MCKAY CORRESPONDENCE AND HILBERT SCHEMES IN DIMENSION THREE

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ABSTRACT. Let G be a nontrivial finite subgroup of $\mathrm{SL}_n(\mathbb{C})$. Suppose that the quotient singularity $\mathbb{C}^n/G$ has a crepant resolution $\pi\colon X \to \mathbb{C}^n/G$ (i.e. $K_X = \mathcal{O}_X$). There is a slightly imprecise conjecture, called the McKay correspondence, stating that there is a relation between the Grothendieck group (or (co)homology group) of X and the representations (or conjugacy classes) of G with a "certain compatibility" between the intersection product and the tensor product (see e.g. [22]). The purpose of this paper is to give more precise formulation of the conjecture when X can be given as a certain variety associated with the Hilbert scheme of points in $\mathbb{C}^n$. We give the proof of this new conjecture for an abelian subgroup G of $\mathrm{SL}_3(\mathbb{C})$.

1. INTRODUCTION

Let G be a nontrivial finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ and let X be the scheme parametrising 0-dimensional subschemes Z of $\mathbb{C}^n$ satisfying the following three conditions:

(1) the length of Z is equal to $\#G$ = the order of G.
(2) Z is invariant under the G-action.
(3) $H^0(\mathcal{O}_Z)$ is the regular representation of G.

This is a union of components (possibly one component) of fixed points of the G-action on the Hilbert scheme of $\#G$-points in $\mathbb{C}^n$. (See [20] for survey on Hilbert schemes of points.) If Z consists of pairwise distinct points, the above conditions mean that Z is a single G-orbit. We have a natural morphism (the Hilbert-Chow morphism) $\pi\colon X \to \mathbb{C}^n/G$ which is a crepant resolution of singularities under certain assumption on G as explained later.

In $n = 2$ (i.e. simple singularities), it was observed by Ginzburg-Kapranov [8] and Ito-Nakamura [10] that X is nonsingular, and in fact, is the minimal resolution of $\mathbb{C}^2/G$ (see also [20, Chapter 4]). The proof is based on the fact that Hilbert schemes of points on $\mathbb{C}^2$ are nonsingular symplectic manifolds. (See e.g., [20, Chapter 1].) Since Hilbert schemes are singular in higher dimensions in general, the proof is not applied to the 3-dimensional case. Hence we were surprised when Nakamura proved that X is nonsingular when G is an abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ [21]. He conjectured

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the same is true for any $G \subset \mathrm{SL}_3(\mathbb{C})$, and it is still an open problem. Anyhow, it seems reasonable to consider X as a first candidate for crepant resolutions of $\mathbb{C}^n/G$.

Now we explain the McKay correspondence. We first recall the 2-dimensional situation. As X is the minimal resolution of $\mathbb{C}^2/G$, it is well-known that the exceptional set consists of projective lines intersecting transversely. Let us denote by C_k the irreducible component. The intersection matrix $C_k \cdot C_l$ is given by the negative of the Cartan matrix. On the other hand, McKay [16] considered the irreducible representations of G and the decomposition of a tensor product

$$Q \otimes \rho_l = \bigoplus_k a_{kl}\rho_k,$$

where $\{\rho_k\}_{k=0}^r$ be the set of isomorphism classes of irreducible representations of G and Q is the 2-dimensional representation given by the inclusion $G \subset \mathrm{SL}_2(\mathbb{C})$. Then he observed that $(2\delta_{kl} - a_{kl})$ is the extended Cartan matrix. The trivial representation, denoted by ρ_0, corresponds to the extra entry added to the finite Cartan matrix, which turns out to be the same as realized by the intersection matrix. The correspondences are summarized as follows:

(a)	finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$	(nontrivial) irreducible representations	decompositions of tensor products
(b)	simple Lie algebra of type ADE	simple roots	(extended) Cartan matrix
(c)	minimal resolution $X \to \mathbb{C}^2/G$	irreducible components of the exceptional set, or a basis of $H_2(X, \mathbb{Z})$	intersection matrix

Gonzales-Sprinberg and Verdier [9] realized the correspondence between (a) and (c) geometrically as follows. Let us consider the diagram

$$X \xleftarrow{p} \mathcal{Z} \xrightarrow{q} \mathbb{C}^n,$$

where $\mathcal{Z} \subset X \times \mathbb{C}^n$ is the universal subscheme and p and q are the projections to the first and second factors. Let us define the *tautological bundle* $\mathcal{R}$ by

$$\mathcal{R} \stackrel{\text{def.}}{=} p_*\mathcal{O}_{\mathcal{Z}}. \tag{1.1}$$

Since $\mathcal{Z}$ has a G-action, each fiber of $\mathcal{R}$ has a structure of a G-module. By (3) in the definition of X, it is the regular representation. We decompose into irreducibles:

$$\mathcal{R} = \bigoplus_k \mathcal{R}_k \otimes \rho_k. \tag{1.2}$$

Then Gonzales-Sprinberg and Verdier observed that

(1) $\{\mathcal{R}_k\}_{k=0}^r$ gives a basis of the Grothendieck group $K(X)$ of algebraic vector bundles over X,
(2) $\{c_1(\mathcal{R}_k)\}_{k\neq 0}$ is the dual basis of $\{[C_k]\}$.

Based on their results and also calculations by Vafa et al. related to the mirror symmetry, Reid conjectured an existence of a similar correspondence between (a) and (c), when X is a crepant resolution of $\mathbb{C}^n/G$ for $G \subset \mathrm{SL}_n(\mathbb{C})$. (See [22] for the history, earlier results, and concrete examples of the McKay correspondence. We do not reproduce them here.)

In this paper, we give more precise formulation of the conjecture when the crepant resolution is given as X defined above, and verify this new conjecture when G is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, or an abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$.

Our new point is to consider the Grothendieck group of bounded complexes of algebraic vector bundles with supports contained in $\pi^{-1}(0)$, denoted by $K^c(X)$. There are natural elements S_k of $K^c(X)$ which are also indexed by irreducible representations as follows. The multiplication of the coordinate functions $(x_1, \dots, x_n)$ on $\mathbb{C}^n$ induces the G-equivariant homomorphism (called *tautological homomorphism*)

$$B : \mathcal{R} \to Q \otimes \mathcal{R},$$

where Q is the n-dimensional representation given by the inclusion $G \subset \mathrm{SL}_n(\mathbb{C})$. It gives rise the complex

(1.3)

$$\mathcal{R} \xrightarrow{d_n} Q \otimes \mathcal{R} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \bigwedge^{n-1} Q \otimes \mathcal{R} \xrightarrow{d_1} \bigwedge^{n} Q \otimes \mathcal{R} = \mathcal{R},$$

where $d_k(\eta) = B \wedge \eta$. This complex decomposes according to (1.2) and consider its transpose

$$S_k : \mathcal{R}_k^\vee \longrightarrow \bigoplus_l a_{kl}^{(n-1)} \mathcal{R}_l^\vee \longrightarrow \cdots \longrightarrow \bigoplus_l a_{kl}^{(1)} \mathcal{R}_l^\vee \longrightarrow \mathcal{R}_k^\vee, \tag{1.4}$$

where the coefficients $a_{kl}^{(i)}$ is determined by

$$\bigwedge^i Q \otimes \rho_l = \bigoplus_k a_{kl}^{(i)} \rho_k. \tag{1.5}$$

This is a generalization of the tensor product considered by McKay.

We will show that $\{\mathcal{R}_k\}_{k=0}^r$ and $\{S_k\}_{k=0}^r$ form dual bases of $K(X)$ and $K^c(X)$ under the above assumption on G. And we conjecture it holds for arbitary $G \subset \mathrm{SL}_3(\mathbb{C})$. Then from this approach, it becomes clear that the intersection product among S_k's are related to the decomposition of the tensor product (see Corollary 5.6 for more precise statement). Thus, our approach gives a 'natural' explanation of the reason why the decomposition of the tensor product is identified with the intersection products in dimension 2. As far as we know, known proofs of this identification in dimension 2 used case-by-case analysis except those given in [15, Appendix] and [18]. Our proof is more natural and can be generalized to higher dimensions.

The most essential ingredient in the proof of our main theorem is a construction of a certain complex (see (4.7)). We conjecture that it gives rise a resolution of the diagonal in $X \times X$ for any $G \subset \mathrm{SL}_3(\mathbb{C})$, and prove it when G is abelian. This complex is an analogue of the Koszul complex on $\mathbb{C}^n$, and consists of vector bundles of forms

$$\bigoplus_a p_1^* E_a \otimes p_2^* F_a,$$

where p_1, $p_2 : X \times X \to X$ are projections into the first and second factor. It is a higher-dimensional generalization of the complex introduced by Kronheimer in his joint work with the second author in 2-dimensional case [15]. The above conjecture was proved there.

If X would be compact, a standard argument (cf. [6, Theorem 2.1]) shows that the Grothendieck group $K(X)$ of vector bundles on X is generated by E_a, either

F_a. However X is not compact, so the argument does not apply to our situation. To overcome this difficulty, we modify the complex to

$$\bigoplus_k p_1^* S_k \otimes p_2^* \mathcal{R}_k,$$

where $\mathcal{R}_k$ and S_k are as above. The original complex and this new complex are connected by a homotopy and defines the same element in the Grothendieck group. This will lead us to our main theorem (Theorem 5.4). The usage of both $K(X)$ and $K^c(X)$ are quite essential in the argument.

Let us comment on our assumption on G. As we explained, we need this assumption to show the exactness of the analogue of the Koszul complex, more precisely, the condition (4.8). When G is abelian, we can use the torus action so that we need to check (4.8) for very specific ideals. The idea to use the torus action is due to Nakamura [21] who used it to prove the smoothness of X.

By the way, the correspondence between (b) and (c) was further developed by the second author [17, 19]. He constructed irreducible integrable representations of the affine Lie algebra on the homology group of moduli spaces of instantons on X. The corresponding result in dimension 3 remains untouched in this paper.

The paper is organized as follows. In §2, we prepare some results on Hilbert schemes of points on $\mathbb{C}^n$ and the fixed point component X of the Hilbert scheme. In §3, we identify X with the moduli space of a certain quiver. In §4, we define the complex on $X \times X$, and show that it gives a resolution of the diagonal Δ under the condition (4.8). In §5, we state our main results on McKay correspondence which is a correspondence between the representation ring $R(G)$ and the Grothendieck group $K(X)$. We also study $K^c(X)$, the Grothendieck group of bounded complexes of algebraic vector bundles over X. In §6, we study 2 dimensional case for the argument in previous section. In §7, we check the condition holds for abelian subgroups $G \subset \mathrm{SL}_3(\mathbb{C})$ and complete the proof for our 3 dimensional McKay correspondence for abelian groups.

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2. Fixed points in Hilbert schemes

In this section, we prepare some preliminary results on Hilbert schemes of points on $\mathbb{C}^n$ and the variety X defined in the introduction.

For a positive integer N, let $\mathrm{Hilb}^N(\mathbb{C}^n)$ be the Hilbert scheme parametrising 0-dimensional subschemes of length N (see [20] for survey on Hilbert schemes of points). In this paper, we shall confuse subschemes and the corresponding ideal of the ring $\mathbb{C}[x_1, \dots, x_n]$. A point in $\mathrm{Hilb}^N(\mathbb{C}^n)$ is either a zero dimensional subscheme $Z \subset \mathbb{C}^n$ or an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$.

Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$. We consider two types of 'quotients' of $\mathbb{C}^n$ divided by G. The first one is the usual set-theoretical quotient $\mathbb{C}^n/G$. It is a subvariety of the Nth ($N = \#G$) symmetric product $S^N(\mathbb{C}^n) = (\mathbb{C}^n)^N/\mathfrak{S}_N$ by the

embedding

$$\mathbb{C}^n/G \ni Gx \longmapsto \sum_{g\in G}[gx] \in S^N(\mathbb{C}^n),$$

where a point in the symmetric product is denoted by a formal sum of points, as usual. The symmetric product is the Chow scheme of N points in $\mathbb{C}^n$ parametrising effective 0-cycles. Hence $\mathbb{C}^n/G$ is the *Chow quotient* in the sense of [13]. It is an irreducible component of the fixed point of the induced G-action on $S^N(\mathbb{C}^n)$.

Another quotient is the *Hilbert quotient* which is obtained by replacing the symmetric product by the Hilbert scheme as follows: Consider the induced G-action on $\mathrm{Hilb}^N(\mathbb{C}^n)$. If $Z \in \mathrm{Hilb}^N(\mathbb{C}^n)$ is a fixed point, then $H^0(\mathcal{O}_Z)$ is a G-module. For example, if Z is a single G-orbit consisting of pairwise distinct N points, $H^0(\mathcal{O}_Z)$ is the regular representation of G. As in §1, let X be the variety parametrising $Z \in (\mathrm{Hilb}^N(\mathbb{C}^n))^G$ such that $H^0(\mathcal{O}_Z)$ is the regular representation of G. The G-module structure is constant on each connected component of the fixed point set $(\mathrm{Hilb}^N(\mathbb{C}^n))^G$. Thus X is a union of components. A priori, it may consists of several irreducible components, but X is irreducible in many cases as we will see later.

We have the *Hilbert-Chow* morphism π from $\mathrm{Hilb}^N(\mathbb{C}^n)$ to $S^N(\mathbb{C}^n)$ defined by

$$\pi\colon \mathrm{Hilb}^N(\mathbb{C}^n) \ni Z \longmapsto \sum_{x\in\mathbb{C}^n} \mathrm{length}(Z_x)[x] \in S^N(\mathbb{C}^n).$$

Take a point Z in X and consider $\pi(Z)$. Since Z is invariant under G, its support consists of a union of G-orbits. However, since constant functions on each orbit form the trivial representation contained in $H^0(\mathcal{O}_Z)$, we only have single G-orbit by the assumption that $H^0(\mathcal{O}_Z)$ is the regular representation. This implies that $\pi(Z)$ is of the form

$$\sum_{g\in G}[gx] = \sum_{y\in Gx} \#(G/G_x)[y],$$

for some $x \in \mathbb{C}^n$. Here Gx denote the G-orbit through x and G_x is the stabilizer of x in G. Hence π maps X to $\mathbb{C}^n/G$. We use the same notation π for the restriction of the map to X for brevity.

The nonsingular locus $(\mathbb{C}^n/G)^{\mathrm{reg}}$ of $\mathbb{C}^n/G$ consists of those orbits Gx with G_x trivial. Since the map π is an isomorphism on $\pi^{-1}((\mathbb{C}^n/G)^{\mathrm{reg}})$, $\pi\colon X \to \mathbb{C}^n/G$ is a resolution of singularities provided X is nonsingular of dimension n and connected.

Remark 2.1. (1) The terminologies, Chow quotients and Hilbert quotients, were introduced by Kapranov [13].

(2) The definition of X given in [8, 10, 21] is slightly different from above. In those papers, X is defined as the irreducible components of $(\mathrm{Hilb}^N(\mathbb{C}^n))^G$ containing G-orbits of cardinality N. In dimension 2, it is known that the above X is smooth and connected, and hence two definitions are same (see [20, 4.4]). We prove the connectedness of X when G is an abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ later. Thus the definition is also the same in this case.

3. Representations of a quiver

In this section, we identify the subvariety X of the Hilbert scheme with the moduli space of a certain quiver. This identification shows that X is a special case of the variety considered by Kronheimer [14] (in dimension 2) and Sardo-Infirri [23]

(in general). We hope that this section will be helpful for the reader to notice that the complex constructed in the next section is a natural generalization of the complex introduced by Kronheimer-Nakajima [15].

Let $A = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the coordinate ring of $\mathbb{C}^n$. Take $I \in X$, or more generally an ideal corresponding to a zero-dimensional subscheme of $\mathbb{C}^n$ of length N. Then A/I is an N-dimensional vector space. If Z is the corresponding subscheme, we have $A/I = H^0(\mathcal{O}_Z)$. The multiplications of the coordinate functions $(x_1, x_2, \dots, x_n)$ induce a map $B = (B_1, B_2, \dots, B_n) \colon A/I \to \mathbb{C}^n \otimes A/I$ by

$$B_\alpha(f \bmod I) \stackrel{\text{def.}}{=} x_\alpha f \bmod I, \qquad (\alpha = 1, 2, \dots, n).$$

It satisfies

$$\operatorname{Hom}(A/I, \textstyle\bigwedge^2 \mathbb{C}^n \otimes A/I) \ni [B \wedge B] = \sum_{\alpha<\beta} [B_\alpha, B_\beta] dx_\alpha \wedge dx_\beta = 0.$$

Let us define $i \colon \mathbb{C} \to A/I$ by $i(\lambda) = \lambda \bmod I$. Then $i(1)$ is a cyclic vector with respect to B_α's, that is there is no proper subspace $S \subsetneq A/I$ which contains $i(1)$ and is invariant under all B_α's.

Conversely if we have an N-dimensional vector space R and homomorphisms $B \colon R \to \mathbb{C}^n \otimes R$, $i \colon \mathbb{C} \to R$ such that $[B \wedge B] = 0$, and $i(1)$ is a cyclic vector with respect to B_α's, we can define an ideal I by

$$I \stackrel{\text{def.}}{=} \{f(x_1, \dots, x_n) \in A \mid f(B_1, \dots, B_n) i(1) = 0\}.$$

Then I defines a 0-dimensional subscheme of $\mathbb{C}^n$ of length N.

Now suppose I is a point in X, i.e. a) it is invariant under the action of G and b) A/I is the regular representations of G. Then the above homomorphisms $B \colon A/I \to Q \otimes A/I$, $i \colon \mathbb{C} \to A/I$ are G-equivariant, where Q is a G-module defined by the inclusion $G \subset \mathrm{SL}_n(\mathbb{C})$, and $\mathbb{C}$ is the trivial G-module. Hence we get

Proposition 3.1. *Let R be the regular representation of G. Then there exists a bijection between X and the quotient space of the homomorphisms $B \in \operatorname{Hom}_G(R, Q \otimes R)$, $i \in \operatorname{Hom}_G(\mathbb{C}, R)$ satisfying*

$$[B \wedge B] = 0, \tag{3.2}$$

$$i(1) \text{ is a cyclic vector with respect to } B_\alpha\text{'s} \tag{3.3}$$

by the action of $\mathrm{GL}_G(R)$, the group of G-equivariant automorphisms of R.

Let us rewrite the above quotient space as a moduli space of representations of a certain quiver. Let $\rho_0, \dots, \rho_r$ be the isomorphism classes of irreducible representations of G, with ρ_0 be the trivial representation. Then the regular representation R decomposes as

$$R = \bigoplus R_k \otimes \rho_k, \tag{3.4}$$

where $R_k = \operatorname{Hom}_G(\rho_k, R)$. Then we have

$$\operatorname{Hom}_G(R, Q \otimes R) = \bigoplus \operatorname{Hom}(R_k, R_l) \otimes \operatorname{Hom}_G(\rho_k, Q \otimes \rho_l) = \bigoplus a_{kl} \operatorname{Hom}(R_k, R_l),$$

where $a_{kl} = a_{kl}^{(1)}$ is given in (1.5). Similarly we have

$$i \in \operatorname{Hom}_G(\mathbb{C}, R) = \operatorname{Hom}(\mathbb{C}, R_0).$$

The group $\mathrm{GL}_G(R)$ of G-equivariant automorphisms of R can be rewritten as

$$\mathrm{GL}_G(R) = \prod_k \mathrm{GL}(R_k).$$

Hence X can be described as

$$\{(B,i) \in \bigoplus a_{kl} \operatorname{Hom}(R_k, R_l) \oplus \operatorname{Hom}(\mathbb{C}, R_0) \mid (3.2), (3.3)\} / \prod_k \mathrm{GL}(R_k). \tag{3.5}$$

This description depends only on a_{kl} and $\dim R_k = \dim \rho_k$.

The *McKay quiver* is the quiver whose vertices are irreducible representation with a_{kl} arrows (possibly 0) from the vertex k to the vertex l. The above space is the framed moduli space of representation of the McKay quiver with the relation corresponding to (3.2). (cf. [18]) (See Figure 3.1.)

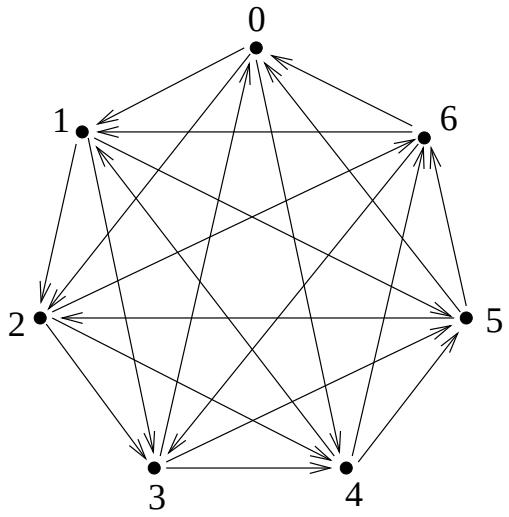

FIGURE 3.1. McKay quiver for $G = \langle\ \mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^4)\ \rangle$ ($\varepsilon = \exp(2\pi i/7)$)

When $n = 2$, the description in Proposition 3.1 is essentially same as Kronheimer's construction of ALE spaces [14]. There are minor differences: First his space depends on a parameter ζ. Our space corresponds his space with a special choice of ζ. Second, we have an extra vector i and take quotient by $\mathrm{GL}_G(R)$, while Kronheimer had no i and took quotients by $\mathrm{GL}_G(R)$/scalar. However, we can always normalize as $i = 1$ by the action of $\mathrm{GL}(R_0) = \mathbb{C}^*$. Hence our quotient is isomorphic to the quotient space of B's by the action of $\prod_{k\neq 0} \mathrm{GL}(R_k) \cong \mathrm{GL}_G(R)$/scalar. Hence our description is same as [14].

Kronheimer's construction was generalized to higher dimensions by Sardo-Infirri [23]. Thus our description coincides with his with a particular parameter.

4. A Koszul complex

In this section we construct a resolution of the diagonal Δ in $X \times X$ following [15, 3.6].

Take I_1, $I_2 \in X$, and consider the corresponding G-equivariant homomorphisms $A/I_1 \to Q \otimes A/I_1$, $A/I_2 \to Q \otimes A/I_2$ as in the previous section. Let us denote them by B^1, B^2. Then consider the following complex of vector spaces:

$$E_n \xrightarrow{d_n} E_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0, \tag{4.1}$$

where

$$E_k \overset{\text{def.}}{=} \operatorname{Hom}_G(A/I_1, \textstyle\bigwedge^{n-k} Q \otimes A/I_2),$$
$$d_k(\eta) \overset{\text{def.}}{=} B^2 \wedge \eta - \eta \wedge B^1 \quad \text{for } \eta \in E_k.$$

The equality $d_{k-1} \circ d_k = 0$ follows from the equation $[B^a \wedge B^a] = 0$ $(a = 1, 2)$.

Lemma 4.2 (cf. [15, 3.8,3.9])**.** (1) *When $I_1 \neq I_2$, d_n is injective and d_1 is surjective.*

(2) *When $I_1 = I_2$, the kernel of d_n is the one-dimensional subspace of scalar endomorphisms and the image of d_1 is the codimension-one subspace of trace-free endomorphisms.*

Proof. First notice that $\bigwedge^n Q$ is the trivial G-module by the assumption $G \subset \mathrm{SL}_n(\mathbb{C})$. Then we have $\bigwedge^{n-1} Q \cong Q^*$, and the statements for d_1 can be proved dually by the same arguments as those for d_n. So we only give the proof on the statements for d_n.

Suppose η is an element in the kernel of d_n. We have

$$\eta B^1 = B^2 \eta. \tag{4.3}$$

This equation implies that the image of η is invariant under B^2.

Since η is G-equivariant, the G-fixed parts are preserved under η. Since A/I_1 and A/I_2 are regular representations, the G-fixed parts consist of constant multiples of $1 \bmod I_1$ and $1 \bmod I_2$ respectively. Hence, we have $\eta(1 \bmod I_1) = \lambda(1 \bmod I_2)$ for some constant λ. Then (4.3) implies

$$\eta(x_1^{k_1} \cdots x_n^{k_n} \bmod I_1) = \lambda x_1^{k_1} \cdots x_n^{k_n} \bmod I_2$$

for any $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$. If $\lambda = 0$, then $\eta = 0$. If $\lambda \neq 0$, then η is surjective. Since A/I_1 and A/I_2 has the same dimension, this implies $I_1 = I_2$ and η is a scalar endomorphism. □

Lemma 4.4 (cf. [15, 3.10])**.** (1) *When $I_1 = I_2$, the homology group $\operatorname{Ker} d_{n-1}/\operatorname{Im} d_n$ is isomorphic to $\operatorname{Hom}_A(I_1, A/I_2)^G$, the G-fixed part of $\operatorname{Hom}_A(I_1, A/I_2)$.*

(2) *When $I_1 \neq I_2$, $\operatorname{Ker} d_{n-1}/\operatorname{Im} d_n$ is isomorphic to $\operatorname{Hom}_A(I_1, A/I_2)^G/\mathbb{C}\Xi$, where Ξ is the composition*

$$\Xi \colon I_1 \hookrightarrow A \to A/I_2.$$

Moreover, in either case, $\operatorname{Ker} d_1/\operatorname{Im} d_2$ is isomorphic to dual space of $\operatorname{Ker} d_{n-1}/\operatorname{Im} d_n$ of the complex with I_1, I_2 are exchanged.

Proof. The statement for the duality between the degree $n-1$ and degree 1 follows exactly as in the proof of Lemma 4.2.

Let $\Phi \in \operatorname{Hom}_A(I_1, A/I_2)^G$. We take an extension $\Psi \colon A \to A/I_2$ as a G-equivariant homomorphism (not necessarily an A-homomorphism). Then we define $\dot{B} = (\dot{B}_1, \ldots, \dot{B}_n) \in \operatorname{Hom}_G(A/I_1, Q \otimes A/I_2)$ by

$$\dot{B}_\alpha(f \bmod I_1) \overset{\text{def.}}{=} \Psi(x_\alpha f) - B_\alpha^2 \Psi(f) \qquad \text{for } \alpha = 1, \ldots, n,\ f \in A.$$

When $f \in I_1$, the right hand side vanishes since Φ is an A-homomorphism. Hence $\dot{B}$ is well-defined.

It is easy to check $\dot{B} \in \operatorname{Ker} d_{n-1}$. Moreover, the ambiguity of the choice of the extension Ψ is compensated by the image of d_n. Hence we have a homomorphism

$$\operatorname{Hom}_A(I_1, A/I_2)^G \to \operatorname{Ker} d_{n-1}/\operatorname{Im} d_n. \tag{4.5}$$

Conversely, if we are given $\dot{B} \in \operatorname{Ker} d_{n-1}$, we define an G-equivariant homomorphism $\Psi\colon A \to A/I_2$ inductively by

$$\begin{cases} \Psi(1) = 1 \bmod I_2 \\ \Psi(x_\alpha f) = B_\alpha^2 \Psi(f) + \dot{B}_\alpha(f \bmod I_1) & \text{for } \alpha = 1, \dots, n,\ f \in A. \end{cases}$$

This is well-defined thanks to the assumption $d_{n-1}\dot{B} = 0$. Moreover, the restriction $\Phi = \Psi|_{I_1}$ is A-linear by the second equation. This argument shows that the map (4.5) is surjective.

Now suppose Φ lies in the kernel of (4.5). Then we can take the extension $\Psi\colon A \to A/I_2$ of Φ so that

$$\Psi(x_\alpha f) = B_\alpha^2 \Psi(f) \qquad \text{for } \alpha = 1, \dots, n,\ f \in A.$$

Since Ψ is G-equivariant, we have $\Psi(1) = \lambda \bmod I_2$ for some constant λ. Then the above equation implies that $\Psi(f) = \lambda f \bmod I_2$ for any $f \in A$. Hence the restriction $\Phi = \Psi|_{I_1}$ is $\lambda\Xi$. If $I_1 = I_2$, then Ξ is zero, and hence we have the assertion. □

Note that $\operatorname{Hom}_A(I, A/I)^G$ is the Zariski tangent space of X at I. In particular, X is nonsingular if and only if $\operatorname{Hom}_A(I, A/I)^G$ has a constant dimension independent of I on each irreducible component.

Corollary 4.6. *Suppose X is nonsingular. Then the bijection given in Proposition 3.1 is an isomorphism. Moreover, the variety P consisting of pairs $(B, i) \in \operatorname{Hom}_G(R, Q \otimes R) \times \operatorname{Hom}_G(\mathbb{C}, R)$ (R is the regular representation of G as before) satisfying both* (3.2), (3.3) *is nonsingular, and the quotient map $P \to P/\operatorname{GL}_G(R) = X$ is a $\operatorname{GL}_G(R)$-principal bundle.*

Proof. Consider the complex (4.1) for $I_1 = I_2$ and $R = A/I_1$. Then d_{n-1} is nothing but the differential of the map $B \mapsto [B \wedge B]$.

The assumption of the smoothness of X implies that $\operatorname{Hom}_A(I_1, A/I_1)^G$, the Zariski tangent space at I_1, has a constant dimension independent of I_1. By Lemma 4.2 and Lemma 4.4, the kernel of d_1 also has a constant dimension. Hence the variety P is nonsingular.

The action of $\operatorname{GL}_G(R)$ on P is free. For, if $g \in \operatorname{GL}_G(R)$ stabilizes B and i, then $\operatorname{Ker}(g-1)$ contains $i(1)$ and is invariant under B. Hence the cyclic vector condition implies $\operatorname{Ker}(g-1) = R$.

Moreover, $P/\operatorname{GL}_G(R)$ is a geometric invariant theory quotient of P by G if we introduce the polarization as in the case of quiver varieties [19]. The cyclic vector condition is the stability in the geometric invariant theory, and hence the quotient map is a $\operatorname{GL}_G(R)$-principal bundle.

Now the map given by the Proposition 3.1 is differentiable and respects tangent spaces. Hence it is an isomorphism. □

Assume that X is nonsingular of dimension n. Then if we vary I_1 in X, A/I_1 forms a holomorphic vector bundle over X. In fact, it is identified with the associated vector bundle $P\times_{\mathrm{GL}_G(R)}R$. Following [15], we denote it by $\mathcal{R}$ and call *tautological vector bundle*. Fibers of $\mathcal{R}$ have structures of G-modules which are isomorphic to the regular representation. The homomorphism $B\in \mathrm{Hom}_G(A/I, Q\otimes A/I)$ defines a G-equivariant holomorphic vector bundle homomorphism $\mathcal{R}\to Q\otimes\mathcal{R}$. This is called *tautological endomorphisms* in [15].

Since A/I is the 0-th cohomology of the structure sheaf of subscheme corresponding to I, we can identify the tautological bundle $\mathcal{R}$ with $p_*\mathcal{O}_{\mathcal{Z}}$ where $\mathcal{Z}\subset X\times\mathbb{C}^3$ is the universal subscheme and $p\colon \mathcal{Z}\to X$ is the first projection. Thus we arrive at the definition (1.1).

As explained in the introduction, the decomposition of the regular representation into irreducible representations (3.4) induces the decomposition of the tautological vector bundle as (1.2). In other words,

$$\mathcal{R}_k = \mathrm{Hom}_G(\rho_k, p_*\mathcal{O}_{\mathcal{Z}}).$$

For brevity, we identify the vector bundle $\mathcal{R}$ with the sheaf of germs of its sections. Then the complex (4.1) induces the following complex of sheaves on $X\times X$:

$$0 \longrightarrow F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{t} \mathcal{O}_\Delta \longrightarrow 0, \tag{4.7}$$

where

$$\begin{aligned} F_k &\overset{\text{def.}}{=} \mathrm{Hom}_G(p_1^*\mathcal{R}, \textstyle\bigwedge^{n-k} Q\otimes p_2^*\mathcal{R}),\\ d_k(\eta) &\overset{\text{def.}}{=} B^2\wedge\eta - \eta\wedge B^1 \quad \text{for } \eta\in F_k,\\ t(\eta) &= \mathrm{tr}(\eta|_\Delta) \quad \text{for } \eta\in F_0. \end{aligned}$$

Here p_a is the projection to the ath factor of $X\times X$. Note that $F_0 = \mathrm{Hom}_G(p_1^*\mathcal{R}, p_2^*\mathcal{R})$ and hence the trace make sense on the diagonal.

When $n=2$, the Hilbert scheme of points is nonsingular by Fogarty [7]. Hence the G-fixed component X is also nonsingular. Thus the assumption of Corollary 4.6 is met. Counting dimensions and using Lemma 4.2, we deduce that $\mathrm{Ker}\, d_1/\mathrm{Im}\, d_2 = 0$ for $I_1\neq I_2$. This and an additional argument shows that the complex (4.7) is exact (see [15, §3] or below). Moreover, the last assertion in Lemma 4.4 says that $\mathrm{Ker}\, d_1/\mathrm{Im}\, d_2$ for $I_1 = I_2$, which is the tangent space of X at I_1, is isomorphic to its dual space. This isomorphism is given by the natural holomorphic symplectic form on X.

We assume $n=3$ hereafter, and consider the following condition:

$$\dim \mathrm{Hom}_A(I_1, A/I_2)^G = \begin{cases} 3, & \text{when } I_1 = I_2\\ 1, & \text{when } I_1\neq I_2. \end{cases} \tag{4.8}$$

When I_1, I_2 are ideals given by distinct points, the above holds. The condition for $I_1 = I_2$ is equivalent to saying that X is nonsingular of dimension 3. And note that similar condition holds for $n=2$ by the above discussion. Those show that the above seems reasonable. And we show the above holds when G is abelian in §7.

Theorem 4.9 (cf. [15, 3.6]). *Under the assumption* (4.8), *the complex* (4.7) *is exact.*

Proof. By the assumption, the complex is exact outside the diagonal Δ. The exactness of (4.7) in degree 0 can be shown exactly as in [15, 3.6]. So we omit the argument.

For the proof of the exactness in degrees other than 0, we use the criterion of Buchsbaum-Eisenbud (see e.g., [5, 20.9]). We need to check (a) $\operatorname{rank} d_k + \operatorname{rank} d_{k+1} = \operatorname{rank} F_k$, and (b) the determinantal ideal of the differential d_k has depth at least k. The first condition holds since the complex is exact on a nonempty open subset. Since the diagonal has codimension 3 in $X \times X$, the determinantal ideal of the differential d_k has depth at least 3. Hence the second condition (in fact, a stronger condition) also holds. □

The following will not used in the other part of this paper, but illustrates the relation between the smoothness and the exactness of the complex (4.7).

Proposition 4.10. *If X is nonsingular, the complex* (4.7) *is exact on the diagonal $\Delta \subset X \times X$.*

Proof. Fix a point $x_0 \in X$ and consider the complex (4.7) at the point (x_0, x_0):

(4.11)
$$(F_3)_{(x_0,x_0)} \xrightarrow{(d_3)_{(x_0,x_0)}} (F_2)_{(x_0,x_0)} \xrightarrow{(d_2)_{(x_0,x_0)}} (F_1)_{(x_0,x_0)} \xrightarrow{(d_1)_{(x_0,x_0)}} (F_0)_{(x_0,x_0)}.$$

By Lemmas 4.2,4.4 together with the smoothness assumption of X, the homology groups of this complex are $\mathbb{C}(\mathrm{Id})$ in degree 0 and 3, the tangent space $T_{x_0}X$ in degree 2, and the cotangent space $T^*_{x_0}X$ in degree 1. We define a trivial vector bundle H_i over the tangent space $T_{x_0}X \times T_{x_0}X$ where the fiber is the ith homology groups of (4.11).

Choosing connections on vector bundles in (4.7), we consider the derivative Dd_k at x_0. Differentiating $d_{k-1} \circ d_k = 0$, we check that Dd_k's induce homomorphisms between homology groups of (4.11). Moreover, they are independent of the choice of the connections. Let us think the homomorphisms as vector bundles homomorphisms between vector bundles H^k by setting the value at $(v, w) \in T_{x_0}X \times T_{x_0}X$ as the derivative Dd_k in the direction (v, w). We simply write Dd_k for these vector bundle homomorphisms. Thus we get

(4.12)
$$0 \longrightarrow H_3 \xrightarrow{Dd_3} H_2 \xrightarrow{Dd_2} H_1 \xrightarrow{Dd_1} H_0$$

Differentiating $d_{k-1} \circ d_k = 0$ twice, one checks that this forms a complex.

Calculating the derivative of each d_k, we find that (4.12) is a part of Koszul complex on $T_{x_0}X \times T_{x_0}X \cong \mathbb{C}^3 \times \mathbb{C}^3$. Namely, if we add the evaluation homomorphism $H_0 \to \mathcal{O}_\Delta$ to the end of (4.12), the complex is the Koszul complex. Now we check the exactness of the original complex (4.7) at (x_0, x_0), except possibly in degree 0. For this we again use the criterion of Buchsbaum-Eisenbud [5]. From the above, the ranks of d_k are

$$\operatorname{rank} d_1 = \operatorname{rank}(d_1)_{(x_0,x_0)} + 1, \quad \operatorname{rank} d_2 = \operatorname{rank}(d_2)_{(x_0,x_0)} + 2,$$
$$\operatorname{rank} d_3 = \operatorname{rank}(d_3)_{(x_0,x_0)} + 1,$$

where $(d_k)_{(x_0,x_0)}$ is what d_k induces on the fibers of F_k over (x_0, x_0). Combining with the above observation on the homology groups of (4.11), we get $\operatorname{rank} d_k + \operatorname{rank} d_{k+1} = \operatorname{rank} F_k$. Moreover, the determinantal ideal of each differential has depth 3. These imply the exactness of the complex as in Theorem 4.9. □

5. McKay correspondence for the K-theory

Given a finite group G acting on a variety Y, we denote by $K_G(Y)$ the Grothendieck group of G-equivariant coherent $\mathcal{O}_Y$-sheaves over Y. When G is the trivial group $\{1\}$, we simply write $K(Y)$ for $K_{\{1\}}(Y)$. This is the ordinary K-group. We denote by $[S]$ the class represented by a G-equivariant sheaf S on Y. But we may drop the bracket when there is no ambiguity. If Y is nonsingular, $K_G(Y)$ is isomorphic to the Grothendieck group of G-equivariant vector bundles (see e.g., [4, Chapter 5]). If $f\colon Y \to Y'$ is a G-equivariant proper morphism, we can define a push-forward $f_*\colon K_G(Y) \to K_G(Y')$ by $f_*([S]) = \sum_i (-1)^i [R^i f_*(S)]$, where $R^i f_*(S)$ is the ith higher direct image sheaf. If $f\colon Y \to Y'$ is a G-equivariant morphism and Y' is smooth, we define the pull-back $f^*\colon K_G(Y') \to K_G(Y)$ as follows: Since Y' is nonsingular, it is enough to define the pull-back for classes represented by G-equivariant *vector bundles*. If E is a G-equivariant vector bundle on Y', then its pull-back $f^*(E)$ is also a G-equivariant vector bundle over Y. Hence we define $f^*([E]) = [f^*(E)]$. We will never use the pull-back homomorphism from singular varieties.

Let us consider a subvariety $\pi^{-1}(0)$ and $K^c(X)$ the Grothendieck group of bounded complexes of algebraic vector bundles over X which are exact outside $\pi^{-1}(0)$. (See [2] for the definition and results used below.) This is isomorphic to the Grothendieck group of coherent sheaves on $\pi^{-1}(0)$, where the isomorphism is given by taking the alternating sum of the homology of the complex:

$$[E_\bullet] = [E_n \to E_{n-1} \to \cdots \to E_1 \to E_0] \longmapsto \sum_{i=0}^{n} (-1)^n [H_i(E_\bullet)].$$

The inverse is given by mapping a sheaf S on $\pi^{-1}(0)$ to its finite resolution by locally free sheaves over X. When we consider push-forward homomorphisms, we represent elements in $K^c(X)$ by sheaves on $\pi^{-1}(0)$. When we consider pull-back homomorphisms, we represent elements by complexes.

There is a natural pairing between $K(X)$ and $K^c(X)$ given by

$$K(X) \times K^c(X) \ni ([E],[S]) \mapsto P_*([E \otimes S]) \in K(\text{point}) \cong \mathbb{Z}, \tag{5.1}$$

where E is a vector bundle on X and S is a sheaf on $\pi^{-1}(0)$ and P is the obvious projection of $\pi^{-1}(0)$ to a point. Note that

(a) $E \otimes S$ is a tensor product of a vector bundle and a sheaf, hence well-defined in the Grothendieck group,
(b) $E \otimes S$ has support contained in $\pi^{-1}(0)$, hence $P_*([E \otimes S])$ can be defined.

Let us consider the complex (1.3) in $n = 3$:

$$\mathcal{R} \xrightarrow{d_3} Q \otimes \mathcal{R} \xrightarrow{d_2} \textstyle\bigwedge^2 Q \otimes \mathcal{R} \xrightarrow{d_1} \bigwedge^3 Q \otimes \mathcal{R} = \mathcal{R}.$$

It is a complex thanks to the equation $[B \wedge B] = 0$.

Lemma 5.2. *The complex* (1.3) *is exact outside* $\pi^{-1}(0)$.

Proof. Take a coordinate system (x_1, x_2, x_3) on $\mathbb{C}^3$ and write $B = (B_1, B_2, B_3)$. Note that the support of the 0-dimensional subscheme corresponding to $[B_1, B_2, B_3, i]$ consists of simultaneous eigenvalues of B_1, B_2, B_3. Hence at least one of B_α's is invertible if $[B_1, B_2, B_3, i]$ is outside of $\pi^{-1}(0)$. Say B_1 is invertible. Now it is clear that d_3 is injective and d_1 is surjective.

Suppose $\eta = (\eta_1, \eta_2, \eta_3)$ is in the kernel of d_2, that is

$$B_1\eta_2 = B_2\eta_1, \quad B_2\eta_3 = B_3\eta_2, \quad B_3\eta_1 = B_1\eta_3.$$

Setting $\xi = B_1^{-1}\eta_1$, we find

$$d_3\xi = (\eta_1, B_2B_1^{-1}\eta_1, B_3B_1^{-1}\eta_1) = (\eta_1, \eta_2, \eta_3),$$

where we have used $[B_1, B_2] = [B_3, B_1] = 0$. This shows that $\operatorname{Ker} d_2 = \operatorname{Im} d_3$. The proof for $\operatorname{Ker} d_1 = \operatorname{Im} d_2$ is same. □

We decompose the complex (1.3) according to (1.2) and denote its transpose by S_k:

$$S_k : \mathcal{R}_k^\vee \longrightarrow \bigoplus_l a_{kl}^{(2)}\mathcal{R}_l^\vee \longrightarrow \bigoplus_l a_{kl}^{(1)}\mathcal{R}_l^\vee \longrightarrow \mathcal{R}_k^\vee.$$

By Lemma 5.2, S_k defines an element in $K^c(X)$.

Now we define the homomorphism from the representation ring $R(G)$ of G to $K(X)$ as follows. Let us consider the diagram

$$X \xleftarrow{\ p\ } \mathcal{Z} \xrightarrow{\ q\ } \mathbb{C}^n,$$

where $\mathcal{Z} \subset X \times \mathbb{C}^n$ is the universal subscheme and p and q are the projections to the first and second factors. Note that the group G acts on $\mathcal{Z}$ and $\mathbb{C}^n$ so that q is G-equivariant. If we let G act on X trivially, p is also G-equivariant. By [4, 5.4.21], the representation ring $R(G)$ is isomorphic to $K_G(\mathbb{C}^n)$ by sending the representation ρ to $\rho \otimes \mathcal{O}_{\mathbb{C}^n}$. We consider the composition of the following homomorphisms in K-theory:

$$\begin{aligned}(5.3)\quad R(G) \xrightarrow{\vee} R(G) \cong K_G(\mathbb{C}^n) \to \\ \xrightarrow{q^*} K_G(\mathcal{Z}) \xrightarrow{p_*} K_G(X) \cong R(G) \otimes_{\mathbb{Z}} K(X) \xrightarrow{\operatorname{Inv}\otimes \operatorname{id}} K(X),\end{aligned}$$

where $\vee$ is sending V to its dual representation $V^\vee$, and $\operatorname{Inv}\colon R(G) \to \mathbb{Z}$ is given by $\operatorname{Inv}(V) = \dim V^G$.

The image of ρ_k under the composition (5.3) is given by

$$\begin{aligned}((\operatorname{Inv}\otimes \operatorname{id}) \circ p_* \circ q^*)(\rho_k^\vee) &= \operatorname{Hom}_G(\rho_0, (p_* \circ q^*)(\rho_k^\vee \otimes \mathcal{O}_{\mathbb{C}^3}))) \\ &= \operatorname{Hom}_G(\rho_k, p_*\mathcal{O}_{\mathcal{Z}}) = \mathcal{R}_k.\end{aligned}$$

The following is one of main results in this paper.

Theorem 5.4. *Suppose G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$. Assume the condition (4.8) holds.*

(1) *The composition $(\operatorname{Inv}\otimes \operatorname{id}) \circ p_* \circ q^* \circ \vee$ in (5.3), which maps the irreducible representation ρ_k to the tautological bundle $\mathcal{R}_k$, gives an isomorphism between $R(G)$ and $K(X)$.*

(2) *The support of the complex S_k is contained in $\pi^{-1}(0)$, and $\{\mathcal{R}_k\}$ and $\{S_k\}$ are dual bases for $K(X)$ and $K^c(X)$, where $K^c(X)$ is the Grothendieck group of bounded complexes of vector bundles with supports contained in $\pi^{-1}(0)$.*

The rest of this section is devoted to the proof of this theorem and to its corollaries. So we assume the condition (4.8) throughout in this section. Hence the complex (4.7) is exact by Theorem 4.9. The assumption (4.8) will be checked for an abelian subgroup $G \subset \mathrm{SL}_3(\mathbb{C})$ in §7. Our proof below works in the 2-dimensional

case with obvious modifications. And the vanishing corresponding to (4.8) is already checked. Thus we have (1) and (2) also in 2-dimensional case.

The statement (1) was conjectured by Reid [22] based on the corresponding result in the 2-dimensional case proved by Gonzales-Sprinberg and Verdier [9]. The statement (2) seems new even in dimension 2. We conjecture that the assumption (4.8) holds for any finite subgroup G of $\mathrm{SL}_3(\mathbb{C})$. Note that our statement makes sense in any dimension provided X is nonsingular, and we conjecture it holds under a reasonable, yet unknown, assumption on G. Remark that in dimension 4, when G is the group of order 2 generated by $\mathrm{diag}(-1,-1,-1,-1)$, the statement of Theorem 5.4 is false while X is nonsingular. In this example, X is not crepant, and the complex is not exact. Thus the smoothness of X and the condition (4.8) are not equivalent at least in dimension 4.

Theorem 5.4 has many interesting applications. First, we prove $K_X = \mathcal{O}_X$ (see Theorem 5.13). When G is an abelian subgroup, this was proved by Nakamura [21] using the description of X as a toric variety. Our proof uses only the above mentioned vanishing of certain homology groups.

The second application is much more interesting. We consider the intersection pairing $(\ ,\ )$ on $K^c(X)$ defined by

$$(S,T) = \langle \theta S, T\rangle, \tag{5.5}$$

where $\theta : K^c(X) \to K(X)$ is the natural homomorphism. Then, we have the following relation between the intersection pairing on $K^c(X)$ and the decomposition of the tensor product.

Corollary 5.6. *Assume the same assumption as in Theorem 5.4. The intersection pairing on $K^c(X)$ and the tensor product decomposition* (1.5) *are related by*

$$(S_k^\vee, S_l) = a_{kl}^{(2)} - a_{kl}^{(1)} = a_{lk}^{(1)} - a_{kl}^{(1)},$$

where $S_k^\vee$ is the dual of S_k:

$$S_k^\vee = -\left[\mathcal{R}_k \longrightarrow \bigoplus_l a_{kl}^{(1)}\mathcal{R}_l \longrightarrow \bigoplus_l a_{kl}^{(2)}\mathcal{R}_l \longrightarrow \mathcal{R}_k\right],$$

and the second equality follows from $\bigwedge^2 Q = Q^*$.

This corollary follows from $\theta S_k^\vee = \sum_l (a_{kl}^{(2)} - a_{kl}^{(1)})\mathcal{R}_l$ and the above theorem.

In dimension 2, the corresponding statement turns out to be

$$(S_k^\vee, S_l) = 2\delta_{kl} - a_{kl}^{(1)}.$$

In §6, we will express S_k in terms of irreducible components C_l. In this way, we recover the identification of the decomposition of tensor products and the intersection pairing explained in the introduction.

Unfortunately we could not give an explicit expression of $S_k^\vee$ in the linear combination of S_l's (or equivalently $\mathcal{R}_k^\vee$ in $\mathcal{R}_l$'s) in general. Thus we could not determine the intersection product in terms of G.

Now we begin the proof of Theorem 5.4. First we show

Theorem 5.7. *$\{\mathcal{R}_k\}_{k=0}^r$ and $\{S_k\}_{k=0}^r$ span $K(X)$ and $K^c(X)$ respectively.*

Proof. Modifying (4.7), we introduce the following complex depending on a parameter s:

$$C_s : F_3 \xrightarrow{d_3^s} F_2 \xrightarrow{d_2^s} F_1 \xrightarrow{d_1^s} F_0, \tag{5.8}$$

where F_k is as in (4.7) and d_k^s is given by

$$d_k^s(\eta) \stackrel{\text{def.}}{=} sB^2 \wedge \eta - \eta \wedge B^1 \quad \text{for } \xi \in F_k\,.$$

This is still complex thanks to the equation $[B^a \wedge B^a] = 0\,(a = 1, 2)$. When $s = 1$, it is the original complex, which gives us a resolution of $\mathcal{O}_\Delta$. If $s \neq 0$, this is nothing but the pull-back of the complex $C_{s=1}$ by the automorphism of $X \times X$ defined by

$$([B^1, i^1], [B^2, i^2]) \mapsto ([B^1, i^1], [sB^2, i^2]). \tag{5.9}$$

When $s = 0$, the complex (5.8) decomposes as

$$\bigoplus_k p_1^* S_k \otimes p_2^* \mathcal{R}_k\,,$$

where p_i is the projection to the ith factor.

Let $\operatorname{Supp} C_s$ be the subvariety on which the complex C_s is not exact. When $s = 1$, $\operatorname{Supp} C_s$ is the diagonal. For $s \neq 0$, $\operatorname{Supp} C_s$ is the pull-back of the diagonal by (5.9). For $s = 0$, $\operatorname{Supp} C_{s=0}$ is contained in $\pi^{-1}(0) \times X$ by Lemma 5.2. In particular, in each case, the restriction of the first projection $p_2 \colon \operatorname{Supp} C_s \to X$ is proper. We consider C_s as a complex on $X \times X \times \mathbb{C}$, pulling back vector bundles F_k and setting the differential d_k^s on $X \times X \times \{s\}$. Then we can define the operator by

$$K(X) \ni E \longmapsto p_{23*}(p_1^* E \otimes C_s) \in K(X \times \mathbb{C}),$$

where $p_{23} \colon X \times X \times \mathbb{C} \to X \times \mathbb{C}$ is the projection to the second and the third factor.

Let $p \colon X \times \mathbb{C} \to X$ be the projection. It is known that $p^* \colon K(X) \to K(X \times \mathbb{C})$ is an isomorphism [3, IX, 1.6]. Let $a_s \colon X \to X \times \mathbb{C}$ denote the embedding given by $a_s(x) = (x, s)$. It satisfies $a_s^* p^* = (p \circ a_s)^* = \mathrm{id}$, and hence a_s^* is independent of s. If we choose $s = 1$, we have $E = a_1^* p_{23*}(p_1^* E \otimes C_s)$ since $C_{s=1}$ is the resolution of the diagonal. Comparing with the pull-back by a_0, we get

$$E = a_0^* p_{23*}(p_1^* E \otimes C_s) = \sum_k \langle E, S_k \rangle \mathcal{R}_k, \tag{5.10}$$

where $\langle\ ,\ \rangle$ is the pairing given by (5.1). In particular, this shows that $\{\mathcal{R}_k\}$ generates $K(X)$ as $\mathbb{Z}$-modules.

Similarly, we consider

$$p_{13*}(p_2^* S \otimes C_s)$$

for $S \in K^c(X) = K(\pi^{-1}(0))$. Since $p_2 \colon \operatorname{Supp} C_s \to X$ is proper, this defines an operator from $K(\pi^{-1}(0))$ to $K(\pi^{-1}(0) \times \mathbb{C})$. The pull-back homomorphism is independent of s as above, hence

$$S = a_0^* p_{13*}(p_2^* S \otimes C_s) = \sum_k \langle \mathcal{R}_k, S \rangle S_k\,.$$

This implies that $\{S_k\}$ generates $K^c(X)$. □

We postpone the proof of the linear independence of $\mathcal{R}_k$ until the end of this section.

The following is the first application of Theorem 5.7

Theorem 5.11. *X is connected.*

Proof. Let $\{X_\alpha\}$ be the set of connected components of X. For any locally free sheaf E (whose rank may change on components), we assign the rank of its restriction to X_α. Then we define the *augmentation*

$$\varepsilon\colon K(X) \to \mathbb{Z}^{\pi_0(X)},$$

where $\pi_0(X)$ denotes the set of connected components of X. This is surjective. However, $K(X)$ is generated by the tautological vector bundles $\mathcal{R}_k$'s which have constant rank over the whole $X = \coprod X_\alpha$. Hence X must be connected. □

As we promised above, we prove that the canonical bundle K_X is trivial as an application of Theorem 5.7. Another ingredient is the Serre duality. For an element $S \in K^c(X)$ represented by a complex

$$E_n \xrightarrow{d_n} E_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0,$$

we define its dual $S^\vee \in K^c(X)$ by

$$(-1)^n \left[E_0^\vee \xrightarrow{{}^t d_0} E_1^\vee \xrightarrow{{}^t d_1} \cdots \xrightarrow{{}^t d_{n-1}} E_{n-1}^\vee \xrightarrow{{}^t d_n} E_n^\vee\right].$$

Then the Serre duality implies that

$$\langle E, S\rangle = -\langle E^\vee \otimes K_X, S^\vee\rangle \tag{5.12}$$

Theorem 5.13. *The canonical bundle K_X is trivial in* $\mathrm{Pic}(X)$.

Proof. Since the composition $\mathrm{Pic}(X) \to K(X) \xrightarrow{\det} \mathrm{Pic}(X)$ is the identity, it is enough to show that $K_X = \mathcal{O}_X$ in $K(X)$.

Substituting $E = \mathcal{R}_0$, $S = S_k^\vee$ into (5.12), we have

$$\langle \mathcal{R}_0, S_k^\vee\rangle = -\langle K_X, S_k\rangle \tag{5.14}$$

where we have used $\mathcal{R}_0 = \mathcal{R}_0^\vee = \mathcal{O}_X$. Combining with (5.10), we get

$$K_X = -\sum_k \langle \mathcal{R}_0, S_k^\vee\rangle \mathcal{R}_k. \tag{5.15}$$

On the other hand, if we replace d_k^s, in the proof of Theorem 5.7, by

$$d_k'^s(\eta) \overset{\text{def.}}{=} B^2 \wedge \eta - s\eta \wedge B^1,$$

we get a homotopy between the complex (4.7) and

$$\bigoplus_k p_1^* \mathcal{R}_k^\vee \otimes p_2^*(-S_k^\vee).$$

By the same argument as above, we obtain

$$E = -\sum_k \langle E, S_k^\vee\rangle \mathcal{R}_k^\vee$$

instead of (5.10). Applying the duality, we have

$$E^\vee = -\sum_k \langle E, S_k^\vee\rangle \mathcal{R}_k.$$

Substituting $E = \mathcal{R}_0$, we have

$$\mathcal{R}_0 = -\sum_k \langle \mathcal{R}_0, S_k^\vee\rangle \mathcal{R}_k.$$

Comparing with (5.15), we get $K_X = \mathcal{R}_0 = \mathcal{O}_X$. □

Thus we get

Corollary 5.16. *X is a crepant resolution of $\mathbb{C}^3/G$.*

Now we relate the representation ring and the cohomology group. Let $\mathrm{ch}\colon K(X) \to H^*(X,\mathbb{Q})$ be the Chern character homomorphism. As we used $K^c(X)$ besides $K(X)$, we need to consider the cohomology group with compact support $H^*_c(X,\mathbb{Q}) \cong H^*(X, X\setminus\pi^{-1}(0),\mathbb{Q})$. The isomorphism can be shown by observing the $\mathbb{C}^*$-action induced by $(x,y,z)\mapsto(tx,ty,tz)$ retracts X to a neighborhood of $\pi^{-1}(0)$. We have the localized Chern character homomorphism $\mathrm{ch}^c\colon K^c(X)\to H^*_c(X,\mathbb{Q})$ defined by Iversen [12].

Theorem 5.17. (1) *The rational cohomology groups $H^*(X,\mathbb{Q})$, $H^*_c(X,\mathbb{Q})$ vanish in odd degrees.*

(2) *$\{\mathrm{ch}(\mathcal{R}_k)\}_{k=0}^r$ and $\{\mathrm{ch}^c(S_k)\}_{k=0}^r$ form dual bases of $H^*(X,\mathbb{Q})$, $H^*_c(X,\mathbb{Q})$ with respect to the pairing*

$$\langle\alpha,\beta\rangle = \int_X \alpha\cup\beta\,\mathrm{Td}(X),$$

where $\mathrm{Td}(X)$ is the Todd class of X.

Proof. The proof proceed exactly as Theorem 5.7 if we apply either the usual Chern character or the localized Chern character. Then we find that $\{\mathrm{ch}(\mathcal{R}_k)\}_{k=0}^r$ and $\{\mathrm{ch}^c(S_k)\}_{k=0}^r$ span $H^*(X,\mathbb{Q})$, $H^*_c(X,\mathbb{Q})$ respectively. Thus we have the assertion (1). Moreover, we know $\dim H^*(X,\mathbb{Q}) = r+1$ by previous results on McKay correspondence [1, 11]. Hence they are bases. Substituting $E=\mathcal{R}_l$ into (5.10), applying the Chern character, and using the Riemann-Roch, we get

$$\mathrm{ch}(\mathcal{R}_l) = \sum_k \langle\mathcal{R}_l, S_k\rangle\,\mathrm{ch}(\mathcal{R}_k) = \sum_k \langle\mathrm{ch}(\mathcal{R}_l),\mathrm{ch}^c(S_k)\rangle\,\mathrm{ch}(\mathcal{R}_k).$$

This shows that $\langle\mathrm{ch}(\mathcal{R}_l),\mathrm{ch}^c(S_k)\rangle = \delta_{kl}$. Thus we have the assertion (2). □

In the course of the proof, we proved the linear independence of $\mathcal{R}_k$ and the equality $\langle\mathcal{R}_l,S_k\rangle=\delta_{kl}$. Thus we have completed the proof of Theorem 5.4.

6. 2-dimensional case

In this section, we study $\mathcal{R}_k$ and S_k in more detail in the 2-dimensional case. Since this case was already studied before [10, 18], we only give a sketch.

The same argument as in the previous section shows that $\{\mathcal{R}_k\}$ and $\{S_k\}$ form dual base of $K(X)$ and $K^c(X)$, where S_k in this case is

$$S_k : \mathcal{R}_k^\vee \xrightarrow{{}^t(\wedge B)} \bigoplus_l a_{kl}\mathcal{R}_l^\vee \xrightarrow{{}^tB} \mathcal{R}_k^\vee. \tag{6.1}$$

First suppose $k\neq 0$. Then the cyclic vector condition implies that ${}^t(\wedge B)$ is injective on each fiber. The other homology groups can be determined by using the following.

Proposition 6.2. (1) *Let C_k denote the subvariety where ${}^tB$ in (6.1) is not surjective. Then C_k is isomorphic to the complex projective line $\mathbb{C}\mathrm{P}^1$.*

(2) *On C_k, the cokernel of ${}^tB$ is isomorphic to $\mathcal{O}_{C_k}(-1)$. (Here -1 means the dual of the hyperplane bundle on $C_k\cong\mathbb{C}\mathrm{P}^1$.)*

(3) *The restriction of the tautological bundle $\mathcal{R}_l$ to C_k is*

$$\begin{cases} \mathcal{O}_{C_k}(1) \oplus (\mathcal{O}_{C_k})^{\oplus \operatorname{rank}\mathcal{R}_l - 1} & \text{if } k = l, \\ (\mathcal{O}_{C_k})^{\oplus \operatorname{rank}\mathcal{R}_l} & \text{if } k \neq l. \end{cases}$$

Thus the homology of S_k vanishes in degree 1, 2 and $\mathcal{O}_{C_k}(-1)$ in degree 0. Hence $S_k \cong \mathcal{O}_{C_k}(-1)$. The equality $\langle \mathcal{R}_l, S_k \rangle = \delta_{kl}$ follows directly in this case.

Next suppose $k = 0$. Instead of considering S_0, we study $S_0^\vee$:

$$S_0 : \mathcal{R}_0 \xrightarrow{B} \bigoplus_l a_{0l}\mathcal{R}_l \xrightarrow{\wedge B} \mathcal{R}_0. \tag{6.3}$$

Then B is injective by the cyclic vector condition.

Proposition 6.4. (1) *The homomorphism $\wedge B$ in* (6.3) *is not surjective exactly on the exceptional set $\pi^{-1}(0)$ of the resolution $X \to \mathbb{C}^2/G$.*

(2) *On the exceptional set, the cokernel of $\wedge B$ is $\mathcal{O}_{\pi^{-1}(0)}$.*

Since $\langle \mathcal{R}_k, S_0 \rangle = \langle \mathcal{R}_k^\vee, S_0^\vee \rangle$ by the Serre duality, we could check $\langle \mathcal{R}_k, S_0 \rangle = \delta_{k0}$ also in this case.

Using the above, it becomes easy to determine $\operatorname{ch}^c(S_k)$. First notice that $\operatorname{Td}(X) = 1$ in this case. Recall also that $H^4_c(X, \mathbb{Q}) = \mathbb{Q}\Omega$ where Ω is the canonical generator satisfying $\int_X \Omega = 1$. Then $\langle \mathcal{R}_0, S_k \rangle = \delta_{k0}$ implies that the degree 4 part of $\operatorname{ch}^c(S_k) = \delta_{k0}\Omega$ because $\operatorname{ch}(\mathcal{R}_0)$ is the canonical generator of $H^0(X, \mathbb{Q})$. By Proposition 6.2, the degree 2 part of $\operatorname{ch}^c(S_k)$ for $k \neq 0$ is the Poincaré dual of $[C_k]$. Since $\operatorname{ch}^c(S_k)$ is a basis of $H^*_c(X, \mathbb{Q})$, this implies that C_k's are irreducible components of the exceptional set. (This can be directly checked by studying C_k's.) Then the equality $\langle \mathcal{R}_k, S_l \rangle = \delta_{kl}$ means that $\{c_1(\mathcal{R}_k)\}_{k \neq 0}$ is the dual basis of $\{[C_k]\}$. This is the second statement of a result of Gonzales-Sprinberg and Verdier explained in the introduction. Finally, considering

$$\begin{aligned} 0 &= \langle \mathcal{R}_k, S_0 \rangle \qquad \text{for } k \neq 0 \\ &= \langle c_1(\mathcal{R}_k), \operatorname{ch}^c(S_0) \rangle + \operatorname{rank}\mathcal{R}_k, \end{aligned}$$

we get

$$\text{degree 2 part of } \operatorname{ch}^c(S_0) = -\sum_{k=1}^r \operatorname{rank}\mathcal{R}_k \, \mathrm{P.D.}[C_k],$$

where P.D. is the Poincaré dual.

Note also that

$$\operatorname{ch}^c(S_k^\vee) = \begin{cases} -\operatorname{ch}^c(S_k) & \text{for } k \neq 0, \\ \sum_{k=1}^r \operatorname{rank}\mathcal{R}_k \, \mathrm{P.D.}[C_k] + \Omega & \text{for } k = 0. \end{cases}$$

This together with $(S_k^\vee, S_l) = (\operatorname{ch}^c(S_k^\vee), \operatorname{ch}^c(S_l)) = 2\delta_{kl} - a_{kl}$ determine the intersection pairing. In particular, we have $(S_k, S_l) = (\mathrm{P.D.}[C_k], \mathrm{P.D.}[C_l]) = a_{kl} - 2\delta_{kl}$ for $k, l \neq 0$. Thus we have checked the identification of the intersection pairing and the decomposition of tensor products without using the classification of simple singularities.

There are possibly many ways to prove Propositions 6.2,6.4, since we have more or less explicit description of X (e.g. [10]). However, we would like to remark that these follows from the theory of the quiver varieties introduced by the second author [17] without the knowledge of the explict description. The variety X is an example of quiver varieties associated with the extended Dynkin diagram. The

second author defined the Hecke correspondence in the product of quiver varieties [ibid., 10.4]. The locus C_k ($k \neq 0$) is an example of Hecke correspondence, where the one factor is X and the other factor is the scheme parametrising 0-dimensional subschemes Z of $\mathbb{C}^2$ such that

(1) Z is invariant under the G-action,
(2) $H^0(\mathcal{O}_Z) \oplus \rho_k$ is the regular representation of G,

which consists of one point. The assertions in Proposition 6.2 follow from [ibid., Lemma 10.10 and its proof]. (We omit details.) Proposition 6.4 is, in fact, much easier to prove, and holds also in 3-dimensional case. Since $\mathcal{R}_0$ is the trivial rank 1 bundle over X, $\wedge B$ is not surjective only when $\wedge B = 0$. If $\wedge B = 0$ at $Z \in X$, B is nilpotent as an endomorphism of $H^0(\mathcal{O}_Z) = A/I$, hence contained in $\pi^{-1}(0)$. Conversely if $Z \in \pi^{-1}(0)$, we have a filtration on $H^0(\mathcal{O}_Z)$, under which B is strictly upper triangular by the Hilbert criterion as in [19]. It implies $\wedge B = 0$.

7. Toric resolution: the case $G \subset \mathrm{SL}_3(\mathbb{C})$ abelian

In this section, we assume G is an abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. It was proved by Nakamura [21] that X is a crepant resolution of $\mathbb{C}^3/G$ under this assumption. (There is also an explanation of Nakamura's proof by Reid [22].)

Theorem 7.1 (Nakamura[21]). *If G is a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$, then X is a crepant resolution of $\mathbb{C}^3/G$.*

In this section, we prove the following which is stronger than the smoothness of X:

Theorem 7.2. *Under the same assumption as Theorem 7.1, the following holds:*

$$\dim \mathrm{Hom}_A(I_1, A/I_2)^G = \begin{cases} 3, & \text{when } I_1 = I_2, \\ 1, & \text{when } I_1 \neq I_2. \end{cases} \tag{7.3}$$

The rest of this section is devoted to the proof of Theorem 7.2. Though the smoothness of X, i.e. (7.3) for $I_1 = I_2$, is contained in Nakamura's result [21], we give its proof for the sake of the reader. Note also that the crepantness of X follows from Theorem 5.13 and the result in this section. Our proof for the smoothness is almost the same as Nakamura's, and the technique (e.g., the use of the diagram $\mathfrak{J}$, Lemma 7.7) is due to him.

By changing coordinates, we may assume G is diagonal. We consider an action of the three dimensional torus on $\mathbb{C}^3$ defined by $t \cdot (x, y, z) = (t_1 x, t_2 y, t_3 z)$ for $t = (t_1, t_2, t_3)$. It induces an action on the Hilbert scheme, commuting with the action of G. Thus this action induces the torus action on X.

In the sequel, the fixed points of the torus action will play the crucial role. These correspond to ideals generated by monomials. First we will classify all these ideals. Then we will check (7.3) for all these ideals. This implies (7.3) for general I_1, I_2 as explained later.

(a). **Classification of fixed points.** The inclusion $G \subset \mathrm{SL}_3(\mathbb{C})$ induces an action of G on the coordinate ring $A = \mathbb{C}[x, y, z]$. It decomposes into the sum of irreducible representations, which are of the forms $\mathbb{C}x^l y^m z^n$ for some l, m, $n \geq 0$. If $I \in X$, then A/I is the regular representation. Hence each irreducible representation appears in A/I with multiplicity one. This implies the following very useful lemma, which will be used throughout in this section.

Lemma 7.4. *Let $I \in X$. Suppose that $x^l y^m z^n$ and $x^{l'} y^{m'} z^{n'}$ are two different monomials which give the isomorphic irreducible representation of G. Then at least one of them is contained in I.*

For the study of ideals, we use the following graphical description of monomials in $A/(xyz)$:

$$\begin{array}{ccccccccc} & \cdots & & & & & & & \\ & & y^2 & & & & & & \\ & & & y & xy & x^2y & & & \\ y^2z^2 & & yz & & 1 & & x & x^2 & x^3 & \cdots \\ & & & z & xz & & & & \\ & & \cdots & & & & & & \end{array}$$

Here xyz is not drawn since it is always in ideals as it gives an isomorphic representation as 1.

Proposition 7.5 (cf. [22, 7.2])**.** *The ideal I which is fixed by the torus action is written as one of the following:*

(A) $$\langle x^{a+d-1},\ y^{b+e-1},\ z^{c+f-1},\ x^a y^e,\ y^b z^f,\ z^c x^d,\ xyz \rangle$$

(B) $$\langle x^{a+d},\ y^{b+e},\ z^{c+f},\ x^a y^e,\ y^b z^f,\ z^c x^d,\ xyz \rangle$$

where $a, b, c, d, e, f > 0$.

Moreover, x^{a+d-1} (resp. x^{a+d}) and $y^{b-1}z^{f-1}$ give the isomorphic representation in type (A) (resp. (B)). Similar conditions hold if we exchange x, y and z.

In above description, there are degenerate cases, for example, $e = 1$ in (A) where $y^b z^f$ is not a generator. But we could determine a, b, c, d, e and f so that the conditions of the isomorphic representations above hold (see Lemma 7.7). We use the above description even in degenerate cases as convention.

The proof of this proposition occupies this subsection. Let I be an ideal which is fixed by the torus action and generated by monomials. By Lemma 7.4, each irreducible representation of G corresponds to the unique monomial in I^c.

Let α, β, γ be the exponents of the generators of $x^\bullet$, $y^\bullet$, $z^\bullet$ respectively.

Lemma 7.6. *Suppose $x^a y^e \in I$, $x^{a-1} y^e \notin I$, and $x^a y^{e-1} \notin I$ for $a, e \geq 1$, i.e. $x^a y^e$ is a generator of I. Then the unique monomial in I^c which has the isomorphic representation as $x^a y^e$ is $z^{\gamma-1}$.*

Proof. Let us consider the monomial φ in I^c which gives the isomorphic representation of G as $x^a y^e$.

If $\varphi = x^p y^q$ with $p \geq 1$, then $x^{p-1} y^q$ gives the isomorphic representation as $x^{a-1} y^e$. From the assumption $x^{a-1} y^e \notin I$ and Lemma 7.4, we have $x^{p-1} y^q \in I$. Therefore $x^p y^q \in I$. This is a contradiction. Exchanging y and z, we can also eliminate the case $\varphi = x^p z^r$ for $p \geq 1$.

Next assume $\varphi = y^q z^r$ with $q \geq 1$. Then $y^{q-1} z^r$ gives the isomorphic representation as $x^a y^{e-1}$. Then we have $y^{q-1} z^r \in I$ by Lemma 7.4. This means that $y^q z^r \in I$ and contradicts with the definition of φ. Thus we have $\varphi = z^r$. Since z^{r+1} gives the isomorphic representation as $x^a y^e$, we have $z^{r+1} \in I$ by Lemma 7.4. Hence z^{r+1} must be a generator of I, i.e. $z^r = z^{\gamma-1}$. □

This lemma implies that I has at most one generator of the form $x^a y^e$ $(a, e \geq 1)$: If $x^a y^e$ and $x^{a'} y^{e'}$ are generators, both $x^{a-1} y^{e-1}$ and $x^{a'-1} y^{e'-1}$ give the isomorphic representation as the generator z^γ. This contradicts with Lemma 7.4.

Although we assume $x^a y^e$ is a generator in the lemma, we can show the following even if there is no generator of that form (i.e. degenerate case).

Lemma 7.7. *The monomial in I^c which has the isomorphic representation as z^γ is of the form $x^{a-1} y^{e-1}$ for some $a, e \geq 1$.*

This lemma can be proved as Lemma 7.6. So we omit the proof.

By above lemmas, the ideal is generated by x^α, y^β, z^γ, $x^a y^e$, $y^b z^f$, $z^c x^d$ and xyz where α, β, γ, a, b, c, d, e, $f > 0$. We draw a diagram by placing hexagons at the monomials in I^c. (See figure 7.1.) Let us call this *junior diagram* $\mathfrak{J}$. (This is called a G-graph in [21].) We may also write the representations corresponding to the monomials. In dimension 2, we had a similar correspondence between idelas and Young diagrams (cf. [20, Chapter 7]).

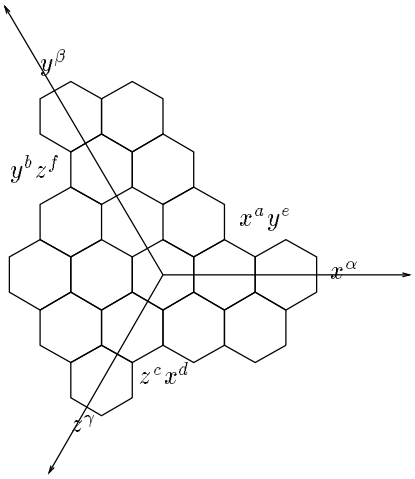

FIGURE 7.1. Junior diagram

Definition 7.8. Let φ be a monomial in $A/(xyz)$. Let φ_0 be the monomial in I^c which has isomorphic representation of G as φ. (It exists and is unique since A/I is the regular representation.) We move the junior diagram $\mathfrak{J}$ by the parallel transport which maps φ_0 to φ, and denote the transported diagram by $\mathfrak{J}(\psi)$. We also call $\mathfrak{J}(\varphi)$ the *junior diagram.*

The parallel transport respects representations corresponding to monomials. Hence each representation appears at the same position in each junior diagram. Each monomials in $\varphi \in A/(xyz)$ belongs to the unique junior diagram $\mathfrak{J}(\varphi)$. Thus each junior diagram is a kind of 'fundamental domain' of $A/(xyz)$.

To complete the proof of Proposition 7.5, we have to show the relation between the exponents of the generators. First we will see the following:

Lemma 7.9. *We have one of the following:*

$$\alpha \leq a + d - 1, \quad \beta \leq b + e - 1, \quad \gamma \leq c + f - 1, \tag{7.10a}$$

$$\alpha \geq a + d, \qquad \beta \geq b + e, \qquad \gamma \geq c + f. \tag{7.10b}$$

Proof. **Step 1**. Assume

$$b + e \leq \beta, \tag{7.11a}$$

and

$$\alpha \leq a + d - 1. \tag{7.11b}$$

Since $x^a y^e$ has the isomorphic representation as $z^{\gamma-1}$, $x^\alpha y^e$ has the isomorphic representation as $z^{\gamma-1}x^{\alpha-a}$, which is in $\mathfrak{J} = \mathfrak{J}(1)$ by (7.11b).

On the other hand, x^α has the isomorphic representation as $y^{b-1}z^{f-1}$. Hence $x^\alpha y^e$ has the isomorphic representation as $y^{b+e-1}z^{f-1}$, which is in $\mathfrak{J}(1)$ by (7.11a). As $b, e \geq 1$, $y^{b+e-1}z^{f-1} \neq x^{\alpha-a}z^{\gamma-1}$. Therefore there are two monomials in $\mathfrak{J}(1)$ which has the isomorphic respresentations as $x^\alpha y^e$. This contradicts with Lemma 7.4. Hence we have either

$$b + e - 1 \geq \beta, \tag{7.12a}$$

or

$$\alpha \geq a + d. \tag{7.12b}$$

Step 2. Suppose $b + e - 1 \geq \beta$. Exchanging x and y in step 1, we have either

$$a + d - 1 \geq \alpha,$$

or

$$\beta \geq b + e.$$

By the assumption, the second case does not occur. Thus we have $a + d - 1 \geq \alpha$. Exchanging x and z, we also have $c+f-1 \geq \gamma$, and hence (7.10a). Similar argument shows that $\alpha \geq a + d$ implies (7.10b). □

Now we complete the proof of Proposition 7.5. First we consider the case (7.10a) and let us show that

$$x^a y^\beta = x^a y^{b+e-1}. \tag{7.13}$$

Since we have $\beta \leq b + e - 1$, it is enough to show that the condition

$$\beta < b + e - 1 \tag{7.14}$$

leads to a contradiction.

Let us study the monomials near $x^{a-1}y^\beta$. Consider the parallel transport mapping $x^{a-1}y^\beta$ to $y^{\beta-e+1}z^\gamma$, which respects representations of G. It maps $x^a y^{\beta+1}$ to $y^{\beta-e+1}z^{\gamma-1}$ which is in $\mathfrak{J}(1)$ by the assumption (7.14). The monomial $x^a y^\beta$ is also mapped to a monomial $y^{\beta-e}z^{\gamma-1} \in \mathfrak{J}(1)$.

On the other hand, $y^{\beta-e+1}z^\gamma$ is not in $\mathfrak{J}(1)$. Hence $x^a y^\beta$ and $x^a y^{\beta+1}$ are not in $\mathfrak{J}(x^{a-1}y^\beta)$. Since $x^{a-1}y^{\beta-1} \in \mathfrak{J}(1)$, it is not in $\mathfrak{J}(x^{a-1}y^\beta)$ either. In summary, for $x^{a-1}y^\beta$, the monomials in the upper right, the right and the lower right are not in $\mathfrak{J}(x^{a-1}y^\beta)$. If φ is the monomial in $\mathfrak{J}(1)$, which has the isomorphic representation as $x^{a-1}y^\beta$, it also has the same property. Hence φ must be $x^{\alpha-1}$.

Thus, x^α has the isomorphic representation as $x^a y^\beta$, and hence as $y^{\beta-e}z^{\gamma-1}$, which is in $\mathfrak{J}(1)$ by (7.14). However, x^α has the isomorphic representation as $y^{b-1}z^{f-1} \in \mathfrak{J}(1)$, hence we must have $\gamma = f$, $\beta - e = b - 1$ by Lemma 7.4. But the latter contradicts with the assumption (7.14).

Therefore the condition (7.13) holds in this case. Exachanging x, y and z, we have

$$\alpha = a+d-1, \quad \beta = b+e-1, \quad \gamma = c+f-1, \tag{7.15}$$

i.e. the ideal is of type (A).

Next consider the case (7.10b). We will show that

$$x^a y^\beta = x^{\alpha-d} y^\beta. \tag{7.16}$$

Since we already have $a \leq \alpha - d$, it is enough to show that

$$a < \alpha - d \tag{7.17}$$

leads to a contradiction. We study the monomials near $x^a y^{\beta-1}$. Consider the parallel transport mapping $x^a y^{\beta-1}$ to $x^{a+d} z^c \notin \mathfrak{J}(1)$. It maps $x^{a+1} y^\beta$ to $x^{a+d} z^{c-1}$, which is in $\mathfrak{J}(1)$ by the assumption (7.17). Thus $x^{a+1} y^\beta$ (and hence $x^a y^\beta$ also) is not in $\mathfrak{J}(x^a y^{\beta-1})$. We have $x^a y^{\beta-1} \notin \mathfrak{J}(1)$ thanks to $\beta - 1 \geq b + e - 1 \geq e$, while we have $x^{a-1} y^{\beta-1} \in \mathfrak{J}(1)$. Hence $x^{a-1} y^{\beta-1}$ is not in $\mathfrak{J}(x^a y^{\beta-1})$ either. By the same argument as above, we conclude that the monomial in $\mathfrak{J}(1)$ which has isomorphic representation as $x^a y^{\beta-1}$ must be $y^{\beta-1}$. Therefore $x^{a-1} y^{\beta-1}$ has the isomorphic representation as $y^\beta z$, and hence as $z^c x^{d-1}$.

However, as $c < \gamma$ by (7.10b), we have $z^c x^{d-1} \in \mathfrak{J}(1)$. Thus both $x^{a-1} y^{\beta-1}$ and $z^c x^{d-1}$ are in $\mathfrak{J}(1)$, and

$$x^{a-1} y^{\beta-1} \neq z^c x^{d-1}$$

as $c \geq 1$. This is a contradiction. Thus we have (7.16). Exchanging x, y and z, we have

$$\alpha = a+d, \quad \beta = b+e, \quad \gamma = c+f, \tag{7.18}$$

i.e. it is of type (B).

Combining the above two cases, we have only two types (A) and (B) for the generator of the ideal I of a fixed point in X. This completes the proof of Proposition 7.5.

(b). **Smoothness.** Using the above description, we obtain the smoothness of X:

Theorem 7.19. *X is nonsingular of dimension 3. Moreover, X is irreducible.*

This subsection is devoted to the proof of this theorem.

Lemma 7.20. *At every fixed point I, the dimension of the Zariski tangent space is three, i.e.*

$$\dim \mathrm{Hom}_A(I, A/I)^G = 3.$$

Proof. Let Φ be a G-equivariant A-homomorphism $\Phi\colon I \to A/I$. Let α, β, γ be the exponents of the generators of $x^\bullet$, $y^\bullet$, $z^\bullet$ as before.

Let us consider the image of generators of I:

$$\begin{aligned}
&\Phi(x^\alpha) = p y^{b-1} z^{f-1} \bmod I, \quad \Phi(x^a y^e) = s z^{\gamma-1} \bmod I,\\
&\Phi(y^\beta) = q z^{c-1} x^{d-1} \bmod I, \quad \Phi(y^b z^f) = t x^{\alpha-1} \bmod I, \quad \Phi(xyz) = v \bmod I,\\
&\Phi(z^\gamma) = r x^{a-1} y^{e-1} \bmod I, \quad \Phi(z^c x^d) = u y^{\beta-1} \bmod I,
\end{aligned}$$

where $p, q, r, s, t, u, v \in \mathbb{C}$. Here we determine the image so that it has the isomorphic representation of G as the generator.

First suppose I is of type (A), i.e. $\alpha = a+d-1$, $\beta = b+e-1$, $\gamma = c+f-1$. Let us consider the image of $x^\alpha y^e = x^{a+d-1}y^e$. We have

$$\Phi(x^{a+d-1}y^e) = py^{b+e-1}z^{f-1} \bmod I = 0$$

as $y^{b+e-1}z^{f-1} \in I$. On the other hand, we have

$$\Phi(x^{a+d-1}y^e) = sx^{d-1}z^{c+f-2} \bmod I.$$

Since $x^{d-1}z^{c+f-2} \notin I$, we get $s = 0$. Exchanging x, y and z, we obtain $t = u = 0$. Considering the image of $x^a y^e z$ in two ways, we similarly get $v = 0$. Therefore the dimension of the Zariski tangent space at I is three.

Next suppose I is of type (B), i.e. $\alpha = a+d$, $\beta = b+e$, $\gamma = c+f$. Then we have

$$\Phi(x^{a+d}y^e) = py^{b+e-1}z^{f-1} \bmod I,$$

and $y^{b+e-1}z^{f-1} \notin I$. But

$$\Phi(x^{a+d}y^e) = sx^d z^{c+f-1} \bmod I = 0$$

as $x^d z^{c+f-1} \in I$. Then we have $p = 0$. After exchanging x, y and z, $q = r = 0$ holds. We also get $v = 0$ as in the case (A). Thus the assertion holds in any cases. □

Lemma 7.21. *X is nonsingular at fixed points of the torus action.*

Proof. Let $I \in X$ be a fixed point of the torus action. First suppose I is of type (A). Let us consider the following defining equation:

$$\begin{aligned} &x^{a+d-1} = \lambda y^{b-1}z^{f-1}, \quad x^a y^e = \lambda\mu z^{c+f-2}, \\ &y^{b+e-1} = \mu z^{c-1}x^{d-1}, \quad y^b z^f = \mu\nu x^{a+d-2}, \quad xyz = \lambda\mu\nu, \\ &z^{c+f-1} = \nu x^{a-1}y^{e-1}, \quad z^c x^d = \nu\lambda y^{b+e-2}. \end{aligned} \tag{7.22}$$

This equations determines an ideal in the neighborhood of the fixed point. ($\lambda = \mu = \nu = 0$ is the fixed point.) If $\lambda\mu\nu \neq 0$, we have

$$\#G = 4 - 2(a+b+c+d+e+f) + ab + ac + ae + bc + bf + dc + de + df + ef$$

distinct solutions of the above equation. Hence it corresponds to a G-orbit consisting of distinct $\#G$-points, and has the Zariski tangent space of dimension 3. In particular, X is of dimension 3 in the neighbourhood of I. Since the Zariski tangent space at I is of dimension 3, it implies that X is nonsingular at I.

Next suppose I is of the type (B). We consider the defining equations

$$\begin{aligned} &x^{a+d} = \nu\lambda y^{b-1}z^{f-1}, \quad x^a y^e = \lambda z^{c+f-1}, \\ &y^{b+e} = \lambda\mu z^{c-1}x^{d-1}, \quad y^b z^f = \mu x^{a+d-1}, \quad xyz = \lambda\mu\nu, \\ &z^{c+f} = \mu\nu x^{a-1}y^{e-1}, \quad z^c x^d = \nu y^{b+e-1}. \end{aligned} \tag{7.23}$$

This equation has

$$\#G = 1 - (a+b+c+d+e+f) + ab + ac + ae + bc + bf + dc + de + df + ef$$

distinct solutions if $stu \neq 0$. The above argument shows that X is nonsingular at I in this case. □

Proof of Theorem 7.19. We take a generic one-parameter subgroup $\lambda\colon \mathbb{C}^* \to (\mathbb{C}^*)^3$ such that $\lambda(t) \to 0$ as $t \to 0$. For any $I \in X$, $\lambda(t)^* I$ converges to a fixed point of the torus action. Thus X is nonsingular at I by Lemma 7.21. Hence X is nonsingular everywhere. The argument also shows that each connected component of X contains fixed points. However, the fixed points are contained the component containing G-orbits of distinct points. Therefore, X must be connected. □

Since X is nonsingular and has an action of 3-dimensional torus with an open dense orbit, we have

Corollary 7.24. *X is a toric variety.*

The coordinate neighbourhoods (7.22),(7.23) are affine charts around fixed points.

The fan corresponding to X are described in [21]. So we do not reproduce it here.

(c). **The case when I_1 and I_2 are in a common affine chart.** Our remaining task is to check (7.3) for I_1 and I_2 are contained in different affine charts of (7.22) or (7.23). In this subsection, we check it when I_1 and I_2 are in a common affine chart given by (7.22). One can check in the case when both are in a chart given by (7.23) in a similar way, so we omit the proof.

Suppose I_1 is given by (7.22) and I_2 is given also by (7.22) with λ, μ, ν are replaced by λ', μ', ν'. (a, b, c, d, e, f are common.) By the assumption $I_1 \neq I_2$, we have $(\lambda, \mu, \nu) \neq (\lambda', \mu', \nu')$. We may assume $\lambda \neq \lambda'$. Let $\Phi\colon I_1 \to A/I_2$ be a G-equivariant A-homomorphism. We determine images of generators:

$$\begin{aligned}
&\Phi(x^{a+d-1} - \lambda y^{b-1} z^{f-1}) = p y^{b-1} z^{f-1} \bmod I_2,\\
&\Phi(y^{b+e-1} - \mu z^{c-1} x^{d-1}) = q z^{c-1} x^{d-1} \bmod I_2,\\
&\Phi(z^{c+f-1} - \nu x^{a-1} y^{e-1}) = r x^{a-1} y^{e-1} \bmod I_2,\\
&\Phi(x^a y^e - \lambda\mu z^{c+f-2}) = s z^{c+f-2} \bmod I_2,\\
&\Phi(y^b z^f - \mu\nu x^{a+d-2}) = t x^{a+d-2} \bmod I_2,\\
&\Phi(z^c x^d - \nu\lambda y^{b+e-2}) = u y^{b+e-2} \bmod I_2,\\
&\Phi(xyz - \lambda\mu\nu) = v \bmod I_2.
\end{aligned}$$

Consider the image of

$$(z^c x^d - \nu\lambda y^{b+e-2}) x^{a-1} = (x^{a+d-1} - \lambda y^{b-1} z^{f-1}) z^c + \lambda (z^{c+f-1} - \nu x^{a-1} y^{e-1}) y^{b-1}.$$

We have

$$\begin{aligned}
&u x^{a-1} y^{b+e-2} \bmod I_2 = p y^{b-1} z^{c+f-1} \bmod I_2 + \lambda r x^{a-1} y^{b+e-2} \bmod I_2\\
&= (\nu' p + \lambda r) x^{a-1} y^{b+e-2} \bmod I_2.
\end{aligned}$$

Since $x^{a-1} y^{b+e-2} \notin I_2$, we have $u = \nu' p + \lambda r$. Exchanging x and z, we get $u = \lambda' r + \nu p$. Thus we have $r = (\nu - \nu')p/(\lambda - \lambda')$. Then exchanging x, y and z, we get $s = \mu' p + \lambda q$, $t = \nu' q + \mu r$, $q = (\mu - \mu')p/(\lambda - \lambda')$.

Next consider the image of

$$\begin{aligned}&(xyz-\lambda\mu\nu)x^{a+d-2}y^{e-1}\\ =\ &(x^{a+d-1}-\lambda y^{b-1}z^{f-1})y^e z\\ &\quad+\lambda(y^{b+e-1}-\mu z^{c-1}x^{d-1})z^f+\lambda\mu(z^{c+f-1}-\nu x^{a-1}y^{e-1})x^{d-1}.\end{aligned}$$

We get

$$\begin{aligned}&v x^{a+d-2}y^{e-1} \bmod I_2\\ =\ &p y^{b+e-1}z^f \bmod I_2+\lambda q z^{c+f-1}x^{d-1} \bmod I_2+\lambda\mu r x^{a+d-2}y^{e-1} \bmod I_2\\ =\ &(\mu'\nu' p+\lambda\nu' q+\lambda\mu r)x^{a+d-2}y^{e-1} \bmod I_2.\end{aligned}$$

Since $x^{a+d-2}y^{e-1}\notin I_2$, we have $v=\mu'\nu' p+\lambda\nu' q+\lambda\mu r$. Thus q, r, s, t, u, v are determined by p. Hence we have $\dim \mathrm{Hom}_A(I_1, A/I_2)^G = 1$ in this case.

(d). **Reduction to the case when I_1, I_2 are fixed points.** Our remaining task is to check (7.3) for $I_1 \neq I_2$. In fact, it is not necessary to check (7.3) for all I_1, I_2 thanks to the torus action. As above, we take a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ and consider the limit of $\lambda(t)^* I_1$, $\lambda(t)^* I_2$ when $t \to 0$. We may assume both converge to fixed points of the torus action. If $\lambda(t)^* I_1$ and $\lambda(t)^* I_2$ converge to the same point, it means that they are contained in a neighborhood of the diagonal for sufficiently small t. This case was treated in the previous subsection. Thus we may assume that $\lambda(t)^* I_1$ and $\lambda(t)^* I_2$ converge to different points if $I_1 \neq I_2$. By the semicontinuity, we have

$$\dim \mathrm{Hom}_A(I_1, A/I_2)^G \le \dim \mathrm{Hom}_A(\lim_{t\to 0}\lambda(t)^* I_1, A/\lim_{t\to 0}\lambda(t)^* I_2))^G.$$

Since the left hand side is bounded by 1 from below (see Lemma 4.4), it is enough to show the right hand side is 1. Thus we may assume I_1 and I_2 are different fixed points of the torus action.

(e). **Classification of the pair of fixed points.** In the remaining of this paper, we assume I_1 and I_2 are different fixed point of the torus action. By Proposition 7.5, we have four possibilities:

(AA) I_1 of type (A), I_2 of type (A),

(BB) I_1 of type (B), I_2 of type (B),

(AB) I_1 of type (A), I_2 of type (B),

(BA) I_1 of type (B), I_2 of type (A).

In order to treat the cases (A) and (B) simultaneously, we write the exponents of generators of $x^\bullet$, $y^\bullet$, $z^\bullet$ by α, β, γ as before. Namely, $\alpha = a+d-1$ (resp. $a+d$) in case (A) (resp. (B)), etc. We put "prime" on the exponents of the generators for the ideal I_2. Namely, a', α', etc. In either cases, we have

(7.25)
$$a+d-1 \le \alpha \le a+d, \quad b+e-1 \le \beta \le b+e, \quad c+f-1 \le \gamma \le c+f,$$
$$a'+d'-1 \le \alpha' \le a'+d', \quad b'+e'-1 \le \beta' \le b'+e', \quad c'+f'-1 \le \gamma' \le c'+f'$$

by Proposition 7.5.

Lemma 7.26. *At least one of three generators x^α, y^β and z^γ of I_1 belongs to the ideal I_2.*

Proof. We assume $x^\alpha, y^\beta, z^\gamma \notin I_2$ and derive a contradiction. From this assumption we have

$$\alpha < \alpha', \quad \beta < \beta', \quad \gamma < \gamma'. \tag{7.27}$$

Since $x^{a-1}y^{e-1}$ and z^γ define the isomorphic representation of G, we have $x^{a-1}y^{e-1} \in I_2$ by Lemma 7.4, as $z^\gamma \notin I_2$ by the assumption. Then one of the following holds:

$$a' \le a-1 \quad \text{and} \quad e' \le e-1, \tag{7.28a}$$

$$a' > a-1 \quad \text{and} \quad \beta' \le e-1, \tag{7.28b}$$

$$\alpha' \le a-1 \quad \text{and} \quad e' > e-1. \tag{7.28c}$$

The case (7.28b) contradicts with (7.27), (7.25) as $\beta < \beta' \le e-1 < \beta$. Similarly we have a contradiction in the case (7.28c). Hence we must have (7.28a). Exchanging x, y and z we have

$$a' \le a-1,\ b' \le b-1,\ c' \le c-1,\ d' \le d-1,\ e' \le e-1,\ f' \le f-1.$$

Combining these with (7.25), we get

$$\alpha' \le a'+d' \le a+d-2 \le \alpha-1 < \alpha'.$$

This is a contradiction. □

Lemma 7.29. *Assume*

$$\beta' \le \beta \quad \textit{and} \quad \gamma' \le \gamma. \tag{7.30}$$

If $x^\alpha = x^{\alpha'}$, then $y^b z^f = y^{b'} z^{f'}$. The same holds if we exchange x, y and z.

Proof. We suppose $y^{b-1}z^{f-1} \ne y^{b'-1}z^{f'-1}$ and lead to a contradiction. Both $y^{b-1}z^{f-1}$ and $y^{b'-1}z^{f'-1}$ give the isomorphic representation as $x^\alpha = x^{\alpha'}$. Since $y^{b-1}z^{f-1} \notin I_1$, $y^{b'-1}z^{f'-1} \notin I_2$, Lemma 7.4 implies

$$y^{b-1}z^{f-1} \in I_2, \quad y^{b'-1}z^{f'-1} \in I_1.$$

From the first condition, we have one of the following:

$$b' \le b-1 \quad \text{and} \quad f' \le f-1, \tag{7.31a}$$

$$b' > b-1 \quad \text{and} \quad \gamma' \le f-1, \tag{7.31b}$$

$$\beta' \le b-1 \quad \text{and} \quad f' > f-1. \tag{7.31c}$$

Similarly the second condition implies the one of the following:

$$b \le b'-1 \quad \text{and} \quad f \le f'-1, \tag{7.32a}$$

$$b > b'-1 \quad \text{and} \quad \gamma \le f'-1, \tag{7.32b}$$

$$\beta \le b'-1 \quad \text{and} \quad f > f'-1. \tag{7.32c}$$

The condition (7.31a) contradicts with (7.32a) as

$$b \le b'-1 \le b-2.$$

The condition (7.31b) (resp. (7.31c)) is not compatible with (7.32a) and (7.25) because

$$\gamma' \le f-1 \le f'-2 \le \gamma'-2 \ (\text{resp. } \beta' \le b-1 \le b'-2 \le \beta'-2).$$

Hence the case (7.32a) does not hold.

And either the condition (7.32b) or (7.32c) contradicts with the conditions (7.25) and (7.30) as

Case (7.32b): $\gamma \le f' - 1 \le \gamma' - 1 \le \gamma - 1$,

Case (7.32c): $\beta \le b' - 1 \le \beta' - 1 \le \beta - 1$.

This completes the proof. □

Lemma 7.33. (1) *When the pair* (I_1, I_2) *is not of the type* (BA), *at least one of* x^α, y^β *and* z^γ *in* I_1 *does not belong to* I_2.

(2) *When the pair* (I_1, I_2) *is of the type* (BA), *one of the following two cases occurs:*

(a) *at least one of* x^α, y^β *and* z^γ *in* I_1 *does not belong to* I_2,

(b) *we have*

$$a = a',\ b = b' - 1,\ c = c',\ d = d',\ e = e',\ f = f' - 1 \tag{7.34}$$

or the one with x, y, z *exchanged. (See Figure 7.5.)*

Proof. Assume that x^α, y^β, $z^\gamma \in I_2$. Then we have

$$\alpha' \le \alpha,\ \beta' \le \beta,\ \gamma' \le \gamma.$$

After exchanging x, y, z if necessary, we may assume either of the following 3 cases occurs:

$$\alpha' < \alpha,\ \beta' < \beta,\ \gamma' \le \gamma, \tag{7.35a}$$

$$\alpha' < \alpha,\ \beta' = \beta,\ \gamma' = \gamma, \tag{7.35b}$$

$$\alpha' = \alpha,\ \beta' = \beta,\ \gamma' = \gamma. \tag{7.35c}$$

In case (7.35a), we have $x^{\alpha'}, y^{\beta'} \notin I_1$, and hence $y^{b'-1}z^{f'-1}$, $z^{c'-1}x^{d'-1} \in I_1$ by Lemma 7.4. From $y^{b'-1}z^{f'-1} \in I_1$, one of the following holds:

$$b \le b' - 1 \quad \text{and} \quad f \le f' - 1, \tag{7.36a}$$

$$b > b' - 1 \quad \text{and} \quad \gamma \le f' - 1, \tag{7.36b}$$

$$\beta \le b' - 1 \quad \text{and} \quad f > f' - 1. \tag{7.36c}$$

The case (7.36b) contradicts with (7.35a) and (7.25) for

$$\gamma' \le \gamma \le f' - 1 < \gamma'.$$

Similarly, (7.36c) contradicts with (7.35a). Hence we must have (7.36a). From the same consideration for $z^{c'-1}x^{d'-1}$, we have $c \le c' - 1$ and $d \le d' - 1$. Therefore we obtain

$$c + f \le c' + f' - 2.$$

However this inequality together with (7.25) contradicts with the assumption $\gamma' \le \gamma$. Thus (7.35a) is excluded.

Next consider the case (7.35b). By Lemma 7.29, we have $a = a', c = c', d = d', e = e'$. We repeat the same argument as in (7.35a) for the condition $x^{\alpha'} \notin I_1$ to get $b \le b' - 1$, $f \le f' - 1$. Combining with (7.25), we get $\beta' = \beta \le b + e \le b' + e' - 1 \le \beta'$. Hence we must have equalities, i.e. $\beta = b + e$, $\beta' = b' + e' - 1$ and $b = b' - 1$. In particular, the pair of the ideals (I_1, I_2) is of type (BA). Similarly we have $f = f' - 1$ by exchanging y and z. Thus we must have (7.34).

Finally, consider the case (7.35c). By Lemma 7.29, we have $a = a'$, $b = b'$, $c = c'$, $d = d'$, $e = e'$, $f = f'$. These together with (7.35c) means $I_1 = I_2$ which is excluded from the beginning. Then we proved the lemma. □

(f). **Division of cases.** Before starting the proof of Theorem 7.2 for $I_1 \neq I_2$, we divide cases by the number of generators x^α, y^β, z^γ belonged in I_2. By Lemma 7.26, the number is either 1, 2, or 3. After exchanging x, y, z, we have

Case A: $x^\alpha \in I_2,\ y^\beta \notin I_2,\ z^\gamma \notin I_2,$

Case B: $x^\alpha \in I_2,\ y^\beta \in I_2,\ z^\gamma \notin I_2,$

Case C: $x^\alpha \in I_2,\ y^\beta \in I_2,\ z^\gamma \in I_2.$

In case A, we have $\alpha' \leq \alpha$, $\beta' > \beta$, $\gamma' > \gamma$. The last two inequalities implies that $y^{\beta'}, z^{\gamma'} \in I_1$. Then we do not have $x^{\alpha'} \in I_1$ since (I_2, I_1) is not of the type Lemma 7.33(2)(b). Hence we must have $x^{\alpha'} \notin I_1$. In summary, we have

$$x^\alpha \in I_2,\ y^\beta \notin I_2,\ z^\gamma \notin I_2,\ x^{\alpha'} \notin I_1,\ y^{\beta'} \in I_1,\ z^{\gamma'} \in I_1.$$

Thus we have

$$\text{Case A} \Longleftrightarrow \alpha > \alpha',\ \beta < \beta',\ \gamma < \gamma'. \tag{7.37}$$

In case B, we have $\alpha' \leq \alpha$, $\beta' \leq \beta$, $\gamma' > \gamma$. We have $z^{\gamma'} \in I_1$. We further separate cases by the number of generators among $x^{\alpha'}$, $y^{\beta'}$ belonged to I_1, i.e. 0, 1, or 2. After exchanging x and y if necessary, we have

Case B1: $x^{\alpha'} \in I_1,\ y^{\beta'} \in I_1,$

Case B2: $x^{\alpha'} \notin I_1,\ y^{\beta'} \in I_1,$

Case B3: $x^{\alpha'} \notin I_1,\ y^{\beta'} \notin I_1.$

Since we have $\alpha \geq \alpha'$ (resp. $\beta \geq \beta'$), $x^{\alpha'} \in I_1$ (resp. $y^{\beta'} \in I_1$) is possible only when $\alpha = \alpha'$ (resp. $\beta = \beta'$). Thus we have

$$\text{Case B1} \Longleftrightarrow \alpha = \alpha',\ \beta = \beta',\ \gamma < \gamma', \tag{7.38}$$

$$\text{Case B2} \Longleftrightarrow \alpha > \alpha',\ \beta = \beta',\ \gamma < \gamma', \tag{7.39}$$

$$\text{Case B3} \Longleftrightarrow \alpha > \alpha',\ \beta > \beta',\ \gamma < \gamma'. \tag{7.40}$$

In case C, we have only one possibility by Lemma 7.33(2).

In fact, it is not necessary to check $\dim \mathrm{Hom}_A(I_1, A/I_2)^G = 1$ all cases: In the complex (4.1), we have seen that the second homology H^2 is the dual space of the first cohomology H^1 of the complex in which I_1 and I_2 are exchanged (see Lemma 4.4). If $I_1 \neq I_2$, then $H^0 = H^3 = 0$ (Lemma 4.2). Since the Euler characteristic of the complex is equals to 0, $H^1 = 0$ implies $H^2 = 0$, and hence $H^1 = 0$ of the complex with I_1 and I_2 exchanged. Therefore it is sufficient to show that $H^1 = 0$ for one of the two pairs (I_1, I_2) and (I_2, I_1). If we exchange I_1 and I_2, then the case A and B3, B1 and C are exchanged. So we only need to consider the cases A, B2, C.

Let $\Phi\colon I_1 \to A/I_2$ be a G-equivariant A-homomorphism. First of all, we have $\Phi(xyz) = 0$ as follows. Since the representation corresponding to xyz is trivial, we

have $\Phi(xyz) = v \bmod I_2$ for some $v \in \mathbb{C}$. Since $I_1 \neq I_2$, there exists $f \in I_1 \setminus I_2$. Then

$$vf \bmod I_2 = f\Phi(xyz) = \Phi(xyz\, f) = xyz\Phi(f) = 0 \bmod I_2.$$

Since $f \notin I_2$, we must have $v = 0$.

(g). **Case A.** First we consider the case A.

Lemma 7.41. (i) *$x^a y^e \in I_2$ and $z^c x^d \in I_2$.*
(ii) *$y^b z^\gamma \notin I_2$ and $y^\beta z^f \notin I_2$.*

Proof. (i) Since $z^{\gamma-1} \notin I_2$ by (7.37), we have $x^a y^e \in I_2$ by Lemma 7.4. Exchanging y and z, we also have $z^c x^d \in I_2$.

(ii) As $x^{\alpha'} \notin I_1$, we have $y^{b'-1} z^{f'-1} \in I_1$ by Lemma 7.4. Then one of the following holds:

(7.42a) $$b \leq b' - 1 \quad \text{and} \quad f \leq f' - 1,$$

(7.42b) $$b > b' - 1 \quad \text{and} \quad \gamma \leq f' - 1,$$

(7.42c) $$\beta \leq b' - 1 \quad \text{and} \quad f > f' - 1.$$

Now we suppose $y^b z^\gamma \in I_2$, then we have one of the following:

(7.43a) $$b' \leq b \quad \text{and} \quad f' \leq \gamma,$$

(7.43b) $$b' > b \quad \text{and} \quad \gamma' \leq \gamma,$$

(7.43c) $$\beta' \leq b \quad \text{and} \quad f' > \gamma.$$

The case (7.43b) contradicts with (7.37) because $\gamma < \gamma' \leq \gamma$. Similarly we have a contradiction in the case (7.43c). Thus we must have (7.43a). However the case (7.43a) is not compatible with any of (7.42a),(7.42b),(7.42c):

Case (7.42a): $$b' \leq b \leq b' - 1,$$

Case (7.42b): $$f' \leq \gamma \leq f' - 1,$$

Case (7.42c): $$\beta \leq b' - 1 \leq b - 1 \leq \beta - 1$$

Therefore we have $y^b z^\gamma \notin I_2$. By exchanging y and z, we also have $y^\beta z^f \notin I_2$. □

Thus we can draw junior diagrams for I_1 and I_2 as Figure 7.2.

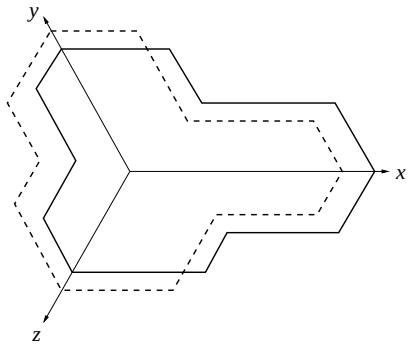

FIGURE 7.2. Case A: I_1 = solid lines, I_2 = dotted lines

Let $\Phi\colon I_1 \to A/I_2$ be a G-equivariant A-homomorphism. By Lemma 7.41 and (7.37), we can determine the image of generators of I_1 as follows:

$$\begin{aligned} &\Phi(x^\alpha) = py^{b-1}z^{f-1} \bmod I_2, && \Phi(x^a y^e) = sz^{\gamma-1} \bmod I_2,\\ &\Phi(y^\beta) = qy^\beta \bmod I_2, && \Phi(y^b z^f) = ty^b z^f \bmod I_2,\\ &\Phi(z^\gamma) = rz^\gamma \bmod I_2, && \Phi(z^c x^d) = uy^{\beta-1} \bmod I_2. \end{aligned}$$

Here we determine the image so that it has the isomorphic representation of G as the generator.

Since $\Phi(y^\beta z^f) = y^\beta \Phi(z^f) = z^f \Phi(y^\beta)$, we have $ty^\beta z^f \bmod I_2 = qy^\beta z^f \bmod I_2$. Thus we obtain $t = q$ by Lemma 7.41(ii). Similarly we have $t = r$ by exchanging y and z.

Consider the image of $z^\gamma x^d$. We have $\Phi(z^\gamma x^d) = rz^\gamma x^d \bmod I_2 = 0$ by Lemma 7.41(i). On the other hand, we have $\Phi(z^\gamma x^d) = uz^{\gamma-c}y^{\beta-1} \bmod I_2$ where $z^{\gamma-c}y^{\beta-1} \notin I_2$ by Lemma 7.41(ii). Therefore we have $u = 0$. Exchanging z and y, we also have $s = 0$.

Next see the image of $x^\alpha z^c$. We have $\Phi(x^\alpha z^c) = x^{\alpha-d}\Phi(z^c x^d) \bmod I_2 = 0$ because $u = 0$. And we have $\Phi(x^\alpha z^c) = py^{b-1}z^{c+f-1} \bmod I_2$ where $y^{b-1}z^{c+f-1} \notin I_2$ by Lemma 7.41(ii). Then we obtain $p = 0$.

Therefore we see $\dim \mathrm{Hom}_A(I_1, A/I_2)^G = 1$ in case A.

(h). **Case B2.** Next we study the case B2.

Lemma 7.44. (i) $y^\beta \in I_2$,
(ii) $x^a y^e \in I_2$ *and* $z^c x^d \in I_2$.
(iii) $f \le f' - 1$.

Proof. (i) Obvious from (7.39).

(ii) We have $x^a y^e \in I_2$ by $z^{\gamma-1} \notin I_2$ with Lemma 7.4. We also have $z^c x^d \in I_2$ since $y^{\beta-1} = y^{\beta'-1} \notin I_2$.

(iii) As $x^{\alpha'} \notin I_1$, we have $y^{b'-1}z^{f'-1} \in I_1$ by Lemma 7.4. As in the proof of Lemma 7.41(ii), we have (7.42a), (7.42b), or (7.42c). But (7.42c) contradicts with (7.39), as $\beta \le b' - 1 \le \beta' - 1 = \beta - 1$. Thus we have either (7.42a) or (7.42b). In either cases, we have the assertion. □

Lemma 7.45. (i) $y^{\beta-1}z^f \notin I_2$.
(ii) $y^{b-1}z^\gamma \notin I_2$.

Proof. (i) If we assume $y^{\beta-1}z^f \in I_2$, then one of the following holds:

(7.46a) $\quad b' \le \beta - 1 \quad \text{and} \quad f' \le f,$

(7.46b) $\quad b' > \beta - 1 \quad \text{and} \quad \gamma' \le f,$

(7.46c) $\quad \beta' \le \beta - 1 \quad \text{and} \quad f' > f.$

(7.46a) contradicts with Lemma 7.44(iii) as $f' \le f \le f' - 1$. (7.46b) is not compatible with (7.39) because $\beta - 1 < b' \le \beta' = \beta$. (7.46c) also contradicts with (7.39) as $\beta' \le \beta - 1 = \beta' - 1$. Thus $y^{\beta-1}z^f \notin I_2$.

(ii) If we assume $y^{b-1}z^\gamma \in I_2$, then one of the following holds:

(7.47a) $\quad b' \le b - 1 \quad \text{and} \quad f' \le \gamma,$

(7.47b) $\quad b' > b - 1 \quad \text{and} \quad \gamma' \le \gamma,$

(7.47c) $\quad \beta' \le b - 1 \quad \text{and} \quad f' > \gamma.$

The condition (7.47a) contradicts with (7.42a) because $b \leq b' - 1 \leq b - 2$. And (7.47a) is not compatible with (7.42b) as $f' \leq \gamma \leq f' - 1$. Thus the case (7.47a) is excluded. The condition (7.47b) (resp. (7.47c)) contradicts with the condition (7.39) as

$$\gamma' \leq \gamma < \gamma' \text{ (resp. } \beta' \leq b - 1 \leq \beta - 1 = \beta' - 1).$$

Thus the proof completes. □

We study the cases $b \neq \beta$ and $b = \beta$ separately. First assume $b \neq \beta$.

Lemma 7.48. $y^b z^\gamma \notin I_2$.

Proof. Now we assume $y^b z^\gamma \in I_2$ and derive a contradiction. From this assumption we have (7.43a), (7.43b) or (7.43c). The cases (7.43a) and (7.43b) lead to contradictions as in Lemma 7.41(ii). Thus we have case (7.43c). However it implies $\beta' \leq b \leq \beta = \beta'$, hence $\beta = b$. This contradicts with the assumption. □

By above, we can draw junior diagrams as Figure 7.3 (if $c = c'$, $d = d'$) or Figure 7.4 (otherwise).

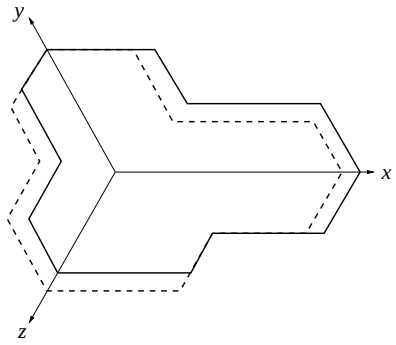

FIGURE 7.3. Case B2 ($c = c'$, $d = d'$)

By above discussions, we can put the images as follows:

$$\Phi(x^\alpha) = p y^{b-1} z^{f-1} \bmod I_2, \quad \Phi(x^a y^e) = s z^{\gamma-1} \bmod I_2,$$
$$\Phi(y^\beta) = q z^{c'-1} x^{d'-1} \bmod I_2, \quad \Phi(y^b z^f) = t y^b z^f \bmod I_2,$$
$$\Phi(z^\gamma) = r z^\gamma \bmod I_2, \quad \Phi(z^c x^d) = u y^{\beta-1} \bmod I_2.$$

Here $\Phi(y^\beta)$ was determined by using $\beta = \beta'$.

As in case A, we have $r = t$ by Lemma 7.48. We have $u = 0$ by Lemmas 7.44(ii) and 7.45(i) as in case A. We also obtain $p = 0$ by the same discussion in case A with Lemmas 7.44(ii) and 7.45(ii).

Now we consider the image of $y^\beta z^f$. We have $\Phi(y^\beta z^f) = t y^\beta z^f \bmod I_2 = 0$ by Lemma 7.44(i). On the other hand, we have $\Phi(y^\beta z^f) = q z^{c'+f-1} x^{d'-1} \bmod I_2$ and $z^{c'+f-1} x^{d'-1} \notin I_2$ because of Lemma 7.44(iii). Thus we get $q = 0$.

For the image of $x^a y^\beta$, we have $\Phi(x^a y^\beta) = x^a \Phi(y^\beta) = 0$ because $q = 0$, while $\Phi(x^a y^\beta) = s y^{\beta-e} z^{\gamma-1} \bmod I_2$ and $y^{\beta-e} z^{\gamma-1} \notin I_2$ by Lemma 7.48. Therefore we obtain $s = 0$ and we see the dimension is also one.

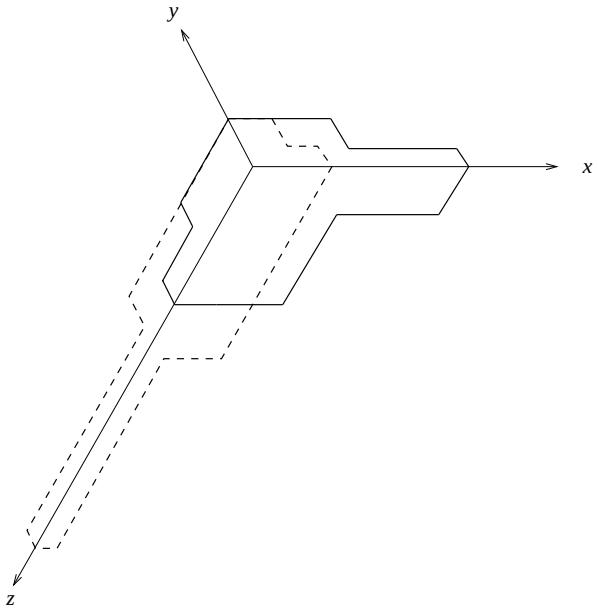

FIGURE 7.4. B2 ($c \neq c'$ or $d \neq d'$)

Next assume $b = \beta$. Then I_1 is of type (A). In this case, neither (7.42a) nor (7.42c) holds because $\beta = b \leq b' - 1 \leq \beta' - 1 = \beta - 1$. Thus the condition (7.42b) always holds, i.e.

$$b > b' - 1 \quad \text{and} \quad \gamma \leq f' - 1. \tag{7.49}$$

The images of the generators are written as follows:

$$\begin{aligned} &\Phi(x^\alpha) = py^{b-1}z^{f-1} \bmod I_2, \quad && \Phi(x^a y^e) = sz^{\gamma-1} \bmod I_2, \\ &\Phi(y^\beta) = qz^{c'-1}x^{d'-1} \bmod I_2, \\ &\Phi(z^\gamma) = rz^\gamma \bmod I_2, && \Phi(z^c x^d) = uy^{\beta-1} \bmod I_2. \end{aligned}$$

We have $u = 0$ and $p = 0$ as in case $b \neq \beta$.

Let us consider the image of $y^\beta z^\gamma$. We have $\Phi(y^\beta z^\gamma) = ry^\beta z^\gamma \bmod I_2 = 0$. On the other hand, we have $\Phi(y^\beta z^\gamma) = qz^{\gamma+c'-1}x^{d'-1} \bmod I_2$ where $z^{\gamma+c'-1}x^{d'-1} \notin I_2$ by (7.49). Therefore we have $q = 0$. Then we can use same discussion to obtain $s = 0$ as in case $b \neq \beta$. Thus we have the assertion in this case.

(i). **Case C.** In case C, we already see the relation between the exponents of the generators of the ideals I_1 and I_2 in Lemma 7.33.

Then we can write as follows:

$$\begin{aligned} &\Phi(x^\alpha) = py^{b-1}z^{f-1} \bmod I_2, && \Phi(x^a y^e) = sz^{\gamma-1} \bmod I_2, \\ &\Phi(y^\beta) = qz^{c-1}x^{d-1} \bmod I_2, && \Phi(y^b z^f) = ty^b z^f \bmod I_2, \\ &\Phi(z^\gamma) = rx^{a-1}y^{e-1} \bmod I_2, && \Phi(z^c x^d) = uy^{\beta-1} \bmod I_2. \end{aligned}$$

By similar discussions as in case A and B2, we obtain $p = q = r = s = u = 0$. Then we have the assertion in case C.

Therefore we proved Theorem 7.2.

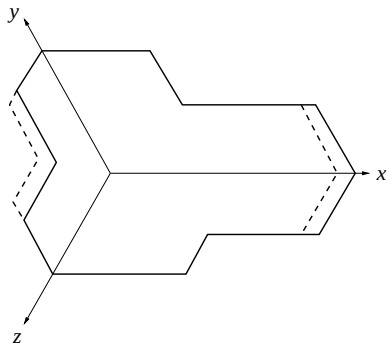

FIGURE 7.5. Case C

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