Instanton Counting: the K-theoretic version

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based on

Nekrasov: hep-th/0206161

Nekrasov + Okounkov: hep-th/0306238

N + Kota Yoshioka: math.AG/0306198, math.AG/0311058,

math.AG/0505553

AMS Summer Institute on Algebraic Geometry
July 25, 2005, Univ. of Washington, Seattle
Additional references

- Braverman: math.AG/0401409
  (affine) Whittaker modules
- Braverman + Etingof: math.AG/0409441
- Göttscbe + N + Yoshioka: in preparation
  Donaldson invariants for projective toric surfaces
History

1994 Many important works on Donaldson invariants.

1994 Seiberg-Witten computed the prepotential of $N=2$ SUSY YM theory (physical counterpart of Donaldson invariants) via periods of Riemann surfaces (SW curve).

1997 Moore-Witten computed Donaldson invariants (blowup formulas, wall-crossing formulas...) via the SW curve.

2002 Nekrasov introduced a partition function $\frac{1}{24}$ `equivariant' Donaldson invariants for $\mathbb{R}^4$.

2003 Seiberg-Witten prepotential from Nekrasov's partition function (Nekrasov-Okounkov, N-Yoshioka).
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Aim of this talk

Study Nekrasov’s partition function for 5d. gauge theory = the K-theoretic version
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It should be related to the $K$-theoretic Donaldson invariants, indices of Dirac operators on instanton moduli spaces. Not defined for general 4-manifolds yet. But we have some nontrivial examples of calculations (→ Göttsche’s talk)
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Hope: Our study of moduli spaces on blowup may be useful.
Framed moduli spaces of instantons on $\mathbb{R}^4$

- $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{> 0}$.
- $M_{0}^{\text{reg}}(n, r)$: framed moduli space of SU($r$)-instantons on $\mathbb{R}^4$ with $c_2 = n$, where the framing is the trivialization of the bundle at $\infty$.

This is noncompact:
- bubbling
- $\exists$ parallel translation symmetry
We kill the first ‘source’ of noncompactness (bubbling) in two ways:

- $M_0(n, r)$: Uhlenbeck (partial) compactification

$$M_0(n, r) = \bigsqcup_{k=0}^{n} M_0^{\text{reg}}(k, r) \times S^{n-k} \mathbb{R}^4.$$
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\]

- $M(n, r)$: Gieseker (partial) compactification, i.e., the framed moduli space of rank $r$ torsion-free sheaves on $\mathbb{P}^2 = \mathbb{R}^4 \cup \ell_\infty$, parametrizing pairs $(E, \varphi)$
  - $E$: a torsion-free sheaf on $\mathbb{P}^2$ with $\text{rk} = r$, $c_2 = n$
  - $\varphi$: $E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing)
• $M(n, r)$: nonsingular hyperKähler manifold of dim. $4nr$ (a holomorphic symplectic manifold)

• $M_0(n, r)$: affine algebraic variety

• $\pi: M(n, r) \to M_0(n, r)$: projective morphism (resolution of singularities) defined by

\[(E, \varphi) \mapsto ((E^\vee \vee, \varphi), \text{Supp}(E^{\vee \vee} / E)).\]

(cf. J. Li, Morgan)

• Example: $M(n, 1) = \text{Hilb}^n(\mathbb{A}^2),$ \quad $M_0(n, 1) = S^n(\mathbb{A}^2)$
**Torus action**

- $T = T^{r-1}$: maximal torus in $\text{SL}(W)$
- $\tilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \sim M(n, r), M_0(n, r)$: torus action
  - $\mathbb{C}^* \times \mathbb{C}^*$ acts via $(x, y) \mapsto (t_1 x, t_2 y)$
  - $T$ acts via the change of the framing
- $\mathbb{C}[M_0(n, r)]$ (the coordinate ring) is a $\tilde{T}$-module. ← our main player
- Similarly $H^i(M(n, r), \mathcal{O})$ and $H^i(M(n, r), E)$ ($E: \tilde{T}$-equivariant sheaf): $\tilde{T}$-modules

**Lemma.** Weight spaces of $\mathbb{C}[M_0(n, r)]$ (and $H^i(M(n, r), E)$) are finite dimensional.
Thus the character makes sense as formal sum of polynomials in $t_1 = e^{e_1},$ $t_2 = e^{e_2},$ $e = (e^{a_1}, \ldots, e^{a_r}) \in T$ \( (\sum_{\alpha=1}^{r} a_{\alpha} = 0) \), i.e.,
\[
\sum t_1^l t_2^m e^{\sum n_{\alpha} a_{\alpha}} \dim \{ v \mid (t_1, t_2, e) \cdot v = t_1^l t_2^m e^{\sum n_{\alpha} a_{\alpha}} v \}
\]
Thus the character makes sense as formal sum of polynomials in $t_1 = e^{\epsilon_1}, t_2 = e^{\epsilon_2}, e = (e^{a_1}, \ldots, e^{a_r}) \in T$ \( \left( \sum_{\alpha=1}^{r} a_{\alpha} = 0 \right) \), i.e.,

\[
\sum t_1^l t_2^m e^{\sum n_{\alpha} a_{\alpha}} \dim \left\{ v \mid (t_1, t_2, e) \cdot v = t_1^l t_2^m e^{\sum n_{\alpha} a_{\alpha}} v \right\}
\]

Define

\[
Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} e^{-rn(\epsilon_1+\epsilon_2)/2} \text{ch}_{\tilde{T}} \mathbb{C}[M_0(n, r)]
\]

vanishing thm.

\[
\sum_{n=0}^{\infty} \Lambda^{2nr} e^{-rn(\epsilon_1+\epsilon_2)/2} \sum_{i} (-1)^i \text{ch}_{\tilde{T}} H^i(M(n, r), \mathcal{O})
\]

(instanton part of Nekrasov’s partition function)

**Problem.** Study $Z_{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; \Lambda)$. 
Example: \( r = 1 \)

\[
\sum_{n=0}^{\infty} \Lambda^{2n} \text{ch}_{\tilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] = \exp \left( \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d} \right)
\]

Thus

\[
\varepsilon_1\varepsilon_2 \log \left( \sum_{n=0}^{\infty} \Lambda^{2n} \text{ch}_{\tilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] \right) = \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{d^3} + o(\varepsilon_1, \varepsilon_2)
\]

as \( \varepsilon_1, \varepsilon_2 \to 0 \).

Nekrasov conjectured the same limiting bahaviour for \( r \geq 2 \).
(Explained later.)
The localization theorem in the equivariant K-theory gives

\[ Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; \Lambda) = \sum_{\bar{Y}} \Lambda^{2r|\bar{Y}|} \frac{1}{\text{ch}_{\mathcal{T}} \left( \Lambda_{-1}^{*} T_{\bar{Y}} \right)} , \]

where \( \bar{Y} = (Y_1, \ldots, Y_r) \) is an \( r \)-tuple of Young diagrams with \( |\bar{Y}| = \sum |Y_{\alpha}| \), and
\[
\operatorname{ch}_T^* \left( \Lambda_{-1} T_\beta^* \right) \\
= \prod_{\alpha, \beta} \prod_{s \in Y_\alpha} \left( 1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha) \right) \\
\times \prod_{t \in Y_\beta} \left( 1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha) \right)
\]

with

\[
\begin{align*}
& a_{Y_\alpha}(s) \\
& l_{Y_\alpha}(s) \\
& s
\end{align*}
\]

\[
\begin{align*}
& Y_\alpha \\
& Y_\beta
\end{align*}
\]
(\(E, \varphi\)) \(\in\) \(M(n, r)\) is fixed by \(T = T^{r-1}\) if and only if it is a direct sum of \(M(n_{\alpha}, 1)\) (\(\sum n_{\alpha} = n\))

\(\therefore W\) decomposes into 1-dim rep’s of \(T\)
Fixed point set $M(n, r)^\tilde{T}$

- $(E, \varphi) \in M(n, r)$ is fixed by $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ ($\sum n_\alpha = n$)
  
  $(\because W$ decomposes into 1-dim rep’s of $T$)

- $M(n_\alpha, 1) = \text{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\iff I_\alpha$ is generated by monomials in $x, y$
  $\iff I_\alpha$ corresponds to a Young diagram $Y_\alpha$

![Young diagram](image)
\[ M(n, r)^{\tilde{T}} \cong \{ \vec{Y} = (Y_1, \ldots, Y_r) \mid \sum |Y_\alpha| = n \} \]

- formula for the character of the tangent space:

\[
\chi_{\tilde{T}}(\bigwedge_{-1}T^*_{\vec{Y}})
= \prod_{\alpha, \beta} \prod_{s \in Y_\alpha} (1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha))
\]

\[
\times \prod_{t \in Y_\beta} (1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha))
\]

follows from the computation of the Ext-group (or via ADHM).
Let $r = 1$. Recall that we have computed $\text{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)]$.

**Corollary.**

$$
\exp \left( \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d} \right)
$$

$$
= \sum \prod_{Y \in Y} \frac{\Lambda^{2|Y|}}{(1 - e^{-l_Y(s)\varepsilon_1 + (1+a_Y(s))\varepsilon_2})(1 - e^{(1+l_Y(s))\varepsilon_1 - a_Y(s)\varepsilon_2})}
$$
Let $r = 1$. Recall that we have computed $\text{ch}_{\bar{T}} \mathbb{C}[S^n(A^2)]$.

**Corollary.**

$$
\exp \left( \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d} \right)
= \sum \prod_{Y \in Y} \frac{\Lambda^{2|Y|}}{(1 - e^{-l_Y(s)\varepsilon_1 + (1+a_Y(s))\varepsilon_2})(1 - e^{(1+l_Y(s))\varepsilon_1 - a_Y(s)\varepsilon_2})}
$$

Purely combinatorial proof: Cauchy formula for Macdonald polynomials, i.e., a generalization of the proof in the previous transparency.

**Remark.** Appearance of Macdonald polynomials are natural in view of Haiman’s work.
Nekrasov Conjecture (2002)

Define the full partition function by

\[ Z(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda) \overset{\text{def.}}{=} Z^\text{pert}(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda)Z^\text{inst}(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda). \]

**Conjecture.** Suppose \( r \geq 2 \).

\[ \varepsilon_1 \varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda) = F_0(\tilde{a}; \Lambda) + o(\varepsilon_1, \varepsilon_2), \]

where \( F_0 \) is the **Seiberg-Witten prepotential**, given by the period integral of certain curves.
Perturbation Part

\[ Z^{\text{pert}}(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda) \overset{\text{def.}}{=} \exp \left( - \sum_{\alpha \neq \beta} \tilde{\gamma}_{\varepsilon_1, \varepsilon_2}(a_\alpha - a_\beta; \Lambda) \right) \]
\[ \gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) \overset{\text{def.}}{=} \frac{1}{2\varepsilon_1 \varepsilon_2} \left( -\frac{1}{6} \left( x + \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \right)^3 + x^2 \log(\Lambda) \right) \]

\[ + \sum_{n \geq 1} \frac{1}{n \left( e^{n\varepsilon_1} - 1 \right) \left( e^{n\varepsilon_2} - 1 \right)} e^{-nx} \]

\[ \tilde{\gamma}_{\varepsilon_1, \varepsilon_2}(x; \Lambda) \overset{\text{def.}}{=} \gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) + \frac{1}{\varepsilon_1 \varepsilon_2} \left( \frac{\pi^2 x}{6} - \zeta(3) \right) \]

\[ + \frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1 \varepsilon_2} \left( x \log(\Lambda) + \frac{\pi^2}{6} \right) + \frac{\varepsilon_1^2 + \varepsilon_2^2 + 3\varepsilon_1 \varepsilon_2}{12\varepsilon_1 \varepsilon_2} \log(\Lambda), \]

\[ Z^\text{pert}(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda) \overset{\text{def.}}{=} \exp \left( - \sum_{\alpha \neq \beta} \tilde{\gamma}_{\varepsilon_1, \varepsilon_2}(a_\alpha - a_\beta; \Lambda) \right) \]
A family of curves (Seiberg-Witten curves) parametrized by \( \vec{X} = (X_1, \ldots, X_r) \) with \( \prod X_\alpha = 1 \):

\[
C_{\vec{U}} : Y^2 = P(X)^2 - 4X^r \Lambda^{2r},
\]

\[
P(X) = X^r + U_1X^{r-1} + \cdots + U_{r-1}X + (-1)^r = \prod_{\alpha=1}^{r} (X - X_\alpha).
\]

\( C_{\vec{U}} \ni (Y, X) \mapsto X \in \mathbb{P}^1 \) gives a structure of hyperelliptic curves. The hyperelliptic involution \( \iota \) is given by \( \iota(Y, X) = (-Y, X) \).

Define the Seiberg-Witten differential (multivalued) by

\[
dS = -\frac{1}{2\pi} \log X \frac{(XP'(X) - \frac{r}{2} P(X))}{YX} dX.
\]
Find branched points $X_{\pm\alpha}$ near $X_{\alpha}$ ($\Lambda$ small). Choose cycles $A_{\alpha}, B_{\alpha}$ ($\alpha = 2, \ldots, r$) as

![Diagram of branched points and cycles](image)

Put

$$a_{\alpha} = \int_{A_{\alpha}} dS, \quad a_{\beta}^{D} = \int_{B_{\beta}} dS$$

Then

$$\exists F_0 : \quad a_{\beta}^{D} = -\frac{1}{2\pi \sqrt{-1}} \frac{\partial F_0}{\partial a_{\beta}}$$
**Blowup equation**

**Theorem (N+Yoshioka).** Assume $r \geq 2$, $0 \leq d \leq r$. Then $Z$ satisfies

\[
Z \approx \frac{1}{(48 \ln \frac{r}{(r-1)})^2} \cdot (\frac{d}{r})^2 \cdot Z_r \approx \frac{1}{(r-2)^2} \cdot Z_r.
\]
Blowup equation

**Theorem (N+Yoshioka).** Assume $r \geq 2$, $0 \leq d \leq r$. Then $Z$ satisfies

$$Z(\varepsilon_1, \varepsilon_2, \bar{a}; \Lambda) = \sum_{\bar{k} \in \mathbb{Z}^{r-1}} \exp \left[ \left( -\frac{(4d - r)(r - 1)}{48} \right) (\varepsilon_1 + \varepsilon_2) \right] \times Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} + \varepsilon_1 \bar{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_1}{4}}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \bar{a} + \varepsilon_2 \bar{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_2}{4}}).$$

- $\mathbb{Z}^{r-1}$ = the weight lattice of SU($r$)
- the equation determines the coefficients of $\Lambda^{2nr}$ recursively.
Theorem (N+Yoshioka). Assume $r \geq 2$, $0 \leq d \leq r$. Then $Z$ satisfies

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$$\times Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \tilde{a} + \varepsilon_1 \vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_1}{4}}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \tilde{a} + \varepsilon_2 \vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_2}{4}}).$$

- $\mathbb{Z}^{r-1}$ = the weight lattice of SU($r$)
- the equation determines the coefficients of $\Lambda^{2nr}$ recursively.
- the equation guarantees $\exists F_0 = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log Z$. 
• it satisfies a differential equation

\[
\exp \left[ -\frac{d^2}{8r^2} \frac{\partial^2 F_0}{(\partial \log \Lambda)^2} \right] \Theta_E (0|\tau)
= \Theta_E \left( -\frac{d}{4\pi \sqrt{-1}r} \frac{\partial^2 F_0}{\partial \log \Lambda \partial \bar{a}} \bigg| \tau \right)
\]

Recently we check that the Seiberg-Witten prepotential satisfies the same equation. As this equation characterizes \( F_0 \), we prove Nekrasov’s conjecture.
Outline of the proof

- \( \widehat{M}(n, k, r) \): the framed moduli spaces on the blowup 
  \( p: \widehat{\mathbb{C}}^2 \to \mathbb{C}^2 \) \( (k = \langle c_1(E), [p^{-1}(0)] \rangle) \).

- Define a morphism \( \widehat{\pi}: \widehat{M}(n, k, r) \to M_0(n, r) \) by 
  \[(E, \varphi) \mapsto (((p_* E) \vee \vee, \Phi); \text{Supp}(p_* E \vee \vee / p_* E) + \text{Supp}(R^1 p_* E))\]
Outline of the proof

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- $\mathcal{O}(\mu(C))$: determinant line bundle in $\widehat{M}(n, k, r)$

- By the study of fixed points of $\widehat{M}(n, k, r)$, the index of $\mathcal{O}(d\mu(C))$ is given by the RHS of Theorem.
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- $\mathcal{O}(\mu(C))$: determinant line bundle in $\widehat{M}(n, k, r)$

- By the study of fixed points of $\widehat{M}(n, k, r)$, the index of $\mathcal{O}(d\mu(C))$ is given by the RHS of Theorem.

- $\mathcal{O}(-\mu(C))$ is $\widehat{\pi}$-nef and $\widehat{\pi}$-big, because it gives the morphism $\widehat{\pi}$. (cf. J.Li)
• Applying Kawamata-Viehweg vanishing theorem + (some arguments) to prove

\[ R^i \hat{\pi}_* \mathcal{O}_{\tilde{M}(n,0,r)}(d\mu(C)) = \begin{cases} 
\mathcal{O}_{M_0(n,r)} & i = 0, \\
0 & i > 0.
\end{cases} \]

for \(0 \leq d < r\).
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\[ R^i \widehat{\pi}_* \mathcal{O}_{\widehat{M}(n,0,r)}(d \mu(C)) = \begin{cases} \mathcal{O}_{M_0(n,r)} & i = 0, \\ 0 & i > 0. \end{cases} \]

for \( 0 \leq d < r \).

**Remark.** This proof is parallel to the earlier proof for homological version except the last part. Vanishing theorem was replaced by a simple dimension counting there.
Put $\varepsilon_1 = -\varepsilon_2 = ig_s$. ($g_s :$ string coupling constant)

**Conjecture.** Expand as

$$\log Z_{\text{inst}}(ig_s, -ig_s, \tilde{a}; \Lambda) = F_0 g_s^{-2} + F_1 g_s^0 + \cdots + F_g g_s^{2g-2} + \cdots.$$ 

Then $F_g$ is the genus $g$ Gromov-Witten invariant for certain noncompact Calabi-Yau 3-fold.

e.g., $r = 2$, Calabi-Yau = canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$
• based on geometric engineering by Katz-Klemm-Vafa (1996)

• Physical proof by Iqbal+Kashani-Poor, hep-th/0212279, hep-th/0306032 (based on earlier ideas by Vafa et al.)

• mathematical proof for $r = 2$ by Zhou, math.AG/0311237.
Via the blowup equation we can show
\[
\log Z(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda)
\]
\[
= \frac{1}{\varepsilon_1\varepsilon_2} F_0(\tilde{a}; \Lambda) + (\varepsilon_1 + \varepsilon_2) H + F_1 + \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1\varepsilon_2} G + \ldots
\]

with
Via the blowup equation we can show

$$\log Z(\varepsilon_1, \varepsilon_2, \tilde{a}; \Lambda)$$

$$= \frac{1}{\varepsilon_1\varepsilon_2}F_0(\tilde{a}; \Lambda) + (\varepsilon_1 + \varepsilon_2)H + F_1 + \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1\varepsilon_2}G + \ldots$$

with

- $H = - \sum_{\alpha < \beta} \pi \sqrt{-1} \frac{a_\alpha - a_\beta}{2}$ (come only from the perturbation part)

- $F_1 = - \log \eta(\tau/2)$, $G = \log \left[ q^{-1/24} \prod_{d=1}^{\infty} (1 - q^{2d-1}) \right]$

(for $r = 2$)

**Remark.** $F_1, G$ appears the wall-crossing formula of Donaldson invariants (and the u-plane integral)