A convergence theorem for Einstein metrics and the ALE spaces

Hiraku Nakajima

Mathematical Institute, Tôhoku University

Introduction

A Riemannian metric g on a manifold is called an Einstein metric, if its Ricci curvature is a constant multiple of g, i.e., $\mathrm{Ric} = kg$ for some constant k. Someone says that it is an equation for the space-time in the relativity, so quite important. However it seems difficult to understand the meaning of this definition itself at first sight. We know its importance only by experiences of seeing that Einstein metrics appeared in many fields of differential geometry. For example, if you read the book by Besse [Be], you know how many fields relate to Einstein metrics. One of the most important results is the existence of Einstein-Kähler metric obtained by Aubin [Au] and Yau [Ya] (shown by Yau when $c_1 = 0$, and by Aubin and Yau independently when $c_1 < 0$). The result had many applications to the algebraic geometry. Recently in works by Narashimhan-Seshadri [NaSe], S.Kobayashi [Kos], Lübke [Lü], Donaldson [Do1, 2], Uhlenbeck-Yau [UY] and others, it was pointed out that an Einstein-Hermitian metric, which is a counterpart of the Einstein-Kähler metric in the gauge theory, has a close relationship with stable holomorphic vector bundles on Kähler manifolds. By their results it becomes possible to study the moduli space of holomorphic vector bundle from the differential geometric view point. (See the textbook by Siu [Si].)

It is believed that Einstein-Kähler metrics also relate to the moduli space of complex structures. Then it is natural to consider its compactification. Anderson [An1] and Bando-Kasue-Nakajima [BaKaNa[1]] obtained a convergence theorem for Einstein metrics when the manifold is 4-dimensional. We used the theory of the Hausdorff distance, which is introduced by Gromov, and developed by many peoples. (It should be noticed that our result is motivated by the study of a degeneration of K3 surfaces by R.Kobayashi-Todorov [KorTo].) In particular, we obtain a compactification of the moduli space when the Ricci curvature is positive, and a completion when nonpositive. As a limit we get a space called an orbifold, which is not a manifold.

From the analytic point of view regarding the Einstein metric as a solution of PDE, one can find many cousins of the convergence theorem in other nonlinear PDE's of elliptic type which have the "the scaling invariance". In these problems, the Palais-Smale condition (C) does not hold, i.e., there may not exist a convergent subsequence in a given sequence of solutions with bounded energy. But one can show that solutions converge outside a finite set thanks to the scaling invariance.

[1] This is a joke by B&O. "Bakana" means foolish in Japanese !

One can find examples in Sacks-Uhlenbeck [SaUh], Uhlenbeck [Uh], Sedlacek [Se], Brezis-Coron [BrCo], Bahri-Coron [BaCo], and so on.

As a bi-product of the convergence theorem, there appears naturally a family of noncompact Ricci-flat manifold, called ALE spaces. (It corresponds to a "bubble" in the case of harmonic maps or Yang-Mills connections.) The ALE stands for *asymptotically locally Euclidean* and means that the metric approximates the standard metric on $\mathbb{R}^4/\Gamma$ in the end. The ALE spaces with hyper-Kähler structures already appeared in the very different context. Physicists, Eguchi-Hanson [EgHa] and Gibbons-Hawking [GiHa] constructed ALE spaces corresponding to cyclic groups. On the other hand, Hitchin [Hi] constructed the same space by using the twistor method. His construction suggested us a relationship to a deformation theory of the simple singularity $\mathbb{C}^2/\Gamma$. Thus he conjectured that there exist similar spaces corresponding to other finite subgroup of $\mathrm{SU}(2)$ (i.e., the binary polyhedral groups). This conjecture was solved affirmatively by Kronheimer using the method called hyper-Kähler quotients. His construction gives a hyper-Kähler structure on each fiber of the semi-universal deformations and their simultaneous resolutions, which were constructed by Brieskorn [Br] and Slodowy [Sl] by using nilpotent varieties of Lie algebras.

ALE spaces with hyper-Kähler structures can be considered as analogues of K3 surfaces in several points. The author [Na2] studied the moduli space of anti-self-dual connections motivated by the study of stable holomorphic vector bundles over K3 surfaces by Mukai [Mu]. The moduli space has a hyper-Kähler structure, and becomes again an ALE space when its dimension is 4. But the base space and the moduli space are not homeomorphic to each other in general, in contrast with the case of K3 surfaces. This result was further developed by Kronheimer-Nakajima [KrNa]. We gave an analogue of the ADHM construction for the anti-self-dual connections on S^4. Our theory relates to the representations of quivers on the extended Dynkin diagrams. All these results show us the richness of the geometry of ALE spaces.

In this article we shall explain about the above results, but we cannot write about moduli spaces of anti-self-dual connections because of limitations of space. I hope other occasion. In §1, we shall introduce the convergence theorem. In §2, we shall see how ALE spaces bubble off. Then in §3, we shall explain the construction of ALE spaces with hyper-Kähler structures. You can read §3 independent of the previous sections, though §2 depends on §1. The statements of theorems are analytic, but we must use the languages of the algebraic geometry to give examples. Please read only the part which the reader has an interest.

1. A convergence theorem for Einstein metrics

As we said in the introduction, the Einstein metric is a solution of a PDE describing the space-time. Here we consider a sequence of Einstein metrics and study its convergence. This means that we study the situation when the space-time is changing and broken. Such convergence theorems appear very often in the study of elliptic partial differential equations, and actually our convergence theorem is strongly motivated by Sacks-Uhlenbeck's theorem on harmonic maps [SaUh] and Uhlenbeck's compactness theorem for Yang-Mills connections [Uh1, 2]. So we first recall the case of harmonic maps, which seems simplest, then proceed to the case of Einstein metrics.

Theorem 1.1 [SaUh]**.** *Let (X, g) be a compact 2-dimensional Riemannian manifold, and let (M, h) be a compact Riemannian manifold of arbitrary dimension. Let $\{f_i : X \to M\}$ be a sequence of harmonic maps with*

$$E(f_i) \stackrel{\text{def.}}{=} \int_X |df_i|^2 dV_g \leq E < \infty,$$

where E is a positive constant independent of i. Then there exists a subsequence $\{j\} \subset \{i\}$ satisfying the following properties:

(1) *There exists a finite set $S = \{x_1, \ldots, x_k\}$ in X such that f_j converges to a map f_∞ in $C^\infty_{\mathrm{loc}}(X \setminus S)$.*

(2) *The limit map $f_\infty : X \setminus S \to M$ extends smoothly to the whole space X.*

The second statement is deduced from the following removable singularities theorem.

Theorem 1.2 [SaUh]**.** *Let D be the 2-disk. If a harmonic map f defined over $D \setminus \{0\}$ satisfies*

$$\int_{D \setminus \{0\}} |df|^2 < \infty,$$

it extends smoothly across the singularity.

Now give an example of convergence. Take the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ for both (X, g) and (M, h), and consider the sequence of rational functions given by $\{f_i(z) = iz\}$[2]. The energy $E(f_i)$ is independent of i. When i goes to infinity, $f_i(z)$ converges to ∞ if $z \neq 0$, but f_i does not converge uniformly in the neighbourhood of 0. The limit is the constant map, so of course extends to the whole space S^2.

If we consider the energy density $|df_j|^2 dV$ as a measure on X in Theorem 1.1, it converges to

$$|df_\infty|^2 dV + \sum_{x \in S} a_x \delta_x,$$

where δ_x is the Dirac measure at x and a_x is a positive constant. That is to say, the energy density becomes concentrated around S and goes to the Dirac measure in the limit. (Remark: We have a similar result when the dimension of X is greater than 2. But the singular set S becomes $n-2$-dimensional, where $n = \dim X$.)

When the target manifold is isometrically embedded in the Euclidean space, the harmonic map equation can be written as $\Delta f + \Pi(df, df) = 0$, where Π is the second fundamental form. It is a non-linear elliptic PDE of second order and the Euler-Lagrange equation for the energy functional. When we study analytic aspects of the harmonic map equation, it is natural to introduce the function space[3] $W^{1,2}$. It is quite essential in the proof of Theorem 1.1, especially for the finiteness of the singular set, that the $p = 2$ is the critical exponent in 2-dimension. (Here we say p is the critical exponent, if any function in the Sobolev space $W^{1,q}$ is continuous if $q > p$, but not continuous when $q = p$.) It is also important that the harmonic

[2] A holomorphic map between Kähler manifolds is harmonic. In fact, it gives a minimum of the energy in its homotopy class.

[3]

$$W^{1,p} \stackrel{\text{def.}}{=} \{f \mid \int_X |f|^p dV + \int_X |df|^p dV < \infty\}$$

map equation has the 'scaling invariance' in 2-dimension. For an Einstein metric g on a 4-manifold X, the counterpart to the energy for the harmonic map is the square of the L^2-norm of the full curvature tensor

$$\int_X |R_g|^2 dV_g.$$

(In fact, the Einstein metric gives the minimum of this functional defined on the set of metrics on X.) The curvature tensor is given by the second order differential of the metric, so one must use $W^{2,2}$ as the function space. In 4-dimension, the exponent $p = 2$ has a similar property as above concerning $W^{2,q}$. So it is natural to expect a similar convergence theorem. Actually we have the following:

Theorem 1.3 [An1, Na1, BaKaNa]. *Let $\{(X_i, g_i)\}$ be a sequence of pairs of 4-dimensional compact manifolds and Einstein metrics with*

$$\operatorname{Ric} g_i = kg_i,\ \operatorname{diam}(X_i, g_i) \leq D,\ \operatorname{vol}(X_i, g_i) \geq V,\ \text{the Euler number of } X_i \leq R,$$

where k, D, V, R is constants independent of i. We also assume that k is normalized to be 0 or ± 1. Then there exists a subsequence $\{j\} \subset \{i\}$ with the following properties:

(1) *$\{(X_j, g_j)\}$ converges to a compact metric space X_∞ in the following sense*[4]*: If we remove a finite set $S = \{x_1, \ldots, x_k\}$ from X_∞, a C^∞-manifold structure and an Einstein metric g_∞ are defined over there. There exists a diffeomorphism F_j from $X_\infty \setminus S$ into X_j such that $F_j^* g_j$ converges to g_∞ in the $C^\infty_{\mathrm{loc}}(X_\infty \setminus S)$-topology.*

(2) *The manifold structure and the Einstein metric g_∞ on $X_\infty \setminus S$ extends to the whole X as an orbifold structure and an orbifold Einstein metric.*

We mean the orbifold structure and the orbifold metric in the following sense:

(a) $X_\infty \setminus S$ is a C^∞-manifold and the restriction of g_∞ is a Riemannian metric.

(b) For each singular point x_n, there exists a neighbourhood N_n such that $N_n \setminus \{x_n\}$ is diffeomorphic to $B^4 \setminus \{0\}/\Gamma$, where B^4 is a 4-dimensional unit ball and Γ is a finite subgroup of $\mathrm{O}(4)$ acting freely on $B^4 \setminus \{0\}$. And if we pull back the metric g_∞ to $B^4 \setminus \{0\}$, it extends smoothly across the singular point 0. (Γ may depends on the singular point x_n.)

As in the case of harmonic maps, we have the removable singularities theorem, but we omit the statement because the conditions are complicated.

The same result under the further assumption on the lower bound of the injectivity radius was independently obtained by Gao [Ga]. In this case, S is an empty set, and the situations like example 1.5 given later do not appear under his assumption.

Remark 1.4. Since we do not give the proof of Theorem at all, the reader may be difficult to understand the meaning of the conditions. The conditions are used to derive the estimates of the isoperimetric constants, are indispensable to obtain the *a priori* estimates for curvatures. From a position of the user of theorem, the conditions are quite natural as:

(1) The assumption k to be 0 or ± 1 is always satisfied if you multiply the metric by an appropriate positive constant. So this condition is not essential.

[4] In fact, the convergence is with respect to the Hausdorff distance introduced by Gromov. We do not talk about the Hausdorff distance here, but note that the recent development of the theory of the Hausdorff convergence is quite essential in the proof of Theorem 1.3.

(2) When $k = 1$, the condition $\operatorname{diam}(X_i, g_i) \leq D$ follows from Myers' theorem.

(3) For a 4-manifold admitting an Einstein metric, the Euler number is given by the (universal) constant multiple of the square integral of the curvature tensor.

(4) The assumption $\dim X_i = 4$ is *not* come from the fact the dimension[5] of the space-time is equal to 4. It comes from the technical reason. In higher dimension, the author conjectures that the sequence converges outside a singular set of codimension 4 if the L^2-norm of the curvature is uniformly bounded.

As in harmonic map case, the measure $|R_{g_j}|^2 dV_{g_j}$ is converges to

$$|R_{g_\infty}|^2 dV_{g_\infty} + \sum_{x \in S} a_x \delta_x$$

in a certain sense. (Since the spaces are changing, the usual convergence does not make sense.) The second term will be explained in the next section. Recall that the curvature measures how the space curves. As $j \to \infty$, the spaces bend around S, and become singularities in the limit. But the singularities are quite simple as finite quotients of manifolds. So it is nothing fearful !

We now give examples.

Example 1.5 (Page[Pa], R.Kobayashi-Todorov[KorTo]). Let X_∞ be an orbifold given by the $\mathbb{Z}/2\mathbb{Z}$-quotient of the complex 2-torus $T = \mathbb{C}^2/\mathbb{Z}^4$, where $-1 \in \mathbb{Z}/2\mathbb{Z}$ acts on T by

$$(z_1, z_2) \mod \mathbb{Z}^4 \mapsto (-z_1, -z_2) \mod \mathbb{Z}^4.$$

The flat metric on T descends to an orbifold metric g_∞ on X_∞, and it is an orbifold Einstein metric. Moreover X_∞ has a structure of a complex manifold (with singularities) since the $\mathbb{Z}/2\mathbb{Z}$-action is holomorphic. Let us take the minimal resolution $\pi\colon X \to X_\infty$. The singularities are sixteen simple singularities of type A_1. The minimal resolution X is called a Kummer surface, and an example of K3 surfaces. Let $S = \{x_1, \dots, x_{16}\}$ be the singular set, and let $E_1, \dots, E_{16}$ be the exceptional sets. These are complex submanifold of X biholomorphic to $\mathbb{C}\mathrm{P}^1$ with the self-intersection number -2. By the solution of the Calabi conjecture by Yau [Ya] we have a unique Ricci-flat Kähler metric in each Kähler class[6]. Take a Kähler class, so a Ricci-flat Kähler metric g_i as follows:

(1) The volume of X with respect to g_i is equal to 1.

(2) The volume of the exceptional set E_n is equal to $1/i$ for $n = 1, \dots, 16$.

It can be shown that the metric g_i converges to $\pi^* g_\infty$ over $X \setminus \cup E_n$, but the condition (2) forces the metric becomes degenerated along E_n as $i \to \infty$. Since $X \setminus E_n$ is diffeomorphic to $X_\infty \setminus S$ via the map π, this gives an example of Theorem 1.3. Note that the limit object is (X_∞, g_∞), not $\pi^* g_\infty$. The limit metric $\pi^* g_\infty$ is degenerate along E_n and the distance between two points in E_n becomes 0. Hence it is more natural to collapse E_n to a point ! (See Figure 1.1.)

FIGURE 1.1. The behaviour of the metrics around the singular point x_1

[5] I mean the dimension of the space-time in relativity, not in the superstring theory.

[6] An Einstein metric with $\mathrm{Ric} = 0$ is called a Ricci-flat metric.

The moduli space of polarized K3 surfaces (i.e., pairs of K3 surfaces and Kähler classes with the volume 1) is known to be an open dense subset $K\Omega$ of

$$\overline{K\Omega} = \Gamma\backslash \mathrm{SO}_o(3,19)/\mathrm{SO}(2) \times \mathrm{SO}(19),$$

where Γ is the group of automorphisms preserving the intersection form on $H^2(X;\mathbb{Z})$. If we add singular K3 surfaces as above to the moduli space, we get the whole space $\overline{K\Omega}$. Thus Theorem 1.3 gives a natural completion of the moduli space (see [KorTo, An2] for detail).

Similarly by using the existence of Einstein-Kähler metric[7] obtained by Aubin and Yau, one can also give examples with $k = -1$. Consider a deformation of projective surfaces of general type $\pi\colon \mathcal{X} \to \Delta$ such that $X_t = \pi^{-1}(t)$ satisfies $c_1(X_t) < 0$ (namely the canonical bundle is ample) when $t \neq 0$, and the central fiber X_0 satisfies $c_1(X_0) \leq 0$ (the canonical bundle is nef), but $c_1(X_0) \not< 0$. When $t \neq 0$, X_t has a unique Einstein-Kähler metric, but X_0 does not. The limit of the Einstein-Kähler metric as $t \to 0$ is the orbifold Einstein-Kähler metric on the canonical model of X_0, that is the surface obtained by collapsing all (-2)-curves in X_0 (see Tsuji [Ts]).

When algebro-geometers talk about the "degeneration", they also consider the cases which are not covered by Theorem 1.3, namely the diameter of (X_i, g_i) may diverge. (For example, in the above case of K3 surfaces, consider a sequence which corresponds a divergent sequence in $K\Omega$.) We do not understand what happens differential geometrically when the diameter goes to infinity. (But see the work of R.Kobayashi [Kor1].) If we consider the moduli space of constant curvature metrics on Riemann surfaces, the limit space is always smooth under the assumption on the diameter. The stable curve does not appear ! On the other hand, when $k = 1$, i.e., when we consider Einstein-Kähler metrics on del Pezzo surfaces[8], the diameter condition follows from Myers' theorem as remark above, so the all conditions in Theorem 1.3 are satisfied. Thus we get the following:

Corollary 1.6. *Fix a underlying differentiable structure $\mathcal{X}$ of the del Pezzo surface X. Let $\mathfrak{M}(\mathcal{X})$ be the moduli space of pairs of complex structures with $c_1 > 0$ and Einstein-Kähler metrics on them. Then adding orbifolds, we can compactify $\mathfrak{M}(\mathcal{X})$.*

Consider only the complex structures, we have a natural map Φ from $\mathfrak{M}(\mathcal{X})$ to the moduli space of complex structures $\mathfrak{H}(\mathcal{X})$. By Bando-Mabuchi's uniqueness theorem of Einstein-Kähler metric [BaMa], the map Φ is injective. In fact, Tian proved the following:

Theorem 1.7 [Ti]**.** *If the automorphism group of the del Pezzo surface X is reductive*[9]*, X has an Einstein-Kähler metric. In other words, the map Φ is bijective unless $\mathcal{X}$ is either one or two points blowing up of $\mathbb{C}\mathrm{P}^2$.*

The proof is given by showing that the image of Φ is nonempty, both open and closed in $\mathfrak{H}(\mathcal{X})$. It is not so difficult to show the openness. The closedness is

[7] We say an Einstein-Kähler metric the Einstein metric which is Kählerian at the same time. Then the first Chern class c_1 of the manifold is given by $k[\omega]$ ($[\omega]$ denotes the Kähler class). In particular, if the complex manifold has an Einstein-Kähler metric, it holds either $c_1 > 0$, $= 0$, or < 0.

[8] A complex surface X with $c_1(X) > 0$ is called a del Pezzo surface. It is known that such a X is either $\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, or blowing up of $\mathbb{C}\mathrm{P}^2$ at r generic points ($1 \leq r \leq 8$).

[9] Matsushima's theorem says if a del Pezzo surface admits an Einstein-Kähler metric, its automorphism group is reductive.

shown by using Theorem 1.3. Suppose that (X_i, g_i) is a sequence in $\mathfrak{M}(\mathcal{X})$ and $\Phi(X_i, g_i) \in \mathfrak{H}(\mathcal{X})$ converges. We want to show that (X_i, g_i) also converges and it holds

$$\lim_{i\to\infty} \Phi(X_i, g_i) = \Phi(\lim_{i\to\infty} (X_i, g_i)).$$

The difficulty relates to the Hausdorff-ness of the moduli space $\mathfrak{H}(\mathcal{X})$. If the jumping of the complex structure would occur, one cannot expect the above equality. Moreover the following problem is still open.

Problem 1.8. *When does a del Pezzo surface with quotient singularities have an Einstein-Kähler orbifold metric and lies in the boundary of the compactification of the moduli space $\mathfrak{M}(\mathcal{X})$ given in Theorem 1.3 ?*

The problem should relate to the Chow stability, but things are not clear yet.[10]

2. Bubbling out of ALE spaces

In order to study the convergence in more detail, we "blow up" the metrics around the singular points by rescaling. This is a standard technique widely used in problems of weak convergence. As before, first study the case of harmonic maps.

Let f_j be a sequence of harmonic maps as in Theorem 1.1, and suppose that f_j does not converges at x, i.e., $x \in S$. Then $|df_j(x)|$ goes to the infinity as $j \to \infty$. Take a normal coordinate system (x_1, x_2) defined on a neighbourhood U of x. Let x_j be a point at which $|df_j|$ attains the maximum in U. We may assume that $\{x_j\}$ converges to x. Define a new coordinate system by $(y_1, y_2) = (|df_j(x_j)|x_1, |df_j(x_j)|x_2)$ and change the metric as $|df_j(x_j)|^2 g$. Remark that the maximum of the energy density of f_j in U is normalized to be 1 with respect to the new metric. The neighbourhood $\{(x_1, x_2) \mid |x_1|^2 + |x_2|^2 < \delta\}$ of x is written as $\{(y_1, y_2) \mid |y_1|^2 + |y_2|^2 < |df_j(x_j)|^2\delta\}$, in the new coordinates, and converges to $\{(y_1, y_2) \in \mathbb{R}^2\}$ as $j \to \infty$. The rescaled metrics converge to the standard metric on $\mathbb{R}^2$. We regard the map f_j defined on the y-plane, and denote by $\widetilde{f}_j$. Then the energy of the $\widetilde{f}_j$ is bounded from the above independent of j. Then one can apply Theorem 1.1 to $\widetilde{f}_j$. In this case, we have a uniform bound on the energy density, so the concentration of the energy does not happen and we get the following:

Theorem 2.1 [SaUh]**.** *There exists a subsequence of $\{\widetilde{f}_j\}$, also denoted by $\{\widetilde{f}_j\}$, which converges to a harmonic map $\widetilde{f}_\infty$ defined on the whole plane $\mathbb{R}^2$ with finite energy.*

Noticing that $\mathbb{R}^2$ and $S^2 \setminus \{p\}$ are conformal to each other, and that the energy and the harmonicity are preserved under a conformal transformation, we can regard $\widetilde{f}_\infty$ as a finite energy harmonic map defined on $S^2 \setminus \{p\}$. Then by the removable singularities theorem, $\widetilde{f}_\infty$ can be extended to the whole S^2. Also note that $\widetilde{f}_\infty$ depends on $x \in S$.

Let use the same technique in the case of Einstein metrics. Let $\{(X_j, g_j)\}$ be a sequence of Einstein metrics as in Theorem 1.3, and take $x \in S$. The absolute value of the curvature $|R_{g_j}|$ diverges to infinity at x. Let r_j be the value of $|R_{g_j}|$ at x. Since x is a point in X, not in X_j, we cannot talk about $|R_{g_j}|$ at x. Precisely

[10] There are progress to this problem after writing the original manuscript. See the references added in translation.

speaking, we mean $|R_{g_j}(x_j)|$, where x_j is a point in X_j and $\{x_j\}$ converges to $x \in X$ in a certain sense (we do not give the precise meaning here to avoid the technical complexity). Now we multiply the metric g_j by r_j. Then the absolute value of the curvature of the new metric $r_j g_j$ is normalized to be 1. This rescaling procedure means "viewing by the microscope". Since the curvatures become uniformly bounded, we can expect the convergence of the manifolds. But the diameter goes to infinity, so the limit space becomes a noncompact manifold. In fact, we have the following:

Theorem 2.2 [Na1]. *Consider the sequence $\{(X_j, r_j g_j, x_j)\}$ of the pairs of Riemannian manifolds and points on it. There exists a subsequence, also denoted by $\{(X_j, r_j g_j, x_j)\}$, which converges*[11] *to a complete Ricci-flat manifold (M, h, o) having the following properties.*

(1) $\displaystyle\int_M |R_h|^2 dV_h < \infty,$

(2) $\operatorname{vol} B(o; r) \geq V r^4$ *for some* $o \in M$, $V > 0$,

where $B(o; r)$ denotes the ball of radius r centered at o. More precisely, there exists a diffeomorphism G_j from M to a neighbourhood of x_j such that $G_j^*(r_j g_j)$ converges to h in $C^\infty_{\mathrm{loc}}(M)$.

We call these phenomena, such as a map from S^2 being cut off from sequences of harmonic maps, a part of the manifold being torn off in our case, "bubbling out".

In the case of harmonic maps, one can regard the limit map $\widetilde{f_\infty}$ as a map from S^2 through the conformal compactification of $\mathbb{R}^2$. The Einstein equation is not preserved under the conformal transformation, so one cannot apply the removable singularities theorem in the same way. However one can modify its proof to this situation to show the following (in fact, the proof becomes more difficult).

Theorem 2.3 [BaKaNa]. *If a 4-dimensional complete Ricci-flat manifold satisfies the conditions* (1) *and* (2) *in Theorem 2.2, then it is ALE of order* 4.

We say a 4-dimensional Riemannian manifold to be ALE of order τ if there exists a compact set $K \subset X$ and a finite subgroup $\Gamma \subset \mathrm{SO}(4)$ and a diffeomorphism (coordinates at infinity) $\mathfrak{X}\colon X \setminus K \to (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma$ such that the following holds in the coordinates $\mathfrak{X}$:

$$|\overbrace{\partial \cdots \partial}^{p \text{ times}}(g_{ij}(x) - \delta_{ij})| = O(|x|^{-p-\tau}) \qquad \text{for } x \in (\mathbb{R}^4 \setminus \overline{B_R})/\Gamma,\ p = 0, 1, 2, 3, \cdots.$$

In the orbifold, the distance sphere around the singular point of sufficiently small radius is diffeomorphic to S^3/Γ, and the distance sphere of sufficiently large radius is diffeomorphic to S^3/Γ in the ALE space. Thus one can understand the ALE space is the counterpart to the orbifold in the category of noncompact manifolds.

Remark 2.4. One cannot drop the condition (2) in Theorem 2.3. There exists a Ricci-flat Kähler metric, called the Taub-NUT metric, defined on $\mathbb{C}^2$. This metric has an asymptotic behaviour called ALF, and the volume of ball of radius r grows as $O(r^3)$ (see [EGH]). We do not have examples of Ricci-flat manifolds satisfying the condition (2) but not satisfying (1).

[11] with respect to the pointed Hausdorff convergence

See what happens in Example 1.5. Take a point y_n in an exceptional set E_n in X which corresponds to a singular point x_n in X_∞. If the value of $|R_{g_i}|$ at y_n is equal to r_i, $(X, r_i g_i)$ converges to an ALE Ricci-flat Kähler manifold (M, h). This M is called the Eguchi-Hanson space and biholomorphic to a cotangent bundle $T^*\mathbb{C}\mathrm{P}^1$ of the complex projective line. ($T^*\mathbb{C}\mathrm{P}^1$ is the minimal resolution of the simple singularity of type A_1. The exceptional set is the zero section.) In this case the limit space is independent of n. In general, when K3 surfaces degenerate to an orbifold with simple singularities, one can obtain a Ricci-flat Kähler metric on the minimal resolution of each singularity in the same way (see [Kor2]).

In Theorem 2.2, 2.3, we rescale the metric so that the curvatures are uniformly bounded. We lose some informations and need more detailed studies in general. This is because several ALE spaces may blow up from one singular point $x \in S \subset X_\infty$. An accurate version of the convergence theorem which includes the above situations was obtained by Bando [Ba][12]. In order to explain his result, we need some terminology and symbols. We say an orbifold metric is ALE when there is no singular set outside a compact set, and there exists a coordinate system $\mathfrak{X}$ at infinity as usual ALE Riemannian manifolds. For such an ALE orbifold (M, h), we denote by $\mathcal{S}(M)$ the set of singular points. By definition, $\mathcal{S}(M)$ is a finite set. We denote by x_j^n the point in X_j which converges to the singular point x^n in X_∞ in the same sense as before. (See the explanation above Theorem 2.2).

We see a neighbourhood of x_j^n by a microscope. In this case, we set the magnification r_j so that

$$\int_{B(x_j^n;R)\setminus B(x_j^n;\frac{1}{\sqrt{r_j}})} |R_{g_j}|^2 dV_{g_j} = \varepsilon,$$

where ε, and R is a (sufficiently small) positive constant. Here $B(x;r)$ denotes the metric ball of radius r centered at x. The similar convergence theorem as 1.3 can be applied to the magnified Riemannian manifold $(X_j, h_j) = (X_j, r_j g_j)$. Since we have

$$\int_{B(x_j^n;\sqrt{r_j}R)\setminus B(x_j^n;1)} |R_{h_j}|^2 dV_{h_j} = \varepsilon$$

in this case, the concentration of the curvatures does not happen in $B(x_j^n; \sqrt{r_j}R) \setminus B(x_j^n; 1)$, hence the limit (M, h) of (X_j, h_j) is an ALE orbifold which does not have singularities outside the ball of radius 1. If we want to know the situation of the convergence, we apply the same argument to (X_j, h_j) in stead of (X_j, g_j). Namely, we multiply h_j again around a singular point of (M, h). This means the magnification is reset bigger. We make the magnification bigger as a singular point occur in the limit ALE Ricci-flat orbifold in this way, then finally no singular points appear and the procedure ends in the finite step. We adjust the following form. The statement becomes complicated, so we give the figure 2.1 which describe the circumstances.

FIGURE 2.1

[12]This paper was dedicated to the late cartoonist Osamu Tezuka.

Theorem 2.5 [Ba]**.** (1) *In the same notation as in Theorem 1.3, (taking a further subsequence $\{j\} \subset \{i\}$ if necessary), there exist a family of pointed ALE Ricci-flat orbifolds and sequences of positive numbers which has a recursive relation as follows*:

$\boxed{1}$ *There exist a pair (M^n, h^n, o^n) of an ALE Ricci-flat orbifold and a point corresponding to each singular point $x^n \in S$ in X_∞ $(n = 1, 2, \cdots, \#S)$ and a divergent sequence $\{r_j^n\}_{j=1,2,\cdots}$ of positive numbers such that*

$$\lim_{j\to\infty}(X_j, r_j^n g_j, x_j^n) = (M^n, h^n, o^n).$$

Here lim *is the convergence in the sense as in Theorem 2.2. Moreover the fundamental group of the end of M^n and the group corresponding to the singular point x^n are isomorphic and their actions on $\mathbb{R}^4$ are the same.*

$\boxed{k-1 \Rightarrow k}$ *Suppose that ALE Ricci-flat orbifold $M^{n_1,n_2,\cdots,n_{k-1}}$ is defined and has a singular point $y^{n_1,n_2,\cdots,n_k} \in \mathcal{S}(M^{n_1,n_2,\cdots,n_{k-1}})$ $(n_k = 1, 2, \cdots, \#\mathcal{S}(M^{n_1,n_2,\cdots,n_{k-1}}))$. There exist a pair $(M^{n_1,n_2,\cdots,n_k}, h^{n_1,n_2,\cdots,n_k}, o^{n_1,n_2,\cdots,n_k})$ of an ALE Ricci-flat orbifold and a point, a sequence $\{r_j^{n_1,n_2,\cdots,n_k}\}_{j=1,2,\cdots}$ of positive numbers, and a sequence $\{x_j^{n_1,n_2,\cdots,n_k}\}_{j=1,2,\cdots}$ of points in X_j such that*

$$\lim_{j\to\infty}(X_j, r_j^{n_1,n_2,\cdots,n_k} g_j, x_j^{n_1,n_2,\cdots,n_k}) = (M^{n_1,n_2,\cdots,n_k}, h^{n_1,n_2,\cdots,n_k}, o^{n_1,n_2,\cdots,n_k}),$$

$$\lim_{j\to\infty}\frac{r_j^{n_1,n_2,\cdots,n_k}}{r_j^{n_1,n_2,\cdots,n_{k-1}}} = \infty.$$

Moreover the fundamental group of the end of $M^{n_1,n_2,\cdots,n_k}$ and the group corresponding to the singular point $y^{n_1,n_2,\cdots,n_k}$ are isomorphic and their actions on $\mathbb{R}^4$ are the same.

(2) *This recursive procedure ends in a finite step, namely the ALE orbifold $M^{n_1,n_2,\cdots,n_k}$ becomes nonsingular for some k. Moreover for the sufficiently large j and sufficiently small R, the ball $B(X_j^n; R)$ is diffeomorphic to a manifold which is obtained from $M^{n_1,n_2,\cdots,n_k} \setminus \mathcal{S}(M^{n_1,n_2,\cdots,n_k})$'s with $n_1 = n$ by attaching a neighbourhood of $y^{n_1,n_2,\cdots,n_k}$ and the end of $M^{n_1,n_2,\cdots,n_k}$.*

(3) *The measure $|R_{g_j}|^2 dV_{g_j}$ converges to*

$$|R_{g_\infty}|^2 dV_{g_\infty} + \sum_{n=1}^{\#S} a_n \delta_{x^n},$$

where a_n is given by

$$a_n = \sum_{(M,h)} \int_M |R_h|^2 dV_h.$$

Here the summation runs over the set of ALE orbifolds $(M, h) = (M^{n_1,n_2,\cdots,n_k}, h^{n_1,n_2,\cdots,n_k})$ with $n_1 = n$.

For a compact Einstein 4-manifold, the L^2-norm of the curvature gives its Euler number. For an ALE Einstein manifold (M, h) we have

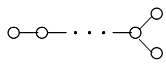
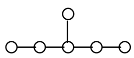
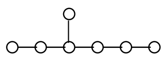
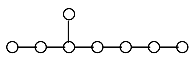
$$\frac{1}{8\pi^2}\int_M |R_h|^2 dV_h = \text{the Euler number of } M - \frac{1}{\text{the order of } \Gamma}.$$

(There are similar formulae for compact orbifolds and ALE orbifold.) We have the following. (For the definition of hyper-Kähler structure, see Sect. 3.)

Proposition 2.6. *Under the same situation as in Theorem 1.3, suppose that (X_i, g_i) has a Kähler (resp. hyper-Kähler) structure. Then the limit space (X_∞, g_∞) and the space $(M^{n_1,n_2,\cdots,n_k}, h^{n_1,n_2,\cdots,n_k})$ appeared in Theorem 2.5 also have a Kähler (resp. hyper-Kähler) structure in the sense of orbifolds.*

3. ALE spaces with hyper-Kähler structures

As we said in the previous section, the ALE spaces bubble out when Einstein metrics degenerate to an orbifold metric. In this way, one obtain existence of ALE spaces. In this section, we shall give a more concrete way to give ALE spaces with hyper-Kähler structures. This is a result by Kronheimer [Kr1, 2]. His result is closely related to the theory of *simple singularities*. A simple singularity is a quotient space $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SU}(2)$, and has been studied for a long time. The classification is known, and it can be realized as a complex hypersurface in $\mathbb{C}^3$. The dual graph of the exceptional set of its minimal resolution is a Dynkin graph with no double edges (i.e., of type A, D, E). (See the table below.) By the theory of Brieskorn [Br] and Slodowy [Sl], one can construct the simple singularity using nilpotent varieties of the corresponding simple Lie algebra. One can also describe the semi-universal deformation and its simultaneous resolution in terms of the Lie algebra. Kronheimer succeeded to construct these varieties using so called *quivers*. Moreover his construction also showed that each fiber has a hyper-Kähler structure.

root system	group	Dynkin graph	hypersurface
A_n	the cyclic group of order $n+1$		$x^{n+1} + yz = 0$
D_n	the binary dihedral group of order $4(n-2)$		$x^2 + y^2 z + z^{n-1} = 0$
E_6	the binary tetrahedral group		$x^2 + y^3 + z^4 = 0$
E_7	the binary octahedral group		$x^2 + y^3 + yz^3 = 0$
E_8	the binary icosahedral group		$x^2 + y^3 + z^5 = 0$

A *hyper-Kähler structure* on a Riemannian manifold (M, g) is, by definition, a triple of almost complex structures (I, J, K) which satisfies a quaternion relation $IJ = -JI = K$ and is parallel with respect to the Levi-Civita connection of g, i.e., $\nabla I = \nabla J = \nabla K = 0$. We call a Riemannian manifold with a hyper-Kähler structure simply the hyper-Kähler manifold. The holonomy group of a hyper-Kähler manifold is contained in $\mathrm{Sp}(n)$, where $n = \frac{1}{4}\dim M$. Each almost complex structure I, J, or K defines a Kähler structure on (M, g). When the manifold is 4-dimensional and simply-connected, a hyper-Kähler structure is nothing but a Ricci-flat Kähler structure thanks to the isomorphism $\mathrm{Sp}(1) = \mathrm{SU}(2)$. For example, a Ricci-flat Kähler metric on a K3 surface gives us a hyper-Kähler structure. Taking a particular almost complex structure I and regarding (M, g) as a Kähler manifold, we have a nowhere vanishing closed $(2,0)$-form $\omega_J + i\omega_K$, where ω_J (resp. ω_K) is the Kähler form associated with J (resp. K). Namely we have a *holomorphic symplectic form*. Conversely if a compact Kähler manifold has a holomorphic symplectic form, there exists a Ricci-flat Kähler metric by the solution of Calabi conjecture, and hence there exists a hyper-Kähler structure.

On the other hand, Hitchin et al. [HKLR] introduced a notion of hyper-Kähler quotients which is an analogue of Marsden-Weinstein quotients for symplectic manifolds, therefore gave a different method to construct hyper-Kähler manifolds. Recall their result.

Suppose that a Lie group G acts on a hyper-Kähler manifold (M, g, I, J, K) preserving the hyper-Kähler structure. Let denote the Lie algebra of G by $\mathfrak{g}$, and its dual space by $\mathfrak{g}^*$. G acts on $\mathfrak{g}^*$ by the coadjoint action. A G-equivariant map $\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^*$ is said to be a *hyper-Kähler moment map* if we have

$$I\,\mathrm{grad}\langle \mu_I, \xi\rangle = J\,\mathrm{grad}\langle \mu_J, \xi\rangle = K\,\mathrm{grad}\langle \mu_K, \xi\rangle = \xi^*$$

for any $\xi \in \mathfrak{g}$. Here $\langle\ ,\ \rangle$ is a natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and ξ^* is a vector field generated by ξ. Let define

$$Z = \{z \in \mathfrak{g}^* \mid \mathrm{Ad}^*_g(z) = z \text{ for any } g \in G.\}$$

Then $\mu^{-1}(\zeta)$ is invariant under the G-action for $\zeta \in \mathbb{R}^3 \otimes Z$. So we can consider the quotient space $X_\zeta = \mu^{-1}(\zeta)/G$.

Theorem 3.1 [HKLR]**.** *Suppose that the G-action on $\mu^{-1}(\zeta)$ is free. Then the quotient space X_ζ is a C^∞ manifold and has a Riemannian metric and a hyper-Kähler structure induced from those on M.*

As an application of the above hyper-Kähler quotient construction, we give ALE spaces. Fix a root system of type A, D, or E. Let $\theta_1, \dots, \theta_r$ be its simple root. Let $\theta_0 = -\sum_i n_i \theta_i$ be the negative of the highest root. Set $n_0 = 1$ for convenience. Draw the extended Dynkin diagram, and put a complex vector space of dimension n_i on each vertex i. Consider linear maps $f_{ij}\colon \mathbb{C}^{n_j} \to \mathbb{C}^{n_i}$ and $f_{ji}\colon \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$ when the vertices i and j are joined by an edge. See the figure for D_5.

Figure 3.1

Let M be the linear space of all such linear maps f_{ij}. Put a hermitian metric on each $\mathbb{C}^{n_i}$ and also the induced metric on M. To make M a quaternion vector space, we introduce J as follows. For joined vertices i and j, we have two linear maps from $\mathbb{C}^{n_i}$ to $\mathbb{C}^{n_j}$ and $\mathbb{C}^{n_j}$ to $\mathbb{C}^{n_i}$. Choose one of them and fix henceforth. When $i \to j$ is chosen, we define J by

$$J(f_{ij}, f_{ji}) = (-f^*_{ji}, f^*_{ij}),$$

where * means the adjoint. Then J satisfies $J^2 = -1$ and is anti-linear with respect to the complex structure. Hence M has a structure of quaternion vector space. There is a natural action on M of a Lie group $G = \prod_{i \neq 0} \mathrm{U}(\mathbb{C}^{n_i})$[13] This action preserves the metric and quaternion linear. Let μ be the hyper-Kähler moment map vanishing at the origin, which exists uniquely. Let Z be as above, and take $\zeta \in \mathbb{R}^3 \otimes Z$. Then the space $X_\zeta = \mu^{-1}(\zeta)/G$ has a hyper-Kähler structure on its nonsingular part. More precisely Kronheimer showed the following:

[13] The action of $\prod \mathrm{U}(\mathbb{C}^{n_i})$ is not appropriate, since $c\,\mathrm{Id} \in \prod \mathrm{U}(\mathbb{C}^{n_i})$ fixes any element in M.

Theorem 3.2 [Kr1]**.** *For a generic ζ, the G-action on $\mu^{-1}(\zeta)$ is free and the quotient space X_ζ is a nonsingular 4-dimensional hyper-Kähler manifold. Moreover X_ζ is diffeomorphic to a minimal resolution of $\mathbb{C}^2/\Gamma$ and the metric is ALE.*

He had a precise description of X_ζ even for non generic ζ, for example, $X_\zeta = \mathbb{C}^2/\Gamma$ if $\zeta = 0$. Write $\zeta = (\zeta_I, \zeta_J, \zeta_K)$, and consider the following family of spaces with $\zeta_I = 0$

$$\bigcup_{\zeta_J + i\zeta_K \in \mathbb{C}\otimes Z} X_{(0,\zeta_J,\zeta_K)}$$

and the natural projection to $\mathbb{C}\otimes Z$. Then it is a semi-universal deformation of $\mathbb{C}^2/\Gamma$ pulled back to the Weyl group covering. For generic ζ_I, one can define a holomorphic map

$$X_{(\zeta_I,\zeta_J,\zeta_K)} \to X_{(0,\zeta_J,\zeta_K)},$$

which is a simultaneous resolution of singularities. When a sequence of generic parameters converges to a non generic parameter, corresponding Einstein metrics converges to an ALE orbifold as in §§1,2.

The geometric meaning of the parameter ζ can be explained in terms of period maps. The exceptional set of the resolution of $\mathbb{C}^2/\Gamma$ is a union of $\mathbb{CP}^1$'s and their intersection matrix is -1 times Cartan matrix of the corresponding Lie algebra. Then for generic ζ, one can identify $H^2(X_\zeta;\mathbb{R})$ with the Cartan subalgebra $\mathfrak{h}$, and an element Σ in $H_2(X_\zeta;\mathbb{Z})$ with $\Sigma\cdot\Sigma = -2$ corresponds to a root. The set $(\mathbb{R}^3\otimes Z)^\circ$ of generic ζ is simply-connected, so the local system $H^2(X_\zeta;\mathbb{R})$ can be trivialized. So taking cohomology classes of three Kähler forms ω_I, ω_J, ω_K on X_ζ, we can define a period map

$$P\colon (\mathbb{R}^3\otimes Z)^\circ \to \mathbb{R}^3\otimes\mathfrak{h}.$$

Then Kronheimer showed that P is induced from an isomorphism between Z and $\mathfrak{h}$, and $\zeta\in\mathbb{R}^3\otimes Z$ is generic (i.e., X_ζ is smooth) if and only if

$$P(\zeta) \notin \bigcup_\theta \mathbb{R}^3\otimes\pi_\theta,$$

where π_θ is a hypersurface defined by the root θ.

He also showed the following classification theorem:

Theorem 3.3 [Kr2]**.** *An ALE spaces with a hyper-Kähler structure is isomorphic to the one constructed by the above construction.*

As above, the ALE space is well-understood when it has a hyper-Kähler structure. But it seems also important to study the spaces with Kähler structures in conjunction with the algebraic geometry. In this case the ALE space is a cyclic quotient of the ALE space with hyper-Kähler structure [Ba]. Using the above classification theorem and an argument in [Ti], we can show the following:

Theorem 3.4. *Let (X,g) be a nonsingular ALE Ricci-flat Kähler 4-manifold. Then the one of the followings holds.*

(1) *(X,g) has a hyper-Kähler structure compatible with the Kähler structure.*

(2) *(X,g) is a quotient of an hyper-Kählerian ALE space $\widetilde{X}$ of type A_n by the cyclic group $\mathbb{Z}/r\mathbb{Z}$. And $n+1$ is a multiple of r.*

As we saw in §§1,2, an ALE space with a Kähler structure bubbles out when Einstein-Kähler metrics converges to an orbifold. For example, consider a deformation of algebraic surfaces $\pi\colon \mathfrak{X} \to \Delta$ and suppose that $X_t = \pi^{-1}(t)$ has an Einstein-Kähler metric g_t when $t \neq 0$. If the diameter of (X_t, g_t) is estimated from the above independent of t, the convergence theorem gives us an Einstein-Kähler orbifold as a limit. The above theorem together with the relation between the singularity and the bubbling off ALE spaces, the singularity of the orbifold must be a simple singularity or a cyclic quotient of a simple singularity of type A_n. Strikingly, the similar result is obtained by using the minimal model theory (see Kawamata [Ka]). If the total space $\mathfrak{X}$ has only terminal singularities and the central fiber X_0 has only orbifold singularities, then the singularities in the central fiber are of type just mentioned.

At the last, we give the following problem:

Problem 3.5. *Are there ALE Ricci-flat* 4*-manifolds without Kähler structures ?*

We do not have such examples at this moment.

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Translated by HIRAKU NAKAJIMA

Aramaki, Aoba-ku, Sendai 980, Japan
E-mail address: nakajima@math.tohoku.ac.jp