C}$

<u>Th.</u> (Jacobson–Morozov + Kostant)

$\rho : sl(2) \longrightarrow \mathfrak{g} \,/\, \text{conj.} \longleftrightarrow$ nilpotent element $e \in \mathfrak{g} \,/\, \text{conj.}$

$$\rho\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e$$

<u>ex</u>

$$e = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & & & 0 \\ n-1 & \ddots & & \\ & 2(n-2) & \ddots & \\ 0 & & n-1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} n-1 & & & 0 \\ & n-3 & & \\ & & \ddots & \\ 0 & & & 1-n \end{bmatrix},$$

$$[e, f] = \begin{bmatrix} n-1 & & & 0 \\ & 2(n-2) & & \\ & & \ddots & \\ & & n-1 & \\ 0 & & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & & 0 \\ & n-1 & & \\ & & 2(n-2) & \\ & & & \ddots \\ 0 & & & n-1 \end{bmatrix}$$

$$k(n-k) - \underbrace{(k-1)(n-k+1)}_{= \, k(n-k) - n + k + k - 1} = n - 2k + 1$$

$$[h, e] = 2e$$

Last time : both $\rho_\pm$ = trivial $(= 1^k)$

$\Longrightarrow$ moduli sp. $\cong T^*GL(k)$

This is true for any $G^{\mathbb{C}}$

Th. (Bielawski)

ρ_- : arbitrary , ρ_+ = trivial

$\Longrightarrow$ moduli space $\cong G^{\mathbb{C}} \times S(\rho_-)$

where $S(\rho_-)$ = Slodowy slice

$= e + Z_{\mathfrak{g}^{\mathbb{C}}}(f)$

$e = \rho_-\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$f = \rho_-\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

ρ_- : trivial $\Rightarrow f = 0 \Rightarrow S(\rho_-) = \mathfrak{g}^{\mathbb{C}}$

recover the previous thm

On Slodowy slice

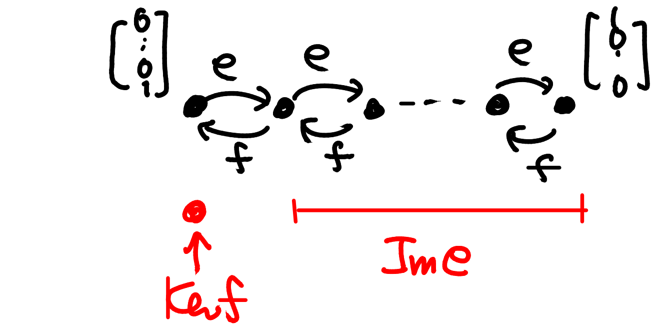
$\mathfrak{g}^{\mathbb{C}} \ni e \qquad T_e \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$

$O(e) =$ nilpotent orbit through e

$T_e O(e) = [\mathfrak{g}^{\mathbb{C}}, e] = \mathrm{Im}(\mathrm{ad}\, e)$

$\mathfrak{g}^{\mathbb{C}} = T_e O(e) \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(f) = \mathrm{Im}(\mathrm{ad}\, e) \oplus \mathrm{Ker}(\mathrm{ad}\, f)$

(∵) $\mathfrak{sl}_2$-rep. theory $\Rightarrow$ irr. rep of $\mathfrak{sl}_2$ looks like



//

$\Rightarrow$ $S(\mathfrak{p}_-) \cap O(e) = \{e\}$ in a nbd of e

$\mathbb{C}^* \curvearrowright S(\mathfrak{p}_-) \qquad t^2 (\underbrace{\mathrm{Ad}(t^{-h})}_{= \rho \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}} (e+s))$

$[h, e] = 2e \Rightarrow e$: fixed

$h|_{Z_{\mathfrak{g}^{\mathbb{C}}}(f)}$: nonpositive eigenvalues

$\Rightarrow$ above action is contracting.

∴ Any pt in $S(\mathfrak{p})$ can be moved to a n.b.d of e.

∴ $S(\mathfrak{p}_-) \cap O(e) = \{e\}$ everywhere

sketch of the proof of Bielawski's Thm

cpx description $\begin{cases} \alpha = \frac{1}{2}(T_0 + i T_1) \\ \beta = \frac{1}{2}(T_2 + i T_3) \end{cases}$

$s=0$ ——— s_+

pole   regular

cpx equ. : $\frac{d}{ds}\beta + 2[\alpha, \beta] = 0$

$-\frac{e}{2} + 2\left[\frac{h}{4}, \frac{e}{2}\right] = 0$

$[h, e] = 2e$ ✓

Step 1° $\{\text{solutions of Nahm's equ}\}/\mathcal{G}_{00} \cong \{\text{solutions of cpx equation}\}/\mathcal{G}_{00}^{\mathbb{C}}$

$\uparrow$ Donaldson's thm (taking limit)

Step 2° $\begin{cases} \operatorname{Res}\alpha = \frac{i}{2}\operatorname{Res} T_1 = \frac{i}{4}\rho\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \frac{h}{4} \\ \operatorname{Res}\beta = \frac{1}{2}(\operatorname{Res} T_2 + i \operatorname{Res} T_3) = \frac{\rho}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e/2 \end{cases}$

Put $\begin{cases} \bar{\alpha} = 2s\alpha \\ \bar{\beta} = 2s\beta \end{cases}$ $s = \frac{1}{2}e^{-2t}$ $s=0 \cdots t=\infty$, $s=s_+ \cdots t$: finite

$$\frac{d\bar{\beta}}{dt} = -2\bar{\beta} + 2[\bar{\alpha}, \bar{\beta}]$$

$$\begin{aligned} \text{LHS} &= 2\left(\frac{ds}{dt}\beta + s\frac{ds}{dt}\frac{d\beta}{ds}\right) \\ &= 2\times(\underbrace{-e^{-2t}}_{2s})(\beta + \underbrace{s\frac{d\beta}{ds}}_{-2[\alpha,\beta]}) \\ &= \text{RHS} \end{aligned}$$

After a cpx gauge transform, we may assume

$\bar{\alpha} = h/2$ (i.e. $\alpha = \frac{h}{4s}$) in a nbd of $t=\infty$

$\Rightarrow \bar{\beta} = e^{-2t}\operatorname{Ad}(e^{th})(\bar{\beta}_0)$ $\exists$ $\bar{\beta}_0 := \bar{\beta}|_{t=0}$ constant

(i.e. $\beta = \operatorname{Ad}((2s)^{-h/2})(\bar{\beta}_0)$)

Decompose :

$\mathfrak{g}^{\mathbb{C}} = \oplus\, \mathfrak{g}^{\mathbb{C}}(n)$ ← eigenvalue of ad h , $\mathfrak{g}^{\mathbb{C}}(<n) = \bigoplus_{m<n} \mathfrak{g}^{\mathbb{C}}(m)$

Need $\overline{\beta} \xrightarrow[(t\to\infty)]{} \cancel{2}e \Rightarrow \overline{\beta}_0 \in \boxed{\cancel{2}e + \mathfrak{g}^{\mathbb{C}}(<2)}$

We can further make a cpx gauge transform $\gamma(t)$ s.t. it does not change $\overline{\alpha} = h$

$$\underbrace{\overline{\alpha}^{\gamma}}_{\overset{\shortparallel}{h}} = \gamma^{-1}\underset{\overset{\shortparallel}{h}}{\overline{\alpha}}\gamma \underset{+}{-} \gamma^{-1}\frac{d\gamma}{dt} \qquad \therefore \frac{d\gamma}{dt} = [h, \gamma]$$

$-\alpha$: connection form

$$\therefore \gamma = e^{th}\gamma_0 e^{-th}$$

Need $\gamma(t) \to 1 \ (t\to\infty) \Rightarrow \gamma_0 \in \exp(\mathfrak{g}^{\mathbb{C}}(<0))$

Then $\overline{\beta}^{\gamma} = \gamma^{-1}\overline{\beta}\gamma = e^{-2t}\,\mathrm{Ad}(e^{th})(\gamma_0^{-1}\overline{\beta}\gamma_0)$

$$\therefore \overline{\beta}_0 \longmapsto \gamma_0^{-1}\overline{\beta}_0\gamma_0$$

<u>Prop.</u> $\exp(\mathfrak{g}^{\mathbb{C}}(<0)) \times S(\rho_-) \xrightarrow[\Phi]{\cong} e + \mathfrak{g}^{\mathbb{C}}(<2)$

$\because$ Take differential at $(1, e)$

$$d\Phi(\xi, z) = [\xi, e] + z$$

$\xi \in \mathfrak{g}^{\mathbb{C}}(<0)$, $z \in Z_{\mathfrak{g}^{\mathbb{C}}}(f)$ $\Rightarrow \Phi$: local isom.

Use $\mathbb{C}^*$-action as before //

$\therefore$ moduli sp. $\cong \underset{\uparrow}{G^{\mathbb{C}}} \times S(\rho)$

value of the gauge transf. g s.t. $(\alpha, \beta)^g$ is of the above form at the other end S_+

Finally we consider

$$\begin{array}{ccc} \circ & \text{———} & \circ \\ \uparrow & & \uparrow \\ \rho_- & & \rho_+ \end{array} \quad : \text{ irreducible rep. of } su(2)$$

<u>Th.</u> (Donaldson)

moduli sp. $\cong$ the space of based maps $\mathbb{P}^1 \xrightarrow[\deg k]{} \mathbb{P}^1$, $\infty \mapsto \infty$

We divide

$$\begin{array}{ccc} \circ & \text{—}\vdots\text{—} & \circ \\ \rho_- & \uparrow & \rho_+ \\ & (X, v) & \end{array}$$

solve ODE from there

<u>Lemma.</u> $G^c \times S(\rho_\pm) \xrightarrow{\cong} \{ (X, v) \in \mathfrak{gl}(k) \times \mathbb{C}^k \mid v : \text{cyclic vectors for } X \}$

value of β at the other end

$\mathbb{C}^k = \mathrm{Span}\langle v, Xv, \dots, X^{k-1} v \rangle$

$$(g, e + z) \mapsto \left(-g(e+z)g^{-1},\ g\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

<u>Rem.</u> $\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$: cyclic vector for $e \Rightarrow$ for $e+z$: small z

$\underset{\mathbb{C}^*}{\Rightarrow}$ for any $e+z$

$\therefore$ LHS $\to$ RHS is defined.

Using this, one can show :

moduli sp. $\cong \{ (X, v_-, v_+) \mid v_- : \text{cyclic},\ {}^t v_+ : \text{cyclic for } {}^t X \} / GL(k)$

$$\begin{tikzcd} & \mathbb{C}^k \arrow[loop above, "X"] \arrow[dr, "v_+"] & \\ \mathbb{C} \arrow[ur, "v_-"] & & \mathbb{C} \end{tikzcd}$$

<u>Prop</u> $\{(X, v_-, v_+) : \text{as above}\}/_{GL(k)} \longrightarrow \{\text{based maps } \mathbb{P}^1 \to \mathbb{P}^1 \text{ of } \deg = k\}$

$$f(z) = v_+ (z - X)^{-1} v_-$$

$$(f(\infty) = 0)$$

quasimaps : $\{(X, v_-, v_+) : \text{arbitary}\} /\!/_{GL(k)}$

$\uparrow$ set of closed orbits