QUIVER VARIETIES AND MCKAY CORRESPONDENCE – LECTURES AT HOKKAIDO UNIVERSITY, 2001 DEC. –

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Lecture Announcement

The theme of lectures is an interplay of representation theory and geometry. We will explain the author's *old* results [N3] in 1995, a geometric construction of affine Lie algebras and their representations. These are constructed by using convolution products on homology groups of so-called *quiver varieties*, which was introduced by the author. But here, we use the language of Hilbert schemes, instead of general quiver varieties, hoping this makes more accecible to readers. Drawbacks are

(1) We can treat only affine Lie algebras, not general Kac-Moody algebras.
(2) We can treat only basic representations, not whole irreducible integrable representations.

But the main feature of the theory still survives.

We also discuss on a construction of Kashiwara's crystal structure on the set of irreducible components of lagrangian subvarieties in Hilbert schemes. This construction is due to Kashiwara-Saito [KS].

Our method is an application of more general technique which has been used successfully in the representation theory during the last several decades (see [CG, G]). It is the construction of algebras by the *convolution product*, defined on homology groups (or their variants) of manifolds. For example, Weyl groups and affine Hecke algebras were constructed by convolutions on homology groups and equivariant K-homology groups of flag varieties (Springer, Borho-MacPherson, Lusztig, Ginzburg, Kazhdan-Lusztig, etc). Also upper triangular parts of quantum enveloping algebras and their canonical bases were constructed by convolutions using perverse sheaves on moduli spaces of representations of quivers (Lusztig).

We also discuss a geometric explanation of McKay correspondence given in [N2]. The McKay correspondence is roughly a correspondence between nontrivial irreducible representations of a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ and irreducible components of the exceptional set of the minimal resolution $X \to \mathbb{C}^2/\Gamma$. The original correspondence was constructed by identifying both with vertices of the Dynkin diagram corresponding to Γ. In [N2] we described irreducible components in terms of representation theory of Γ, where the minimal resolution was identified with a special quiver variety. Here we use the language of Hilbert schemes also. The explanation turns out to be same as one found afterward independently by Ito-Nakamura [IN].

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References

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhähser, 1997.

[G] V. Ginzburg, *Geometric methods in representation theory of Hecke algebras and quantum groups*, in "Representation theories and algebraic geometry" (Montreal, PQ, 1997), 127–183, Kluwer Acad. 1998, (math.AG/9802004).

[IN] Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, in New trends in algebraic geometry, Proc. Europian Algebraic Geometry Conf., Warwick 1996, Cambridge University Press, 1999, 151–233.

[Lecture] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, **18**, American Mathematical Society, 1999.

[N1] ———, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[N2] ———, *Varieties associated with quivers*, in *Representation theory of algebras and related topics*, CMS conference proceedings **19**, AMS (1996) 139–157.

[N3] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[KS] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), no. 1, 9–36.

Contents

1. Convolution product

The first two sections are general theory of convolution products.

1.1. **General definition.** Let X, Y be finite sets. Let $\mathcal{F}(X)$, $\mathcal{F}(Y)$ the vector space of $\mathbb{C}$-valued functions on X, Y. If a $\mathbb{C}$-valued fuction $K(x,y)$ on $X \times Y$ is given, we can define an operator $\mathcal{F}(Y) \to \mathcal{F}(X)$ by

$$\mathcal{F}(Y) \ni f(y) \longmapsto (K * f)(x) \overset{\text{def.}}{=} \sum_{y \in Y} K(x,y) f(y) \in \mathcal{F}(X).$$

This is called the *convolution product* of K and f.

Suppose X, Y, Z are finite sets. For given functions $K(x,y)$ on $X \times Y$ and $K'(y,z)$ on $Y \times Z$, we consider the composition of operators given by the convolution products:

$$K * (K' * f)(x) = \sum_{y \in Y} K(x,y) \left(\sum_{z \in Z} K'(y,z) f(z) \right).$$

This is equal to

$$\sum_{z \in Z} \left(\sum_{y \in Y} K(x,y) K'(y,z) \right) f(z).$$

Hence if we define $(K * K')(x,z) \overset{\text{def.}}{=} \sum_{y \in Y} K(x,y) K'(y,z)$, then the composite of operators is given again by the convolution product.

If we take $X = Y$, then the vector space of $\mathbb{C}$-valued functions on $X \times X$, which we denote by $\mathcal{F}(X \times X)$ is an algebra by the above convolution product $K * K'$. It is clearly associative:

$$(K * K') * K'' = K * (K' * K'').$$

And the unit is given by the characteristic function of the diagonal Δ_X:

$$K * \Delta_X = \Delta_X * K = K, \qquad \text{where } \Delta_X(x,x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The algebra $\mathcal{F}(X \times X)$ has a natural representation. Namely $\mathcal{F}(X)$ under the convolution product !

Example 1.1 (Trivial)**.** Suppose $\#X = n$. Then $\mathcal{F}(X \times X)$ is the matrix algebra of $n \times n$ matrices. $\mathcal{F}(X)$ is the vector representation.

This example means that we need to consider a subalgebra of $\mathcal{F}(X \times X)$ in order to get an interersting algebra.

1.2. **Iwahori-Hecke algebra (due to Iwahori).** The Iwahori-Hecke algebra $\mathcal{H}_q$ is a q-analogue of the group ring of the Weyl group W associated with a complex simple Lie algebra $\mathfrak{g}$. Here q is an indeterminate (parameter). We consider the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the Weyl group W is $\mathbb{Z}/2\mathbb{Z}$. The Iwahori-Hecke algebra $\mathcal{H}_q$ is the $\mathbb{C}[q,q^{-1}]$-algebra with generator T and the defining relation

$$(T - q)(T + 1) = 0.$$

Note that the relation reduces to $T^2 = 1$ if $q = 1$.

Let $k = \mathbb{F}_q$ be the finite field of q elements. We consider the projective line $\mathbb{P}^1(k)$ of k, the space of 1-dimensional subspaces of k^2. We consider a natural action of $\mathrm{SL}_2(k)$ on $\mathbb{P}^1(k)$, and the diagonal action on the product $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. Let $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$ be the vector space of $\mathbb{C}$-valued functions on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ which is invariant under the $\mathrm{SL}_2(k)$-action.

By the following elementary result, $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$ is an associative algebra (with unit) by the convolution.

Lemma 1.2. *Suppose a group G acts on X. Let $\mathcal{F}(X \times X)^G$ be the vector space of functions on $X \times X$ invariant under the diagonal action of G. Then it is a subalgebra of $\mathcal{F}(X \times X)$ with respect to multiplication given by the convolution.*

The vector space $\mathcal{F}(X \times X)^G$ has a base given by characteristic functions of G-orbits in $X \times X$. In our case $X = \mathbb{P}^1(k)$, $G = \mathrm{SL}_2(k)$, it is easy to see that the diagonal action has two orbits: the diagonal Δ and the complement of the diagonal $U \overset{\text{def.}}{=} \mathbb{P}^1(k) \times \mathbb{P}^1(k) \setminus \Delta$. Let us denote the characteristic functions by the same notation: Δ and U. In order to identify the algebra $\mathcal{F}(\mathbb{P}^1(k) \times \mathbb{P}^1(k))^{\mathrm{SL}_2(k)}$, it is enough to compute the convolution products $\Delta * \Delta$, $\Delta * U$, $U * \Delta$, $U * U$. But the first three are trivial. Δ is unit, so $\Delta * \Delta = \Delta$, $\Delta * U = U$, $U * \Delta = U$. Let us compute the last one:

$$(U * U)(x,z) = \sum_{y \in \mathbb{P}^1(k)} U(x,y)U(y,z) = \#\{y \in \mathbb{P}^1(k) \mid y \neq x, y \neq z\}$$
$$= \begin{cases} q-1 & \text{if } x \neq y, \\ q & \text{if } x = y, \end{cases}$$

where we have used $\#\mathbb{P}^1(k) = q+1$. Thus we finally get

$$U * U = (q-1)U + q\Delta,$$

Or, equivalently

$$(U - q\Delta) * (U + \Delta) = 0.$$

After the substitution $U \to T$, $\Delta \to 1$, this is the defining relation of the Iwahori-Hecke algebra for $\mathfrak{sl}_2$.

This example can be generalized to the case of arbitary Iwahori-Hecke algebra associated with a complex simple Lie algebra $\mathfrak{g}$, by considering the flag variety G/B instead of $\mathbb{P}^1(k)$.

1.3. **The quantum universal enveloping algebra $\mathbf{U}_v(\mathfrak{sl}_2)$ (due to Beilinson-Lusztig-MacPherson).** Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It is the complex Lie algebra generated by

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the defining relation

$$[h,e] = 2e, \quad [h,f] = 2f, \quad [e,f] = h.$$

The universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_2)$ is the associative algebra with generators e, f, h and the same defining relation, where $[x,y]$ is understood as $xy - yx$. Geometrically $\mathfrak{sl}_2(\mathbb{C})$ is the space of left invariant vector fields on the Lie group $\mathrm{SL}_2(\mathbb{C})$, and $\mathbf{U}(\mathfrak{sl}_2)$ is the ring of invariant differential operators on $\mathrm{SL}_2(\mathbb{C})$. (A vector field is a 1st order differential operator.)

Let us define a q-analogue of $\mathbf{U}(\mathfrak{sl}_2)$, called the quantum enveloping algebra of Drinfeld-Jimbo, attached to $\mathfrak{sl}_2$. Let v be an indeterminate. (We will use the finite field $\mathbb{F}_q$ again, and the parameter v will be given by $\sqrt{q}$.)

Let us introduce v-intergers:

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

Let $\mathbf{U}_v(\mathfrak{sl}_2)$ be the associative $\mathbb{C}(v)$-algebra with generators e, f, $k^{\pm}$ and the defining relations

$$kk^{-1} = k^{-1}k = 1,$$
$$kek^{-1} = v^2 e, \qquad kfk^{-1} = v^{-2} f, \tag{1.3}$$
$$ef - fe = \frac{k - k^{-1}}{v - v^{-1}}.$$

Heuristically we can think $k = v^h$. If we make $v \to 1$, then we recover the defining relations of $\mathfrak{sl}_2$.

The representation theory (finite dimensional representations) of $\mathbf{U}_v(\mathfrak{sl}_2)$ is known to be the same as that of $\mathfrak{sl}_2(\mathbb{C})$. In particular, we have the unique irreducible representation of dimension $N+1$ for each $N \in \mathbb{Z}_{\geq 0}$. It is realized on the space of polynomials in x with degree $\leq N$ as

$$kx^d \overset{\text{def.}}{=} v^{N-2d} x^d, \quad ex^d \overset{\text{def.}}{=} \begin{cases} [N-d+1]_v \, x^{d-1} & \text{if } d > 0, \\ 0 & \text{if } d = 0, \end{cases},$$
$$fx^d \overset{\text{def.}}{=} \begin{cases} [d+1]_v \, x^{d+1} & \text{if } d < N, \\ 0 & \text{if } d = N. \end{cases}$$

(The defining relation (1.3) follows from the identity $[N-d]_v[d+1]_v - [d]_v[N-d+1]_v = [N-2d]_v$.)

We give a geometric realization of $\mathbf{U}_v(\mathfrak{sl}_2)$, which is nothing to do with the ring of differential operators on $\mathrm{SL}_2(\mathbb{C})$. In fact, the Lie group $\mathrm{SL}_2(\mathbb{C})$ is absent in the following construction.

Let $k = \mathbb{F}_q$ be the finite field of q elements. Set $v = \sqrt{q}$. Fix a positive integer N. Let $\mathcal{G}$ be the Grassmann variety of all subspaces of k^N. It is a disjoint union of $\mathcal{G}_d$ with $0 \leq d \leq N$, where $\mathcal{G}_d$ is the Grassmann variety of d-dimensional subspace of k^N. We consider the action of $\mathrm{GL}_N(k)$ on $\mathcal{G}$ and the diagonal action on $\mathcal{G} \times \mathcal{G}$. Then the vector space $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ of $\mathbb{C}$-valued $\mathrm{GL}_N(k)$-invariant functions on $\mathcal{G} \times \mathcal{G}$ is an associative algebra by the convolution.

It has a base given by the characteristic functions of orbits.

Lemma 1.4. *The $\mathrm{GL}_n(k)$-orbits in $\mathcal{G} \times \mathcal{G}$ are parametrized by 2×2-matrices*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with entries $a_{ij} \in \mathbb{Z}_{\geq 0}$ satisfying $a_{11} + a_{12} + a_{21} + a_{22} = N$. The corresponding orbit is the set of pairs (V, V') of subspaces of k^N with

$$\dim(V \cap V') = a_{11}, \quad \dim(V/V \cap V') = a_{12},$$
$$\dim(V'/V \cap V') = a_{21}, \quad \dim(k^N/V + V') = a_{22}.$$

Let Δ_d denote the diagonal in $\mathcal{G}_d \times \mathcal{G}_d$. The corresponding matrix is $\begin{bmatrix} d & 0 \\ 0 & N-d \end{bmatrix}$. Let e_d be the orbit

$$\{(V, V') \mid V \subset V', \dim V = d-1, \dim V' = d\}.$$

The corresponding matrix is $\begin{bmatrix} d-1 & 0 \\ 1 & N-d \end{bmatrix}$. Exchanging the role of V and V', we also define f_d:

$$f_d = \{(V, V') \mid V \supset V', \dim V = d+1, \dim V' = d\} \longleftrightarrow \begin{bmatrix} d & 0 \\ 1 & N-d-1 \end{bmatrix}.$$

Let us denote the characteristic functions by the same symbol as orbits. Let $\widetilde{e} \overset{\text{def.}}{=} \sum_{d=0}^{N} e_d$, $\widetilde{f} \overset{\text{def.}}{=} \sum_{d=0}^{N} f_d$. We compute the convolution products of these functions. The followings are obvious:

$$\Delta_d * \Delta_{d'} = \delta_{dd'}\Delta_d, \quad \Delta_d * \widetilde{e} = \widetilde{e} * \Delta_{d+1}, \quad \Delta_d * \widetilde{f} = \widetilde{f} * \Delta_{d-1}.$$

Let us compute the commutator $[\widetilde{e}, \widetilde{f}] = \widetilde{e} * \widetilde{f} - \widetilde{f} * \widetilde{e}$. We have

$$\begin{aligned}(\widetilde{e} * \widetilde{f})(V, V') &= \#\{V'' \subset k^N \mid V \subset V'' \supset V', \dim V'' = \dim V + 1 = \dim V' + 1\},\\ (\widetilde{f} * \widetilde{e})(V, V') &= \#\{V''' \subset k^N \mid V \supset V''' \subset V', \dim V''' = \dim V - 1 = \dim V' - 1\}.\end{aligned}$$

In particular, the both are 0 unless $\dim V = \dim V'$ and $\dim V \cap V' = \dim V - 1$ (or equivalently $\dim V + V' = \dim V + 1$). Moreover, if $V \neq V'$, then we must have $V'' = V + V'$, $V''' = V \cap V'$. This means that the both functions take values 1 on this pair (V, V'). The only remaining case is $V = V'$. We have

$$\begin{aligned}(\widetilde{e} * \widetilde{f})(V, V) &= \#\{V'' \subset k^N \mid V \subset V'', \dim V'' = \dim V + 1\} = \#\mathbb{P}(k^N/V),\\ (\widetilde{f} * \widetilde{e})(V, V) &= \#\{V''' \subset k^N \mid V \supset V''', \dim V''' = \dim V - 1\} = \#\mathbb{P}(V^*),\end{aligned}$$

where $\mathbb{P}(\ )$ is the projective space of 1-dimensional subspace of a given vector space. We have

$$\#\mathbb{P}(k^N/V) = 1 + q + q^2 + \cdots + q^{N - \dim V - 1}, \quad \#\mathbb{P}(V^*) = 1 + q + q^2 + \cdots + q^{\dim V - 1}.$$

Thus we have

$$\begin{aligned}&v^{1-N}(\widetilde{e} * \widetilde{f})(V, V) - v^{1-N}(\widetilde{f} * \widetilde{e})(V, V) = v^{1-N}\left(v^{2\dim V} + v^{2\dim V + 2} + \cdots + v^{2N - 2\dim V - 2}\right)\\ &= v^{2\dim V - N + 1} + v^{2\dim V - N - 1} + \cdots + v^{N - 2\dim V - 1} = [N - 2\dim V]_v.\end{aligned}$$

We define

$$e \overset{\text{def.}}{=} \sum_d v^{d-N} e_d, \qquad f \overset{\text{def.}}{=} \sum_d v^{-d} f_d,$$

and also define

$$k = \sum_d v^{N-2d}\Delta_d.$$

Then the above computation means that e, f, k satisfy the defining relation (1.3) with parameter $v = \sqrt{q}$. Thus we have an algebra homomorphism

$$\Phi\colon \mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2) \to \mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}.$$

This cannot be an isomorphism since $\dim \mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)} < \infty$, while $\dim \mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2) = \infty$. However one can show that

Proposition 1.5. *The homomorphism Φ is surjective. So $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ is a quotient of $\mathbf{U}_{v=\sqrt{q}}(\mathfrak{sl}_2)$ divided by a two-sided ideal I_N.*

Consider the constant function c_d on $\mathcal{G}_d$. Then we have

$$\begin{aligned}
k * c_d &= v^{N-2d} c_d, \\
(e * c_d)(V) &= v^{d-N} \#\{V' \mid V \subset V', \dim V + 1 = \dim V' = d\} \\
&= \begin{cases} 0 & \text{if } \dim V \neq d-1, \\ v^{d-N} \#\mathbb{P}(k^N/V) & \text{otherwise,} \end{cases} \\
&= [N-d+1]_v \; c_{d-1}(V) \\
(f * c_d)(V) &= v^{-d} \#\{V' \mid V \supset V', \dim V - 1 = \dim V' = d\} \\
&= \begin{cases} 0 & \text{if } \dim V \neq d+1, \\ v^{-d} \#\mathbb{P}(V^*) & \text{otherwise,} \end{cases} \\
&= [d+1]_v \; c_{d+1}(V).
\end{aligned}$$

These equations mean that the representation $\mathcal{F}(\mathcal{G})^{\mathrm{GL}_N(k)}$ of $\mathcal{F}(\mathcal{G} \times \mathcal{G})^{\mathrm{GL}_N(k)}$ is isomorphic to the $(N+1)$-dimensional irreducible representation of $\mathbf{U}_v(\mathfrak{sl}_2)$ via the homomorphism Φ.

This example can be generalized to the quantized universal enveloping algebra $\mathbf{U}_v(\mathfrak{sl}_n)$ by considering the n-step partial flag varieties

$$\{0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^N\}$$

instead of the Grassmann variety. Here the dimensions of V_i is not fixed as above. So the above variety is a disjoint union of varieties of various dimensions.

However, a generalization of this example to $\mathbf{U}_v(\mathfrak{g})$ for arbitary $\mathfrak{g}$ is still open, even for $\mathfrak{g}$ of classical type.

2. Convolution on Borel-Moore homology

2.1. **Convolution on cohomology.** We can replace finite sets by the Euclidean space $\mathbb{R}^m$, the summation over the finite set by the integration in the definition of the convolution product. Namely, if $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ with coordinates $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$, then a given function $K(x_1, \dots, x_m, y_1, \dots, y_n)$ on $X \times Y$ defines an operator from the space of functions on Y to the space of functions X by the formula

$$(K * f)(x_1, \dots, x_m) \overset{\text{def.}}{=} \int_{\mathbb{R}^n} K(x_1, \dots, x_m, y_1, \dots, y_n)\, f(y_1, \dots, y_n)\, dy_1 \dots dy_n.$$

The Fourier transform is an example of an operator given by the convolution. Strictly speaking, we must impose some restrictions on functions to have convergence of the integration.

We can further replace $\mathbb{R}^m$, $\mathbb{R}^n$ by oriented C^∞-manifolds and functions by differential forms on the manifolds. Let M and N be oriented C^∞-manifolds. Let $A^*(M)$, $A^*(N)$ be the vector space of all (complex valued) differential form on M and N. If a differential form K on $M \times N$ is given, we want to define an operator $A^*(N) \to A^*(M)$ by

$$A^*(N) \ni \alpha \longmapsto K * \alpha \overset{\text{def.}}{=} \int_N K \wedge \alpha \in A^*(M). \tag{2.1}$$

More precisely, $K \wedge \alpha$ is the exterior product of K and the pullback of α to $M \times N$, and $\int_N$ is the integration of $K \wedge \alpha$ over each $\{x\} \times N$. If M and N are compact, the integration is well-defined. We assume this condition for a moment. However it will be too restrictive for our later purpose.

The composite of this convolution is again a convolution of this type: If K (resp. K') is a differential form on $M \times N$ (resp. $N \times O$) with the above condition, then we have

$$K * (K' * \alpha) = \int_O \left(\int_N K \wedge K' \right) \wedge \alpha.$$

The space $A^*(M)$ of differential forms is too big. We work on the de Rham cohomology group $H^*(M)$, which is by definition, the space of closed forms modulo the space of exact forms:

$$H^k(M) \overset{\text{def.}}{=} \frac{\{\alpha \in A^k(M) \mid d\alpha = 0\}}{\{d\beta \in A^k(M) \mid \beta \in A^{k-1}(M)\}}.$$

(We always consider the cohomology group with complex coefficients. So we do not write $H^k(M, \mathbb{C})$. Moreover, all results which we will use on cohomology can be found in a standard textbook, e.g., [3].)

If K is a closed p-form on $M \times N$, then we have

$$d_M \left(\int_N K \wedge \alpha \right) = \int_N d_{M \times N} (K \wedge \alpha) = \pm \int_N K \wedge d_N \alpha,$$

where we put the suffix to the exterior differential operator in order to emphasize the manifold where the relevant differential form is defined. In particular, the convolution product maps closed (resp. exact) forms to closed (resp. exact) forms. Therefore, we have a well-defined operator

$$K * \cdot \colon H^*(N) \to H^*(M). \tag{2.2}$$

Let us consider the degree more precisely. If α is a k-form, then $K \wedge \alpha$ is $(k + p)$-form, so $\int_N K \wedge \alpha$ is $(k + p - \dim N)$-form.

Moreover, if K is written as $K = d_{M \times N} F$, then the operator on the de Rham cohomology group is 0 as

$$\int_N K \wedge \alpha = \int_N (d_{M \times N} F) \wedge \alpha = \int_N d_{M \times N} (F \wedge \alpha) = 0,$$

where we have used the Stokes theorem in the last equality. This means that the operator (2.2) depends only on the class in

$$[K] \in H^*(M \times N).$$

Take $M = N$. Then the cohomology group $H^*(M \times M)$ has a structure of an associative algebra by the convolution.

Example 2.3. Suppose that M is a compact oriented C^∞-manifold as above. By the Künneth isomorphism $H^*(M\times M) \cong H^*(M)\otimes H^*(M)$, together with the Poincaré duality $\left(H^k(M)\right)^* \cong H^{\dim M-k}(M)$, the algebra $H^*(M \times M)$ is isomorphic to the matrix algebra $\operatorname{End}(H^*(M))$.

2.2. **Borel-Moore homology.** As illustrated by above example, the condition that M is compact is restrictive, and we do not get an interesting algebra by the convolution product on cohomology groups.

If we carefully see the definition (2.1), we find that it is enough to impose the following:

the restriction of the projection $M \times N \to M$ to the support of K is proper.

Recall that a continuous map between topological spaces is *proper*, if the inverse image of a compact set is again compact. Then the above integration is convergent. Thus the operator is well-defined.

In our later examples, we have the following situation: Let Z be a fixed closed subset $Z \subset M \times N$ such that

the restriction of the projection $M \times N \to M$ to Z is proper.

Then we consider a variant of the de Rham cohomology group

$$\frac{\{K \mid d_{M\times N}K = 0, \text{the support of } K \text{ is contained in a small neighbourhood of } Z\}}{\{d_{M\times N}F \mid \text{the support of } F \text{ is contained in a small neighbourhood of } Z\}}.$$

Then the operator $H^*(N) \to H^*(M)$ is well-defined. Namely the integration is convergent, and the result is independent of the choice of the representative in the above coset.

The above definition is a little bit naive. A rigorous defintion is given by the relative cohomology group

$$H^*(M \times N, M \times N \setminus Z),$$

which is, by definition, the cohomology groups of the following complex:

$$\cdots \to A^k(M \times N) \oplus A^{k-1}(M \times N \setminus Z) \xrightarrow{\left[\begin{smallmatrix} d & 0 \\ j^* & -d \end{smallmatrix}\right]} A^{k+1}(M \times N) \oplus A^k(M \times N \setminus Z) \to \cdots,$$

where $j\colon M \times N \setminus Z \to M \times N$ is the inclusion. For most of our purpose, the above naive definition is sufficient.

For the study of the convolution product, it is more natural to consider the above cohomology group than the usual cohomology group. The above is (a variant of) the so-called Borel-Moore homology group. We give the definition and list its properties.

When X is a topological space which can be embedded as a closed subset in an oriented C^∞-manifold M, we define

$$H_k(X) \overset{\text{def.}}{=} H^{\dim M-k}(M, M \setminus X).$$

The relative cohomology group is defined as above. (**NB**: We will never use the ordinary homology group. So there is no confusion in the notation.)

We must check that the right hand side is independent of the choice of M. Let us study it in an example.

$$H_k(\mathbb{R}^n) = H^{n-k}(\mathbb{R}^n) = \begin{cases} 0 & \text{if } k \neq n, \\ \mathbb{C} & \text{if } k = n. \end{cases}$$

In particular, our Borel-Moore homology is different from the usual homology. Consider the embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as a linear subspace. So we might define as

$$H_k(\mathbb{R}^n) = H^{n+1-k}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \mathbb{R}^n).$$

Let us check that this gives us the same answer. By the Künneth theorem, the above is equal to

$$\bigoplus_{p+q=n+1-k} H^p(\mathbb{R}^n) \otimes H^q(\mathbb{R}, \mathbb{R} \setminus \{0\}).$$

So the assertion follows from

Lemma 2.4.

$$H^q(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \begin{cases} 0 & \textit{if } q \neq 1, \\ \mathbb{C} & \textit{if } q = 1. \end{cases}$$

Since the proof is so simple. We give it.

Proof. Obviously the cohomology group vanishes unless $q = 0, 1$. Consider the case $q = 0$ first. Let $j\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the inclusion. By the above definition of the relative cohomology group, a class is represented by a closed form α such that $j^*\alpha = 0$. A closed 0-form on $\mathbb{R}$ is nothing but a constant function. And $j^*\alpha = 0$ means that the constant must be 0.

Next consider the case $q = 1$. A 1-form α on $\mathbb{R}$ is written as

$$\alpha = f(x)dx.$$

By the definition of the relative cohomology group, $H^1(\mathbb{R}, \mathbb{R} \setminus \{0\})$ is represented by a pair $(f(x)dx, g(x))$ of 1-form on $\mathbb{R}$ and a function on $\mathbb{R} \setminus \{0\}$ such that $dg = j^*f(x)dx$, i.e., $g'(x) = f(x)$ for $x \in \mathbb{R}\setminus\{0\}$. If there exists a function $F(x)$ on $\mathbb{R}$ such that $F'(x) = f(x)$, $j^*F(x) = g(x)$, then the class $(f(x)dx, g(x))$ is zero. We define a map

$$H^1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \ni (f(x)dx, g(x)) \longmapsto \int_{-\varepsilon}^{\varepsilon} f(x)dx - (g(\varepsilon) - g(-\varepsilon)) \in \mathbb{C},$$

where ε is a positive real number. It is independent of the choice of the representative of the class. Namely, if $F'(x) = f(x)$, $j^*F(x) = g(x)$ for some $F(x)$, then the above is 0. Moreover, since $g'(x) = f(x)$ outside $\{0\}$, the above is independent of ε.

Obviously the map is linear and surjective. Let us show that is is injective. Define

$$F(x) = \int_{-\varepsilon}^{x} f(t)dt + g(-\varepsilon).$$

It defines a function on $\mathbb{R}$ and satisfies $d\beta = f(x)dx$. It satisfies $F(-\varepsilon) = g(-\varepsilon)$. If $(f(x)dx, g(x))$ is contained in the kernel of the above homomorphism, then it means that $F(\varepsilon) = g(\varepsilon)$. Then $(f(x)dx, g(x)) = dF$, so it is 0 as a cohomology class. □

Note that we can take a representative $(f(x)dx, g(x))$ so that its support is contained in a given arbitrary small neighbourhood of 0. In this sense, we recover the naive definition.

Our canonical isomorphism

$$H^{n-k}(\mathbb{R}^n) \to H^{n+1-k}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$$

is given by

$$[\alpha] \longmapsto [\alpha \wedge f(x_{n+1})dx_{n+1}, \alpha \wedge g(x_{n+1})] = [\alpha] \wedge [f(x_{n+1})dx_{n+1}, g(x_{n+1})],$$

where $(f(x)dx, g)$ is a class such that

$$\int_{-\varepsilon}^{\varepsilon} f(x)dx - (g(\varepsilon) - g(-\varepsilon)) = 1.$$

This $[f(x_{n+1})dx_{n+1}, g(x_{n+1})]$ is an example of the *Thom class*.

Theorem 2.5. *If E is an oriented C^∞ vector bundle over a C^∞-manifold M of rank r, then there exists a unique class $\Phi \in H^r(E, E \setminus M)$ such that*

$$\int_{E_x} \Phi = 1$$

for each fiber E_x of E. Here M is embedded in E as the 0-section.

This class is called the *Thom class* of E. And as above, the support of Φ is contained in arbitrary small neighbourhood of M.

If S is an oriented closed submanifold of M, then its tubular neighbourhood is diffeomorphic to the normal bundle $N_{S/M}$. We can consider the Thom class of $N_{S/M}$ as a class of $H^{\operatorname{codim} S}(M, M \setminus S)$. If X is a closed subset of S, then the homomorphism

$$H^{\dim S - k}(S, S \setminus X) \ni \alpha \mapsto \alpha \wedge \Phi \in H^{\dim M - k}(M, M \setminus X)$$

is an isomorphism. This means that two definitions of the Borel-Moore homology group $H_k(X)$, one using S and the other using M, are canonically isomorphic. Based on this result, one can prove

Proposition 2.6. *The Borel-Moore homology group $H_k(X) = H^{\dim M - k}(M, M \setminus X)$ is independent of the choice of the umbient manifold M.*

We list up properties of the Borel-Moore homology, which we will use later.

(**Fundamental class of manifolds**) Suppose M is a connected oriented C^∞ manifold. Then

$$H_k(M) = H^{\dim M - k}(M).$$

If $k = \dim M$, then a constant function on M with value 1 is a generator of $H^0(M)$. We call the corresponding element in $H_{\dim M}(M)$ the *fundamental class* of M, and denote it by $[M]$. Note that it is always nonzero. If M is not necessarily connected, its fundamental class is defined as a sum of the fundamental classes of connected components.

If S is an oriented submanifold of M, then the fundamental class $[S]$ is identified with the Thom class of the normal bundle under the two realization of the Borel-Moore homology:

$$\begin{array}{ccccc} H^0(S) & \ni 1 & \longleftrightarrow & \Phi \in & H^{\operatorname{codim} S}(M, M \setminus S) \\ \| & \updownarrow & & \updownarrow & \| \\ H_{\dim S}(S) & \ni [S] & = & [S] \in & H_{\dim S}(S) \end{array}$$

(**Pull-back with support**) Suppose that M and N are oriented C^∞ manifolds with $\dim M = m$, $\dim N = n$, and $f\colon M \to N$ is a smooth map. If $X \subset M$, $Y \subset N$ are closed subsets with $f^{-1}(Y) \subset X$, then we have a homomorphism

$$f^*\colon H_k(Y) \to H_{k-n+m}(X)$$

as a composite

$$H^{n-k}(N, N \setminus Y) \xrightarrow{f^*} H^{n-k}(M, M \setminus f^{-1}(Y)) \to H^{m-(k-n+m)}(M, M \setminus X).$$

This map depends on manifolds M, N, f. A continuous map $\bar{f}\colon X \to Y$ does *not* necessarily induce a homomorphism $\bar{f}^*\colon H_k(Y) \to H_{k-n+m}(X)$.

In particular, we consider the following situation:

- X is an open subset of Y,
- Y is a closed subset of an oriented C^∞-manifold N.

Then we take $M = N \setminus (Y \setminus X)$, which is an open submanifold of N containing X as a closed subset. Then we have a homomorphism

$$H_k(Y) \to H_k(X).$$

(**Pushforward**) (See also Remark 2.11 below) Suppose $f\colon X \to Y$ is a *proper* continuous map. Then we have a homomorphism

$$f_*\colon H_k(X) \to H_k(Y).$$

This is defined as follows. Suppose that X (resp. Y) is embedded in $(0,1)^m$ (resp. $\mathbb{R}^n$) as a closed subset. Then the composition

$$X \xrightarrow{f\times i} Y \times (0,1)^m \to Y \times [0,1]^m \to \mathbb{R}^n \times \mathbb{R}^m$$

is a closed embedding. The properness of f is used to show that the image is closed. Thus

$$H_k(X) = H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus X).$$

We have a map

$$H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus X) \to H^{m+n-k}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^m \setminus Y \times [0,1]^m),$$

i.e.,

$$H_k(X) \to H_k(Y \times [0,1]^m).$$

By the Künneth theorem, we have $H_k(Y \times [0,1]^m) = \bigoplus_{p+q=k} H_p(Y) \otimes H_q([0,1]^m)$. But it is easy to see

$$H_q([0,1]^m) \cong \begin{cases} 0 & \text{if } q \neq 0, \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

The isomorphism for $q = 0$ is given by

$$H_0([0,1]^m) = H^m(\mathbb{R}^m, \mathbb{R}^m \setminus [0,1]^m) \ni [\alpha, \beta] \mapsto \int_{\mathbb{R}^m} \alpha \in \mathbb{C},$$

where we suppose α, β have support contained in a neighourhood of $[0,1]^m$ as before.

Thus we have a homomorphism

$$f_*\colon H_k(X) \to H_k(Y).$$

Exercise 2.7. Show that f_* is independent of various choices. Show that $(g \circ f)_* = g_* \circ f_*$.

If X is compact, then the projection $P\colon X \to$ point is proper. Thus we have a map $P_*\colon H_0(X) \to H_0(\text{point})$. But $H_0(\text{point})$ is isomorphic to $\mathbb{C}$, where the constant function on point with value 1 corresponds to 1 in $\mathbb{C}$. This map is identified with

$$H_0(X) = H^{\dim M}(M, M \setminus X) \ni [\alpha, \beta] \mapsto \int_M \alpha \in \mathbb{C},$$

where we take the representative $[\alpha, \beta]$ so that its support is contained in a small neighbourhood of X.

Exercise 2.8. Check the above assertion from the definition.

(**Long exact sequence**) Let U be an open set of X, and $Y = X \setminus U$ be the complement. Let $i\colon Y \to X$, $j\colon U \to X$ be inclusions. We have a long exact sequence

$$\cdots \to H_k(Y) \xrightarrow{i_*} H_k(X) \xrightarrow{j^*} H_k(U) \xrightarrow{\delta^*} H_{k-1}(Y) \to \cdots,$$

where δ^* is the boundary homomorphism.

(**Intersection with support**) Suppose X, Y are closed subsets of an oriented C^∞ manifold M with $\dim M = m$. Then we can define a cap product (in M)

$$\cap\colon H_k(X) \otimes H_l(Y) \to H_{k+l-m}(X \cap Y)$$

from a cup product in the relative cohomology:

$$\cup\colon H^k(M, M \setminus X) \otimes H^l(M, M \setminus Y) \to H^{k+l}(M, M \setminus (X \cap Y)).$$

Note that this depends on the umbient space M.

Exercise 2.9. Suppose that X and Y are oriented submanifolds of M. Assume that they intersect transversally. Namely, $T_xX + T_xY = T_xM$ for all $x \in X \cap Y$. Then $X \cap Y$ is an oriented manifold with dimension $\dim X + \dim Y - \dim M$, where the orientation is induced from that of X and Y. We have the following formula:

$$[X] \cap [Y] = [X \cap Y]$$

in $H_{\dim X + \dim Y - \dim M}(X \cap Y)$.

(**Self-intersection and Euler class**) We suppose X, Y are oriented submanifolds of M. We want to compute the intersection product $[X] \cap [Y]$ without assuming the intersection is transverse. The most extreme case is when $X = Y$. In this case $[X] \cap [X]$ is called *self-intersection*. Let $\Phi \in H^{\operatorname{codim} X}(M, M \setminus X)$ be the Thom class of the normal bundle. Let $\vartheta\colon H^*(M, M \setminus X) \to H^*(M)$ be the natural homomorphism, and let $i\colon X \to M$ be the inclusion. Then it is easy to check that $[X] \cap [X]$ is identified with $i^*\vartheta\Phi$ under the isomorphism $H^{\operatorname{codim} X}(X) \cong H_{\dim X - \operatorname{codim} X}(X)$. In general, the pullback of the Thom class of an oriented vector bundle E is called the *Euler class* of E. Thus $i^*\vartheta\Phi$ is the Euler class of the normal bundle.

If i' is a small pertubation of $i\colon X \to M$, then i and i' is homotopic, so the class $i^*\vartheta\Phi$ is equal to $i'^*\vartheta\Phi$. Using the above argument backwords, we find

$$i'^*\vartheta\Phi = j_*\left([i'X] \cap [X]\right),$$

where $j\colon i'X \cap X \to X$ is the inclusion. We can choose i' so that $i'X$ and X is transversal. Then the right hand side is $j_*([i'X \cap X])$. Combining all these discussions, we get

$$[X] \cap [X] = [i'X \cap X].$$

(**Fundamental class of subvarieties**) Let M be a complex manifold, and $X \subset M$ be a closed subvariety (not necessarily irreducible) with $\dim_{\mathbb{C}} X = n$.

Proposition 2.10. *We have $H_k(X) = 0$ for $k > 2n$ and $H_{2n}(X)$ has a base corresponding to irreducible components of X of dimension n.*

Proof. If X is nonsingular, this is obvious from $H_k(X) = H^{2n-k}(X)$. We prove the assertion for general case by induction on $\dim_{\mathbb{C}} X$. The case $\dim_{\mathbb{C}} X = 0$ is obvious. We have a closed subvariety $Z \subset X$ with $\dim Z < n$ such that and $X \setminus Z$ is nonsingular, of pure dimension n. We consider the long exact sequence

$$\cdots \to H_k(Z) \xrightarrow{i_*} H_k(X) \xrightarrow{j^*} H_k(X \setminus Z) \xrightarrow{\delta^*} H_{k-1}(Z) \to \cdots.$$

By the induction hypothesis, $H_k(Z) = 0$ if $k > 2(n-1)$. And we have $H_k(X \setminus Z) = 0$ if $k > 2n$, and $H_{2n}(X \setminus Z)$ has a base given by fundamental classes of its connected components. Since the connected components of $X \setminus Z$ are the irreducible components of X with dimension n, we get the assertion. □

If X is irreducible, we denote by $[X]$ the class in $H_{2n}(X)$ given by the above lemma, and call it the *fundamental class*. If X is not irreducible, its fundamental class is the sum of fundamental classes of irreducible components of dimension n.

Remark 2.11. It is known that our Borel-Moore homology group $H_k(X)$ is isomorphic to homology group of infinite singular chains with locally finite support. More precisely, a formal *infinite* singular chains $\sum_i a_i\sigma_i$, where σ_i is a simplex, $a_i \in \mathbb{C}$, is called *locally finite*, if for any compact subset $D \subset X$ there are only finitely many nonzero a_i such that $D \cap \operatorname{Supp} \sigma_i \neq \emptyset$. One can define the boudary operator exactly as in the usual *finite* singular chains. It preserves the locally finiteness condition, so one can define the associated homology group. It is canonically isomorphic to our $H_k(X)$.

Moreover, it is clear that a *proper* continuous map $f\colon X \to Y$ induces a homomorphism $f_*\colon H_k(X) \to H_k(Y)$ exactly as in the case of usual homology groups, since the locally finiteness condition is preserved under the proper map f.

2.3. **Lagrangian construction of the Weyl group (due to Ginzburg).** Let M_1, M_2, M_3 be oriented C^∞ manifolds with dim $M_i = m_i$. Let $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$ be closed subsets satisfying

> the restrictions of the projections $M_1 \times M_2 \to M_1$, $M_2 \times M_3 \to M_2$ to Z_{12}, Z_{23} are proper.

Let $p_{12}\colon M_1 \times M_2 \times M_3 \to M_1 \times M_2$, etc, be the projection. Then we can define the convolution product by

$$\begin{aligned} &H_k(Z_{12}) \otimes H_l(Z_{23}) \ni K \otimes K' \\ &\qquad\qquad \longmapsto p_{13*}\left(p_{12}^* K \cap p_{23}^* K'\right) \in H_{k+l-m_2}(p_{13}(Z_{12} \times M_3 \cap M_1 \times Z_{23})). \end{aligned}$$

More precisely, we take the cup product of $p_{12}^* K \in H_{k+m_3}(Z_{12} \times M_3)$ and $p_{23}^* K' \in H_{l+m_1}(M_1 \times Z_{23})$ in $M_1 \times M_2 \times M_3$. Then the restriction of p_{13} to $Z_{12} \times M_3 \cap M_1 \times Z_{23}$ is proper by the above condition. Thus the pushforward is well-defined. Note that $p_{13}(Z_{12} \times M_3 \cap M_1 \times Z_{23})$ is a closed subset of $M_1 \times M_3$.

Let $M = T^*\mathbb{P}^1(\mathbb{C})$, the cotangent bundle of the complex projective line. It is the set of pairs

$$T^*\mathbb{P}^1(\mathbb{C}) = \{(V, \xi) \in \mathbb{P}^1(\mathbb{C}) \times \operatorname{End}(\mathbb{C}^2) \mid \xi(V) = 0, \xi(\mathbb{C}^2) \subset V\}.$$

Note that ξ is nilpotent by the condition. We define the Steinberg variety

$$Z \overset{\text{def.}}{=} \{(V_1, V_2, \xi) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \operatorname{End}(\mathbb{C}^2) \mid (V_1, \xi), (V_2, \xi) \in T^*\mathbb{P}^1(\mathbb{C})\}.$$

It is a closed subvariety in $T^*\mathbb{P}^1(\mathbb{C}) \times T^*\mathbb{P}^1(\mathbb{C})$. If $\xi \neq 0$, then $V_1 = V_2 = \operatorname{Ker}\xi$. Thus it is contained in the diagonal of $T^*\mathbb{P}^1(\mathbb{C}) \times T^*\mathbb{P}^1(\mathbb{C})$. Thus Z is a union of two 2-dimensional complex submanifolds

$$\Delta_{T^*\mathbb{P}^1(\mathbb{C})} \cup \left(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})\right),$$

where $\mathbb{P}^1(\mathbb{C})$ is contained in $T^*\mathbb{P}^1(\mathbb{C})$ as $\xi = 0$ (0-section). Thus

$$H_4(Z) = \mathbb{C}[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] \oplus \mathbb{C}[\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})].$$

By the definition, the map $Z \to T^*\mathbb{P}^1(\mathbb{C})$ is proper. Hence we have the convolution product on $H_4(Z)$:

$$H_4(Z) \times H_4(Z) \ni (K, K') \longmapsto p_{13*}(p_{12}^* K \cap p_{23}^* K') \in H_4(Z),$$

where we should notice $p_{13}(Z \times M \cap M \times Z) = Z$.

Theorem 2.12. *$H_4(Z)$ is isomorphic to the group ring $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ of the Weyl group $\mathbb{Z}/2\mathbb{Z}$ of $\mathfrak{sl}_2$.*

Proof. Let us compute the convolution product

$$\begin{gathered} [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] * [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}], \quad [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] * [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})], \\ [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] * [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}], \quad [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] * [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})]. \end{gathered}$$

The first threes are easy. The intersections are transversal, and we easily get

$$\begin{gathered} [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] * [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] = [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}], \\ [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] * [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] = [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] * [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}] = [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})]. \end{gathered}$$

Namely $[\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$ is the unit. This holds in general.

Let us consider the last one. We have

$$\begin{aligned}[\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] * [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] &= p_{13*}\left([\mathbb{P}^1(\mathbb{C})] \times \left([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]\right) \times [\mathbb{P}^1(\mathbb{C})]\right) \\ &= P_*\left([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]\right) [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})],\end{aligned}$$

where $[\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})]$ is the intersection product in $M_2 = T^*(\mathbb{P}^1(\mathbb{C}))$, and $P\colon \mathbb{P}^1 \to \text{point}$ is the projection to the single point. So $P_*([\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})])$ is an element in $H_0(\text{point})$. But it is considered as a real number by the isomorphism $H_0(\text{point}) \cong \mathbb{C}$.

Exercise 2.13. Compute the self-intersection $[\mathbb{P}^1(\mathbb{C})]$ in $T^*\mathbb{P}^1(\mathbb{C})$:

$$[\mathbb{P}^1(\mathbb{C})] \cap [\mathbb{P}^1(\mathbb{C})] = -2[\text{point}],$$

where point is the fundamental class of a point in $[\mathbb{P}^1(\mathbb{C})]$. (It is independent of the choice of points.)

By this exercise, $T \overset{\text{def.}}{=} [\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})] + [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$ satisfies $T^2 = [\Delta_{T^*\mathbb{P}^1(\mathbb{C})}]$. Thus we get the assertion. □

Remark 2.14. The result of this section and that in §1.2 is deeply connected. The result of §1.2 can be reformulated by using SL_2-invariant mixed perverse sheaves on $\mathbb{P}^1$. Functions appeared in §1.2 are traces of the Frobenius homomorphism on stalks of perverse sheaves on rational points. Forgetting the mixed structure, one can formulate the result on equivariant D-modules on the complex manifold $\mathbb{P}^1(\mathbb{C})$ (the Riemann-Hilbert correspondence). It gives the group ring $\mathbb{Z}[W]$, the specialization of the Hecke algebra $\mathcal{H}_q$ at $q = 1$. There is a natural passage from D-modules to cycles in cotangent bundles, i.e., characteristic cycles.

Exercise 2.15 (See [16]). By considering the cotangent bundle of the Grassmann variety, construct $\mathbf{U}(\mathfrak{sl}_2)$.

3. Hilbert schemes of points

3.1. **Definition.** In this subsection, we define the Hilbert scheme of points on the complex plane, and study its geometric properties. We do not use a general construction due to Grothendieck, and give an elementary treatment which works only our special case.

First we do not restrct ourselves to the case when dimension is 2. Let X be the N-dimensional complex affine space $\mathbb{C}^N$. We define the Hilbert scheme of points by

$$X^{[n]} \overset{\text{def.}}{=} \{I \mid I \text{ is an ideal of } \mathbb{C}[x_1,\dots,x_N] \text{ with } \dim_{\mathbb{C}} \mathbb{C}[x_1,\dots,x_N]/I = n\}.$$

So far, we consider $X^{[n]}$ just a set. It can be considered also as

$$X^{[n]} = \{A_Z \mid A_Z \text{ is a quotient ring of } \mathbb{C}[x_1,\dots,x_N] \text{ with } \dim_{\mathbb{C}} A_Z = n\}.$$

The correspondence is given by

$$0 \to I \to \mathbb{C}[x_1,\dots,x_N] \to A_Z \to 0.$$

(The notation A_Z is borrowed from algebraic geometry. Z is a 0-dimensional subscheme of $\mathbb{C}^N$, and A_Z is the coordinate ring of Z.)

The Hilbert scheme $X^{[n]}$ is related to the symmetric product S^nX in the following way. If we have distinct n points $p_1, \dots, p_n$ in X, then it defines both a point in S^nX and a point in $X^{[n]}$. In fact, if we set

$$I \overset{\text{def.}}{=} \{f \in \mathbb{C}[x_1,\dots,x_N] \mid f \text{ vanishes at } p_1, \dots, p_n\},$$

it is an ideal with $\dim_{\mathbb{C}} \mathbb{C}[x_1,\dots,x_N]/I = n$. (The correspondence subscheme is of course, $Z = \{p_1,\dots,p_n\}$.

However, the difference occurs if some points collide. Consider the case $n = 2$. In this case, there are two types of ideals in $X^{[2]}$. The first type is an ideal given by two distinct points p, q. The other type is an ideal given by

$$I = \{f \mid f(p) = 0,\ df_p(v) = 0\} \tag{3.1}$$

for some point $p \in X$ and nonzero tangent vector $v \in T_pX$. This ideal is a limit of ideals of the first type when q approaches to p. And the information of the direction in which q approaches to p is remembered in I. In the symmetric product, the limit is simply $2p$, and this information is lost. When the number of points is greater than 2, much more complicated ideals will occur.

Exercise 3.2. Show that the Hilbert scheme $X^{[n]}$ coincides with the symmetric product S^nX when the dimension of the base space is 1.

We give a *matrix description* of $X^{[n]}$. Let $V \overset{\text{def.}}{=} \mathbb{C}[x_1,\dots,x_N]/I$, considered as a vector space. We define linear operators B_i on V by

$$B_i(f \bmod I) \overset{\text{def.}}{=} x_i f \bmod I.$$

We define a vector $v \in V$ as $v \overset{\text{def.}}{=} 1 \bmod I$. Then it is clear that they satisfy the following properties

(3.3.1) $[B_i, B_j] = 0$,
(3.3.2) v is a cyclic vector, i.e., if a subspace $S \subset V$ contains v and is invariant under B_i's, then it must be the whole space V.

Conversely, if a vector space V and such $(B_1,\dots,B_N,v)$ is given, we can define an ideal I as a kernel of a surjective homomorphism

$$\mathbb{C}[x_1,\dots,x_N] \ni f(x_1,\dots,x_N) \longmapsto f(B_1,\dots,B_N)v \in V.$$

Here $f(B_1, \dots, B_N)$ makes sense since $[B_i, B_j] = 0$. Moreover, the surjectivity follows from the cyclicity of v. Thus I is a point in $X^{[n]}$. This I is not changed under the action of $\mathrm{GL}(V)$ given by

$$(B_1, \dots, B_N, v) \longmapsto (gB_1g^{-1}, \dots, gB_Ng^{-1}, gv).$$

Moreover, it is easy to check that these maps are mutually inverse. We have a set-theoretical bijection

$$X^{[n]} \longleftrightarrow \{(B_1, \dots, B_N, v) \mid (3.3.1), (3.3.2)\} \;/\, \mathrm{GL}(V).$$

When a $\mathrm{GL}(V)$-orbit through $(B_1, \dots, B_N, v)$ is considered as a point in $X^{[n]}$, we denote it by $[(B_1, \dots, B_N, v)]$.

For example, consider the case $n = 2$. Since $[B_i, B_j] = 0$, we can make B_i's simultaneously into upper triangular matrices as

$$B_1 = \begin{bmatrix} x_1 & a_1 \\ 0 & y_1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} x_2 & a_2 \\ 0 & y_2 \end{bmatrix}, \qquad \dots .$$

If $(x_1, x_2, \dots, x_N) \neq (y_1, y_2, \dots, y_N)$, then we can simultaneously diagonalize all B_i's. This case corresponds to the ideal given by distinct two points. Suppose $(x_1, x_2, \dots, x_N) = (y_1, y_2, \dots, y_N)$. Then the cyclicity implies that $(a_1, a_2, \dots, a_N) \neq (0, 0, \dots, 0)$. Now it is not difficult to see that this case corresponds to an ideal of type (3.1) with $v = (a_1, a_2, \dots, a_N)$.

From now on we assume $N = 2$.

Theorem 3.4. *$X^{[n]}$ is a nonsingular complex manifold of dimension $2n$.*

The following proof will be not used later. So an uninterested reader can safely skip it.

Proof. Let $\widetilde{X}^{[n]} \overset{\text{def.}}{=} \{(B_1, B_2, v) \mid (3.3.1), (3.3.2)\}$, i.e., $X^{[n]} = \widetilde{X}^{[n]} / \mathrm{GL}(V)$.

Step 1. We first show that $\widetilde{X}^{[n]}$ is a nonsingular complex manifold of dimension $2n + n^2$. Let

$$\mu\colon \operatorname{End}(V) \times \operatorname{End}(V) \times V \to \operatorname{End}(V)$$

be a map defined by

$$\mu(B_1, B_2, v) = [B_1, B_2].$$

Then $\widetilde{X}^{[n]}$ is an open subset of $\mu^{-1}(0)$. The differential of μ at (B_1, B_2, v) is given by

$$d\mu(\delta B_1, \delta B_2, \delta v) = [B_1, \delta B_2] + [\delta B_1, B_2].$$

It is enough to show that the cokernel of $d\mu$ is dimension n for any $(B_1, B_2, v) \in \widetilde{X}^{[n]}$. We identify the dual space of $\operatorname{End}(V)$ with itself by the inner product given by trace. Then the cokernel of $d\mu$ is

$$\begin{aligned} &\{C \in \operatorname{End}(V) \mid \operatorname{tr}(C d\mu(\delta B_1, \delta B_2, \delta v)) = 0 \text{ for all } (\delta B_1, \delta B_2, \delta v)\} \\ = &\{C \in \operatorname{End}(V) \mid [B_1, C] = [B_2, C] = 0\}. \end{aligned}$$

This space is isomorphic to V under the map $C \mapsto Cv$ thanks to the conditions (3.3.1), (3.3.2). This completes the step 1.

Step 2. Next we show that the action of $\mathrm{GL}(V)$ on $\widetilde{X}^{[n]}$ is free. Suppose that $g \in \mathrm{GL}(V)$ stabilizes (B_1, B_2, v), i.e.,

$$gB_1g^{-1} = B_1,\ gB_2g^{-1} = B_2,\ gv = v.$$

Then $S \overset{\text{def.}}{=} \operatorname{Ker}(g - 1)$ is a subspace of V which is invariant under B_1, B_2 and contains v. Hence $S = V$ by the condition (3.3.2). Thus we have $g = 1$.

Step 3. We show that every $\mathrm{GL}(V)$-orbit in $\widetilde{X}^{[n]}$ is closed. In fact, the closure of an orbit is a union of orbits, but any orbit cannot be contained in the closure of another orbit since both have the dimension $\dim \mathrm{GL}(V)$ by Step 2. In particular, the quotient space $\widetilde{X}^{[n]}/\,\mathrm{GL}(V)$ is Hausdorff.

Step 4. We show that a bijection

$$\mathrm{GL}(V)\times \widetilde{X}^{[n]} \to \Gamma \overset{\text{def.}}{=} \{(x,gx)\in \widetilde{X}^{[n]}\times \widetilde{X}^{[n]} \mid g\in \mathrm{GL}(V)\}$$

is a homeomorphism. Thus we want to show that the inverse of the map is continuous. Suppose

$$\lim_{i\to\infty}((B_{1,i},B_{2,i},v_i),(g_iB_{1,i}g_i^{-1},g_iB_{2,i}g_i^{-1},g_iv_i)) = ((B_1,B_2,v),(gB_1g^{-1},gB_2g^{-1},gv)).$$

We need to show that g_i converges to g. Set

$$B'_{1,i}=g_iB_{1,i}g_i^{-1},\quad B'_{2,i}=g_iB_{2,i}g_i^{-1},\quad v'_i=g_iv_i.$$

We have

$$g_iB_{1,i}=B'_{1,i}g_i,\quad g_iB_{2,i}=B'_{2,i}g_i.$$

We consider $h_i=g_i/\|g_i\|$. Then $\|h_i\|=1$, so we may assume that h_i converges to an endomorphism $h\in\mathrm{End}(V)$ with $\|h\|=1$ if we replace h_i by a subsequence. Therefore, we have

$$hB_1=gB_1g^{-1}\,h,\quad hB_2=gB_2g^{-1}\,h.$$

Suppose $\|g_i\|\to\infty$. Then

$$hv=\lim_{i\to\infty}h_iv_i=\lim_{i\to\infty}\frac{1}{\|g_i\|}v'_i=0.$$

This means the kernel of h contains v and invariant under B_1, B_2. Thus $h=0$ by (3.3.2). This contradicts with $\|h\|=1$. Therefore $\|g_i\|$ is bounded, and may assume g_i converges to $g'\in\mathrm{End}(V)$. As above, we have

$$g'B_1=gB_1g^{-1}\,g',\quad g'B_2=gB_2g^{-1}\,g',\quad g'v=gv.$$

By Step 2 (we do not need the invertibility of g'), we have $g=g'$. This completes the proof of Step 4.

Step 5. The rest of the proof is a standard argument (see e.g., [40, Theorem 2.9.10]). So we explain it only briefly.

Take $(B_1,B_2,v)\in\widetilde{X}^{[n]}$. Consider the deformation complex at (B_1,B_2,v):

$$\mathrm{End}(V)\xrightarrow{\iota}\mathrm{End}(V)\times\mathrm{End}(V)\times V\xrightarrow{d\mu}\mathrm{End}(V),$$

where ι is the differential of the $\mathrm{GL}(V)$-action, i.e.,

$$\iota(\xi)=([\xi,B_1],[\xi,B_2],\xi v).$$

By the arugment in Step 1, we know that ι is injective. We can take a submanifold S of $\widetilde{X}^{[n]}$ passing through x such that

(1) its tangent space T_xS is complementary to $\operatorname{Im}\iota$,
(2) $\mathrm{GL}(V)\cdot S$ is an open subset of $\widetilde{X}^{[n]}$ and the map $\mathrm{GL}(V)\times S\to\mathrm{GL}(V)\cdot S$ is an isomorphism of complex manifolds.

We can give a structure of a complex manifold to the quotient space $\widetilde{X}^{[n]}/\,\mathrm{GL}(V)=X^{[n]}$ so that the natural map $\{1\}\times S\to\mathrm{GL}(V)\cdot S\to X^{[n]}$ is an isomorphism onto an open set of $X^{[n]}$. □

Remark 3.5. (1) Theorem 3.4 is originally due to Fogarty [13]. Our proof here is completely different, and somehow similar to the construction of a moduli space in the gauge theory.

(2) It is known that $X^{[n]}$ is a hyperKähler manifold [Lecture, §3]. This result is based on the matrix description of $X^{[n]}$, a well-known correspondence between the GIT quotient and the symplectic quotient, plus a *misterious* lemma in linear algebra. An existence of

a holomorphic symplectic form (weaker than the existence of a hyperKähler structure) was proved by Beauville [2] earlier.

3.2. **The Hilbert-Chow morphism and the punctual Hilbert scheme.** Let $S^n X$ be the nth symmetric product of $X = \mathbb{C}^2$. It is an orbifold, locally isomorphic to an open set of the Euclidean space divided by an action of a finite group. In particular, it has a natural topology and complex structure. It is an affine algebraic variety, whose coordinate ring is $\mathbb{C}[\lambda_1, \mu_1, \dots, \lambda_n, \mu_n]^{\mathfrak{S}_n}$. It is known that the ring is generated by $\sum_i \lambda_i^p \mu_i^q$ for various p, q.

The symmetric product has a natural stratification indexed by partitions of n:

$$S^n X = \bigsqcup_\lambda S^n_\lambda X, \qquad \text{where } S^n_\lambda X = \left\{ \sum_i \lambda_i x_i \in S^n X \;\middle|\; x_i \neq x_j (i \neq j) \right\}.$$

For example, if $\lambda = (1^n) = (1, \dots, 1)$, then $S^n_{(1^n)} X$ is the open set consisting of distinct n points. It is a nonsingular locus of $S^n X$, i.e., $S^n X$ has singularities along the complement $S^n X \setminus S^n_{(1^n)} X$. The other extreme is $\lambda = (n)$. Then $S^n_{(n)} X$ is the set of points with multiplicity n. Hence $S^n_{(n)} X$ is isomorphic to X.

Let $[(B_1, B_2, v)] \in X^{[n]}$. Since $[B_1, B_2] = 0$, we can make B_1 and B_2 simultaneously into upper triangular matrices as

$$B_1 = \begin{bmatrix} \lambda_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \mu_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_n \end{bmatrix}.$$

We define a map $\pi\colon X^{[n]} \to S^n X$ by

$$\pi([B_1, B_2, v]) = (\lambda_1, \mu_1) + \dots + (\lambda_n, \mu_n).$$

From the above remark on the coordinate ring of $S^n X$, it is clear that this is a morphism between complex analytic varieties. It is called the *Hilbert-Chow morphism*. If $[(B_1, B_2, v)]$ corresponds to an ideal given by distinct n points, then it is easy to see that the corresponding matrices B_1, B_2 are simultaneously diagonalizable, and the eigenvalues are the given points. This shows that π is an isomorphism on an open set consisting of ideals given by distinct points, i.e., $\pi^{-1}(S^n_{(1^n)} X)$.

The other extreme is the inverse image of a point in $S^n_{(n)} X$. We define

$$X_0^{[n]} \overset{\text{def.}}{=} \pi^{-1}(n0), \qquad X_*^{[n]} \overset{\text{def.}}{=} \pi^{-1}(S^n_{(n)} X)$$

where 0 is the origin of $X = \mathbb{C}^2$. The formar $X_0^{[n]}$ is called the *punctual Hilbert scheme*. These are closed subvarieties of $X^{[n]}$ and we have $X_*^{[n]} = X_0^{[n]} \times X$. If $n = 1$, $X_0^{[1]} = \{0\}$. If $n = 2$, $X_0^{[2]} \cong \mathbb{P}^1$ by the description explained in §3.1. The inverse image of the other points can be easily described. If $C \in S^n_\lambda X$, then

$$\pi^{-1}(C) \cong X_0^{[\lambda_1]} \times X_0^{[\lambda_2]} \times \cdots, \tag{3.6}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a partition of n.

It is known that

Theorem 3.7. *If $n \neq 0$, $X_0^{[n]}$ is an $(n-1)$-dimensional irreducible subvariety.* (*If $n = 0$, we understand $X_0^{[n]} = \{point\}$.*)

This result is due to Briançon [4] (see also Iarrobino [19]).

Theorem 3.8. *$\pi\colon X^{[n]} \to S^n X$ is a resolution of singularities. Namely, π is a proper surjective morphism such that*

(1) *$X^{[n]}$ is nonsingular,*

(2) *π is an isomorphism on $\pi^{-1}(S^n_{(1^n)} X)$.*

(3) $\pi^{-1}(S^n_{(1^n)}X)$ *is a dense subset in* $X^{[n]}$.

Moreover, $X^{[n]}$ *is irreducible.*

Proof. We do not prove that π is proper yet. For example, it becomes obvious if we consider the Hilbert scheme $(\mathbb{P}^2)^{[n]}$ of points on projective plane. Then $(\mathbb{P}^2)^{[n]}$ is a projective variety since it is a subvariety of the Grassmann manifold. In particular, it is compact. We can define the Hilbert-Chow morphism $\pi\colon (\mathbb{P}^2)^{[n]} \to S^n(\mathbb{P}^2)$. It is clearly proper. Then $X^{[n]} = \pi^{-1}(S^nX)$, where S^nX is an open subset of $S^n\mathbb{P}^2$. Thus the properness is clear.

Another way to show the properness is to identify S^nX with the quotient in the geometric invariant theory:

$$S^nX = \mu^{-1}(0)/\!\!/ \operatorname{GL}(V).$$

Then the properness follows from a general theory in the geometric invariant theory. This argument is necessary for *quiver varieties*.

Now we check other conditions. The surjectivity of π is clear. The remaining one is the condition (3). We have proved that $X^{[n]}$ has dimension $2n$. Thus the condition (3) follows if we show that $\dim \pi^{-1}(S^nX \setminus S^n_{(1^n)}X) < 2n$. And this follows easily from the previous theorem and (3.6). Since it is clear that $S^n_{(1^n)}X$ is connected, it also implies that $X^{[n]}$ is connected. □

Remark 3.9. For the proof of this theorem, the full strength of Theorem 3.7 is not necessary. It is enough to prove the weaker statement 'there is only one $(n-1)$-dimensional irreducible component in $X_0^{[n]}$'. In fact, even weaker statement $\dim X_0^{[n]} \le n-1$ is enough. There is a very simple proof of this statement based on the symplectic geometry on $X^{[n]}$ ([Lecture, 1.13]).

Remark 3.10. Using Theorem 3.7, one can show that

$$\dim \pi^{-1}(C) = n - \#\text{ of nonzero entries in } \lambda = \frac{1}{2}\operatorname{codim} S^n_\lambda X^n, \quad C \in S^n_\lambda X^n.$$

Thus the map π is *semi-small*. This immediately gives us a formula of Betti numbers of $X^{[n]}$. (See [Lecture, Chapter 5].)

4. Quiver varieties (or Γ-Hilbert schemes)

Quiver varieties were introduced in [N1]. They arised a natural generalization of moduli spaces of anti-self-dual connections of the so-called ALE spaces, studied in [29]. However, we give a different geometric description in these lectures.

4.1. **Γ-fixed point set.** Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. The classification of such subgroups has been well-known to us, since they are symmtry groups of regular polytopes via the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The classification table is the following:

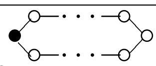
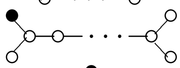
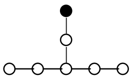
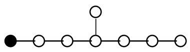
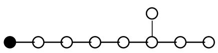
type	affine Dynkin graph	group
A_n $(n \geq 0)$		$\left\{ \left[\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right] \,\middle|\, \varepsilon^{n+1} = 1 \right\}$, the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$
D_n $(n \geq 4)$		the binary dihedral group of order $4(n-1)$
E_6		the binary tetrahedral group
E_7		the binary octahedral group
E_8		the binary icosahedral group

Table 1. finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ and affine Dynkin diagrams

The link between this classification and that of simple simply-laced complex Lie algebra of type ADE will be explained soon.

The group Γ acts on the complex plane $X = \mathbb{C}^2$, and also on the Hilbert scheme $X^{[n]}$. We want to consider the fixed point variety

$$\left(X^{[n]}\right)^\Gamma = \{I \in X^{[n]} \mid \gamma \cdot I = I \text{ for any } \gamma \in \Gamma\}.$$

If $I \in \left(X^{[n]}\right)^\Gamma$, then $\mathbb{C}[x,y]/I$ is a Γ-module.

Since $X^{[n]}$ is smooth and Γ is finite, the fixed point set $\left(X^{[n]}\right)^\Gamma$ is a union of nonsingular submanifolds (of various dimensions). The Γ-module structure of $\mathbb{C}[x,y]/I$ is constant along each connected component of $\left(X^{[n]}\right)^\Gamma$. For a given isomorphism class $\mathbf{v}$ we set

$$X(\mathbf{v}) \overset{\text{def.}}{=} \left\{ I \in \left(X^{[n]}\right)^\Gamma \,\middle|\, \mathbb{C}[x,y]/I \cong V \right\}, \quad \text{where } n = \dim V,$$

where V is a Γ-module in the isomorphism class $\mathbf{v}$. A priori, this is a union of connected components of $\left(X^{[n]}\right)^\Gamma$. However, a stronger result is known: $X(\mathbf{v})$ is connected. This follows from a general result for quiver varieties by Crawley-Boevey [9].

Example 4.1. This example was first given by Kronheimer [28] (in a slightly different language), and was a starting point of later works on quiver varieties [29, N1, N3]. It was *re*discovered later by Ginzburg-Kapranov (unpublished), and then Ito-Nakamura [IN] independently.

Consider the ideal I of functions vanishing at points in a free Γ-orbit. The action of Γ on X is free outside the origin 0, therefore the orbit consists of $\#\Gamma$-elements. The Γ-module $\mathbb{C}[x,y]/I$ is the regular representation of Γ. Let R be the regular representation of Γ, and $\mathbf{r}$ its the isomorphism class. Then $X(\mathbf{r})$ of the fixed point set in the Hilbert scheme $X^{[n]}$ $(n = \#\Gamma)$, which contains ideals consisting of functions vanishing on a Γ-orbit. The corresponding fixed point set $(S^n X)^\Gamma$ in the symmetric product is isomorphic to X/Γ in this case. Thus we have a proper morphism $\pi\colon X(\mathbf{r}) \to X/\Gamma$, which is a resolution of singularities. This is easy to check. In fact, it is an isomorphism on $\pi^{-1}((S^n_{(1^n)} X)^\Gamma) = \pi^{-1}((X \setminus \{0\})/\Gamma)$. And $X(\mathbf{r})$ is connected, so the complement is lower dimensional. (There is a very simple proof of the connectedness for this $X(\mathbf{r})$. See [Lecture, Chatper 4].)

Since $X(\mathbf{r})$ is a symplectic manifold, its canonical bundle is trivial. So $X(\mathbf{r})$ is the so-called *minimal resolution* of X/Γ. Such a resolution is unique, and has been studied from various points of view (much before the theory of quiver varieties is developed). In particular, it is known that the inverse image $\pi^{-1}(0)$ of the origin 0 under π is a union $\bigcup C_i$ of projective lines, whose intersection graph is a Dynkin graph of type ADE. (Delete the black vertex from the affine Dynkin graph in the table.) In other words, the intersection matrix $C_i \cdot C_j$ is (-1) times the Cartan matrix of type ADE. This is a reason why the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ is related to the classification of simple Lie algebras.

In fact, it is possible to study the exceptional fiber $\pi^{-1}(0)$ in the language of Hilbert schemes, or quiver varieties. See Example 6.3.

4.2. **simple Lie algebras and root systems.** We briefly recall the theory of simple Lie algebras and their root systems.

Let $\mathfrak{g}$ be a Lie algebra defined over $\mathbb{C}$. It is said *simple* if $\dim \mathfrak{g} > 1$ and it contains no nontrivial ideals. Such a Lie algebra is known to be classified by its *root system*.

A simple Lie algbra contains a maximal abelian subalgebra, unique up to an inner automorphism. Such a maximal abelian subalgebra is called a *Cartan subalgebra*, and denoted by $\mathfrak{h}$. It is known that the adjoint action of an element $h \in \mathfrak{h}$ is semisimple. Therefore we have a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \langle \alpha, h\rangle x \text{ for all } h \in \mathfrak{h}\}.$$

When $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, α is called a *root*. The set of roots is denoted by Δ. It is known that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$ for $\alpha \in \Delta$. It is known that if α is a root, then $-\alpha$ is also a root. Since $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ are 1-dimensional, the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is at most 1-dimensional. It is known that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is, in fact, 1-dimensional. It is also known that the subalgebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2$. We can choose $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ so that the standard relation for $\mathfrak{sl}_2$ holds: $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$, $[x_\alpha, y_\alpha] = h_\alpha$. We normalize h_α by $\langle \alpha, h_\alpha\rangle = 2$.

For a root $\alpha \in \Delta$, we define a linear automorphism s_α of $\mathfrak{h}^*$ by $s_\alpha(\xi) = \xi - \langle \xi, h_\alpha\rangle\alpha$. The *Weyl group* $W \subset \mathrm{GL}(\mathfrak{h}^*)$ of $\mathfrak{g}$ is the group generated by s_α's ($\alpha \in \Delta$). It is known that W permutes roots.

The lattice $Q \subset \mathfrak{h}^*$ generated by roots is called the *root lattice*. Let $\mathfrak{h}_{\mathbb{R}} \overset{\text{def.}}{=} \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}$. We have $\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}$. Connected components of $\mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha\in\Delta}\{h \mid \alpha(h) = 0\}$ are called (open) *Weyl chambers*. It is known that the Weyl group W acts simply transitively on the set of Weyl chambers. Once a Weyl chamber $\mathcal{W}$ is chosen, we define the set Δ_+ of *positive roots* by $\Delta_+ \overset{\text{def.}}{=} \{\alpha \in \Delta \mid \alpha \geq 0 \text{ on } \mathcal{W}\}$. We say a postive root α is *indecomposable* if it is not written as $\alpha = \beta_1 + \beta_2$ with β_1, $\beta_2 \in \Delta_+$. The indecomposable positive roots are called *simple*. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. It is known that Π gives a base of $\mathfrak{h}^*$, and any root β is written as $\beta = \sum_{i=1}^n m_i\alpha_i$ with integral coefficients $m_i \in \mathbb{Z}$ all nonnegative or all nonpositive. Therefore, $\Delta = \Delta_+ \sqcup -\Delta_+$. We define the *Cartan matrix* of $\mathfrak{g}$ by $(c_{ij})_{i,j=1,\dots,n}$ with $c_{ij} = \langle \alpha_j, h_{\alpha_i}\rangle$. We define a partial ordering $\leq$ on $\mathfrak{h}^*$ by setting $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots or $\mu = \lambda$. There exists a unique $\theta \in \Delta_+$ which is maximal with respect to $\leq$. It is called the *highest root*.

It is known that $c_{ij}c_{ji} = 0$, 1, 2, or 3 for $i \neq j$. Define the Dynkin diagram of $\mathfrak{g}$ as follows. The vertices correspond to simple roots $\alpha_i \in \Pi$. If $|c_{ij}| \geq |c_{ji}|$, two vertices α_i and α_j are connected by $|c_{ij}|$ lines, and these lines are equipped with an arrow pointing toward α_i if $|c_{ij}| > 1$. The Lie algebra $\mathfrak{g}$ is said *simply-laced* if $c_{ij} = 0$ or 1.

The reconstruction of $\mathfrak{g}$ from the Cartan matrix will be explained in §5 in the context of Kac-Moody Lie algebras.

Example 4.2. Consider $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We choose a standard basis of $\mathbb{C}^{n+1}$ so that $\mathfrak{g}$ is the Lie algebra of tracefree matrices. Let $e_{i,j}$ be the $(n+1) \times (n+1)$ whose (i,j)th entry is

1 and all other entries are 0. Then $\{e_{i,j}(i \neq j), e_{i,i} - e_{i+1,i+1}(i = 1, \dots n)\}$ is a base of $\mathfrak{g}$. The subalgebra of all tracefree diagonal matrices is a Cartan subalgebra. Let us denote it by $\mathfrak{h}$. We define $v_i \in \mathfrak{h}^*$ $(i = 1, \dots, n+1)$ by $\langle v_i, e_{j,j} - e_{j+1,j+1}\rangle = \delta_{i,j} - \delta_{i,j+1}$. We have $\mathfrak{h}^* = \bigoplus_{i=1}^{n+1} \mathbb{C}v_i \big/ (v_1 + \cdots + v_{n+1} = 0)$. The root spaces are

$$\mathfrak{g}_\alpha = \begin{cases} \mathfrak{h} & \text{if } \alpha = 0, \\ \mathbb{C}e_{i,j} & \text{if } \alpha = v_i - v_j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

So $\Delta = \{v_i - v_j \mid i \neq j\}$. Let $x_{v_i - v_j} = e_{i,j} \in \mathfrak{g}_{v_i - v_j}$, $y_{v_i - v_j} = e_{j,i} \in \mathfrak{g}_{v_j - v_i}$, $h_{v_i - v_j} = e_{i,i} - e_{j,j} \in [\mathfrak{g}_{v_i - v_j}, \mathfrak{g}_{v_j - v_i}]$. Then $x_{v_i - v_j}$, $y_{v_i - v_j}$, $h_{v_i - v_j}$ satisfy the standard relation for $\mathfrak{sl}_2$. The normalization condition $\langle v_i - v_j, h_{v_i - v_j}\rangle = 2$ is satisfied. The reflection $s_{v_i - v_j}$ corresponding to a root $v_i - v_j$ exchanges v_i and $v_j \in \mathfrak{h}^*$ and fixes the other v_k. Therefore the Weyl group W is the symmetric group S_{n+1} on v_i's. The subspace $\mathfrak{h}_\mathbb{R}$ consists of all *real* tracefree $(n+1)\times(n+1)$ diagonal matrices. We take a Weyl chamber $\mathcal{W}$ given by

$$\mathcal{W} = \{\operatorname{diag}(\lambda_1, \dots, \lambda_{n+1}) \in \mathfrak{h}_\mathbb{R} \mid \lambda_i > \lambda_{i+1} \quad (i = 1, \dots, n)\}.$$

Then we have $\Delta_+ = \{v_i - v_j \mid i < j\}$ and $\Pi = \{\alpha_i \mid i = 1, \dots, n\}$ with $\alpha_i \overset{\text{def.}}{=} v_i - v_{i+1}$. We have $\Delta_+ = \{\alpha_i + \cdots + \alpha_j \mid i \leq j\}$. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

The highest root is $v_1 - v_{n+1} = \alpha_1 + \cdots + \alpha_n$. The Dynkin diagram is the graph in Table 1.

4.3. **Affine Cartan matrix.** The Cartan matrix $\mathbf{C} = (c_{ij})$ of a simple complex Lie algebra appeared in the previous section has the following properties:

(1) it is indecomposable (i.e., the corresponding Dynkin diagram is connected),
(2) $c_{ii} = 2$ for all i,
(3) $c_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
(4) $c_{ij} = 0 \iff c_{ji} = 0$,
(5) all principal minors of $\mathbf{C}$ are positive.

Here principal minors are determinants of the matrices $(c_{ij})_{i,j\in S}$ for subsets $S \subset \{1, \dots, n\}$. When (c_{ij}) is symmetric, i.e., of type ADE, we have

(5s) $\mathbf{C}$ is positive definite.

In fact, it is known that the properties (1) $\sim$ (5) ((1) $\sim$ (4) and (5s) for symmetric matrices) characterizes Cartan matrices of simple Lie algebras.

Affine Cartan matrices are defined as matrices satisfying (1) $\sim$ (4) and

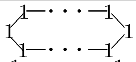
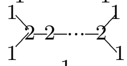
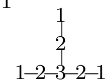
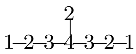
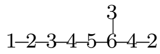
(5)′ all *proper* principal minors of $\mathbf{C}$ are positive and $\det \mathbf{C} = 0$,

or for symmetric matrices

(5s)′ $\mathbf{C}$ is positive semidefinite and $\det \mathbf{C} = 0$.

The classification of affine Cartan matrices is known (see [22, §4]). Symmetric affine Cartan matrices (more precisely, corresponding Dynkin diagrams) are ones given in Table 1.

Suppose that $\mathbf{C} = (c_{ij})$ is an $(n+1)\times(n+1)$-matrix. We let indices i, j run from 0 to n and 0 corresponds to the black vertex. There is a vector $\delta \in \mathbb{Z}^{n+1}$ such that $\mathbf{C}\delta = 0$ and entries of δ are relatively prime, since $\mathbf{C}$ is rank n. Such a vector is unique up to $\pm$. In fact,

type	entries of δ
A_n $(n \geq 0)$	
D_n $(n \geq 4)$	
E_6	
E_7	
E_8	

it is known that all entries are positive or negative. So we take δ for the positive one. For symmetric case, the entries are given by the following table.

From the table, one can see that the entry of δ corresponding to the vertex 0 (the black vertex) is always 1. Moreover, it is known that the other entries of δ are the coefficients of the highest root of the corresponding simple Lie algebra $\mathfrak{g}$:

$$\delta = {}^t(1, a_1, \dots, a_n), \qquad \theta = \sum_{i=1}^n a_i \alpha_i.$$

4.4. **McKay correspondence.** McKay obtained more direct connection between finite subgroups Γ of $\mathrm{SL}_2(\mathbb{C})$ and the Dynkin diagrams [33].

Let ρ_0, ... , ρ_n be (the isomorphism classes of) irreducible representations of Γ, where ρ_0 is the trivial representation. Let Q be the 2-dimensional representation defined by the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ as above. Then we define nonnegative integers a_{ij} by

$$Q \otimes \rho_i = \bigoplus_j \rho_j^{\oplus a_{ij}},$$

i.e., $a_{ij} = \dim \mathrm{Hom}(\rho_j, Q \otimes \rho_i)^\Gamma$. Since Q is isomorphic to its dual Q^*, we see that $a_{ij} = a_{ji}$. Then we define a graph as follows. The vertices are irreducible representations ρ_i. We draw $a_{ij} = a_{ji}$ edges between the vertices ρ_i and ρ_j. A remarkable observation due to McKay is that this graph is of type affine ADE. The black vertex corresponds to the trivial representation ρ_0. Moreover, the graph obtained by removing the black vertex is same as one given by the intersection of irreducible components of the exceptional set. The original McKay's argument was based on the explicit calculation of characters of irreducible representations of Γ. Although it is completely rigorous, it remains misterious why such a result holds. There was a geometric approach to prove this assertion, due to Gonzalez-Sprinberg and Verdier [17], and also its generalization to the case $\Gamma \subset \mathrm{SL}_3(\mathbb{C})$ [20, 5]. We will come back to two dimensional case later.

The McKay observation gives us another description of $X(\mathbf{v})$. Let

$$V = \bigoplus V_i \otimes \rho_i$$

be decomposition of the Γ-module V, i.e., $V_i = \mathrm{Hom}(\rho_i, V)^\Gamma$. This notation will be used for any Γ-module V hereafter. We consider the matrix description (B_1, B_2, v) for a point in $X(\mathbf{v})$. Since $v \in V$ is given by 1 mod I, it is fixed by the Γ-action, i.e., $v \in V_0 \otimes \rho_0$. We take a base for ρ_0, and identify ρ_0 with $\mathbb{C}$. So v is an element in V_0.

We consider the pair (B_1, B_2) as an element of $\mathrm{Hom}(V, Q \otimes V)$, and denote it by B. It is clear that B is contained in $(Q \otimes \mathrm{Hom}(V, V))^\Gamma$. We have

$$(Q \otimes \mathrm{Hom}(V, V))^\Gamma = \bigoplus_{i,j} \mathrm{Hom}(V_j, V_i) \otimes \mathrm{Hom}(\rho_j, Q \otimes \rho_i)^\Gamma.$$

Choose and fix a base for $\mathrm{Hom}(\rho_j, Q \otimes \rho_i)^\Gamma$ for each pair (i, j). (In fact, if the graph is not $\widetilde{A}_1$, then the space is at most one dimensional.) Collecting the bases for all i, j, we denote the union by H. To each $h \in H$, we associate an *oriented* edge in the affine Dynkin diagram from the vertex l to k, if h is an element of the base of $\mathrm{Hom}(\rho_j, Q \otimes \rho_i)^\Gamma$. In this case, we denote i by $\mathrm{in}(h)$, j by $\mathrm{out}(h)$. For every edge in the affine Dynkin diagram, we can attach *two* orientations. In particular, the number of oriented edges is twice the number of unoriented edges. We decompose B as

$$B = \bigoplus B_h \otimes h, \qquad \text{where } B_h \in \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}).$$

The figure 1 represents the data, when Γ is of type A_n, where an oriented edge h is denoted by $\mathrm{in}(h), \mathrm{out}(h)$.

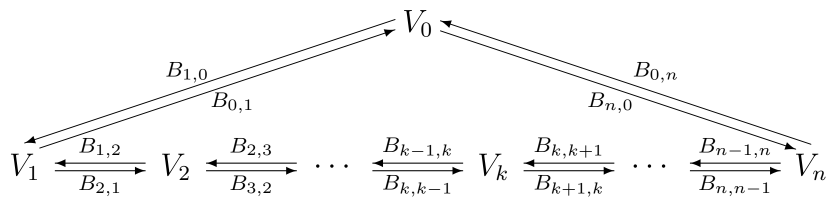


FIGURE 1

An oriented graph is called a *quiver*. This description for Example 4.1, i.e., V is the regular representation, is Kronheimer's construction [28]. The description for a general V is a special case of the quiver variety in [N1].

From now on, we identify the isomorphism class $\mathbf{v}$ of a Γ-module with a vector in $\mathbb{Z}_{\geq 0}^{n+1}$ as

$$\mathbf{v} = {}^t(\dim V_0, \dots, \dim V_n), \qquad \text{where } V = \bigoplus V_i \otimes \rho_i.$$

We also denote components of $\mathbf{v}$ as $\mathbf{v} = {}^t(v_0, \dots, v_n)$.

************ We assume Γ is *not* trivial subgroup $\{1\}$ hereafter. ************

4.5. **Lagrangian subvarieties.** Let $\pi\colon X(\mathbf{v}) \to (S^{\dim V} X)^\Gamma$ be the restriction of the Hilbert-Chow morphism. Since π is proper, the inverse image $\pi^{-1}(0)$ is a compact subvariety. (Here 0 means $\dim V \cdot [0]$, the 0-dimensional cycle at the origin with multiplicity $\dim V$.) It can be shown that it is homotopic to $X(\mathbf{v})$, so $H_*(X(\mathbf{v}), \mathbb{C})$ is isomorphic to $H_*(\pi^{-1}(0), \mathbb{C})$. We introduce the notation:

$$\mathfrak{L}(\mathbf{v}) \overset{\text{def.}}{=} \pi^{-1}(0) \subset X(\mathbf{v}).$$

We have $\mathfrak{L}(\mathbf{v}) = X(\mathbf{v}) \cap X_0^{[\dim V]}$, where $X_0^{[\dim V]}$ is the punctual Hilbert scheme.

We have

Theorem 4.3. $\mathfrak{L}(\mathbf{v})$ *is a lagrangian subvariety in* $X(\mathbf{v})$*. In particular, it is middle dimensional.*

We will give a proof, due to Lusztig, which is different from the original one [N3], during our study of the crystal structure on the set of irreducible components of $\bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v})$.

When $X(\mathbf{r})$ is the minimal resolution of X/Γ as in an example above, $\mathfrak{L}(\mathbf{r})$ is the union of projective lines. So the result can be checked directly.

4.6. **Roots in terms of quiver varieties.** A correspondence between roots of a Lie algebra and indecomposable representations of a quiver was first observed by Gabriel [14] and further studied by Kac [23]. We study a correspondence from the view point of quiver varieties in this subsection.

Proposition 4.4. *Let* $\mathbf{C} = (2\delta_{ij} - a_{ij})_{i,j=0,\dots,n}$ *be the Cartan matrix of type affine ADE. Let* v_0 *be the* 0*th component of* $\mathbf{v}$*, and* $(\ ,\ )$ *be the standard inner product on* $\mathbb{Z}^{n+1}$*. Then*

$$\dim_{\mathbb{C}} X(\mathbf{v}) = 2v_0 - (\mathbf{v}, \mathbf{C}\mathbf{v}).$$

Proof. Considering the Γ-invariance in the proof of Theorem 3.4, we have

$$T_{[(B,v)]}X(\mathbf{v}) = \operatorname{Ker} d\mu / \operatorname{Im} \iota,$$

where

$$\operatorname{End}(V)^\Gamma \xrightarrow{\iota} (Q \otimes \operatorname{End}(V) \oplus V)^\Gamma \xrightarrow{d\mu} \operatorname{End}(V)^\Gamma,$$

where $(\ )^\Gamma$ means the Γ-invariant part. Also ι is injective and the cokernel of $d\mu$ is isomorphic to V^Γ. Therefore

$$\dim X(\mathbf{v}) = \dim (Q \otimes \operatorname{End}(V))^\Gamma + 2\dim V^\Gamma - 2\dim \operatorname{End}(V)^\Gamma.$$

Using the decomposition $V = \bigoplus_{i=0}^n V_i \otimes \rho_i$, we find

$$\operatorname{End}(V)^\Gamma = \bigoplus_{i=0}^n \operatorname{End}(V_i), \qquad (Q \otimes \operatorname{End}(V))^\Gamma = \bigoplus_{i,j=0}^n a_{ij} \operatorname{Hom}(V_i, V_j), \qquad V^\Gamma = V_0.$$

Now the assertion follows. □

Exercise 4.5. It should be possible to prove this dimension formula using the equivariant version of Hirzebruch-Riemann-Roch formula (i.e., Lefschetz formula), without using quiver variety description. A similar calculation can be found in [29, Appendix]. But the details are not worked out yet.

From this dimension formula, an importance of the bilinear form $(\cdot, \mathbf{C}\cdot)$ on $\mathbb{Z}^{n+1}$ is clear. The following is well-known (see [22, §4,§5])

(1) $(\mathbf{v}, \mathbf{C}\mathbf{v}) \in 2\mathbb{Z}_{\geq 0}$,
(2) $(\mathbf{v}, \mathbf{C}\mathbf{v}) = 0$ if and only if $\mathbf{v} \in \mathbb{Z}\delta$, (this property characterize δ up to $\pm$),
(3) the $\mathbb{Z}^n$-part of δ is the highest root θ of $\mathfrak{g}$.
(4) if $v_0 = 1$, then $(\mathbf{v}, \mathbf{C}\mathbf{v}) = 2$ if and only if $\delta - \mathbf{v}$, considered as a vector in $\mathbb{Z}^n$, is a root of $\mathfrak{g}$,

(4) is a consequence of the property of the bilinear form associated with the Cartan matrix of finite type.

Consider Example 4.1. We have $\dim V_i = \dim \rho_i$, when V is the regular representation. Therefore the dimension vector $\mathbf{r}$ of V is equal to ${}^t(\dim \rho_0, \dots, \dim \rho_n)$. We have $2 = \dim X(\mathbf{r}) = 2 - (\mathbf{r}, \mathbf{C}\mathbf{r})$, and therefore $(\mathbf{r}, \mathbf{C}\mathbf{r}) = 0$. By the property (2) above, we have $\mathbf{r} \in \mathbb{Z}\delta$. Since $\dim \rho_0 = 1$, we have $\mathbf{r} = \delta$.

Theorem 4.6. *Consider the case* $v_0 = \dim V^\Gamma = 1$*. Let* $\mathbf{v}' = {}^t(\dim V_1, \dots, \dim V_n) \in \mathbb{Z}^n$*. Let* $\theta \in \mathbb{Z}^n$ *be the highest root. Then* $X(\mathbf{v})$ *is nonempty if and only if* $\theta - \mathbf{v}'$ *is a positive root or* 0*. Moreover, in the first case* $X(\mathbf{v})$ *is a single point, while the second case is the minimal resolution of* $\mathbb{C}^2/\Gamma$*.*

Proof. If $X(\mathbf{v})$ is nonempty, then

$$0 \leq \dim X(\mathbf{v}) = 2 - (\mathbf{v}, \mathbf{C}\mathbf{v}).$$

Therefore the necessity assertion follows from the above mentioned property of the bilinear form.

For the converse, we must show that $X(\mathbf{v})$ is a single point if $\theta - \mathbf{v}'$ is a root. This can be proved by using the theory of the 'reflection functors' for quiver varieties, which says

- $X(\mathbf{v}_1)$ and $X(\mathbf{v}_2)$ are diffeomorphic if $\theta - \mathbf{v}_1'$ and $\theta - \mathbf{v}_2'$ are in the same Weyl group orbit.

Then it is enough to show the assertion for the special case $\mathbf{v}' = 0$, i.e., V is the trivial representation. Then $X(\mathbf{v})$ is a single point corresponding to the maximal ideal at the origin.

There is also a proof which reduce the assertion to Gabriel's theorem (see e.g., [8].)

Here we give another argument which shows that $X(\mathbf{v})$ is a single point, provided $X(\mathbf{v})$ is nonempty, due to Mukai [34]. Let I, I' be points in $X(\mathbf{v})$, and let Z, Z' be the corresponding 0-dimensional subschemes. Then we have

$$\dim \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)^\Gamma = \dim \operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)^\Gamma = 1, \quad \dim \operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)^\Gamma = \dim X(\mathbf{v}) = 0.$$

By the equivariant version of the Riemman-Roch theorem (Lefschetz theorem), the Euler characteristic $\sum_i (-1)^i \dim \operatorname{Ext}^i(\mathcal{O}_Z, \mathcal{O}_{Z'})^\Gamma$ is independent of Z, Z', so is equal to 2. Therefore, either $\operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_{Z'})$ or $\operatorname{Ext}^2(\mathcal{O}_Z, \mathcal{O}_{Z'}) = \operatorname{Hom}(\mathcal{O}_{Z'}, \mathcal{O}_Z)^*$ is nonzero. In either case, a nonzero homomorphism $\mathcal{O}_Z \to \mathcal{O}_{Z'}$ or $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is easily seen to be an isomorphism.

Probably it should be possible to write down the ideal in $X(\mathbf{v})$ *explicitly* as in [IN], and prove the assertion in a different way. But so far, I do not know..... □

Theorem 4.6 implies an existence of an isomorphism of vector spaces:

$$\bigoplus_{\mathbf{v}: v_0 = 1} H_{\mathrm{mid}}(X(\mathbf{v}), \mathbb{C}) \cong \mathfrak{h} \oplus \bigoplus_{\alpha \text{ is a root}} \mathfrak{g}_\alpha = \mathfrak{g},$$

where mid means the middle degree, i.e., mid $= \dim_{\mathbb{C}} X(\mathbf{v}) = 0$ or 2.

We will give a Lie algebra structure on the left hand side in §6. In fact, we drop unnatural condition $v_0 = 1$ and embed it to the basic representation of the corresponding untwisted affine Lie algebra.

4.7. **Some cohomology groups.** Let $I \in X(\mathbf{v})$. We will study several cohomology groups later. In this subsection, we describe them in terms of the matrix data (B, v).

Let us introduce the following notation.

$$\begin{aligned}
&A_X = \mathbb{C}[x, y] \quad \text{(the coordinate ring of } X = \mathbb{C}^2\text{)},\\
&A_0 = \mathbb{C} \quad \text{(the coordinate ring of the origin)},\\
&A_X \to A_0; \quad f(x,y) \to f(0,0),\\
&\mathfrak{m} = \operatorname{Ker}(A_X \to A_0) \quad \text{(the maximal ideal corresponding to 0)}.
\end{aligned}$$

For $(B, v) \in X(\mathbf{v})$, we introduce two complexes:

$$V \xrightarrow{\sigma} Q \otimes V \oplus \rho_0 \xrightarrow{\tau} \textstyle\bigwedge^2 Q \otimes V \cong V, \tag{4.7}$$

$$V \xrightarrow{\sigma'} Q \otimes V \xrightarrow{\tau'} \textstyle\bigwedge^2 Q \otimes V \cong V, \tag{4.8}$$

where

$$\sigma(v) = \begin{bmatrix} Bv \\ 0 \end{bmatrix}, \qquad \tau\left(\begin{bmatrix} C \\ a \end{bmatrix}\right) = B \wedge C + av,$$
$$\sigma'(v) = Bv, \qquad \tau'(C) = B \wedge C.$$

Let us rewrite the first complex in terms of the ideal I:

$$\begin{array}{ccccc} \mathbb{C}[x,y]/I & \xrightarrow{\sigma} & Q\otimes\mathbb{C}[x,y]/I\oplus\rho_0 & \xrightarrow{\tau} & \mathbb{C}[x,y]/I \\ f & \mapsto & \begin{bmatrix} xf \bmod I \\ yf \bmod I \\ 0 \end{bmatrix} & & \\ & & \begin{bmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{bmatrix} & \mapsto & (xf_2 - yf_1 + a) \bmod I, \end{array}$$

where $a\in\rho_0$ is considered as a scalar. The second complex also has a similar description.

Lemma 4.9. *Let $I\in X^{[n]}$ and let A_Z be the corresponding quotient. We have*

$$\operatorname{Ker}\sigma \cong \operatorname{Hom}_{A_X}(A_0, A_Z), \qquad \operatorname{Ker}\tau/\operatorname{Im}\sigma \cong I/\mathfrak{m}I,$$
$$\operatorname{Ker}\sigma' \cong \operatorname{Hom}_{A_X}(A_0, A_Z), \qquad \operatorname{Ker}\tau'/\operatorname{Im}\sigma' = \operatorname{Ext}^1_{A_X}(A_0, A_Z), \qquad \operatorname{Coker}\tau' = \operatorname{Ext}^2_{A_X}(A_0, A_Z).$$

Proof. Since $\operatorname{Ker}\sigma = \operatorname{Ker}\sigma'$, the first equation follows from the third equation.

Let us prove the second equation. We define a map $\operatorname{Ker}\tau\to I/\mathfrak{m}I$ by

$$\begin{bmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{bmatrix} \longmapsto (xf_2 - yf_1 + a) \bmod \mathfrak{m}I.$$

Since $xf_2 - yf_1 + a$ is contained in I if $\left[\begin{smallmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{smallmatrix}\right]$ is in $\operatorname{Ker}\tau$, the right hand side makes a sense. Moreover, if we take another representative, i.e., if we replace f_1, f_2 by f_1+g_1, f_2+g_2 with $g_1, g_2\in I$, the right hand side is unchanged since

$$xg_2 - yg_1 \in \mathfrak{m}I.$$

It is also clear that $\operatorname{Im}\sigma$ is mapped to 0. Therefore we have an induced map $\operatorname{Ker}\tau/\operatorname{Im}\sigma\to I/\mathfrak{m}I$.

Suppose that $\left[\begin{smallmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{smallmatrix}\right]\in\operatorname{Ker}\tau$ is mapped to 0 in $I/\mathfrak{m}I$, i.e., $xf_2 - yf_1 + a\in\mathfrak{m}I$. We have $xf_2 - yf_1 + a = xg_1 + yg_2$ for some $g_1, g_2\in I$. Therefore, a must be 0. Let us write $x(f_2 - g_1) = y(g_2 + f_1) = xyf$. Then we have

$$f_1 = xf - g_2, \qquad f_2 = yf + g_1.$$

Therefore $\left[\begin{smallmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{smallmatrix}\right]\in\operatorname{Ker}\tau$ is contained in $\operatorname{Im}\sigma$. Therefore the map $\operatorname{Ker}\tau/\operatorname{Im}\sigma\to I/\mathfrak{m}I$ is injective.

Let us show the map is surjective. Suppose $g\in I$ is given. We can write it

$$g = xf_2 - yf_1 + a$$

for some $a\in\mathbb{C}$, $f_1, f_2\in\mathbb{C}[x,y]$. Then $\left[\begin{smallmatrix} f_1 \bmod I \\ f_2 \bmod I \\ a \end{smallmatrix}\right]$ is mapped to $g \bmod I = 0$ by τ, i.e., it is contained in $\operatorname{Ker}\tau$.

Next we turn to equation in the second row. We have the Koszul resolution

$$0\to A_X\to Q\otimes_{\mathbb{C}} A_X\to A_X\to A_0\to 0.$$

Therefore $\operatorname{Ext}^i_{A_X}(A_0, A_Z)$ is the ith cohomology group of

$$\operatorname{Hom}_{A_X}(A_X, A_Z)\to\operatorname{Hom}_{A_X}(Q\otimes_{\mathbb{C}} A_X, A_Z)\to\operatorname{Hom}_{A_X}(A_X, A_Z).$$

This is nothing but the complex (4.8). □

Corollary 4.10. *We have the following equality in the representation ring of* Γ:

$$I/\mathfrak{m}I - \operatorname{Hom}_{A_X}(A_0, A_Z) = Q \otimes V - V^{\oplus 2} + \rho_0. \tag{4.11}$$

Proof. By the above lemma, the left hand side is equal to

$$\operatorname{Ker}\tau / \operatorname{Im}\sigma - \operatorname{Ker}\sigma.$$

The cyclic vector condition (3.3.2) implies τ is surjective. Now the result follows. □

Remark 4.12. Let us rewrite the left hand side of (4.11). From the exact sequence $0 \to I \to A_X \to A_Z \to 0$, we have

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{A_X}(A_0, I) &\longrightarrow \operatorname{Hom}_{A_X}(A_0, A_X) \longrightarrow \operatorname{Hom}_{A_X}(A_0, A_Z) \\ \longrightarrow \operatorname{Ext}^1_{A_X}(A_0, I) &\longrightarrow \operatorname{Ext}^1_{A_X}(A_0, A_X) \longrightarrow \operatorname{Ext}^1_{A_X}(A_0, A_Z) \\ \longrightarrow \operatorname{Ext}^2_{A_X}(A_0, I) &\longrightarrow \operatorname{Ext}^2_{A_X}(A_0, A_X) \longrightarrow \operatorname{Ext}^2_{A_X}(A_0, A_Z) \longrightarrow 0. \end{aligned}$$

It is easy to show $\operatorname{Ext}^i_{A_X}(A_0, A_X) = 0$ for $i = 0, 1$ and $\operatorname{Ext}^2_{A_X}(A_0, A_X) = \rho_0$ by using the argument in the proof of Lemma 4.9. Therefore

$$\operatorname{Ext}^2_{A_X}(A_0, I) = \operatorname{Ext}^1_{A_X}(A_0, A_Z) - \operatorname{Ext}^2_{A_X}(A_0, A_Z) + \rho_0$$

in the representation ring of Γ. Again by the argument in the proof of Lemma 4.9, we have

$$\operatorname{Ext}^2_{A_X}(A_0, I) \cong I/\mathfrak{m}I.$$

Therefore we have

$$I/\mathfrak{m}I - \operatorname{Hom}_{A_X}(A_0, A_Z) = -\sum_{i=0}^{2} (-1)^i \operatorname{Ext}^i_{A_X}(A_0, A_Z) + \rho_0.$$

Exercise 4.13. Probably it is possible to give a direct proof of this corollary by using Lefschetz fixed point theorem.

4.8. **Tautological bundles.** Each point I in $X(\mathbf{v})$ defines a Γ-module $\mathbb{C}[x,y]/I$. When I is moved, we get a vector bundle with a fiberwise Γ-module structure. By abuse of notation, we denote this vector bundle by V and decompose it as $V = \bigoplus_i V_i \otimes \rho_i$. The homomorphisms B and v on $\mathbb{C}[x,y]/I$ can be considered as vector bundle homomorphisms:

$$B\colon V \to Q \otimes V, \qquad v\colon \mathcal{O}_{X(\mathbf{v})} \to V.$$

We call them *tautological sections*.

The complexes (4.7, 4.8) above can be considered as complexes of vector bundles, when we move point $[(B, v)] \in X(\mathbf{v})$.

4.9. **Bases in K-theory and correspondences in derived categories.** The rest of this section will be needed only for the explanation of McKay correspondence. So readers who only have interest in representations of affine Lie algbras can safely skip them. Moreover, we use K-groups and derived categories of coherent sheaves, which we will not review.

We decompose the complex (4.8) according to the Γ-module decomposition:

$$V_i \to \bigoplus_j a_{ij} V_j \to V_i.$$

We denote this complex by S_i.

We consider the special case $\mathbf{v} = \mathbf{r} = \delta$. Let $K(X(\mathbf{r}))$ (resp. $K_c(X(\mathbf{r}))$) is the Grothendieck group of the abelian category of coherent sheaves (resp. coherent sheaves with supports contained in the exceptional set) on $X(\mathbf{r})$. Since $X(\mathbf{r})$ is nonsingular, $K(X(\mathbf{r}))$ (resp. $K_c(X(\mathbf{r}))$) is isomorphic to the Grothendieck group of the abelian category of algebraic vector bundles (resp. the derived category of bounded complexes of vector bundles whose cohomology groups have supports in the exceptional set) on $X(\mathbf{r})$. It is easy to see the complex S_i has support in the exceptional set, and defines an element in $K_c(X(\mathbf{r}))$.

Theorem 4.14 ([17, 20]). *$\{V_i\}_{i=0,\dots,n}$ and $\{S_i\}_{i=0,\dots,n}$ form bases of $K(X(\mathbf{r}))$ and $K_c(X(\mathbf{r}))$ respectively. Moreover, they are dual to each other with respect to the natural pairing*

$$\langle\ ,\ \rangle\colon K_c(X(\mathbf{r}))\otimes K(X(\mathbf{r}))\to\mathbb{Z}$$

$$[S]\otimes[E]\longmapsto\sum_{i=0}^{2}(-1)^i\dim H^i(S\otimes E^*)=\sum_{i=0}^{2}(-1)^i\dim\operatorname{Ext}^i_{\mathcal{O}_{X(\mathbf{r})}}(E,S),$$

where E^* is the dual vector bundle of E.

(In [20] the pairing was defined without taking dual, and S_i was replaced by its transpose.)

Corollary 4.15. *Let $\theta\colon K_c(X(\mathbf{r}))\to K(X(\mathbf{r}))$ be the natural homomorphism, forgetting the support condition. Then we have*

$$\langle\theta S_i,S_j\rangle=2\delta_{ij}-a_{ij}.$$

This is a consequence of $\theta S_i=\sum_j(2\delta_{ij}-a_{ij})V_j$. This corollary gives a geometric explanation of McKay correspondence. The identification of the intersection product with the tensor product multiplicities is manifest ! Moreover, we will identify S_i with a sheaf on the exceptional divisor, and recover the original form of McKay correspondence. (See Example 6.3.)

In stead of reproducing the original proof of Theorem 4.14, let us reformulate the result in terms of derived categories following [24, 5].

Let $\mathcal{Z}\subset X(\mathbf{r})\times\mathbb{C}^2$ be the *universal subscheme* for $X(\mathbf{r})$. We have two projections

$$X(\mathbf{r})\xleftarrow{p_1}X(\mathbf{r})\times\mathbb{C}^2\xrightarrow{p_2}\mathbb{C}^2.$$

Then the bundle R is nothing but the direct image sheaf $p_{1*}\mathcal{O}_{\mathcal{Z}}$. Since the restriction of p_1 to $\mathcal{Z}$ is flat and finite, it follows that $p_{1*}\mathcal{O}_{\mathcal{Z}}$ is locally free, i.e., gives a vector bundle. The tautological section B comes from the multiplication x, $y\colon\mathcal{O}_{\mathcal{Z}}\to\mathcal{O}_{\mathcal{Z}}$, and v comes from the natural homomorphism $\mathcal{O}_{X(\mathbf{v})\times\mathbb{C}^2}\to\mathcal{O}_{\mathcal{Z}}$.

These define a convolution operator between derived categories. Let $D(X(\mathbf{r}))$ be the derived category of coherent sheaves on $X(\mathbf{r})$ and $D^\Gamma(\mathbb{C}^2)$ be the derived category of Γ-equivariant coherent sheaves on $\mathbb{C}^2$. We define a convolution operator $\Psi\colon D^\Gamma(\mathbb{C}^2)\to D(X(\mathbf{r}))$ by

$$\Psi(\bullet)=\left[\mathbf{R}p_{1*}\left(\mathcal{O}_{\mathcal{Z}}\otimes^{\mathbf{L}}_{\mathcal{O}_{X(\mathbf{v})\times\mathbb{C}^2}}p_2^*(\bullet)\right)\right]^\Gamma.$$

(In [5], $\mathcal{O}_{\mathcal{Z}}^\vee$ (see below) was used in stead of $\mathcal{O}_{\mathcal{Z}}$.) We have $V_i=\Psi(\mathcal{O}_{\mathbb{C}^2}\otimes\rho_i)$. The same formula also defines an operator between derived categories with compact supports $\Psi\colon D^\Gamma_c(\mathbb{C}^2)\to D_c(X(\mathbf{r}))$. We have $S_i=\Psi(\mathcal{O}_0\otimes\rho_i)$. This follows from the Koszul resolution of $\mathcal{O}_0$:

$$0\to\bigwedge^2Q^*\otimes\mathcal{O}_{\mathbb{C}^2}\to Q^*\otimes\mathcal{O}_{\mathbb{C}^2}\to\mathcal{O}_{\mathbb{C}^2}\to\mathcal{O}_0\to0,$$

giving by the interior product by (x,y). Note that $Q^*=Q$ and $\bigwedge^2Q=\rho_0$.

Theorem 4.16 ([24, 5]). *Ψ is an equivalence of categories (for both arbitary support and compact support).*

Since an equivalence of derived categories induces an isomorphism of the Grothendieck groups, this theorem implies the previous theorem together with the following statements:

- $K^G(\mathbb{C}^2)$ has a base $\{\mathcal{O}_{\mathbb{C}^2}\otimes\rho_i\}_{i=0,\dots,n}$.
- $K^G_c(\mathbb{C}^2)$ has a base $\{\mathcal{O}_0\otimes\rho_i\}_{i=0,\dots,n}$.

Note also that a similar map gives a description of a framed moduli space of vector bundles over $X(\mathbf{r})$ in terms of representations of the corresponding affine quiver [29]. This result was obtained much earlier.

We consider the right adjoint $\Phi\colon D_c(X(\mathbf{r}))\to D^\Gamma_c(\mathbb{C}^2)$ of Ψ. A standard calculcation shows

$$\Phi(\bullet)=\mathbf{R}p_{2*}\left(\mathcal{O}_{\mathcal{Z}}^\vee\otimes^{\mathbf{L}}_{\mathcal{O}_{X(\mathbf{r})\times\mathbb{C}^2}}p_1^*(\bullet\otimes\rho_0)\right)[2],$$

where $\mathcal{O}_{\mathcal{Z}}^{\vee} = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X(\mathbf{r})\times\mathbb{C}^2}}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{X(\mathbf{r})\times\mathbb{C}^2})$. Since Ψ is an equivalence, we should have $\Psi\circ\Phi$ is identity. By a standard calculation for convolution, we have

$$\Psi\circ\Phi(\bullet) = \mathbf{R}P_{1*}\left(\mathcal{K}\otimes^{\mathbf{L}} P_2^*(\bullet)\right),$$

where

$$\begin{array}{ccccc} X(\mathbf{r})\times\mathbb{C}^2 & \xleftarrow{p_{12}} & X(\mathbf{r})\times\mathbb{C}^2\times X(\mathbf{r}) & \xrightarrow{p_{23}} & \mathbb{C}^2\times X(\mathbf{r}) \\ & & \Big\downarrow{\scriptstyle p_{13}} & & \\ X(\mathbf{r}) & \xleftarrow{P_1} & X(\mathbf{r})\times X(\mathbf{r}) & \xrightarrow{P_2} & X(\mathbf{r}), \end{array}$$

and

$$\mathcal{K} = \mathbf{R}p_{13*}\left(p_{12}^*\mathcal{O}_{\mathcal{Z}}^{\vee}\otimes^{\mathbf{L}} p_{23}^*\mathcal{O}_{\mathcal{Z}}\right)^{\Gamma}.$$

In fact, we can give an explicit description of $\mathcal{K}$. We denote the tautological bundle of the first (resp. second) factor by V^1 (resp. V^2). Tautological sections are denoted by B^1, v^1, B^2, v^2. Then $\mathcal{K}$ is equal to the following complex of vector bundles over $X(\mathbf{r})\times X(\mathbf{r})$:

$$\operatorname{Hom}(V^1, \textstyle\bigwedge^2 Q^*\otimes V^2)^{\Gamma} \to \operatorname{Hom}(V^1, Q^*\otimes V^2)^{\Gamma} \to \operatorname{Hom}(V^1, V^2)^{\Gamma}, \tag{4.17}$$

where differentials are given by $\eta\mapsto B^2\wedge\eta - \eta\wedge B^1$. (Note that $Q^* = Q$ and $\bigwedge^2 Q = \rho_0$.) Now, it is not difficult to show that this gives a locally free resolution of the diagonal $\mathcal{O}_{\Delta}$ [29, §3] or [20, §4]. Here the map

$$\operatorname{Hom}(V^1, V^2)^{\Gamma} \to \mathcal{O}_{\Delta}$$

is given by $\eta\mapsto \operatorname{tr}(\eta|_{\Delta})$.

In [20], we only use the complex (4.17) to deduce the isomorphism of K-groups. In [N2], we prove a further weaker statement. Using (4.17), we calculate the intersection product in the cohomology group.

Note that we do *not* have similar description of derived categories or K-groups for general quiver varieties $X(\mathbf{v})$. When $\mathbf{v} = n\mathbf{r}$ for some $n\in\mathbb{Z}_{>0}$, the derived category $D(X(n\mathbf{r}))$ should be equivalent to isomorphic to $D^{\Gamma_n}(X^n)$, where Γ_n is the nth wreath product of Γ by the result of Haiman [18].

At the end of this section, let us remark a similarlity between above isomorphism of $D(X(\mathbf{r}))$ and the description of framed anti-self-dual connections on $X(\mathbf{r})$ in [29]. A description of holomorphic vector bundles on $\mathbb{P}^2$ in terms of 'monads' was given by Barth. This description can be also applied to framed vector bundles, where the framing means an isomorphism between $E|_{\ell_\infty}$ to a trivial vector bundle on ℓ_∞, where ℓ_∞ is a line in $\mathbb{P}^2$. (See [Lecture, Chapter 2].) The description turns out to be the same as the ADHM description of framed anti-self-dual connections on S^4 as noticed by Donaldson [10], where the framing means a trivialization of E_∞, where ∞ is a point in S^4. The identification of two descriptions can be naturally explained if one works over noncompact space $\mathbb{P}^2\setminus\ell_\infty = \mathbb{C}^2 = \mathbb{R}^4 = S^4\setminus\{\infty\}$, as noticed by Donaldson [11]. (See also Nahm [35] and Corrigan-Goddard [7] for the earlier approach for the ADHM desciption over $\mathbb{R}^4$.)

The ADHM description of framed anti-self-dual connections or monad description of framed holomorphic vector bundles can be easily adapted to Γ-equivariant connections/vector bundles. Then the descriptions [29] of framed anti-self-dual connections and framed holomorphic vector bundles on $X(\mathbf{r})$ is obtained as 'deformations' of descriptions of Γ-equivariant ones. Here the 'deformation' means the stability condition imposed there is changed. (Or the value of the moment map is changed under the correspondence of symplectic quotients and geometric invariant theory quotients.) Therefore, two framed moduli spaces, one for holomorphic vector bundles over $X(\mathbf{r})$ and one for Γ-equivariant holomorphic vector bundles over X are very similar. (In fact, they are birational under a mild condition.) This is, of course, similar to the statement $D^{\Gamma}(\mathbb{C}^2) \cong D(X(\mathbf{r}))$. However, logically, the equivalence of derived category does not imply the result of [29] and [29] does not imply the equivalence.

5. Affine Lie algebra

We briefly recall the theory of untwisted affine Lie algebras in this section. See [22] for more detail.

5.1. **Definition.** The untwisted *affine Lie algebra* $\widehat{\mathfrak{g}}$ associated with a complex simple Lie algebra $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with the Lie algebra structure given by

$$\begin{gathered}[\widehat{\mathfrak{g}}, K] = 0,\\ [X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n} + m\delta_{m+n,0}(X, Y)K,\\ [d, X \otimes z^m] = mX \otimes z^m,\end{gathered}$$

where (X, Y) is the Killing form of $\mathfrak{g}$. Note that $\widehat{\mathfrak{g}}$ contains $\mathfrak{g}$ as a Lie subalgebra by $\mathfrak{g} \ni X \mapsto X \otimes 1 \in \widehat{\mathfrak{g}}$.

We use an alternative description of $\widehat{\mathfrak{g}}$, as an example of a Kac-Moody Lie algebra. $\widehat{\mathfrak{g}}$ has generators e_i, f_i, h_i $(i = 0, 1, \dots, \operatorname{rank}\mathfrak{g})$, d and defining relations

$$[h_i, h_j] = 0, \qquad [h_i, d] = 0 \tag{5.1}$$

$$[h_i, e_j] = c_{ji}e_j, \qquad [h_i, f_j] = -c_{ji}f_j, \tag{5.2}$$

$$[d, e_i] = \delta_{0i}e_i, \qquad [d, f_i] = -\delta_{0i}f_i \tag{5.3}$$

$$[e_i, f_j] = \delta_{ij}h_i, \tag{5.4}$$

$$(\operatorname{ad} e_i)^{1-c_{ij}}\, e_j = 0, \qquad (\operatorname{ad} f_i)^{1-c_{ij}}\, f_j = 0 \qquad \text{for } i \neq j. \tag{5.5}$$

Here c_{ij} is the affine Cartan matrix. A subalgebra generated by e_i, f_i, h_i $(i \neq 0)$ is isomorphic to $\mathfrak{g}$. The isomorphism between this description and the above description is given by

$$\begin{gathered} e_i \longleftrightarrow E_i \otimes 1, \qquad f_i \longleftrightarrow F_i \otimes 1, \qquad h_i \longleftrightarrow H_i \otimes 1, \qquad \text{for } i \neq 0,\\ e_0 \longleftrightarrow E_\theta \otimes z, \qquad f_0 \longleftrightarrow F_\theta \otimes z^{-1}, \qquad h_0 \longleftrightarrow [E_\theta, F_\theta] \otimes 1 + (E_\theta, F_\theta)\, K,\end{gathered}$$

where θ is the highest root of $\mathfrak{g}$, and E_θ, F_θ are suitably normalized elements in the root spaces $\mathfrak{g}_{-\theta}$, $\mathfrak{g}_\theta$ respectively. Moreover, we denote the elements e_i, f_i, h_i $(i \neq 0)$ by E_i, F_i, H_i respectively when they are considered as elements of $\mathfrak{g}$.

Remark 5.6. The element d is called *the degree operator*. The subalgebra generated by e_i, f_i, h_i is also called an *affine Lie algebra* in some literature. It is denoted by $\mathfrak{g}'(A)$ in [22, §1.5].

Let $\mathfrak{h} = \bigoplus \mathbb{C}h_i \oplus \mathbb{C}d \subset \widehat{\mathfrak{g}}$. It is an abelian subalgebra, called the *Cartan subalgebra* of $\widehat{\mathfrak{g}}$.

We define $\alpha_i \in \mathfrak{h}^*$ by

$$\langle \alpha_i, h_j \rangle = c_{ji}, \qquad \langle \alpha_i, d \rangle = \delta_{0i}.$$

The α_i are called *simple roots*.

5.2. **Integrable representations.** A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition*: $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \{v \in V \mid hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\}.$$

A weight module V is called a *highest weight module* of highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v_\Lambda \in V$, called a *highest weight vector* such that

$$\begin{aligned} &e_i v_\Lambda = 0 \quad \text{for all } i,\\ &hv_\Lambda = \langle \Lambda, h \rangle v_\Lambda \quad \text{for all } h \in \mathfrak{h},\\ &V = \mathbf{U}(\widehat{\mathfrak{g}})v_\Lambda.\end{aligned}$$

For each $\Lambda \in \mathfrak{h}^*$, there exists a unique (up to isomorphism) irreducible highest weight module, denoteed by $L(\Lambda)$.

A weight module V is called *integrable* if all e_i and f_i are locally nilpotent on V. Integrable modules are counterparts of finite dimensional modules of $\mathfrak{g}$. We have the following result.

Theorem 5.7. (1) *The irreducible highest weight module $L(\Lambda)$ is integrable if and only if Λ satisfies $\langle \Lambda, h_i \rangle \in \mathbb{Z}_{\geq 0}$. (A weight Λ satisfying this condition is called* dominant.)

(2) *An integrable highest weight module V is automatically irreducbile.*

For the proof, see [22, §10].

Remark 5.8. A integrable highest weight module $L(\Lambda)$ is not *finite dimensional* unless $\Lambda = 0$.

In the next section, we give a construction of an integrable highest weight module using quiver varieties.

6. Quiver varieties and affine Lie algebras

6.1. **Hecke correspondences and Nested Hilbert schemes.** A subvariety

$$X^{[n,n+1]} \stackrel{\text{def.}}{=} \{(I_1, I_2) \in X^{[n]} \times X^{[n+1]} \mid I_1 \supset I_2\}$$

of $X^{[n]} \times X^{[n+1]}$ is called a *nested Hilbert scheme*. A remarkable feature of the nested Hilbert scheme is that it is *nonsingular* of dimension $2n+2$. If we define a similar subvariety in $X^{[n]} \times X^{[n+k]}$ is *singular* for $k > 1$ [6, 39].

We want to consider similar subvarieties for fixed point sets with respect to the action of a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ as in the previous section. For nested Hilbert schemes, I_1, I_2 are Γ-invariant. For Hecke correspondences, V and S are Γ-modules. The condition $\dim I_1/I_2 = 1$ is *not* natural in this setting, we suppose I_1/I_2 is *irreducible* Γ*-modules*. Therefore we can define subvarieties for each ρ_i:

$$\mathfrak{P}_i(\mathbf{v}) \stackrel{\text{def.}}{=} \{(I_1, I_2) \in X(\mathbf{v} - \mathbf{e}_i) \times X(\mathbf{v}) \mid I_1 \supset I_2\},$$

where $\mathbf{e}_i$ is the ith coordinate vector of $\mathbb{Z}^{n+1}$. Note that I_1/I_2 is trivial as a module of $\mathbb{C}[x,y]$, i.e., x, y act by 0. For x, y act by $\pi(I_1) - \pi(I_2)$, but this must be 0 since it is fixed by Γ. We call this variety *a Hecke correspondence*. Let us denote two projections by p_1, p_2:

$$X(\mathbf{v} - \mathbf{e}_i) \xleftarrow{p_1} \mathfrak{P}_i(\mathbf{v}) \xrightarrow{p_2} X(\mathbf{v}).$$

The following result was proved in [N3, §5].

Theorem 6.1. $\mathfrak{P}_i(\mathbf{v})$ *is a nonsingular complex manifold. Moreover, it is lagrangian in* $X(\mathbf{v} - \mathbf{e}_i) \times X(\mathbf{v})$ *if* $\Gamma \neq \{1\}$.

Sketch of Proof. Let us consider $\mathbb{C}[x,y]/I_1$ and $\mathbb{C}[x,y]/I_2$ as vector bundles over $X(\mathbf{v} - \mathbf{e}_i)$ and $X(\mathbf{v})$ respectively. We denote them by V^1, V^2. We define a complex of vector bundles over $X(\mathbf{v} - \mathbf{e}_i) \times X(\mathbf{v})$ by

$$\operatorname{Hom}(V^2, V^1)^\Gamma \xrightarrow{\alpha} \begin{matrix} \operatorname{Hom}(V^2, Q \otimes V^1)^\Gamma \\ \oplus \\ V^1_0 \\ \oplus \\ (V^2_0)^* \end{matrix} \xrightarrow{\beta} \begin{matrix} \operatorname{Hom}(V^2, V^1)^\Gamma \\ \oplus \\ \mathcal{O}_{X(\mathbf{v}-\mathbf{e}_i)\times X(\mathbf{v})} \end{matrix},$$

where

$$\alpha(\xi) = \begin{bmatrix} B^1\xi - \xi B^2 \\ -\xi(v^2) \\ 0 \end{bmatrix},$$

$$\beta\left(\begin{bmatrix} C \\ a \\ b \end{bmatrix}\right) = \begin{bmatrix} B^2 \wedge C + C \wedge B^1 + v^1 \otimes b \\ \langle b, v^2 \rangle \end{bmatrix}$$

Here B^1, B^2, v^1, v^2 are tautologial homomorphisms

$$V^1 \xrightarrow{B^1} Q \otimes V^1, \quad V^2 \xrightarrow{B^2} Q \otimes V^2, \quad \mathcal{O} \xrightarrow{v^1} V^1_0, \quad \mathcal{O} \xrightarrow{v^2} V^2_0.$$

Then one can show that α is injective and β is surjective on each fiber (as linear maps between vector spaces). Therefore $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ is a vector bundle over $X(\mathbf{v} - \mathbf{e}_i) \times X(\mathbf{v})$. Let us define a section s by

$$s = \begin{bmatrix} 0 \\ -v^1 \\ 0 \end{bmatrix}.$$

Then s vanishes exactly on $\mathfrak{P}_i(\mathbf{v})$. In fact, if s vanishes, there exists ξ, λ such that

$$B^1\xi - \xi B^2 = 0, \quad \xi(v^2) = v^1.$$

From these equations, ξ is surjective. Returning back to the definition $V^1 = \mathbb{C}[x,y]/I_1$, $V^2 = \mathbb{C}[x,y]/I_2$, we find that $I_1 \supset I_2$, i.e., $(I_1, I_2) \in \mathfrak{P}_i(\mathbf{v})$. So it is enough to show that the derivative of s is surjective on $\mathfrak{P}_i(\mathbf{v})$. □

Proposition 6.2. (1) *Let $I_2 \in X(\mathbf{v})$ and let A_{Z_2} be the corresponding quotient. Then we have*

$$p_2^{-1}(I_2) \cong \mathbb{P}(\operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_{Z_2})^\Gamma),$$

where the right hand side is the projective space of one-dimensional subspaces of $\operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_{Z_2})^\Gamma$.

(2) *Let $I_1 \in X(\mathbf{v} - \mathbf{e}_i)$ and let A_{Z_1} be the corresponding quotient. Then we have*

$$p_1^{-1}(I_1) \cong \mathbb{P}(\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma),$$

where the right hand side is the projective space of one-dimensional subspace of $\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma$.

Proof. We first give a proof using algebraic geometry, and then give a proof which uses only linear algebra.

(1) If $(I_1, I_2) \in \mathfrak{P}_i(\mathbf{v})$, we have an exact sequence

$$0 \to A_0 \otimes \rho_i \to A_{Z_2} \to A_{Z_1} \to 0.$$

The first homomorphism gives an element of $\mathbb{P}(\operatorname{Hom}(A_0 \otimes \rho_i, A_{Z_2})^\Gamma)$. Conversely suppose that a nonzero homomorphism $A_0 \otimes \rho_i \to A_{Z_2}$ is given. Then it must be injective, since A_0 is generated by 1. Therefore we can define I_1 so that we get the above exact sequence, i.e., $A_{Z_1} = A_{Z_2}/A_0 \otimes \rho_i$. Therefore we get a point $(I_1, I_2) \in \mathfrak{P}_i(\mathbf{v})$.

(2) First note that there exists a natural isomorphism $\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \mathbb{C}) \cong \operatorname{Hom}_{A_X}(I_1, A_0)$. If $(I_1, I_2) \in \mathfrak{P}_i(\mathbf{v})$, then we have an exact sequence

$$0 \to I_2 \to I_1 \to A_0 \otimes \rho_i \to 0.$$

Therefore the last homomorphism gives an element of $\mathbb{P}(\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma)$. Conversely suppose that a nonzero Γ-equivariant homomorphism $I_1 \to A_0 \otimes \rho_i$ is given. It must be surjective since there is no nontrivial submodule in ρ_i. Therefore we can define I_2 so that we have the above exact sequence, i.e., $I_2 = \operatorname{Ker}(I_1 \to A_0 \otimes \rho_i)$.

Now we give a down-to-earth proof. (For general quiver varieties, we do not have the first proof.)

Let

$$\operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_{Z_2}) = \operatorname{Ker}\sigma = \bigoplus_i (\operatorname{Ker}\sigma)_i \otimes \rho_i,$$

$$I_1/\mathfrak{m}I_1 = \operatorname{Ker}\tau/\operatorname{Im}\sigma = \bigoplus_i (\operatorname{Ker}\tau/\operatorname{Im}\sigma)_i \otimes \rho_i$$

be the decomposition of $\operatorname{Ker}\sigma$ and $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ into irreducible Γ-modules.

Let $(I_1, I_2) \in \mathfrak{P}_i(\mathbf{v})$ and let $V^1 = \mathbb{C}[x,y]/I_1$, $V^2 = \mathbb{C}[x,y]/I_2$. From the definition, we have a commutative diagram of Γ-modules

$$\begin{array}{ccccc}
\rho_i & \longrightarrow & Q \otimes \rho_i & \longrightarrow & \rho_i \\
\downarrow & & \downarrow & & \downarrow \\
V^2 & \xrightarrow{\sigma_2} & Q \otimes V^2 \oplus \rho_0 & \xrightarrow{\tau_2} & V^2 \\
\downarrow & & \downarrow & & \downarrow \\
V^1 & \xrightarrow{\sigma_1} & Q \otimes V^1 \oplus \rho_0 & \xrightarrow{\tau_1} & V^1,
\end{array}$$

where the vertical arrows are natural projections.

(1) Let us apply $\operatorname{Hom}(\rho_i, \ )^\Gamma$ to each term. Since $\operatorname{Hom}(\rho_i, Q\otimes\rho_i)^\Gamma = 0$, we have $\operatorname{Hom}(\rho_i, \operatorname{Ker}(V^2 \to V^1)) \subset K_i$ from the left end row of the diagram:

$$\begin{array}{ccccc} \mathbb{C} & & 0 & & \mathbb{C} \\ \downarrow & & & & \downarrow \\ V_i^2 & \xrightarrow{\sigma_2} & (Q\otimes V^2 \oplus \rho_0)_i & \xrightarrow{\tau_2} & V_i^2 \\ \downarrow & & \downarrow\cong & & \downarrow \\ V_i^1 & \xrightarrow{\sigma_1} & (Q\otimes V^1 \oplus \rho_0)_i & \xrightarrow{\tau_1} & V_i^1 \end{array}$$

Therefore, we have a one-dimensional subspace in K_i, corresponding to (I_1, I_2).

Conversely if a one-dimensional subspace L in K_i is given, then we define I_1 so that

$$V_j^1 = \begin{cases} V_j^2 & \text{if } j \neq i, \\ V_i^2/L & \text{if } j = i. \end{cases}$$

(2) We study the right end column of the commutative diagram. Then we have

$$\operatorname{Ker}(V^2 \to V^1)_i \cong (\operatorname{Ker}\tau_1)_i/(\operatorname{Ker}\tau_2)_i.$$

Therefore we have a surjection $(\operatorname{Ker}\tau_1)_i \to \operatorname{Ker}(V^2 \to V^1)_i$ which factors through as

$$(\operatorname{Ker}\tau_1/\operatorname{Im}\sigma_1)_i \twoheadrightarrow \operatorname{Ker}(V^2 \to V^1)_i.$$

Conversely if a codimension one subspace S of $(\operatorname{Ker}\tau_1/\operatorname{Im}\sigma_1)_i$ is given, we define I_2 so that

$$V_j^2 = \begin{cases} V_j^1 & \text{if } j \neq i, \\ (Q\otimes V^1 \oplus \rho_0)_i/S & \text{if } j = i. \end{cases}$$

□

The above proposition implies

$$\begin{aligned} \mathfrak{P}_i(\mathbf{v}) &= \left\{(I_2, S) \,\middle|\, I_2 \in X(\mathbf{v}), S \in \mathbb{P}(\operatorname{Hom}_{A_X}(A_0\otimes\rho_i, A_{Z_2})^\Gamma)\right\} \\ &= \left\{(I_1, S') \,\middle|\, I_1 \in X(\mathbf{v}-\mathbf{e}_i), S' \in \mathbb{P}(\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma)\right\}. \end{aligned}$$

This shows that $\mathfrak{P}_i(\mathbf{v})$ is an analogue of a moduli space of *coherent system* [30].

Example 6.3. This example was first appeared in [N2, Theorem 5.10] (in a slightly different terminology) and was *re*discovered independently in [IN].

Consider $\mathfrak{P}_i(\mathbf{r})$ when $\mathbf{r}$ is (the isomorphism class of) the regular representation as in Example 4.1. By Theorem 4.6, $X(\mathbf{r}-\mathbf{e}_i)$ is a single point if $i \neq 0$. It is obvious that $X(\mathbf{r}-\mathbf{e}_i)$ is empty if $i = 0$. So we assume $i \neq 0$ from now. We consider $\mathfrak{P}_i(\mathbf{r})$ as a subvariety in $X(\mathbf{r})$. By Proposition 6.2, $\mathfrak{P}_i(\mathbf{r})$ is isomorphic to a projective space. Moreover, it is a projective line, since $\mathfrak{P}_i(\mathbf{r})$ is an lagrangian subvariety in $X(\mathbf{r})$. Since it is compact, it must be contained in $\mathfrak{L}(\mathbf{r})$. By Proposition 6.2, we have

$$I \in \mathfrak{P}_i(\mathbf{r}) \iff \dim \operatorname{Hom}_{A_X}(A_0\otimes\rho_i, A_Z)^\Gamma \neq 0,$$

where $A_Z = \mathbb{C}[x,y]/I$. Furthermore, from (4.11), we have

$$\begin{aligned} \dim \operatorname{Hom}_{A_X}(A_0\otimes\rho_i, A_Z)^\Gamma \neq 0 &\iff \dim \operatorname{Hom}_{\mathbb{C}}(\rho_i, I/\mathfrak{m}I)^\Gamma \neq 0 \\ &\iff I/\mathfrak{m}I \text{ contains a direct summand } \rho_i. \end{aligned}$$

If I is contained in $\pi^{-1}(0) \subset X(\mathbf{r})$, then multiplication by x, y on $\mathbb{C}[x,y]/I$ are nilpotent, so $\operatorname{Hom}_{A_X}(A_0, A_Z) \neq 0$. Therefore $\dim \operatorname{Hom}_{A_X}(A_0\otimes\rho_i, A_Z)^\Gamma \neq 0$ for some i. It means that $\pi^{-1}(0) = \bigcup_{i=1}^n \mathfrak{P}_i(\mathbf{r})$. By [N3, Lemma 9.8], $\mathfrak{P}_i(\mathbf{r})$ and $\mathfrak{P}_j(\mathbf{r})$ intersect transversely if $i \neq j$ and $\mathfrak{P}_i(\mathbf{r}) \cap \mathfrak{P}_j(\mathbf{r}) \neq \emptyset$. In particular, $\mathfrak{P}_i(\mathbf{r}) \neq \mathfrak{P}_j(\mathbf{r})$ for $i \neq j$. Therefore $\{\mathfrak{P}_i(\mathbf{r})\}_{i=1,\dots n}$ are irreducible components of $\pi^{-1}(0)$.

Let us show that the following formula of the intersection matrix

$$([\mathfrak{P}_i(\mathbf{r})], [\mathfrak{P}_j(\mathbf{r})]) = -c_{ij}, \tag{6.4}$$

which gives a geometric proof of the McKay correspondence, i.e., the identification of the intersection matrix and the tensor product multiplicities. Let $I \in X(\mathbf{r})$ and $V = \mathbb{C}[x,y]/I$. Recall that we have defined a complex S_i, the ρ_i-component of (4.7):

$$V_i \xrightarrow{\sigma_i} \bigoplus_j V_j^{\oplus a_{ij}} \xrightarrow{\tau_i} V_i,$$

where σ_i, τ_i is the ρ_i-component of σ, τ. Since τ is surjective, we have a homomorphism between vector bundles:

$$V_i \xrightarrow{\sigma_i} \operatorname{Ker}\tau_i.$$

The above discussion shows that

$$\begin{aligned} I \in \mathfrak{P}_i(\mathbf{r}) &\iff \sigma_i \text{ is } \textit{not} \text{ an isomorphism} \\ &\iff \det\sigma_i = 0. \end{aligned}$$

Therefore we have

$$c_1(S_i) = c_1(\operatorname{Ker}\tau_i) - c_1(V_i) = -\sum_j c_{ij}c_1(V_j) = m_i(\text{the Poincaré dual of } [\mathfrak{P}_i(\mathbf{r})])$$

for some $m_i \in \mathbb{Z}_{>0}$. By Theorem 4.14 (more precisely, we apply the Chern character homomorphism $\operatorname{ch}\colon K(X(\mathbf{r})) \to H^*(X(\mathbf{r}),\mathbb{Q})$ and the local Chern character homomorphism $\operatorname{ch}\colon K_c(X(\mathbf{r})) \to H_c^*(X(\mathbf{r}),\mathbb{Q})$ to deduce the corresponding result for cohomology groups), we have

$$\langle c_1(V_j), c_1(S_i)\rangle = \delta_{ij}.$$

In particular, we have

$$\langle c_1(V_i), [\mathfrak{P}_i(\mathbf{r})]\rangle = \frac{1}{m_i}.$$

Since the left hand side is an integer, we have $m_i = 1$. Using the above formula again, we get (6.4) as

$$([\mathfrak{P}_i(\mathbf{r})], [\mathfrak{P}_j(\mathbf{r})]) = -\sum_k c_{ik}\langle c_1(V_k), [\mathfrak{P}_k(\mathbf{r})]\rangle = -c_{ij}.$$

Since $\mathfrak{P}_i(\mathbf{r})$ and $\mathfrak{P}_j(\mathbf{r})$ intersect transversely if they intersect, we have

$$\begin{aligned} ([\mathfrak{P}_i(\mathbf{r})], [\mathfrak{P}_j(\mathbf{r})]) \neq 0 &\iff \mathfrak{P}_i(\mathbf{r}) \cap \mathfrak{P}_j(\mathbf{r}) \neq \emptyset \\ &\iff \exists I \in X(\mathbf{r}) \text{ such that } I/\mathfrak{m}I \supset \rho_i \oplus \rho_j \end{aligned}$$

for $i \neq j$. (An explicit form of such an I was given in [IN].)

NB. The original argument in [N2] was slightly different. We use the fact that the vector bundle V_i extends to the 1-point compactification $\overline{X(\mathbf{r})}$ of $X(\mathbf{r})$, which is a differentiable orbifold (see [29]). Therefore the Chern classes of V_i makes a sense as a class in $H^*(\overline{X(\mathbf{r})},\mathbb{R})$. Then the locally free resolution of $\mathcal{O}_\Delta$, mentioned in §4.9, implies

$$\sum_j c_{ij}\int_{\overline{X(\mathbf{r})}} c_1(V_j)\wedge c_1(V_k) = \delta_{ik}.$$

Then we can define $\mathfrak{P}_i(\mathbf{r})$ as a zero locus of $\det\sigma_i$. From the above formula, we have $([\mathfrak{P}_i(\mathbf{r})], [\mathfrak{P}_j(\mathbf{r})]) = -c_{ij}$. (There is a small gap in [N2]. We defined $\mathfrak{P}_i(\mathbf{r})$ by the zero locus of $\det\sigma_i$ and claimed that it is isomorphic to $\mathbb{P}^1$ by the adjunction formula and $[\mathfrak{P}_i(\mathbf{r})]\cdot[\mathfrak{P}_i(\mathbf{r})] = -2$. But we need to show the irreducibility and the smoothness of $\mathfrak{P}_i(\mathbf{r})$. These statements followed from [N3] as above.)

6.2. **Convolution.** Let

$$Z(\mathbf{v}^1,\mathbf{v}^2) \overset{\text{def.}}{=} \{(I^1,I^2) \in X(\mathbf{v}^1)\times X(\mathbf{v}^2) \mid \pi(I^1)=\pi(I^2)\}.$$

We need an explanation for the equality $\pi(I^1)=\pi(I^2)$. The left hand side is an element of $(S^{\dim V^1}X)^\Gamma$, while the right hand side is of $(S^{\dim V^2}X)^\Gamma$. For $m\le n$, we have an inclusion $(S^m X)^\Gamma \to (S^n X)^\Gamma$ defined by $C\mapsto C+(n-m)0$. We denote by $(S^\infty X)^\Gamma$ the direct limit $\lim_{n\to\infty}(S^n X)^\Gamma$. The notation π in the above equality is the composition of the previous π and the inclusion $(S^{\dim V^1}X)^\Gamma \to (S^\infty X)^\Gamma$, $(S^{\dim V^2}X)^\Gamma \to (S^\infty X)^\Gamma$. So both hand sides are elements of $(S^\infty X)^\Gamma$, and the equality makes sense.

Let $p_{ij}\colon X(\mathbf{v}^1)\times X(\mathbf{v}^2)\times X(\mathbf{v}^3)\to X(\mathbf{v}^i)\times X(\mathbf{v}^j)$ be the projection. We define the convolution

$$*\colon H_*(Z(\mathbf{v}^1,\mathbf{v}^2))\otimes H_*(Z(\mathbf{v}^2,\mathbf{v}^3)) \to H_*(Z(\mathbf{v}^1,\mathbf{v}^3))$$

by

$$c*c' \overset{\text{def.}}{=} p_{13*}\left(p_{12}^*c\cap p_{23}^*c'\right) \qquad c\in H_*(Z(\mathbf{v}^1,\mathbf{v}^2)),\ c'\in H_*(Z(\mathbf{v}^2,\mathbf{v}^3)).$$

Let us check that this is well-defined. We have

$$\begin{aligned} &p_{12}^*c \in H_*(p_{12}^{-1}(Z(\mathbf{v}^1,\mathbf{v}^2))), \quad p_{12}^{-1}(Z(\mathbf{v}^1,\mathbf{v}^2)) = \{(I_1,I_2,I_3)\mid \pi(I_1)=\pi(I_2)\},\\ &p_{23}^*c' \in H_*(p_{23}^{-1}(Z(\mathbf{v}^2,\mathbf{v}^3))), \quad p_{23}^{-1}(Z(\mathbf{v}^2,\mathbf{v}^3)) = \{(I_1,I_2,I_3)\mid \pi(I_2)=\pi(I_3)\}.\end{aligned}$$

Therefore

$$\begin{gathered} p_{12}^*c\cap p_{23}^*c' \in H_*(p_{12}^{-1}(Z(\mathbf{v}^1,\mathbf{v}^2))\cap p_{23}^{-1}(Z(\mathbf{v}^2,\mathbf{v}^3))),\\ p_{12}^{-1}(Z(\mathbf{v}^1,\mathbf{v}^2))\cap p_{23}^{-1}(Z(\mathbf{v}^2,\mathbf{v}^3)) = \{(I_1,I_2,I_3)\mid \pi(I_1)=\pi(I_2)=\pi(I_3)\}.\end{gathered}$$

Finally the restriction of p_{13} to $p_{12}^{-1}(Z(\mathbf{v}^1,\mathbf{v}^2))\cap p_{23}^{-1}(Z(\mathbf{v}^2,\mathbf{v}^3))$ is proper, and the image is contained in $Z(\mathbf{v}^1,\mathbf{v}^3)$. Thus the convolution is well-defined.

We will be intereseted in the case when degree is middle:

$$H_{d^1+d^2}(Z(\mathbf{v}^1,\mathbf{v}^2))\otimes H_{d^2+d^3}(Z(\mathbf{v}^2,\mathbf{v}^3)) \to H_{d^1+d^3}(Z(\mathbf{v}^1,\mathbf{v}^3)),$$

where $d^i=\dim X(\mathbf{v}^i)$. For abuse of notation, we denote these degrees by 'top', although they are different for different components.

Let

$$H_{\text{top}}(Z) \overset{\text{def.}}{=} \prod_{\mathbf{v}^1,\mathbf{v}^2}{}' H_{\text{top}}(Z(\mathbf{v}^1,\mathbf{v}^2)),$$

($\mathbf{v}^1$, $\mathbf{v}^2$ run all pairs of isomorphism classes of Γ-modules) be the subspace of the direct product $\prod_{\mathbf{v}^1,\mathbf{v}^2} H_{d^1+d^2}(Z(\mathbf{v}^1,\mathbf{v}^2))$ consisting elements $(c_{\mathbf{v}^1,\mathbf{v}^2})$ such that

- for fixed $\mathbf{v}^1$, $c_{\mathbf{v}^1,\mathbf{v}^2}=0$ for all but finitely many choices of $\mathbf{v}^2$,
- for fixed $\mathbf{v}^2$, $c_{\mathbf{v}^1,\mathbf{v}^2}=0$ for all but finitely many choices of $\mathbf{v}^1$.

Then the convolution is well-defined on $H_{\text{top}}(Z)$, which becomes an associative algebra. The unit is the product of diagonals.

Note also that the convolution defines

$$H_{d^1+d^2}(Z(\mathbf{v}^1,\mathbf{v}^2))\otimes H_{d^2}(\mathfrak{L}(\mathbf{v}^2)) \to H_{d^1}(\mathfrak{L}(\mathbf{v}^1)).$$

We also denote these degrees by 'top'. The direct sum

$$H_{\text{top}}(\mathfrak{L}) \overset{\text{def.}}{=} \bigoplus_{\mathbf{v}^1} H_{\text{top}}(\mathfrak{L}(\mathbf{v}^1))$$

is a representation of $H_{\text{top}}(Z)$.

In this subsection, we define an algebra homomorphism $\mathbf{U}(\widehat{\mathfrak{g}}) \to H_{\text{top}}(Z)$. We define the image of generators and check the defining relations. We set

$$\begin{gathered} d \mapsto \prod_{\mathbf{v}} (-v_0) \left[\Delta_{X(\mathbf{v})}\right], \\ h_i \mapsto \prod_{\mathbf{v}} (\delta_{0i} - \sum_j c_{ij} v_j) \left[\Delta_{X(\mathbf{v})}\right], \\ e_i \mapsto \prod_{\mathbf{v}} [\mathfrak{P}_i(\mathbf{v})], \qquad f_i \mapsto (-1)^{r(\mathbf{v})} \prod_{\mathbf{v}} [\omega\mathfrak{P}_i(\mathbf{v})] \end{gathered}$$

where $\Delta_{X(\mathbf{v})}$ is the diagonal in $X(\mathbf{v}) \times X(\mathbf{v})$, $\omega\colon X(\mathbf{v}-\mathbf{e}_i) \times X(\mathbf{v}) \to X(\mathbf{v}) \times X(\mathbf{v}-\mathbf{e}_i)$ is the interchange of the factors, and $r(\mathbf{v}) = \frac{1}{2}(\dim X(\mathbf{v}-\mathbf{e}_i) - \dim X(\mathbf{v}))$.

Theorem 6.5. *The above assignment extends* (*uniquely*) *to an algebra homomorphism* $\mathbf{U}(\widehat{\mathfrak{g}}) \to H_{\text{top}}(Z)$.

It is clear that d and h_i's make a commuting family. Thus we have the relation (5.1).
Since

$$\left[\Delta_{X(\mathbf{v}-\mathbf{e}_i)}\right] * [\mathfrak{P}_i(\mathbf{v})] = [\mathfrak{P}_i(\mathbf{v})] * \left[\Delta_{X(\mathbf{v})}\right],$$

the relations (5.2, 5.3) follow.

Thus the relations (5.4, 5.5) are remained to be checked.

Lemma 6.6. *For a fixed* V, $e_i^N \left[\Delta_{X(\mathbf{v})}\right]$ *and* $f_i^N \left[\Delta_{X(\mathbf{v})}\right]$ *are* 0 *for sufficiently large* N. *In particular, the operators* e_i, f_i *are locally nilpotent on* $H_{\text{top}}(\mathfrak{L})$.

Proof. The first case is obvious since

$$e_i^N \left[\Delta_{X(\mathbf{v})}\right] \in H_{\text{top}}(Z(\mathbf{v}-N\mathbf{e}_i, \mathbf{v}))$$

and $X(\mathbf{v}-N\mathbf{e}_i) = \emptyset$ if N is greater than v_i.

The second case follows from the assertion that if

$$\delta_{i0} - \sum_j c_{ij} v_j + v_i = \delta_{i0} - \sum_{j:j\neq i} c_{ij} v_j - v_i < 0$$

then $X(\mathbf{v}) = \emptyset$. This assertion follows from the surjectivity of τ, as its ρ_i-component

$$(\delta_{i0}\mathbb{C}) \oplus \bigoplus_{j:j\neq i} V_j^{\oplus(-c_{ij})} \to V_i$$

must be surjective. □

It is known that the relation (5.5) follows the rest of relations and the property in Lemma 6.6 (see e.g., §3.3 of [22]). Thus the only remaining relation is (5.4). We explain only the key point in the proof. See the original paper [N3] for the complete proof. We consider $\left[\Delta_{X(\mathbf{v}^1)}\right] e_i f_j$ and $\left[\Delta_{X(\mathbf{v}^1)}\right] f_j e_i$. Let us consider two triple products

$$X(\mathbf{v}^1) \times X(\mathbf{v}^2) \times X(\mathbf{v}^3), \qquad X(\mathbf{v}^1) \times X({}'\mathbf{v}^2) \times X(\mathbf{v}^3),$$

where

$$\mathbf{v}^2 = \mathbf{v}^1 + \mathbf{e}_i = \mathbf{v}^3 + \mathbf{e}_j, \quad {}'\mathbf{v}^2 = \mathbf{v}^1 - \mathbf{e}_j = \mathbf{v}^3 - \mathbf{e}_i.$$

Note that these equations are compatible since

$$\mathbf{v}^2 = {}'\mathbf{v}^2 + \mathbf{e}_i + \mathbf{e}_j.$$

Let p_{ij} be the projection as usual. Then we have

$$\begin{aligned} \left[\Delta_{X(\mathbf{v}^1)}\right] e_i f_j &= \pm p_{13*}\left(p_{12}^* \left[\mathfrak{P}_i(\mathbf{v}^2)\right] \cap p_{23}^* \left[\omega\mathfrak{P}_j(\mathbf{v}^2)\right]\right), \\ \left[\Delta_{X(\mathbf{v}^1)}\right] f_j e_i &= \pm p_{13*}\left(p_{12'}^* \left[\omega\mathfrak{P}_j(\mathbf{v}^1)\right] \cap p_{2'3}^* \left[\mathfrak{P}_i(\mathbf{v}^3)\right]\right). \end{aligned}$$

Let us consider the set theoretical interesections:

$$(6.7)\qquad \begin{aligned} p_{12}^{-1}(\mathfrak{P}_i(\mathbf{v}^2)) \cap p_{23}^{-1}(\omega\mathfrak{P}_j(\mathbf{v}^2)) &= \{(I_1, I_2, I_3) \mid I_1 \supset I_2 \subset I_3\}, \\ p_{12'}^{-1}(\omega\mathfrak{P}_j(\mathbf{v}^1)) \cap p_{2'3}^{-1}(\mathfrak{P}_i(\mathbf{v}^3)) &= \{(I_1, I_2', I_3) \mid I_1 \subset I_2' \supset I_3\}. \end{aligned}$$

The crusial observation is the following: if $I_1 \neq I_3$, I_2 and I_2' are determined by I_1, I_3 as

$$I_2 = I_1 \cap I_3, \qquad I_2' = I_1 + I_3.$$

Moreover, $I_1 \cap I_3 \in X(\mathbf{v}^2)$ if and only if $I_1 + I_3 \in X('\mathbf{v}^2)$. Let U be the open subset given by $X(\mathbf{v}_1) \times X(\mathbf{v}_3)$ given by $I_1 \neq I_3$. If $i \neq j$, then $U = \emptyset$. The above means that on the open set $p_{13}^{-1}(U)$, the intersections (6.7) and their images under the projection p_{13} are all isomorphic. Let us draw a picture when $\Gamma = \{1\}$ and all I_i are ideals of functions vanishing at distinct points, although the case $\Gamma = \{1\}$ is excluded from our discussion:

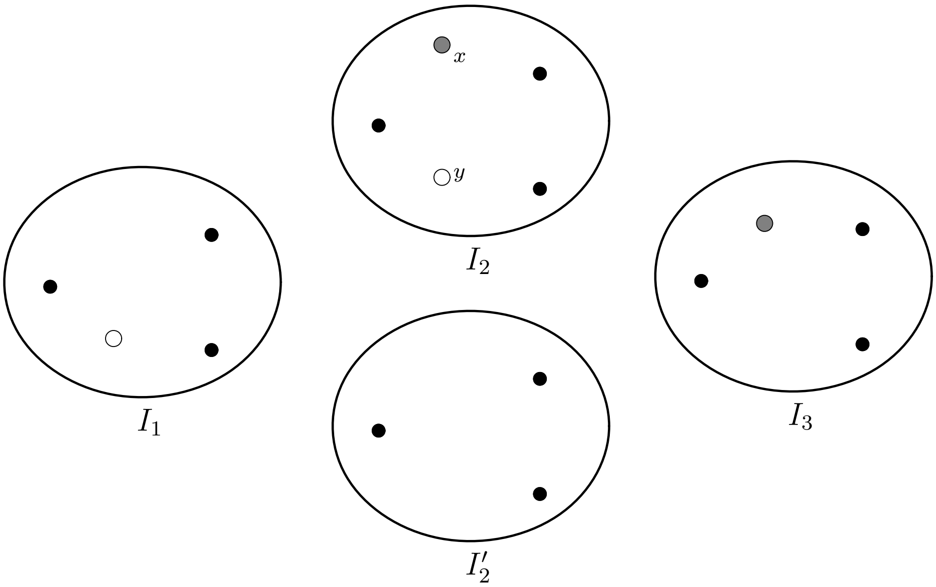


FIGURE 2. Correspondence between $I_1 \supset I_2 \subset I_3$ and $I_1 \subset I_2' \supset I_3$

One can check the transversality of the intersections on $p_{13}^{-1}(U)$ (see [N3, Appendix]). This result was mentioned in Example 6.3. If $j\colon U \to X(\mathbf{v}^1) \times X(\mathbf{v}^3)$ denotes the inclusion, we get

$$j^* \left[\Delta_{X(\mathbf{v}^1)}\right] e_i f_j = j^* \left[\Delta_{X(\mathbf{v}^1)}\right] f_j e_i.$$

Thus we have checked the relation (5.4) for $i \neq j$.

Consider the case $i = j$. By the above and the long exact sequence in the homology groups, we know that $\left[\Delta_{X(\mathbf{v}^1)}\right] e_i f_j - \left[\Delta_{X(\mathbf{v}^1)}\right] f_j e_i$ is contained in the image of

$$H_{\mathrm{top}}(\Delta_{X(\mathbf{v}^1)}) \to H_{\mathrm{top}}(Z).$$

Since $X(\mathbf{v}^1)$ is connected and has dimension equal to 'top', we have

$$e_i f_j - f_j e_i = c_{\mathbf{v}} \left[\Delta_{X(\mathbf{v})}\right]$$

for some constant $c_{\mathbf{v}} \in \mathbb{Z}$. The last step in the proof is the calculation of a self-intersection product to compute the constant $c_{\mathbf{v}}$. For this, see [N3, §9].

7. Lagrangian subvarieties and a crystal structure

7.1. **Crystal.** Let us review the notion of Kashiwara's crystals briefly. See [25, KS] for detail. Let

$$I \overset{\text{def.}}{=} \{0, 1, \dots, n\} \quad \text{(the index set of simple roots)},$$
$$P^\vee \overset{\text{def.}}{=} \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d, \qquad P \overset{\text{def.}}{=} \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, P^\vee \rangle \in \mathbb{Z}\}.$$

Definition 7.1. A *crystal* $\mathcal{B}$ associated with the affine Lie algebra $\widehat{\mathfrak{g}}$ is a set together with maps $\mathrm{wt}\colon \mathcal{B} \to P$, $\varepsilon_i, \varphi_i\colon \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\}$, $\widetilde{e}_i, \widetilde{f}_i\colon \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ $(i \in I)$ satisfying the following properties

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle, \tag{7.2a}$$

$$\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i,\ \varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1,\ \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1, \quad \text{if } \widetilde{e}_i b \in \mathcal{B}, \tag{7.2b}$$

$$\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i,\ \varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1,\ \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1, \quad \text{if } \widetilde{f}_i b \in \mathcal{B}, \tag{7.2c}$$

$$b' = \widetilde{f}_i b \iff b = \widetilde{e}_i b' \quad \text{for } b, b' \in \mathcal{B}, \tag{7.2d}$$

$$\text{if } \varphi_i(b) = -\infty \text{ for } b \in \mathcal{B}, \text{ then } \widetilde{e}_i b = \widetilde{f}_i b = 0 \tag{7.2e}$$

We set $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}(b) \rangle$.

The crystal can be defined for finite dimensional Lie algebras $\mathfrak{g}$ (in fact, for any Kac-Moody Lie algebra). We give an example corresponding to the irreducible $(n+1)$-dimensional representation of $\mathfrak{sl}_2$ (see §1.3).

Example 7.3. Let $\mathfrak{g} = \mathfrak{sl}_2$, $I = \{1\}$, $P^\vee = \mathbb{Z}h$, $P = \mathbb{Z}\Lambda$, where $\langle h, \Lambda \rangle = 1$. Let $\mathcal{B} \overset{\text{def.}}{=} \{b(0), b(1), \dots, b(n)\}$. We define

$$\mathrm{wt}(b(k)) = (n - 2k)\Lambda,$$
$$\varepsilon(b(k)) = k, \quad \varphi(b(k)) = n - k,$$
$$\widetilde{f}(b(k)) = \begin{cases} b(k+1) & \text{if } k \neq n, \\ 0 & \text{otherwise}, \end{cases}$$
$$\widetilde{e}(b(k)) = \begin{cases} b(k-1) & \text{if } k \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

where we omit the suffix i.

We give simple examples, which does not come from representations.

Example 7.4. (1) For all $i \in I$, we define the crystal $\mathcal{B}_i$ as follows:

$$\mathcal{B}_i = \{b_i(n) \mid n \in \mathbb{Z}\},$$
$$\mathrm{wt}(b_i(n)) = n\alpha_i, \quad \varphi_i(b_i(n)) = n, \quad \varepsilon_i(b_i(n)) = -n,$$
$$\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \quad (i \neq j),$$
$$\widetilde{e}_i(b_i(n)) = b_i(n+1), \quad \widetilde{f}_i(b_i(n)) = b_i(n-1),$$
$$\widetilde{e}_j(b_i(n)) = \widetilde{f}_j(b_i(n)) = 0 \quad (i \neq j).$$

(2) For $\lambda \in P^+$, we define the crystal T_λ by

$$T_\lambda = \{t_\lambda\},$$
$$\mathrm{wt}(t_\lambda) = \lambda, \quad \varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty,$$
$$\widetilde{e}_i(t_\lambda) = \widetilde{f}_i(t_\lambda) = 0.$$

A crystal $\mathcal{B}$ is called *normal* if

$$\varepsilon_i(b) = \max\{n \mid \widetilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \mid \widetilde{f}_i^n b \neq 0\}.$$

For given two crystals $\mathcal{B}_1$, $\mathcal{B}_2$, a *morphism* ψ of crystal from $\mathcal{B}_1$ to $\mathcal{B}_2$ is a map $\mathcal{B}_1 \sqcup \{0\} \to \mathcal{B}_2 \sqcup \{0\}$ satisfying $\psi(0) = 0$ and the following conditions for all $b \in \mathcal{B}_1$, $i \in I$:

(7.5a) $$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b),\ \varepsilon_i(\psi(b)) = \varepsilon_i(b),\ \varphi_i(\psi(b)) = \varphi_i(b) \quad \text{if } \psi(b) \in \mathcal{B}_2,$$

(7.5b) $$\widetilde{e}_i\psi(b) = \psi(\widetilde{e}_i b) \quad \text{if } \psi(b) \in \mathcal{B}_2,\ \widetilde{e}_i b \in \mathcal{B}_1,$$

(7.5c) $$\widetilde{f}_i\psi(b) = \psi(\widetilde{f}_i b) \quad \text{if } \psi(b) \in \mathcal{B}_2,\ \widetilde{f}_i b \in \mathcal{B}_1.$$

A morphism ψ is called *strict* if ψ commutes with $\widetilde{e}_i$, $\widetilde{f}_i$ for all $i \in I$ without any restriction. A morphism ψ is called an *embedding* if ψ is an injective map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$.

Definition 7.6. The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ with maps defined by

(7.7a) $$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

(7.7b) $$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)),$$

(7.7c) $$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2)),$$

(7.7d) $$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{otherwise,} \end{cases}$$

(7.7e) $$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

Here (b_1, b_2) is denoted by $b_1 \otimes b_2$ and $0 \otimes b_2$, $b_1 \otimes 0$ are identified with 0.

It is easy to check that these satisfy the axioms in Definition 7.1. It is also easy to check that the tensor product of two normal crystals is again normal.

It is easy to check $(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3 = \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3)$. We denote it by $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$. Similarly we can define $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$.

The crystal was introduced by abstracting the notion of crystal bases constructed by Kashiwara [25]. Thus we have the following examples of crystals.

Example 7.8. (1) The lower half $\mathbf{U}_q(\mathfrak{g})^-$ of the quantized universal enveloping algebra has a base which has a structure of the crystal. Let $\mathcal{B}(\infty)$ denote this crystal. Let b_0 be the vector corresponding to $1 \in \mathbf{U}_q(\mathfrak{g})^-$.

(2) Similarly the simple $\mathbf{U}_q(\mathfrak{g})$-module $L(\Lambda)$ with highest weight λ has a base which has a structure of the crystal. Let $\mathcal{B}(\Lambda)$ denote this crystal. Let b_λ denote the highest weight vector considered as an element of $B(\Lambda)$. It is known that $\mathcal{B}(\Lambda)$ is normal. It is also known that the map

$$\pi\colon \mathcal{B}(\infty) \otimes T_\Lambda \ni \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} b_0 \otimes t_\Lambda \longmapsto \widetilde{f}_{i_1} \dots \widetilde{f}_{i_n} b_\Lambda \in \mathcal{B}(\Lambda) \sqcup \{0\}$$

is well-defined and is a strict morphism. Furthermore, $L(\Lambda_1) \otimes L(\Lambda_2)$ has a base which has a structure of crystal isomorphic to $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_2)$.

Remark that the character of $L(\Lambda)$ is given by

$$\operatorname{ch} L(\Lambda) \overset{\text{def.}}{=} \sum_\lambda \dim L(\Lambda)_\lambda e^\lambda = \sum_{b \in \mathcal{B}(\Lambda)} e^{\mathrm{wt}(b)}.$$

We also have the tensor product decomposition (generalized Littlewood-Rechiardson rule):

$$L(\Lambda_1) \otimes L(\Lambda_2) = \bigoplus L(\mathrm{wt}(b_1) + \mathrm{wt}(b_2)),$$

where the summation runs over all $b_1 \otimes b_2 \in \mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_2)$ such that $\varepsilon_i(b_1 \otimes b_2) = 0$ for all $i \in I$.

7.2. **A crystal structure on the set of irreducible components of the Lagrangian subvarieties.** In §6.1 we have studied the Hecke correspondences, pairs $I_1 \supset I_2$ of ideals such that $I_1/I_2 \cong \rho_i$. In this subsection we study the following generalization of the Hecke correspondence:

$$\mathfrak{P}_i^{(r)}(\mathbf{v}) \overset{\text{def.}}{=} \{(I_1, I_2) \in X(\mathbf{v} - r\mathbf{e}_i) \times X(\mathbf{v}) \mid I_1 \supset I_2\}.$$

The quotient I_1/I_2 is isomorphic to $\rho_i^{\oplus r}$. Let us denote two projections by p_1, p_2:

$$X(\mathbf{v} - r\mathbf{e}_i) \xleftarrow{p_1} \mathfrak{P}_i^{(r)}(\mathbf{v}) \xrightarrow{p_2} X(\mathbf{v}).$$

We have the following as in Proposition 6.2:

Proposition 7.9. (1) *Let $I_2 \in X(\mathbf{v})$ and let A_{Z_2} be the corresponding quotient. Then we have*

$$p_2^{-1}(I_2) \cong \operatorname{Gr}_r(\operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_{Z_2})^\Gamma),$$

where the right hand side is the Grassmann manifold of r-dimensional subspaces of $\operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_{Z_2})^\Gamma$.

(2) *Let $I_1 \in X(\mathbf{v} - r\mathbf{e}_i)$ and let A_{Z_1} be the corresponding quotient. Then we have*

$$p_1^{-1}(I_1) \cong \operatorname{Gr}_r(\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma),$$

where the right hand side is the Grassmann manifold of r-dimensional subspace of $\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)^\Gamma$.

Let

$$X_{i;r}(\mathbf{v}) \overset{\text{def.}}{=} \{I \in X(\mathbf{v}) \mid \dim \operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_Z)^\Gamma = r\}.$$

It is a locally closed subvariety since $\bigcup_{s: s \le r} X_{i;s}(\mathbf{v})$ is an open subset of $X(\mathbf{v})$. Let us define a map

$$p\colon X_{i;r}(\mathbf{v}) \to X_{i;0}(\mathbf{v} - r\mathbf{e}_i)$$

as follows. We consider the natural homomorphism

$$\begin{array}{ccc} \operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_Z)^\Gamma \otimes (A_0 \otimes \rho_i) & \longrightarrow & A_Z \\ \xi \otimes g & \mapsto & \xi(g) \end{array}$$

This is obviously injective. Then $I' = p(I)$ is given by so that the corresponding quotient A_X/I' is the cokernel of this homomorphism. By the construction, we have $(p(I), I) \in \mathfrak{P}_i^{(r)}(\mathbf{v})$.

Lemma 7.10. *When we move $I_1 \in X_{i;0}(\mathbf{v} - r\mathbf{e}_i)$, $\operatorname{Hom}_{\mathbb{C}}(I_1/\mathfrak{m}I_1, \rho_i)$ forms a vector bundle of rank*

$$\delta_{0i} - \sum_j c_{ij}(\dim V_j - r\delta_{ij}) = (\mathbf{e}_i, \mathbf{e}_0 - \mathbf{C}(\mathbf{v} - r\mathbf{e}_i)).$$

The first statement follows from (4.11).

It is clear that $\pi(p(I)) = \pi(I) - r[0]$, where $[0]$ is the 0-dimensional cycle given by the origin. Therefore, the restriction of the Grassmann bundle to $\mathfrak{L}(\mathbf{v} - r\mathbf{e}_i)$ gives us

$$p\colon X_{i;r}(\mathbf{v}) \cap \mathfrak{L}(\mathbf{v}) \to X_{i;0}(\mathbf{v} - r\mathbf{e}_i) \cap \mathfrak{L}(\mathbf{v} - r\mathbf{e}_i),$$

which is still isomorphic to a Grassmann bundle. Let $\mathfrak{L}_{i;r}(\mathbf{v}) \overset{\text{def.}}{=} X_{i;r}(\mathbf{v}) \cap \mathfrak{L}(\mathbf{v})$.

Lemma 7.11. (1) *We have $\mathfrak{L}(\mathbf{v}) = \bigcup_{i,r>0} \mathfrak{L}_{i;r}(\mathbf{v})$.*

(2) *We have $\frac{1}{2}\dim X(\mathbf{v} - r\mathbf{e}_i) + \dim(\text{fiber of } p) = \frac{1}{2}\dim X(\mathbf{v})$.*

Proof. (1) The assertion means $\dim \operatorname{Hom}_{A_X}(A_0 \otimes \rho_i, A_Z)^\Gamma > 0$ for $I \in \mathfrak{L}(\mathbf{v})$. This is equivalent to $\dim \operatorname{Hom}_{A_X}(A_0, A_Z) \neq 0$. Multiplications by x, y on A_Z (operators B_1, B_2) are nilpotent since $I \in \mathfrak{L}(\mathbf{v})$. Therefore the statement is clear.

(2) The left hand side is

$$
\begin{aligned}
&(v_0 - r\delta_{i0}) - \frac{1}{2}(\mathbf{v} - r\mathbf{e}_i, \mathbf{C}(\mathbf{v} - r\mathbf{e}_i)) + r\{(\mathbf{e}_i, \mathbf{e}_0 - \mathbf{C}(\mathbf{v} - r\mathbf{e}_i)) - r\} \\
&= v_0 - r\delta_{i0} - \frac{1}{2}(\mathbf{v}, \mathbf{C}\mathbf{v}) + r(\mathbf{e}_i, \mathbf{C}\mathbf{v}) - \frac{1}{2}r^2(\mathbf{e}_i, \mathbf{C}\mathbf{e}_i) + r\delta_{i0} - r(\mathbf{e}_i, \mathbf{C}\mathbf{v}) + r^2(\mathbf{e}_i, \mathbf{C}\mathbf{e}_i) - r^2 \\
&= v_0 - \frac{1}{2}(\mathbf{v}, \mathbf{C}\mathbf{v}),
\end{aligned}
$$

which is equal to the right hand side. □

The second statement means that the dimension of the fiber is just half of the difference of dimensions of total space. This remarkable observation is due to Lusztig.

Let us show that $\dim \mathfrak{L}_{i;r}(\mathbf{v})$ is equal to the half of $\dim X(\mathbf{v})$ by induction, by using this observation. A little bit more effort shows that $\mathfrak{L}_{i;r}(\mathbf{v})$ is a lagrangian subvariety. We omit the proof of this part.

When $V = 0$, then $\mathfrak{L}(0) = X(0)$ is a point. So the assertion is obvious. Assume that we have $\dim \mathfrak{L}_{i;r}(\mathbf{v}') = \frac{1}{2} \dim X(\mathbf{v}')$ if $\dim V' < \dim V$. If $V \neq 0$, then $(\operatorname{Ker} \sigma)_i \neq 0$ for some i for any point in $\mathfrak{L}(\mathbf{v})$. That is

$$
\mathfrak{L}(\mathbf{v}) = \bigcup_{i \in I, r \neq 0} \mathfrak{L}_{i;r}(\mathbf{v})
$$

By the induction hypothesis and the above observation, $\mathfrak{L}_{i;r}(\mathbf{v})$ is a half-dimensional subvariety. Since the above is a finite union, the total set $\mathfrak{L}(\mathbf{v})$ is also half-dimensional. Since $\mathfrak{L}_{i;0}(\mathbf{v})$ is an open subset of $\mathfrak{L}(\mathbf{v})$, it is also half-dimensional. This completes the induction.

By Lemma 7.10 we have the following sequence of Grassmann bundles over $\mathfrak{L}_{i;0}(\mathbf{v})$:

$$
\begin{array}{ccccccccc}
\mathfrak{L}_{i;0}(\mathbf{v}) & \mathfrak{L}_{i;1}(\mathbf{v}+\mathbf{e}_i) & \cdots & \mathfrak{L}_{i;r}(\mathbf{v}+r\mathbf{e}_i) & \cdots & \mathfrak{L}_{i;r_{\max}}(\mathbf{v}+r_{\max}\mathbf{e}_i) \\
\Big\| & \downarrow & \cdots & \downarrow & \cdots & \Big\| \\
\mathfrak{L}_{i;0}(\mathbf{v}) & \mathfrak{L}_{i;0}(\mathbf{v}) & \cdots & \mathfrak{L}_{i;0}(\mathbf{v}) & \cdots & \mathfrak{L}_{i;0}(\mathbf{v}),
\end{array}
\tag{7.12}
$$

where $r_{\max} = \operatorname{rank}(\operatorname{Ker}\tau/\operatorname{Im}\sigma)_i = (\mathbf{e}_i, \mathbf{e}_0 - \mathbf{C}\mathbf{v})$.

We consider this as the $(r_{\max}+1)$-dimensional irreducible representations of $\mathfrak{sl}_2$ and defines a crystal structure on the set $\bigsqcup_{\mathbf{v}} \operatorname{Irr} \mathfrak{L}(\mathbf{v})$ of irreducible components of $\bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v})$. (Compare §1.3 and Example 7.3.)

Let Y be an irreducible component of $\mathfrak{L}(\mathbf{v})$. We define $\operatorname{wt}(Y)$ by

$$
\sum_i \left(\delta_{i0} - \sum_j c_{ij} \dim V_j \right) \Lambda_i.
$$

We define $\varepsilon_i(Y)$ so that

$$
\begin{aligned}
\varepsilon_i(Y) &= \dim(\operatorname{Ker}\sigma)_i \quad \text{for a generic point } I \text{ in } Y, \\
&= \min_{I \in Y} \dim(\operatorname{Ker}\sigma)_i.
\end{aligned}
$$

As we remarked above, $\varepsilon_i(Y) > 0$ for some i if $V \neq 0$. We set $\varphi_i(Y) = \varepsilon_i(Y) + \langle \operatorname{wt}(Y), h_i \rangle$.

Let $r \overset{\text{def.}}{=} \varepsilon_i(Y)$. A nonempty open subset of Y is contained in $\mathfrak{L}_{i;r}(\mathbf{v}+r\mathbf{e}_i)$ in (7.12) for some $i, r, \mathbf{v}$. We define an irreducible component Y' of $\mathfrak{L}(\mathbf{v})$ by

$$
Y' \overset{\text{def.}}{=} \overline{p\,(Y \cap \mathfrak{L}_{i;r}(\mathbf{v}+r\mathbf{e}_i))},
$$

where p is the projection of the Grassmann bundle above. We have

$$\varepsilon_i(Y') = 0$$

Conversely, we can recover Y from Y' as

$$Y = \overline{p^{-1}\left(Y' \cap \mathfrak{L}_{i;0}(\mathbf{v})\right)}.$$

Therefore we have a bijection

$$\{Y \in \operatorname{Irr}\mathfrak{L}(\mathbf{v} + r\mathbf{e}_i) \mid \varepsilon_i(Y) = r\} \longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v}) \mid \varepsilon_i(Y') = 0\}.$$

Using above observation, we want to define maps

$$\widetilde{e}_i, \widetilde{f}_i \colon \bigsqcup \operatorname{Irr}\mathfrak{L}(\mathbf{v}) \to \bigsqcup \operatorname{Irr}\mathfrak{L}(\mathbf{v}) \sqcup \{0\}.$$

If $\varepsilon_i(Y) = 0$, then we define $\widetilde{e}_i Y = 0$. Otherwise, we define $\widetilde{e}_i Y$ as the image of Y under the composition of bijections

$$\begin{aligned}\{Y \in \operatorname{Irr}\mathfrak{L}(\mathbf{v} + r\mathbf{e}_i) \mid \varepsilon_i(Y) = r\} &\longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v}) \mid \varepsilon_i(Y') = 0\} \\ &\longleftrightarrow \{Y'' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v} + (r-1)\mathbf{e}_i) \mid \varepsilon_i(Y') = r-1\},\end{aligned}$$

where the latter bijection is again given by the Grassmann bundle.

Similarly we define $\widetilde{f}_i Y$ as the image of Y under the composition of bijections

$$\begin{aligned}\{Y \in \operatorname{Irr}\mathfrak{L}(\mathbf{v} + r\mathbf{e}_i) \mid \varepsilon_i(Y) = r\} &\longleftrightarrow \{Y' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v}) \mid \varepsilon_i(Y') = 0\} \\ &\longleftrightarrow \{Y''' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v} + (r+1)\mathbf{e}_i) \mid \varepsilon_i(Y') = r+1\}.\end{aligned}$$

If $r = r_{\max}$, then the latter bijection does not exist. So we set $\widetilde{f}_i Y = 0$ in this case.

Theorem 7.13. *The above ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ on $\bigsqcup_{\mathbf{v}} \operatorname{Irr}\mathfrak{L}(\mathbf{v})$ is a crystal.*

Using the exact sequence in homology groups, it is not difficult to show that

$$f_i\,[Y] = c\left[\widetilde{f}_i Y\right] + \sum_{Y' : \varepsilon_i(Y) > \varepsilon_i(Y)+1} c_{Y'}\,[Y']$$

for some constants c, $c_{Y'}$. (Use the open set $\bigcup_{s : s \le \varepsilon(Y)} X_{i;s}(\mathbf{v})$.) In order to determine the constant c, we pullback both hand sides to $\bigcup_{s : s \le \varepsilon(Y)+1} X_{i;s}(V, W)$. In the right hand side, only $c\left[\widetilde{f}_i Y\right]$ survives. Then it is not difficult to determine c by using the self-intersection formula. It is given by $\pm(\varepsilon(Y) + 1)$.

Using the above formula, we prove that $H_{\mathrm{top}}(\mathfrak{L})$ is a highest weight module by induction on $\dim V$ and ε_i. If $\mathbf{v} = 0$, $\mathfrak{L}(0) = X(0)$ is a point. We have nothing to prove. Let Y be an irreducible component of $\mathfrak{L}(\mathbf{v})$. There exists i such that $\varepsilon_i(Y) > 0$. Suppose that we already know that

(1) if $\dim V' < \dim V$, then $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v}'))$ is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [X(0)]$.

(2) if $Y' \in \operatorname{Irr}\mathfrak{L}(\mathbf{v})$ satisfies $\varepsilon_i(Y') > \varepsilon_i(Y)$, then $[Y']$ is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [X(0)]$.

Since the value of ε_i on $\operatorname{Irr}\mathfrak{L}(\mathbf{v})$ is bounded from above, we may assume the second condition by the descending induction. By the above formula, we have

$$f_i\,[\widetilde{e}_i Y] = \pm\varepsilon_i(Y)[Y] + \sum_{Y' : \varepsilon_i(Y) > \varepsilon_i(Y)} c_{Y'}\,[Y'].$$

By (1), the left hand side is contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [X(0)]$. By (2), terms in the right hand side, except $\pm\varepsilon_i(Y)[Y]$ are contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [X(0)]$. Therefore $[Y]$ is also contained in $\mathbf{U}(\widehat{\mathfrak{g}}) \cdot [X(0)]$. This completes the proof.

Remark 7.14. It is known that the crystal defined above (the definition is due to Lusztig) is isomorphic to the crystal of the highest weight module of the quantum affine algebra. See [KS, 31, 38] for the proof.

7.3. **Sheaves on $K3$ surfaces.** The Grassmann bundle structures between stratifications of moduli spaces of sheaves on $K3$ surfaces have been studied by Yoshioka [42], Markman [32] and Kawai-Yoshioka [27]. Let us briefly review their results.

Let X be a $K3$ surface. Let $(H^*(X,\mathbb{Z}),\langle\ ,\ \rangle)$ be the Mukai lattice. Namely, it is the integral cohomology group with the pairing

$$\langle x, y\rangle = -\int_X x_0 \cup y_2 - x_1 \cup y_1 + x_2 \cup y_0,$$

where $x = x_0 + x_1 + x_2$, $y = y_0 + y_1 + y_2$ with $x_i, y_i \in H^{2i}(X,\mathbb{Z})$. We also set $x^\vee = x_0 - x_1 + x_2$. If E is a coherent sheaf on X (more generally, an object of the derived category of coherent sheaves on X), we define its *Mukai vector* $\mathbf{v}(E)$ by

$$\mathbf{v}(E) \overset{\text{def.}}{=} \operatorname{ch}(E)\sqrt{\operatorname{td}_X} = \operatorname{ch}(E)(1+\omega),$$

where $\omega \in H^4(X,\mathbb{Z})$ is the fundamental class. Since $\langle\ ,\ \rangle$ is even, $\mathbf{v}(E)$ has a value in $H^*(X,\mathbb{Z})$. The Riemann-Roch formula says

$$\sum_{i=0}^{2} \dim \operatorname{Ext}^i(E,F) = -\langle \mathbf{v}(E), \mathbf{v}(F)\rangle$$

for coherent sheaves E, F on X. Moreover, the Serre duality says the natural pairing (Yoneda product)

$$\operatorname{Ext}^i(E,F)\otimes \operatorname{Ext}^{2-i}(F,E) \to \mathbb{C}; \alpha\otimes\beta \mapsto \operatorname{tr}(\alpha\wedge\beta) \in H^2(X,\mathcal{O}_X) \cong \mathbb{C}$$

is *non-degenerate*, where we have used $K_X = \mathcal{O}_X$.

Let us take and fix an ample line bundle H over X. Let $\mathfrak{M}(\mathbf{v})$ be the moduli space of H-stable sheaves E with $\mathbf{v}(E) = \mathbf{v}$. (We omit the notation H.) By the fundamental result of Mukai, $\mathfrak{M}(\mathbf{v})$ is a nonsingular symplectic manifold of dimension $2 + \langle \mathbf{v}, \mathbf{v}\rangle$. This is an analog of our $X(\mathbf{v})$. The analog of the sheaf $\mathcal{O}_0 \otimes \rho_i$ is a *rigid vector bundle* E_0 on $K3$, i.e., a stable vector bundle E_0 with $\operatorname{Ext}^1(E_0,E_0) = 0$. This has $\dim\mathfrak{M}(\mathbf{v}(E_0)) = 0$. In fact, Mukai shows a stronger result: $\mathfrak{M}(\mathbf{v}(E_0)) = \{E_0\}$. Yoshioka defined a stratification on moduli of sheaves by

$$\mathfrak{M}_{E_0;r}(\mathbf{v}) = \{E \in \mathfrak{M}(\mathbf{v}) \mid \dim \operatorname{Hom}(E_0, E) = r\}.$$

(This is an analog of the Brill-Noether locus.) In [loc. cit.] it was shown that under certain conditions on $\mathbf{v}$ and the polarization H, there exists a natural homomorphism

$$p\colon \mathfrak{M}_{E_0;r}(\mathbf{v}) \to \mathfrak{M}_{E_0;0}(\mathbf{v}'),$$

which is a Grassmann bundle. Here $\mathbf{v}' = \mathbf{v} - r\mathbf{v}(E_0)$ and $\mathbf{v}(E_0)$ is the Mukai vector of E_0. When the rank of $\mathbf{v}'$ is negative, the right hand side must be understood properly. This will be explained later. So first consider the case the rank of $\mathbf{v}'$ is nonnegative. The map is defined by assigning E' to E with

$$0 \to \operatorname{Hom}(E_0,E)\otimes E_0 \to E \to E' \to 0.$$

Taking the long exact sequence for $\operatorname{Hom}(E_0,\bullet)$, we get $\operatorname{Hom}(E_0,E') = 0$. Therefore $E' \in \mathfrak{M}_{E_0;0}(\mathbf{v}')$. (For the study of the stability condition, see the original papers.) The corresponding extension class is an element e of

$$\operatorname{Ext}^1(E', \operatorname{Hom}(E_0,E)\otimes E_0) \cong \operatorname{Hom}(\operatorname{Hom}(E_0,E)^*, \operatorname{Ext}^1(E',E_0)).$$

The corresponding homomorphism $\operatorname{Hom}(E_0,E)^* \to \operatorname{Ext}^1(E',E_0)$ is injective and defines a r-dimensional subspace in $\operatorname{Ext}^1(E',E_0)$. This gives us a Grassmann bundle structure on $p\colon \mathfrak{M}_{E_0;r}(\mathbf{v}) \to \mathfrak{M}_{E_0;0}(\mathbf{v}')$.

Next consider the case $\mathbf{v}'$ is negative. Since the ranks of coherent sheaves are always non-negative, we must interpret $\mathfrak{M}(\mathbf{v}')$ suitably. It turns out that the reasonable choice is

$$\mathfrak{M}(\mathbf{v}') \stackrel{\text{def.}}{=} \mathfrak{M}(-\mathbf{v}'^{\vee}),$$
$$\mathfrak{M}_{E_0;r}(\mathbf{v}') \stackrel{\text{def.}}{=} \mathfrak{M}_{E_0^{\vee};r+\langle \mathbf{v}(E_0),\mathbf{v}'\rangle}(-\mathbf{v}'^{\vee})$$

where $\vee$ is the involution on $H^*(X,\mathbb{Z})$ given above and $E_0^{\vee}$ is the dual vector bundle of E_0. Under the assumption (which is not mentioned here), we have $\operatorname{Ext}^2(E_0^{\vee}, E') = 0$, therefore Riemann-Roch says

$$\dim \operatorname{Hom}(E_0^{\vee}, E') - \dim \operatorname{Ext}^1(E_0^{\vee}, E') = -\langle \mathbf{v}(E_0)^{\vee}, -\mathbf{v}'^{\vee}\rangle = \langle \mathbf{v}(E_0), \mathbf{v}'\rangle$$

for $E' \in \mathfrak{M}(\mathbf{v}')$. Hence

$$\mathfrak{M}_{E_0;r}(\mathbf{v}') = \left\{E' \in \mathfrak{M}(-\mathbf{v}'^{\vee}) \,\middle|\, \dim \operatorname{Ext}^1(E_0^{\vee}, E') = r\right\}.$$

Now the map is defined by assigning E' to E with

$$0 \to E^{\vee} \to \operatorname{Hom}(E_0, E)^* \otimes E_0^{\vee} \to E' \to 0.$$

Here $E^{\vee}$ is the dual vector bundle of E. (If E is not locally free, we need a slight modification.) Taking the long exact sequence for $\operatorname{Hom}(E_0^{\vee}, \bullet)$, we get $\operatorname{Ext}^1(E_0^{\vee}, E') = 0$. The above short exact sequence defines a homomorphism

$$\operatorname{Hom}(E_0, E)^* \cong \operatorname{Hom}(E_0, E)^* \otimes \operatorname{Hom}(E_0^{\vee}, E_0^{\vee}) \to \operatorname{Hom}(E_0^{\vee}, E').$$

This is injective and defines an r-dimensional subspace in $\operatorname{Hom}(E_0^{\vee}, E')$. This gives us the Grassmann bundle on p.

As an application of the Grassmann bundle, we can construct $\mathfrak{sl}_2$-actions on the cohomology group of moduli spaces corresponding to rigid sheaves on $K3$ (paper in preparation).

References

[1] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for quantum groups*, Duke Math. J. **61** (1990), 655–675.

[2] A. Beauville, *Variété kählriennes dont la première classe de Chern est nulle*, J. of Differential Geom. **18** (1983), 755–782.

[3] R. Bott and L.W. Tu, Differential forms in algebraic topology, GTM **82**, Springer-Verlag, 1982.

[4] J. Briançon, *Description de* $\operatorname{Hilb}^n \mathbb{C}\{x,y\}$, Invent. Math. **41** (1977), 45–89.

[5] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), 535–554.

[6] J. Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math. **183** (1998), 39–90.

[7] E. Corrigan and P. Goddard, *Construction of instanton and monopole solutions and reciprocity*, Jour. of Phys. **154** (1984), 253–279.

[8] W. Crawley-Boevey, *On the exceptional fibres of Kleinian singularities*, Amer. J. Math., **122** (2000), 1027–1037.

[9] ———, *Geometry of the moment map for representations of quivers*, Composito Math., **126** (2001), 257–293.

[10] S.K. Donaldson, *Instantons and geometric invariant theory*, Comm. in Math. Phys. **93** (1984), 453–460.

[11] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990.

[12] G. Ellingsrud and S.A. Strømme, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. **87** (1987), 343–352.

[13] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

[14] P. Gabriel, *Unzerlegbare Darstellungen. I.*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309.

[15] V. Ginzburg, *$\mathfrak{G}$-modules, Springer's representations and bivariant Chern classes*, Adv. in Math. **61** (1986), 1–48.

[16] ———, *Lagrangian construction of the enveloping algebra* $U(\mathfrak{sl}_n)$, C.R. Acad. Sci. Paris Sér I Math. **312** (1991), 907–912.

[17] G. Gonzalez-Springberg and J.L. Verdier, *Construction géometrique de la correspondence de McKay*, Ann. Sci. École Norm. Sup. **16** (1983), 409–449.

[18] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), 941–1006.
[19] A. Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **188**, 1977.
[20] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Topology **39** (2000), 1155–1191.
[21] N. Iwahori, *On the structure of the Hecke ring of a Chevalley group over a finite field*, J. Fac. Sci. Univ. Tokyo Sec 1A, **10** (1964), 215–236.
[22] V.G. Kac, Infinite dimensional Lie algebras (3rd Ed.), Cambridge Univ. Press 1990.
[23] ———, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57–92.
[24] M. Kapranov and E. Vasserot, *Kleinian singulatires, derived categories and Hall algebras*, Math. Ann. **316** (2000), 565–576.
[25] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. **63** (1991), 465–516.
[26] ———, *The crystal base and Littelmann's refine Demazure character formula*, Duke Math. **71** (1993), 839–858.
[27] T. Kawai and K. Yoshioka, *String partition functions and infinite products*, Adv. Theor. Math. Phys. **4** (2000), 397–485.
[28] P.B. Kronheimer, *The construction of ALE spaces as a hyper-Kähler quotients*, J. Differential Geom. **29** (1989) 665–683.
[29] P.B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), 263–307.
[30] J. Le Potier, *Systèmes cohérents et structures de niveau*, Astérisque, **214** (1993)
[31] A. Malkin, *Tensor product varieties and crystals. ADE case*, preprint, math.AG/0103025.
[32] E. Markman, *Brill-Noether duality for moduli spaces of sheaves on $K3$ surfaces*, J. Algebraic Geom. **10** (2001), 623–694.
[33] J. McKay, *Graphs, singularities and finite groups*, in Proc. Sympos. Pure Math. **37**, Amer. Math. Soc., 1980, 183–186.
[34] S. Mukai, *On the moduli space of bundles on $K3$ surfaces. I.*, in "Vector bundles on algebraic varieties (Bombay, 1984)", Tata Inst. Fund. Res. Stud. Math., **11**, Tata Inst. Fund. Res., Bombay, 1987, 341–413.
[35] W. Nahm, *Self-dual monopoles and calorons*, in "Group theoretical methods in physics (Trieste, 1983)", Lecture Notes in Phys., **201**, Springer, Berlin, 1984. 189–200.
[36] H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), 145–238.
[37] ———, *t–analogue of the q–characters of finite dimensional representations of quantum affine algebras*, in "Physics and Combinatorics", Proceedings of the Nagoya 2000 International Workshop, World Scientific, 2001, 195–218.
[38] ———, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), 399–449.
[39] A. Tikhomirov, ???
[40] V.S. Varadarajan, Lie groups, Lie algebras, and their representations, GTM **102**, Springer-Verlag, 1984.
[41] W. Wang, *Equivariant K-theory, wreath products, and Heisenberg algebra*, Duke Math., **103** (2000), 1–23.
[42] K. Yoshioka, *Some examples of Mukai's reflections on $K3$ surfaces*, J. Reine Angew. Math. **515** (1999), 97–123.

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