Finite dimensional representations of quantum affine algebras

Hiraku Nakajima
Kyoto University

2000 Aug. 23

Quantum affine algebras and quantum loop algebras

- $\mathfrak{g}$: simple Lie algebra of type ADE $/\mathbb{C}$
- $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$: its loop algebra
- $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$: quantum loop algebra
  = quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$
  without central extension & degree operator
  = $\mathbb{C}(q)$-algebra generated by q^h, $e_{k,r}$, $f_{k,r}$, $h_{k,n}$
  ($h \in P^*$, $k \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$)
  with certain relations
- $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$: specialized quantum loop algebra ($\varepsilon \in \mathbb{C}^*$)

Assume $\varepsilon \notin \sqrt{1}$ today.

$\langle q^h, h_{k,\pm m}\rangle \subset \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$: commutative subalgebra

Let $\psi_k^\pm(z) \overset{\text{def.}}{=} q^{\pm h_k} \exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty} h_{k,\pm m} z^{\mp m}\right)$.

Theorem 1 (Chari-Pressley) Any simple finite dimensional module M of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ has a vector $m_0 \in M$ (called an *l-highest weight vector*) such that

$$e_{k,r} m_0 = 0 \quad \text{for all } k,\ r, \qquad \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \cdot m_0 = M$$

$$\psi_k^\pm(z) m_0 = \varepsilon^{\deg P_k} \left(\frac{P_k(\varepsilon^{-1}/z)}{P_k(\varepsilon/z)}\right)^{\pm} m_0 \quad \text{for all } k$$

for some polynomial $P_k(u)$ with $P_k(0) = 1$.

Moreover, the I-tuple of polynomials $P = (P_k(u))_{k\in I}$ parametrizes simple modules. (Drinfeld polynomials)

For P as above, let L_P denote the corresponding simple module.

- an *l-weight space* (of a $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module) : a simultaneous generalized eigenspace of the commutative subalgebra $\langle q^h, h_{k,\pm m}\rangle \subset \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.

Theorem 2 (Frenkel-Reshetikhin) Any finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M is a direct sum of l-weight spaces. Moreover, the eigenvalues of $\psi_k^\pm(z)$ are written in the form

$$\varepsilon^{\deg R_k} \left(\frac{R_k(\varepsilon^{-1}/z)}{R_k(\varepsilon/z)}\right)^{\pm}$$

for some rational function

$$R_k(u) = \frac{\prod_{i=1}^{s_k}(1 - a_{ki}u)}{\prod_{j=1}^{r_k}(1 - b_{kj}u)}.$$

Definition 3 (Frenkel-Reshetikhin's q-character)

$$\chi_q(M) \stackrel{\text{def.}}{=} \sum_{\text{eigenvalues}} \text{multiplicity} \times \prod_{k\in I} \prod_{i=1}^{s_k} \prod_{j=1}^{r_k} Y_{k,a_{ki}} Y_{k,b_{kj}}^{-1}$$

Theorem 4 (FR) (1) χ_q defines an injective ring homomorphism from the Grothendieck ring $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules to $\mathbb{Z}[Y_{k,a}^{\pm}]_{k\in I, a\in\mathbb{C}^*}$ (a ring of laurent polynomials of infinitely many variables).

(2) If we compose a map $Y_{k,a}^{\pm} \mapsto t_k^{\pm}$ (forgetting 'spectral parameters'), it gives the (usual) character of the restriction of M to a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module.

Problem 5 Compute $\chi_q(L_P)$ for all simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module L_P.

Definition 6 A monomial $\prod_{k\in I} \prod_{i=1}^{s_k} \prod_{j=1}^{r_k} Y_{k,a_{ki}} Y_{k,b_{kj}}^{-1}$ appearing in the q-character is called *l-dominant* if $r_k = 0$ for all k, i.e., a product of positive powers.

Remark 7 (1) $\{L_P\}_P$ forms a basis of $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.

(2) $\chi_q(L_P)$ contains an l-dominant monomial (denoted by m_P) with multiplicity 1 corresponding to the l-highest weight vector.

Corollary 8 If $\chi_q(M)$ contains only one l-dominant monomial, then M is simple.

For $\mathfrak{g} = \mathfrak{sl}_n$, we have an evaluation homomorphism $\mathrm{ev}_a \colon \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathbf{U}_\varepsilon(\mathfrak{g})$ corresponding to $\mathbf{L}\mathfrak{g} \to \mathfrak{g}$; $z \mapsto a$ (Jimbo). Pullbacks of simple $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules are simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules.

Example 9 $\mathfrak{g} = \mathfrak{sl}_2$, $M_a = \mathrm{ev}_a^*(\text{2-dim. module})$

$$\chi_q(M_a) = Y_{1,a} + Y_{1,a\varepsilon^2}^{-1} \qquad (P(u) = 1 - ua).$$

Since χ_q is a ring hom., we have

$$\begin{aligned}\chi_q(M_a \otimes M_b) &= \left(Y_{1,a} + Y_{1,a\varepsilon^2}^{-1}\right)\left(Y_{1,b} + Y_{1,b\varepsilon^2}^{-1}\right) \\ &= Y_{1,a}Y_{1,b} + Y_{1,a\varepsilon^2}^{-1}Y_{1,b} + Y_{1,a}Y_{1,b\varepsilon^2}^{-1} + Y_{1,a\varepsilon^2}^{-1}Y_{1,b\varepsilon^2}^{-1}.\end{aligned}$$

If $b \neq a\varepsilon^2, a\varepsilon^{-2}$, $M_a \otimes M_b$ is simple ($P(u) = (1-ua)(1-ub)$).

If $b = a\varepsilon^2$ or $a\varepsilon^{-2}$, then the second term or the third term $= 1$. Thus,

$$\chi_q(M_a \otimes M_{a\varepsilon^2}) = \chi_q(\mathrm{ev}_a^*(\text{3-dim. mod.}) \oplus \text{trivial mod.})$$

Definition 10 We say L_P a *l-fundamental representation* if

$$P_k(u) = \begin{cases} 1 - su & \text{if } k = k_0, \\ 1 & \text{otherwise,} \end{cases}$$

for some $s \in \mathbb{C}^*$ and $k_0 \in I$. We denote L_P by $L(\Lambda_{k_0})_s$.

Definition 11 (N. + Varagnolo-Vasserot) Let

$$M(s; (k_\alpha), (n_\alpha)) = L(\Lambda_{k_1})_{\varepsilon^{n_1}s} \otimes L(\Lambda_{k_2})_{\varepsilon^{n_2}s} \otimes \cdots,$$

for $s \in \mathbb{C}^*$, $k_\alpha \in I$, $n_\alpha \in \mathbb{Z}$ with $n_1 \geq n_2 \geq \cdots$.

A *standard module* M is a tensor product

$$M(s^1; (k_\alpha^1), (n_\alpha^1)) \otimes M(s^2; (k_\alpha^2), (n_\alpha^2)) \otimes \cdots$$

with $s^i/s^j \notin \varepsilon^{\mathbb{Z}}$ for $i \neq j$. (This is independent of the ordering of the tensor product.)

The standard modules are parametrized by P as for simple modules. Let us denote it by M_P.

Properties:

(1) $\{M_P\}$ is a basis of $\operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.

(2) M_P has an l-highest weight vector ($\Longrightarrow$ 'cyclicity conjecture' of Akasaka-Kashiwara).

(3) M_P depends 'continuously' on P.

(4) for a generic P, $M_P \cong L_P$.

The originally M_P was defined as the (specialized) equivariant K-group of lagrangian subvariety of the quiver variety. The identification of the above tensor product was proved by Varagnolo-Vasserot.

Using the geometry of quiver varieties, I can define a certain homomorphism (of additive groups)

$$\chi_{q,t} \colon \operatorname{Rep} \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathbb{Z}[t, t^{-1}][Y_{k,a}^{\pm}]_{k \in I, a \in \mathbb{C}^*}$$

such that $\chi_{q,t=1} = \chi_q$. (**t-analogue of q-character**)

Goal: $\exists$ A combinatorial algorithm to compute $\chi_{q,t}(L_P)$ for all simple modules L_P.

Step 1. Compute $\chi_{q,t}$ for all l-fundamental rep's, by generalizing Frenkel-Mukhin's algorithm for χ_q of l-fundamental rep's.

Step 2. Compute $\chi_{q,t}(M_P)$ for all standard modules M_P. (Tensor product = modified multiplication.)

Step 3. Express the multiplicity $[M_Q : L_P]$ in terms of $\chi_{q,t}(M_Q)$. (This was done essentially in [N. JAMS, to appear]).

(Conjectural) alternative definition

Let M_P be a standard module, $M_{(\Psi^\pm)}$ be a simultaneous generalized eigenspace of $\psi_k^\pm(z)$ with eigenvalue $\Psi_k^\pm(z)$ ($(\Psi^\pm) = (\Psi_k^\pm(z))_{k\in I}$). Define a filtration on $M_{(\Psi^\pm)}$ by

$$\begin{aligned} 0 = M_{(\Psi^\pm)}^{-1} \subset M_{(\Psi^\pm)}^0 &\stackrel{\text{def.}}{=} \bigcap_k \operatorname{Ker}(\psi_k^\pm(z) - \Psi_k^\pm(z)\,\mathrm{id}) \\ \subset M_{(\Psi^\pm)}^1 &\stackrel{\text{def.}}{=} \bigcap_k \operatorname{Ker}(\psi_k^\pm(z) - \Psi_k^\pm(z)\,\mathrm{id})^2 \subset \cdots \end{aligned}$$

Conjecture 12

$$\chi_{q,t}(M_P) = \sum_{(\Psi^\pm)} t^{c(\Psi^\pm)} \sum_n t^{2n} \dim M_{(\Psi^\pm)}^{n+1}/M_{(\Psi^\pm)}^n \, m_{(\Psi^\pm)},$$

where $c(\Psi^\pm)$ is an integer (determined explicitly from $(\Psi^\pm)$), and $m_{(\Psi^\pm)}$ is a monomial corresponding to $(\Psi^\pm)$.

Step 3

Recall that $\chi_q(L_P)$ contains a monomial m_P corresponding to the highest weight vector. Let

$$c_{PQ}(t) \stackrel{\text{def.}}{=} \text{coeff. of } m_P \text{ in } \chi_{q,t}(M_Q).$$

Then $(c_{PQ}(t))_{P,Q}$ is upper-triangular (with respect to a certain ordering on P, Q's) and $c_{PP}(t) = 1$.

Let

- $(c^{PQ}(t)) \stackrel{\text{def.}}{=} (c_{PQ}(t))^{-1}$ (inverse matrix)
- $u_{PQ}(t) \stackrel{\text{def.}}{=} \sum_R c^{PR}(t^{-1}) c_{RQ}(t)$.

Let $\overline{}$ be the involution on $\mathbb{Z}[t, t^{-1}]$ given by $t^\pm \mapsto t^\mp$.

Lemma 13 (Lusztig) There exists a unique solution $Z_{PQ}(t) \in \mathbb{Z}[t^{-1}]$ $(P \leq Q)$ of

$$Z_{PQ}(t) = \sum_{R:P\leq R\leq Q} u_{PR}(t)\overline{Z_{RQ}(t)},$$

$$Z_{PP}(t) = 1, \quad Z_{PQ}(t) \in t^{-1}\mathbb{Z}[t^{-1}] \text{ for } P < Q.$$

(This Lemma was used many times by Lusztig, e.g., construction of canonical basis for ADE, a tensor product, etc.)

Theorem 14

$$[M_Q : L_P] = Z_{PQ}(1)$$

This is an analogue of Kazhdan-Lusztig conjecture expressing the multiplicities of simple modules in Verma modules via Kazhdan-Lusztig polynomials.

Example 15 $\mathfrak{g} = \mathfrak{sl}_2$, M_a as before. By steps 1,2 (or a direct argument) we have

$$\chi_{q,t}(M_{a\varepsilon^2} \otimes M_a) = Y_{1,a}Y_{1,a\varepsilon^2} + Y_{1,a}Y^{-1}_{1,a\varepsilon^4} + Y^{-1}_{1,a\varepsilon^2}Y^{-1}_{1,a\varepsilon^4} + t^{-1}1.$$

Then

$$Z_{PQ}(t) = t^{-1}, \qquad Z_{PP}(t) = Z_{QQ}(t) = 1,$$

where $P(u) = 1$ (trivial module), $Q(u) = (1 - au)(1 - a\varepsilon^2 u)$ (i.e., $M_Q = M_{a\varepsilon^2} \otimes M_a$).

Kirillov-Reshetikhin modules

A simple module L_P is called a *Kirillov-Reshetikhin module* if P is given by

$$P_k(u) = \begin{cases} (1-su)(1-s\varepsilon^2 u)\cdots(1-s\varepsilon^{2n}u) & \text{if } k = k_0, \\ 1 & \text{otherwise,} \end{cases}$$

for some $s \in \mathbb{C}^*$, $n \in \mathbb{Z}_{\geq 0}$ and $k_0 \in I$.

Conjecture 16 (true for type A_n) If P is as above (and Q is arbitary), then $Z_{QP}(t)$ is either 0 or t^s for some (explicitly determined) $s \in Z_{\leq 0}$.

This is related the smoothness of the closure of a certain stratum of quiver varieties.

Step 1

Let $A_{k,a} \overset{\text{def.}}{=} Y_{k,a\varepsilon} Y_{k,a\varepsilon^{-1}} \prod_{l:a_{kl}=1} Y_{l,a}^{-1}$.

Theorem 17 (1) Let $Y_{k,b}^{(n)} \overset{\text{def.}}{=} Y_{k,b}^{n} \sum_{r=0}^{n} \left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_t A_{k,b\varepsilon}^{-r}$, where $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_t$ is the t-binomial coefficient. For each $k \in I$, $\chi_{q,t}(M_P)$ is expressed as a linear combination of

$$Y_{k,b_1}^{(n_1)} Y_{k,b_2}^{(n_2)} \cdots \qquad (b_i \in \mathbb{C}^*,\ n_i \in \mathbb{Z}_{>0} \text{ with } b_i \neq b_j \text{ for } i \neq j),$$

where coefficients are in $\mathbb{Z}[t, t^{-1}][Y_{l,a}^{\pm}]_{l \neq k, a \in \mathbb{C}^*}$.

(2) If L_P is an l-fundamental representation (and hence $M_P = L_P$), then $\chi_{q,t}(M_P)$ contains no l-dominant monomials other than m_P and the condition above uniquely determines $\chi_{q,t}(M_P)$.

Remark 18 This result for $t = 1$ is due to Frenkel-Mukhin.

Example 19 $\mathfrak{g} = \mathfrak{sl}_4$, $\chi_{q,t}(L(\Lambda_2)_1) =$

$$\begin{array}{ccccccc}
Y_{2,1} & \xrightarrow{A_{2,\varepsilon}^{-1}} & Y_{1,\varepsilon}Y_{2,\varepsilon^2}^{-1}Y_{3,\varepsilon} & \xrightarrow{A_{1,\varepsilon^2}^{-1}} & Y_{1,\varepsilon^3}^{-1}Y_{3,\varepsilon} & & \\
 & & \downarrow{\scriptstyle A_{3,\varepsilon^2}^{-1}} & & \downarrow{\scriptstyle A_{3,\varepsilon^2}^{-1}} & & \\
 & & Y_{1,\varepsilon}Y_{3,\varepsilon^3}^{-1} & \xrightarrow{A_{1,\varepsilon^2}^{-1}} & Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon^2}Y_{3,\varepsilon^3}^{-1} & \xrightarrow{A_{2,\varepsilon^3}^{-1}} & Y_{2,\varepsilon^4}^{-1}
\end{array}$$

For $\mathfrak{g} = \mathfrak{sl}_n$, the coeff. of $\chi_{q,t}(L(\Lambda_k)_a)$ are all 1. (More generally, if the coeff. of α_k in the highest root is 1, then the same holds. The proof is obvious if we use quiver varieties.) Thus
$\chi_{q,t}(L(\Lambda_k)_a) = \chi_{q,t=1}(L(\Lambda_k)_a)$.

This implies that $L(\Lambda_k)_a$ is also simple as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module.

Example 20 $\mathfrak{g} = D_4$, $\chi_{q,t}(L(\Lambda_4)_1) =$

$$\begin{gathered}
Y_{4,1} \\
+Y_{1,\varepsilon}Y_{2,\varepsilon}Y_{3,\varepsilon}Y_{4,\varepsilon^2}^{-1} \\
+Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon}Y_{3,\varepsilon} + Y_{1,\varepsilon}Y_{2,\varepsilon^3}^{-1}Y_{3,\varepsilon} + Y_{1,\varepsilon}Y_{2,\varepsilon}Y_{3,\varepsilon^3}^{-1} \\
+Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon^3}^{-1}Y_{3,\varepsilon}Y_{4,\varepsilon^4} + Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon^3}Y_{3,\varepsilon^3}^{-1}Y_{4,\varepsilon^4} + Y_{1,\varepsilon}Y_{2,\varepsilon^3}^{-1}Y_{3,\varepsilon^3}^{-1}Y_{4,\varepsilon^4} \\
+Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon^3}^{-1}Y_{3,\varepsilon^3}^{-1}Y_{4,\varepsilon^2}^{2} + Y_{1,\varepsilon}Y_{1,\varepsilon^3}Y_{4,\varepsilon^4}^{-1} + Y_{2,\varepsilon}Y_{2,\varepsilon^3}Y_{4,\varepsilon^4}^{-1} + Y_{3,\varepsilon}Y_{3,\varepsilon^3}Y_{4,\varepsilon^4}^{-1} \\
+(t+t^{-1})Y_{4,\varepsilon^2}Y_{4,\varepsilon^4}^{-1} + Y_{1,\varepsilon}Y_{1,\varepsilon^5}^{-1} + Y_{2,\varepsilon}Y_{2,\varepsilon^5}^{-1} + Y_{3,\varepsilon}Y_{3,\varepsilon^5}^{-1} \\
+Y_{1,\varepsilon^3}Y_{2,\varepsilon^3}Y_{3,\varepsilon^3}Y_{4,\varepsilon^4}^{-2} + Y_{1,\varepsilon^3}^{-1}Y_{1,\varepsilon^5}^{-1}Y_{4,\varepsilon^2} + Y_{2,\varepsilon^3}^{-1}Y_{2,\varepsilon^5}^{-1}Y_{2,\varepsilon^2} + Y_{3,\varepsilon^3}^{-1}Y_{3,\varepsilon^5}^{-1}Y_{4,\varepsilon^2} \\
+Y_{1,\varepsilon^5}^{-1}Y_{2,\varepsilon^3}Y_{3,\varepsilon^3}Y_{4,\varepsilon^4}^{-1} + Y_{1,\varepsilon^3}Y_{2,\varepsilon^5}^{-1}Y_{3,\varepsilon^3}Y_{4,\varepsilon^4}^{-1} + Y_{1,\varepsilon^3}Y_{2,\varepsilon^3}Y_{3,\varepsilon^5}^{-1}Y_{4,\varepsilon^4}^{-1} \\
+Y_{1,\varepsilon^5}^{-1}Y_{2,\varepsilon^5}^{-1}Y_{3,\varepsilon^3} + Y_{1,\varepsilon^5}^{-1}Y_{2,\varepsilon^3}Y_{3,\varepsilon^5}^{-1} + Y_{1,\varepsilon^3}Y_{2,\varepsilon^5}^{-1}Y_{3,\varepsilon^5}^{-1} \\
+Y_{1,\varepsilon^5}^{-1}Y_{2,\varepsilon^5}^{-1}Y_{3,\varepsilon^5}^{-1}Y_{4,\varepsilon^4} \\
+Y_{4,\varepsilon^6}^{-1}
\end{gathered}$$

Step 2

Let $M_P = M(s^1; (k^1_\alpha), (n^1_\alpha)) \otimes M(s^2; (k^2_\alpha), (n^2_\alpha)) \otimes \cdots$ be a standard module.

Lemma 21

$$\chi_{q,t}(M_P) = \chi_{q,t}(M(s^1; (k^1_\alpha), (n^1_\alpha)))\chi_{q,t}(M(s^2; (k^2_\alpha), (n^2_\alpha)))\cdots$$

if $s^i/s^j \notin \varepsilon^{\mathbb{Z}}$ for $i \neq j$.

Thus it is enough to study

$$\chi_{q,t}(M(s; (k_\alpha), (n_\alpha))) = \chi_{q,t}(L(\Lambda_{k_1})_{\varepsilon^{n_1} s} \otimes L(\Lambda_{k_2})_{\varepsilon^{n_2} s} \otimes \cdots).$$

Let $\chi_{q,t}(L(\Lambda_{k_\alpha})_{\varepsilon^{n_\alpha} s})) = \sum_{r_\alpha} a_{m_\alpha, r_\alpha}(t) m_{\alpha, r_\alpha}$, where m_{α,r_α} is a monomial in $Y^{\pm}_{k,a}$ and $a_{m_\alpha,r_\alpha}(t) \in \mathbb{Z}[t, t^{-1}]$ is its coefficient.

If $t = 1$, $\chi_{q,1}$ is a ring hom., hence we have

$$\chi_{q,1}(M(a; (k_\alpha), (n_\alpha))) = \sum_{r_1, r_2, \dots} \prod_\alpha a_{m_r, r_\alpha}(1) m_{\alpha, r_\alpha}$$

Theorem 22 There exists a nonnegative integer $d(m_{\alpha,r_\alpha}, m_{\beta,r_\beta})$ which is computable from m_{α,r_α}, n_α and m_{β,r_β}, n_β such that

$$\chi_{q,t}(M(a; (k_\alpha), (n_\alpha))) = \sum_{r_1, r_2, \dots} t^{\sum_{\alpha,\beta} \pm d(m_{\alpha,r_\alpha}, m_{\beta,r_\beta})} \prod_\alpha a_{m_r, r_\alpha}(t) m_{\alpha, r_\alpha},$$

where the sign is $-$ if $\alpha \leq \beta$, $+$ otherwise.

Example 23 For $\mathfrak{g} = \mathfrak{sl}_2$, we have $d(Y_{1,a}, Y^{-1}_{1,a\varepsilon^2}) = 1$, $d(Y_{1,a}, Y^{-1}_{1,a}) = 1$ and all others are 0. Then for example, if $P = (1 - au)^n$, we get

$$\chi_{q,t}(M_P) = \sum_{r=0}^{n} \left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]_t Y^{n-r}_{1,a} Y^{-r}_{1,a\varepsilon^2}$$

from $\chi_{q,t}(L(\Lambda_1)_a) = Y_{1,a} + Y^{-1}_{1,a\varepsilon^2}$.

We also have

$$\chi_{q,t}(M_{a\varepsilon^2} \otimes M_a) = Y_{1,a}Y_{1,a\varepsilon^2} + Y_{1,a}Y^{-1}_{1,a\varepsilon^4} + Y^{-1}_{1,a\varepsilon^2}Y^{-1}_{1,a\varepsilon^4} + t^{-1}Y_{1,a\varepsilon^2}Y^{-1}_{1,a\varepsilon^2}.$$