§ Notation, etc.

- $G = SL(r)$

  $\pi : \widetilde{\mathcal{U}}^d_r \longrightarrow \mathcal{U}^d_G$ resolution of singularities

  $\widetilde{\mathcal{U}}^d_r$ ↑ Gieseker sp. ↑ symplectic, $\dim = 2dr$

  $\mathcal{U}^d_G$ ↑ Uhlenbeck sp. general G later

  (E, φ) or $B_1 \circlearrowright \mathbb{C}^d \circlearrowleft B_2$, $\quad J \downarrow \uparrow I$, $\quad \mathbb{C}^r$

- $r = 1$ $\quad \widetilde{\mathcal{U}}^d_1 = \mathrm{Hilb}^d \mathbb{C}^2$, $\quad \mathcal{U}^d_G = \mathcal{U}^d_{\{e\}} = S^d \mathbb{A}^2$
- group action

  $\mathbb{G} = G \times (\mathbb{C}^*)^2 \curvearrowright \widetilde{\mathcal{U}}^d_r, \mathcal{U}^d_G$

  $\cup\ \mathbb{T} = T \times (\mathbb{C}^*)^2$

  $H^{[*]}_{\mathbb{G}(,c)}(\widetilde{\mathcal{U}}^d_r)$, $H^{[*]}_{\mathbb{T}(,c)}(\widetilde{\mathcal{U}}^d_r)$: equivariant cohomology

  cpt supports

  degree shifted by $2dr$

  [ ] : omitted here after

- Heisenberg action

$$\widetilde{\mathcal{U}}^d_r \times \widetilde{\mathcal{U}}^{d+n}_r \times \mathbb{A}^2 \supset P_n := \{ (E_1, E_2, x) \mid E_1 \supset E_2,\ \mathrm{Supp}(E_1/E_2) = \{x\} \}$$

P_n ↑ lagrangian

$$P_n \to \widetilde{\mathcal{U}}^{d+n}_r, \qquad P_n \to \widetilde{\mathcal{U}}^d_r \times \mathbb{A}^2$$

$$\alpha \in \underset{\mathbb{T}}{H^*_{\mathbb{T},c}}(\mathbb{A}^2) \rightsquigarrow P^\Delta_{-n}(\alpha) : \underset{\mathbb{T},c}{H^*_{\mathbb{T}}}(\widetilde{\mathcal{U}}^d_r) \longrightarrow \underset{\mathbb{T},c}{H^{*+\deg\alpha}_{\mathbb{T}}}(\widetilde{\mathcal{U}}^{d+n}_r)$$

creation operator

$$P^\Delta_n(\alpha) = \text{its adjoint } \underset{\mathbb{T}}{H^*_{\mathbb{T},c}}(\widetilde{\mathcal{U}}^{d+n}_r) \to \underset{\mathbb{T}}{H^{*+\deg\alpha}_{\mathbb{T},c}}(\widetilde{\mathcal{U}}^d_r)$$

inner product : $\underbrace{(-1)^{\dim X/2}}_{\text{sign convention}} \int_X \bullet \cup \bullet$

Fact ($r=1$, N, Grojnowski, r: general Baranovsky)

$$[P^{\Delta}_m(\alpha), P^{\Delta}_n(\beta)] = \langle \alpha, \beta \rangle m \delta_{m+n,0} \times r$$

(Heisenberg relation)

$r=1$ $\bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d) \mathfrak{z}^d$ has $\dim_{(\text{over } H^*_{(\mathbb{C}^*)^2}(pt))} = \prod_{d=1}^{\infty} \frac{1}{1-\mathfrak{z}^d}$,

$\therefore$ Heis. rep. is irreducible. dim Fock

Q. If $r>1$, the representation is not irreducible.
So a larger algebra should act.
What algebra?

Why larger?

Consider $T \curvearrowright \tilde{\mathcal{U}}^d_r$ fixed pts $= (E, \varphi)$

$$\begin{array}{c} \| \\ I_1 \oplus \cdots \oplus I_r,\ \varphi_1 \oplus \cdots \oplus \varphi_r \end{array}$$

$$\therefore \tilde{\mathcal{U}}^d_r = \coprod_{d_1 + \cdots + d_r = r} \mathrm{Hilb}^{d_i} \mathbb{A}^2$$

$$\therefore \bigoplus_d H^*_{\mathbb{T}}(\tilde{\mathcal{U}}^d_r) \otimes \mathrm{Frac}(H^*_T(pt)) \cong \left[\bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d) \otimes \mathrm{Frac}\right]^{\otimes r}$$

r copy of Fock space!

Therefore the bigger algebra should be $\sim (\text{Heisenberg})^{\otimes r}$.

But it does not act on $\bigoplus H^*_{\mathbb{T}}(\tilde{\mathcal{U}}^d_r)$.
only after $\otimes$ Frac.

Correct Answer : $W(\mathfrak{gl}_r) = W(\mathfrak{sl}_r) \oplus \mathrm{Heis}$.

above

(Schiffmann-Vasserot

Maulik-Okounkov

§ Stable envelop (Maulik-Okounkov)

∘ situation

$\pi: X \longrightarrow X_0$ — resolution

symplectic — affine variety

- $\mathbb{T}$ - equivariant
- $T \subset \mathbb{T}$ preserving symp. form

ex. $X = \widetilde{\mathcal{U}}^d_r \longrightarrow X_0 = \mathcal{U}^d_G$, $\mathbb{T} \supset T$ as above

$X = T^*\mathcal{B} \longrightarrow X_0 = \mathcal{N}$, $\mathbb{T} = T \times \mathbb{C}^* \supset T$

$X \overset{i}{\hookleftarrow} X^T = \bigsqcup F_\alpha$ $\qquad F_\alpha$: symplectic

dim $2dr$

$$i^* : H^{[*]}_{\mathbb{T}}(X) \longrightarrow H^{[* + \mathrm{codim}_{\mathbb{C}} X^T]}_{\mathbb{T}}(X^T) = \bigoplus_\alpha H^{[* + \mathrm{codim} F_\alpha]}_{\mathbb{T}}(F_\alpha)$$

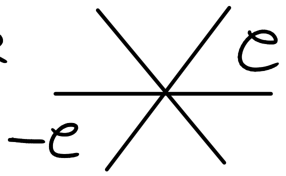
Want

$[*]_X = * + \dim X \longrightarrow * + \dim X = [*]_{X^T} + \mathrm{codim} X^T$

∘ chamber structure on the space of 1PS's

$\mathrm{Hom}(\mathbb{C}^*, T)$ has "root" hyperplanes $\quad \rho : X^{\rho(\mathbb{C}^*)} \neq X^T$

$\mathrm{Hom}(\mathbb{C}^*, T) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathrm{Lie}\, T)_{\mathbb{R}}$



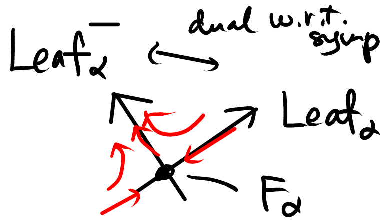
$-\mathfrak{C}$: opposite

Choose a chamber $\mathfrak{C}$ and $\rho : \mathbb{C}^* \to T \in \mathfrak{C}$

Define $X_{(0)} \supset \mathcal{A}_{X_{(0)}} := \{ x \mid \lim_{t \to 0} \underset{\shortparallel\ \rho(t)}{t} \cdot x \text{ exists} \} \longrightarrow X^T_{(0)}$

attracting set $\qquad$ depending only on $\mathfrak{C}$

∘ $X \supset \mathrm{Leaf}_\alpha := \{ x \mid \lim_{t \to 0} t \cdot x \in F_\alpha \} \xrightarrow[\text{vectr bdle}]{} F_\alpha$ (BB decomp.)



$TX|_{F_\alpha} = \bigoplus_{\substack{\text{wt} \\ \text{w.r.t. } \rho}} E(m)$ $\qquad \mathrm{Leaf}^{\pm} = \bigoplus_{m > 0} E(\pm m)$

$TF_\alpha = E(0)$

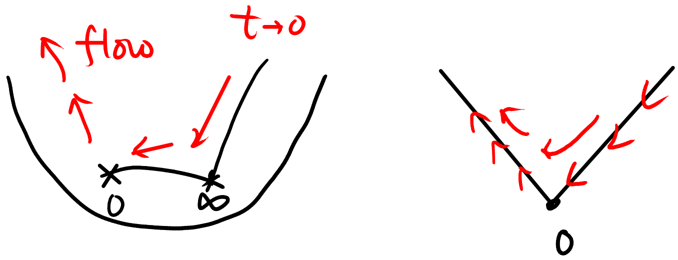
$E(m) \leftrightarrow E(-m)$: dual

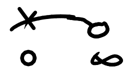
Rem $\quad X \times F_\alpha \supset \mathrm{Leaf}_\alpha \quad$ lagrangian

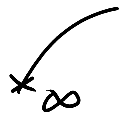
Ex $T^*\mathbb{P}^1 \longrightarrow \mathbb{C}^2/\pm$ $\qquad (T^*\mathbb{P}^1)^T = \{0, \infty\}$

$\circlearrowleft T = \mathbb{C}^*$ $\qquad (z_1, z_2)/\pm 1$

$\mapsto (tz_1, t^{-1}z_2)$



$\therefore$ Leaf$_0$ = 

Leaf$_\infty$ = 

Rem. Two topologies on $\mathcal{A}_{X_0} = \coprod \mathrm{Leaf}_\alpha$

— disjoint

— $\mathcal{A}_{X_0} \subset X_0$ induced topology

Same for X_0 (affine), different for X

- Define a partial order
  $\alpha \geq \beta \Leftrightarrow \overline{\mathrm{Leaf}_\alpha} \cap F_\beta \neq \phi$ $\qquad$ (above example $0 \geq \infty$

- Define $\deg_T$
  $T \curvearrowright X^T$ trivial $\qquad \therefore \quad H_T^*(X^A) = H^*(X^A) \otimes \underbrace{H_T^*(pt)}_{\cong}$

  $\deg_T$ is defined $\longrightarrow$ $\mathbb{C}[\mathrm{Lie}\,T]$
  (Lie T : deg = 2) $\qquad$ polynomial ring

- Steinberg type variety

$$Z_{\mathcal{J}} := \mathcal{A}_X \times_{X_0^T} X^T \subset X \times X^T \qquad \text{lagrangian subvar.}$$

Consider $\mathcal{A}_X \subset X$ closed subvariety

- $F_\beta \times_{X_0^T} F_\alpha$: lagrangian

<u>Fact</u> $F_\beta \to \pi_1(F_\beta)$ $F_\alpha \to \pi_2(F_\alpha)$ semismall

$\subset X_0^T \supset$ S_ν common stratum

$$F_\beta \times_{X_0^T} F_\alpha = \bigcup_\nu \pi_1^{-1}(S_\nu) \times_{S_\nu} \pi_2^{-1}(S_\nu)$$

dim S_ν + dim fiber of π_1 over S_ν

+ " π_2

$$= \dim S_\nu + \frac{1}{2}(\dim F_\beta - \dim S_\nu) + \frac{1}{2}(\dim F_\alpha - \dim S_\nu)$$
$$= \frac{1}{2}(\dim F_\alpha + \dim F_\beta)$$

- $Z_{\mathfrak{g}} = \bigcup \overline{\mathrm{Leaf}_\beta}$ | irr. comp. of $F_\beta \times_{X_0^T} F_\alpha$

This is the description of the irreducible components of $Z_{\mathfrak{g}}$

<u>Th</u> [MO] top!

$\exists 1\ \mathcal{L} = \mathcal{L}_C \in H_{\cancel{[0]}}^T(Z_{\mathfrak{g}}) \left(= H_T^{[0]}(X \times X^T; X \times X^T \setminus Z_{\mathfrak{g}})\right)$

s.t. (1) $\mathcal{L}|_{X \times F_\alpha}$ is supported on $\bigcup_{\beta \leqq \alpha} \overline{\mathrm{Leaf}_\beta \times_{X_0^T} F_\alpha}$

(2) $i_{\alpha,\alpha} : F_\alpha \times F_\alpha \to X \times X^T$

$$i_{\alpha,\alpha}^* \mathcal{L} = \pm e(\mathrm{Leaf}_\alpha^-) \cap [\Delta_{F_\alpha}]$$

$\uparrow$ vector b'dle over F_α

$*$ normal b'dle of $\mathrm{Leaf}_\alpha^+ = \mathrm{Leaf}_\alpha^-$

(3) $i_{\beta,\alpha} : F_\beta \times F_\alpha \to X \times X^T$ $(\beta < \alpha)$

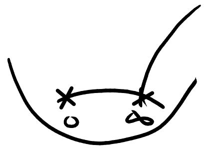
$$\deg_T i_{\beta,\alpha}^* \mathcal{L} < \frac{1}{2} \operatorname{codim} F_\beta$$

(In fact, $i_{\beta,\alpha}^* \mathcal{L} = 0$)

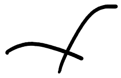
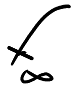
$*$ rk $\mathrm{Leaf}_\beta^\pm = \frac{1}{2} \operatorname{codim} F_\beta$

$\mathcal{L}$ is called the stable envelop.

Ex. $X = T^*\mathbb{P}^1 \curvearrowleft T$



$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\infty$$

$\mathcal{L}_0 = \pm$  $\times \{0\}$ $\quad \mathcal{L}_\infty = \pm$  $\times \{\infty\}$

- polarization (sign)

$$X \supset F_\alpha \qquad N_{F_\alpha/X} = \underbrace{N^+ \oplus N^-}_{\text{dual}}$$

$$\therefore (-1)^{\mathrm{codim}/2} e(N_{F_\alpha/X}) = e(N^+)^2$$

On the other hand, X and F_α are cotangent b'dles

$$X \doteqdot T^*M \qquad T(T^*M) = TM \oplus T^*M$$

$$(-1)^{\dim/2} e(TX) = e(TM)^2 \quad \Rightarrow e(TM) = \pm e(N^+)$$

in T-equiv. coh.

In many situations ($\widetilde{\mathcal{U}}_r^d$, $T^*\mathcal{B}$), we have a prefered choice of $\sqrt{e(N)}$. We define $\pm$ accordingly.

proof) Uniqueness

$$\mathcal{L} = \sum_\alpha \mathcal{L}_\alpha \qquad \text{according to } X^T = \coprod F_\alpha$$

$$\mathcal{L}_\alpha = \sum_{\beta \leq \alpha} \sum_{\substack{Z:\mathrm{irr} \\ F_\beta \times_{X_0^T} F_\alpha}} a_Z \cdot [\overline{\mathrm{Leaf}_\beta|_Z}] \qquad a_Z \in \mathbb{Z}$$

$$(1): \beta = \alpha \longrightarrow Z = \Delta_{F_\alpha} \subset F_\alpha \times_{X_0^T} F_\alpha \quad \& \quad a_Z = \pm 1$$

$$N_{\mathrm{Leaf}_\alpha/X} = \mathrm{Leaf}_\alpha^- \Rightarrow i_{\alpha,\alpha}^* \mathcal{L}_\alpha = \pm e(\mathrm{Leaf}_\alpha^-) \cap [\Delta_{F_\alpha}]$$

Suppose two $\mathcal{L}_\alpha^1, \mathcal{L}_\alpha^2$ $\quad \mathcal{L}_\alpha^1 - \mathcal{L}_\alpha^2 = \sum_{\beta<\alpha} \sum_Z a'_Z [\overline{\mathrm{Leaf}_\beta|_Z}]$

Take a maximum β_0 among $a'_Z \neq 0$ for some Z

$\Rightarrow$ other terms do not contribute to $i_{\beta_0,\alpha}^*$

$$\therefore \quad i_{\beta_0,\alpha}^* (\mathcal{L}_\alpha^1 - \mathcal{L}_\alpha^2) = \sum_Z a'_Z \underbrace{e(\mathrm{Leaf}_\beta^-)}_{\substack{\text{invertible} \\ \text{in } H_T^*}} \cap [Z]$$

$(2) \Rightarrow a'_Z = 0 \ \forall Z \qquad$ contradiction!

existence

Fact X, X_0 has a 1 parameter deformation

$$
\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "p"'] & \mathcal{X}_0 \arrow[dl] \\
\Delta &
\end{tikzcd}
$$

s.t. $X_t = p^{-1}(t)$ is affine

& $X_t \xrightarrow{\cong} X_{0,t}$ for $t \neq 0$

T-action extends to $\mathcal{X}$ (NB. $\mathbb{T}$-action does not extend)

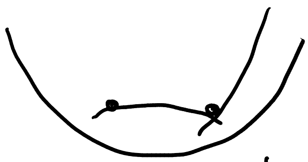
$\widetilde{\mathfrak{M}}^d_r$ $\longrightarrow$ deform the moment map equ.

$$[B_1, B_2] + IJ = t\,\mathrm{id}$$

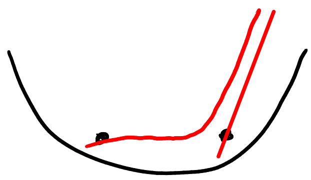
$\mathbb{C}^* \times \mathbb{C}^*$-action does not preserve $\uparrow$

Now $\left(\begin{array}{l} F_\alpha \\ \mathrm{Leaf}_\alpha \end{array} \right.$ also deformed

Leaf_α : closed for $t \neq 0$



$t=0$



$t \neq 0$

$[\mathrm{Leaf}_\alpha]$ at $t \neq 0$

$$\mathcal{L}_\alpha := \lim_{t \to 0} [\mathrm{Leaf}_\alpha]$$

$$i_{\beta,\alpha}^* \mathrm{Leaf}_\alpha = 0 \quad \Rightarrow \quad i_{\beta,\alpha}^* \mathcal{L}_\alpha = 0$$

Proof finish

@ transpose is given by $-C$

$\mathbb{T} \supset T$ $\qquad t \in \mathbb{T}$ commutes with T

$\qquad\qquad \Rightarrow Z_\sigma$: invariant under $\mathbb{T}$

$\therefore \quad \mathcal{L} \in H_0^T(Z_\sigma) = H_0^{\mathbb{T}}(Z_\sigma)$ (as it is spanned by irr. comp.)

$\mathcal{L}_\varepsilon$ defines an operator

$$\begin{array}{ccc} & \mathcal{L}_\varepsilon & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ \mathcal{A}_X & & X^T \end{array} \qquad \mathrm{Stab}_\varepsilon : H^*_{\mathbb{T}}(X^T) \longrightarrow H^{\mathbb{T}}_*(\mathcal{A}_X) = H^*_{\mathbb{T}}(X; X \setminus \mathcal{A}_X)$$

$$\alpha \longmapsto p_{1*}(p_2^*\alpha \cap \mathcal{L}_\varepsilon)$$

(p_1 : proper)

$$i^*\mathrm{Stab}_\varepsilon : H_*^{\mathbb{T}}(X^T) = \underbrace{H_*^{\mathbb{T}/T}(X^T)}_{} \underset{H^*_{\mathbb{T}/T}(pt)}{\otimes} H^*_{\mathbb{T}}(pt), \qquad X^T = \bigsqcup F_\alpha$$

uppertriangular

& $\quad$ diagonal $= e(\mathrm{Leaf}_\alpha^-) \cup$

$\qquad\qquad\qquad$ top part $=$ $\mathbb{T}$ wts $\neq 0$

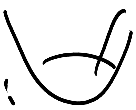
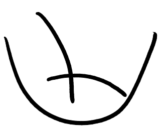
$\therefore$ $\mathrm{Stab}_\varepsilon$ becomes invertible after $\otimes \underbrace{\mathrm{Frac}\, H^*_{\mathbb{T}}(pt)}_{= \mathbb{Q}(\mathrm{Lie}\,\mathbb{T})}$

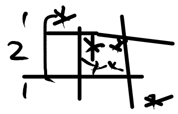
§ R-matrix and Yangian

Def. $\quad R_{\varepsilon',\varepsilon} = \mathrm{Stab}_{\varepsilon'}^{-1} \circ \mathrm{Stab}_\varepsilon$

R-matrix $\qquad \in \mathrm{End}(H^*_{\mathbb{T}}(X^T) \otimes \mathbb{Q}(\mathrm{Lie}\,\mathbb{T}))$

$$\begin{tikzcd} H^*_{\mathbb{T}}(X^T) \arrow[r, "\mathrm{Stab}_\varepsilon", "\cong"'] \arrow[rrrr, bend right, "R_{\varepsilon',\varepsilon}"'] & H_*^{\mathbb{T}}(\mathcal{A}_X^\varepsilon) \arrow[r] & H_*^{\mathbb{T}}(X) & H_*^{\mathbb{T}}(\mathcal{A}_X^{\varepsilon'}) \arrow[l] & H^*_{\mathbb{T}}(X^T) \arrow[l] \end{tikzcd}$$

Ex. $\quad X = T^*\mathbb{P}^1$ $\quad \varepsilon$:  $\qquad \varepsilon^{op}$  $\qquad X^T = \{0, \infty\}$



$R_{\varepsilon^{op},\varepsilon}$ = block of Yang's R-matrix

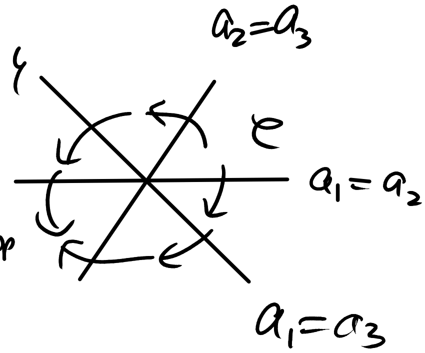
$(Y(\mathfrak{sl}_2)\ \ \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad R = 1 - \frac{P}{u} \qquad P = \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji}$

☆ Yang-Baxter equation

Suppose T : 2 dim $\left(\begin{matrix}\cap \\ SL_3\end{matrix}\right)$

$\mathrm{Lie}\,T = \{a_1+a_2+a_3=0\}$

Suppose chambers



$$\underline{R_{12}}R_{13}R_{23} = R_{23}R_{13}\underline{R_{12}}$$

depending only on $\left(\begin{matrix} \boxed{*} & 0 \\ 0 & * \end{matrix}\right) \subset T$

☆ Baranovsky's Heisenberg operators preserves $H_*^{\mathbb{T}}(\mathcal{J})$

Under
$$H_*^{\mathbb{T}}(\mathcal{J}) \cong \bigoplus_d H_*^{\mathbb{T}}((\tilde{\mathcal{U}}_r^d)^T)$$
$$= \Big\{\bigoplus_d H_*^{\mathbb{T}}(\mathrm{Hilb}^d)\Big\}^{\otimes r}$$

__Prop__. It is given by the diagonal Heisenberg, i.e.,
$$P_n^{\Delta}(\alpha) = \sum_{i=1}^{r} \mathrm{id}\otimes\cdots\otimes \underbrace{P_n(\alpha)}_{i\text{th factor}}\otimes \mathrm{id}\otimes\cdots\otimes\mathrm{id}$$

☆ R-matrix intertwines $P_n^{\Delta}(\alpha)$ by definition

$$\begin{tikzcd} H_{\mathbb{T}}^*(X^T) \arrow[r, "\mathrm{Stab}_e", "\cong"'] \arrow[rrrr, bend right, "R_{e,e'}"'] & H_*^{\mathbb{T}}(\mathcal{A}_X^e) \arrow[r] & H_*^{\mathbb{T}}(X) & H_*^{\mathbb{T}}(\mathcal{A}_X^{e'}) \arrow[l] & H_{\mathbb{T}}^*(X^T) \arrow[l] \end{tikzcd}$$

commute by definition

__Rem__. $r=1$ $\quad \underset{\substack{\| \\ P_n\otimes 1}}{P_n^{(1)}} + \underset{\substack{\| \\ 1\otimes P_n}}{P_n^{(2)}}$, but not $P_n^{(1)} - P_n^{(2)}$

$c_1(E)$ commutes with R

E: tautological bdle $\rightsquigarrow$ Virasoro alg

※ R-matrix $\rightsquigarrow$ Yangian RTT construction

Fadeev – Reshetikhin – Takhadzhyan

- $T^*\mathbb{P}^1$ (more generally T^*Gr = quiver variety of type A) $\rightsquigarrow Y(\mathfrak{sl}_2)$ recover N, Varagnolo. $U_q(\mathfrak{g})$
- $\tilde{\mathcal{U}}^d_r \rightsquigarrow Y(\hat{\mathfrak{gl}}_1)$ $\overset{\shortparallel}{\text{Heis.}}$ new!

<u>Data</u> (a simpler version)

① F : vector space / $\mathbb{K}$: ring $\supset \mathbb{Q}$

e.g. $F = \bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d)$, $\mathbb{K} = H^*_{(\mathbb{C}^*)^2}(pt)$

② $R(u) \in \mathrm{End}(F\otimes F)(u)$; a matrix-valued rational fct in u

s.t. YBE is satisfied

normalization $R(\infty) = 1$

$R(u) - 1$ divisible by $\hbar \in \mathbb{K}$

$\curvearrowleft (\mathbb{C}^*)^2/\mathbb{C}^*_{hyp}$.

<u>Construction</u>

$W \equiv W_p := F[u_1] \otimes \cdots \otimes F[u_p]$

$R_{F,W} := R(u-u_p) \cdots R(u-u_1)$ $\quad F \otimes F[u_1] \otimes \cdots \otimes F[u_p]$

$\in \mathrm{End}(F\otimes W) = \mathrm{End}\,F \otimes \mathrm{End}\,W$

Define $Y \subset \prod_p \mathrm{End}_{\mathbb{K}[u_1,\cdots,u_p]} W$ as an algebra generated by coefficients of the following elements $\times(\frac{1}{\hbar})$

— Choose a base of F (possibly ∞-dim'l)

$t_{ij}(u) := \big((i,j)\text{-matrix entry of } R_{F,W}\big) \in \prod_p \mathrm{End}_{\mathbb{K}[u_1,\cdots,u_p]} W[[u^{-1}]]$

$$\frac{1}{u-u_i} = 1 + \frac{u_i}{u} + \frac{u_i^2}{u^2} + \cdots \quad \rightarrow \text{polynomial in } u_i$$

$T(u) := (t_{ij}(u)) \in \mathrm{End}(F) \otimes Y[[u^{-1}]]$

YB equ. $\Longrightarrow$ T satisfies the RTT relation

$$T_2(u_2)T_1(u_1)R(u_1-u_2) = R(u_1-u_2)T_1(u_1)T_2(u_2)$$

- Yang's R-matrix $\Longrightarrow$ $Y(\mathfrak{gl}_2)$
- Hilb $=:$ $Y(\hat{\mathfrak{gl}}_1)$
  (definition)

∘ What is R concretely ?

$r=2$

$G=SL_2$ $\quad \widetilde{\mathcal{U}}_2^d \supset (\widetilde{\mathcal{U}}_2^d)^{\mathbb{C}^*} = \coprod_{d=d_1+d_2} \mathrm{Hilb}^{d_1} \times \mathrm{Hilb}^{d_2}$

$\mathfrak{h} = \{(a_1,a_2) \mid a_1+a_2=0\}$ $\quad \therefore\ R \circlearrowright \left(\bigoplus H_T^*(\mathrm{Hilb}^{d_1}) \otimes \mathbb{Q}(\mathrm{Lie}\,T)\right)^{\otimes 2}$ $\curvearrowleft$ irreducible $P_n^{(1)}, P_n^{(2)}$

$\mathrm{Fock}^{\otimes 2}$

commute

$P_n^{\Delta} = P_n^{(1)} + P_n^{(2)}$

$P_n^{-} = P_n^{(1)} - P_n^{(2)}$

$\therefore$ R is written by P_n^-

<u>Lemma</u> $\quad R$ is determined by its matrix elements on $\mathrm{vac} \otimes \mathrm{Fock} \leftrightsquigarrow \mathrm{Hilb}^0 \times \mathrm{Hilb}^d$

Feigin-Fuchs Heis. $\to$ Virasoro

- $P_n^- := P_n^-(1)$ $\left(\rightsquigarrow \text{comm} \quad \langle 1,1\rangle_{\mathbb{C}^2} = \frac{1}{\varepsilon_1\varepsilon_2}\right)$

- Put $P_0^- := \frac{1}{\varepsilon_1\varepsilon_2}(a^1 - a^2 - (\varepsilon_1+\varepsilon_2))$ as convention

$$L_n := -\frac{1}{4}\sum_m :P_m^- P_{n-m}^-: - \frac{n+1}{2}\,\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2} P_n^- \qquad H^0_{(\mathbb{C}^*)^2}(pt)$$

$\Longrightarrow$ L_n satisfies the Virasoro relation

$$[L_m, L_n] = (m-n)L_{m+n} + \left(1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)\delta_{m,-n}\frac{m^3-m}{12}$$

$$L_0|\mathrm{vac}\rangle = -\frac{1}{4}\left(\frac{(a_1-a_2)^2}{\varepsilon_1\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)|\mathrm{vac}\rangle$$

highest wt

$a_1 \leftrightarrow a_2$ invariant $\therefore$ $\mathrm{Fock}^- \circlearrowleft \exists 1$ reflection op.

Virasoro intertwiner s.t. $|\mathrm{vac}\rangle \mapsto |\mathrm{vac}\rangle$

<u>Th.</u> $R =$ reflection operator