Convergence of anti-self-dual metrics

Conference on Variational Problems in Differential Geometry and Partial Differential Equations

Hiraku Nakajima

Mathematical Institute, Tôhoku University

Let (X, g) be a compact oriented Riemannian 4-manifold. The $*$ operator takes $\Lambda^2 = \Lambda^2 T^* X$ to itself and we have $** = 1$. Its ± 1 eigenspace is denoted by $\Lambda^{\pm}$. The curvature tensor R_g, viewed as an endomorphism on Λ^2, has the following matrix expression:

$$R_g = \begin{pmatrix} W_g^+ + \frac{1}{12} S_g & \mathrm{Ric}_g^0 \\ (\mathrm{Ric}_g^0)^* & W^- + \frac{1}{12} S_g \end{pmatrix},$$

where $\mathrm{Ric}_g^0 \in \mathrm{Hom}(\Lambda^+, \Lambda^-)$ is the trace-free part of the Ricci curvature, S_g is the scalar curvature, and $W_g^{\pm}$ is the (anti-)self-dual part of the Weyl tensor. The metric g is called *Einstein*, if $\mathrm{Ric}_g^0 = 0$, and *anti-self-dual*, if $W_g^+ = 0$.

It is natural to introduce the moduli spaces of these metrics, i.e., the quotient space of all such metrics by the group of diffeomorphisms. The local structures of these spaces (e.g. C^∞-structures) are discussed by N.Koiso (for Einstein), M.Itoh, A.King and D.Kotchick (for anti-self-dual). We want to study the global structure of the moduli space, especially the problem of the compactness. For Einstein metrics, this problem was studied by M.Anderson [An1] and by myself [Na] (see also [BKN]) and we have a satisfactory answer, at least when the scalar curvature is positive. So in this talk, I want to discuss about anti-self-dual metrics. Similar problem is recently studied by K. Akutagawa [Ak] for conformally flat metrics. If the metric is conformally flat, it is anti-self-dual. Our discussion is exactly parallel to the case of Einstein metrics, once we use Akutagawa's volume estimate (Lemma 2 below).

Before we enter the general theory, we first give examples:

Example (1) (O.Kobayashi, R.Schoen)**.** There exists a family of conformally flat metrics g_t on $S^1 \times S^3$ $(t \geq 1)$ such that the diameter of $S^1 \times \{x\}$ goes to 0 as $t \to \infty$ for a point $x \in S^3$. For other point $y \neq x$ the diameter of the slice $S^1 \times \{y\}$ is bounded from below. The limit space is the quotient space of S^4 which identifies the north pole and the south pole.

Supported in part by Grant-in-Aid for Scientific Reserch (No. 03740014), Ministry of Education, Science and Culture, Japan

Example (2) (C. LeBrun)**.** There exists a family of anti-self-dual metrics g_t on the connected sum $\overline{\mathbb{C}\mathrm{P}^2}\#\overline{\mathbb{C}\mathrm{P}^2}$ $(0 < t < 1)$ of two orientation reveresed complex projective surfaces such that the space converges to two $\overline{\mathbb{C}\mathrm{P}^2}$'s joined at a point as $t \to 0$, and to two orbifolds (each is the compactification of the Eguchi-Hanson gravitational instanton) joined at the singular point.

Now we want to discuss the behaviour of a genaral family of anti-self-dual metrics. The first problem is that the anti-self-duality of the metric depends only on the conformal class. So there is an action of the group of all positive functions on the moduli space. To divide out this action, we use the solution of the Yamabe problem by R.Schoen. Let μ be the Yamabe functional given by

$$\mu(g) \overset{\text{def.}}{=} \frac{\int_X R_g dV_g}{\mathrm{vol}(X,g)^{1/2}}.$$

Then we consider the following space:

$$\mathfrak{Asd}(X) \overset{\text{def.}}{=} \{g \mid g \text{ satisfies the following three conditions.}\} / \operatorname{Diff}_+(X),$$

(1) $W_g^+ = 0$,
(2) $\mathrm{vol}(X,g) = 1$,
(3) $\mu(g) = \inf\{\mu(h) \mid h \text{ is conformal to } g\}$.

Here $\operatorname{Diff}_+(X)$ is the group of orientation-preserving diffeomorphisms. Note that the minimizer of μ may not be unique in a given conformal class, but the space of the minimizer is compact in C^∞-topology unless $X = S^4$.

If g minimizes the Yamabe functional μ in its conformal class, then the scalar curvature S_g is a constant function. It follows that the Levi-Civita connection on Λ^+ is a Yang-Mills connection. So it is natural to try to apply Uhlenbeck's weak compactness theorem for Yang-Mills connection, especially the *a priori* estimate for the curvature. The trouble is that the base metric is not fixed and changing. By a careful check of her proof, we can obtain the desired estimate depending only on the Sobolev constants of the metrics. Thus it is natural to consider the following class for constants C_0, S_0:

$$\mathfrak{Asd}(X, C_0, S_0) \overset{\text{def.}}{=} \{[g] \in \mathfrak{Asd}(X) \mid g \text{ satisfies the following (1), (2).}\}$$

(1) $S_g \geq S_0$,
(2) the following Sobolev inequality holds for any function f:

$$(*) \qquad \left(\int_X |f|^4 dv_g \right)^{1/2} \leq \frac{1}{C_0} \int_X |df|_g^2 dv_g + \int_X |f|^2 dv_g,$$

where $|\cdot|_g$ is the norm for the cotangent vector given by the metric g.

The Yamabe functional and the Sobolev constant relate in the following way:

Lemma 1. *Suppose that g attains the minimum of the Yamabe functional μ in its conformal class and $\mu(g) > 0$. Then we can take $C_0 = const.\mu(g)$ in the above Sobolev inequality.*

The following Lemma was first obtained by K.Akutagawa [Ak] in his study of the convergence of conformally-flat metrics.

Lemma 2 (Akutagawa). *Let B_r be a metric ball of radius r in a compact Riemannian 4-manifold (X, g) satisfying the above Sobolev inequality (*) with some constant C_0. Then there exist constants r_0, V_0 depending only on C_0 such that the following holds if $r \leq r_0$:*

$$\mathrm{vol}(B_r, g) \geq V_0 r^4.$$

Proof (due to S.Bando). The absolute value of a harmonic function defined on a ball is estimated by (its L^2-norm)r^{-2}. This is proved by the Moser iteration method, and the constant depends only on the Sobolev constant. Apply this inequality to the constant function ! □

Corollary 3. *Let (X, g) be a compact Riemannian manifold of unit volume. Suppose that the Sobolev inequality (*) with constant C_0. Then the diameter of (X, g) is bounded from above by a constant depending only on C_0.*

Using the fact that the Levi-Civita connection is a Yang-Mills connection, one can prove the following as in the case of Einstein metrics:

Proposition 4. *Let g be an anti-self-dual metric defined on a metric ball B_r with a constant scalar curvature S_g. Let $p > 2$. Then there exists a positive constant ε depending on p, S_g and the costant C_0 in the Sobolev inequality in (*) such that if*

$$\int_{B_r} |R_g|^2 dv_g \leq \varepsilon,$$

then

$$r^{2-4/p} \left(\int_{B_{r/2}} |\operatorname{Ric}|^p dv_g \right)^{1/p} \leq C \left(\int_{B_r} |R_g|^2 dv_g \right)^{1/2},$$

where C is a constant depending only on p, S_g, C_0.

Using Anderson's method [An2], we have a bound on the L^p-norm of the full curvature tensor and can take a harmonic coordinate system under the smallness assumption on the L^2-norm of the curvarture. Combining with Lemma 2, we get the following:

Theorem 5. *Let X be a compact oriented 4-manifold. Suppose that constants C_0, S_0 are given. Let $[g_i]$ be a sequence in $\mathfrak{Asd}(X, C_0, S_0)$. Then either of the following two cases must be hold.*

(1) $\mathrm{diam}(X, g_i) \to 0$;

(2) *there exist a subsequence $\{j\} \subset \{i\}$ and a compact metric space X_∞ with positive diameter which contains a finite subset set $S = \{x_1, \ldots, x_k\} \subset X_\infty$ with the following properties:*

(2.a) *$X_\infty \setminus S$ has a structure of a C^∞-manifold and an anti-self-dual metric g_∞ which is compatible with the distance on $X_\infty \setminus S$;*

(2.b) *for every compact set $K \subset X_\infty \setminus S$, there exists an into diffeomorphism $F_j \colon K \to X$ for each j such that $F_j^* g_j$ converges to g_∞ on K.*

It seems very likely that the bounds on S_0, C_0 are necessary. Otherwise, the limit space may have the infinite diameter.

For Einstein metrics, the case (1) does not appear thanks to the Bishop volume comparison: If B_r is a ball in an Einstein 4-manifold with $\mathrm{Ric} = 3\lambda$, then

$$\mathrm{vol}(B_r) \le \mathrm{vol}(B_r^\lambda),$$

where B_r^λ is a metric ball of the radius r in the space form of constant scalar curvature λ.

I do not know whether the case (1) occurs actually or not.

For Einstein metrics, the limit space is an orbifold: a singular point has a neighbourhood homeomorphic to the cone $C(S^3/\Gamma)$ with $\Gamma \subset \mathrm{SO}(4)$ a finite subgroup, and the metric extends across the origin after pull back to the universal covering of $C(S^3/\Gamma) \setminus \{0\}$. If we try to adapt the proof of [BKN] to anti-self-dual metrics with constant scalar curvature, we encounter the difficulty caused by the lack of the volume. So we only have the following weak result:

Proposition 6. *Let X_∞ be the metric space appeared in Theorem 5. Suppose that*

(♠) *for any singular point $x \in X_\infty$, the volume of the metric ball centered at x can be estimated as $\mathrm{vol}(B_r(x)) \le Cr^4$ for any small r.*

Then X_∞ has a structure of a (generalized) orbifold: each singular point has a neighbourhood homeomorphic to the finite union of the cones $C(S^3/\Gamma)$ with $\Gamma \subset \mathrm{SO}(4)$ a finite group, joined at the vertex. The metric g_∞ extends as a (generalized) orbifold metric.

For a noncompact version, we have

Proposition 7. *Let (M, g) be a noncompact 4-dimensional anti-self-dual scalar-flat manifold with*

(1) $V_0 r^4 \le \mathrm{vol}(B_r) \le V_1 r^4$ *for any* $r > 0$

(2) $\int_M |R_g|^2 dv_g < \infty$

Then (M, g) is ALE: in each end the metric approximates the Euclidean metric $\mathbb{R}^4/\Gamma$.

Again by the lack of the volume comparison, we must assume the upper bound of the volume, which we do not need for Ricci-flat Einstein metrics. If one can drop this condition, then we have the desired results: the case (1) does not occur in Theorem 5, and the condition ♠ is not necessary in Proposition 6.

References

[Ak] K. Akutagawa, *Yamabe metrics of positive scalar curvature and conformally flat manifolds*, preprint (1992).

[An1] M. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. **2** (1989), 455–490.

[An2] ———, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math..

[BKN] S. Bando, A.Kasue and H.Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), 313–349.

[Na] H. Nakajima, *Hausdorff convergence of Einstein 4-manifolds*, J. of Fac. of Sci., the Univ. of Tokyo **35** (1988), 411–424.

Aramaki, Aoba-ku, Sendai 980, Japan

E-mail address: nakajima@math.tohoku.ac.jp