Geometric construction of representations of affine algebras

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International Congress of Mathematicians
Beijing, China
2002/8/26

My research is on the crossing of

- Geometry = moduli spaces of coherent sheaves (instantons) on minimal resolutions of simple singularities $\mathbf{C}^2/\Gamma$ (ALE spaces)
- Representation theory = affine Lie algebras, quantum affine (or toroidal) algebras

The connection is very different from the conventional geometric intuition of affine Lie algebras. (e.g., the conformal field theory)

Applications in both fields

- Generating functions of signature (middle degree homology groups) of moduli spaces are modular forms.

  (Vafa-Witten : S-duality)

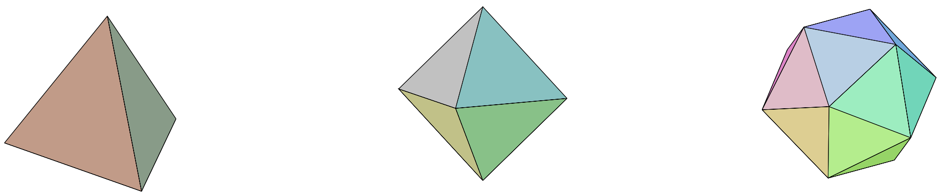
- Character formulae for representations of quantum affine algebras

Today I will explain mainly about the connection, and briefly about applications.

$\Gamma \subset \mathrm{SL}_2(\mathbf{C})$: a nontrivial finite subgroup

Classification (Klein)

$$\boxed{\begin{array}{c} \left\{ \left[\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right] \,\middle|\, \varepsilon^{n+1} = 1 \right\}, \text{ the cyclic group } \mathbf{Z}/(n+1)\mathbf{Z} \ (n \geq 1) \\ \text{the binary dihedral group of order } 4(n-1) \ (n \geq 4) \\ \text{the binary tetrahedral/octahedral/icosahedral group} \end{array}}$$



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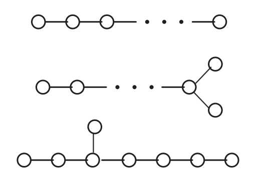
$\left\{\left[\begin{smallmatrix}\varepsilon & 0 \\ 0 & \varepsilon^{-1}\end{smallmatrix}\right] \,\middle|\, \varepsilon^{n+1}=1\right\}$, the cyclic group $\mathbf{Z}/(n+1)\mathbf{Z}$ $(n \geq 1)$
the binary dihedral group of order $4(n-1)$ $(n \geq 4)$
the binary tetrahedral/octahedral/icosahedral group

$\Longleftrightarrow$ (1. DuVal, 2. McKay)

Classification of simple Lie algebras (simply-laced)

A_n : $\mathfrak{sl}_{n+1}(\mathbf{C})$	
D_n $(n \geq 4)$: $\mathfrak{so}_{2n}(\mathbf{C})$	
E_n $(n = 6, 7, 8)$: exceptional Lie algebras	

Correspondence 1 = DuVal

- $0 \in \mathbf{C}^2/\Gamma$: simple singularity
- $\pi : M \to \mathbf{C}^2/\Gamma$: minimal resolution
- Exceptional set $\pi^{-1}(0) = \bigcup C_i$ $(C_i \cong \mathbf{P}^1)$
- Draw a diagram by the rule

  (1) vertex $\circ \longleftrightarrow C_i$,

  (2) $\underset{C_i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{C_j}{\circ} \Longleftrightarrow C_i \cap C_j \neq \emptyset$.

  $\Longrightarrow$ a Dynkin diagram of type ADE.

Correspondence 2 = McKay

- $\{\rho_0, \rho_1, \dots, \rho_n\}$: irreducible representations of Γ ($\rho_0 =$ trivial)
- Q : 2-dimensional representation given by $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$
- Tensor product decompsosition $Q \otimes \rho_i = \bigoplus_j a_{ij}\rho_j$.
- Draw a diagram by the rule

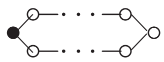
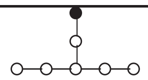
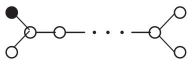
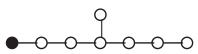
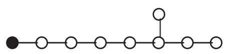
(1) vertex $\circ \longleftrightarrow \rho_i$,

(2) $\underset{\rho_i}{\circ}\!=\!=\!\underset{\rho_j}{\circ}$ $(a_{ij} = 2)$, $\underset{\rho_i}{\circ}\!-\!\!-\!\underset{\rho_j}{\circ}$ $(a_{ij} = 1)$, $\underset{\rho_i}{\circ}\quad\underset{\rho_j}{\circ}$ $(a_{ij} = 0)$

(**NB** $a_{ij} = a_{ji}$).

$\Longrightarrow$ an affine Dynkin diagram of type ADE.

Remove $\rho_0 \Longrightarrow$ the ordinary Dynkin diagram.

type	affine Dynkin graph		
A_n		E_6	
D_n		E_7	
		E_8	

(finite dimensional) simple Lie algebra :

$$\mathfrak{g} = \langle e_i, f_i, h_i \rangle_{i=1,\dots,n} \Big/ \left\{ \begin{matrix} [h_i,h_j]=0, \quad [h_i,e_j]=c_{ij}e_j \\ [e_i,f_j]=\delta_{ij}h_i, \quad \text{etc.} \end{matrix} \right\}$$

affine Lie algebra : $\widehat{\mathfrak{g}} = \langle e_i, f_i, h_i, d \rangle_{i=0,\dots,n}$/similar relations

Another realization : $\widehat{\mathfrak{g}} \cong \mathfrak{g} \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$

$$[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)c \quad \text{etc.}$$

Vertices of a Dynkin diagram are identified with simple roots, simple coroots, ...

- DuVal correspondence $\Longrightarrow$

$$\text{the dual root lattice } Q^\vee \text{ of } \mathfrak{g} \xrightarrow{\sim} H_2(\pi^{-1}(0), \mathbf{Z})$$

$$\text{simple coroots } h_i \longmapsto [C_i]$$

$$\text{Cartan matrix} \longleftrightarrow -\text{intersection matrix}$$

- McKay correspondence $\Longrightarrow$

$$\text{the root lattice } \widehat{Q} \text{ of } \widehat{\mathfrak{g}} \xrightarrow{\sim} \text{the representation ring } R(\Gamma) \text{ of } \Gamma$$

$$\text{simple roots } \alpha_i \longmapsto [\rho_i].$$

$$\text{affine Cartan matrix} \longleftrightarrow \text{the pairing}(\bullet, \textstyle\bigwedge_{-1} Q \otimes \bullet)$$

$$\textstyle\bigwedge_{-1} Q = \bigwedge^0 Q - \bigwedge^1 Q + \bigwedge^2 Q = 2\rho_0 - Q$$

NB. Gonzalez-Sprinberg and Verdier's direct correspondence

- $K(M) \overset{\text{def.}}{=}$ the Grothendieck ring of algebraic vector bundles on M

$$\begin{aligned} R(\Gamma) &\cong K(M) \\ \rho &\longmapsto \pi^*(\mathcal{O}_{\mathbf{C}^2} \otimes \rho)^\Gamma/\text{torsion} \end{aligned}$$

- $\mathcal{R}_i \overset{\text{def.}}{=} \text{image of } \rho_i \Longrightarrow \langle c_1(\mathcal{R}_i), [C_j]\rangle = \delta_{ij}$

Since Lie algebras are reconstructed from Dynkin diagrams, the correspondence is OK. But people have been asked

Question. Are there more direct correspondence between Γ and $\mathfrak{g}$?

Earlier Answer by Brieskorn, Slodowy

One can construct $\mathbf{C}^2/\Gamma$ in $\mathfrak{g}$.

New answer. Construct representations of the affine Lie algebra $\widehat{\mathfrak{g}}$ (and more) via varieties related to Γ.

Hilbert schemes of points

$\mathrm{Hilb}^n(\mathbf{C}^2)$: the Hilbert scheme of n points in $\mathbf{C}^2$

$= \{I \subset \mathbf{C}[x,y] \mid I \text{ is an ideal}, \dim \mathbf{C}[x,y]/I = n\}$

e.g., $I = \{f \mid f(p_1) = \cdots = f(p_n) = 0\}$ $(p_i \neq p_j,\, i \neq j)$

$S^n(\mathbf{C}^2) = \mathbf{C}^{2n}/S_n$: nth symmetric product of $\mathbf{C}^2$

$\pi\colon \mathrm{Hilb}^n(\mathbf{C}^2) \to S^n(\mathbf{C}^2)$: Hilbert-Chow morphism (proper)

Facts. (1) $\mathrm{Hilb}^n(\mathbf{C}^2)$ is a resolution of singularities of $S^n(\mathbf{C}^2)$. It is nonsingular of dimension $2n$. (Fogarty)

(2) $\mathrm{Hilb}^n(\mathbf{C}^2)$ has a holomorphic symplectic form. (Beauville)

(3) $\mathrm{Hilb}^n(\mathbf{C}^2)$ has a hyper-Kähler structure. (N)

Quiver varieties (or Γ-Hilbert schemes)

$\Gamma \curvearrowright \mathbf{C}^2 \Longrightarrow \Gamma \curvearrowright \mathrm{Hilb}^n(\mathbf{C}^2)$

$\mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma$: the fixed point set $=$ Γ-invariant ideals

$M(\mathbf{v}) \overset{\text{def.}}{=} \left\{ I \in \mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma \;\middle|\; [\mathbf{C}[x,y]/I] = \mathbf{v} \right\}$: quiver variety

($\mathbf{v}$ = an isomorphism class of a representation of $\Gamma \in R(\Gamma)$)

$\pi\colon M(\mathbf{v}) \to \left(S^n(\mathbf{C}^2)\right)^\Gamma$: restriction of Hilbert-Chow morphism

Proposition. (1) $M(\mathbf{v})$ is nonsingular of dimension $2(\rho_0, \mathbf{v}) - (\mathbf{v}, \bigwedge_{-1} Q \otimes \mathbf{v})$.

(2) $M(\mathbf{v})$ is connected. (Crawley-Boevey)

Examples. (1) (trivial). $M(0) = \{I = \mathbf{C}[x,y]\}$: a single point

(2) (trivial). $M(\rho_0) = \{\mathfrak{m}_0 \subset \mathbf{C}[x,y]\}$: a single point

(3) (Kronheimer, Ginzburg-Kapranov, Ito-Nakamura).

$\mathbf{v}$ = regular representation

$\Longrightarrow$
- $M(\mathbf{v})$ = the minimal resolution of $\mathbf{C}^2/\Gamma$
- $\pi\colon M(\mathbf{v}) \to (S^{\#\Gamma}(\mathbf{C}^2))^\Gamma = \mathbf{C}^2/\Gamma$
- $M(\mathbf{v}) \setminus \pi^{-1}(0) \cong \mathbf{C}^2 \setminus \{0\}/\Gamma$
  = ideals corresponding to free Γ-orbits

A geometric construction of an affine Lie algebra

- $M \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} M(\mathbf{v}) = \bigsqcup_{n} \operatorname{Hilb}^n (\mathbf{C}^2)^\Gamma$
- $M_0(\infty) \overset{\text{def.}}{=} \bigcup_n S^n(\mathbf{C}^2)^\Gamma$,

  where $S^n(\mathbf{C}^2)^\Gamma \subset S^{n+1}(\mathbf{C}^2)^\Gamma$ by $C \mapsto C + [0]$
- $L(\mathbf{v}) = M(\mathbf{v}) \cap \pi^{-1}(0)$ $(\pi \colon M(\mathbf{v}) \to (S^n(\mathbf{C}^2))^\Gamma)$
- $L \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} L(\mathbf{v})$
- $Z \overset{\text{def.}}{=} M \times_{M_0(\infty)} M = \{(I_1, I_2) \mid \pi(I_1) - \pi(I_2) \in \mathbf{Z}[0]\}$

  $= \bigsqcup_{\mathbf{v}^1, \mathbf{v}^2} Z(\mathbf{v}^1, \mathbf{v}^2) \quad (Z(\mathbf{v}^1, \mathbf{v}^2) = Z \cap M(\mathbf{v}^1) \times M(\mathbf{v}^2))$

Proposition. (1) $L(\mathbf{v})$ is a lagrangian subvariety of $M(\mathbf{v})$.

(2) $Z(\mathbf{v}^1, \mathbf{v}^2)$ is a lagrangian subvariety in $M(\mathbf{v}^1) \times M(\mathbf{v}^2)$.

Consider convolution product

$$*\colon H_{\mathrm{top}}(Z,\mathbf{C}) \otimes H_{\mathrm{top}}(Z,\mathbf{C}) \to H_{\mathrm{top}}(Z,\mathbf{C})$$

by $c * c' = p_{13*}\left(p_{12}^*(c) \cap p_{23}^*(c')\right)$. ($\cap$ is taken in $M \times M \times M$.)

Well-defined on the subspace

$$H_{\mathrm{top}}(Z,\mathbf{C}) \subset \prod_{\mathbf{v}^1,\mathbf{v}^2} H_{2\dim Z(\mathbf{v}^1,\mathbf{v}^2)}(Z(\mathbf{v}^1,\mathbf{v}^2),\mathbf{C})$$

consisting of elements $(F_{\mathbf{v}^1,\mathbf{v}^2})$ such that

(1) for fixed $\mathbf{v}^1$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ except finitely many $\mathbf{v}^2$,

(2) for fixed $\mathbf{v}^2$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ except finitely many $\mathbf{v}^1$.

$\Longrightarrow H_{\mathrm{top}}(Z,\mathbf{C})$ is an associative algebra with unit $[\Delta]$.

$H_{\mathrm{top}}(L,\mathbf{C}) \overset{\text{def.}}{=} \bigoplus_{\mathbf{v}} H_{2\dim L(\mathbf{v})}(L(\mathbf{v}),\mathbf{C})$.

$\Longrightarrow H_{\mathrm{top}}(L,\mathbf{C})$ is its module again by the convolution product.

Theorem. $\mathbf{U}(\widehat{\mathfrak{g}})$ = the universal enveloping algebra of the untwisted affine algebra $\widehat{\mathfrak{g}}$ corresponding to Γ.

(1) $\mathbf{U}(\widehat{\mathfrak{g}}) \xrightarrow{\exists} H_{\mathrm{top}}(Z,\mathbf{C})$: algebra homomorphism

(2) $H_{\mathrm{top}}(L,\mathbf{C})$ is the basic representation (in particular irreducible).

Remarks. (1) Motivated by a) Ringel, Lusztig's works on a construction of upper half parts of quantum enveloping algebras, and b) Ginzburg's work on a construction of the group ring of Weyl groups via convolution (Springer representations)

(2) If we replace $\mathrm{Hilb}^n(\mathbf{C}^2)$ by its higher rank analog, we can construct arbitrary integrable highest weight representations. (See proceeding for detail.)

(3) Case $\Gamma = 1$ (excluded).

- Heisenberg algebra $\xrightarrow{\exists} H_*(Z, \mathbf{C})$
- $H_*(L, \mathbf{C})$: Fock space representation

Proof of (1). Define the image of generators of $\mathbf{U}(\widehat{\mathfrak{g}})$ and check the defining relations.

$$\mathbf{U}(\widehat{\mathfrak{g}}) = \langle e_i, f_i, h_i, d\rangle_{i=0,\dots,n} \Big/ \left\{ \begin{matrix} [h_i,h_j]=0, & [h_i,e_j]=c_{ij}e_j \\ [e_i,f_j]=\delta_{ij}h_i, & \text{etc.} \end{matrix} \right\}$$

Image of $h_i = \sum_{\mathbf{v}} (\delta_{0i} - (\rho_i, \mathbf{C}\otimes\mathbf{v}))[\Delta M(\mathbf{v})]$.

Image of $d = \sum_{\mathbf{v}} (\rho_0, \mathbf{v})[\Delta(M(\mathbf{v}))]$.

Image of e_i = [Hecke correspondence]

$$\sum_{\mathbf{v}} [\{ (I_1, I_2) \in M(\mathbf{v}) \times M(\mathbf{v}+\rho_i) \mid I_1 \supset I_2 \}] .$$

Image of $e_i = \sum \pm [I_1 \leftrightarrow I_2]$

Proof of (2) is based on the structure of a crystal (in the sense of Kashiwara) on the set of irreducible components of L. (Defined by Lusztig, identified with the crystal of the basic representation of the quantum affine algebra by Kashiwara-Saito)

$$\widetilde{e}_i, \widetilde{f}_i \colon \operatorname{Irr} L \to \operatorname{Irr} L \sqcup \{0\}$$

$$\varepsilon_i(\Lambda) = \max\{n \mid \widetilde{e}_i^n \Lambda \neq 0\}, \qquad \varphi_i(\Lambda) = \max\{n \mid \widetilde{f}_i^n \Lambda \neq 0\}$$

$$\Longrightarrow f_i[\Lambda] = \pm r[\widetilde{f}_i \Lambda] + \sum_{\Lambda' : \varepsilon_i(\Lambda') > r} c_{\Lambda'}[\Lambda'] \qquad (r = \varepsilon_i(\Lambda) + 1)$$

NB. Base $\{\operatorname{Irr} L\}$ is different from the specialization of canonical base at $q = 1$.

Lower degree homology groups

For each $d \in \mathbf{Z}$,

$$H_{\mathrm{top}-d}(L, \mathbf{C}) \overset{\text{def.}}{=} \bigoplus_{\mathbf{v}} H_{2\dim L(\mathbf{v}) - d}(L(\mathbf{v}), \mathbf{C})$$

is also a module of $H_{\mathrm{top}}(Z, \mathbf{C})$.

$\Longrightarrow$ It is a direct sum of basic representations. Multiplicity can be explicitly given.

NB. For higher rank analogs, various integrable highest weight modules occur. Multiplicities are given by dimensions of certain intersection homology groups. There is an algorithm to compute them (but not closed formulas).

A quantum toroidal algebra

- Toroidal Lie algebra $\mathbf{L}\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}} \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]$: loop algebra of an affine Lie algebra

× It is not a Kac-Moody Lie algebra.

× Drinfeld-Jimbo's definition of a quantum enveloping algebra does not apply to $\mathbf{L}\widehat{\mathfrak{g}}$.

- Mimic Drinfeld's new realization of a quantum affine algebra instead.

  $=$ q-analog of the loop realization of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathbf{L}\mathfrak{g} \oplus \mathbf{C}c \oplus \mathbf{C}d$

$\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$: quantum toroidal algebra

$= \mathbf{Q}(q)$-algebra with generators
$e_{i,r}, f_{i,r}, q^h, h_{i,m} \left(i = 0, \dots, n, r \in \mathbf{Z}, h \in \widehat{P}^\vee, m \in \mathbf{Z} \setminus \{0\}\right)$

$+$ defining relations (e.g., q^h, $h_{i,m}$: commutes)

$\mathbf{U}_q(\widehat{\mathfrak{g}}) \subset \mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$: subalgebra generated by $e_{i,0}$, $f_{i,0}$, q^h

V : l–integrable representations $\overset{\text{def}}{\Longleftrightarrow}$

(1) weight space decomposition w.r.t $\mathbf{U}_q(\widehat{\mathfrak{g}})$

(2) $e_{i,r}$, $f_{i,r}$ are locally nilpotent on V

V : l–highest weight representations $\overset{\text{def}}{\Longleftrightarrow}$ $\exists m_0 \in V$ such that

(1) $e_{i,r} m_0 = 0$, $V = \mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}}) \cdot m_0$

(2) Simultaneous eigenvector for q^h, $h_{i,m}$.

Geometric construction of $U_q(\mathbf{L}\widehat{\mathfrak{g}})$

- $\mathbf{C}^* \curvearrowright \mathbf{C}^2 : (x, y) \mapsto (qx, qy)$, commutes with Γ-action
  $\Longrightarrow \mathbf{C}^* \curvearrowright M, Z, L$
- $K^{\mathbf{C}^*}(Z)$ = the Grothendieck group of $\mathbf{C}^*$-equivariant coherent sheaves on Z
- $K^{\mathbf{C}^*}(Z)$: an associative algebra over $R(\mathbf{C}^*) = \mathbf{Z}[q, q^{-1}]$ by the convolution product
- $K^{\mathbf{C}^*}(L)$: its representation.

Theorem. $\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}})$: certain $\mathbf{Z}[q, q^{-1}]$-subalgebra (conjecturally an integral form)

(1) $\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}}) \xrightarrow{\exists} K^{\mathbf{C}^*}(Z)/$ torsion : algebra homomorphism

(2) $K^{\mathbf{C}^*}(L) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Q}(q)$ irreducible.

Remarks. (1) Kazhdan-Lusztig, Ginzburg : Corresponding result for affine Hecke algebras. (Deligne-Langlands conjecture)

(2) Ginzburg-Vasserot : Corresponding result for $\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\,\mathfrak{sl}_{n+1})$. (Use n-step flag varieties instead of quiver varieties.)

(3) Varagnolo-Vasserot, Saito-Takemura-Uglov : Algebraic construction of the above module when $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Brief Comments on higher rank case

Specialize at $q = \varepsilon \in \mathbf{C}^*$.

- A product $\prod_{i=0}^{n} \mathrm{GL}(n_i, \mathbf{C})$ acts on Z.
- Irreducible l-integral representations

  $\longleftrightarrow$ Polynomials $P_i(u)$ $(i = 0, \dots, n)$ with $P_i(0) = 1$.
  (Analog of Drinfeld polynomials)

  $\longleftrightarrow$ Semisimple elements $s \in \prod_{i=0}^{n} \mathrm{GL}(n_i, \mathbf{C})$
- $L(P)$ = irreducible representation

  $M(P)$ = specialization of $K^{\prod \mathrm{GL}(n_i, \mathbf{C}) \times \mathbf{C}^*}(L)$ at (s, ε).

  $\Longrightarrow$ $\exists$ algorithm computing $[M(P) : L(Q)]$.

  (analog of Kazhdan-Lusztig polynomials)