Instanton Counting

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based on

Nekrasov : hep-th/0206161

Nekrasov + Okounkov : hep-th/0306238

N + Kota Yoshioka : math.AG/0306198, math.AG/0311058, math.AG/0505553

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Additional references

- Braverman : math.AG/0401409

  (affine) Whittaker modules

- Braverman + Etingof :math.AG/0409441

- Göttsche + N + Yoshioka : in preparation

  Donaldson invariants for projective toric surfaces

Framed moduli spaces of instantons on $\mathbb{R}^4$

- $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$.
- $M_0^{\mathrm{reg}}(n, r)$: framed moduli space of $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4$ with $c_2 = n$, where the framing is the trivialization of the bundle at ∞.

  This is noncompact:

  – bubbling

  – $\exists$ parallel translation symmetry

We kill the first ‘source’ of noncompactness (bubbling) in two ways:

- $M_0(n,r)$: Uhlenbeck (partial) compactification

$$M_0(n,r) = \bigsqcup_{k=0}^{n} M_0^{\mathrm{reg}}(k,r) \times S^{n-k}\mathbb{R}^4.$$

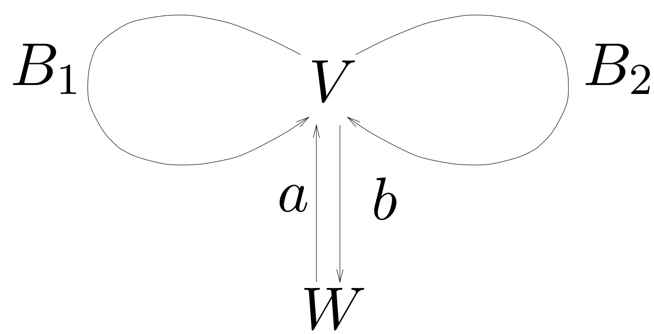
- $M(n,r)$: Gieseker (partial) compactification, i.e., the framed moduli space of rank r torsion-free sheaves on $\mathbb{P}^2 = \mathbb{R}^4 \cup \ell_\infty$, parametrizing pairs (E,φ)
  - E : a torsion-free sheaf on $\mathbb{P}^2$ with $\mathrm{rk} = r$, $c_2 = n$
  - $\varphi\colon E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing)

- $M(n,r)$: nonsingular hyperKähler manifold of dim. $4nr$ (a holomorphic symplectic manifold)
- $M_0(n,r)$: affine algebraic variety
- $\pi\colon M(n,r) \to M_0(n,r)$: projective morphism (resolution of singularities) defined by

$$(E,\varphi) \mapsto ((E^{\vee\vee},\varphi), \operatorname{Supp}(E^{\vee\vee}/E)).$$

(cf. J. Li, Morgan)

- These are quiver varieties for the Jordan quiver
  - V, W : cpx vector spaces with $\dim V = n$, $\dim W = r$
  - $\mathbb{M}(n,r) = \operatorname{End} V \oplus \operatorname{End} V \oplus \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W)$
  - $\mu\colon \mathbb{M}(n,r) \to \operatorname{End}(V)$; $\mu(B_1, B_2, a, b) = [B_1, B_2] + ab$



  - $M_0(n,r) = \mu^{-1}(0) /\!\!/ \operatorname{GL}(V)$ (affine GIT quotient)
  - $M(n,r) = \mu^{-1}(0)^{\text{stable}} / \operatorname{GL}(V)$
  - stable $\overset{\text{def.}}{\iff} \nexists S \subsetneq V$ with $B_\alpha(S) \subset S$, $\operatorname{Im} a \subset S$

Example $r = 1$: Hilbert scheme of points

Theorem. $M(n,1) = \mathrm{Hilb}^n(\mathbb{A}^2), \quad M_0(n,1) = S^n(\mathbb{A}^2)$

$\mathrm{Hilb}^n(\mathbb{A}^2)$: Hilbert scheme of n points in the affine plane $\mathbb{A}^2$

$S^n(\mathbb{A}^2)$: symmetric product (unordered n points with mult.)

Sketch of Proof

- $\mathrm{Hilb}^n(\mathbb{A}^2) = \{I \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n\}$
- Set $V = \mathbb{C}[x,y]/I$
  $B_1, B_2 = \times x, \times y$, $a(1) = 1 \mod I$, $b = 0$

◦ $S^n(\mathbb{A}^2) \to M_0(n,1)$ is induced by $\mathbb{A}^{2n} \to \mathbb{M}(n,1)$:
$(B_1, B_2, a, b) = (\mathrm{diag}(x_1, \ldots, x_n), \mathrm{diag}(y_1, \ldots, y_n), 0, 0)$

Torus action

- $T = T^{r-1}$: maximal torus in $\mathrm{SL}(W)$
- $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \curvearrowright M(n,r), M_0(n,r)$: torus action
  $(B_1, B_2, a, b) \longmapsto (t_1 B_1, t_2 B_2, a e^{-1}, t_1 t_2 e b)$
  $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, $e \in T$
- $\mathbb{C}[M_0(n,r)]$ (the coordinate ring of $M_0(n,r)$) is a $\widetilde{T}$-module.
- Similarly $H^i(M(n,r), \mathcal{O})$ and $H^i(M(n,r), E)$ (E : $\widetilde{T}$-equivariant sheaf) : $\widetilde{T}$-modules

Lemma. *Weight spaces of $\mathbb{C}[M_0(n,r)]$ (and $H^i(M(n,r), E)$) are finite dimensional.*

Let $t_1 = e^{\varepsilon_1}$, $t_2 = e^{\varepsilon_2}$, $e = (e^{a_1}, \dots, e^{a_r}) \in T$ $(\sum_{\alpha=1}^{r} a_\alpha = 0)$.

(identified with characters)

Define

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[M_0(n,r)]$$

$$\overset{\text{vanishing thm.}}{=} \sum_{n=0}^{\infty} \Lambda^{2nr} \sum_i (-1)^i \operatorname{ch}_{\widetilde{T}} H^i(M(n,r), \mathcal{O})$$

(**instanton part of Nekrasov's partition function**)

Problem. *Study* $Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$.

Example : $r = 1$

$$\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] = \exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d} \right)$$

Thus

$$\varepsilon_1 \varepsilon_2 \log \left(\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] \right) = \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{d^3} + \cdots \quad (\text{as } \varepsilon_1, \varepsilon_2 \to 0)$$

Nekrasov conjectured the same limiting bahaviour for $r \geq 2$.

(Explained later.)

A reformulation in terms of equivariant K-theory

Theorem (Combinatorial Expression).

$$Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{\vec{Y}} \Lambda^{2r|\vec{Y}|} \frac{1}{\mathrm{ch}_{\widetilde{T}} \left(\bigwedge_{-1} T^*_{\vec{Y}} \right)},$$

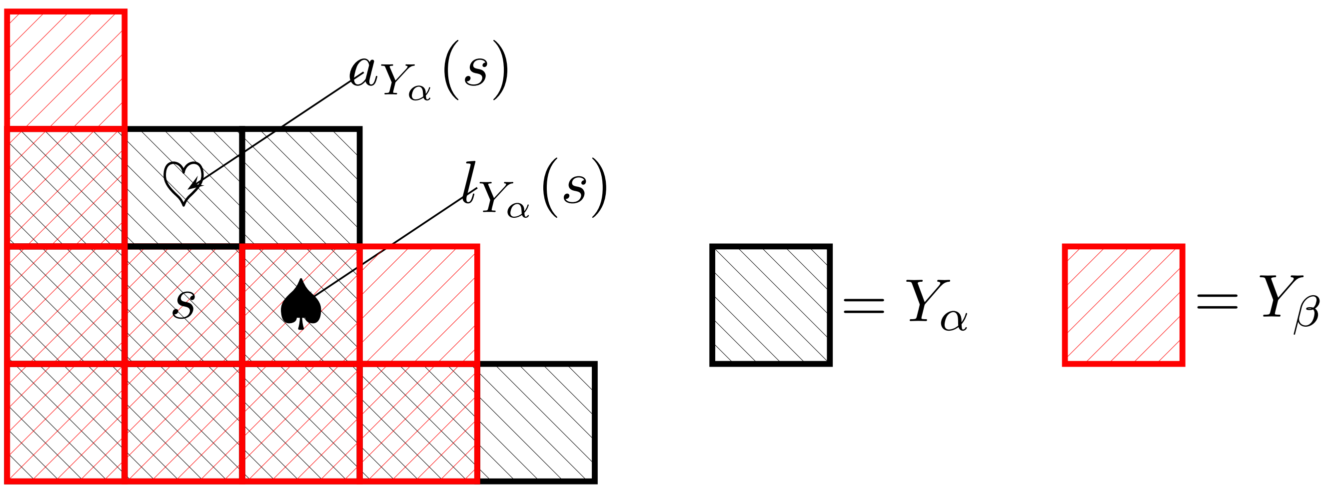
where $\vec{Y} = (Y_1, \dots, Y_r)$ is an r-tuple of Young diagrams with $|\vec{Y}| = \sum |Y_\alpha|$, and

$$\mathrm{ch}_{\widetilde{T}}\left(\bigwedge\nolimits_{-1} T^*_{\vec{Y}}\right)$$

$$= \prod_{\alpha,\beta}\prod_{s\in Y_\alpha}\left(1-\exp(-l_{Y_\beta}(s)\varepsilon_1 + (1+a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha)\right)$$

$$\times \prod_{t\in Y_\beta}\left(1-\exp((1+l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha)\right)$$

with

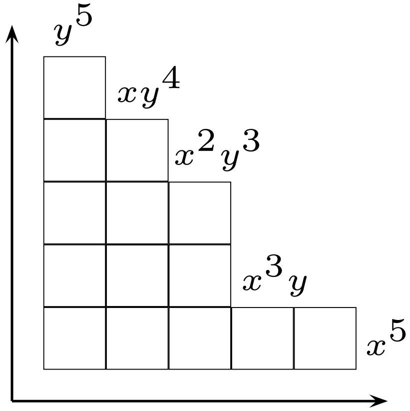


Fixed point set $M(n,r)^{\widetilde{T}}$

- $[B_1, B_2, a, b] \in M(n,r)$ is fixed by $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$

  ($\because W$ decomposes into 1-dim rep's of T)

- $M(n_\alpha, 1) = \mathrm{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\iff$ I_α is generated by monomials in x, y
  $\iff$ I_α corresponds to a Young diagram Y_α



- $M(n,r)^{\widetilde{T}} \cong \{\vec{Y} = (Y_1, \dots, Y_r) \mid \sum |Y_\alpha| = n\}$
- formula for the character of the tangent space:

$$
\begin{aligned}
&\mathrm{ch}_{\widetilde{T}}\left(\textstyle\bigwedge_{-1} T^*_{\vec{Y}}\right) \\
&\quad = \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} \big(1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha)\big) \\
&\qquad\qquad \times \prod_{t \in Y_\beta} \big(1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha)\big)
\end{aligned}
$$

follows from the computation of the Ext-group (or via (B_1, B_2, a, b)).

Let $r = 1$. Recall that we have computed $\mathrm{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)]$.

Corollary.

$$\exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d}\right)$$

$$= \sum_{Y} \prod_{s \in Y} \frac{\Lambda^{2|Y|}}{(1 - e^{-l_Y(s)\varepsilon_1 + (1+a_Y(s))\varepsilon_2})(1 - e^{(1+l_Y(s))\varepsilon_1 - a_Y(s)\varepsilon_2})}$$

Purely combinatorial proof : Cauchy formula for Macdonald polynomials, i.e., a generalization of the proof in the previous transparency.

Remark. *Appearance of Macdonald polynomials are natural in view of Haiman's work.*

Perturbation Part

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \stackrel{\text{def.}}{=} \frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{1}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2\log(\Lambda)\right)$$

$$+\sum_{n\geq 1}\frac{1}{n}\frac{e^{-nx}}{(e^{n\varepsilon_1}-1)(e^{n\varepsilon_2}-1)},$$

$$\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \stackrel{\text{def.}}{=} \gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) + \frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6}-\zeta(3)\right)$$

$$+\frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log(\Lambda)+\frac{\pi^2}{6}\right)+\frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log(\Lambda),$$

$$Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\text{def.}}{=} \exp\left(-\sum_{\alpha\neq\beta}\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha-a_\beta;\Lambda)\right)$$

Nekrasov Conjecture (2002)

Define the full partition function by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \stackrel{\text{def.}}{=} Z^{\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda).$$

Conjecture. *Suppose $r \geq 2$.*

$$\varepsilon_1 \varepsilon_2 Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = F_0 + \dots,$$

where F_0 is the **Seiberg-Witten prepotential**, *given by the period integral of certain curves.*

Seiberg-Witten geometry

A family of curves (*Seiberg-Witten curves*) parametrized by $\vec{X} = (X_1, \ldots, X_r)$ with $\prod X_\alpha = 1$:

$$C_{\vec{U}} : Y^2 = P(X)^2 - 4X^r \Lambda^{2r},$$

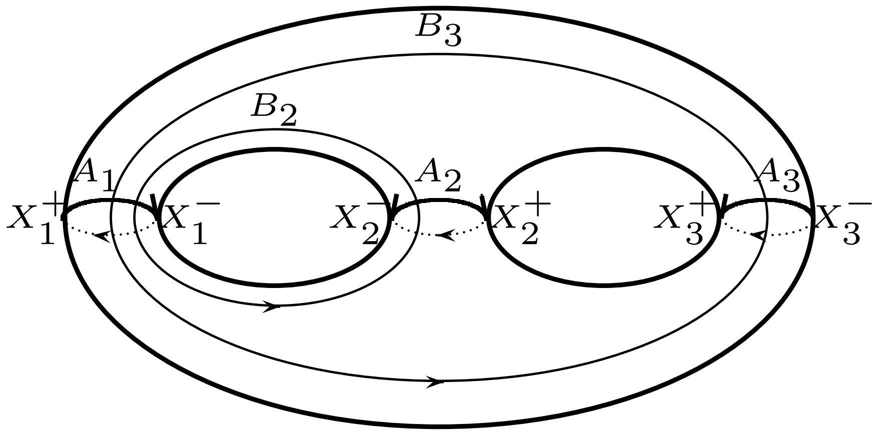
$$P(X) = X^r + U_1 X^{r-1} + \cdots + U_{r-1} X + (-1)^r = \prod_{\alpha=1}^{r} (X - X_\alpha).$$

$C_{\vec{U}} \ni (Y, X) \mapsto X \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(Y, X) = (-Y, X)$.

Define the *Seiberg-Witten differential* (multivalued) by

$$dS = -\frac{1}{2\pi} \log X \frac{(XP'(X) - \frac{r}{2} P(X))dX}{YX}.$$

Find branched points $X_\alpha^\pm$ near X_α (Λ small). Choose cycles A_α, B_α ($\alpha = 2, \ldots, r$) as



Put

$$a_\alpha = \int_{A_\alpha} dS, \qquad a_\beta^D = \int_{B_\beta} dS$$

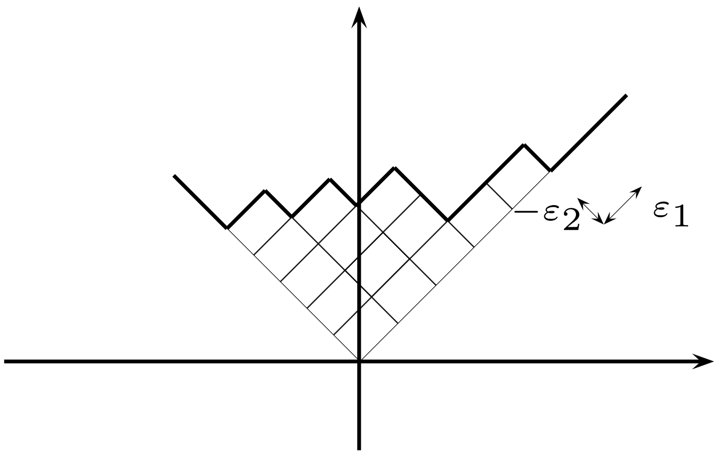
Then

$$\exists F_0 : \quad a_\beta^D = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial F_0}{\partial a_\beta}$$

Random partition

Theorem (Nekrasov-Okounkov). *Conjecture is true.*

Rotate the Young diagram 45° and consider it as a graph of a funtion.

$f_Y(x) =$ 

It satisfies

$$f(x) = |x| \quad \text{for } |x| \gg 0, \qquad |f(x) - f(y)| \leq |x - y|.$$

Put

$$f_{\vec{a},\vec{Y}}(x \mid \varepsilon_1, \varepsilon_2) = \sum_{\alpha=1}^{r} f_{Y_\alpha}(x - a_\alpha \mid \varepsilon_1, \varepsilon_2).$$

Then

- $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ is a sum over the space of such functions.
- Consider it as a mesure on the space of Lipschitz functions.

As $\varepsilon_1, \varepsilon_2 \to 0$, these measure converge to the delta measure at a single function f_*.

- Analysis of $f_* \longrightarrow$ SW geometry

Blowup equation – another proof

Theorem (N+Yoshioka). *Assume $r \geq 2$. Then Z satisfies*

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{\vec{k}\in\mathbb{Z}^{r-1}} \exp\left[\left(-\frac{(4d-r)(r-1)}{48}\right)(\varepsilon_1+\varepsilon_2)\right]$$

$$\times Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1\vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_1}{4}}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2\vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_2}{4}}).$$

- $\mathbb{Z}^{r-1}$ = the weight lattice of $\mathrm{SU}(r)$
- the equation determines the coefficients of Λ^{2nr} recursively.
- the equation guarantees $\exists F_0 = \lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \log Z$.

- it satifies a differential equation

$$\exp\left[-\frac{d^2}{8r^2}\frac{\partial^2 F_0}{(\partial\log\Lambda)^2}\right]\Theta_E\left(0\middle|\,\tau\right)$$
$$= \Theta_E\left(-\frac{d}{4\pi\sqrt{-1}r}\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\middle|\,\tau\right)$$

We can show that the Seiberg-Witten prepotential satisfies the same equation. As this equation characterizes F_0, we prove Nekrasov's conjecture.