Donaldson invariants of toric surfaces via instanton counting

Hiraku Nakajima

Department of Mathematics, Kyoto University

joint work with Lothar Göttsche and Kota Yoshioka

(work in progress)

Mathematical Aspects of String Theory
July 30, 2004, Kyoto Univ.

References for instanton counting

- Nekrasov : hep-th/0206161

  Instanton counting conjecture was proposed

- Nekrasov + Okounkov : hep-th/0306238
- N + Yoshioka : math.AG/0306198, math.AG/0311058

  Solved affirmatively by two groups independently

- Braverman : math.AG/0401409

  (affine) Whittaker modules

- .... and many others

Aim of this talk

Find the relation between

Nekrasov's partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

(= equivariant Donaldson invariants for $\mathbb{R}^4$)

$\longleftrightarrow$ usual Donaldson invariants for a cpt 4-mfd X

Aim of this talk

Find the relation between

Nekrasov's partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

(= equivariant Donaldson invariants for $\mathbb{R}^4$)

$\longleftrightarrow$ usual Donaldson invariants for a cpt 4-mfd X

Answer in Physics

$$\text{Donaldson inv.} \xrightarrow[t\to\infty]{(X,tg)} \text{Seiberg-Witten inv.} + \text{local contrib.}$$

$$\begin{tikzcd}
\text{Nekrasov part. func. } Z \arrow[d, "Z=\exp(\frac{F_0}{\varepsilon_1\varepsilon_2}+\dots)"'] & \text{wall-crossing formula} \\
\text{Seiberg-Witten prep. } F_0 \arrow[r, "\text{Moore-Witten}", "u\text{-plane integral}"'] & \text{Donaldson inv. for } b_+ = 1 \arrow[u, "\text{regularization of the integral}"']
\end{tikzcd}$$

Aim of this talk

Find the relation between

Nekrasov's partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

(= equivariant Donaldson invariants for $\mathbb{R}^4$)

$\longleftrightarrow$ usual Donaldson invariants for a cpt 4-mfd X

Answer in Physics / Answer by [GNY]

$$\text{Donaldson inv.} \xrightarrow[t\to\infty]{(X,tg)} \text{Seiberg-Witten inv.} + \text{local contrib.}$$

$$\begin{tikzcd}
\text{Nekrasov part. func. } Z \arrow[r, "\text{[GNY]}", "\text{Bott's formula}"'] \arrow[d, "Z=\exp(\frac{F_0}{\varepsilon_1\varepsilon_2}+\dots)"'] & \text{wall-crossing formula} \arrow[d, shift right, "\substack{\text{vanishing on}\\ \text{a chamber}}"'] \\
\text{Seiberg-Witten prep. } F_0 \arrow[r, "\text{Moore-Witten}", "u\text{-plane integral}"'] & \text{Donaldson inv. for } b_+ = 1 \arrow[u, shift right, "\substack{\text{regularization}\\ \text{of the integral}}"']
\end{tikzcd}$$

A naive question :

Why do not we calculate (equivariant) Donaldson invariants for toric surfaces directly, using Bott's formula (as in mirror computation by Givental) ?

A naive question :

Why do not we calculate (equivariant) Donaldson invariants for toric surfaces directly, using Bott's formula (as in mirror computation by Givental) ?

Nekrasov Yes, we can calculate ! Integrate over $\vec{a}$.

A naive question :

Why do not we calculate (equivariant) Donaldson invariants for toric surfaces directly, using Bott's formula (as in mirror computation by Givental) ?

Nekrasov Yes, we can calculate ! Integrate over $\vec{a}$.

GKN No ! The fixed points are not *isolated* in general.

A naive question :

Why do not we calculate (equivariant) Donaldson invariants for toric surfaces directly, using Bott's formula (as in mirror computation by Givental) ?

Nekrasov Yes, we can calculate ! Integrate over $\vec{a}$.

GKN No ! The fixed points are not *isolated* in general.

- Nekrasov must choose the contour of the integral....
- The wall-crossing formula is enough, as we have a vanishing theorem on a certain chamber.

NB. For $\mathbb{P}^2$ ($r = 2$), there is no wall and a direct computation is possible. $\Longrightarrow$ nontrivial identity on Z

Quick Review of Donaldson invariants

- (X, g) : cpt, oriented, simply-conn., Riem. 4-mfd
- $P \to X$: U(2)- (or SO(3)-)principal bundle
- $c_1 = c_1(P)$, $c_2 = c_2(P)$: Chern classes
- $M = M_g(c_1, c_2)$: moduli of instantons (Uhlenbeck cptfication)
- $\mathcal{E} \to X \times M$: universal bundle
- $\mu(\bullet) = \left(c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2\right)/\bullet \colon H_*(X) \to H^*(M)$

Let

$$\Phi^g_{c_1,c_2}(x) \overset{\text{def.}}{=} \int_M \exp\left(\mu(\alpha) + x\mu(p)\right), \quad \alpha \in H_2(X), p \in H_0(X)$$

- $b_2^+ > 1 \Longrightarrow$ independent of g
- $b_2^+ = 1 \Longrightarrow$ depend on g, but only on

$$\omega(g) \in H^2(X)^+/\mathbb{R}_{>0} = \{\omega \in H^2(X) \mid \omega^2 > 0\}/\mathbb{R}_{>0} = \mathcal{H} \sqcup (-\mathcal{H})$$

- where $\omega(g)$: self-dual harmonic form with $\|\omega(g)\| = 1$ unique up to sign ($\longleftrightarrow$ orientation of M)

Calculation of $\Phi^g_{c_1,c_2}$ was *difficult*........

- $b_2^+ > 1 \Longrightarrow$ independent of g
- $b_2^+ = 1 \Longrightarrow$ depend on g, but only on

$$\omega(g) \in H^2(X)^+/\mathbb{R}_{>0} = \{\omega \in H^2(X) \mid \omega^2 > 0\}/\mathbb{R}_{>0} = \mathcal{H} \sqcup (-\mathcal{H})$$

- where $\omega(g)$: self-dual harmonic form with $\|\omega(g)\| = 1$ unique up to sign ($\longleftrightarrow$ orientation of M)

Calculation of $\Phi^g_{c_1,c_2}$ was *difficult*........

1994 Donaldson invariants are determined by Seiberg-Witten invariants, which are much easier to calculate !

Wall crossing formula

- $W^\xi = \{\omega \in H^2(X)^+ \,|\, \xi \cdot \omega = 0\}$: wall defined by $\xi \in \frac{1}{2}H^2(X,\mathbb{Z})$
- $\xi = c_1(L) - \frac{1}{2}c_1 \longrightarrow$ a reducible (i.e., U(1)-) instanton L

  $\Longrightarrow$ a singularity of $M = M_{c_1,c_2}$
- $[L] \in M \Longrightarrow \xi - \frac{1}{2}c_1 \in H^2(X,\mathbb{Z})$, $4c_2 - c_1^2 + 3 \leq \xi^2 < 0$

  $\Longrightarrow$ # of walls are locally finite
- $\Phi^g_{c_1,c_2}$ is constant when $\omega(g)$ moves in a chamber $\mathcal{C}$: a connected component of $H^2(X)^+ \setminus \bigcup W^\xi$

Theorem (Kotschick-Morgan '94). $\exists \delta^{\xi}_{c_2}$ *s.t.*

$$\Phi^{g_1}_{c_1,c_2} - \Phi^{g_2}_{c_1,c_2} = 1^{C^2/8} \sum_{\xi} (-1)^{(\xi - C/2)C} \delta^{\xi}_{c_2}$$

Kotschick-Morgan conjecture : $\delta^{\xi}_{c_2}|_{\mathrm{Sym}\, H_2(X)}$ is

- a polynomial in ξ and the intersection form Q_X
- with coeff's depend only on ξ, c_2, homotopy type of X

Remark. If $c_1 \not\equiv 0$ (2), $\exists$ chamber $\mathcal{C}$ s.t. $\Phi^{\mathcal{C}}_{c_1,c_2} \equiv 0$.

If $c_1 \equiv 0$, $\exists$ a similar result (Göttsche-Zagier)

1995 Göttsche computed $\delta^\xi = \sum_{c_2} \delta^\xi_{c_2}$ explicitly in terms of modular forms, assuming KM conj.

1997 Moore-Witten : Derive Göttsche's formula from the u-plane integral

Our goal today :

δ^ξ can be expressed via Nekrasov's partition function

1995 Göttsche computed $\delta^\xi = \sum_{c_2} \delta^\xi_{c_2}$ explicitly in terms of modular forms, assuming KM conj.

1997 Moore-Witten : Derive Göttsche's formula from the u-plane integral

Our goal today :

δ^ξ can be expressed via Nekrasov's partition function

In fact, we only need F_0 (genus 0, Seiberg-Witten prepotential) and A, B (genus 1) parts. This reminds us Moore-Witten's computation. (**NB**. We must keep ε_1, ε_2 independent.)

Higher terms do *not* contribute to Donaldson invariants.

But .....

GNY's approach naturally *defines*

- local wall-crossing density s.t.
  - expressed in terms of the curvature
  - its integral over X gives the wall-crossing formula (toric case)

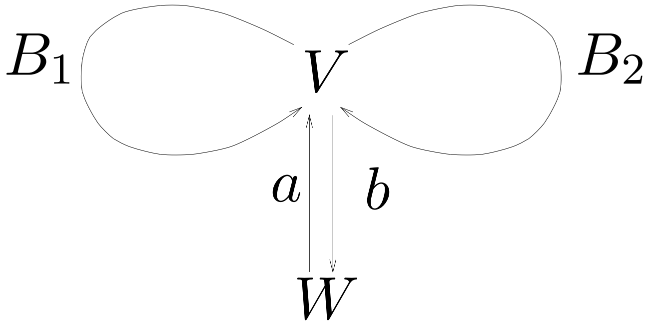
cf. local Atiyah-Singer index theorem via the heat equation approach

Problem. Justify the 'local density' via Donaldson invariants for *families*.

Framed moduli spaces of instantons on $\mathbb{R}^4$

- $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$. ($r = 2$ later)
- $M(n,r)$, $M_0(n,r)$: Gieseker/Uhlenbeck (partial) compactification of framed moduli spaces of SU(r)-instantons on $\mathbb{R}^4$
- These are quiver varieties for the Jordan quiver
  - $M(n,r)$: nonsingular hyperKähler manifold of dim. $4nr$ (a holomorphic symplectic manifold)
  - $M_0(n,r)$: affine algebraic variety
  - $\pi\colon M(n,r) \to M_0(n,r)$: projective morphism (resolution of singularities)

- V, W : cpx vector spaces with $\dim V = n$, $\dim W = r$
- $\mathbb{M}(n,r) = \operatorname{End} V \oplus \operatorname{End} V \oplus \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W)$
- $\mu\colon \mathbb{M}(n,r) \to \operatorname{End}(V)$; $\mu(B_1, B_2, a, b) = [B_1, B_2] + ab$



- $M_0(n,r) = \mu^{-1}(0) /\!\!/ \operatorname{GL}(V)$ (affine GIT quotient)
- $M(n,r) = \mu^{-1}(0)^{\text{stable}} / \operatorname{GL}(V)$
- stable $\overset{\text{def.}}{\iff}$ $\nexists S \subsetneq V$ with $B_\alpha(S) \subset S$, $\operatorname{Im} a \subset S$

Example $r = 1$: Hilbert scheme of points

Theorem. $M(n,1) = \mathrm{Hilb}^n(\mathbb{A}^2), \quad M_0(n,1) = S^n(\mathbb{A}^2)$

$\mathrm{Hilb}^n(\mathbb{A}^2)$: Hilbert scheme of n points in the affine plane $\mathbb{A}^2$

$S^n(\mathbb{A}^2)$: symmetric product (unordered n points with mult.)

Sketch of Proof

- $\mathrm{Hilb}^n(\mathbb{A}^2) = \{I \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n\}$
- Set $V = \mathbb{C}[x,y]/I$
  $B_1, B_2 = \times x, \times y$, $a(1) = 1 \mod I$, $b = 0$

○ $S^n(\mathbb{A}^2) \to M_0(n,1)$ is induced by $\mathbb{A}^{2n} \to \mathbb{M}(n,1)$:
$(B_1, B_2, a, b) = (\mathrm{diag}(x_1, \ldots, x_n), \mathrm{diag}(y_1, \ldots, y_n), 0, 0)$

Torus action and equivariant homology group

- $T = T^{r-1}$: maximal torus in $\mathrm{SL}(W)$
- $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \curvearrowright M(n,r), M_0(n,r)$: torus action
  $(B_1, B_2, a, b) \longmapsto (t_1 B_1, t_2 B_2, a e^{-1}, t_1 t_2 e b)$
  $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, $e \in T$
- $H_*^{\widetilde{T}}(M(r,n))$, $H_*^{\widetilde{T}}(M_0(r,n))$: equivariant (Borel-Moore) homology groups
- modules over S : symmetric power of $\mathrm{Lie}(\widetilde{T})^* = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_\alpha] = H^*_{\widetilde{T}}(\mathrm{pt})$ $(\sum a_\alpha = 0)$
- $[M(r,n)]$, $[M_0(r,n)]$: fundamental classes
- $\mathfrak{S}$: quotient field of S

Theorem (Localization). *Let* ι_0 *be the inclusion of the fixed point set* $M_0(n,r)^{\widetilde{T}}$ *in* $M_0(n,r)$. *Then*

$$H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathfrak{S} \xleftarrow[\cong]{\iota_{0*}} H_*^{\widetilde{T}}(M_0(n,r)^{\widetilde{T}}) \otimes_S \mathfrak{S}.$$

The same holds for $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$.

Observation. $M_0(n,r)^{\widetilde{T}} = \{0\}$, *so* $\mathrm{RHS} = \mathfrak{S}$.

Theorem (Localization). *Let* ι_0 *be the inclusion of the fixed point set* $M_0(n,r)^{\widetilde{T}}$ *in* $M_0(n,r)$*. Then*

$$H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathfrak{S} \underset{\cong}{\xleftarrow{\iota_{0*}}} H_*^{\widetilde{T}}(M_0(n,r)^{\widetilde{T}}) \otimes_S \mathfrak{S}.$$

The same holds for $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$*.*

Observation. $M_0(n,r)^{\widetilde{T}} = \{0\}$*, so* RHS $= \mathfrak{S}$*.*

Define

$$\begin{aligned} Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) &= \sum_{n=0}^{\infty} \Lambda^{2nr}(\iota_{0*})^{-1}[M_0(n,r)] \\ &= \sum_{n=0}^{\infty} \Lambda^{2nr}(\iota_{0*})^{-1}\pi_*[M(n,r)] \end{aligned}$$

(**instanton part of Nekrasov's partition function**)

- On the other hand, $M(n,r)^{\widetilde{T}}$ is a finite set, parametrized by r-tuple of Young diagrams $\{\vec{Y} = (Y_1, \dots, Y_r)\}$ (explained below)

- On the other hand, $M(n,r)^{\widetilde{T}}$ is a finite set, parametrized by r-tuple of Young diagrams $\{\vec{Y} = (Y_1, \dots, Y_r)\}$ (explained below)
- $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$: inclusion

$\Longrightarrow$

$$\begin{tikzcd}
{[M(n,r)] \in H_*^{\widetilde{T}}(M(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_*)^{-1}"'] \arrow[d, "\pi_*"'] & \bigoplus_{\vec{Y}} \mathcal{S} \arrow[d, "\sum_{\vec{Y}}"] \\
{[M_0(n,r)] \in H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_{0*})^{-1}"'] & \mathcal{S}
\end{tikzcd}$$

- On the other hand, $M(n,r)^{\widetilde{T}}$ is a finite set, parametrized by r-tuple of Young diagrams $\{\vec{Y} = (Y_1, \dots, Y_r)\}$ (explained below)
- $\iota \colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$: inclusion

$\Longrightarrow$

$$\begin{tikzcd}
{[M(n,r)] \in H_*^{\widetilde{T}}(M(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_*)^{-1}"'] \arrow[d, "\pi_*"'] & \bigoplus_{\vec{Y}} \mathcal{S} \arrow[d, "\sum_{\vec{Y}}"] \\
{[M_0(n,r)] \in H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_{0*})^{-1}"'] & \mathcal{S}
\end{tikzcd}$$

As $M(n,r)$ is smooth, we have an explicit formula:

$$(\iota_*)^{-1}[M(n,r)] = \bigoplus_{\vec{Y}} \frac{1}{\operatorname{Euler}(T_{\vec{Y}})}$$

where

- $\operatorname{Euler}(T_{\vec{Y}})$: equivariant Euler class of $T_{\vec{Y}} \in H^*_{\widetilde{T}}(\{\vec{Y}\})$

Fixed point set $M(n,r)^{\widetilde{T}}$

- $[B_1, B_2, a, b] \in M(n,r)$ is fixed by the first factor $T = T^{r-1}$ $\Longleftrightarrow$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$

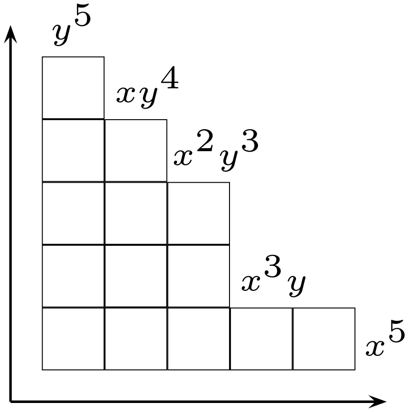
  ($\because W$ decomposes into 1-dim rep's of T)

Fixed point set $M(n,r)^{\widetilde{T}}$

- $[B_1, B_2, a, b] \in M(n,r)$ is fixed by the first factor $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$

  ($\because$ W decomposes into 1-dim rep's of T)

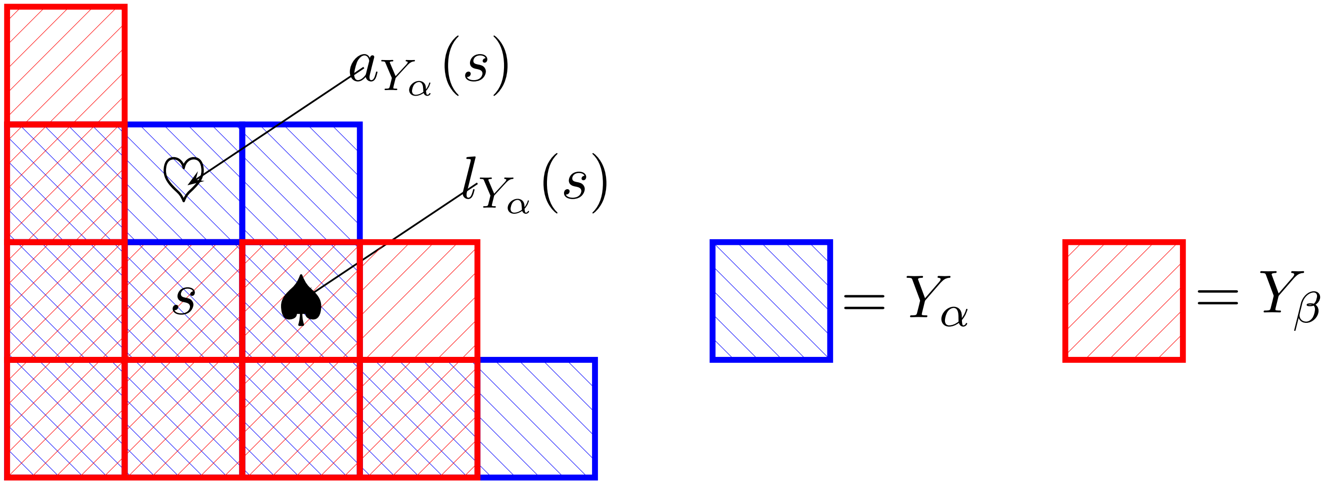
- $M(n_\alpha, 1) = \mathrm{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\iff$ I_α is generated by monomials in x, y
  $\iff$ I_α corresponds to a Young diagram Y_α



- $M(n,r)^{\widetilde{T}} \cong \{\vec{Y} = (Y_1, \ldots, Y_r) \mid \sum |Y_\alpha| = n\}$
- formula for the equivariant Euler class of the tangent space:

$$\begin{aligned}
&\operatorname{Euler}(T_{\vec{Y}}) \\
&\quad = \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} \left(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha\right) \\
&\qquad \times \prod_{t \in Y_\beta} \left((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha\right)
\end{aligned}$$

where



Example

Let $r = 1$. Put $\varepsilon_1 = -\varepsilon_2$. We have

$$(\iota_{0*})^{-1}[M_0(n,1)] = \sum_{|Y|=n} (-\frac{1}{\varepsilon_1})^{2|Y|} \prod_{s\in Y} \frac{1}{h(s)^2}.$$

The hook length formula says

$$\prod_{s\in Y} \frac{1}{h(s)} = \frac{\dim R_Y}{n!},$$

where R_Y is the irreducible representation of S_n associated with Y. Note

$$\sum_{|Y|=n} \dim R_Y^2 = n!$$

Therefore

$$(\iota_{0*})^{-1}[M_0(n,1)] = \frac{1}{n!}(-\frac{1}{\varepsilon_1^2})^n.$$

This can be proven directly by Bott's formula for orbifolds.

Perturbation Part

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda) \overset{\text{def.}}{=} \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{\varepsilon_1 t}-1)(e^{\varepsilon_2 t}-1)}.$$

$$Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \overset{\text{def.}}{=} \exp\left(-\sum_{\alpha\neq\beta} \widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta;\Lambda)\right)$$

Define the full partition function by

$$Z(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) \overset{\text{def.}}{=} Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda) Z^{\text{inst}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda).$$

Main Result

- X : projective toric surface
- $X^{\mathbb{C}^* \times \mathbb{C}^*} = \{p_i\}$: fixed point set
- (x_i, y_i) : toric coordinate around p_i
- $w(x_i)$, $w(y_i)$: weights of the $\mathbb{C}^* \times \mathbb{C}^*$-action (linear functions in ε_1, ε_2)
- $\iota_{p_i} \colon \{p_i\} \to X$: inclusion
- $\alpha, p \in H_*^{\mathbb{C}^* \times \mathbb{C}^*}(X)$, $\xi \in H^2_{\mathbb{C}^* \times \mathbb{C}^*}(X)$: equivariant lifts of the previous $\alpha, p \in H_*(X)$, $\xi \in H^2(X)$.

The wall crossing term δ_ξ has a natural equivariant lift, as moduli spaces have $\mathbb{C}^* \times \mathbb{C}^*$-action and the universal bundle is equivariant. (**equivariant Donaldson invariants**)

Theorem. The *equivariant* wall-crossing term is given by

$$\delta_\xi\left(\exp(\alpha+px)\right) = \operatorname*{res}_{t=0}\Bigg[\frac{dt}{t^2}(-1)^{\chi(L)}$$

$$\times \exp\left(\frac{1}{24}\Big(\langle\alpha, c_1(X)^2 + c_2(X)\rangle + x\langle p, c_1^2(X) + c_2(X)\rangle\Big)\right)$$

$$\times \prod_i Z(w(x_i), w(y_i), -\frac{1}{2}\left(\frac{1}{t} - \iota_{p_i}^*\xi\right)\Lambda \exp(\frac{1}{4}\iota_{p_i}^*(\alpha+px)))\Bigg]$$

'Almost' immediate from Ellingsrud-Göttsche's result:

wall-crossing formula in terms of Hilbert schemes

Recover Göttsche's formula ....

Expand as

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1,\varepsilon_2,\vec{a};\mathfrak{q})$$

$$= F_0 + (\varepsilon_1+\varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2+\varepsilon_2^2}{3}B + \cdots$$

Recover Göttsche's formula ....

Expand as

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q})$$

$$= F_0 + (\varepsilon_1 + \varepsilon_2)H + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B + \cdots$$

(1) no pole

(2) constant term F_0 = **Seiberg-Witten prepotential**

(3) H comes only from the perturbation part

(4) A, B (genus 1 part) can be written in terms of SW curves ($r = 2$)

(1), (2) = Nekrasov's conjecture : [NY, NO]

(3), (4) : [NY]

Plug the expansion into the equivariant wall-crossing term:

$$\prod_i Z(w(x_i), w(y_i), -\frac{1}{2}(\frac{1}{t} - \iota_{p_i}^*(\xi)); \Lambda \exp(\frac{1}{4}\iota_{p_i}^*(\alpha + xp)))$$

Plug the expansion into the equivariant wall-crossing term:

$$\prod_i Z(w(x_i), w(y_i), -\frac{1}{2}(\frac{1}{t} - \iota_{p_i}^*(\xi)); \Lambda \exp(\frac{1}{4}\iota_{p_i}^*(\alpha + xp)))$$

$$= \exp\Bigg[\sum_i \frac{1}{w(x_i)w(y_i)}\Big(F_0 + \frac{\partial F_0}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2} + \frac{\partial F_0}{\partial \log\Lambda}\frac{\iota_{p_i}^*(\alpha + px)}{4}$$

$$+\frac{\partial^2 F_0}{(\partial a)^2}\frac{\iota_{p_i}^*(\xi)^2}{8} + \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\frac{\iota_{p_i}^*(\xi)\iota_{p_i}^*(\alpha + px)}{8} + \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\frac{\iota_{p_i}^*(\alpha + px)^2}{16}$$

$$+ (w(x_i) + w(y_i))\left(H + \frac{\partial H}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2}\right)$$

$$+w(x_i)w(y_i)A + \frac{w(x_i)^2 + w(y_i)^2}{3}B + \dots\Big)\Bigg]$$

Plug the expansion into the equivariant wall-crossing term:

$$\prod_i Z(w(x_i), w(y_i), -\frac{1}{2}(\frac{1}{t} - \iota_{p_i}^*(\xi)); \Lambda \exp(\frac{1}{4}\iota_{p_i}^*(\alpha + xp)))$$

$$= \exp\Bigg[\sum_i \frac{1}{w(x_i)w(y_i)}\Bigg(F_0 + \frac{\partial F_0}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2} + \frac{\partial F_0}{\partial \log\Lambda}\frac{\iota_{p_i}^*(\alpha + px)}{4}$$

$$+\frac{\partial^2 F_0}{(\partial a)^2}\frac{\iota_{p_i}^*(\xi)^2}{8} + \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\frac{\iota_{p_i}^*(\xi)\iota_{p_i}^*(\alpha + px)}{8} + \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\frac{\iota_{p_i}^*(\alpha + px)^2}{16}$$

$$+(w(x_i) + w(y_i))\left(H + \frac{\partial H}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2}\right)$$

$$+w(x_i)w(y_i)A + \frac{w(x_i)^2 + w(y_i)^2}{3}B + \dots\Bigg)\Bigg]$$

Apply Bott's formula for X to this !

Plug the expansion into the equivariant wall-crossing term:

$$\prod_i Z(w(x_i), w(y_i), -\frac{1}{2}(\frac{1}{t} - \iota_{p_i}^*(\xi)); \Lambda \exp(\frac{1}{4}\iota_{p_i}^*(\alpha + xp)))$$

$$= \exp\Bigg[\sum_i \frac{1}{w(x_i)w(y_i)}\Bigg(F_0 + \frac{\partial F_0}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2} + \frac{\partial F_0}{\partial \log\Lambda}\frac{\iota_{p_i}^*(\alpha + px)}{4}$$

$$+\frac{\partial^2 F_0}{(\partial a)^2}\frac{\iota_{p_i}^*(\xi)^2}{8} + \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\frac{\iota_{p_i}^*(\xi)\iota_{p_i}^*(\alpha + px)}{8} + \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\frac{\iota_{p_i}^*(\alpha + px)^2}{16}$$

$$+ (w(x_i) + w(y_i))\left(H + \frac{\partial H}{\partial a}\frac{\iota_{p_i}^*(\xi)}{2}\right)$$

$$+w(x_i)w(y_i)A + \frac{w(x_i)^2 + w(y_i)^2}{3}B + \dots\Bigg)\Bigg]$$

Apply Bott's formula for X to this !

$$= \exp\Bigg[\frac{\partial F_0}{\partial \log\Lambda}\frac{x}{4} + \frac{\partial^2 F_0}{\partial a^2}\int_X \frac{\xi^2}{8} + \frac{\partial^2 F_0}{\partial a \partial \log\Lambda}\int_X \frac{\xi(\alpha + px)}{8}$$

$$+ \frac{\partial^2 F_0}{(\partial \log\Lambda)^2}\int_X \frac{(\alpha + px)^2}{16} + \frac{\partial H}{\partial a}\int_X \frac{c_1(X)\xi}{8} + A\chi + B\sigma + \cdots\Bigg]$$

Seiberg-Witten geometry

A family of curves (*Seiberg-Witten curves*) parametrized by $\vec{u} = (u_2, \ldots, u_r)$:

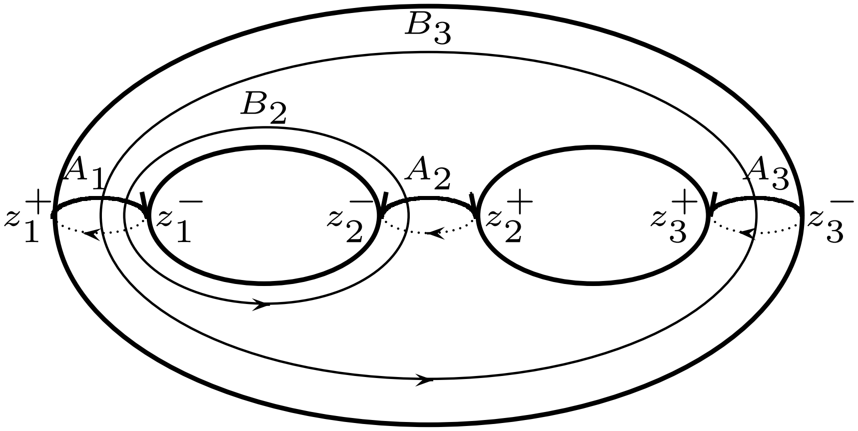
$$C_{\vec{u}} : y^2 = P(z)^2 - 4\Lambda^{2r}, \qquad P(z) = z^r + u_2 z^{r-2} + \cdots + u_r.$$

$C_{\vec{u}} \ni (y, z) \mapsto z \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(y, z) = (-y, z)$.

Define the *Seiberg-Witten differential* (multivalued) by

$$dS = -\frac{1}{2\pi} \frac{zP'(z)dz}{y}.$$

Find branched points $z_\alpha^\pm$ near z_α (roots of $P(z) = 0$) (Λ small).
Choose cycles A_α, B_α ($\alpha = 2, \ldots, r$) as



Put

$$a_\alpha = \int_{A_\alpha} dS, \qquad a_\beta^D = 2\pi\sqrt{-1}\int_{B_\beta} dS$$

Then

$$\exists F_0 : \quad a_\beta^D = -\frac{\partial F_0}{\partial a_\beta}$$

Suppose $r = 2$. Let τ be the period of the Seiberg-Witten (elliptic curve).

$$\frac{1}{4}\frac{\partial F_0}{\partial \log \Lambda} = -u, \quad \frac{1}{4}\frac{\partial^2 F_0}{\partial \log \Lambda \partial a} = -\frac{\partial u}{\partial a}$$

$$\frac{1}{16}\frac{\partial^2 F_0}{(\partial \log \Lambda)^2} = \frac{1}{24}\left(\frac{du}{da}\right)^2 E_2(\tau) - \frac{1}{6}u \quad \text{(contact term)}$$

$$\frac{\partial^2 F_0}{\partial a^2} = -2\pi\sqrt{-1}\tau$$

$$A = \frac{1}{2}\log\left(\frac{\sqrt{-1}}{\Lambda}\frac{du}{da}\right), \quad B = \frac{1}{8}\log\left(\frac{4(u^2 - 4\Lambda^4)}{\Lambda^4}\right).$$