Cells in quantum affine algebras

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- $\mathfrak{g}$: affine Lie algebra

  (e.g., untwisted $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Q}[z, z^{-1}] \oplus \mathbb{Q}c \oplus \mathbb{Q}d$)

- $\mathfrak{h} = \bigoplus_i \mathbb{Q}h_i \oplus \mathbb{Q}d$: Cartan subalgebra

- $P^* \subset \mathfrak{h}$: dual weight lattice

- $P = \mathrm{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z}) \subset \mathfrak{h}^*$: weight lattice

- $(\ ,\ )$: nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ s.t. $(\delta, \lambda) = \langle c, \lambda \rangle$

- $\mathrm{cl}\colon \mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{Q}\delta$

- $P_{\mathrm{cl}} = \mathrm{cl}(P)$, $P^0 = \{\lambda \mid \langle c, \lambda \rangle = 0\}$, $P^0_{\mathrm{cl}} = \mathrm{cl}(P^0)$

NB. $\mathfrak{g}$ untwisted affine Lie algebra of $\mathfrak{g}_0$

$\Longrightarrow P^0_{\mathrm{cl}}$: weight lattice of $\mathfrak{g}_0$

- $q_s = q^{\min_i(\alpha_i,\alpha_i)/2}$
- $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$: quantum enveloping algebra (QEA) with P as the weight lattice (defined over $\mathbb{Q}(q_s)$)
- $\mathbf{U}'$: QEA with P_{cl} as the weight lattice, i.e., $\mathbf{U}' \subset \mathbf{U}$ (without q^d)
- $\mathbf{U}'_0$: QEA with P^0_{cl} as the weight lattice, i.e., $\mathbf{U}' \twoheadrightarrow \mathbf{U}'_0$ ($q^c = 1$, i.e., level 0)

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Modified version of U'_0

- $\tilde{\mathbf{U}} = \bigoplus_{\lambda \in P^0_{\mathrm{cl}}} \mathbf{U}'_0 a_\lambda$: modified enveloping algebra (of level 0)
- a_λ : projection to the weight space with weight $\lambda \in P^0_{\mathrm{cl}}$
- $\tilde{\mathcal{B}} = \bigoplus_{\lambda \in P^0_{\mathrm{cl}}} \mathcal{B}(\mathbf{U}'_0 a_\lambda)$: its global crystal basis

Kashiwara extremal weight modules

Definition 1. Let V be an integrable $\mathbf{U}$-module. A vector $v \in V$ with weight λ is called **extremal** if

$$\begin{cases} E_i T_w v = 0 & \text{if } \langle h_i, w\lambda \rangle \geq 0, \\ F_i T_w v = 0 & \text{if } \langle h_i, w\lambda \rangle \leq 0. \end{cases}$$

for all $w \in \hat{W}$ (affine Weyl group).

$\exists$ similar definition for regular crystals.

Theorem (Kashiwara 1994). (1) $\exists$ universal extremal weight module $V(\lambda)$ having an extremal weight vector $v_\lambda \in V(\lambda)$ of weight λ.

(2) For $w \in \hat{W}$, $\exists$ a **U**-module isomorphism $V(\lambda) \cong V(w\lambda)$, respecting the global crystal bases. (The inverse image of $v_{w\lambda}$ is $q_s^\star T_w v_\lambda$.)

(3) $V(\lambda)$ has a global crystal basis $\mathcal{B}(\lambda)$, s.t. $\tilde{\mathcal{B}}$ is mapped to $\mathcal{B}(\lambda) \sqcup \{0\}$.

In fact, Kashiwara gives a combinatorial description of elements of $\tilde{\mathcal{B}}$ mapped to 0.

Quiver varieties

Assume $\mathfrak{g}$ is symmetric and λ is (level 0) dominant.

Consider quiver varieties:

- $\mathfrak{M}(\lambda) = \bigsqcup_{\mu \le \lambda} \mathfrak{M}(\mu, \lambda) \supset \mathfrak{L}(\lambda) = \bigsqcup_{\mu \le \lambda} \mathfrak{L}(\mu, \lambda)$
- $\mathfrak{M}_0(\infty, \lambda) = \bigcup_{\mu \in P_{\mathrm{cl}}^{0,+},\ \mu \le \lambda} \mathfrak{M}_0(\mu, \lambda)$
- $\pi \colon \mathfrak{M}(\lambda) \to \mathfrak{M}_0(\infty, \lambda)$
- $G_\lambda = \prod_i \mathrm{GL}(\lambda_i)$ $(\lambda = \sum_i \lambda_i \varpi_i)$
- $K^{G_\lambda \times \mathbb{C}^*}(\mathfrak{L}(\lambda))$: equivariant K-homology
- it is a module over $R(G_\lambda \times \mathbb{C}^*) = R(G_\lambda)[q, q^{-1}]$

Theorem (N). One can construct a $\mathbf{U}_0'$-module strucutre on $K^{G_\lambda \times \mathbb{C}^*}(\mathfrak{L}(\lambda))$ via the convolution product.

In particular, it commutes with the $R(G_\lambda)[q, q^{-1}]$-structure.

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- $\mathbf{U}'_0$-str. on LHS is given in Drinfeld realization, while that on RHS is given in Chevalley generators.
- The existence of $R(G_\lambda)$-module structure on $V(\lambda)$ is non-trivial.
- $\pm$\{global basis\} has a characterization as bar-invariant + almost orthogonal. The bar involution and (,) can be defined via geometry. (Lusztig's conjectural signed basis for equivariant K-theory)

  But the existence is not proved via geometry so far.

  The bar involution is defined via Grothendieck-Serre duality + an 'opposition' $^\vee\colon \mathfrak{M}(\lambda) \to \mathfrak{M}(\lambda)$.

Remarks.

(1) An irr. f.d. representation of $(\mathbf{U}_0')_{q_s=\varepsilon}$: unique simple quotient of $V(\lambda)\otimes_\chi \mathbb{C}$ where $\chi\colon R(G_\lambda\times\mathbb{C}^*)\to\mathbb{C}$ with $\chi(q_s)=\varepsilon$.

After the specialization $q_s=\varepsilon$:

$\{$iso. classes of irr. rep.'s$\}\leftrightarrow\{\chi\colon R(G_\lambda)\to\mathbb{C}\}/$conjugate
$\leftrightarrow\{s\in G_\lambda\mid \text{semisimple}\}\leftrightarrow\{$Drinfeld poly.'s$\}$

(2) If $\mu\in W\lambda$, $\mathfrak{M}(\mu,\lambda)=\mathfrak{L}(\mu,\lambda)=$ point.

Hence $K^{G_\lambda\times\mathbb{C}^*}(\mathfrak{L}(\mu,\lambda))\cong R(G_\lambda)[q,q^{-1}]$.

The global basis elements in this weight space are

$$\{\text{irr. rep's of } G_\lambda\}\times\mathcal{O}_{\text{point}}.$$

$(\mathcal{O}_{\text{point}}\in K^{G_\lambda\times\mathbb{C}^*}(\mathfrak{L}(\lambda,\lambda))$ is the vector v_λ.)

These are the 'extremal' vectors in $V(\lambda)$. In particular, other elements in the crystal basis are connected to one of those by Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$.

Several points of the proof

- Show the assertion for $\lambda = \varpi_i$ (level 0 fundamental representation) : easy, thanks to irreducibility

  NB. $V(\varpi_i)$ has a $R(\mathbb{C}^*) = \mathbb{Z}[z^{\pm}]$-module structure. (Kashiwara)

- $K^{G_\lambda \times \mathbb{C}^*}(\mathfrak{L}(\lambda)) \subset K^{T_\lambda \times \mathbb{C}^*}(\mathfrak{L}(\lambda)) \approx \bigotimes_i K^{\mathbb{C}^* \times \mathbb{C}^*}(\mathfrak{L}(\varpi_i))^{\otimes \lambda_i}$.

  '$\approx$' becomes '$\cong$' if

  - tensor the quotient field of $R(T_\lambda)$, or
  - replace $\mathfrak{L}(\lambda)$ by a 'tensor product variety' $\mathfrak{Z}(\lambda)$

- Prove similar statements for $V(\lambda)$:

  - $V(\lambda) \subset \breve{V}(\lambda) \subset \widetilde{V}(\lambda) \overset{\text{def.}}{=} \bigotimes_{i \neq 0} V(\varpi_i)^{\otimes \lambda_i}$
  - where $\breve{V}(\lambda) \overset{\text{def.}}{=} \mathbf{U}[z_{i,\nu}^{\pm}]\widetilde{v}_\lambda \subset \widetilde{V}(\lambda)$ with $\widetilde{v}_\lambda = \bigotimes v_{\varpi_i}^{\otimes \lambda_i}$
  - $V(\lambda) \cong \mathbf{U}\widetilde{v}_\lambda$

level 0 fundamental representations

Choose $0 \in I$ with $a_0 = 1$. For $i \neq 0$, let

$$\varpi_i = \begin{cases} \Lambda_i - a_i^\vee \Lambda_0 & \text{if } (\mathfrak{g}, i) \neq (A_{2n}^{(2)}, n) \\ 2\Lambda_n - \Lambda_0 & \text{otherwise} \end{cases}$$

(level 0 fundamental weight)

Theorem (Kashiwara). (1) $\exists z_i \in \mathrm{Aut}_{\mathbf{U}'}(V(\varpi_i))$ of weight $d_i\delta$ ($d_i = \max(1, (\alpha_i, \alpha_i)/2)$) s.t. $z_i v_{\varpi_i} = v_{\varpi_i + d_i\delta}$
(2) $V(\varpi_i) \cong \mathbb{Q}(q_s)[z_i, z_i^{-1}] \otimes W(\varpi_i)$, where $W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i)$ (finite dimensional $\mathbf{U}'$-module).

Two-sided cells

- A : associative algebra over a commutative ring R
- $\mathcal{B}$: a basis of A as an R-module
- $\mathcal{F} \stackrel{\text{def.}}{=} \{K \subset \mathcal{B} \mid \mathrm{Span}(K) \text{ is a two-sided ideal}\}$
- $b' \preceq b$ $(b, b' \in \mathcal{B}) \stackrel{\text{def.}}{\Longleftrightarrow} \forall K \in \mathcal{F}$ with $b \in K \Rightarrow b' \in K$.
- $b \sim b' \stackrel{\text{def.}}{\Longleftrightarrow} b \preceq b'$ and $b' \preceq b$.
- two-sided cells in $\mathcal{B}$: equivalence classes for $\sim$
- left (right) cells : two-sided ideal $\rightarrow$ right (left) ideal

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If $\mathcal{B}$ is *generic*, then $\mathcal{F} = \{\mathcal{B}\}$ and $\exists$ only one cell.

We should choose a *good* basis, e.g., Kazhdan-Lusztig basis for the Iwahori-Hecke algebra.

Recall that the modified enveloping algebra $\tilde{\mathbf{U}}$ has a global basis (Lusztig).

- $P_{\mathrm{cl}}^{0,+}$: (level 0) dominant weights, $\leq$: dominance order
- $\tilde{\mathbf{U}}[^{\geq}\lambda] \overset{\text{def.}}{=} \{x \in \tilde{\mathbf{U}} \mid \ x \text{ acts by } 0 \text{ on } V(\mu) \text{ with } \mu \not\geq \lambda\}$ (two-sided ideal)
- example : $a_\lambda \in \tilde{\mathbf{U}}[^{\geq}\lambda]$, as all weights of $V(\mu)$ are $\leq \mu$.
- $\tilde{\mathbf{U}}[\lambda] \overset{\text{def.}}{=} \tilde{\mathbf{U}}[^{\geq}\lambda]/\tilde{\mathbf{U}}[^{>}\lambda]$
- $\tilde{\mathcal{B}}[\lambda] \overset{\text{def.}}{=} \{b \in \tilde{\mathcal{B}} \mid b \in \tilde{\mathbf{U}}[^{\geq}\lambda], b|_{V(\lambda)} \neq 0\}$

NB. Similar definitions for $\tilde{\mathbf{U}}$ of finite type were given by Lusztig. ('Based modules')

Theorem (Beck+N.). (1) $\tilde{\mathcal{B}}[\lambda]$ descends to a global crystal basis of $\tilde{\mathbf{U}}[\lambda]$ under $\tilde{\mathbf{U}}[^{\geq}\lambda] \to \tilde{\mathbf{U}}[\lambda]$.
(2) $\tilde{\mathcal{B}}[\lambda]$ is a two-sided cell of $\tilde{\mathcal{B}}$, and all two-sided cells are of this form.

By definition, $\tilde{\mathbf{U}}[\lambda] \hookrightarrow \mathrm{End}(V(\lambda))$. It is natural to expect that $\tilde{\mathbf{U}}[\lambda]$ is similar to the matrix algebra. It is true for finite types (Lusztig). For affine types, is 'almost' true...

$\tilde{\mathrm{U}}$ at $q = 0$ (different from Kashiwara's $q = 0$)

- Let $\mathfrak{L}(\tilde{\mathbf{U}}[\lambda])$ be $\mathbb{Z}[q_s]$-submodule of $\tilde{\mathbf{U}}[\lambda]$ generated by $\tilde{\mathcal{B}}[\lambda]$
- Define $a(b) \in \mathbb{Z} \sqcup \{\infty\}$ as $\min\{n \mid q_s^n b \mathfrak{L}(\tilde{\mathbf{U}}[\lambda]) \subset \mathfrak{L}(\tilde{\mathbf{U}}[\lambda])\}$.

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The **property P1** holds:

(1) $a(b) < \infty$

(2) $\forall \mu$, the restriction $a|_{\tilde{\mathcal{B}}[\lambda] \cap \mathcal{B}(\mathbf{U}a_\mu)}$ is constant.

NB. $a|_{\tilde{\mathcal{B}}[\lambda] \cap \mathcal{B}(\mathbf{U}a_\mu)} = \dim \mathfrak{L}(\mu, \lambda)$.

- Let $c_{bb'}^{b''}(q)$: structure constant of $\tilde{\mathbf{U}}[\lambda]$ w.r.t. the basis $\tilde{\mathcal{B}}[\lambda]$
- $\widehat{b} \stackrel{\text{def.}}{=} q_s^{a(b)} b$: normalized basis element

We have

$$\widehat{b}\widehat{b'} = \sum_{b'' \in \tilde{\mathcal{B}}[\lambda]} q_s^{a(b)} c_{bb'}^{b''} \widehat{b''}$$

(We used that we may assume $a(b') = a(b'')$.)

- By the definition of a, we have $q_s^{a(b)} c_{bb'}^{b''} \in \mathbb{Z}[q_s]$.
- Therefore, $\tilde{\mathbf{U}}[\lambda]^- \stackrel{\text{def.}}{=} \bigoplus \mathbb{Z}[q_s]\widehat{b}$ is closed under the multiplication
- Let $\tilde{\mathbf{U}}[\lambda]_0 \stackrel{\text{def.}}{=} \tilde{\mathbf{U}}[\lambda]^- / q_s \tilde{\mathbf{U}}[\lambda]^-$ (asymptotic algebra)

- t_b : image of $\widehat{b}$, a $\mathbb{Z}$-basis of $\tilde{\mathbf{U}}[\lambda]_0$
- structure constant = constant part of $q_s^{a(b)} c_{bb'}^{b''}$

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We have the **property P2** (existence of the generalized unit):

$^{\exists}$ a finite set $\mathcal{D}_{\tilde{\mathcal{B}}[\lambda]} \subset \tilde{\mathcal{B}}[\lambda]$ s.t. $t_d t_{d'} = \delta_{dd'} t_d$ for $d, d' \in \mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}$

and $^{\forall} b \in \tilde{\mathcal{B}}[\lambda]$ $^{\exists 1} d, d' \in \mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}$ s.t. $t_b = t_d t_b t_{d'}$

NB. $\mathcal{D}_{\tilde{\mathcal{B}}[\lambda]} \subset \tilde{\mathcal{B}}[\lambda]$ can be naturally identified with the global basis of a finite dimensional module $\bigotimes_i W(\varpi_i)^{\otimes \lambda_i}$.

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and $\forall b \in \tilde{\mathcal{B}}[\lambda]$ $\exists 1 d, d' \in \mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}$ s.t. $t_b = t_d t_b t_{d'}$

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We also have the **property P3**:

$\Phi \colon \tilde{\mathbf{U}}[\lambda] \to \tilde{\mathbf{U}}[\lambda]_0 \otimes \mathbb{Q}(q_s)$ defined by
$\Phi(b) = \sum_{d \in \mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}, b' \in \tilde{\mathcal{B}}[\lambda]} c_{bd}^{b'} t_{b'}$ is an algebra homomorphism.

- $G_\lambda = \prod_{i \in I_0} GL(\lambda_i)$ $(\lambda = \sum \lambda_i \varpi_i)$
- $\operatorname{Irr} G_\lambda$: the set of (iso. classes of) irreducible rep's.
  $c_{ss'}^{s''} = \dim \operatorname{Hom}(s'', s \otimes s')$
- $T_\lambda \overset{\text{def.}}{=} \{(d_1, s, d_2) \mid d_i \in \mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}, s \in \operatorname{Irr} G_\lambda\}$
- J_λ : a free $\mathbb{Z}$-module with basis T_λ with the multiplication :

$$(d_1, s, d_2)(d_1', s', d_2') = \sum_{s''} c_{ss'}^{s''} \delta_{d_2, d_1'} (d_1, s'', d_2')$$

J_λ is a matrix algebra over the represenatation ring of G_λ (size $= \#\mathcal{D}_{\tilde{\mathcal{B}}[\lambda]}$), and T_λ is its basis.

Theorem (Beck+N.). $(\tilde{\mathbf{U}}[\lambda]_0, \tilde{\mathcal{B}}[\lambda])$ is isomorphic to (J_λ, T_λ) as a based ring.

NB. A similar result was conjectured for cells of an affine Hecke algebra by Lusztig. A big progress was made by Bezrukavnikov+Ostrik.

Problems

- Prove the existence of the crystal basis in a geometric way. (cf. Bezrukavnikov's talk.)
- Define Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ in a geometry way.
- Try to extend these to quantum toroidal algebras. (K-theory for quiver varieties of affine types.)