

# Cluster algebras and canonical base

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## Project

Relate a cluster algebra  $\mathcal{A}$  to Lusztig's canonical base / perverse sheaves on the spaces of quiver representations, or related spaces

## Goal

$\mathcal{A}$  has a (dual) canonical base  $\mathbb{B}$  containing all cluster monomials.  
(In fact, in our example today,  $\mathbb{B} = \{\text{cluster monomials}\}$ )

- Cor.
- positivity of Laurent expansions with respect to any seed.  $\leftarrow$  cluster alg. side
  - factorization of dual canonical base elements.  $\leftarrow$  canonical base side

## Why?

- Original motivation (Fomin-Zelevinsky)
- Canonical base elements should reflect various properties of quiver representations.  
So, want to relate canonical base / tilting theory  
cluster category
- Also gives a monoidal categorification (Hernandez-Leclerc),  
as the canonical base is the set of simple objects in an abelian category.



Other cluster variables (and monomials) are given by Caldero-Chatron formula

Let  $W = W(1) \oplus W(2) \oplus \dots \oplus W(l)$  be a graded vector space over  $\mathbb{C}$   
 Let  $w_i := \dim W(i)$ . ^ not  $l+1$

Then

$$y[W] = \frac{1}{y_1^{w_1} \dots y_l^{w_l}} \sum_{\nu} \text{Euler}(\text{Gr}_{\nu}(x)) \prod_i y_i^{\nu_{i-1} + w_{i+1} - \nu_{i+1}}$$

where  $\text{Gr}_{\nu}(x) =$  quiver Grassmann for a general representation  $x$   
 such that the underlying vector space  $= W$ ,  
 and  $\nu \in \mathbb{Z}_{\geq 0}^l$  dimension of submodules.

$x$ : indecomposable  $\iff y[W] = y[\alpha]$  is a cluster variable.

I use this CC formula to show that cluster monomials correspond to perverse sheaves on the space of quiver rep's.

## Graded quiver varieties (of type $A_1$ )

Consider the opposite quiver  $\alpha_1 \leftarrow \alpha_2 \leftarrow \dots \leftarrow \alpha_l \leftarrow \alpha_{l+1}$

$W = W(1) \oplus W(2) \oplus \dots \oplus W(l+1)$ : graded vector space over  $\mathbb{C}$

$$\mathbb{E}_W := \bigoplus_i \text{Hom}(W(i+1), W(i)) \ni x = \bigoplus_{i=1}^l x_i \quad W(1) \xleftarrow{x_1} W(2) \xleftarrow{x_2} W(3) \xleftarrow{\dots} W(l+1)$$

$\hookrightarrow G_W = \prod \text{GL}(W(i))$  the space of quiver representations with bases

We introduce a closed subvariety: (this is an affine graded quiver variety of type  $A_1$ )

$$M_0^\circ(W) := \{ x \in \mathbb{E}_W \mid x^2 = 0 \} \subset \mathbb{E}_W$$

We will study  $G_W$ -invariant (constructible)  $\mathbb{Z}$ -valued functions on  $M_0^\circ(W) \subset \mathbb{E}_W$

Let  $K(Q_W) =$  the set of all such functions.

NB. In the original article, I used constructible sheaves, instead of functions. This is necessary even here for the proof of our main result. But in this exposition, I suppose the audience is not familiar with sheaves, and use functions instead.

This has a drawback. I cannot explain what are perverse sheaves. I will only say they are nice constructible functions...

\*  $K(Q_w)$  has a basis  $\{1_{\mathcal{O}(x)}\}$  consisting of characteristic functions of orbits  $\mathcal{O}(x)$  through  $x$ .

Later we will move  $W$ . So we change the notation  $1_{\mathcal{O}(x)}$  to  $M_W(\mathcal{V})$ , where  $\mathcal{V}$  is the graded vector space, defined by  $\mathcal{V} = \text{Im } x$ .

\*  $K(Q_w)$  has a **nicer** basis  $\{IC_W(\mathcal{V})\}$ , given by the simple perverse sheaf associated with  $\mathcal{O}(x)$ .

I don't explain what perverse sheaves are. We have

$$IC_W(\mathcal{V}) = M_W(\mathcal{V}) + \sum a_{\mathcal{V}'} M_W(\mathcal{V}')$$

with  $M_W(\mathcal{V}')$  corresponding to an orbit in the **closure** of  $\mathcal{O}(x)$ .

As notation suggests  $K(Q_w)$  is the Grothendieck ring of an additive category  $Q_w$ . In fact,  $\text{Hom}(K(Q_w), \mathbb{Z})$  is the module category of a **quasihereditary algebra**  $A_w$ . The dual of  $\{IC_W(\mathcal{V})\}$  is a base given by simple modules.

The definition of  $A_w$  is geometric, and in this particular case it is probably possible to give a presentation. But I don't know how to do in general.

★  $\bigoplus_W K(Q_W)$  has a structure of cocommutative coalgebra:

Fix  $W \twoheadrightarrow W'$  and set  $W^2 := \text{Ker}$ . Consider the diagram

$$\begin{array}{ccc}
 & & \mathcal{Z}_0(W'; W^2) := \{ x \in M_0^\bullet(W) \mid x(W^2) \subset W^2 \} \xrightarrow{\iota} M_0^\bullet(W) \\
 & \swarrow k & \\
 M_0^\bullet(W') \times M_0^\bullet(W^2) & & 
 \end{array}$$

Define  $K(Q_W) \xrightarrow{\Delta} K(Q_{W'}) \otimes K(Q_{W^2})$  by  $\Delta\psi := k! \iota^* \psi$

$$\text{where } (k! \psi)(x) = \sum_{m \in \mathbb{Z}} m \cdot \text{Euler}(k^{-1}(x) \cap \psi(m))$$

NB.  $\bigoplus_W K(Q_W)$  is a **co-subalgebra** of Lusztig's construction of  $U(\mathfrak{N})$ .

★ We introduce an equivalence relation  $\sim$  on the set of  $\bigcup_W \{ IC_W(V) \mid V \}$  generated by  $IC_W(V) \sim IC_{W'}(V')$  where  $W' = \text{Ker } x / \text{Im } x$ .

Then we define  $\mathbf{R} = \{ (f_W) \in \prod_W \text{Hom}(K(Q_W), \mathbb{Z}) \mid \langle f_W, IC_W(V) \rangle = \langle f_{W'}, IC_{W'}(V') \rangle \}$   
if  $IC_W(V) \sim IC_{W'}(V')$ .

One can show  $\mathbf{R}$  is compatible with the comultiplication  $\Delta$ .  
 Therefore  $\mathbf{R}$  is an algebra.

It has a base dual to  $\{ [IC_W(0)] \mid W: \text{graded vector space} \}$ .  
 Denote it by  $\{ L(W) \}$ .

I mentioned that  $K(Q_W)^* = K(\text{mod } A_W)$ . The idea for  $\sim$  comes from the fact that there exists a Hopf algebra  $\mathcal{U}(\hat{U}_q(\mathfrak{sl}_2))$ : quantum affine  $\mathfrak{sl}_2$  in our example) and a family of homomorphisms

$$\begin{array}{ccc}
 \mathcal{U} & \twoheadrightarrow & A_W \\
 & \searrow & \downarrow \\
 & & A_{W'}
 \end{array}
 \quad \text{compatible with } \Delta : \quad
 \begin{array}{ccc}
 \mathcal{U} & \longrightarrow & A_W \\
 \Delta \downarrow & \curvearrowright & \downarrow \Delta \\
 \mathcal{U} \otimes \mathcal{U} & \longrightarrow & A_{W'} \otimes A_{W''}
 \end{array}$$

We have  $\mathbf{R} \cong K(\text{mod } \mathcal{U})$ , and  $L(W)$  is the class of a simple module.

Thus we have a monoidal categorification of  $\mathbf{R}$ .

Goal  $\mathbf{R} \cong \mathcal{A}$  so that  $L(W) \leftrightarrow$  a cluster monomial corresponding to a **generic** representation of  $\mathbb{F}_W$

In order to relate  $\mathbf{R}$  with the cluster algebra, we introduce several other spaces:

$$\mathbb{F}_W^* := \text{dual space to } \mathbb{F}_W = \bigoplus_i \text{Hom}(W(i), W(i+1)) \quad W(1) \xrightarrow{x_1^*} W(2) \xrightarrow{x_2^*} \dots \xrightarrow{x_l^*} W(l+1)$$

Choose another graded vector space  $V = V(1) \oplus \dots \oplus V(l+1)$ .

$$\text{Gr}_V(W) := \{ (x^*, S \subset W) \mid S \cong V, x^*(S) \subset S \}$$

$\uparrow$   
 I-graded subsp

$$x^* = \begin{array}{c|c} S & W/S \\ \hline * & * \\ \hline 0 & * \end{array}$$

$$\downarrow \pi^\perp$$

$$\mathbb{F}_W^*$$

fiber of  $\pi^\perp =$  quiver Grassmann

$\text{Gr}_V(W)$  is a vector bundle over the product  $\prod_i \text{Gr}(v_i, W_i)$  of (usual) Grassmann manifolds. ( $v_i = \dim V(i)$ )

It is a subbundle of a trivial bundle  $\mathbb{F}_W^* \times \prod_i \text{Gr}(v_i, W_i)$ .

We consider annihilator:

$$m(\mathcal{V}, W) := \{ (x, S \subset W) \in \mathbb{F}_W^* \times \prod_i \text{Gr}(v_i, W_i) \mid \langle x, x^* \rangle = 0 \quad \forall x_{s,t}^* \text{ s.t. } x^*(S) \subset S \}$$

$$= \{ (x, S \subset W) \mid \text{Im } x \subset S \subset \text{Ker } x \} \quad x = \begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}$$

Let  $\pi: M^\bullet(\mathcal{V}, \mathcal{W}) \rightarrow \mathbb{A}^n_{\mathbb{Q}} : \text{natural projection}$

Note  $x^2=0$  if  $x \in \text{image of } \pi$ .

Thus  $\pi: M^\bullet(\mathcal{V}, \mathcal{W}) \rightarrow M_0^\bullet(\mathcal{W})$ .

These are *graded free varieties* of type  $A_1$ .  
nonsingular / affine

NB, 
$$\begin{array}{c} x_i \qquad \qquad x_{i+1} \\ W^{(i-2)} \leftarrow W^{(i)} \leftarrow W^{(i+1)} \\ \qquad \qquad \cup \\ \text{Im } x_{i+1} \subset \mathcal{U}^{(i)} \subset \text{Ker } x_i \end{array}$$

$\therefore \text{fiber of } \pi = \prod_i \text{Gr}(n_i - r_k x_{i+1}, \frac{\text{Ker } x_i}{\text{Im } x_{i+1}})$

Let  $\pi_{\mathcal{W}}(\mathcal{V}) := \pi!(1_{M^\bullet(\mathcal{V}, \mathcal{W})}) \in K(\mathbb{Q}_{\mathcal{W}})$ .

Lemma  $\text{ch}: \mathbb{R} \rightarrow \sum_{\cup}^*$   $\star = \text{all graded vector spaces } \mathcal{V} = \sum_{\geq 0}^{\mathcal{R}+1}$   
 $L(\mathcal{W}) \mapsto \langle \pi_{\mathcal{W}}(\mathcal{V}), L(\mathcal{W}) \rangle$  is injective.

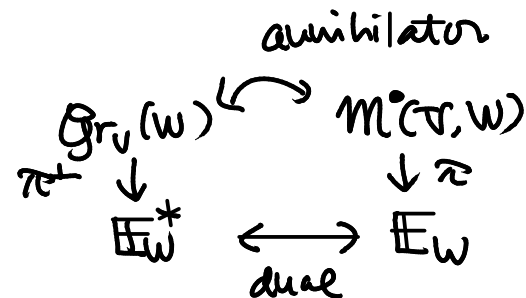
Therefore it is enough to calculate

$$\begin{aligned} \langle \pi_w(\mathcal{V}), L(w) \rangle &= \text{coeff. of } IC_w(0) \text{ in } \pi_!(1_{M(\mathcal{V}, w)}) \\ &= m_0 \text{ where } \pi_!(1_{M(\mathcal{V}, w)}) = \sum_{\mathcal{V}'} m_{\mathcal{V}'} IC_w(\mathcal{V}') \end{aligned}$$

### Key Observation

$\pi_w(\mathcal{V})$  is related to CC formula via Fourier transform  $\mathbb{F}: \text{Func}(\mathbb{E}_V) \rightarrow \text{Func}(\mathbb{E}_V^*)$ .

$$\mathbb{F}(\pi_w(\mathcal{V})) = \pi_!^{\dagger}(1_{G_{\mathcal{V}}(w)})$$



Recall  $\pi_!^{\dagger}(1_{G_{\mathcal{V}}(w)})(x^*) = \text{Euler}(G_{\mathcal{V}}(x^*))$ .  
 $\uparrow$  quiver Grassmann

If  $x^*$  is general, RHS appears in CC formula.

The Fourier transform  $\mathbb{F}$  is defined by  $\mathbb{F}(\varphi) = p_2!(p_1^*\varphi \cdot 1_P)$

where  $\begin{array}{ccc} & \mathbb{E}_V \times \mathbb{E}_V^* & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{E}_V & & \mathbb{E}_V^* \end{array} \supset P = \{ (x, x^*) \mid \langle x, x^* \rangle \leq 0 \}$

It is known that  $\Phi$  maps a simple perverse sheaf to a simple perv. sheaf.

$$\therefore \langle \pi_w(\mathcal{T}), L(w) \rangle = \text{coeff. of } \pi_1^\perp(1_{g_{r_V}(w)}) \text{ in } \Phi(IC_w(0))$$

Now  $IC_w(0) = 1_{\text{pt}}$  (and/or  $\delta$ -function)

$$\therefore \text{Fourier transform } \Phi(IC_w(0)) = 1_{\mathbb{A}_w^*} \quad (\text{constant function}).$$

All other  $\Phi(IC_w(\mathcal{T}')) = \text{char. func. of an orbit} + \sum \text{smaller}$   
 have smaller support  $\neq \mathbb{A}_w^*$

$$\Rightarrow \langle \pi_w(\mathcal{T}), L(w) \rangle = \pi_1^\perp(1_{g_{r_V}(w)})(x^*)$$

$\uparrow$   
 general element

$\Rightarrow$   
 CC formula  $L(w)$  is a cluster monomial.