Instanton Counting :

the K-theoretic version

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based on

Nekrasov : hep-th/0206161

Nekrasov + Okounkov : hep-th/0306238

N + Kota Yoshioka : math.AG/0306198, math.AG/0311058, math.AG/0505553

AMS Summer Institute on Algebraic Geometry
July 25, 2005, Univ. of Washington, Seattle

Additional references

- Braverman : math.AG/0401409

  (affine) Whittaker modules

- Braverman + Etingof :math.AG/0409441

- Göttsche + N + Yoshioka : in preparation

  Donaldson invariants for projective toric surfaces

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1994 Seiberg-Witten computed the *prepotential* of $N = 2$ SUSY YM theory (physical counterpart of Donaldson invariants) via periods of Riemann surfaces (SW curve).

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2002 Nekrasov introduced a partition function $\approx$ 'equivariant' Donaldon invariants for $\mathbb{R}^4$

2003 Seiberg-Witten prepotential from Nekrasov's partition function (Nekrasov-Okounkov, N-Yoshioka)

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Hope : Our study of moduli spaces on blowup may be useful.

Framed moduli spaces of instantons on $\mathbb{R}^4$

- $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$.
- $M_0^{\mathrm{reg}}(n, r)$: framed moduli space of $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4$ with $c_2 = n$, where the framing is the trivialization of the bundle at ∞.

  This is noncompact:

  - bubbling
  - $\exists$ parallel translation symmetry

We kill the first 'source' of noncompactness (bubbling) in two ways:

- $M_0(n,r)$: Uhlenbeck (partial) compactification

$$M_0(n,r) = \bigsqcup_{k=0}^{n} M_0^{\mathrm{reg}}(k,r) \times S^{n-k}\mathbb{R}^4.$$

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- $M(n,r)$: Gieseker (partial) compactification, i.e., the framed moduli space of rank r torsion-free sheaves on $\mathbb{P}^2 = \mathbb{R}^4 \cup \ell_\infty$, parametrizing pairs (E, φ)
  - E : a torsion-free sheaf on $\mathbb{P}^2$ with $\mathrm{rk} = r$, $c_2 = n$
  - φ: $E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing)

- $M(n,r)$: nonsingular hyperKähler manifold of dim. $4nr$ (a holomorphic symplectic manifold)
- $M_0(n,r)$: affine algebraic variety
- $\pi\colon M(n,r) \to M_0(n,r)$: projective morphism (resolution of singularities) defined by

$$(E,\varphi) \mapsto ((E^{\vee\vee},\varphi), \mathrm{Supp}(E^{\vee\vee}/E)).$$

(cf. J. Li, Morgan)

- Example : $M(n,1) = \mathrm{Hilb}^n(\mathbb{A}^2), \quad M_0(n,1) = S^n(\mathbb{A}^2)$

Torus action

- $T = T^{r-1}$: maximal torus in $\mathrm{SL}(W)$
- $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \curvearrowright M(n,r), M_0(n,r)$: torus action
  - $\mathbb{C}^* \times \mathbb{C}^*$ acts via $(x,y) \mapsto (t_1 x, t_2 y)$
  - T acts via the change of the framing
- $\mathbb{C}[M_0(n,r)]$ (the coordinate ring) is a $\widetilde{T}$-module.
  $\leftarrow$ our main player
- Similarly $H^i(M(n,r),\mathcal{O})$ and $H^i(M(n,r),E)$
  (E : $\widetilde{T}$-equivariant sheaf) : $\widetilde{T}$-modules

Lemma. *Weight spaces of $\mathbb{C}[M_0(n,r)]$ (and $H^i(M(n,r),E)$) are finite dimensional.*

Thus the character makes sense as formal sum of polynomials in $t_1 = e^{\varepsilon_1}$, $t_2 = e^{\varepsilon_2}$, $e = (e^{a_1}, \dots, e^{a_r}) \in T \quad (\sum_{\alpha=1}^{r} a_\alpha = 0)$, i.e.,

$$\sum t_1^l t_2^m e^{\sum n_\alpha a_\alpha} \dim \left\{ v \,\middle|\, (t_1, t_2, e) \cdot v = t_1^l t_2^m e^{\sum n_\alpha a_\alpha} v \right\}$$

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Define

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} e^{-rn(\varepsilon_1+\varepsilon_2)/2} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[M_0(n,r)]$$

$$\overset{\text{vanishing thm.}}{=} \sum_{n=0}^{\infty} \Lambda^{2nr} e^{-rn(\varepsilon_1+\varepsilon_2)/2} \sum_i (-1)^i \operatorname{ch}_{\widetilde{T}} H^i(M(n,r), \mathcal{O})$$

(**instanton part of Nekrasov's partition function**)

Problem. *Study* $Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$.

Example : $r = 1$

$$\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] = \exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1-e^{d\varepsilon_1})(1-e^{d\varepsilon_2})d} \right)$$

Thus

$$\varepsilon_1\varepsilon_2 \log\left(\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] \right) = \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{d^3} + o(\varepsilon_1, \varepsilon_2)$$

as $\varepsilon_1, \varepsilon_2 \to 0$.

Nekrasov conjectured the same limiting bahaviour for $r \geq 2$.

(Explained later.)

Combinatorial Expression

The localization theorem in the equivariant K-theory gives

Theorem.

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{\vec{Y}} \Lambda^{2r|\vec{Y}|} \frac{1}{\mathrm{ch}_{\widetilde{T}} \left(\bigwedge_{-1} T^*_{\vec{Y}} \right)},$$

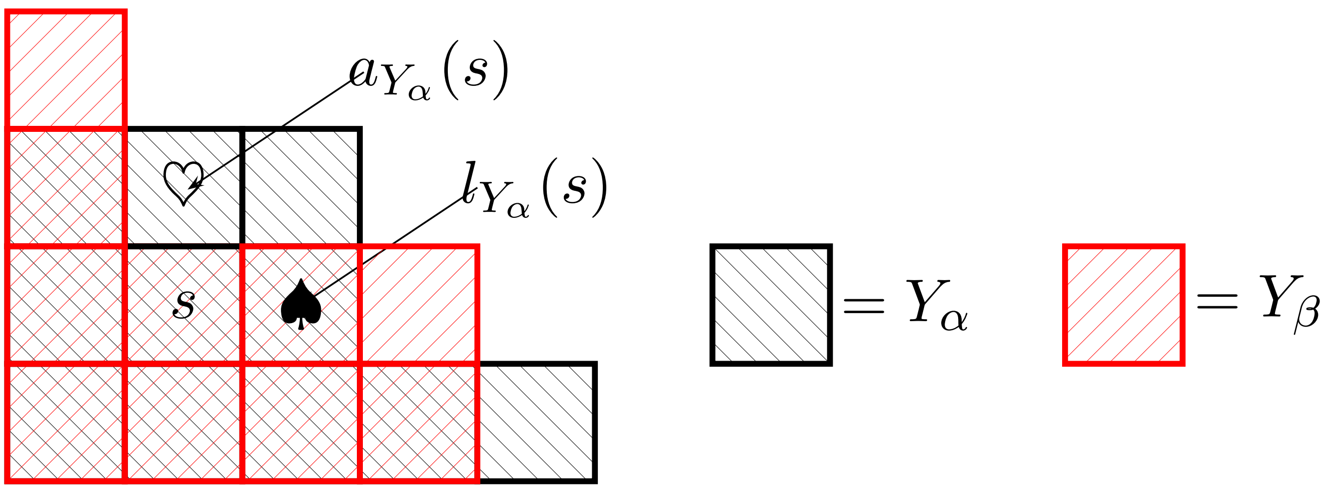
where $\vec{Y} = (Y_1, \ldots, Y_r)$ *is an* r*-tuple of Young diagrams with* $|\vec{Y}| = \sum |Y_\alpha|$*, and*

$$\mathrm{ch}_{\widetilde{T}}\left(\bigwedge\nolimits_{-1} T^*_{\vec{Y}}\right)$$

$$= \prod_{\alpha,\beta} \prod_{s\in Y_\alpha} \left(1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha)\right)$$

$$\times \prod_{t\in Y_\beta} \left(1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha)\right)$$

with



Fixed point set $M(n,r)^{\widetilde{T}}$

- $(E,\varphi) \in M(n,r)$ is fixed by $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$

  ($\because$ W decomposes into 1-dim rep's of T)

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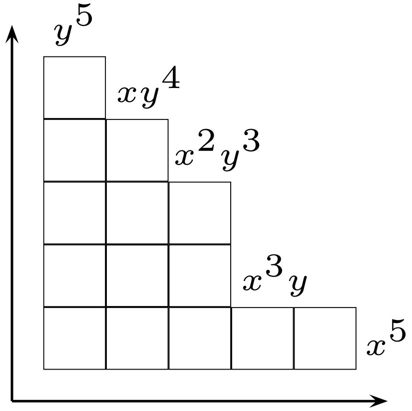
  $\Longleftrightarrow$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$
  ($\because$ W decomposes into 1-dim rep's of T)

- $M(n_\alpha, 1) = \mathrm{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\Longleftrightarrow$ I_α is generated by monomials in x, y
  $\Longleftrightarrow$ I_α corresponds to a Young diagram Y_α



- $M(n,r)^{\widetilde{T}} \cong \{\vec{Y} = (Y_1, \dots, Y_r) \mid \sum |Y_\alpha| = n\}$
- formula for the character of the tangent space:

$$\begin{aligned} &\mathrm{ch}_{\widetilde{T}} \left(\textstyle\bigwedge_{-1} T^*_{\vec{Y}} \right) \\ &\quad = \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} \left(1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha)\right) \\ &\qquad\qquad \times \prod_{t \in Y_\beta} \left(1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha)\right) \end{aligned}$$

follows from the computation of the Ext-group (or via ADHM).

Let $r = 1$. Recall that we have computed $\mathrm{ch}_{\widetilde{T}}\,\mathbb{C}[S^n(\mathbb{A}^2)]$.

Corollary.

$$\exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1-e^{d\varepsilon_1})(1-e^{d\varepsilon_2})d}\right)$$

$$= \sum_{Y} \prod_{s \in Y} \frac{\Lambda^{2|Y|}}{(1-e^{-l_Y(s)\varepsilon_1+(1+a_Y(s))\varepsilon_2})(1-e^{(1+l_Y(s))\varepsilon_1-a_Y(s)\varepsilon_2})}$$

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Purely combinatorial proof : Cauchy formula for Macdonald polynomials, i.e., a generalization of the proof in the previous transparency.

Remark. *Appearance of Macdonald polynomials are natural in view of Haiman's work.*

Nekrasov Conjecture (2002)

Define the full partition function by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \stackrel{\text{def.}}{=} Z^{\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda).$$

Conjecture. *Suppose* $r \geq 2$*.*

$$\varepsilon_1 \varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = F_0(\vec{a}; \Lambda) + o(\varepsilon_1, \varepsilon_2),$$

where F_0 *is the* **Seiberg-Witten prepotential**, *given by the period integral of certain curves.*

Perturbation Part

$$Z^{\mathrm{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \stackrel{\mathrm{def.}}{=} \exp\left(-\sum_{\alpha \neq \beta} \widetilde{\gamma}_{\varepsilon_1, \varepsilon_2}(a_\alpha - a_\beta; \Lambda) \right)$$

Perturbation Part

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \stackrel{\text{def.}}{=} \frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{1}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3 + x^2\log(\Lambda)\right)$$

$$+\sum_{n\geq 1}\frac{1}{n}\frac{e^{-nx}}{(e^{n\varepsilon_1}-1)(e^{n\varepsilon_2}-1)},$$

$$\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \stackrel{\text{def.}}{=} \gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) + \frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6} - \zeta(3)\right)$$

$$+\frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log(\Lambda)+\frac{\pi^2}{6}\right) + \frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log(\Lambda),$$

$$Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \stackrel{\text{def.}}{=} \exp\left(-\sum_{\alpha\neq\beta}\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta;\Lambda)\right)$$

Seiberg-Witten geometry

A family of curves (*Seiberg-Witten curves*) parametrized by $\vec{X} = (X_1, \ldots, X_r)$ with $\prod X_\alpha = 1$:

$$C_{\vec{U}} : Y^2 = P(X)^2 - 4X^r \Lambda^{2r},$$

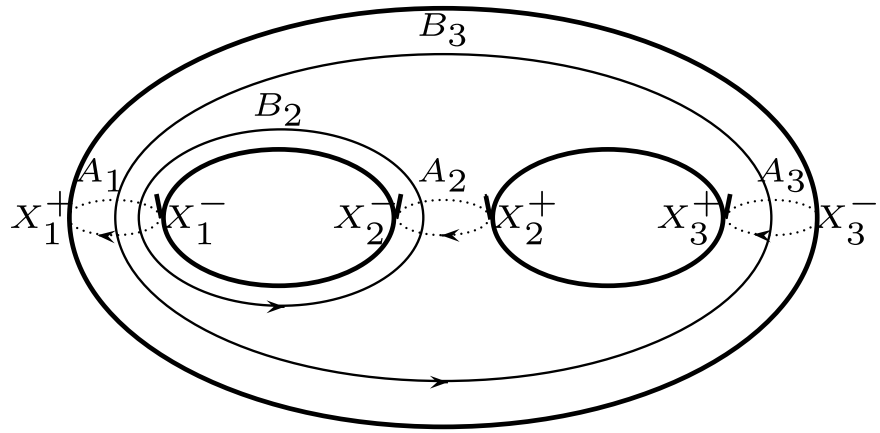
$$P(X) = X^r + U_1 X^{r-1} + \cdots + U_{r-1} X + (-1)^r = \prod_{\alpha=1}^{r} (X - X_\alpha).$$

$C_{\vec{U}} \ni (Y, X) \mapsto X \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(Y, X) = (-Y, X)$.

Define the *Seiberg-Witten differential* (multivalued) by

$$dS = -\frac{1}{2\pi} \log X \frac{(XP'(X) - \frac{r}{2} P(X))dX}{YX}.$$

Find branched points $X_\alpha^\pm$ near X_α (Λ small). Choose cycles A_α, B_α $(\alpha = 2, \ldots, r)$ as



Put

$$a_\alpha = \int_{A_\alpha} dS, \qquad a_\beta^D = \int_{B_\beta} dS$$

Then

$$\exists F_0 : \quad a_\beta^D = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial F_0}{\partial a_\beta}$$

Blowup equation

Theorem (N+Yoshioka). *Assume* $r \geq 2$, $0 \leq d \leq r$. *Then* Z *satisfies*

Blowup equation

Theorem (N+Yoshioka). *Assume $r \geq 2$, $0 \leq d \leq r$. Then Z satisfies*

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{\vec{k} \in \mathbb{Z}^{r-1}} \exp\left[\left(-\frac{(4d-r)(r-1)}{48}\right)(\varepsilon_1 + \varepsilon_2)\right]$$

$$\times Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_1}{4}}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}; \Lambda e^{\frac{d}{2r} - \frac{\varepsilon_2}{4}}).$$

- $\mathbb{Z}^{r-1}$ = the weight lattice of $\mathrm{SU}(r)$
- the equation determines the coefficients of Λ^{2nr} recursively.

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- $\mathbb{Z}^{r-1}$ = the weight lattice of $\mathrm{SU}(r)$
- the equation determines the coefficients of Λ^{2nr} recursively.
- the equation guarantees $\exists F_0 = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log Z$.

- it satifies a differential equation

$$\exp\left[-\frac{d^2}{8r^2}\frac{\partial^2 F_0}{(\partial\log\Lambda)^2}\right]\Theta_E\left(0\right|\tau) = \Theta_E\left(-\frac{d}{4\pi\sqrt{-1}r}\frac{\partial^2 F_0}{\partial\log\Lambda\partial\vec{a}}\right|\tau\bigg)$$

Recently we check that the Seiberg-Witten prepotential satisfies the same equation. As this equation characterizes F_0, we prove Nekrasov's conjecture.

Outline of the proof

- $\widehat{M}(n,k,r)$: the framed moduli spaces on the blowup $p\colon \widehat{\mathbb{C}}^2 \to \mathbb{C}^2$ $(k = \langle c_1(E), [p^{-1}(0)]\rangle)$.
- Define a morphism $\widehat{\pi}\colon \widehat{M}(n,k,r) \to M_0(n,r)$ by $(E,\varphi) \mapsto \big(((p_*E)^{\vee\vee},\Phi), \mathrm{Supp}(p_*E^{\vee\vee}/p_*E) + \mathrm{Supp}(R^1p_*E)\big)$

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- $\mathcal{O}(\mu(C))$: determinant line bundle in $\widehat{M}(n,k,r)$
- By the study of fixed points of $\widehat{M}(n,k,r)$, the index of $\mathcal{O}(d\mu(C))$ is given by the RHS of Theorem.

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- By the study of fixed points of $\widehat{M}(n,k,r)$, the index of $\mathcal{O}(d\mu(C))$ is given by the RHS of Theorem.
- $\mathcal{O}(-\mu(C))$ is $\widehat{\pi}$-nef and $\widehat{\pi}$-big, because it gives the morphism $\widehat{\pi}$. (cf. J.Li)

- Applying Kawamata-Viehweg vanishing theorem + (some arguments) to prove

$$R^i\widehat{\pi}_*\mathcal{O}_{\widehat{M}(n,0,r)}(d\mu(C)) = \begin{cases} \mathcal{O}_{M_0(n,r)} & i = 0, \\ 0 & i > 0. \end{cases}$$

for $0 \leq d < r$.

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Remark. *This proof is parallel to the earlier proof for homological version except the last part. Vanishing theorem was replaced by a simple dimension counting there.*

Nekrasov Conjecture (2002) - Part 2

Put $\varepsilon_1 = -\varepsilon_2 = ig_s$. ($g_s$: string coupling constant)

Conjecture. *Expand as*

$$\log Z^{\mathrm{inst}}(ig_s, -ig_s, \vec{a}; \Lambda) = F_0 g_s^{-2} + F_1 g_s^0 + \cdots + F_g g_s^{2g-2} + \cdots .$$

Then F_g is the genus g Gromov-Witten invariant for certain noncompact Calabi-Yau 3-fold.

e.g., $r = 2$, Calabi-Yau = canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$

- based on geometric engineering by Katz-Klemm-Vafa (1996)
- Physical proof by Iqbal+Kashani-Poor, hep-th/0212279, hep-th/0306032 (based on earlier ideas by Vafa et al.)
- mathematical proof for $r = 2$ by Zhou, math.AG/0311237.

Via the blowup equation we can show

$$\log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$$

$$= \frac{1}{\varepsilon_1 \varepsilon_2} F_0(\vec{a}; \Lambda) + (\varepsilon_1 + \varepsilon_2) H + F_1 + \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1 \varepsilon_2} G + \dots$$

with

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with

- $H = -\sum_{\alpha < \beta} \pi \sqrt{-1} \frac{a_\alpha - a_\beta}{2}$ (come only from the perturbation part)
- $F_1 = -\log \eta(\tau/2)$, $G = \log \left[q^{-1/24} \prod_{d=1}^{\infty} (1 - q^{2d-1}) \right]$ (for $r = 2$)

Remark. *F_1, G appears the wall-crossing formula of Donaldson invariants (and the u-plane integral)*