Instanton Counting and Donaldson invariants

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based on
Nekrasov : hep-th/0206161
N + Kota Yoshioka : math.AG/0306198, math.AG/0311058, math.AG/0505553
Lothar Göttsche + N + Y : math.AG/0606180

Additional references

- Nekrasov + Okounkov : hep-th/0306238

  (another proof of Nekrasov's conjecture based on random partitions)
- Braverman : math.AG/0401409

  (affine) Whittaker modules
- Braverman + Etingof :math.AG/0409441

  (yet another proof)
- Takuro Mochizuki : math.AG/0210211

  (wall crossing formula for general walls)

History

$\sim$**1994** Many important works on Donaldson invariants

1994 Seiberg-Witten computed the *prepotential* of $N = 2$ SUSY YM theory (physical counterpart of Donaldson invariants) via periods of Riemann surfaces (SW curve).

1997 Moore-Witten computed Donaldson invariants (blowup formulas, wall-crossing formulas...) via the SW curve.

2002 Nekrasov introduced a partition function $\approx$ 'equivariant' Donaldon invariants for $\mathbb{R}^4$

2003 Seiberg-Witten prepotential from Nekrasov's partition function (Nekrasov-Okounkov, N-Yoshioka)

Aim of talks

1. Nekrasov's partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$
2. Relation between

   $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ ('equivariant Donaldson invariant for $\mathbb{R}^4$')
   $\longleftrightarrow$ Donaldson invariants for a cpt 4-mfd (proj. surf.) X

where

$\varepsilon_1, \varepsilon_2$: basis of $\operatorname{Lie} T^2$ (acting on $\mathbb{R}^4 = \mathbb{C}^2$)

$\vec{a} = (a_1, \dots, a_r)$ with $\sum a_\alpha = 0$

: basis of $\operatorname{Lie} T^{r-1}$ (max. torus of the gauge group $\mathrm{SU}(r)$.

Λ : formal variable for the instanton numbers

Alg. Geom. is very powerful for the calculation of invariant ....

Physics vs Math.

Donaldson inv. $\xrightarrow[t\to\infty]{(X,tg)}$ Seiberg-Witten inv. + local contrib.

Nekrasov part. func. Z $\xrightarrow[\text{fixed point formula + cobordism argument}]{\text{[GNY]+[Mochizuki]}}$ wall-crossing formula

$Z=\exp(\frac{F_0}{\varepsilon_1\varepsilon_2}+\dots)$ ↓ [NY],[NO] | vanishing on a chamber ↓ ↑ regularization of the integral

Seiberg-Witten prep. F_0 $\xrightarrow[u\text{-plane integral}]{\text{[Moore-Witten]}}$ Donaldson inv. for $b_+=1$

[GNY]+[Mochizuki] : More precisely,

1. Describe wall-crossing formula as an integral over Hilbert schemes.
2. Show the integral is 'universal'.
3. Compute the integral for toric surfaces via fixed point formula

Quick Review of Donaldson invariants

- (X,g) : cpt, oriented, simply-conn., Riem. 4-mfd
- $P\to X$: $\mathrm{U}(2)$- (or $\mathrm{SO}(3)$-)principal bundle
- $c_1=c_1(P)$, $c_2=c_2(P)$: Chern classes
- $M_0^{\mathrm{reg}}=M_{g,0}^{\mathrm{reg}}(c_1,c_2)$: moduli of instantons
- $M_0=M_{g,0}(c_1,c_2)=\bigsqcup M_{g,0}^{\mathrm{reg}}(c_1,c_2-k)\times S^kX$ (Uhlenbeck cptfication)
- M_0^{reg} is a C^∞ mfd. of expected dimension $2d=8c_2-2c_1^2-3(1+b_+)$ for a generic metric g
- the fundamental class $[M_0]$ can be defined if $c_1\neq 0$ or exp. dim. $2d>4c_2=\dim_{\mathbb{R}}(\{\theta\}\times S^{c_2}X)$ (stable range)

Review of Donaldson invariants – cont'd.

- $\mathcal{E} \to X \times M_0^{\text{reg}}$: universal bundle
- $\mu(\bullet) = \left(c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2\right)/\bullet\colon H_*(X) \to H^*(M_0^{\text{reg}})$
- $\mu(\alpha)$ ($\alpha \in H_2(X)$) extends to M_0
- $\mu(p)$ ($p \in H_0(X)$) extends to $M_0 \setminus \{\theta\} \times S^{c_2}X$

Let

$$\Phi^g_{c_1,c_2}(\exp(\alpha z + px)) \stackrel{\text{def.}}{=} \int_{M_0} \exp\left(z\mu(\alpha) + x\mu(p)\right)$$
$$\alpha \in H_2(X), p \in H_0(X)$$

We first define this in the stable range (i.e. $\mu(\alpha)$ appears $\geq \frac{3b_+ + 5}{4}$ times, and then extend it by the blow-up formula.

Algebro-geometric approach

- X : (simply conn.) projective surface
- H : ample line bundle
- $\mu(E) = \frac{1}{\operatorname{rank} E}\int_X c_1(E) \cup H$: slope
- $p_E(n) = \frac{1}{\operatorname{rank} E}\chi(E(nH))$: normalized Hilbert polynom.
- E is μ-(semi)stable $\stackrel{\text{def.}}{\Longleftrightarrow} \mu(F) < (\leq)\mu(E)$ for $\forall F \subset E$ with $0 < \operatorname{rank} F < \operatorname{rank} E$
- E is H-(semi)stable $\stackrel{\text{def.}}{\Longleftrightarrow} p_F(n) < (\leq) p_E(n)$ ($n \gg 0$) for $\forall F \subset E$ with $0 < \operatorname{rank} F < \operatorname{rank} E$
- μ-stable $\Rightarrow$ H-stable $\Rightarrow$ H-semistable $\Rightarrow$ μ-semistable

Algebro-geometric approach - cont'd

- $M^{\text{reg}}_{H,0}(c_1, c_2)$: moduli space of μ-stable rank 2 holo. vect. bundles E with $c_1(E) = c_1$, $c_2(E) = c_2$
- $M = M_H(c_1, c_2)$: moduli space of H-semistable sheaves
- $M^{\text{reg}}_{H,0}(c_1, c_2) \subset M_H(c_1, c_2)$ (Gieseker-Maruyama cptfication)
- $M_H(c_1, c_2)$ is of expected dimension if $c_2 \gg 0$

Let $g =$ Hodge metric with class H

- $M^{\text{reg}}_{g,0}(c_1, c_2)$ (uncpt'd moduli sp.) $= M^{\text{reg}}_{H,0}(c_1, c_2)$

  (Donaldson) (Hitchin-Kobayashi corr.)
- $\pi\colon M_H(c_1, c_2) \to M_{g,0}(c_1, c_2)$: cont. map (J.Li)

Algebro-geometric approach – cont'd.

Then (Morgan, J. Li)

$$\Phi^g_{c_1,c_2}(\exp(\alpha z + px)) = \int_{M_H(c_1,c_2)} \exp\left(z\mu(\alpha) + x\mu(p)\right)$$

$$\alpha \in H_2(X), p \in H_0(X)$$

Two approaches to define inv. for arb. c_2

- Use blowup formula
- Virtual fundamental class (Mochizuki)

Question 1. *Do two approaches give the same answer ?*

Metric dependence

Return to a C^∞ 4-mfd.

- $b_2^+ > 1 \Longrightarrow$ independent of g
- $b_2^+ = 1 \Longrightarrow$ depend on g, but only on

$$\omega(g) \in H^2(X)^+/\mathbb{R}_{>0} = \{\omega \in H^2(X) \mid \omega^2 > 0\}/\mathbb{R}_{>0} = \mathcal{H} \sqcup (-\mathcal{H})$$

- where $\omega(g)$: self-dual harmonic form with $\|\omega(g)\| = 1$ unique up to sign ($\longleftrightarrow$ orientation of M)

Calculation of $\Phi^g_{c_1,c_2}$ was *difficult*........

1994 Donaldson invariants are determined by Seiberg-Witten invariants, which are much easier to calculate !

Wall-crossing formula

- $W^\xi = \{\omega \in H^2(X)^+ \mid \xi \cdot \omega = 0\}$: wall defined by $\xi \in H^2(X, \mathbb{Z})$ s.t. $c_1 \equiv \xi \mod 2$
- $\omega(g) \in W^\xi$ $\Longrightarrow \exists$ a reducible instanton $L_+ \oplus L_-$ with $c_1(L_\pm) = \frac{c_1 \pm \xi}{2}$
- $[L] + \sum m_i p_i$ may occur M_0.
- This happens only when $\begin{cases} \xi \equiv c_1 \mod 2 \\ 4c_2 - c_1^2 \geq -\xi^2 > 0 \end{cases}$

$\Longrightarrow \#$ of walls are locally finite

- $\Phi^g_{c_1,c_2}$ is constant when $\omega(g)$ moves in a chamber $\mathcal{C}_{c_1,c_2}$: a connected component of $H^2(X)^+ \setminus \bigcup W^\xi$

Kotschick-Morgan conjecture

Fact (Kotschick-Morgan '94). $\exists \delta^{\xi}_{c_2}$ *s.t.*

$$\Phi^{g_1}_{c_1,c_2} - \Phi^{g_2}_{c_1,c_2} = 1^{C^2/8} \sum_{\xi} (-1)^{(\xi - C/2)C} \delta^{\xi}_{c_2}$$

Kotschick-Morgan conjecture : $\delta^{\xi}_{c_2}|_{\mathrm{Sym}\, H_2(X)}$ is

- a polynomial in ξ and the intersection form Q_X
- with coeff's depend only on ξ, c_2, homotopy type of X

Remark. If $c_1 \not\equiv 0\ (2)$, $\exists$ chamber $\mathcal{C}$ s.t. $\Phi^{\mathcal{C}}_{c_1,c_2} \equiv 0$.
If $c_1 \equiv 0$, $\exists$ a similar result (Göttsche-Zagier)

Göttsche's computation

1995 Göttsche computed $\delta^{\xi} = \sum_{c_2} \delta^{\xi}_{c_2}$ explicitly in terms of modular forms, assuming KM conj.

1997 Moore-Witten : Derive Göttsche's formula from the u-plane integral

Our goal today :

δ^{ξ} can be expressed via Nekrasov's partition function

There are several peoples (Feehan-Leness, Chen) announcing/proving KM conjecture. Their approach is differential geometric which ours is algebro-geomtric. I do not check their approach in detail. Their approach only yields KM conj., not Göttsche's formula.

Framed moduli spaces of instantons on $\mathbb{R}^4$

- $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$. ($r = 2$ later)
- $M_0^{\text{reg}}(n,r)$: framed moduli space of $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4$ with $c_2 = n$, where the framing is the trivialization of the bundle at ∞.

This space is noncompact:

- bubbling
- $\exists$ parallel translation symmetry

Two partial compactifications

We kill the first 'source' of noncompactness (bubbling) in two ways:

- $M_0(n,r)$: Uhlenbeck (partial) compactification

$$M_0(n,r) = \bigsqcup_{k=0}^{n} M_0^{\text{reg}}(k,r) \times S^{n-k}\mathbb{R}^4.$$

- $M(n,r)$: Gieseker (partial) compactification, i.e., the framed moduli space of rank r torsion-free sheaves E on $\mathbb{P}^2 = \mathbb{R}^4 \cup \ell_\infty$
  - E : a torsion-free sheaf on $\mathbb{P}^2$ with $\mathrm{rk} = r$, $c_2 = n$
  - $\varphi\colon E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing)

Morphism from Gieseker to Uhlenbeck

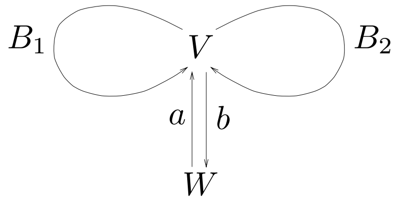
- $M(n,r)$: nonsingular hyperKähler manifold of dim. $4nr$ (a holomorphic symplectic manifold)
- $M_0(n,r)$: affine algebraic variety
- $\pi\colon M(n,r) \to M_0(n,r)$: projective morphism (resolution of singularities) defined by

$$(E,\varphi) \mapsto ((E^{\vee\vee},\varphi), \operatorname{Supp}(E^{\vee\vee}/E)).$$

(cf. J. Li, Morgan)

Quiver varieties for the Jordan quiver

- V, W : cpx vector sp.'s with $\dim V = n$, $\dim W = r$
- $\mathbb{M}(n,r) = \operatorname{End} V \oplus \operatorname{End} V \oplus \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W)$
- $\mu\colon \mathbb{M}(n,r) \to \operatorname{End}(V)$; $\mu(B_1,B_2,a,b) = [B_1,B_2] + ab$



- $M_0(n,r) = \mu^{-1}(0) /\!\!/ \operatorname{GL}(V)$ (affine GIT quotient)
- $M(n,r) = \mu^{-1}(0)^{\text{stable}} / \operatorname{GL}(V)$
- stable $\overset{\text{def.}}{\Longleftrightarrow}$ $\nexists S \subsetneq V$ with $B_\alpha(S) \subset S$, $\operatorname{Im} a \subset S$

Example $r = 1$ *: Hilbert scheme of points*

Theorem. $M(n,1) = (\mathbb{A}^2)^{[n]}$, $\quad M_0(n,1) = S^n(\mathbb{A}^2)$

$(\mathbb{A}^2)^{[n]}$: Hilbert scheme of n points in the affine plane $\mathbb{A}^2$

$S^n(\mathbb{A}^2)$: symmetric product (unordered n points with mult.)

Sketch of Proof

- $(\mathbb{A}^2)^{[n]} = \{I \subset \mathbb{C}[x,y] \text{ ideal} \mid \dim \mathbb{C}[x,y]/I = n\}$
- Set $V = \mathbb{C}[x,y]/I$
  $B_1, B_2 = \times x, \times y$, $a(1) = 1 \mod I$, $b = 0$
- ◦ $S^n(\mathbb{A}^2) \to M_0(n,1)$ is induced by $\mathbb{A}^{2n} \to \mathbb{M}(n,1)$:
  $(B_1, B_2, a, b) = (\mathrm{diag}(x_1, \dots, x_n), \mathrm{diag}(y_1, \dots, y_n), 0, 0)$

Torus action and equivariant homology group

- $T = T^{r-1}$: maximal torus in $\mathrm{SL}(W)$
- $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \curvearrowright M(n,r), M_0(n,r)$: torus action
  - $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{C}^2$ and T acts by the change of the framing
  - $(B_1, B_2, a, b) \longmapsto (t_1 B_1, t_2 B_2, a e^{-1}, t_1 t_2 e b)$
    $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, $e \in T$
- $H_*^{\widetilde{T}}(M(r,n))$, $H_*^{\widetilde{T}}(M_0(r,n))$: equivariant (Borel-Moore) homology groups
- modules over S : symmetric power of
  $\mathrm{Lie}(\widetilde{T})^* = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_\alpha] = H^*_{\widetilde{T}}(\mathrm{pt})$ ($\sum a_\alpha = 0$)
- $[M(r,n)]$, $[M_0(r,n)]$: fundamental classes
- $\mathfrak{S}$: quotient field of S

Instanton part of Nekrasov's partition function

Fact (Localization). *Let ι_0 be the inclusion of the fixed point set $M_0(n,r)^{\widetilde{T}}$ in $M_0(n,r)$. Then*

$$H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathfrak{S} \xleftarrow[\cong]{\iota_{0*}} H_*^{\widetilde{T}}(M_0(n,r)^{\widetilde{T}}) \otimes_S \mathfrak{S}.$$

The same holds for $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$.

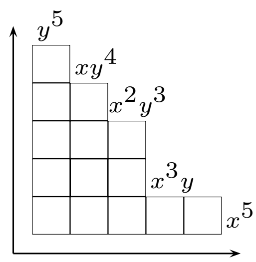
Observation. $M_0(n,r)^{\widetilde{T}} = \{0\}$, *so* $\mathrm{RHS} = \mathfrak{S}$.

Define

$$\begin{aligned} Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &= \sum_{n=0}^{\infty} \Lambda^{2nr} (\iota_{0*})^{-1}[M_0(n,r)] \\ &= \sum_{n=0}^{\infty} \Lambda^{2nr} (\iota_{0*})^{-1}\pi_*[M(n,r)] \end{aligned}$$

Fixed point set $M(n,r)^{\widetilde{T}}$

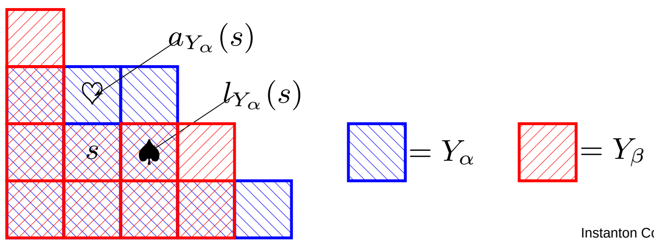
- $(E,\varphi) \in M(n,r)$ is fixed by the first factor $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ ($\sum n_\alpha = n$)
  ($\because$ W decomposes into 1-dim rep's of T)
- $M(n_\alpha,1) = \mathrm{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\iff$ I_α is generated by monomials in x, y
  $\iff$ I_α corresponds to a Young diagram Y_α



- $M(n,r)^{\widetilde{T}} \cong \{\vec{Y} = (Y_1, \dots, Y_r) \mid \sum |Y_\alpha| = n\}$
- the tangent space
  $T_{\vec{Y}} = \mathrm{Ext}^1(E, E(-\ell_\infty)) = \bigoplus_{\alpha,\beta} \mathrm{Ext}^1(I_\alpha, I_\beta(-\ell_\infty))$
- its equivariant Euler class

$$\begin{aligned}\mathrm{Euler}\,(T_{\vec{Y}}) = \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} &\left(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha\right) \\ \times \prod_{t \in Y_\beta} &\left((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha\right)\end{aligned}$$

where



Combinatorial expression

- $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$: inclusion

$\Longrightarrow$

$$\begin{array}{ccc} [M(n,r)] \in H_*^{\widetilde{T}}(M(n,r)) \otimes_S \mathcal{S} & \xrightarrow[(\iota_*)^{-1}]{\cong} & \bigoplus_{\vec{Y}} \mathcal{S} \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \sum_{\vec{Y}}} \\ [M_0(n,r)] \in H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathcal{S} & \xrightarrow[(\iota_{0*})^{-1}]{\cong} & \mathcal{S} \end{array}$$

As $M(n,r)$ is smooth, we have an explicit formula:

$$(\iota_*)^{-1}[M(n,r)] = \bigoplus_{\vec{Y}} \frac{1}{\mathrm{Euler}\,(T_{\vec{Y}})}$$

where $\mathrm{Euler}\,(T_{\vec{Y}})$: equivariant Euler class of $T_{\vec{Y}} \in H^*_{\widetilde{T}}(\{\vec{Y}\})$

Combinatorial expression – cont’d.

$$Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = \sum_{\vec{Y}} \frac{\Lambda^{2r\sum|Y_\alpha|}}{\text{Euler}\left(T_{\vec{Y}}\right)}$$

$$= \sum_{\vec{Y}} \Lambda^{2r\sum|Y_\alpha|}$$

$$\times \prod_{\alpha,\beta}\prod_{s\in Y_\alpha} \frac{1}{\left(-l_{Y_\beta}(s)\varepsilon_1 + (1+a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha\right)}$$

$$\times \prod_{t\in Y_\beta} \frac{1}{\left((1+l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha\right)}$$

This is purely combinatorial expression !

Example $r=1$, Hilbert scheme

Let $r=1$. Put $\varepsilon_1 = -\varepsilon_2$. We have

$$(\iota_{0*})^{-1}[M_0(n,1)] = \sum_{|Y|=n} (-\frac{1}{\varepsilon_1})^{2|Y|} \prod_{s\in Y} \frac{1}{h(s)^2}.$$

The hook length formula says

$$\prod_{s\in Y}\frac{1}{h(s)} = \frac{\dim R_Y}{n!},$$

where R_Y is the irreducible representation of S_n associated with Y. Note

$$\sum_{|Y|=n} \dim R_Y^2 = n!$$

Therefore

$$(\iota_{0*})^{-1}[M_0(n,1)] = \frac{1}{n!}(-\frac{1}{\varepsilon_1^2})^n.$$

This can be proven directly by Bott’s formula for orbifolds.

Perturbation Part

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \overset{\text{def.}}{=} \frac{d}{ds}\Big|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)}.$$

$$Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \overset{\text{def.}}{=} \exp\left(-\sum_{\alpha\neq\beta}\gamma_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta;\Lambda)\right)$$

Define the full partition function by

$$Z(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \overset{\text{def.}}{=} Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

Nekrasov Conjecture (2002) - Part 1

Conjecture. *Suppose* $r \geq 2$*.*

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) = F_0 + O(\varepsilon_1,\varepsilon_2),$$

where F_0 *is the* **Seiberg-Witten prepotential***, given by the period integral of certain curves.*

Remark. ($r = 1$)

$$Z^{\text{inst}}(\varepsilon_1,\varepsilon_2;\Lambda) = \sum_{n=0}^\infty \frac{\Lambda^{2n}}{n!(\varepsilon_1\varepsilon_2)^n} = \exp(\frac{\Lambda^2}{\varepsilon_1\varepsilon_2}).$$

Therefore

$$\varepsilon_1\varepsilon_2 \log Z^{\text{inst}}(\varepsilon_1,\varepsilon_2;\Lambda) = \Lambda^2.$$

Seiberg-Witten geometry

A family of curves (*Seiberg-Witten curves*) parametrized by $\vec{u} = (u_2, \dots, u_r)$:

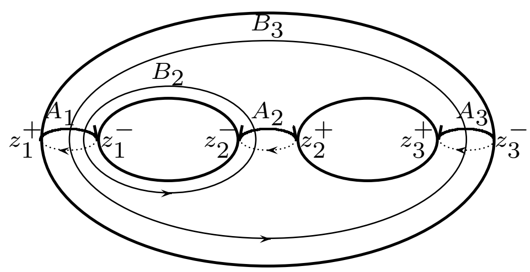
$$C_{\vec{u}} : y^2 = P(z)^2 - 4\Lambda^{2r}, \quad P(z) = z^r + u_2 z^{r-2} + \cdots + u_r.$$

$C_{\vec{u}} \ni (y, z) \mapsto z \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(y, z) = (-y, z)$.
Define the *Seiberg-Witten differential* (multivalued) by

$$dS = -\frac{1}{2\pi} \frac{zP'(z)dz}{y}.$$

Seiberg-Witten geometry — cntd.

Find branched points $z_\alpha^\pm$ near z_α (roots of $P(z) = 0$) (Λ small). Choose cycles A_α, B_α ($\alpha = 2, \dots, r$) as



Put

$$a_\alpha = \int_{A_\alpha} dS, \qquad a_\beta^D = \int_{B_\beta} dS$$

Then (Seiberg-Witten prepotential)

$$\exists F_0 : \quad a_\beta^D = -2\pi\sqrt{-1}\frac{\partial F_0}{\partial a_\beta}$$

Analogy with mirror symmetry

- Mirror symmetry

  A-model Gromov-Witten invariants

  B-model periods

- Nekrasov's conjecture

  A-model Partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

  B-model Seiberg-Witten prepotential F_0

Nekrasov Conjecture - Part 2

Put $\varepsilon_1 = -\varepsilon_2 = ig_s$. ($g_s$: string coupling constant)

Conjecture. *Expand as*

$$\log Z(ig_s, -ig_s, \vec{a}; \Lambda) = F_0 g_s^{-2} + F_1 g_s^0 + \cdots + F_g g_s^{2g-2} + \cdots .$$

Then F_g is (a limit of) the genus g Gromov-Witten invariant for certain noncompact Calabi-Yau 3-fold.

e.g., $r = 2$, Calabi-Yau = canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$

- based on geometric engineering by Katz-Klemm-Vafa (1996)
- Example of topological vertex
- Physical proof by Iqbal+Kashani-Poor : hep-th/0212279, hep-th/0306032, Eguchi-Kanno : hep-th/0310235.

- mathematical proof
  - $r = 2$ by Zhou, math.AG/0311237
  - general r by Li-Liu-Liu-Zhou math.AG/0408426 + recent work by Maulik-Okounkov-Pandharipande.

Then $F_0 = (\text{SW prepotential})$ is a consequence of the 'local mirror symmetry' (at least for $r = 2$).

Remark. *We can expand as*

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = F_0 + (\varepsilon_1 + \varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B + \cdots$$

H, A, B *also play roles in Donaldson invariants. (But no higher terms.)*

Main Result 1

Theorem. (1) [NY],[NO],[BE] Nekrasov's conjecture (part 1) is true.
(2) [NY] ($r = 2$)

$$H = \pi\sqrt{-1}a, \quad A = \frac{1}{2}\log\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right),$$

$$B = \frac{1}{8}\log\left(\frac{4(u^2 - 4\Lambda^4)}{\Lambda^4}\right)$$

where

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = F_0 + (\varepsilon_1 + \varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B + \cdots$$

Blowup equation

The main result is a consequence of the following equation:

$$\sum_{\vec{k}\in\mathbb{Z}^r:\sum k_\alpha=0} \exp\left[-\frac{t(r-1)(\varepsilon_1+\varepsilon_2)}{12}\right] \times Z(\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+\varepsilon_1\vec{k};\Lambda e^{\varepsilon_1 t/2r}) \times Z(\varepsilon_1-\varepsilon_2,\varepsilon_2,\vec{a}+\varepsilon_2\vec{k};\Lambda e^{\varepsilon_2 t/2r}) = Z(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)+O(t^{2r})$$

Take coeff's of t^d $(0\le d\le 2r-1)$ in LHS.

$\Longrightarrow$ nontrivial constraints on Z.

They determine the coeff's of Λ in Z recursively starting from the perturbation part.

Contact term equation

Taking ε_1, $\varepsilon_2 \to 0$, we get

$$\left(\Lambda\frac{\partial}{\partial\Lambda}\right)^2 F_0 = \frac{\sqrt{-1}}{\pi}\sum_{\alpha,\beta=2}^{r}\frac{\partial}{\partial a_\alpha}\left(\Lambda\frac{\partial}{\partial\Lambda}F_0\right)\frac{\partial}{\partial a_\beta}\left(\Lambda\frac{\partial}{\partial\Lambda}F_0\right) \times \frac{\partial}{\partial\tau_{\alpha\beta}}\log\Theta_E(0|\tau),$$

where

- $\tau_{\alpha\beta} = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial^2 F_0}{\partial a_\alpha\partial a_\beta}$: period of SW curve
- Θ_E : theta function with the characteristic E

This equation determines the coeff. of Λ in F_0 recursively starting from the perturbation part.

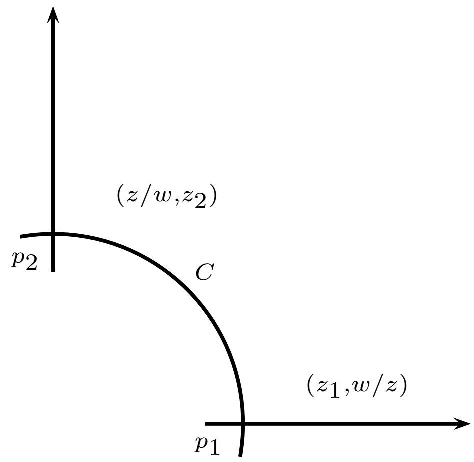
The SW prepotential satisfies the same equation.

$\Longrightarrow$ They must be the same !

Consider the blowup at the origin

$$\widehat{\mathbb{C}}^2 = \{(z_1, z_2, [z : w]) \mid z_1 w = z_2 z\} \xrightarrow{p} \mathbb{C}^2$$

$$C = \{(0, 0, [z : w]) \mid [z : w] \in \mathbb{P}^1\} \quad \text{(except. div.)}$$



$\widehat{M}(k, n, r) = \{(E, \varphi)\}$ framed moduli space on blowup

- E : torsion free sheaf on $\widehat{\mathbb{P}}^2$, $\operatorname{rank} E = r$, $\langle c_1(E), C\rangle = -k$, $c_2(E) - \frac{r-1}{2r} c_1(E)^2 = n$
- $\varphi\colon E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing)

Idea : Compare $\widehat{M}(k, n, r)$ and $M(n, r)$!

Proposition. *Normalize k so that $0 \le k < r$.*
$\exists$ *projective morphism* $\widehat{\pi}\colon \widehat{M}(k, n, r) \to M_0(r, n - \frac{k(r-k)}{2r})$ *given by*

$$(E, \varphi) \mapsto \left(p_* E^{\vee\vee}, \varphi, \operatorname{Supp}(p_* E^{\vee\vee}/p_* E) + \operatorname{Supp}(R^1 p_* E)\right).$$

e.g. $k = 0$, $\widehat{\pi}$ is birational and an isom. on $p^{-1}(M_0^{\mathrm{reg}}(r, n))$.

Torus action on blowup

$\widetilde{T} \curvearrowright \widehat{M}(r,k,n)$

Proposition. $\widehat{M}(r,k,n)^{\widetilde{T}}$ *is parametrized by*

$$\{(\vec{k},\vec{Y}^1,\vec{Y}^2) \mid \sum k_\alpha = k, |\vec{Y}^1| + |\vec{Y}^2| + \frac{1}{2r}\sum_{\alpha<\beta}(k_\alpha - k_\beta)^2 = n\}$$

Proof. $(E,\varphi) = (I_1(k_1C),\varphi_1) \oplus \cdots \oplus (I_r(k_rC),\varphi_r)$ and I_α is an T^2-equivariant ideal.

$\Longrightarrow \mathcal{O}_{\widehat{\mathbb{C}}^2}/I_\alpha$ is supported at $\{p_1,p_2\} = (\widehat{\mathbb{C}}^2)^{\widetilde{T}}$ and corresponds to a pair of r-tuples of Young diagrams. □

Tangent space

The tangent space of the moduli space is given by the extension

$$\mathrm{Ext}^1(E,E(-\ell_\infty)) = \bigoplus_{\alpha,\beta}\mathrm{Ext}^1(I_\alpha(k_\alpha C), I_\beta(k_\beta C - \ell_\infty)).$$

We have $I_\alpha = I_\alpha^1 \cap I_\alpha^2$ (Supp$(\mathcal{O}/I_\alpha^a) = \{p_a\}$ with $a = 1,2$). Then

$$\begin{aligned}\mathrm{Ext}^1(E,E(-\ell_\infty)) &= H^1(\mathcal{O}((k_\beta - k_\alpha)C - \ell_\infty)) \\ &+ \mathcal{O}((k_\beta - k_\alpha)C)|_{p_1} \otimes \mathrm{Ext}^1(I_\alpha^1, I_\beta^1(-\ell_\infty)) \\ &+ \mathcal{O}((k_\beta - k_\alpha)C)|_{p_2} \otimes \mathrm{Ext}^1(I_\alpha^2, I_\beta^2(-\ell_\infty))\end{aligned}$$

These are the same as tangent space of $M(r,n)$ with shifts of variables $\varepsilon_1 \to \varepsilon_1 - \varepsilon_2$, $\varepsilon_2 \to \varepsilon_2 - \varepsilon_1$ resp.

μ-class

- $\mathcal{E} \to \widehat{\mathbb{P}}^2 \times \widehat{M}(r,k,n)$: (equivariant) universal sheaf
- $\mu(C) = \left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2\right) / [C] \in H^2_{\widetilde{T}}(\widehat{M}(r,k,n))$: (equivariant) μ-class

Proposition.

$$\begin{aligned}&\mu(C)|_{(\vec{k},\vec{Y}^1,\vec{Y}^2)} \\ &= |Y^1|\varepsilon_1 + |Y^2|\varepsilon_2 \\ &\qquad + \frac{1}{2r}\sum_{\alpha<\beta} 2(k_\alpha - k_\beta)(a_\alpha - a_\beta) + (k_\alpha - k_\beta)^2(\varepsilon_1 + \varepsilon_2)\end{aligned}$$

The blowup formula

Combining all these, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} \Lambda^{2rn} \int_{\widehat{M}(r,0,n)} \left(\exp(t\mu(C)) \cap [\widehat{M}(r,0,n)]\right) \\ &= \sum_{\vec{k}} \exp\left[t\left(\frac{1}{2r}\sum(\vec{k},\vec{a}) + \frac{(\vec{k},\vec{k})}{2}(\varepsilon_1+\varepsilon_2)\right)\right] \\ &\qquad \times \frac{\Lambda^{\frac{1}{2}(\vec{k},\vec{k})/4r}}{\prod_{\alpha,\beta} \mathrm{Euler}(e^{a_\beta - a_\alpha} H^1(\mathcal{O}((k_\beta - k_\alpha)C - \ell_\infty)))} \\ &\qquad \times Z^{\text{inst}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1\vec{k}; \Lambda e^{t\varepsilon_1/2r}) \\ &\qquad\qquad \times Z^{\text{inst}}(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2\vec{k}; \Lambda e^{t\varepsilon_2/2r}).\end{aligned}$$

where $\int_{\widehat{M}(r,0,n)} = \iota_{0*}^{-1}\widehat{\pi}_* =$ sum over the fixed points.

Dimension Gap

Proposition. $\widehat{\pi}_*(\mu(C)^d \cap [\widehat{M}(r,0,n)]) = 0$ *for* $1 \le d \le 2r-1$.

Proof. Note

$$\widehat{\pi}_*(\mu(C)^d \cap [\widehat{M}(r,0,n)]) \in H_{4rn-2d}(M_0(r,n)).$$

Let $S = \overline{\{0\} \times M_0^{\mathrm{reg}}(r,n-1))}$.

- $\operatorname{codim}_{\mathbb{C}} S = 2r$
$$\implies H_{4rn-2d}(M_0(r,n)) \cong H_{4rn-2d}(M_0(r,n) \setminus S).$$
- $\mu(C)$ is trivial on $\widehat{\pi}^{-1}(M_0(r,n) \setminus S)$.

□

Combining this vanishing with the blowup formula, we get the blowup equation !

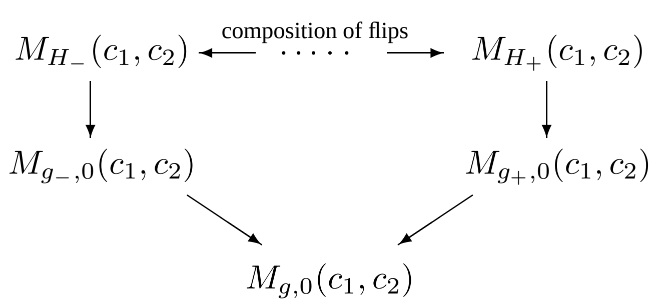
Wall-crossing term via Hilbert schemes

- X : projective surface with $b_+ = 1$, $\pi_1(X) = 1$

We use the alg-geometric definition of Donaldson invariants.

- $H \in W$, H_+ and H_- are separated by W.
- g, g_+, g_- : corr. Kähler metrics

Then



$$0 \to I_{Z_1}(\frac{c_1+\xi}{2}) \to E \to I_{Z_2}(\frac{c_1-\xi}{2}) \to 0 \quad \begin{pmatrix} Z_1 \in X^{[l]} \\ Z_2 \in X^{[m]} \end{pmatrix}$$

$$\overset{\text{Replaced}}{\Longrightarrow} \quad 0 \leftarrow I_{Z_1}(\frac{c_1+\xi}{2}) \leftarrow E' \leftarrow I_{Z_2}(\frac{c_1-\xi}{2}) \leftarrow 0$$

Wall-crossing term via Hilbert schemes – cont'd.

- Suppose W is good $\overset{\text{def.}}{\iff} |K_X + \xi'| = \emptyset$ for $\forall \xi'$ s.t. $W = W^{\xi'}$.
  $\Longrightarrow$ moduli sp's are smooth near the above E, E'

Fact (Friedman-Qin,Ellingsrud-Göttsche).

$$\delta^\xi(\exp(\alpha z + px)) = \Bigg[\sum_{n\geq 0} \Lambda^{4n-\xi^2-3} \times \int_{(X\sqcup X)^{[n]}} \frac{\exp\left(-\left[\mathrm{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + \mathrm{ch}(\mathcal{I}_2)e^{\frac{t-\xi}{2}}\right]_2 / (\alpha z + px)\right)}{c^t(\mathrm{Ext}^1_{\pi_2}(\mathcal{I}_1, \mathcal{I}_2(-\xi)) c^{-t}(\mathrm{Ext}^1_{\pi_2}(\mathcal{I}_2, \mathcal{I}_1(\xi))}\Bigg]_{t^{-1}},$$

where

- $\mathcal{I}_1, \mathcal{I}_2$ universal sheaf for $(X \sqcup X)^{[n]} = \bigsqcup_{l+m=n} X^{[l]} \times X^{[m]}$
- $c^t(E) = \sum c_i(E) t^{r-i}$ for $r = \mathrm{rank}\, E$

Universality of the formula

Let δ^ξ_t be the inside of $[\bullet]_{t^{-1}}$ in the formula.

Theorem (GNY). $\exists\ A_i \in \mathbb{Q}((t^{-1}))[[\Lambda]]$ ($i = 1, \ldots 8$) indep. of X, ξ s.t.

$$(-1)^{\chi(\mathcal{O}_X)+\xi(\xi-K_X)/2} t^{-\xi^2-2\chi(\mathcal{O}_X)} \Lambda^{\xi^2+3\chi(\mathcal{O}_X)} \delta^\xi_t(\exp(\alpha z + px))$$
$$= \exp\Big[\xi^2 A_1 + \xi c_1(X) A_2 + c_1(X)^2 A_3 + c_2(X) A_4 + z\alpha \cdot \xi A_5 + z\alpha \cdot c_1(X) A_6 + z^2\alpha^2 A_7 + x A_8\Big]$$

- To determine A_i, it is enough to compute δ^ξ_t for toric surfaces X.

Proof of the universality

- The proof is similar to

  Fact (Ellingsrud-Göttsche-Lehn). The complex cobordism class of $X^{[n]}$ depends only on the cpx cob. class of X.

- The essential point is to consider ($X_2 = X \sqcup X$, $\alpha = 1, 2$)

$$\begin{array}{ccccc} X & \xleftarrow{\rho_\alpha} & X_{2,\alpha}^{[n,n+1]} & \xrightarrow{\psi_\alpha} & X_2^{[n+1]} \\ & & \Big\downarrow \phi_\alpha & & \\ & & X_2^{[n]} & & \end{array}$$

where

$$X_{2,\alpha}^{[n,n+1]} = \left\{ (Z, Z') \in X_2^{[n]} \times X_2^{[n+1]} \,\middle|\, \begin{array}{l} Z \subset Z' \\ Z' \setminus Z \text{ is in } \alpha^{\text{th}}\text{-factor} \end{array} \right\}$$

- Intersection on $X_2^{[n+1]} \rightsquigarrow$ Intersection on $X_2^{[n]} \times X$

Main Result 2

X : projective toric surface with fixed points $p_1, \dots, p_\chi$

(x_i, y_i) : toric coord. around p_i with weights $w(x_i), w(y_i)$

ξ, α, x : take equivariant lifts $\widetilde{\delta}^\xi$: the equivariant lift of δ^ξ

Theorem (GNY).

$$\widetilde{\delta}^\xi(\exp(\alpha z + px))$$

disapper in non-equivariant limit

$$= \frac{1}{\Lambda}\Bigg[\exp\left(\frac{1}{2}\operatorname{Todd}_2(X)(\alpha z + px)\right) \times \prod_{i=1}^{\chi} Z(w(x_i), w(y_i), \frac{t - \iota_{p_i}^*\xi}{2}; \Lambda e^{\iota_{p_i}^*(\alpha z + px)/4})\Bigg]_{t^{-1}}$$

product over the fixed point set

It is easy to show

$$\widetilde{\delta}^{\xi}_t(\exp(\alpha z + px)) = \frac{\prod_i Z^{\text{inst}}(w(x_i), w(y_i), \frac{t-\iota^*_{p_i}\xi}{2}; \Lambda e^{\iota^*_{p_i}(\alpha z+px)/4})}{\Lambda^{\xi^2+3} c^{-t}(H^1(X,L))\, c^t(H^1(X,L^\vee))}.$$

Then we show that the denominator can be absorbed into the perturabation term.

Recall $\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = F_0 + (\varepsilon_1+\varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} B + \cdots$

Then $\prod_i Z(w(x_i), w(y_i), \frac{t-\iota^*_{p_i}(\xi)}{2}; \Lambda \exp(\frac{\iota^*_{p_i}(\alpha z + px)}{4}))$

$$\begin{aligned}
= \exp\Bigg[\sum_i \frac{1}{w(x_i)w(y_i)}\Bigg(& F_0 - \frac{\partial F_0}{\partial a}\frac{\iota^*_{p_i}(\xi)}{2} + \frac{\partial F_0}{\partial \log\Lambda}\frac{\iota^*_{p_i}(\alpha + px)}{4} + \frac{\partial^2 F_0}{(\partial a)^2}\frac{\iota^*_{p_i}(\xi)^2}{8} \\
&- \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\frac{\iota^*_{p_i}(\xi)\iota^*_{p_i}(\alpha z + px)}{8} + \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\frac{\iota^*_{p_i}(\alpha z + px)^2}{16} \\
&+ (w(x_i)+w(y_i))\left(H - \frac{\partial H}{\partial a}\frac{\iota^*_{p_i}(\xi)}{2}\right) \\
&+ w(x_i)w(y_i)A + \frac{w(x_i)^2 + w(y_i)^2}{3}B + \dots\Bigg)\Bigg]
\end{aligned}$$

Apply fixed point formula for X to this !

$$\begin{aligned}
= \exp\Bigg[& \frac{\partial F_0}{\partial \log\Lambda}\int_X \frac{x}{4} + \frac{\partial^2 F_0}{\partial a^2}\int_X \frac{\xi^2}{8} - \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\int_X \frac{\xi(\alpha z + px)}{8} \\
&+ \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\int_X \frac{(\alpha z + px)^2}{16} - \frac{\partial H}{\partial a}\int_X \frac{c_1(X)\xi}{8} + A\chi + B\sigma + \cdots\Bigg]
\end{aligned}$$

Substituting Main Result 1(2), we get Göttsche's formula:

$$\delta^\xi(\exp(\alpha z + px)) =$$
$$\sqrt{-1}^{\int_X \xi K_X - 1} \operatorname*{res}_{q=0}\left[q^{-\frac{1}{2}\int_X (\frac{\xi}{2})^2} \exp\left(\frac{du}{da} z \int_X \alpha \cup \frac{\xi}{2} + T z^2 \int_X \alpha^2 - ux\right) \times \left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right)^3 \theta_{01}^{\sigma+8} \frac{dq}{q}\right],$$

where

$$q = e^{2\pi\sqrt{-1}\tau}, \quad u = -\frac{\theta_{00}^4 + \theta_{10}^4}{\theta_{00}^2\theta_{10}^2}\Lambda^2, \quad \frac{du}{da} = \frac{2\sqrt{-1}\Lambda}{\theta_{00}\theta_{11}}, \quad T = \frac{1}{24}\left(\frac{du}{da}\right)^2 E_2 - \frac{u}{6}.$$

Generalizations & Problems # 1

Higher terms of Nekrasov partition function do *not* contribute to Donaldson invariants.
But .....
GNY's approach naturally *defines*

- local wall-crossing density s.t.
  - expressed in terms of the curvature
  - its integral over X gives the wall-crossing formula (toric case)

cf. local Atiyah-Singer index theorem via the heat equation approach

Problem. Justify the 'local density' via Donaldson invariants for *families*.

Generalizations & Problems # 2

Let $\vec{\tau} = (\tau_1, \tau_2, \cdots)$ be a vector of formal variables.
We consider the partition function with the higher Chern classes:

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda, \vec{\tau}) = \sum_{n=0}^{\infty} \Lambda^{2rn} \int_{M(r,n)} \exp\left(\sum_{p=1}^{\infty} \tau_p \operatorname{ch}_{p+1}(\mathcal{E})/[\mathbb{C}^2]\right)$$

The case $r = 1$, this $=$ the full GW invariants for $\mathbb{P}^1$.
For $r \geq 2$

Theorem (NY). $\left.\dfrac{\partial(\varepsilon_1\varepsilon_2 \log Z^{\text{inst}})}{\partial \tau_{p-1}}\right|_{\substack{\varepsilon_1=\varepsilon_2=0 \\ \vec{\tau}=0}}$ $(p = 2, \ldots, r)$ are essentially u_p in the SW curve.

Problem. (1) Justify $\operatorname{ch}_p(\mathcal{E})$ with $p \geq r+1$ in a diff. geom. way.
(2) Study this for $r \geq 2$.

Generalizations & Problems # 3

$\exists$ K-theoretic generalization of the instanton counting and alg-geom. def. of Donaldson invariants (holo. Euler characteristic).
In Nekrasov's conjecture - part 2, F_g is the GW inv. for a local toric CY, not its limit.

Problem. Define K-theoretic Donaldson inv. in a diff. geom. way.